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ORIGINALARBEIT/ORIGINAL ARTICLE

Colouring a Constant in Figural Growing Patterns as an Approach to Understanding the Structure of Linear Functions: an Exploratory Study with German Primary School Children

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Abstract In addition to basic calculation skills, mathematical competencies that focus on understanding relationships and structures are also gaining importance in primary school education. Functional relationships are a key component of this. Currently, in Germany only proportional relationships are routinely addressed. The project presented in this paper investigates a possible approach to non-proportional linear functions via figural growing patterns. The approach used here is to directly prompt the children to colour an additive constant in the pattern in order to make the rate of change visible at the same time. Although colouring approaches are worked on in previous studies, the colouring of a constant approach is novel and, moreover, primary school children have not been engaged with it before now. The results of an exploratory study with over 200 third and fourth graders indicate that the approach can be fruitful and that highly significant correlations and medium size effects between colouring and functional thinking strategies can be revealed. However, the children's functional thinking strategies demonstrate that the project approach of colouring a constant does not necessarily result in explicit functional thinking. It may therefore be worthwhile to pursue the approach in more depth in follow-up studies.

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Einfärben einer Konstanten in wachsenden Figurenfolgen als Zugang zum Verständnis der Struktur linearer Funktionen: Eine explorative Studie mit deutschen Grundschulkindern

Zusammenfassung Neben grundlegenden Rechenfertigkeiten gewinnen auch mathematische Kompetenzen, die sich auf das Verstehen von Zusammenhängen und Strukturen konzentrieren, in der Grundschulbildung zunehmend an Bedeutung. Funktionale Zusammenhänge sind dabei ein wesentlicher Baustein. In Deutschland werden aktuell oft nur proportionale Zusammenhänge thematisiert. Das vorliegende Projekt untersucht einen möglichen Zugang zu nicht-proportionalen, linearen Funktionen über Muster aus wachsenden Figurenfolgen. Hierbei wird der Ansatz genutzt, eine additive Konstante im Muster zu färben, um gleichzeitig die Änderungsrate sichtbar zu machen. Dieser Ansatz ist neu, da vorliegende Studien zwar Ideen des Einfärbens, aber nicht des Färbens einer Konstanten untersucht haben und die Studien zudem bisher nicht mit Grundschulkindern durchgeführt wurden. Ergebnisse einer explorativen Studie mit über 200 Dritt- und Viertklässlern deuten an, dass der Zugang fruchtbar sein kann und hoch signifikante Korrelationen und mittlere Effekte zwischen Färben und funktionalen Denkweisen aufgezeigt werden können. Gleichwohl wird an den von den Kindern genutzten funktionalen Denkstrategien deutlich, dass der Projektzugang des Einfärbens einer Konstanten nicht unbedingt in expliziten funktionalen Denkweisen mündet, und es sich daher lohnen könnte, den Ansatz in Folgestudien vertieft zu verfolgen.

Schlüsselwörter Wachsende Figurenfolgen · Funktionales Denken · Einfärben-Ansatz · Algebraisches Denken · Primarstufe

1 Introduction

Blanton and Kaput suggested as early as 2011, that “the increasingly complex mathematics of the 21st century will require children to have a type of elementary school experience that goes beyond arithmetic and computational fluency to attend to the deeper underlying structure of mathematics” (Blanton and Kaput 2011, p. 6). One of these structures in mathematics are linear functional relationships. The current German primary school standards for mathematics include this topic under the title “pattern, structures, and functional relationships” (KMK 2022).

However, linear functional relationships appear rather occasionally in primary school mathematics textbooks or classes in Germany. The focus is more on proportional relationships as one special type of linear functions. Where tasks on non-proportional linear functions can be found, they are represented numerically in number sequences. If the representation of functional relationships in figural growing patterns is used in German mathematics textbooks at all, these tasks address special figurate numbers like triangle, rectangle or quadratic numbers, which represent quadratic functional relationships. Figural representations of linear functions in growing patterns are usually missing. Hence, figural growing patterns of non-proportional linear functional relationships, which are used in this study, offer a novel approach for German primary school children to understanding non-proportional linear functional relationships.

Functional thinking becomes apparent in various thinking strategies (e.g. recursive, explicit; cf. section 2.1). Moreover, functional thinking is—besides, inter alia, generalised arithmetic and equations—one important branch of algebraic thinking in early algebra (e.g. Blanton et al. 2019; Kaput 2008; Kieran 2018). Various studies underline the potential of tasks on figural growing patterns as discrete representations of functional relationships on the set of natural numbers (e.g. Akinwunmi 2012; Cooper and Warren 2008; Radford 2011; Twohill 2018; Warren 2005; Wilkie 2016). Patterning activities (finding next, near, and far figures) have the potential to pave the way for understanding the structure of functional relationships and for explicit descriptions or even algebraic expressions of these relationships.

Understanding the structure of non-proportional linear functional relationships $n \to an + b$ in $\mathbb{N}$ means to identify the additive component (constant b) and the multiplicative component (rate of change a multiplied by the independent variable n). This understanding reveals itself in explicit thinking strategies used by the children. Vice versa explicit description can only be performed, if the components of the function are identified and the structure is understood. Studies with adults and secondary school children (e.g. Strømskag 2015; Nilsson and Eckert 2019) indicate that tasks, in which the independent variable n is coloured in the figures of growing patterns, are supportive to develop explicit thinking strategies. Yet, comparable studies addressing primary school children do not currently exist.

The exploratory study reported in this paper was carried out within the ZADIE project (Steinweg et al. 2023). The project aims to support understanding of non-proportional linear functions represented in figural growing patterns by directly prompting the children to colour constant elements (in each figure). This approach to understanding the mathematical structure is new in two aspects: (I) a constant (b) and not the independent variable n is coloured, and (II) the colouring is not given by the teacher or researcher but as an activity to the children to spot constant elements in each figure and to colour them. Through this novel approach children are supported to perceive the structure of the pattern by "seeing the general in the particular" (Mason and Pimm 1984) and, in this way, the pattern acts as a generic example of a linear functional relationship.

This paper documents results of a short-term exploration which examines how far the colouring approach supports explicit functional thinking strategies, to thereby promote understanding of the structure of non-proportional linear functions. The findings presented here resulted from the analysis of individually completed written responses by the participating children ZADIE project worksheets (Steinweg et al. 2023). Since the children's unedited original works constituted the database, individual functional thinking strategies were available for investigation.

2 Theoretical Framework

Tasks on functional relationships offer children the opportunity to experience variables not only as unknowns or indeterminates, but rather in terms of the variable aspect of varying values (Freudenthal 1983; Küchemann 1978) in functional relationships (Ursini and Trigueros 2001). To enable children access to the concept of a variable, it is important to use suitable representations of functional relationships in school approaches to the mathematical structure of (linear) functions, in which figural growing patterns are one possible representation. The colouring approach, focused on in this paper, might serve as one of many catalysts to support children's understanding of the mathematical properties of functional relationships and to develop their functional thinking towards explicit thinking strategies.

2.1 Functional Thinking Strategies with Growing Patterns

Working on growing patterns (or other representations of functional relationships) is intended to promote functional thinking, which is key for many mathematical problems. Stephens et al. (2017) define functional thinking "as the process of building, describing, and reasoning with and about functions" (p. 144).

For many decades studies have identified strategies spontaneously used by children in tasks on growing patterns (e.g. Stacey 1989). The development of functional thinking can emerge in fundamentally different strategies (e.g. Smith 2008), which can be labelled as recursive, co-variation, and explicit or correspondence:

- *recursive*: Consecutive values of the dependent variables are calculated by attending to the difference of the values from one figure to the next (difference sequence). It is possible to determine the next (successive) value. A connection to the respective position in the sequence (independent variable) is not established.
- *co-variation*: Two (or more) successive pairs are considered in both changes, i.e. attention is focused on the change of the independent variable (position) and the simultaneous change in the dependent variable (value). The focus is on the rate of change in a local perspective.
- *explicit* or *correspondence*: A general explicit rule for the functional relationship of all pairs can be given, i.e. the identification of a constant (if given) in interplay with covariant change is considered. The general relationship of all pairs between independent position and dependent values is thus described as a rate of change and constant. The determination of values at arbitrary positions is directly possible.

Wilkie (2016) emphasises this fundamental difference between co-variation and correspondence:

> Co-variation highlights the relationship between successive items in a pattern—also known as a local rule (Mason 1996)—and correspondence describes the relationship between two quantities or variables (usually the item/term

position number in the pattern/sequence and a quantifiable aspect of the item/ term itself—also known as a direct or closed or relational rule) (Wilkie 2016, p. 252).

The co-variative strategy focuses on the rate of change, i.e. the factor in the multiplicative component of the structure of the non-proportional linear function, thus ignoring the additive component, the constant. The co-variative strategy carries the risk of overgeneralising (Richter 2014) the relationship discovered for two pairs to all pairs and transferring this local rule unchecked into a general rule. For example, it can happen that it is mistakenly assumed that the discovered relationship can be transferred additively or multiplicatively, as is the case with proportional structures only (cf. section 2.2).

To encourage explicit functional thinking strategies, which enable children to become aware of the components of the mathematical structure of non-proportional linear functions, it is important to provide challenging and supportive tasks that scaffold the development of functional thinking in mathematics lessons.

2.2 Approaches to Functional Relationships Via Different Representations

In German schools, linear functions are usually first introduced in secondary school using representations of symbolic expression $x \rightarrow ax + b$, which consists of multiplicative and additive elements, value tables, and plots of function graphs in the coordinate system. Children learn how the parameters a and b affect linear graphs, i.e. the parameter a indicates the rate of change or gradient (slope) and the constant b the dependent value for the independent argument zero (y-axes intercept).

In primary school linear functions are not communicated through symbolic expression or linear graphs in a coordinate system. Representations, that have been used successfully with primary school children—and thus shown to be accessible for this age group—are numerical representations such as number sequences, tables, input-output machines (e.g. Ng 2018; Moss and London McNab 2011) or figural representations of growing patterns (e.g. Akinwunmi 2012; Warren 2005).

In German primary schools, linear functional relationships are usually presented as a particular type of proportional relationship (constant $b = 0$). Typical tasks focus on proportional relationships between quantities like price—number of items and only these examples are explicitly mentioned in the mathematics education standards for primary schools (KMK 2022). However, it should be noted that specific mathematical characteristics like the option to add values, e.g. adding the 2nd and 3rd value to get the 5th one (addition property), or to multiply values, e.g. multiply the 4th value by ten to get the 40th value (multiplication property), only hold for proportional relationships and must not be generalised on linear functions, where these may incorporate a non-zero constant. It should be also kept in mind that recursive strategies work for proportional functional relationships, for which the identified sequence of differences between the values (always +3) can be generalised in a valid rule $n \rightarrow 3n$, whereas this functional thinking strategy does not hold for linear functions with $b \neq 0$. Consequently, to understand non-proportional linear functions it is necessary that children not only rely on recursive strategies but broaden their repertoire of function thinking strategies by adding explicit strategies.

Indeed, if the focus is exclusively on proportional functions, tendencies to overgeneralise properties of proportional relationships can also be found in linear or even quadratic functions. For example, De Bock et al. (2002) document in responses of secondary school students an “alarmingly strong tendency to apply proportional reasoning ‘everywhere’” (p. 314). Also, Richter (2014, p. 58) refers to a wide range of studies that reveal these tendencies of proportional overgeneralisation in learner’s solutions on functional tasks at the end of lower secondary school and sometimes even beyond (e.g. De Bock et al. 2010; Van Dooren and Greer 2010).

The focus on proportional relationships exclusively is therefore problematic when considering connectivity in the learning process across all school grades, as it limits opportunities to develop functional thinking. International studies have demonstrated that functional thinking can be addressed in a broader sense in primary school mathematics. Tasks on patterns and especially figural growing pattern, as already mentioned above, have been shown to be a suitable and promising context (e.g. Akinwunmi 2012; Cooper and Warren 2008; Radford 2011; Twohill 2018; Warren 2005; Wilkie 2016).

2.3 Functional Relationships in Figural Growing Patterns

Growing patterns can be seen as figural representations of functions over the set of natural numbers $\mathbb{N} \rightarrow \mathbb{N}$. A spatial arrangement of elements—squares, dots, matchsticks, etc.—grows infinitely from one figure to the next (left-total). The growing pattern represents “a functional relationship between position and numerical value of an element” (Strømskag 2015, p. 475). The position of the figures in the pattern indicates the independent variable whereas the value of the dependent variable equals the quantity of elements (right-unique).

Consequently, exploring a given growing pattern requires one to interpret “a regularity that involves the linkage of two different structures: one spatial and the other numerical” (Radford 2011, p. 19). A semiotic perspective highlights the challenging nature of this interpretation because figural (spatial) and numerical representations as two different “sign systems” (Warren and Cooper 2008, p. 178) need to be merged. Hence, it is important to represent both in an accessible way for the children. For example, Cooper and Warren (2008) recommend the use of “step cards” (p. 34) in growing patterns (Fig. 1). In their research, they used tiles to create the pattern figures and “small cards with position numbers” (Warren and Cooper, 2008, p. 177) were placed with each figure to indicate the position in the pattern. In this representation, the binary relation of the independent and the dependent variable is

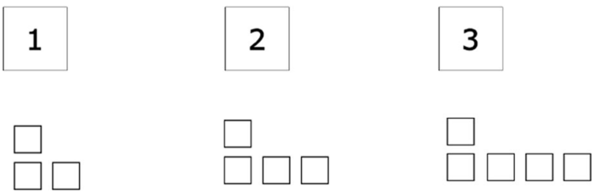


Fig. 1 A growing pattern as a spatial arrangement of squares, with the position of figures indexed by number cards

made visually accessible for the children. Figural growing patterns allow children to view a given or established relationship holistically. Vollrath (2014, p. 119) refers to a ‘view as a whole’, in which one “no longer considers individual pairs of values, but the set of all pairs of values” [translated by the authors].

The challenges in discovering the relationship between the figures in spatial arrangements and the numerical position in a growing pattern result in a broad range of reactions to tasks on growing patterns. Twohill (2018; 2025) identified four crucial growth points in patterning:

> recognise commonality or regularity and continue the pattern (*next* figure), verbally describe *some* aspects of the pattern,
> find *near and next* figures with reasoning, verbally describe the pattern,
> find *near and far* figures with reasoning, verbally describe the pattern *explicitly*,
> generate *far* figures by using the rule, describe *the rule in symbolic* notation[1].
> (Twohill et al. 2025)

These activities aim to support explicit functional thinking strategies and exploit the opportunities of patterning in figural representations to enable children to understand the structure of (linear) functions.

In numerical representations like tables or sequences, the total value of the dependent variable in a linear function is already calculated and the structure of the linear function is therefore hidden in the numerical values. In contrast, the figural representation allows children to recognise the algebraic structure of the two components, i.e. the multiplicative structure which combines the rate of change with the independent variable (*an*) and the additive constant (*b*) in $n \to an + b$. This approach resonates with Hewitt’s recommendation to “never carry out any arithmetic” without it being visible (Hewitt 2019).

Mielicki et al. (2021) showed in studies with children at lower secondary level for tasks involving non-proportional functions, that figural growing patterns have a positive effect on solution rates compared to purely numerical representations (number sequences, tables). Also, the findings of El Mouhayar and Jurdak (2016) provide further and enlightening information on this decision, whether to provide numerical or figural representations to foster explicit (correspondence) functional thinking strategies. In their study they “select a range of tasks that involved students in generalizing immediate, near and far pattern generalization types” (El Mouhayar and Jurdak 2016, p. 201) of figural growing patterns and analysed strategies of children from grade 4 to 11. In the representations El Mouhayar and Jurdak (2016) provided the number of elements underneath each figure. Hence, the representation allows thinking strategies which refer to the figural arrangements (“figural reasoning”) or to the numerical values only (“numerical reasoning”). They conclude, “students using numerical reasoning approach are more likely to use recursive strategy whereas students using figural reasoning are more likely to use functional strategy” (El Mouhayar and Jurdak 2016, p. 213–214); whereby this ‘functional strategy’ equates to explicit or correspondence functional thinking (Smith 2008; cf. section 2.1).

[1] Symbolic notation—here included in the last growth point—constitutes a further challenge for primary school children. The study presented here accepts verbally given explicit rules as equivalent.

These findings are in line with Lannin (2005), who summarized: “The value of situations that allowed for geometric schemes that connected the rule with a visual representation … increased the likelihood of student success” (p. 252). Therefore, it can be concluded that growing pattern tasks are fruitful for understanding functional relationships and offer advantages over numerical representations of functional relationships in supporting children’s explicit functional thinking. The study presented in the paper, therefore, focuses on figural growing patterns. One documented approach to foster explicit (correspondence) functional thinking strategies is to colour the figures in a given growing pattern which will be discussed in more detail in the next section.

2.4 Colouring Approaches in Figural Growing Patterns

Figural growing patterns facilitate the circling, marking or colouring of elements of given figures, to clearly identify the structure in order to find an explicit rule or generalisation of the function. Nilsson and Eckert (2019) have shown how colour-coding can focus secondary school children’s attention to the relationship between an algebraic expression and visual structures of the pattern. The study used figural growing patterns, which are coloured by the researchers to focus the participants’ attention on the independent variable n and to identify the appropriate symbolic expression out of four options.

> The color-coding prompted the students to change perspective of n, from a number variable (how many threes to multiply) to a visual unit in the structure of a figure. In particular, the analysis shows how implementing coloring in pattern generalization task, supported students in discerning and making sense of n in the relationship between n in the structure of a general expression and n in the structure of a figure. (Nilsson and Eckert 2019, p. 620)

Steinweg (2019) asked primary school children (grade 3 and 4) to identify connections between a figural growing pattern, in which a constant is coloured by the researcher, with the symbolic expression with letter variables. Although the primary school children were given some time to explore the pattern beforehand by themselves (find the next and a far figure), this study reveals that this prior knowledge cannot be derived as a sufficient condition for understanding the expression. In this study, none of the participating children was able to interpret the variable n in the multiplicative component of the expression.

While both previously described studies used given colouring, in contrast Strømskag (2015; also c.f. Strømskag Måsøval 2011) asked the adult participants in her project to “decompose” (colour/circle/mark) given patterns by themselves.

> The point is to express the number of components of each partition of an element as a *function* of the element’s position in the shape pattern. These arithmetical expressions are used to express the total number of components of the element. (Strømskag 2015, p. 475)

Giving learners the opportunity to interpret the figural representations on their owns is in line with Rivera (2013), who emphasises perceptual diversity: “That is, especially in patterning activity, what one considers meaningfully invariant and/or changing may likely be unrecognizable or trivial to another" (p. 5). In non-proportional linear functional relationships, one needs to "make a choice between what counts as the same and the different" (Radford 2008, p. 84). At best (from a mathematical perspective) the 'same' in each figure corresponds with the constant additive parameter and the 'change' refers to the multiplicative parameter, i.e. the factor by which the independent variable is multiplied. The individual perception of a given growing pattern may be quite different however, because

> ... students are tremendously creative. However, nothing guarantees that their idiosyncratic [individual] procedures and ideas will necessarily converge with the cultural ones conveyed by the mathematics curriculum [generally accepted mathematical agreements]. (Radford 2008, p. 87)

2.5 Project Approach to Support Explicit Functional Thinking and Research Questions

In summary, the following conclusions can be drawn from the findings of the studies outlined above:

- Explicit functional thinking, which generalises the correspondence between independent and dependent variable (section 2.1), indicates understanding the structure of the given function. For non-proportional linear functions this means to identify an additive component (constant) and the corresponding multiplicative component (product of rate of change and independent variable).
- German primary school children typically meet proportional relationships and linear functions, if at all, in the representation of number sequences (section 2.2), which do not include independent variables, and thereby do not facilitate the children in developing explicit functional thinking, which is necessary to understand the structure of linear functions.
- Studies proved representations in figural growing patterns as advantageous over numerical representations in that they pave the way to understand the mathematical structure indicated by explicit functional thinking (section 2.3).
- Colouring approaches in figural growing patterns (section 2.4) with secondary school children and adults showed promising results to enable the participants to connect symbolic expressions with the figures and, therefore, to identify the structural components of the (linear) function. A study with primary school children has shown that it is almost impossible for them to interpret a *given* colouring in the symbolic representation accordingly. The opportunity to approach and understand the structure of non-proportional linear functions by colouring the figures in growing patterns on their owns has not yet been investigated with primary school children.

The exploratory study presented in this paper responds to the gaps in research identified above and intends to address these gaps as a sub-study of the ZADIE project (Steinweg et al. 2023). This study involves German primary school children, using the representation of figural growing patterns for non-proportional linear functions, and a colouring approach to support explicit functional thinking strategies.

In the ZADIE project (Steinweg et al. 2023) tasks are especially designed to support generalisations and explicit functional thinking, through the use of figural representations of growing patterns. To find an explicit rule for linear growing patterns, the figures need to be (mentally) decomposed (cf. Strømskag 2015) into the different constituent parts, i.e. the additive expression (constant b) and the multiplicative expression (ax).

Instead of spotting the variable n (Nilsson and Eckert 2019), in the ZADIE project-design (Steinweg et al. 2023) the children are asked to identify a constant (b) in the figures. This is possible, because the constant b is the value of the expression $n=0$, "which is included in all other values" (Richter 2014, p. 49, translation by the authors). In figural representations, children can exploit this property by marking or colouring the spotted constant included in each figure of a growing pattern. We expect that not all children will colour arrangements of elements as constant which equal b. Also from a mathematical perspective, other constants are certainly possible, leading to variations of equivalent expression of the function.

Typical activities with growing patterns (cf. section 2.3) provide opportunities to develop functional thinking by enabling the children to understand functional relationships, and to find an explicit description or rule. However, the step towards explicit description of functions in their mathematical structure can be challenging. The colouring activity, proposed in this paper, therefore provides a key step towards identifying the mathematical structure, which cannot be discovered without recognising the constant elements in each figure.

In our project, children are *explicitly* prompted to spot *constant elements*, to colour these elements in each figure, and to describe the relationship between the position number and the uncoloured elements of the figures. The deliberately focused act of colouring is designed to lead to the discovery of the mathematical structure of the functional relationship (Akinwunmi and Steinweg 2022) and to therefore support explicit (correspondence) functional thinking.

Explicit rules (correspondence functional thinking) can be given in verbal descriptions or condensed in symbolic expressions. It is important to emphasise that letter variables and symbolic expressions are usually not used in German primary schools. Hence, these representations (symbolic letter expressions) are not expected or addressed. Rather, generalisations of the pattern through verbal descriptions and rules are considered appropriate in line with Akinwunmi (2012), who documented various opportunities to verbally generalise patterns, i.e. in conditional sentences, as quasi-variables, as generic examples and so forth (also cf. Akinwunmi and Steinweg 2024).

This paper reports on a short-term exploratory study with German primary school children that addresses the overarching question:

> *To what extent does the approach of colouring a constant in figural growing patterns support primary school children in demonstrating explicit thinking strategies in linear functional relationships?*

To answer the research question, we investigated the following:

- what functional thinking strategies do children use in the description tasks (far figures, relationship between the changing elements and the position number, or generalisations in explicit rules),
- to what extent do the children embrace the idea of colouring a constant in the figural growing pattern, and
- if and how does the children's choice of colouring of a certain constant predict the functional thinking strategy the children adopt in the description of relationship between the uncoloured changing elements and the position number.
- whether there are significant correlations between colouring a certain constant and the functional thinking strategies used in the description tasks (by means of Spearman's rank correlation).
- whether and to what extent the colouring of a constant not only co-relates to but statistically affects the functional thinking strategies used in description tasks (by means of paired samples t-tests (Cohen's d)).
- whether there is a relationship between the school grade of the children and the choice of a constant or the functional thinking strategies adopted in the description tasks.
  As children attending both 3rd and 4th grade completed the tasks, we thought it pertinent to investigate whether the tasks are generally suitable for primary school children, and whether 4th graders, who presumably have more prior (mathematical) knowledge, respond differently to 3rd graders.

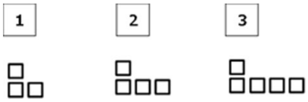
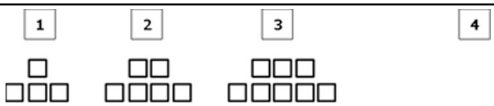
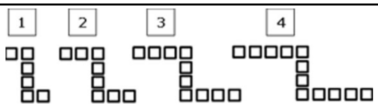
3 Methodology and Design

In this exploratory study functional relationships are represented in growing patterns of squares or dots. The spatial arrangement of single objects constitutes the figures. The squares in the pattern are deliberately not shaded, to allow children to shadow or colour them.

Each figure in the growing pattern is accompanied by a number card indicating the position in line with Warren and Cooper (2008; Fig. 1). The number cards allow the use of natural numbers rather than ordinals (1st, 2nd, etc.) for the independent variable. The children can, therefore, use the natural number on the card in calculations and expressions straight away.

The short-term intervention includes four project lessons, designed and described in more detail by Steinweg et al. (2023). The first lesson aims to familiarise the children with growing patterns and stresses the idea of the position number, represented by index cards/number cards. The big idea in this first lesson is that the quantity of squares in the figure is not the (position) number of the figure. The second lesson uses the common approach to prompt the children to generalise the pattern by asking for near (7th) and far figures (947th). The third lesson introduces the children to the idea of colouring in those elements in each figure which are constant throughout the pattern. The task is twofold: (I) identifying and colouring constant element(s) and (II) describing a relationship between the position number and the uncoloured elements in each figure. And finally in the fourth lesson the children are asked to

Table 1 Figural growing patterns and worksheet tasks of lesson 2, 3, and 4 analysed in this paper

Lesson	Pattern	Task
Lesson 2		Describe the 947th figure. Explain your thinking
Lesson 3		What stays the same? Colour the constant squares. Draw the 4th figure and colour the constant squares. Explain your thinking. Can you see a link to the position number in the squares that have *not* been coloured? Explain your thinking
Lesson 4		Can you think of a rule for this pattern? Explain your thinking

find a (general) explicit rule, which includes identifying the two important elements (constant and change) of an explicit expression by colouring.

Since we focus, in this paper, on the colouring approach and the functional thinking strategies which children adopt in their generalisations of patterns, we give insight into children's responses to four different tasks (Table 1) from three project lessons: describing a far figure (task from lesson 2), colouring a constant and describing the relationship between uncoloured elements of the figures and the number on the position cards (both tasks from lesson 3), giving a rule (task from lesson 4).

The intervention took place in ten different 3rd and 4th grade classes (five of each) as it is helpful for the children for developing explicit thinking strategies to already have some knowledge of multiplication tables. In total up to 219 children attended the intervention. Each project lesson was held on a different day so the number of participating children varied due to illness or absences.

The data used in the analysis were the children's responses to the worksheet tasks outlined in Table 1. The children completed the worksheets individually in written form and the unedited sheets were handed in. In the documented exploratory lessons, the teachers did not give an example task to be collectively solved beforehand, i.e. no strategies were discussed nor promoted by the teacher. Hence, the spontaneous responses individually given during the first encounter with the tasks were available for analysis.

In the descriptive analysis step, the written answers given by the children are inductively categorised using qualitative content analysis (Mayring 2022). In a second step categories of each task indicating similar (functional) thinking strategies are collapsed into four substantially different types of functional thinking strategies indicated by the answers (Table 2). The coding was carried out by one of the authors. If the assignment of one of the brief answers (cf. section 4.1) seemed ambiguous, the case was discussed within the research group.

Table 2 Analysis categories of the description tasks illustrated by representative examples

	Describing far figure (947th) $n \to n+2$	*Describing* *rate of change* $n \to 2n+2$	*Describing* rule $n \to 2n+4$
3_EX	Full description like "2 and 947"	Full relationship between uncoloured squares and position number like "double the number"	Full general rule like "double number *and* add 4"
2_PEX	Part of a full description, like description mentions 2 but not adding of 947	The number is identified once like "the upper row is the number"	Part of the general rule is mentioned "double number" *or* "in each figure are 4"
1_RE	Recursive (one more) *or* drawing (attempts)	Recursive like "always two more"	Recursive like "always two more"
0_IN	Incomprehensible or no answer	Incomprehensible, false, or no answer	Incomprehensible or no answer

The categories 3_EX (explicit) and 2_PEX (partially explicit) indicate explicit functional thinking because they expect to see a general structure in the given pattern (at least partially). The category 1_RE indicated recursive thinking.

Since the colourings are at the heart of the project approach, it is necessary to categorise the various constants identified and coloured by the participating children in lesson 3 ($n \to 2n+2$). Four categories (also cf. Fig. 2) are deduced inductively from the analysis of the data.

- 3_b: colouring any 2 squares
- 2_b + 1: colouring any 3 squares
- 1_1st: colouring the 1st figure (identified in each figure)
- 0_no: inconsistent, incomprehensible or no colouring

From a mathematical perspective, the different constants may lead to different but valid explicit functional descriptions/generalisations. Ideally, the choice of which squares to colour in the first figure should allow for identification of 'one' set of elements remaining in the first figure, 'two' sets in the second figure and so on, as One and Two are the cardinal number respectively of these figures. The choice of a certain constant influences children's further steps in their thinking because generalisations are derived differently:

- If the children chose to colour 2 squares (category 3_b) as constant in each figure, the quantity of the uncoloured squares equals the double of the figure number respectively. Once discovered, this multiplicative relationship is easy to describe in an explicit generalisation. The corresponding symbolic expression would be $n \to 2n+2$.
- If the children chose to colour 3 squares (category 2_b + 1) as constant, the quantity of uncoloured squares in each figure does not equal a multiple of the figure number but one less than the double. This means, the children need to (implicitly) respect that the chosen constant is one element larger than b and balance +1 by –1

in a description of an explicit generalisation referring to the independent variable n. In a symbolic expression this detour would look like $n \to 2n - 1 + 3$.

- If the children chose to colour the 1st figure (category 1_1st) as constant, the coloured four squares in each figure of the pattern leave the growing difference in relation to the first figure (2 in the 2nd figure, 4 in in the 3rd figure, 6 in the 4th figure and so on) uncoloured. The quantity of uncoloured elements equals the double of the previous figure number (predecessor of the position number). Consequently, the children need to refer to the number of the previous figure in an explicit generalisation. In a symbolic expression this would mean not to refer to n but (n–1): $n \to 2(n-1) + 4$.

The primary school children participating in this exploration do not have the ability to use symbolic expressions with letters. Hence, the children need to find equivalent verbal—sometime elaborate—description. In this way, the task to describe explicit generalisations of the rate of change or a general rule, is more challenging for the children who chose to colour the 1st figure or any 3 squares as constant (category 1_1st or 2_b + 1) than for the ones colouring 2 squares (category 3_b).

4 Research Findings

4.1 Descriptive Results

From the children's worksheets (lesson 2–4, Table 1), their written solutions of the description tasks were analysed under the strategies listed in Table 2, that is 3_EX, 2_PEX, 1_RE or 0_IN. The choice of constants coloured was analysed and assigned to the categories 3_b, 2_b + 1, 1_1st or 0_no described above in section 3.

In the following, we present the percentage of children using each of these strategies when working on the [name of project] description tasks (finding far figures, establishing a relationship between the changing elements and the position number, or generalisations in explicit rules) (first four columns in Table 3) and we identify how far the idea of colouring a constant was embraced by the children in completion of the tasks and to what extent which constant were coloured (last two columns in Table 3).

For clarity, in this section, we present examples of the children applying each of the four strategies and colourings tabulated in Table 3.

4.1.1 Task 1 (Lesson 2) Far Figure of $n \to n + 2$

Nearly half of the children (n = 104 of 212) were able to give a full *description of a far figure* (947th) of the growing pattern $n \to n + 2$ which can be interpreted as indication of explicit functional thinking (3_EX) such as:[2]

[2] All translations of children's responses are as literal as possible without grammatical adjustments by the authors.

Table 3 Overall results of strategies used and colourings chosen in the four tasks

%	Describing far figure	Describing rate of change	Describing rule	%	Colouring constant
3_EX	49.06	16.67	22.28	3_b	26.47
2_PEX	16.51	27.45	25.74	2_b+1	5.88
1_RE	16.51	27.45	44.06	1_1st	46.08
0_IN	17.92	28.43	7.92	0_no	21.57
n	212	204	202	n	204

- "The 947 is 947 boxes to the right and 2 boxes to the top. Thus, nothing new." (D19)
- "948 below 1 above" (K3)

The other categories are approximately equally shared by the other half. 35 children (16.51%) described parts of the description in a partially explicit way (2_PEX), such as:

- "it is 948 boxes wide" (P21)
- "below are 948 boxes" (H20)

Again 35 children (16.51%) described the growth by recursive means (recursive functional thinking) such as drawing (attempts) of the 947th figure or verbal description of the growth, for example:

- "it always gets bigger by one right stone" (H17)
- "it is always 1 more till it is eventually 947" (D4)

The remaining 38 children (17.92%) gave no answer or indicated their thoughts in statements such as:

- "you need to draw it" (D6)
- "look it up on the internet" (D23)
- "I cannot calculate this task this is inhuman" (K4)

Some children already used colours to structure the figures without any instruction in the task.

4.1.2 Task 2 (Lesson 3) Colouring a Constant $n \to 2n + 2$

Following on from the lesson 2, in lesson 3 the children were advised to *colour constant elements* in the growing pattern $n \to 2n + 2$. Figure 2 presents an example for each colouring category. Slightly more than one fifth (21.57%) of the participating children (44 of 204) used no colouring or coloured various boxes inconsistently throughout the pattern (category 0_no). The largest percentages (46.08%) of the children (94) coloured the first figure in each of the given figures (category 1_1st). Twelve others (5.88%) coloured 3 squares, which equals b+1 (category 2_b+1),—from which most children (9 of 12) coloured an (inverse) L-shape in each figure of the pattern (Fig. 2). 54 children (26.47%) coloured two squares in

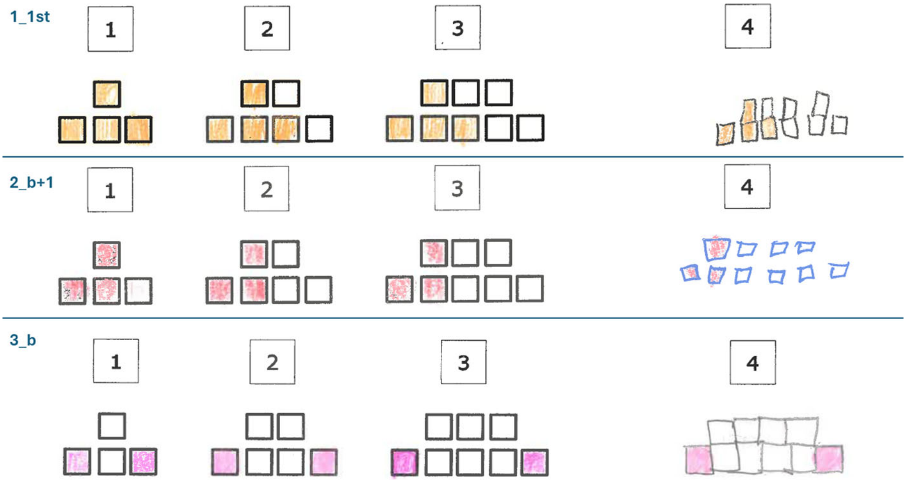


Fig. 2 Examples of the three different colouring categories for $n \to 2n + 2$

each pattern, which correspond to the parameter b = 2 in the symbolic expression (category 3_b).

4.1.3 Task 3 (Lesson 3) Describing Rate of Change $n \to 2n + 2$

Still in lesson 3 on the same worksheet as the colouring (task 2) described above and the same growing pattern $n \to 2n + 2$, the children were asked to give a *description of the rate of change*. The children were directly prompted to relate the uncoloured elements of each figure to the value of the number card (position). The underlying idea is to thereby support explicit functional thinking.

58 (28.43% of the children) gave no, an incomprehensible, or false description (0_IN). 56 children (27.45%) described recursive relationships between the figures (1_RE) such as:

- "in each pattern there will be down and up one more" (M20)
- "there will be always 2 more" (S16)

Another 56 children (27.45%) described parts of the relationship explicitly (2_PEX)—mainly the perceiving of the (position) number given on the card in one part of the figure like:

- "the upper squares are as much as the number" (P5)
- "by number 1 is above 1 box by number 2 2 by 3 3 and so forth" (S8)

An explicit description of a valid relationship between the uncoloured squares and the (position) number was given by 34 (16.67%) children (3_EX). Inevitably, the descriptions differ depending on which colouring of constant elements was used in task 2 (see above): If 2 squares were coloured as constant the uncoloured equal $2n$ in an explicit description, if an (inverse) L-shape of 3 squares was coloured in each figure the uncoloured squares equal $2n$–1, and if the 1st figure (4 squares) was regarded as constant the uncoloured squares equal $2(n–1)$ or in other words the double of the (position) number of the previous figure. Examples of full descriptions:

- "the white ones are always the double of the position number" (B4)
- "the squares in the middle have above and below each always the quantity of the number" (L7)
- "it is always one less than the double of the number" (D1; L-shape coloured)
- "if you double the number and then take away 2 you get the quantity of remaining boxes" (A2; 1st figure coloured)

4.1.4 *Task 4 (Lesson 4) Describing a Rule $n \rightarrow 2n + 4$*

In lesson 4 the figural growing pattern $n \rightarrow 2n + 4$ is provided on the worksheet (Table 1) and the children were prompted to find a (general) rule for the pattern. The colouring idea is not explicitly asked for but can be used by the children spontaneously. In fact, 70.8% of the children (202) working on the lesson 4 worksheets used colouring. 40.6% (82 of 202) of the children coloured the 4 vertical squares in the middle of each figure whilst other colouring ideas like colouring the first figure (14.4%) occurred much lesser than in the previous lesson.

However, the task to *give a rule* as an explicit generalisation of the pattern structure led to a recursive strategy (1_RE) by the largest percentage (44.06%) of the children (89 of 202) participating in this lesson.

- "it is like a Z and in each number one box in the front and one box in the rear is attached to it" (D2)
- "always one to the left and to the right" (P10)

Only 16 of the 202 children gave no or an incomprehensible answer (0_IN), like

- "you can turn the dice so that they build a cuboid" (A21)

Nearly half of the children formulated an explicit rule which includes at least one of the structural elements needed. 52 children (25.74%) formulated rules about parts of the relationship (2_PEX), i.e. the description explicitly explains a relationship to the position number or describes the constant, for example:

- "The position number says the boxes which are not coloured" (W16)
- "in the middle the boxes facing upwards stay the same" (A13)

and another 45 children (22.28%) described the relationship in an explicit strategy (3_EX), for example:

- "it is aways 4 and the number is at the top and the bottom" (K23)
- "one has to add the card [number] with itself and then +4" (A19)

4.2 Interplay of Colouring Choices and Functional Thinking Strategies

The two tasks from lesson 3 (task 2 and task 3 described above in section 4.1), which refer to the same figural growing pattern ($n \rightarrow n + 2$), offer the option to analyse possible relationships between colouring choices and the functional thinking

Table 4 Cross-tabulation of the variables colouring and description strategy of $n \to n+2$

$n \to 2n+2$		description				total
		0_IN	1_RE	2_PEX	3_EX	
colouring	0_no	15	14	13	2	44
	1_1st	26	33	27	8	94
	2_b+1	3	2	5	2	12
	3_b	14	7	11	22	54
total		58	56	56	34	204

strategy adopted in the description of the rate of change. The interrelation between the *colouring* (the squares each child chose to colour as constant) and the *description* strategies (whether explicit, partially explicit, or recursive) is displayed in a cross tabulation (Table 4). As can be seen each field in the matrix is filled, i.e. any colouring choice may lead to any description strategy and vice versa.

The individual movement profiles from colouring (a constant) to description strategies (of the rate of change) are analysed in more depth and result in the following findings.

We examine to what extent the children's colouring choices predict the use of description strategies in order to analyse whether the group of children who used a certain colouring of a constant prefer a certain strategy in the description of the changing objects:

- The group of children with no plausible reaction in colouring (cat. 0_no) used roughly evenly divided strategies 0_IN (34.1%), 1_RE (31.8%), and 2_PEX (29.5%) in the description. Two of these children were able to give a full explicit description (3_EX).
- The largest share (41.6%) of children who coloured an (inverse) L-shape in the pattern (cat. 2_b + 1), find a partially explicit description (2_PEX) whilst 3 of them (25%) found no plausible description (0_IN), 2 (16.6%) gave a recursive (1_RE), and another 2 a full explicit description (3_EX).
- As illustrated in Fig. 3, nearly one third of the children who coloured the complete 1st figure as constant in each figure (cat. 1_1st) applied each of the description strategies 0_IN (27.7%), 1_RE (35.1%), and 2_PEX (28.7%). 8.5% gave an explicit description (3_EX) including both components of the linear function.
- The largest share (40.7%) of the children who coloured the constant 2 squares (cat 3_b), used explicit thinking in the description of the change (3_EX), a fifth of the group moved to partially explicit (20.4%) strategies (2_PEX) and a quarter gave no or inconsistent (0_IN) description (25.9%). Few of these children switched to recursive (1_RE) descriptions strategies (13.0%).

Looking at the interplay of colouring choices and descriptions strategies in reverse, it becomes evident that the group of children who gave explicit descriptions of the change (3_EX) mainly stem from the group of children who coloured 2 constant squares (65%), nearly a quarter from colouring 1_1st, whereas very few came from the other colouring choices (two each from 2_b + 1 or 0_no) (Fig. 4).

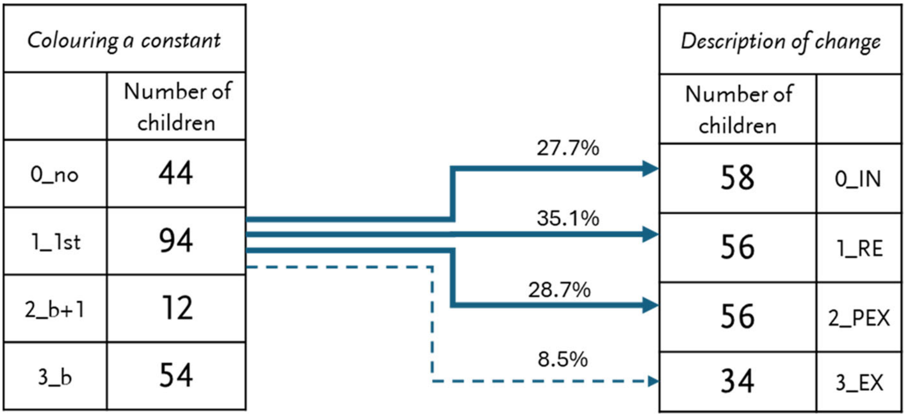


Fig. 3 Example of movements from colouring choice 1_1st (1st figure coloured as constant) towards description strategies of the same growing pattern $n \rightarrow n + 2$ in the same lesson 3

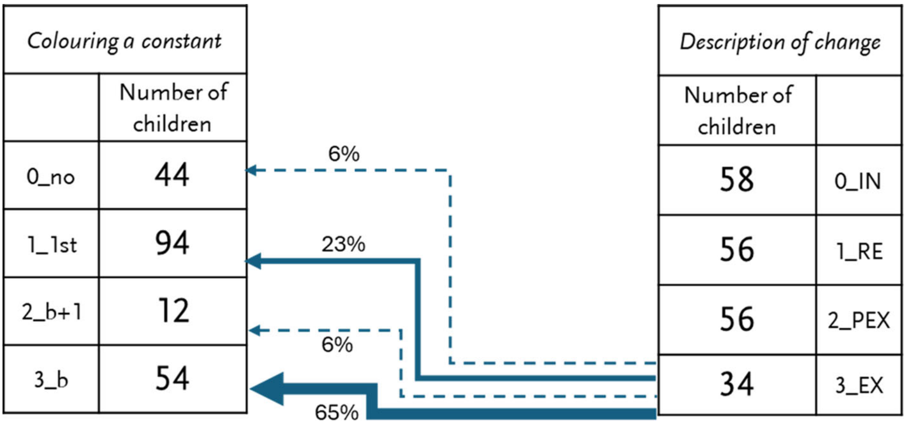


Fig. 4 Example of colouring choices origins of explicit descriptions strategies (3_EX) of the same growing pattern $n \rightarrow n + 2$ in the same lesson 3

The children who could describe the rate of change partially explicitly (2_PEX) mainly (48%) used 1_1st colouring, 23% coloured 0_no, 20% coloured the constant 3_b, and 9% stem from the group of children who coloured 2_b + 1.

4.3 Correlations Between Colouring a Constant and Description Tasks

Normality cannot be assumed across the responses of the children participating in the project, and the categories are ordinally scored. Thus, Spearman's rank correlation is chosen to explore possible correlations. In line with the focus of this paper Spearman's rank correlation is computed to assess the relationship between the *colouring a constant* variable and the strategies adopted in their written work on the three *descriptions* tasks (Table 5).

Table 5 Spearman's rank correlation between the variable *colouring* and the three *description* tasks

			Describing far figure	Describing rate of change	Describing rule
Spearman-Rho	Colouring constant	Correlation coefficient	0.200**	0.236**	0.192**
		Sig. (2-tailed)	0.005	0.001	0.007
		N	200	204	199

**. The correlation is significant at the 0.01 level (2-tailed).

Table 6 Spearman's rank correlation between the variable *grade* and *all four tasks*

			Describing far figure	Colouring constant	Describing rate of change	Describing rule
Spearman-Rho	Grade/ school year	Correlation coefficient	0.268**	0.094	0.153*	0.230**
		Sig. (2-tailed)	0.000	0.182	0.029	0.001
		N	212	204	204	202

**. The correlation is significant at the 0.01 level (2-tailed).
*. The correlation is significant at the 0.05 level (2-tailed).

The data analysis shows a positive 2-tailed significant Spearman's rank correlation

- between the two variables *colouring a constant* and *describing the far figure*: $\rho\,(198) = .20$, $p = .005$,
- between the two variables *colouring a constant* and *describing the rate of change*: $\rho\,(202) = .24$, $p = .001$,
- between the two variables *colouring a constant* and *describing a rule*: $\rho\,(197) = .19$, $p = .007$.

Furthermore, we explored whether the school grade—indicating previous mathematical experiences at least in respect of duration—of the participating children correlates with the strategies used in the tasks. The Spearman's rank correlation between *grade* and strategies used in the four tasks focused on in this paper (Table 6).

The data analysis shows a positive 2-tailed significant Spearman's rank correlation

- between the two variables *grade* and *describing the far figure*: $\rho\,(210) = .27$, $p = .000$,
- between the two variables *grade* and *describing the rate of change:* $\rho\,(202) = .24$, $p = .001$,
- between the two variables *grade* and *describing a rule*:$\rho\,(200) = .23$, $p = .001$.

The analysis reveals one exception: Between the two variables *grade* and *colouring a constant* no significant correlation can be shown: $\rho\,(202) = .09$, $p = .0182$.

4.4 Effects of Colouring a Constant On Description Tasks

The project lessons, explored in the study reported here, were designed on the assumption and hope that *colouring constant elements* in each figure of the growing pattern affect the *description of the change* and the *description of a rule*. From a mathematical perspective different constants may lead to different but valid descriptions (expressions) (section 3). We kept this fact in mind in the next step of analysis, in which we categorised the responses dichotomously, i.e. any consistent colouring (1) versus no or inconsistent colouring (0) against at least partially explicit descriptions (1) versus recursive, incomprehensible, false, or no descriptions (0).

We began by focussing on lesson 3 because the colouring and the description tasks related to the same figural growing pattern (Table 1) and possible effects were particularly interesting. 204 children participated in lesson 3 in which the main design idea of colouring was introduced. The data was analysed as to whether the children embraced the idea of *colouring* a constant in each figure in a consistent way and whether the children used at least partially explicit strategies in the *description of the rate of change* in the growing pattern $n \to 2n + 2$. The effect (Cohen's d) was analysed by means of paired samples t-tests. The large sample allowed us to ignore the normality assumption for paired samples t-tests.

The paired samples t-test revealed a significant difference in scores between *colouring*$(M = 0.78, SD = 0.412)$ and *description of change*$(M = 0.44, SD = 0.498), t(203) = 8.01, p = 0.000$. The effect size, as measured by Cohen's $d = 0.61$, indicating a medium effect.

Furthermore, we analysed whether the *colouring* of a constant in each figure in the growing pattern $n \to 2n + 2$in lesson 3 affects strategies used in *describing a rule* of another growing pattern $(n \to 2n + 4)$ in the next lesson.

The paired samples t-test of 199 children who participated in both lessons reveals a significant difference in scores between *colouring*$(M = 0.79, SD = 0.409)$ and *description of a rule*$(M = 0.48, SD = 0.501), t(198) = 7.16, p = 0.000$. The effect size, as measured by Cohen's $d = 0.61$, indicating a medium effect.

5 Discussion and Limitations

The paper addresses a colouring approach to functional thinking in patterning activities on non-proportional linear growing patterns. German primary school children are neither familiar with linear growing patterns in figural representations nor with tasks on non-proportional linear relationships. In fact, this is a great opportunity for our research, because the findings do not interfere with prior knowledge on growing patterns nor with established approaches and expected solution strategies introduced in the daily classroom. Accordingly, the children participating in this study do not follow any presumed expectations in their reactions and functional thinking strategies. For these reasons we rate the children's responses to the tasks in this study as highly authentic.

This short-term exploratory intervention aims to support explicit functional thinking which indicates understanding of the structure of non-proportional linear functions. As reported in the descriptive results (section 4.1), nearly two thirds (65.6%) of the participating children showed at least partially explicit functional thinking in the description of far figures. However, this percentage dropped slightly in the tasks on describing the rate of change (44.1%) and a general rule (48.0%). This percentage also shows that nearly a third, or more than half, of all participating children in the project are unable to identify and describe at least one part of the structural components of the function represented in the figural growing pattern. Within this latter group of children, the majority (62.2% in description of the rate of change and 91.8% in the description of a rule) used recursive strategies. Over all groups of children and all description tasks, the share of children who showed recursive functional thinking in their descriptions rises from 16.5% (far figure) to 27.5% (rate of change) up to 44.1% (rule). This preference for recursive strategies mirrors their prior school experiences in typical German maths classes and is in line with the findings of the referred studies with secondary school students.

The fact that nearly 80% (Table 3 in section 4.1) embraced the approach of colouring a valid constant—even though the idea was completely new to them—is encouraging. The choice of a constant impacts the degree of challenges met in seeking to identify a relationship between the uncoloured elements of the figure and the number of the figure, to thereby find generalisations in an explicit functional thinking (cf. section 3). However, the colouring choice does not define which description strategies are used by the children (section 4.2). In our study all combinations can be found in the cross-tabulation of choice of a constant and functional thinking strategy used later in the description of the same pattern (Table 4).

Having said this, our analysis also shows that the choice of which elements are spotted as the same in each figure influenced the children's functional thinking strategies adopted in description of the relationship between the uncoloured elements of the figure and the position number (section 4.2). The biggest group of the children who coloured 3 squares (category 2_b + 1) or 2 squares (category 3_b) in the figural growing pattern $n \rightarrow 2n + 2$ used at least partially explicit thinking strategies in the description of change (58.3% resp. 61.1%). In contrast, children who coloured the first figure as constant in each figure (category 1_1st), no elements or inconsistent parts (category 0_no) used (partially) explicit thinking only in a good third of the cases (2_PEX 34.1% resp. 3_EX 37.2%). These results can be confirmed in the vice versa crosscheck, i.e. what kind of colouring have the children who gave a full or partially explicit description used (section 4.2).

In this project the choice to colour the whole first figure as constant led to an explicit description in only a few cases. Most of the children, who gave an explicit description, chose to colour the quantity of elements, that equals the parameter b in the symbolic expression $n \rightarrow an + b$. In this project, this colouring strategy is most efficient for an explicit description.

As an aside, the study results indicate that the figural arrangement of the growing pattern likely influenced the choice of a constant. Even though two squares in the linear relationship $n \rightarrow 2n + 2$ visually stand out in the figural representation like a left and right appendix and it might seem tempting for the children to choose these two squares as constant, most of the children decided to colour the 1st figure or at least an (inverse) L-shape of side-by-side squares as constant (Fig. 2). In contrast the

4 side-by-side squares as vertical centre line in the middle of each figure in lesson 4 ($n \rightarrow 2n + 4$) were coloured as constant by 40.6% of the participating children and only 14.4% chose the 1st figure as constant. Whether the spatial arrangement of the figures or the prior knowledge the children gained in lesson 3, in which they first encounter the idea to colour a constant, causes these differences, cannot be deduced from this study. Thus, whether different representations of the same linear function or different constants in comparable spatial arrangements of figures in a growing pattern affect the colouring choices and functional thinking strategies used, needs further research.

Although no linear relation can be documented in the descriptive results, positive significant (at the 0.01 level) 2-tailed correlations can be stated as results in the relationship between the variable *colouring a constant* and the functional thinking strategies the children adopted in the three *descriptions* tasks (Table 5 in section 4.3).

Moreover, we analysed whether the school grade correlates with the children's choices made in colouring and functional thinking strategies adopted by the children in the description tasks. We found a positive 2-tailed (highly) significant Spearman's rank correlation between the variables *school grade* and the *description tasks* (describing the far figure; describing the rate of change; and describing a rule), whereas no correlation can be found between *school grade* and *colouring* choices (Table 6 in section 4.3). These findings may indicate broader previous (school) experiences with mathematical tasks or possibly with other patterns in general of the 4th graders, although the colouring choices do not correlate with the school grade. The findings of our project are limited in that there is no information available regarding previous experiences with patterning tasks. Future long-term studies may look at this relation even in a broader sense (e.g. including repeating patterns) from kindergarten onwards (Wijns et al. 2019) or control for the influence of other variables like socio-economic status or gender. The data we analysed in this study consist of verbal description in written documents. Consequently, our findings potentially reflect greater fluency in communicating thinking through writing in more senior grades. Notwithstanding, it could be that the children would have used more or other functional thinking strategies in oral discussions or interviews. The missing correlation between colouring and school grade reported above, highlights this critical fact. This result also indicates that the colouring approach seems to be suitable for primary school children.

In addition, statistically relevant effects between colouring and strategies used in the description tasks can be shown in this exploratory study (section 4.4). Paired samples t-tests (Cohen's d) reveal medium size effects ($d = 0.61$) between *colouring* and the *description of change* as well as between *colouring* and *description of a rule*. These findings indicate that the colouring not only affects the description of the uncoloured elements in relation to the position number of the same pattern, but the strategies used to describe the rule for another pattern that was presented in the lesson on the next day. The influence of colouring a consistent constant on (at least partially) explicit thinking strategies displayed by the children can be proven in this project. The interplay between the colouring approach and the development of functional thinking therefore opens an interesting field for further qualitative and quantitative research.

Our results reveal that the colouring approach is promising and suitable for primary school children. We reported results of a short-term exploration of tasks designed for four lessons of which two address the colouring approach. Since identifying a particular constant, which is coherent with the explicit expression (e.g. 2 in the functional relationship $n \to n + 2$), mainly led to explicit functional thinking—like our findings in line with El Mouhayar and Jurdak (2016) showed—mathematics lessons should put emphasis on the colouring activity as an important pivotal point for further conceptual development. Future studies need to give the children more time to explore the idea of different growing patterns with various representations and different constants and to work on the effects of colouring in more depths.

The impact of figural representations on classroom interactions and lesson discussions could also be a promising focus for further studies. As coloured representations of growing patterns can be seen as diagrams in a semiotic perspective, established teaching ideas from this field (e.g. Ott 2020) could be adapted in daily classroom routines and long-term further studies. The teaching-learning-settings need to give enough time for the children not only to find suitable items to colour but also to discuss the individual choice of colouring and alternative options with others (children and teacher). It is important for the children to think through the task more than once to allow shifts of attention (Mason et al. 2009) and to encourage "a process that integrates learners' experience with the act of making sense of their experience" (Voutsina et al. 2019, p. 71).

The individual ideas and examples of descriptions cited in the findings illustrate various opportunities to prompt further questions and to scaffold more efficient functional thinking in line with Lannin (2005);

> In addition, the role of the teacher was instrumental in referring students back to the context of the situation. When students were asked to relate their rule to a diagram or to the situation, this often provoked students to develop further connections between the context and the rules. (p. 252)

The data we analysed in this study consist of individually completed work sheets in written form. Hence, the findings reported here cannot be related to the actual course of events in the project lessons, because the classroom interaction is not captured by video- or audio-documents. Consequently, the findings are not controlled against the variable 'teacher', which means effects of different teacher's instructions or variations in how the children were supported. However, the role of the teacher can be assumed as crucial in these processes, as other studies have already shown (e.g. Mc Auliffe 2013, Strømskag Måsøval 2011). Teachers need both proficiency in the domain of functional thinking and in understanding children's thinking and the challenges they might encounter (e.g. Wilkie 2016, Twohill et al. 2025). Video studies on classroom discussions between the children and between the children and the teacher can shed more light on these important aspects in fostering children's functional thinking.

This study was not accompanied by interviews with the children. Consequently, further studies could focus on the missing correlation (between colouring and school grade) and enable children to verbally comment on their colouring choices and on the descriptions of relationships they can derive from the colouring. Beyond that, the missing correlation between school grade and colouring characterises the colouring approach as providing low-threshold access and is therefore particularly appropriate for young children—especially in multilingual classrooms.

Last but not least, the limitation to linear relationships should be considered. The colouring a constant approach does not apply to quadratic functional relationships as in sequences of triangular or square numbers. Unlike linear functions, these so called 'figurate numbers' are quite commonly addressed in figural representations of growing patterns in German textbooks. However, the complex quadratic functional relationship is not focused on and children are not expected to generalise or use explicit thinking strategies. Only recursive approaches and recursive functional thinking strategies are supported by the tasks. These limited experiences with growing patterns, in turn, prevents children from having the opportunity to develop explicit functional thinking (correspondence). In contrast, linear growing patterns have the added advantage that primary school children can fully explore and understand the mathematical structures.

6 Conclusion

The ZADIE project aims to support explicit functional thinking. The project approach is to directly prompt the children to colour a constant and therefore to draw children's attention to the structural components. This awareness is regarded as a possible key step to understanding non-proportional linear functions. Previous studies proved that typical patterning activities (finding next, near, and far figures) are not entirely sufficient to support primary school children's ability to decompose the given values in an additive constant and a multiplicative product of rate of change and independent variable (figure number), and to thereby understand linear relationships. The colouring approach could serve as one of many catalysts to support and promote children's understanding of functions.

The project approach of colouring is taken up by almost 80% of the participating children in lesson 3, where the children first met this idea. Even though the direct prompt to colour constant elements is missing in the lesson 4 tasks, over 70% used colourings. The percentage of children who gave no or incomprehensible answers drops to under 8% in the last lesson. On the one hand these results show that the tasks are suitable for primary school children, and that the children adopted the colouring idea. On the other hand, the findings indicate quite a high level of motivation of the participating children to find (any) answer and that the children are not hesitant to try to give (any) responses—even though growing patterns are a completely new and unfamiliar context for them. The finding indicates that the ZADIE project tasks provide a reasonable approach to functional thinking for primary school children.

The main focus of this approach is colouring constant elements in the figures of a non-proportional growing pattern. The children's choices of which elements of each figure they colour as constant were broadly diverse. Each colouring could guide the children to a valid description, though some choices are more challenging than others. The tasks are designed to support explicit functional thinking strategies. Respectively, 44.1% and 48% of the children used at least partially explicit thinking strategies in each of lesson 3 and 4. This indicates for the participating German children, the tendency to use the presumably familiar thinking strategy of recursive thinking is persistent. The short time span of the brief exploration documented here is certainly not enough to make explicit thinking a habit.

Nevertheless, the reported development in tendencies towards explicit functional thinking via the colouring approach show highly significant correlations and medium size effects between colouring and at least partially explicit thinking strategies used in descriptions tasks. Thus, this approach seems to be promising and may sensibly enhance the current curriculum used in German primary school mathematics classes, and elsewhere. This and more detailed hypotheses can be derived from the findings of this exploratory study and need to be evaluated in controlled group designs with pre- and post-tests in future follow-up studies.

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Declarations

Conflict of interest S. Mc Auliffe, A. Steinweg and A. Twohill declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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