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Modeling and Analysis of Communication Systems Based on Computational Methods for Markov Chains

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Abstract—**We describe a computational approach for modeling and analyzing modern communication systems based on numerical methods for Markov chains. Advanced direct and iterative procedures for the calculation of the stationary distribution of a homogeneous discrete- or continuous-time Markov chain with finite state space are presented. They are implemented in a convenient software tool for interactive modeling and performance evaluation of communication systems. The versatility of these algorithms is illustrated by their application to Markovian queueing models derived from telecommunication networks.**

I. Introduction

MODELING of communication systems by means of homogeneous discrete- or continuous-time Markov chains is a current technique that has grown out of Erlang's pioneering work [13] in this field. Erlang's investigations have initiated a rapid development of probability theory and its applications.

Recently, considerable attention has been devoted to a new branch of this theory, namely, the analysis of Markov chains by numerical methods (cf. [1]). From a probabilistic point of view, the problem of computing the steady-state distribution of an ergodic homogeneous Markov chain has already been solved since the stationary distribution may be calculated by the solution of a system of linear equations called steady-state balance equations. Therefore, it is not very attractive to probability theorists any more. In practice, however, the application of the theory of Markov chains to the field of modeling and performance analysis of telecommunication systems raises a lot of difficulties. Considering Markovian models of practical interest, analytic solutions of the steady-state distributions of the Markov chains are rarely available.

On the other hand, nowadays we have to cope with the investigation of complex, distributed technical systems such as telecommunication networks within an ISDN environment or SPC exchanges. Although Markov modeling techniques are still useful, the construction of explicit analytical solutions of the relevant model characteristics such as steady-state distributions are normally not possible any more. Therefore, numerical solution methods for Markov chains are the only feasible approach besides simulation which is a rather expensive method.

Regarding the successful development of Markovian techniques in the past, we are convinced that the Markovian approach still deserves our attention, despite all difficulties concerning its application in practice. Moreover, the software techniques for the construction of modeling tools are available at present, and numerical methods for computing either transient or stationary characteristics of a model are provided to an abundant extent by numerical mathematics. Furthermore, the knowledge of principles applied to the decomposition and aggregation of models has been improved. Last but not least, an adequate hardware and software environment is now available on the desks of teletraffic engineers and network designers. Workstations with graphical interfaces provide the basis for the interactive construction of models and the graphical representation of the calculated results.

From a practical point of view, it is desirable that numerical techniques for Markov chains are supported by convenient software tools which facilitate the use of the various algorithms. Although there are several software packages such as NUMAS or QNAP2 which offer numerical solution techniques for Markovian queueing models, these software tools are often based on old-fashioned methods such as the deflation approach (cf. [88], [90], [89], [16], [125]).

We intended to overcome this deficit. For this reason, a software package called MACOM was developed by the Deutsche Bundespost Telekom and the Computer Science Department at the University of Dortmund (cf. [91], [92]). Its capabilities are tailored to the requirements arising from the performance evaluation of modern communication systems. The only analysis techniques employed are efficient numerical procedures for the solution of finite Markov chains. The package can cope with large models comprising up to 100 000 states. It is implemented in C and running on a SUN3 workstation.

MACOM provides the user with a predefined Markovian model world describing modern telecommunication networks with adaptive routing schemes and advanced congestion-control mechanisms. The structure of

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its modeling components guarantees the Markovian nature of a specified model. The basic elements consist of links, specific routing elements, called conexes, offering probabilistic and state-dependent routing, queueing stations with multiple, homogeneous servers and queues with finite capacities, different service disciplines (infinite server, processor sharing, random, nonpreemptive priority), and general Coxian service-time distributions, as well as versatile arrival streams such as phase-type renewal and Markov-modulated Poisson processes. Hence, the resulting class of queueing models includes Markovian queueing networks with state-dependent routing which do not possess a steady-state distribution of product form.

The model world of MACOM is provided by means of a graphical user interface, i.e., MACOM models are specified by a graphical editor (see Fig. 1). The resulting network graph is interactively constructed from predefined elements, more specifically, sources, sinks, loss exits, stations, conexes, and links (see Fig. 2). Moreover, further textual information is added to the model elements by means of pop-up menus which are activated by a mouse. The text comprises the types of the distributions (Coxian, Erlang, hyperexponential, exponential) of the service and arrival process and the parameters of a station, i.e., its name, the service discipline, the number of processors, the service speed, which may depend on the population, and the capacity of the station defined as the number of customers which can be held simultaneously. Furthermore, the attributes of the routing elements have to be described.

Advanced direct and iterative methods for the computation of the steady-state vector of a continuous-time Markov chain (CTMC) (cf. [45], [35], [57], [7], [71]) constitute the numerical solver of MACOM. Furthermore, the package offers the evaluation of different performance measures of the constructed models which are specified by the user.

The tool MACOM combines state-of-the-art techniques for systems modeling and analysis by finite Markov chains, software methods for the specification of models by graphical means and for the enhanced support of the user by menu techniques, and the advanced technology of modern workstations (cf. [109], [110], [92]).

In this paper we present the advanced numerical methods which are employed for the analysis of the constructed Markovian models in the tool MACOM. We feel that there is no elementary, modern guide on computational methods for Markov chains and their applications. Therefore, it is our main objective to summarize some recommended algorithms from the literature within a unifying mathematical framework and to illustrate their application to several models derived from actual telecommunication networks. This paper does not claim to present new mathematical results or to give a comprehensive survey of the issue. Readers who are interested in the mathematical background of the proposed algorithms are referred to Barker's excellent survey [5] (see also [71], [99]). Although this paper's references cannot be exhaustive, it is comprehensive enough to provide sources from the literature for further study of the subject.

The paper is organized as follows: Section II provides the mathematical basis for calculating the steady-state distribution of a finite Markov chain and a classification of computational methods. In Section III we present the most important direct methods. Section IV is concerned with advanced point and block iterative procedures based on matrix splittings for singular M-matrices. Last but not least, the proposed methods are applied to some teletraffic models arising from the analysis of modern telecommunication networks.

II. The Mathematical Background of Computational Methods

In practice, telecommunication system behavior is often described by Markovian models with discrete valued, finite state spaces $S = \{1, 2, \cdots, n\}$ (see Section V and [68], [57], [83]). It is a major objective to calculate the steady-state distributions of such discrete- or continuous-time Markov chains. For this purpose, computational solution methods must be employed if the models do not belong to special classes such as product-form networks (cf. [60]) or matrix-geometric models (cf. [94]) since analytical solutions are rarely available.

In this section, we provide the mathematical background of these computational methods. It is based on a unifying mathematical framework derived from the theory of nonnegative matrices and M-matrices (cf. [9]). Furthermore, we show that, from the computational point of view, discrete- and continuous-time Markov chains may be treated in the same way.

We assume the reader to be familiar with the theory of homogeneous discrete- and continuous-time Markov chains with finite state spaces, abbreviated DTMC and CTMC, respectively, to the extent of the books of Heyman and Sobel [54, ch. 7, 8] and Kemeny and Snell [62]. Furthermore, we shall adopt the terminology of Heyman and Sobel and we use the following notation of Berman and Plemmons [9, ch. 2, p. 26] w.r.t. vector and matrix orderings: Let $x \in \mathbb{R}^n$, then $x >> 0 \Leftrightarrow x_i > 0$ for each $i \in \{1, \cdots, n\}$, $x > 0 \Leftrightarrow x_i \geq 0$ for each $i \in \{1, \cdots, n\}$ and $x_j > 0$ for some $j \in \{1, \cdots, n\}$, $x \geq 0 \Leftrightarrow x_i \geq 0$ for each $i \in \{1, \cdots, n\}$.

For an irreducible DTMC $\{Y_n, n \geq 0\}$ with a *finite* state space $S = \{1, 2, \cdots, n\}$ and an irreducible transition probability matrix (t.p.m.) P, a solution of the stated problem is provided by Markov's well-known theorem (cf. [54, corollary 7-4, p. 231], [24, corollary 2.11, p. 153]). The stationary distribution π is the unique solution of the linear system

$$z^t = z^t P, \qquad \sum_{i \in S} z_i = 1, \qquad z_i \geq 0 \qquad \text{for all } i \in S. \tag{2.1}$$

Considering an irreducible CTMC $\{X(t), t \geq 0\}$ with finite state space S and the conservative generator matrix

$Q = (q_{ij})$, its steady-state distribution p is determined by the solution of the linear system

$$x'Q = 0, \quad \sum_{i \in S} x_i = 1, \quad x_i \geq 0 \quad \text{for all } i \in S \tag{2.2}$$

(cf. [54, theorem 8-6, p. 304]).

Moreover, it is well known that the existence of these steady-state distributions can be proved by algebraic results, only (cf. [38, §13.7, 431ff], [9, theorem 8.3.11]). In this context, a major mathematical tool is provided by the famous theorem of Perron–Frobenius for nonnegative matrices (cf. [9, ch. 2], [38, §13.2, p. 397f], [112, theorem 1.5, p. 20], [24, appendix, sec. 4], [123, theorem 2.1, p. 30]). This set of nonnegative matrices, especially the subset of probability matrices, is very important in computational probability theory.

It is well known that under appropriate regularity conditions (i.e., for conservative, uniformizable MC's—cf. [54, §8-7]) homogeneous, continuous-time and discrete-time Markov chains with denumerable state spaces satisfy a close stochastic relation. There is a natural way to associate an embedded DTMC $\{Y_n, n \geq 0\}$, called the *jump chain*, with each standard CTMC possessing a generator matrix Q. This transformation of a CTMC into its associated embedded DTMC is called *uniformization* or *randomization* (cf. [56, p. 228], [60, p. 3], [54, pp. 290, 310]). It is obtained from the CTMC by observing its states only immediately after a state transition. Thus the transition probabilities $p_{ij} = P(Y_{n+1} = j \mid Y_n = i)$ of Y_n are just the conditional probabilities of a transition of $X(t)$ from state i to state j given that a transition occurs, i.e.,

$$p_{ij} = \begin{cases} q_{ij}/q_i & i \neq j \\ 0 & i = j \end{cases}$$

where $q_i = -q_{ii} = \Sigma_{j:j \neq i} q_{ij}$. Hence, the t.p.m. $P = (p_{ij})$ of $\{Y_n\}$ is defined by

$$P = D^{-1} \cdot B \tag{2.3}$$

where $Q = (q_{ij}) = -D + B$ is a decomposition into the negative diagonal part of Q, $D = -\text{diag}\,(q_{ii})_{i=1,\cdots,n} = \text{diag}\,(q_i)_{i=1,\cdots,n}$, and the nonnegative off-diagonal elements B. Here, we introduce the convention that diag $(x) = D$ generates a diagonal matrix from a vector x by setting $D_{ii} = x_i$. Moreover, let e denote the vector of all ones.

If z is the stationary distribution of the jump chain satisfying $e'D^{-1}z = \Sigma_j z_j/q_j$, we conclude from (2.3) that $p = D^{-1}z/(e'D^{-1}z)$ is a probability vector for which the relation $0 = p'Q = p'(B - D) = z'(D^{-1}B - I)/(e'D^{-1}z)$ holds.

If the parameters q_j of the exponential sojourn times in the states do not depend on j, i.e., $q_j = \lambda$, then the jump epochs of the embedded chain form a Poisson process with rate λ and the equilibrium distribution of the jump chain coincides with the steady-state vector of the CTMC (cf. [54, theorem 8-5, p. 303]).

A unifying mathematical framework of computational methods for Markov chains is based on the theory of M-matrices (cf. [9, ch. 6]). A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called *M-matrix* if $a_{ij} \leq 0$ holds for all i, j with $i \neq j$ and if A can be represented in the form $A = sI - B$, $s > 0$, $B \geq 0$, $s \geq \rho(B)$, where $\rho(B)$ is the spectral radius of B and I denotes the identity matrix. A is a regular M-matrix if $s > \rho(B)$ holds and a singular M-matrix for $s = \rho(B)$. An important subset of the class of singular M-matrices is given by the *Q-matrices*. A Q-matrix $Q = (q_{ij}) \in \mathbb{R}^{n \times n}$ satisfies the conditions $q_{ij} \leq 0$ for all i, j with $i \neq j$ and $\Sigma_{i=1}^{n} q_{ij} = 0$ for all j with $1 \leq j \leq n$. Thus the negative transpose $A = -Q'$ of the generator matrix Q of a CTMC is a Q-matrix (cf. [101]). Furthermore, the matrix $A = I - P'$ is also a singular M-matrix with zero column sums, i.e., a Q-matrix, if P is the t.p.m. of a DTMC.

It is our objective to calculate the steady-state vector of an irreducible Markov chain with finite state space. As we have already seen, it is sufficient to study a system of linear equations of the form

$$A \cdot x = 0, \quad A \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n \tag{2.4}$$

where A is an *irreducible Q-matrix*. It is necessary to compute a positive solution x of (2.4) that satisfies the additional normalization condition

$$e' \cdot x = \sum_{i=1}^{n} x_i = 1 \tag{2.5}$$

where e denotes the vector of all ones. If A is reducible, the problem may be reduced to this case by transforming A into the well-known block-triangular canonical form (cf. [9, p. 39], [38, §13.4]). From the mathematical theory we know that the problem (2.4), (2.5) has a unique positive solution (cf. [9, theorem 6.4.16, p. 156]). Furthermore, it can be shown that $\lambda = 0$ is a simple eigenvalue of A with one-dimensional eigenspaces spanned by a positive right and left eigenvector, respectively. In the following, unless otherwise stated, we assume A to be an irreducible Q-matrix.

A simple solution technique for the singular system (2.4) is provided by the transformation into a regular system of linear equations. An approach frequently used is the substitution of the normalization condition (2.5) for the last equation of (2.4) which yields an inhomogeneous system $A' \cdot x = (0, \cdots, 0, 1)'$ with a unique positive solution x. An alternative method is given by a deflation technique proposed by Kemeny [61] (see also [62, VIII, p. 211 ff], [51]). It is a rank-1 modification that may be applied to any matrix A with a simple eigenvalue 0.

Regarding the issue of calculating the steady-state distribution of a Markov chain, computational methods proposed in the literature may be classified according to the representation of the problem and the numerical technique applied for its solution. We divide the procedures into the following categories:

TABLE I
OVERVIEW OF COMPUTATIONAL METHODS

Category	Methods	Algorithms	References
Direct methods	Gaussian elimination techniques	Crout algorithm	[43, p. 58]
		LU decomposition	[34], [37], [36],
		for singular M-matrices	[9, Cor. 6.4.17, p. 157]
		Grammann's algorithm	[45], [44], [113]
	rank reduction techniques	block LU algorithms	[67], [39], [49], [43, Sec. 5.5], [72], [104], [41, §12.4], [128]
		blockwise bordering	[41, §12.3]
	deflation techniques		[61], [60], [97], [51]
	QR method		[43, Sec. 7.5, p.228], [42]
	inverse iteration		[57], [58], [116]
Iterative methods	procedures based on matrix splittings for singular M-matrices and their accelerated variants including semi-iterative methods	Jacobi procedure	[6], [57], [87]
		Gauss-Seidel procedure	[6], [57], [117]
		JOR	[14]
		SOR	[117], [46]
		SSOR	[31]
		block SOR	[57], [58]
		A/D algorithms	[22], [18], [66], [106], [108], [70]
		Chebyshev algorithm	[29], [57], [123], [47]
	conjugate gradient techniques	CG procedure	[105], [19], [20], [21]
		ICCG	[19], [31, Sec. 3, p.45ff]
	projective methods (Galerkin methods)	power method	[127], [116], [16], [57], [6]
		simultaneous iteration	[116], [115]
		Lanczos' procedure and Arnoldi's procedure	[102], [16]
		iterative A / D schemes	[22], [52]
	multigrid algorithms	AMG	[11, Sec. 10, p. 257ff], [118]

TABLE II
CRITERIA FOR THE CHOICE OF A PROCEDURE

Criteria	Recommended Methods: Gaussian elimination techniques	Recommended Methods: Iterative procedures based on splittings
Properties of Q-matrix A		
order n	$n \leq 500$	$n > 500$
structure:		
dense	recommended	suitable
banded	for small bandwidth	for large bandwidth
block structured	for additional structure	generally recommended
sparse	-	recommended
spectral properties:		
well-conditioned	suitable	suitable
ill-conditioned including NCD-type	-	recommended
Properties of the procedure		
algorithmic complexity:	$s = \max_{1 \leq i \leq n}\{\#\{A_{ij} \neq 0, 1 \leq j \leq n\}\}$	
w.r.t. time	$O(2/3n^3)$	$O(n \times s)$ per step
w.r.t. memory	$O(n^2)$	$O(n \times s)$
ease and efficiency of the implementation:		
algorithmic structure	matrix-vector scheme	matrix-vector scheme
data structures	matrices, vectors	matrices, vectors or sparse storage scheme
vectorisable or parallel executable	yes	yes
accuracy aspects:		
stability w.r.t. rounding errors	yes	yes
mathematically founded termination test	-	available

I) Procedures based on a representation by a system of linear equations

$$0 = A \cdot x, \qquad A \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n \tag{2.6}$$

given $A = -Q^t$ or $A = I - P^t$ where Q is the irreducible generator matrix of a CTMC and P the irreducible t.p.m. of a DTMC with n states.

II) Procedures based on a representation by an eigenvalue problem

$$x = T \cdot x \tag{2.7}$$

given $T = P^t$ or $T = I + Q^t D^{-1} = B^t D^{-1}$ with $Q = -D + B$ and its negative diagonal part $D = -\text{diag}\ (Q_{ii})_{i=1,\cdots,n}$ or $D = \lambda I$, $\lambda \geq \max_i (-Q_{ii})$ (uniformization procedure for CTMC's).

In the following, we restrict our attention to procedures of the first category since they are employed in MACOM only. The computational procedures may be divided into the classes of direct and iterative methods. An overview is provided by Table I.

The choice of an appropriate algorithm is directed by the properties of the Q-matrix A and the properties of the selected procedure. Some criteria are listed in Table II. The advantages and disadvantages of direct and iterative procedures are discussed by Evans [31, sec. 3, p. 45f] (cf. also [121]). Computational aspects concerning the efficient implementation of iterative methods on a vector computer are treated by Kincaid *et al.* [63].

III. DIRECT METHODS

In this section we study the solution of the homogeneous system of linear equations (2.4) by direct methods, i.e., variants of the Gaussian elimination technique. After the solution phase, vector x must be normalized such that (2.5) is satisfied. Subsequently, we shall show that it is not necessary to proceed to a regular system by deflation or rank-1 modification techniques which are proposed in the literature (cf. [97], [89]).

Solution techniques from numerical mathematics that reduce the order n of the linear system (2.4) are very often discussed in the literature (cf. [41], [45], [53], [57], [64], [67], [72], [77], [97], [104], [113], [128]). All these procedures are based on a block decomposition of the irreducible Q-matrix A

$$A = \begin{pmatrix} A_{11} & \vdots & A_{12} \\ \cdots & \cdots & \cdots \\ A_{21} & \vdots & A_{22} \end{pmatrix} \tag{3.1}$$

with $A_{11} \in \mathbb{R}^{k_1 \times k_1}$, $A_{12} \in \mathbb{R}^{k_1 \times k_2}$, $A_{21} \in \mathbb{R}^{k_2 \times k_1}$, $A_{22} \in \mathbb{R}^{k_2 \times k_2}$, and $k_2 \geq 2$. Obviously, A_{11} and A_{22} are regular M-matrices and A_{12} and A_{21} are nonpositive.

Based on this decomposition, a block elimination scheme may be applied to reduce the solution of $Ax = 0$ to the analysis of smaller subproblems. This rank-reduction technique is recommended if the structure of matrix A can be exploited (cf. [50], [57]).

A. Block LU-Algorithm

A classical numerical approach is given by the following block Gaussian elimination procedure (cf. [43, sec. 5.5, p. 110ff], [72], [104], [67], [39]):

1) Choose a nontrivial partition

$$A^{(1)} = A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

of the irreducible Q-matrix A.

2) Compute a factorization $A_{11}^{(1)} = L_{11}^{(1)} U_{11}^{(1)}$ by Gaussian elimination without pivoting.

3) Calculate a solution $R^{(2)}$ of $R^{(2)} L_{11}^{(1)} U_{11}^{(1)} = -A_{21}^{(1)}$ by forward elimination and back substitution (taking into account $R^{(2)} \geq 0$, $e^t R^{(2)} = e^t$ as accuracy check).

4) Set

$$A_{22}^{(2)} = A_{22}^{(1)} + R^{(2)}A_{12}^{(1)}. \tag{3.2}$$

5) Solve

$$A_{22}^{(2)}x^{(2)} = 0. \tag{3.3}$$

6) Compute a solution vector $x^{(1)}$ of $A_{11}^{(1)}x^{(1)} = L_{11}^{(1)}U_{11}^{(1)}x^{(1)} = (-A_{12}^{(1)})x^{(2)}$ by forward elimination and back substitution.

7) Normalize the resulting vector

$$x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}$$

by setting $p = x/(e^t x)$.

□

Obviously, one step of this elimination scheme yields $A^{(1)} = A = L^{(1)}U^{(1)}$ with the regular M-matrix

$$L^{(1)} = \begin{pmatrix} I & 0 \\ A_{21}^{(1)}A_{11}^{(1)^{-1}} & I \end{pmatrix}$$

and the singular M-matrix

$$A^{(2)} = U^{(1)} = \begin{pmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ 0 & A_{22}^{(2)} \end{pmatrix}$$

where the irreducible Q-matrix $A_{22}^{(2)} = A_{22} + (-A_{21}A_{11}^{-1})A_{12}$ is called the *Schur complement* of A_{11} (cf. [43, P4.2-3, p. 58]). If we perform k_1 steps of the normal Gaussian elimination procedure on matrix A, the resulting submatrix $A_{22}^{(k_1+1)}$ obviously coincides with the Schur complement (3.2) (cf. [43, P4.2-3, p. 58]).

As $L^{(1)}$ is a regular matrix, the solution of $A^{(1)}x = 0$ is equivalent to that of $A^{(2)}x = 0$. In order to solve the linear system (3.3), the same reduction step may be applied to the resulting submatrix $A_{22}^{(2)}$ or any other direct or iterative solution technique for singular linear systems can be employed, for instance, the direct-iterative approach of Funderlic and Plemmons [35]. The successive application of this reduction process is stopped after $m - 1$ steps, yielding a submatrix $A_{22}^{(m)}$ of order l with rank $l - 1$. Then, we fix one component of the solution vector $x^{(m)}$ resulting from the decomposition of x according to the block partition of A, for instance, its last one $x_l^{(m)} = 1$, and solve the linear system $A_{22}^{(m)}x^{(m)} = 0$. The back substitution yields the other component vectors $x^{(m-1)}, \cdots, x^{(1)}$ of x.

The computation of $A_{22}^{(2)}$ may be improved by exploiting its Q-matrix structure. As $R^{(2)}$ is nonnegative, $A_{12}^{(1)} \leq 0$ and $A_{22}^{(1)}$ is a Q-matrix, the off-diagonal elements of $A_{22}^{(2)}$ are computed from nonpositive elements only, whereas the diagonal elements are given by the negative column sums of the off-diagonal elements. Hence, there is no loss of significance due to cancellation errors while performing these additions. This computation of $A_{22}^{(2)}$ according to (3.2) is a rather stable process. Considering the propagation of rounding errors, the crucial step of the algorithm is the matrix inversion, which is necessary to calculate $R^{(2)}$, and has to be performed very carefully. Le Boudec [78], for instance, proposes to use a special inversion algorithm for strictly substochastic matrices. It is based on the representation of the inverse by a power series. In conclusion, we recommend either to avoid successive reduction or to apply only a few steps of this block LU-algorithm exploiting the structure of A as mentioned by Kaufman [57] or Le Boudec [78].

A major advantage of the proposed block LU-algorithm is its inherent probabilistic interpretation (cf. Schassberger [104], Grassmann *et al.* [45], Grassmann *et al.* [44], Kohlas [64], Gaver *et al.* [39], Krieger [71]).

B. Grassmann's Algorithm

Choosing a block decomposition of A with $k_1 = 1$, we may successively apply the proposed block elimination procedure to the generated block matrices $A_{22}^{(i)}$. If we calculate the negative values of its diagonal elements by means of relation $e^t A_{22}^{(i)} = 0$ as sum of the off-diagonal elements, we obtain a numerical procedure that does not contain any subtractions or additions of numbers with different signs. Thus, the algorithm avoids cancellation errors, a main difficulty of Gaussian elimination, although it coincides with the result of the Gaussian elimination without pivoting applied to A. This procedure is due to Grassmann *et al.* [45] and called *Grassmann's algorithm*. Grassmann *et al.* [45] proved the feasibility of this approach for DTMC's by probabilistic arguments. Furthermore, its application to CTMC's has been studied in [44] (see also [64], [72]). The corresponding algorithm reads as follows:

Assumptions: Given a homogeneous, irreducible Markov chain with a finite state space $S = \{1, \cdots, n\}$ set $A = Q^t \in \mathbb{R}^{n \times n}$ for a CTMC or $A = P^t - I$ for a DTMC.

Algorithm:

1) Gaussian elimination:

```
For k = 1 to n − 1 do
    diag = Σ_{i=k+1}^{n} A_ik
    For j = k + 1 to n do
        A_kj = A_kj/diag
    Endfor
    For j = k + 1 to n do
        For i = k + 1 to n do
            A_ij = A_ij + A_ik A_kj
    Endfor
Endfor
```

2) Norm = 1

$x_n = 1$

3) Back substitution:

```
For i = n − 1 to 1 do
  x_i = Σ_{k=i+1}^{n} A_ik · x_k
    Norm = Norm + x_i
Endfor
```

4) Normalization:

For $k = 1$ to n do
$x_k = x_k/\text{Norm}$
Endfor

□

The proposed algorithm requires $O(2/3n^3)$ flops and $O(n^2)$ amount of memory for computing the steady-state vector x. Moreover, it has the advantage of preserving the band structure of a matrix. An implementation based on a sparse matrix storage scheme is also possible for banded or sparse matrices. The storage scheme should support access to matrix elements by rows and columns (cf. [27]).

C. LU-Factorization of Singular M-Matrices

From the mathematical literature it is known that only in the last elimination step of the Gaussian elimination without pivoting, a zero pivot occurs if we decompose an irreducible singular M-matrix (cf. [37, cor. 1], [9, cor 6.4.17]). Therefore, the Crout algorithm may be employed for the factorization of a diagonally dominant Q-matrix specifying the steady-state distribution of a Markov chain (cf. [34]). Let us summarize this algorithm, called direct factorization method (cf. [50], [51]):

1) Decompose the irreducible Q-matrix $A \in \mathbb{R}^{n \times n}$ by Gaussian elimination without pivoting (Crout algorithm) according to Funderlic, Mankin [34]:
For $k = 1, 2, \cdots, n-1$ do

1.1 $\tilde{L}$: $A_{ik} = A_{ik} - \Sigma_{m=1}^{k-1} A_{im} \cdot A_{mk}$, $i = k, \cdots, n$
1.2 $\tilde{U}_D$: $A_{ki} = A_{kk}^{-1} \cdot (A_{ki} - \Sigma_{m=1}^{k-1} A_{km} \cdot A_{mi})$, $i = k+1, \cdots, n$

2) Transform $\tilde{U}_D$ into a regular matrix U_D: $A_{nn} = 1$

3) Solve $U_D \cdot x = e_n$ by back substitution:

3.1 $x_n = 1$
3.2 $x_i = (-\Sigma_{j=i+1}^{n} A_{ij} \cdot x_j)$, $i = n-1, \cdots, 1$.

4) Normalize the resulting vector x:

4.1 $a = 1/\Sigma_{j=1}^{n} x_j$
4.2 $x_i = a \cdot x_i$, $i = 1, \cdots, n$

□

D. Further Reading

Further block LU procedures for block tridiagonal matrices A have been studied by Basharin (cf. [41, §12, p. 390ff]), Wikarski [128], Kramer [67], Krieger [72], Herzog *et al.* [53], and Gaver *et al.* [39]. Meyer (cf. [85], [86], [80]) has recently extended the concept of block LU factorization to nearly decomposable systems which are also known as nearly completely decomposable (NCD) models (cf. [25]). His method, called stochastic complementation, uncouples a Markov chain in several smaller independent chains. It is well suited for parallel computation (cf. [80]). Other variants of rank-reduction techniques that exploit especially the sparsity structure of matrix A and properties of separable matrices are discussed by Kaufman [57].

Results concerning the factorization of a (singular) M-matrix A into a (singular) upper triangular M-matrix U having the same rank as A and a regular, lower triangular M-matrix L are well known in numerical mathematics (cf. [32], [9, theorem 6.2.3, p. 135], [76], [34], [37], [124, theorem 3, p. 182], [2]). Although a singular M-matrix A does not possess an LU decomposition in general (see [124, (1.15), p. 183]), it is known that a factorization into M-matrices always exists for a permuted version PAP^t (see Kuo [76, theorem 2], also [9, theorem 6.4.18], [37, cor. 1, p. 106]). In general, the LU decomposition into M-matrices is not unique. In the special case of an irreducible singular matrix, however, uniqueness of the factorization may be established.

It is well known that, even in the case of a singular linear system, this LU factorization is a stable method (cf. [36], [50, theorem 1], [51]). Furthermore, Barlow [8] has shown that a transformation of the singular system (2.4) into a regular inhomogeneous system by specifying a component of the solution vector x and deleting the corresponding row of A will not improve the condition of the linear system. Similar observations based on extensive numerical comparisons have been reported by Harrod and Plemmons [50] and Kaufman [57].

An alternative direct method is given by the QR-factorization algorithm for singular M-matrices. Information about this procedure and its probabilistic relevance is provided by Golub and Meyer [42]. From the computational point of view, however, this technique is not superior to the factorization methods mentioned above.

Further applications of LU-factorization algorithms are studied in the context of combined direct-iterative methods such as incomplete factorization procedures. The interested reader is referred to Funderlic and Plemmons [35] as well as Buoni [15] and Meijerink and van der Vorst [84].

E. Applicability of Direct Methods

In conclusion, we recommend the application of either the proposed LU-factorization method or Grassmann's algorithm, which has the additional advantage of avoiding cancellation errors. Both methods have a similar complexity of $O(2/3n^3)$ flops and normally provide comparable accuracy for small matrices (see Table III—cf. [72]). The application of these direct methods is only limited by memory constraints and rounding errors. But they are suitable for computing the stationary distribution of small Markov chains with up to 500 states if the latter do not belong to the special class of nearly completely decomposable systems.

IV. Iterative Methods

We may also employ iterative methods to solve the linear system (2.4) for a given irreducible, always unsymmetric Q-matrix $A \in \mathbb{R}^{n \times n}$. These methods are preferable for banded, sparse or large generator matrices with more

TABLE III
COMPARISON OF GRASSMANN'S ALGORITHM AND THE DIRECT LU FACTORIZATION METHOD BASED ON COMPLETELY FILLED GENERATOR MATRICES WITH $N = 200$ STATES WHOSE ELEMENTS ARE GENERATED AT RANDOM. THE ERRORS ARE DEFINED IN TERMS OF THE MAXIMUM NORM OF THE STEADY-STATE VECTORS. THE DATA ARISE FROM A FORTRAN REAL* 8 IMPLEMENTATION ON AN IBM 3090 Vf. THE EXACT DATA OF THE STEADY-STATE VECTORS HAVE BEEN CALCULATED BY MEANS OF THE PACKAGE ACRITH [55] WHICH PROVIDES HIGH-ACCURACY ARITHMETIC

Grassmann's algorithm			Direct factorisation method		
correct digits	absolute error	relative error	correct digits	absolute error	relative error
14	0.38945D-15	0.65454D-13	13	0.25127D-14	0.42232D-12
14	0.36950D-15	0.65013D-13	13	0.23610D-14	0.41541D-12
14	0.37383D-15	0.65811D-13	13	0.24173D-14	0.42556D-12
14	0.37990D-15	0.65179D-13	13	0.24061D-14	0.41280D-12
14	0.38424D-15	0.66186D-13	13	0.24251D-14	0.41761D-12
14	0.35475D-15	0.61330D-13	13	0.22560D-14	0.39003D-12
14	0.38077D-15	0.65494D-13	13	0.24789D-14	0.42638D-12
14	0.37904D-15	0.65627D-13	13	0.24035D-14	0.41614D-12
14	0.39291D-15	0.68606D-13	13	0.24954D-14	0.43572D-12
14	0.38598D-15	0.66331D-13	13	0.24859D-14	0.42377D-12

than 1000 states. To apply an iterative procedure, its convergence must be guaranteed. As A is singular now, a straightforward application of standard results developed for regular matrices is impossible. Based on the unifying framework of M-matrices, the required generalization of the classical convergence theory to singular M-matrices has fortunately been developed over the last decade (cf. [105], [101], [93], [14], [57], [71]).

Usually the classical iteration procedures are based on an additive decomposition of matrix A, called *matrix splitting*. It has the form $A = M - N$, $M \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{n \times n}$ with a regular matrix M yielding the *iteration matrix* $T = M^{-1} \cdot N$. If $N \geq 0$ and $M^{-1} \geq 0$, then $T = I - M^{-1}A \geq 0$ holds and the splitting is called *regular splitting*. Note that an M-matrix M satisfies $M^{-1} \geq 0$. The corresponding regular splittings are called *M-splittings* (cf. [105, def. 2.3, p. 410]). Then an iteration procedure may be defined by $M \cdot x^{(k+1)} = N \cdot x^{(k)}$ or

$$x^{(k+1)} = T \cdot x^{(k)} \qquad k = 0, \cdots, \tag{4.1}$$

respectively. Furthermore, it is easy to see that $Ax = 0$ is equivalent to $x = Tx$ (cf. [57, lemma 4.1]), i.e., each nontrivial solution of the homogeneous system (2.4) is also a right eigenvector corresponding to the maximal eigenvalue $\rho(T) = 1$ of T and vice versa.

A. Point Iteration Methods

In the following, let $A = D - L - U$ be a *point splitting* into a diagonal matrix D, a strictly lower triangular matrix L, and a strictly upper triangular matrix U, respectively.

1) The Point Jacobi Procedure: The *Jacobi procedure*

$$Dx^{(m+1)} = (L + U)x^{(m)} \qquad m = 0, 1, \cdots \tag{4.2}$$

results from the M-splitting $M = D$, $N = L + U$. As the diagonal D of an irreducible M-matrix is positive ([9, theorem 6.4.16, p. 156]), this splitting with the Jacobi iteration matrix $J = D^{-1}(L + U)$ is well defined. Obviously, the Jacobi procedure is equivalent to the power method applied to $D^{-1}A = I - D^{-1}(L + U)$.

It is known that any eigenvalues on the unit circle apart from $\rho(T) = 1$ prevent the convergence of the scheme (4.1) derived from any regular splitting $A = M - N$ of an irreducible, singular M-matrix. On the other hand, there is the simple idea of shifting these eigenvalues into the unit circle by an appropriate transformation $T_\alpha = \alpha T + (1 - \alpha)I = I - \alpha M^{-1}A$, $0 < \alpha < 1$, of T called stationary first-order Richardson extrapolation (cf. Kemeny and Snell [62, ch. V: Ergodic Markov Chains, theorem 5.1.1, p. 99], Kaufman [57, p. 540], and Neumann and Plemmons [93, p. 273 ff]). Then the resulting *extrapolated Jacobi method* (JOR) $x^{(k+1)} = J_\alpha \cdot x^{(k)} = [\alpha J + (1 - \alpha)I]^{k+1} \cdot x^{(0)}$ converges to a nontrivial solution of $Ax = 0$, which depends on $x^{(0)}$, for each $0 < \alpha < 1$ and each initial vector $x^{(0)} >> 0$ (cf. [14, theorem 3.4, p. 191], [93, p. 273 ff]).

Furthermore, in some special cases the optimal extrapolation parameter α has been determined by Neumann and Plemmons [93, theorem 7, p. 275] (cf. [9, theorem 8.4.32, p. 234], [46], [47]).

2) The Point Gauss-Seidel Procedure: There are two possible Gauss-Seidel iteration procedures for the solution of the homogeneous system (2.4): the *forward Gauss-Seidel iteration*

$$(D - L)x^{(m+1)} = Ux^{(m)} \qquad m = 0, 1, \cdots \tag{4.3}$$

with the iteration matrix $T_L = (D - L)^{-1}U$ and the *backward Gauss-Seidel iteration*

$$(D - U)x^{(m+1)} = Lx^{(m)} \qquad m = 0, 1, \cdots \tag{4.4}$$

with the iteration matrix $T_U = (D - U)^{-1}L$. Note that the Gauss-Seidel splitting of an irreducible M-matrix is an M-splitting.

Simple examples reveal that the convergence of these schemes is not guaranteed for arbitrary M-matrices A. But it can be shown that a Gauss-Seidel scheme is convergent, too, if the iteration matrix T (the index of T is omitted) has no further eigenvalues on the unit circle apart from the maximal eigenvalue $\rho(T) = 1$, i.e., if $\delta(T) = \max\{|\lambda| : \lambda \in \sigma(T), \lambda \neq 1\} < 1$ holds (cf. [71, §6.1], [6], [105], [101]). Here, $\sigma(T)$ denotes the spectrum of T. This criterion may be verified *a priori* by a simple analysis of the zero structure of A (cf. [6, theorem 1, p. 394]). But there are some simple sufficient conditions for the convergence of the point Gauss-Seidel procedures which can easily be verified by the inspection of the zero structure of matrix A, too (cf. [6, cor. 1, p. 394], [105, cor 3.6], [101, cor. 2]).

Recall that a sequence $\alpha = (i_0, i_1, \cdots, i_l, i_0)$ of distinct nodes (except the end point) in the *directed matrix graph* $\Gamma(A)$ associated with A is called a *monotone cycle* if $\alpha_1 = (i_0, i_1, \cdots, i_l)$ is monotone with either $i_l < i_0$ for decreasing α_1 or $i_l > i_0$ for increasing α_1 (cf. [101], [6]).

Result 1: If the directed graph $\Gamma(A)$ of the irreducible, singular M-matrix A has a monotone decreasing *cycle,*

then the forward *Gauss-Seidel iteration converges for each* $x^{(0)}$.

If $\Gamma(A)$ *has a* monotone increasing *cycle, then the* backward *Gauss-Seidel iteration converges for each* $x^{(0)}$.

We state another useful criterion for matrices with symmetric zero structures (cf. [35], [57]). These matrices A have the property that $A_{ij} \neq 0$ for $i \neq j$ implies $A_{ji} \neq 0$.

Result 2: If the irreducible M-matrix A has a symmetric zero structure, both the forward and backward Gauss-Seidel iterations are convergent.

To avoid expensive divisions during the execution of the Jacobi or Gauss-Seidel procedure, we prefer to apply the schemes to the irreducible M-matrix $A \cdot D^{-1} = I - \tilde{L} \cdot D^{-1} - \tilde{U} \cdot D^{-1} = I - L - U$. It is derived by scaling the columns of A by means of the diagonal matrix $D^{-1} > 0$ (cf. [117], [87]). This transformation is equivalent to the transition from a CTMC to its embedded jump chain with the t.p.m. $P = I + D^{-1} \cdot Q = (L + U)^t$. The scaling of A is associated with a scaling of the steady-state distribution π: $A \cdot D^{-1} \cdot (D \cdot \pi) = 0$, $x = D \cdot \pi$. Thus, this steady-state vector may be calculated from the vector x resulting after convergence of the iterative scheme by the modified normalization $\pi = D^{-1} \cdot x / (e^t \cdot D^{-1} \cdot x)$ provided that $x > 0$ holds. An efficient implementation of an accelerated version of this Gauss-Seidel scheme has been described by Stewart and Goyal [117].

B. The Block Gauss-Seidel Procedure and R-Regular Splittings

An important class of iterative methods is based on a block partition $A = (A_{ij})_{1 \le i,j \le p}$, $p > 1$, of a given Q-matrix A. Regarding the corresponding block iterative schemes, Rose [101] has established some convergence results. They are based on a generalization of the block Gauss-Seidel splitting technique, called *R-regular splitting* (cf. [101, p. 138]). Constructing the splitting by this technique, appropriate LU decomposition schemes resulting from special factorization methods for sparse matrices, e.g., an incomplete LU decomposition, can be taken into consideration. A variant of the corresponding iterative algorithm which is accelerated by relaxation reads as follows.

Assumptions: Let $A \in \mathbb{R}^{n \times n}$ be the Q-matrix associated with a Markov chain. Define a block partition $A = (A_{ij})_{1 \le i,j \le p}$, given $p > 1$, and extend it to the steady-state vector x specified by (2.4), (2.5). Choose an R-regular block splitting $A = M - N = (D - D(N)) - (L + L(N)) - U(N)$ which has the following properties:

(1) $D = \operatorname{diag}(D_{ii})_{1 \le i \le p}$ and $D(N)$ are block-diagonal matrices with $D(N) \ge 0$. L and $L(N)$ are strictly lower block-triangular matrices such that $L \ge 0$, $L(N) \ge 0$. $U(N)$ is a strictly upper block-triangular matrix with $U(N) \ge 0$.

(2) $D_{ii}^{-1} >> 0$ for $1 \le i \le p$. (If $0 < \omega < 1$ holds, only $D_{ii}^{-1} > 0$ is required.)

(3) $M = D - L$ is a lower block-triangular matrix.

(4) $N = L(N) + U(N) + D(N) \ge 0$.

(5) $A_0 = D - L - U(N)$ is irreducible.

(6) The block matrix graph $\Gamma(A_0) = (V, E)$ has a monotone decreasing cycle, this is a sequence $c = (i_1, i_2, \cdots, i_l, i_1)$ of adjacent nodes with the property $i_l \neq i_1$ and $i_j \ge i_{j+1}$, $1 \le j \le l - 1$. Recall that the block matrix graph $\Gamma(A_0) = (V, E)$ is a directed matrix graph with nodes $V = \{V_i \mid 1 \le i \le p\}$ and directed edges $(V_i, V_j) \in E$. V_i results from the partition of the index set $\{1, \cdots, n\}$ according to the block partition. $(V_i, V_j) \in E$ iff $A_{ij} \neq 0$, that means there are indices $l \in V_i$, $m \in V_j$ such that $(l, m) \in E_{\Gamma(A)}$ is an edge in the matrix graph of A.

Algorithm:

1) Initialization:
Select an initial vector $x^{(0)} >> 0$, e.g., $x^{(0)} = M^{-1}e/n$, $0 < \epsilon < 1$, and an appropriate relaxation parameter $0 < \omega < 2$. Set $k = 0$.

2) Iteration step:

For $i = 1$ to p do

$$\text{Solve } D_{ii} \cdot x_i^{(k+1)} = \Sigma_{j=1}^{i-1} L_{ij} \cdot x_j^{(k+1)} + \Sigma_{j=1}^{p} N_{ij} \cdot x_j^{(k)}$$

$$x_i^{(k+1)} = (1 - \omega) \cdot x_i^{(k)} + \omega \cdot x_i^{(k+1)}$$

endfor

(Let $\Sigma_{j=1}^{0} \equiv 0$.)

3) Convergence test:

If $\|x^{(k+1)} - x^{(k)}\| / \|x^{(k)}\| < \epsilon$ then
goto step 4
else
goto step 2
endif

4) Normalization:

Normalize the resulting vector

$$x^{(k+1)} = \begin{pmatrix} x_1^{(k+1)} \\ \vdots \\ x_p^{(k+1)} \end{pmatrix} > 0$$

by setting $x = x^{(k+1)} / \|x^{(k+1)}\|_1$.

□

Obviously, every R-regular splitting is also a regular splitting. If all diagonal blocks D_{ii} are M-matrices, it is an M-splitting. Without loss of generality we assume that D_{ii} are irreducible regular M-matrices which implies (2) and the irreducibility of A (cf. [101]). The monotony of a cycle (6) existing due to the irreducibility of A can be enforced by an appropriate block permutation (cf. [101, prop. 3, p. 138]).

The importance of R-regular splittings stems from Rose's result [101, theorem 1, p. 138] that each R-regular splitting $A = M - N$ of an irreducible, singular M-matrix

$A \in \mathbb{R}^{n \times n}$, e.g., the Q-matrix of an irreducible Markov chain, generates a convergent iterative scheme.

Employing this special splitting technique, Rose has also shown that there always exists a convergent block Gauss–Seidel splitting of the irreducible Q-matrix corresponding to an irreducible Markov chain with appropriately ordered states (cf. [101, theorem 1, p. 138]). In this context, the following sufficient convergence criterion may often be employed in practice (cf. [101, cor. 2, p. 139]).

Result 3: Let $A \in \mathbb{R}^{n \times n}$ be an irreducible, singular M-matrix, e.g., a Q-matrix, and $A = M - N = (D - L) - U(N)$ a block Gauss-Seidel splitting with irreducible block matrices D_{ii} along the diagonal of M. Furthermore, suppose there exist block matrices $A_{ij} \neq 0$ and $A_{ji} \neq 0$ for $i, j \in \{1, \cdots, n\}$, $i \neq j$. Then the block Gauss-Seidel method converges.

Other sufficient convergence criteria concerning the block Jacobi and block Gauss–Seidel method as well as their implementation by a two-step iterative procedure derived from an A/D scheme are provided by Courtois [26] (see Section IV-C). These methods are particularly suitable for NCD models.

C. The IAD Procedure

In this section we present a universal iterative solution procedure which can be used as a building block of an analysis program for Markov chains. Let $A = M - N$ be a regular splitting of the irreducible Q-matrix associated with an irreducible finite Markov chain. Denote the nonnegative iteration matrix by $J = M^{-1}N$ and the corresponding extrapolated iteration matrix by $J_\omega = (1 - \omega)I + \omega J$, $0 < \omega < 1$. A standard convergence-acceleration technique besides the relaxation method mentioned previously is given by the insertion of some aggregation/disaggregation (A/D) steps during the iteration (cf. [106], [108], [52], [22], [23], [90]). In order to guarantee convergence of the scheme in this case (cf. [108, theorem 4, p. 328]), we have to define a fallback procedure $x^{(k+1)} = T \cdot x^{(k)}$. It is based on a stochastic matrix T that converges to the normalized eigenvector x^* corresponding to the eigenvalue $\rho(T) = 1$ which is related to the stationary distribution p of the Markov chain by some transformation (see (4.8)—cf. [87, p. 126]).

Usually, neither J nor J_ω is stochastic. In order to construct a stochastic iteration matrix, we proceed to a nonnegative matrix T, called dual iteration matrix, by a similarity transformation $T = M \cdot J \cdot M^{-1} = N \cdot M^{-1}$ or $T_\omega = M \cdot J_\omega \cdot M^{-1} = (1 - \omega)I + \omega T$, $0 < \omega < 1$. Then T_ω, $0 < \omega \leq 1$, is column stochastic and $\| T_\omega \|_1 = \rho(T_\omega) = 1$ holds. Furthermore, T_ω, $0 < \omega < 1$, determines a convergent scheme and $\rho(T_\omega) = 1$ is a simple eigenvalue. Subsequently, the subscript ω will be omitted. Let $x^* > 0$ denote the unique normalized eigenvector corresponding to the spectral radius $\rho(T) = 1$.

We choose a partition $\Gamma = \{J_1, \cdots, J_m\}$ of the state space $S = \{1, \cdots, n\}$ into $m \geq 2$ disjoint sets J_i with $n_i \geq 1$ elements each. Without loss of generality, we assume the elements of these sets to be enumerated in a consecutive order such that $i < j$ holds if $i \in J_l$, $j \in J_k$, $l < k$. Furthermore, w.l.g. let T and $x^{*t} = (x_1^{*t}, \cdots, x_m^{*t})$ be arranged according to this state space partition and ordering. Following the approach of Chatelin and Miranker (cf. [23], [52], [22]), we define an aggregation matrix $R \in \mathbb{R}^{m \times n}$ by

$$R_{ij} = \begin{cases} 1 & \text{if } j \in J_i \\ 0 & \text{otherwise} \end{cases} \qquad 1 \leq i \leq m,\ 1 \leq j \leq n. \tag{4.5}$$

For a fixed probability vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} > 0,$$

i.e., $e^t \cdot x = 1$, the prolongation matrix $P_{(x)} \in \mathbb{R}^{n \times m}$ is given by

$$P_{(x)ij} = \begin{cases} (y_j)_i & \text{if } i \in J_j \\ 0 & \text{otherwise} \end{cases} \qquad 1 \leq i \leq n,\ 1 \leq j \leq m \tag{4.6}$$

where the vector

$$y = y_{(x)} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

is defined for $j \in \{1, \cdots, m\}$ as follows:

$$\mathbb{R}^{n_j} \ni y_{(x)_j} = \begin{cases} x_j / \alpha_{(x)_j} & \text{if } x_j > 0 \\ 1/n_j \cdot e & \text{if } x_j = 0 \end{cases}$$

$$\text{with } \alpha_{(x)_j} = e^t \cdot x_j$$

$y_{(x)_j}$ comprises the conditional probabilities of the states in the aggregate j and y is called the vector of the intra-aggregate probabilities. Based on these matrices

$$R = \begin{pmatrix} e^t & 0 & \cdots & 0 \\ 0 & e^t & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^t \end{pmatrix} > 0,$$

$$P_{(x)} = \begin{pmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_m \end{pmatrix} > 0$$

we define a stochastic aggregated iteration matrix by

$$B = B_{(x)} = R \cdot T \cdot P_{(x)} \in \mathbb{R}^{m \times m}. \qquad (4.7)$$

In a way, B describes the behavior of a new system approximating the original one. It results from the aggregation of the original state space S into a smaller set $\{1, \cdots, m\}$ of macrostates determined by the partition $\Gamma = \{J_1, \cdots, J_m\}$. Hence, there exists a probability vector $\alpha = \alpha_{(x)} = (\alpha_{(x)_1}, \cdots, \alpha_{(x)_m})^t > 0$ in $\mathbb{R}^m$ satisfying $B_{(x)} \cdot \alpha_{(x)} = \alpha_{(x)}$ and $e^t \cdot \alpha_{(x)} = 1$. α is called the vector of interaggregate probabilities. This steady-state information about the aggregated system is extended to the whole state space by the generation of the disaggregated probability vector $\hat{x}_{(x)} \in \mathbb{R}^n$ defined by $\hat{x}_{(x)} = P_{(x)} \cdot \alpha_{(x)} \geq 0$. It can be used as an approximation of the required steady-state vector.

Regarding this approach, we may construct a corresponding iterative A/D algorithm. It is based on a scheme developed by Schweitzer and Kindle (cf. [108, p. 326f]) and called *dual IAD-algorithm*.

Assumption: Let $A = M - N$ be a regular splitting of the irreducible Q-matrix $A \in \mathbb{R}^{n \times n}$ with an extrapolated dual iteration matrix $T = I - \omega A M^{-1}$ for some $\omega \in (0, 1)$. Choose a partition $\Gamma = \{J_1, \cdots, J_m\}$ of $\{1, \cdots, n\}$ into $m \geq 2$ disjoint sets. Define $r(x) = \|(I - T) \cdot x\|_1$, $x \in \mathbb{R}^n$.

Algorithm:

1) Initialization:

Select an initial vector $x^{(0)} >> 0$, $e^t x^{(0)} = 1$, and three real numbers $0 < \epsilon, c_1, c_2 < 1$. Construct the matrices $P_{(\cdot)} \in \mathbb{R}^{n \times m}$, $R \in \mathbb{R}^{m \times n}$, $T \in \mathbb{R}^{n \times n}$ and $B_{(\cdot)} \in \mathbb{R}^{m \times m}$ according to (4.6), (4.5), (4.7). Set $k = 0$.

2) A/D step:

Solve $\quad B_{(x^{(k)})} \cdot \alpha_{(x^{(k)})} = \alpha_{(x^{(k)})}$
subject to $\quad e^t \cdot \alpha_{(x^{(k)})} = 1, \ \alpha_{(x^{(k)})} > 0$
and compute $\quad \bar{x} = P_{(x^{(k)})} \cdot \alpha_{(x^{(k)})}$

3) Iteration step:

Compute $\quad x^{(k+1)} = T \cdot \bar{x}$

4) Convergence test:

If $\quad r(\bar{x}) \leq c_1 \cdot r(x^{(k)})$
then $\quad$ goto step 5
else $\quad$ compute $x^{(k+1)} = T^m \cdot x^{(k)}$
with $m = m(x^{(k)}) \in \mathbb{N}$ such that
$r(x^{(k+1)}) \leq c_2 \cdot r(x^{(k)})$
endif

5) Termination test:

If $\quad \|x^{(k+1)} - x^{(k)}\|_1 / \|x^{(k)}\|_1 = \|x^{(k+1)} - x^{(k)}\|_1 < \epsilon$
then $\quad$ goto step 6
else $\quad k = k + 1$
goto step 2
endif

6) Normalization:

$$p = \frac{M^{-1} \cdot x^{(k+1)}}{e^t \cdot M^{-1} \cdot x^{(k+1)}} \qquad (4.8)$$

Given the irreducible Q-matrix A associated with an irreducible finite Markov chain, we choose a regular splitting $A = M - N$, e.g., an M-splitting like the point or block Gauss–Seidel splitting $A = (D - L) - U$. Then it can be shown that the proposed dual IAD-algorithm converges to the steady-state distribution p of the Markov chain for any initial vector $x^{(0)} >> 0$ with $e^t x^{(0)} = 1$. Moreover, $r(x^{(k+1)}) \leq \max(c_1, c_2) \cdot r(x^{(k)})$, $k \geq 0$ holds (cf. [108, theorem 4], [70]).

D. Further Reading

As in the case of regular matrices, all iterative methods for singular systems are variants of the power method ([43, sec. 7.3, p. 209]). Considering the splitting $A = M - N$ of a regular matrix A, its convergence is guaranteed if the spectral radius of $T = M^{-1}N$ satisfies $\rho(T) < 1$ (cf. Varga [123, 3.2, p. 61ff]). In the case of singular systems, however, other conditions are necessary to guarantee the convergence of the scheme (4.1) because the spectral radius $\rho(T)$ of T is equal to 1 (cf. [9, ex. 6.4.9, p. 152, lemma 7.6.9, p. 197], [101, prop. 1, p. 136]).

Necessary and sufficient conditions guaranteeing the convergence of standard iterative schemes applied to singular M-matrices, like the Jacobi or Gauss–Seidel procedure, have been derived by Rose [101], Schneider [105], Barker and Plemmons [6], and Barker and Yang [7] among others.

Considering the block or point Gauss–Seidel splitting $A = (D - L) - U$ of an irreducible M-matrix, it is convenient to accelerate the Gauss–Seidel procedure by applying standard over- or underrelaxation techniques (cf. [57], [123], [6]) or by employing a semi-iterative technique such as the stationary or nonstationary Chebyshev method or Eiermann's stationary fourth-order scheme (cf. [123, §5], [29], [96], [30, lemma 8.4, p. 28], [7], [46], [47]). The main difficulty w.r.t. these procedures is the determination of "optimal" relaxation parameters. As there is no *a priori* information about the location of the eigenvalues of the iteration matrices, heuristic procedures estimating approximately optimal parameters seem to be the only practicable approach (cf. [48, §9.5, p. 223ff], [117], [111], [46], [47]).

The convergence of the Jacobi underrelaxation scheme has been established by Varga *et al.* [14, theorem 3.4, p. 191] who generalized the classical theorem of Stein–Rosenberg [123, theorems 3.15, 3.16, p. 90ff] to singular M-matrices A with positive diagonal elements. Furthermore, these authors have shown that the Gauss–Seidel underrelaxation scheme with the iteration matrix $T_\omega = (D - \omega L)^{-1}((1 - \omega)D + \omega U)$, $0 < \omega < 1$, is also convergent if it is applied to an irreducible, singular M-matrix. A corresponding result was also proved by

Barker and Plemmons [6, cor. 3, p. 395]. Therefore, both methods may be employed for the solution of (2.4).

Applying a continuity argument, we conclude that the accelerated iteration procedure associated with an R-regular splitting converges for $\omega > 1$ if the scheme of the R-regular splitting converges for $\omega = 1$ (cf. [123, p. 109]). But suitable formulas for the determination of the optimal relaxation parameter $\omega \in (1, 2)$ only exist for special cases (cf. [123, §4.3], [48, §9], [46], [47]).

Semi-iterative variants of the Gauss-Seidel procedure for singular M-matrices have been studied by Kaufman [57], Eiermann, Varga, and Niethammer [30], and Eiermann, Marek, and Niethammer [29] among others. For details the reader is referred to these articles and the references therein. We especially recommend the survey article of Eiermann *et al.* [29].

Regarding the iterative A/D procedure, an error analysis concerning the error reduction of an A/D step and the gain of an iteration step following the A/D step has been provided by Krieger [70]. Other A/D methods have been studied by Courtois [25], Vantilborgh [122], Koury *et al.* [66], Haviv [52], Cao and Stewart [18], Chatelin [22], Sumita *et al.* [119], Schweitzer [106], Schweitzer and Kindle [108]. Especially Chatelin [22], Schweitzer [107], and Haviv [52] provide excellent surveys of this topic. The reader interested in this subject is referred to their contributions.

Last but not least, it is worthwhile to mention that both the power method and the point Jacobi procedure have a stochastic interpretation (cf. [87, p. 122], [71, p. 56]). Moreover, the point Gauss-Seidel procedure has a probabilistic interpretation, too. It has been pointed out by Mitra and Tsoucas [87] for the first time. Krieger (cf. [71, p. 60], [70, §4.3]) has described an equivalent, rather simple approach based on Grassmann's state space reduction technique (cf. [45]) which is also applicable to the block Gauss-Seidel procedure.

E. Applicability of Iterative Methods

In conclusion, we recommend to apply variants of the dual IAD-algorithm as basic iterative procedures. They should be based on an appropriate regular splitting $A = M - N$ which yields an efficiently solvable linear system $Mx^{(k+1)} = Nx^{(k)}$ such as the block or point Gauss-Seidel splitting or an R-regular splitting. Furthermore, a careful choice of the relaxation parameter $\omega \in (0, 2)$ is required which may be supported by a heuristic procedure (cf. [48, §9.5, p. 223ff]). During the iteration process, the insertion of a few expensive, but effective A/D steps is useful, especially in the initial phase. Both the splitting and the partition of the state space should be adapted to the natural structure of a model and its associated Q-matrix A.

Basically, a procedure of this kind can be applied to all banded, sparse or large matrices and to all NCD-type models.

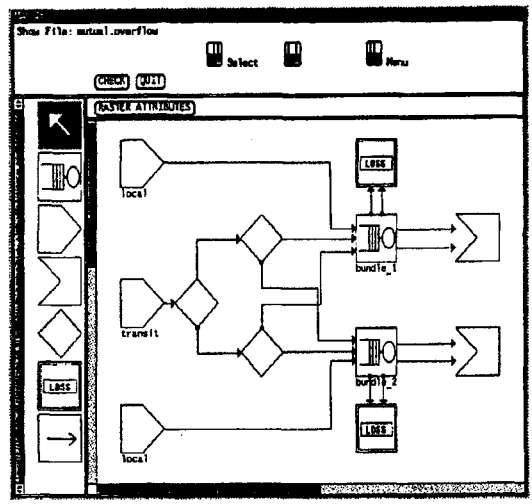

Fig. 1. The graphical user interface showing a mutual overflow model.

V. The Application of Numerical Methods to Markovian Models

In this section we illustrate the versatility of the presented computational approach by the application of some numerical methods to Markovian models which are derived from modern telecommunication networks.

A. A Model of Alternative Routing

In the last decade, considerable attention has been devoted to the analysis of advanced routing schemes in circuit-switched digital networks based on efficient modern signaling systems such as CCITT CCS No. 7. There are several interesting teletraffic models which are the building blocks of a network analysis (cf. [4], [17], [28], [40], [59], [68], [3], [130]).

The analysis of a circuit-switched network with alternative routing is a classical issue of teletraffic theory (cf. [129], [120]). In this context, a well-known model is provided by a single, fully available trunk group that carries both direct traffic and multiple overflow traffic streams (cf. [10], [65], [12]). This model describes a part of the network where several origin-destination pairs $O - D_1, \cdots, O - D_n$ share the same overflow trunk group $O - T$ which carries additional direct traffic (cf. Fig. 3). The direct traffic streams corresponding to $O - D_i$ overflow to the common trunk group $O - T$ with n trunks if the corresponding direct routes are blocked.

If the offered streams are modeled by independent Poisson processes and the call holding times are mutually independent, exponentially distributed random variables with a common mean, no explicit analytical representation of the steady-state distribution of the resulting Markovian model exists for more than one overflow stream.

For one overflow stream, however, its steady-state distribution has been derived by Brockmeyer [12]. Recently Pearce [98] has generalized the corresponding results. Kosten [65] derived a partial analytical representation of the steady-state distribution based on a reduction approach. But it has no completely closed form since some terms have to be calculated from the solution of a system of linear equations.

If the overflow streams are approximated by special recurrent point processes with hyperexponentially distributed interarrival times known as interrupted Poisson processes (IPP), an analytical solution of the steady-state distribution of this model has been provided by Kuczura among others (cf. [75], [100]).

Recently, Meier-Hellstern [83] has presented a numerical solution based on an MMPP/$M/n/n$ loss model. It exploits the representation of an IPP by a Markov-modulated Poisson process (MMPP) and the property that the superposition of independent MMPP's is still an MMPP (cf. [95], [83], [82]).

Considering this model, we want to demonstrate the use of an iterative method based on an R-regular splitting. We assume that the common overflow trunk group consists of n trunks. The number of busy trunks at time $t \geq 0$ in this fully available group will be denoted by $X(t)$. The offered load consists of a Poisson stream (stream 0) with intensity $\lambda_0 > 0$ and two overflow streams (1 and 2). These streams are assumed to be independent of each other. The mean call holding time is denoted by $1/\mu$.

Following a standard approach in teletraffic theory (cf. [82], [83]), the overflow streams are modeled by two interrupted Poisson processes (IPP's) resulting from a two-moment approximation. They will be represented by two mutually independent Markov-modulated Poisson processes (MMPP's 1 and 2) with the generator matrices

$$Q_1 = \begin{pmatrix} -\gamma_1 & \gamma_1 \\ \omega_1 & -\omega_1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} -\gamma_2 & \gamma_2 \\ \omega_2 & -\omega_2 \end{pmatrix}$$

and the rate vectors

$$\hat{\lambda} = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}, \quad \hat{\lambda} = \begin{pmatrix} \lambda_2 \\ 0 \end{pmatrix}$$

(cf. [82], [83], [81], [95], [74]). Here, $\lambda_i > 0$ is the intensity of the Poisson process associated with the IPP stream i, $1/\gamma_i$ its mean on-time and $1/\omega_i$ its mean off-time, $i \in \{1, 2\}$ (cf. [74, p. 438]). Obviously, each Poisson stream may be regarded as an MMPP, too.

Let us denote the phase of the controlling CTMC in the MMPP representation of the IPP $i \in \{1, 2\}$ at time $t \geq 0$ by $Y_i(t)$. Its associated irreducible generator matrix is Q_i. As the superposition of independent MMPP's is still an MMPP, the Markovian environment resulting from the composition of the arrival streams is given by $Y(t) = (Y_1(t), Y_2(t))$. It possesses the irreducible generator matrix

$$Q = Q_1 \oplus Q_2 = \begin{pmatrix} -\gamma_1 - \gamma_2 & \gamma_2 & \gamma_1 & 0 \\ \omega_2 & -\gamma_1 - \omega_2 & 0 & \gamma_1 \\ \omega_1 & 0 & -\omega_1 - \gamma_2 & \gamma_2 \\ 0 & \omega_1 & \omega_2 & -\omega_1 - \omega_2 \end{pmatrix} \in \mathbb{R}^{m \times m},$$

$m = 4$. Its states $Y(t) = (k_1, k_2) \equiv k$ will be ordered lexicographically and enumerated by integers $k \in \{1, 2, 3, 4\}$. The rate vector of the resulting MMPP is given by $\hat{\lambda} = (\lambda_0 + \lambda_1 + \lambda_2, \lambda_0 + \lambda_1, \lambda_0 + \lambda_2, \lambda_0)' >> 0$.

Then the model of alternative routing may be regarded as $M/M/n/n$ loss system in a Markovian environment. It can be described by an irreducible CTMC $Z(t) = (X(t), Y(t))$, $t \geq 0$, with a finite state space $S = \{(i, k) \mid 1 \leq k \leq 4; 0 \leq i \leq n\}$.

If we suppose a lexicographical ordering of states, the negative transpose $A = -\tilde{Q}' \in \mathbb{R}^{L \times L}$, $L = m \cdot (n + 1)$, of the irreducible generator matrix $\tilde{Q}$ associated with $Z(t)$ has the following block tridiagonal structure:

$$A = \begin{pmatrix} -Q' + \Lambda & -1\mu I_4 & 0 & \cdots & \cdots & 0 \\ -\Lambda & -Q' + 1\mu I_4 + \Lambda & -2\mu I_4 & \ddots & \cdots & \vdots \\ 0 & -\Lambda & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \ddots & \ddots & -n\mu I_4 \\ 0 & \cdots & \cdots & 0 & -\Lambda & -Q' + n\mu I_4 \end{pmatrix}. \tag{5.1}$$

Here, $\Lambda_0 = \lambda_0 I_4$, $\Lambda_1 = \text{diag}(\lambda_1, \lambda_1, 0, 0)$, $\Lambda_2 = \text{diag}(\lambda_2, 0, \lambda_2, 0)$ are the arrival rate matrices of the Poisson stream and the two IPP's. $\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2$ is the arrival rate matrix of the MMPP resulting from their superposition and I_k denotes the identity matrix of order k.

Taking advantage of the block structure of A, the steady-state vector π of $Z(t)$ may be computed either by a block iterative scheme derived from an R-regular splitting of A or by an accelerated point iterative scheme such as JOR or SOR (cf. [71], [101], [83]).

Such an R-regular splitting technique has been suggested by Meier-Hellstern [82, §3], [83], but without a rigorous mathematical proof of the convergence of the resulting iterative scheme. The proposed procedure is based on the following block splitting $A = M - N$:

$$M = \begin{pmatrix} -Q' + \Lambda + n\mu I_4 & 0 & \cdots & \cdots & 0 \\ -\Lambda & -Q' + \Lambda + n\mu I_4 & \ddots & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\Lambda & -Q' + \Lambda + n\mu I_4 \end{pmatrix}$$

$$= D - L \tag{5.2}$$

$$N = \begin{pmatrix} n\mu I_4 & 1\mu I_4 & 0 & \cdots & 0 \\ 0 & (n-1)\mu I_4 & 2\mu I_4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & 1\mu I_4 & n\mu I_4 \\ 0 & \cdots & \cdots & 0 & \Lambda \end{pmatrix}.$$

$$= L(N) + U(N) + D(N) \geq 0 \tag{5.3}$$

Obviously, $L(N) = 0$, $D = (D_{ii})_{i=0,\ldots,n} = I_{n+1} \otimes (-Q' + \Lambda + n\mu I_4)$ and $D(N) = \text{diag}(n\mu e', (n-1)\mu e', \cdots, 1\mu e', \hat{\Lambda}') \geq 0$,

$$L = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \ddots & \cdots & \vdots \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \otimes \Lambda \geq 0,$$

$$U(N) = \begin{pmatrix} 0 & 1\mu & 0 & \cdots & 0 \\ \vdots & \ddots & 2\mu & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & \ddots & n\mu \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \otimes I_4 \geq 0$$

hold, where e denotes the vector of all ones. $\otimes$ is the Kronecker product of two matrices defined by $A \otimes B = (A_{ij} \cdot B)$ (cf. [94, p. 53]). Thus, $D_{ii} = -Q' + \Lambda + n\mu I_4$ is an irreducible, column diagonally dominant, regular M-matrix. Hence, $D_{ii}^{-1} >> 0$ holds (cf. [9, theorem 6.2.7, p. 141]).

The matrix $A_0 = D - L - U(N)$ has the same zero structure as A. Thus, it is an irreducible, block tridiagonal M-matrix. The corresponding block matrix graph $\Gamma(A_0)$ possesses a monotone decreasing cycle, for instance, [1, 0, 1] since $(A_0)_{10} = -\Lambda \neq 0$ and $(A_0)_{01} = -\mu I_4 \neq 0$ hold.

Therefore, the proposed block splitting $A = M - N$ defined by (5.2), (5.3) is an R-regular splitting. Moreover, it is an M-splitting since the diagonal blocks of M are regular M-matrices. Hence, we conclude from Rose's convergence theorem [101, theorem 1, p. 138] that the resulting block iterative procedure $x^{(k+1)} = M^{-1}Nx^{(k)}$, $k = 0, 1, \cdots$ converges for each initial vector $x^{(0)} >> 0$ to the steady-state vector π of the CTMC $Z(t)$ if the iteration vector $x^{(k+1)}$ is normalized after convergence.

Of course, this algorithm may be accelerated by inserting several aggregation/disaggregation steps during the iteration according to the IAD scheme (see Sections IV-C and V-B).

In comparison with the block Gauss-Seidel scheme, the proposed R-regular splitting procedure has the advantage that all diagonal blocks of the matrix M are identical. Therefore, it is necessary to decompose only one small matrix D_{ii} and to store its inverse during the iteration process. The resulting algorithm is well suited for an imple-

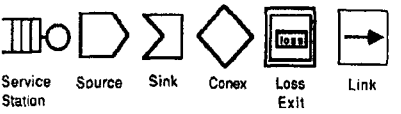

Fig. 2. The elements of the MACOM model world.

mentation on a vector processor. Experimental results of the scheme have been provided by Meier-Hellstern [82].

B. A Loss System with Mutual Overflow and External Traffic

The next example concerns the study of an adaptive routing strategy in circuit-switched networks. We consider a network consisting of a local exchange (EX_0) and two exchanges of the long-distance network (EX_1, EX_2) which are connected to each other by two both-way trunk groups. The traffic outgoing from the local exchange is split into two portions and each portion is offered to an outgoing group. These partial traffic streams are routed according to an adaptive routing scheme called mutual overflow routing (cf. [68], [69], [79], [33]). Additionally each trunk group carries external traffic (cf. Fig. 4).

The network may be described by a loss system composed of two fully available trunk groups called systems 1 and 2 with N_1 and N_2 lines. Two originating traffic streams (streams 2 and 3) representing the portions of the outgoing traffic and two incoming external traffic streams (streams 1 and 4) are offered to the loss system. These arrival processes are modeled by mutually independent Poisson processes with positive rates λ_2, λ_3 and λ_1, λ_4.

The external traffic streams 1 and 4 offered to systems 1 and 2 follow a random hunting scheme for free lines. Their calls are lost without further impact on the system if the corresponding trunk group is busy upon arrival. The outgoing streams 2 and 3 follow a mutual overflow routing scheme. This means that, upon arrival at system 1, a call of flow 2, for instance, is searching for a free line. If possible, a free trunk is selected in a random manner and occupied. If system 1 is busy and there are free lines in system 2, the incoming call of flow 2 will immediately overflow to system 2 upon arrival and occupy a line selected at random. If both systems are busy, the call will be blocked and lost without further impact on the system (lost calls cleared).

Call holding times are considered to be mutually independent, exponentially distributed random variables with a common finite mean $1/\mu$. They are also assumed to be independent of the arrival processes.

It is obvious that the occupation process of both trunk groups in this loss system can be modeled by an irreducible CTMC $X(t) = (X_1(t), X_2(t))$, $t \geq 0$ with finite state space $S = \{(i, j) \mid 0 \leq i \leq N_1;\ 0 \leq j \leq N_2\}$, where the state variables $X_i(t)$, $i = 1, 2$, denote the number of busy trunks in group i at time t.

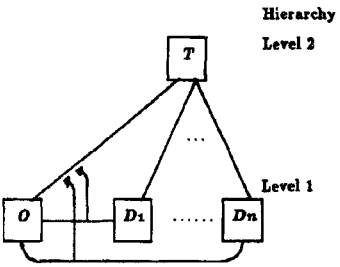

Fig. 3. Part of a 2-level circuit-switched network with alternative routing.

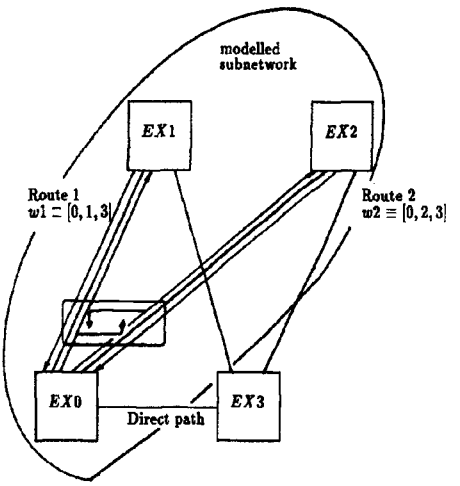

Fig. 4. Network with mutual overflow routing.

We assume a lexicographical ordering of states and denote the negative transpose of the corresponding generator matrix Q of $X(t)$ by $A = -Q^t \in \mathbb{R}^{N \times N}$, $N = (N_1 + 1) \cdot (N_2 + 1)$. It is an irreducible Q-matrix and has a block tridiagonal structure

$$A = \begin{pmatrix} B_0 & C_0 & 0 & \cdots & \cdots & \cdots & 0 \\ D_1 & B_0 & C_1 & \ddots & \cdots & \cdots & \vdots \\ 0 & D_2 & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & B_k & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & \ddots & \cdots & \ddots & C_{N_1-1} \\ 0 & \cdots & \cdots & \cdots & 0 & D_{N_1} & B_{N_1} \end{pmatrix} \in \mathbb{R}^{N \times N} \tag{5.4}$$

with irreducible, tridiagonal M-matrices

$$B_0 = \begin{pmatrix} a & -1\mu & 0 & \cdots & \cdots & \cdots & 0 \\ -a_2 & a+1\mu & -2\mu & \ddots & \cdots & \cdots & \vdots \\ 0 & -a_2 & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & a+k\mu & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & -N_2\mu \\ 0 & \cdots & \cdots & \cdots & 0 & -a_2 & a_3+N_2\mu \end{pmatrix},$$

$$B_{N_1} = \begin{pmatrix} a_4 & -1\mu & 0 & \cdots & \cdots & \cdots & 0 \\ -a_4 & a_4+1\mu & -2\mu & \ddots & \cdots & \cdots & \vdots \\ 0 & -a_4 & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & a+k\mu & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & \ddots & \ddots & a_4+(N_2-1)\mu & -N_2\mu \\ 0 & \cdots & \cdots & \cdots & 0 & -a_4 & N_2\mu \end{pmatrix} + N_1\mu \cdot I,$$

$B_i = B_0 + i\mu \cdot I$, $i = 0, \cdots, N_1 - 1$, of order $N_2 + 1$ along the diagonal. The off-diagonal matrices $D_i = -\operatorname{diag}(a_1, \cdots, a_1, a_3)$, $i = 1, \cdots, N_1$, $C_i = -(i+1)\mu \cdot I$, $i = 0, \cdots, N_1 - 1$, have diagonal structures. Here, we have set $a_1 = \lambda_1 + \lambda_2 > 0$, $a_2 = \lambda_3 + \lambda_4 > 0$, $a_3 = \lambda_1 + \lambda_2 + \lambda_3 \geq a_1$, $a_4 = \lambda_2 + \lambda_3 + \lambda_4 \geq a_2$, $a = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a_1 + a_2$ and λ_i denotes the intensity of the offered Poisson stream $i \in \{1, \cdots, 4\}$. $1/\mu$ is the mean call holding time and I the identity matrix.

Hence, A is a consistently ordered 2-cyclic Q-matrix w.r.t. this block partition ([123, p. 102]). Moreover, it can be shown that A is also a consistently ordered 2-cyclic matrix w.r.t. the point partition and that it has property A (cf. [71], [123, p. 187], [101], [9]). Thus, the point Jacobi matrix $J = D^{-1}(L + U)$ is cyclic of index 2, i.e., the greatest common divisor of the lengths of all proper cycles in the matrix graph of A is 2. According to [6, prop. 1, p. 392] the Jacobi procedure is not convergent (see also [9, theorem 2.2.30, p. 35], [71, theorem 29, p. 56]), whereas the corresponding JOR and SOR procedures with the iteration matrices $J_\omega = (1-\omega)I + \omega J$ and $L_\omega = (D - \omega L)^{-1}((1-\omega)D + \omega U)$ are convergent for each relaxation parameter $\omega \in (0, 1)$ (see [7, theorem 3.9], [71, p. 73], [6, cor. 3, p. 395], [14, theorem 3.4, p. 191]).

As A has a symmetric zero structure, we conclude from Result 2 that the point Gauss–Seidel procedure is convergent, too (see also [101, cor. 2, p. 139], [105, cor. 3.8, p. 417], [71, theorem 35, p. 66]). Taking into account $M_{21} = (D - L)_{21} = -a_2 < 0$ and $N_{12} = U_{12} = \mu > 0$, this result also follows immediately from Result 1. This point iteration may be accelerated by applying the standard overrelaxation technique (cf. [57], [117]) or a semi-iterative technique such as the stationary or nonstationary Chebyshev method or Eiermann's stationary fourth-order scheme (cf. [123, §5], [29], [96], [30, lemma 8.4, p. 28], [7], [46], [47]). The main difficulty w.r.t. these procedures concerns the determination of "optimal" relaxation parameters ω. As there is no *a priori* information about the location of the eigenvalues of the iteration matrices, heuristic procedures estimating approximately optimal parameters are the only practicable approach (cf. [48, §9.5, p. 223ff]. [111], [46], [47]). But in the case of a consistently ordered 2-cyclic matrix the well-known relation $(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2$ (see [123, theorem 4.3, p. 106]) between the eigenvalues μ and λ of the (block) Jacobi and SOR iteration matrix may be exploited (cf. [123, §4.3, p. 109]).

An alternative is provided by block iterative schemes such as the block Gauss–Seidel procedure or its modified versions based on Rose's R-regular splitting (see [83], [73]). According to Result 3, the block Gauss–Seidel scheme derived from the given block tridiagonal structure (5.4) of A is convergent since the diagonal blocks are irreducible, regular M-matrices.

All methods may be combined with A/D steps if Schweitzer's IAD procedure is used (see Section IV-C—cf. [90], [66], [119]). The transformation of each block of A into a single scalar is a very natural way of aggregation. This procedure corresponds to the aggregation of each macrostate $\{(i, j) \mid 0 \leq j \leq N_2\}$, $0 \leq i \leq N_1$, into

TABLE IV
COMPARISON OF THE SOLUTION METHODS BASED ON THE LOSS SYSTEM WITH MUTUAL OVERFLOW AND EXTERNAL TRAFFIC WITH $\lambda_1 = 40$, $\lambda_2 = 30$, $\lambda_3 = 60$, $\lambda_4 = 10$, $\mu = 1$, $N_1 = 30$, $N_2 = 60$. THE MODEL HAS $N = 1891$ STATES AND THE GENERATOR MATRIX HAS 9271 NONZERO ELEMENTS. THE REQUIRED ACCURACY OF THE POINT ITERATION PROCEDURES IS $\epsilon = 10^{-6}$. THE DATA ARE COMPUTED BY THE PACKAGE MACOM ON A SUN 3/470 WITH FLOATING POINT ACCELERATOR

Method	Relaxation parameter	Number of A/D steps	Number of iterations	Time for the solution in sec
SOR	1.2	-	180	16.8
	1.3	-	124	11
	1.4	-	144	13
	dynamic	-	100	9.1
SOR-A/D	1.0	10	102	16.2
	1.2	10	116	16.1
	1.3	2	104	13.9
	dynamic	3	140	19.6
JOR-A/D	0.9	51	216	38.3
	dynamic	45	201	33
JOR	0.9	-	> 500	-
Grassmann's algorithm		Fill-in 1.37 %		67
direct LU-factorisation		Fill-in 1.37 %		64.8

a single state $\tilde{i}$. In this way, a simple birth-death process is generated. Its aggregated steady-state distribution α may be calculated from an explicit analytical representation in an efficient way.

A comparison of different solution techniques yields the results shown in Table IV.

All experiments performed so far confirm our view that the JOR and SOR procedures with optional aggregation-disaggregation are suitable solution techniques for large telecommunication models. Similar observations have been reported by Kaufman [57], [58], too. Furthermore, Chan's theoretical investigations [20], [21] have proved that SOR methods are superior to CG-type procedures and projection techniques if models of this type are considered. In conclusion, we recommend to use the proposed direct and iterative methods as universal procedures for finite Markov chains.

VI. CONCLUSIONS AND PERSPECTIVES

We have presented a computational approach for modeling and analyzing advanced communication systems based on numerical solution techniques for finite Markov chains. From a practical point of view, it is desirable that these solution techniques are supported by convenient software tools which facilitate the use of the various algorithms. For this reason, a software package called MACOM has been developed.

MACOM provides the user with a predefined Markovian model world describing modern telecommunication networks with adaptive routing schemes and advanced congestion-control mechanisms. The package is endowed with a user-friendly graphical interface that facilitates the interactive specification of models. A Markov chain is automatically generated from this graphical specification and its steady-state distribution is computed by advanced direct or iterative numerical procedures. In this context, we have discussed the algebraic background of some direct and iterative solution methods which may be employed to calculate the stationary distribution of an irreducible, homogeneous, discrete- or continuous-time Markov chain with finite state space. The direct LU-factorization method, Grassmann's algorithm, and the point iteration methods JOR and SOR with optional aggregation-disaggregation according to the IAD scheme constitute the numerical solver of the package MACOM. At present, the incorporation of block solution techniques such as the block LU-factorization and the block SOR procedure is considered. Regarding the analysis, one of these methods may be applied to investigate the Markovian queueing models describing advanced telecommunication systems.

The package also offers the evaluation of different performance characteristics of a model which are specified by the user. Moreover, it supports the representation of the calculated results.

MACOM can cope with large Markovian models. The versatility of the computational approach implemented by MACOM has been illustrated by the application of the proposed algorithms to some examples arising from modern telecommunication networks (cf. [57], [69], [70], [83]).

At present, a prototype of the package is available on a SUN3 workstation. It is employed for modeling and performance analysis of adaptive routing schemes in circuit-switched networks. Further development of the tool includes the improvement of the graphical specification and the representation of results, as well as the extension of the model world.

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