Doktorarbeit

Investigating distribution changes and testing dependence under local stationarity with applications to financial time series

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Abstract

Since many decades, a lot of methods have been proposed which are capable of investigating (strictly) stationary time series. However, it is expectable that many time series of practical interest (like those that underlie stock market data) are non-stationary. This motivates to consider locally stationary processes, which allow to model not just a stationary behaviour but also gradual distribution changes over a regarded time period.

In this thesis, some new tools for exploring a quite general class of locally stationary processes, which fulfil pretty weak moment conditions, are introduced and applied to log returns of several stock prices. Concretely, instruments are proposed that detect deviations between the distributions of the stationary approximations belonging to a locally stationary process and that also measure how large such deviations are. In addition, a new consistent level-alpha test for independence is introduced, which aims to reveal whether dependences between the stationary approximations of two locally stationary processes exist. Moreover, it is shown that the present methods also allow to formulate approximate statements about deviations between the distributions of the random variables contained in a locally stationary process and about dependences between two locally stationary processes.

In detail, at first, the class of locally stationary Bernoulli shift processes (which underlies the present thesis) and belonging characteristic functions as well as their estimators are introduced. Next, two L^2-distance-based measures that quantify deviations between the distributions of the stationary approximations belonging to a locally stationary process are proposed, which are based on the previously defined characteristic functions. However, these characteristic functions are commonly unknown in practise, such that confidence intervals for both measures are estimated in order to quantify the intensity of deviations between distributions in applications. Therefor, in a first step, empirical versions of both measures are proposed, which are constructed by using estimators for the underlying characteristic functions. Next, in a second step, for each measure, the asymptotic distribution of the (with an appropriate rate of convergence scaled) difference between this measure and its empirical version is stated. Since these limiting distributions depend on parameters which are commonly unknown in practise, dependent wild bootstrap procedures are introduced in a third step that approximate them. Overall, combining these steps yields suitable estimators for the confidence intervals for both measures. Furthermore, based on the empirical measures, consistent level-alpha tests are constructed that aim to detect whether deviations between the distributions of the stationary approximations belonging to a locally stationary process exist, whereby the associated p-values are estimated by an appropriate dependent wild bootstrap procedure. In addition, the first change point in the distributions of the stationary approximations is estimated by using these empirical measures.

Subsequently, a consistent level-alpha test is proposed that allows to reveal whether the stationary approximations of two locally stationary time series dependent on each other within arbitrary but fixed time periods. The belonging test statistic is constructed by using empirical characteristic functions in combination with an L^2-distance and the associated p-values are estimated by a dependent wild bootstrap procedure.

Furthermore, the finite sample behaviour of the instruments introduced in this thesis is evaluated by simulation studies and these tools are applied to log returns of several stocks, whereby the results are interpreted from an economic perspective.

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1. Introduction

1.1. General introduction to local stationarity and aims of the present thesis

Since their first formal description in [47, Khintchine (1934)], (strictly) stationary processes play a fundamental role in time series analysis, whereby several tools for investigating them were introduced in the past. However, as stated in [35, Hirukawa (2006), p. 114], empirical studies show that many time series data have non-stationary behaviour. For instance, modelling the world population over the last centuries by a stationary time series does not seem to be adequate. To give another example, it is expectable that many financial time series (like stock prices of listed companies or from them derived log returns) are affected by global financial/economical crises, energy price changes or other time-dependent economic developments, such that describing financial time series by stationary processes may be inappropriate.
These and several other applications indicate that evolving methods for analyzing non-stationary processes is of high importance. However, constructing consistent estimators for features (like the time-dependent expectation, variance or characteristic function) of a general non-stationary time series is often not possible because all of the random variables contained in such a process may own very different distributions with various moments and no useful connection between these distributions. A common way to solve this issues is to consider a locally stationary framework, that is more general than a stationary one and for which evolving a comprehensive estimation and hypothesis testing theory is possible. Intuitively spoken, locally stationary time series are processes which rely on the property that the distributions of the contained random variables do not differ much from each other within short time periods but may vary substantially over long periods. Thereby, on the one hand, many belonging features (like the time-dependent expectation, variance or characteristic function) can be estimated appropriately by using kernel-based estimators that respect the time-dependence of this feature (see e. g. [74, Vogt (2012)], [14, Dahlhaus (2012)] as well as [40, Jentsch et al. (2020a)] for some of these investigations) and, on the other hand, many practical examples that are characterized by gradual changes over time are conceivable. For instance, in [1, Adak (1998)], speech signals as well as earthquake data are modelled by locally stationary time series and in [68, Shiraishi and Taniguchi (2007)], a locally stationary framework is used to describe returns of investment portfolios. Moreover, in [29, Fryzlewicz et al. (2006)], financial data (like a stock index, a currency exchange rate or a share price) are investigated by modelling the belonging log returns series as a certain non-stationary triangular array, which can be regarded as a special case of a locally stationary process (for details, see (1) in [29, Fryzlewicz et al. (2006), p. 688]).
Originally, locally stationary processes were introduced in [61, Priestley (1965)], where they are defining by time-dependent spectral representations, which generalize the spectral representations of stationary processes. In contrast, in the present thesis, locally stationary processes are not expressed by spectral representations but another approach is used to define them, which is similar to that in [76, Vogt and Dette (2015), p. 717] and introduced in the following:
Assume that $\{X_{t,T} : t \in \{1, \dots, T\}\}_{T=1}^{\infty}$ is a (triangular) array of $\mathbb{R}^d$-valued random variables (with $d \in \mathbb{N}$) which lives on a probability space $(\Omega, \Sigma, \mathbb{P})$. Moreover, it is supposed for all $u \in [0,1]$ that $\mathbb{R}^d$-valued (strictly) stationary processes $\{\widetilde{X}_t(u) : t \in \mathbb{Z}\}$ and arrays $\{U_{t,T}(u) : t \in \{1, \dots, T\}\}_{T=1}^{\infty}$ of positive random variables exist on $(\Omega, \Sigma, \mathbb{P})$, which fulfil $\mathbb{E}\left[U_{t,T}(u)^{\mathfrak{n}}\right] < C$ for some $\mathfrak{n} > 0$ as well as $C < \infty$ (whereby $\mathfrak{n}$ and C are independent of t, T as well as u). In addition, the following property should hold for all $t \in \{1, \dots, T\}$, $T \in \mathbb{N}$, $u \in [0,1]$ and for the ℓ^1-norm on $\mathbb{R}^d$, which is denoted as $|\cdot|_1$:

$$\left|X_{t,T} - \widetilde{X}_t(u)\right|_1 \leqslant \left(\left|\frac{t}{T} - u\right| + \frac{1}{T}\right) U_{t,T}(u) \quad \text{a. s.} \tag{1.1}$$

Then, $\{X_{t,T} : t \in \{1, \dots, T\}\}_{T=1}^{\infty}$ is called locally stationary process and, for each $u \in [0,1]$, the process $\{\widetilde{X}_t(u) : t \in \mathbb{Z}\}$ is called stationary approximation of this locally stationary process.
Thereby, $\sup_{t=1,\dots,T} \mathbb{E}\left[\left|X_{t,T} - \widetilde{X}_t(t/T)\right|_1^{\mathfrak{n}}\right]$ vanishes asymptotically for $T \to \infty$, which motivates why

$\{\widetilde{X}_t(u) : t \in \mathbb{Z}\}$ is called stationary approximation.
Further, one observes that this definition is based on rescaling the time in the sense that $t/T, u \in [0,1]$, such that the theory for $T \to \infty$ belonging to locally stationary processes is based on infill asymptotic (cf. [14, Dahlhaus (2012)]). This projection of the points in time $t \in \{1, \ldots, T\}$ into the interval $[0,1]$ (which is also proposed in, e.g., [13, Dahlhaus (1996)], [74, Vogt (2012)] and [40, Jentsch et al. (2020a)]) is advantageous for evolving a comprehensive theory of locally stationary processes because $[0,1]$ is a bounded compact interval.
The present thesis investigates locally stationary processes that can be expressed by Bernoulli shifts, such that these processes are called locally stationary Bernoulli shift processes (which are formally introduced in Section 2.1). This class of locally stationary processes also underlies [62, Richter (2016)] as well as [16, Dahlhaus et al. (2019)] and contains many linear, non-linear and, in particular, also several recursively defined locally stationary processes (as mentioned in [62, Richter (2016), p. 14]).

The first aim of the present work is to evolve instruments that investigate distribution changes of $\widetilde{X}_0(u)$ in dependence of varying u. Concretely, two measures that quantify how intensive the distribution of $\widetilde{X}_0(u)$ changes within the rescaled time period $[\mathfrak{U}_0, \mathfrak{U}_1]$ with arbitrary but fixed $\mathfrak{U}_0, \mathfrak{U}_1$ are introduced, estimated and empirical confidence intervals for these measures are constructed. Derived from estimators for these measures, consistent level-alpha tests are proposed which investigate whether $\widetilde{X}_0(u)$ owns the same distribution as $\widetilde{X}_0(v)$ for all $u, v \in [\mathfrak{U}_0, \mathfrak{U}_1]$. In addition, the rescaled point in time at which the characteristic function of $\widetilde{X}_0(u)$ changes (smoothly) for the first time in dependence of $u \in [0,1]$ is estimated.
The second aim of the present thesis is to construct a consistent level-alpha test based on characteristic functions which decides for (arbitrary, non-empty, fixed as well as finite) sets $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$ and the stationary approximations $\{\widetilde{X}_t^{[1]}(u) : t \in \mathbb{Z}\}$ as well as $\{\widetilde{X}_t^{[2]}(u) : t \in \mathbb{Z}\}$ of two locally stationary processes $\{X_{t,T}^{[1]} : t \in \{1, \ldots, T\}\}_{T=1}^{\infty}$ and $\{X_{t,T}^{[2]} : t \in \{1, \ldots, T\}\}_{T=1}^{\infty}$, respectively, that live on the same probability space, whether $\big(\widetilde{X}_{-\mathfrak{d}}^{[1]}(u)\big)_{\mathfrak{d} \in \mathfrak{D}_1}$ is (stochastically) independent of $\big(\widetilde{X}_{-\mathfrak{d}}^{[2]}(u)\big)_{\mathfrak{d} \in \mathfrak{D}_2}$ for all $u \in [0,1]$ against the full alternative. (Formulating this test problem based on points in time $-\mathfrak{d}$ which belong to the present (for $\mathfrak{d} = 0$) and the past (in the case $\mathfrak{d} > 0$) will prove to be useful in regard of the structure of the locally stationary processes considered in this work.)
Concerning these aims, it should be noted that the research presented in this thesis is focused on investigating the stationary approximations instead of locally stationary processes themselves, which is beneficial from a methodical point of view because this allows to formulate the null hypothesis and alternatives which underlie the consistent level-alpha tests constructed in the present work independently of the number of observations T. However, it is explained in the course of the present publication that the proposed tools allow to formulate for large given numbers of observations T approximative statements that concern the similarity of the distributions of the random variables $X_{1,T}, \ldots, X_{T,T}$ as well as the dependence between the time series $\big(X_{t,T}^{[1]}\big)_{t=1}^{T}$ and $\big(X_{t,T}^{[2]}\big)_{t=1}^{T}$, respectively. Moreover, it is worth mentioning that focusing on exploring properties of the stationary approximations is consistent with the literature (see e. g. [60, Paparoditis (2009)], [19, Dette et al. (2011)], [76, Vogt and Dette (2015)], [65, Schmidt et al. (2021)] and [4, Beering (2021)]).
Further, the instruments introduced in the present work are constructed based on characteristic functions because, as explained in the following, this is advantageous for detecting as well as quantifying distribution changes and for testing for independence over other methods that determine some distribution features (like expectations, distribution functions, density functions) or dependence properties (e. g., covariance functions, distribution functions, density functions). Concretely, in contrast to the expectation and variance, the characteristic function depicts all properties of the distribution of a random variable. In addition, characteristic functions can detect all kinds of dependences between two random variables, which is not possible by using covariances. Further, taking characteristic functions avoids to assume that density functions have to exist and, in contrast to distribution functions, all characteristic functions are continuous (even uniformly continuous), which allows to construct the above-mentioned tools for measuring distribution changes and testing for independence based on integrated L^2-distances (regardless whether the underlying random variables are continuously distributed). Furthermore, it is well-known that many approaches which are derived from density estimation own weaker convergence rates than similar ones that are constructed by estimating characteristic functions.

In order to motivate the instruments introduced in the present thesis, it is explained in the following two sections how they allow to explore questions of high practical relevance and how they differ from existing methods which fulfil similar purposes.

1.2. Motivation for investigating distribution changes

In many applications, it is of interest to analyze whether and how intensive the distributions of the random variables contained in a time series depend on time. To give some examples: One likes to investigate for a certain geographical region whether the distribution that underlies the amount of precipitation on day t changes within a season, i. e., for varying t. Thereby, it may also be important to answer whether the intensity of such changes is higher in one season than in another because this may contribute to agricultural risk management. Moreover, investigating whether and how strong the distributions of the frequency and the intensity of earthquakes in a volcanic region have changed in the last time is interesting because such changes may be an indicator for an upcoming eruption since, according to [66, Seropian et al. (2021), p. 1], earthquakes may trigger volcanic activity. Further, analyzing for each species in an ecosystem whether and how intensive the distribution of the number of animals of this species has changed over time allows to evaluate how stable the structure of the fauna in this ecosystem is. Such an analysis may contribute to decide for the regarded ecosystem whether actions for nature conservation are necessary. Quantifying the degree of distribution changes also has various economic applications. For example, it is of interest to check whether the productivity of an employee is related to his working experience (e. g., measured in total working hours) and whether this dependence is stronger in the first five years of pursuing this profession than in the next five years, which may be the case due to learning effects (that often decrease over time). Thereby, such a quantification of changes in the working-experience depending productivity distribution can contribute to decide how much impact the working experience should have on the salary of an employee. E. g., if the distribution of the productivity of an employee (almost) does not change between the last and next determination of his salary, it will not be justified that working experience gotten by this employee in the meanwhile contributes to the next salary negotiation. Further, it is of interest to evaluate different investment portfolios in a fixed time period based on the distribution change intensity of their (daily) returns. Thereby, a stronger time-dependence of the return distribution may indicate that the belonging portfolio reacts more intensively to events which occurred in the investigated time period. In addition, the question rises whether the degree of distribution changes of the returns of an investment portfolio is higher after a certain event (like the Coronavirus outbreak) than before. This may be the case because such events can trigger various crises. For instance, it is expectable that the share price of a company with many small shareholders whose sales are mainly driven by storefronts changes gradually during a pandemic because the shareholders react differently to curfew laws and other regulations. E. g., some risk averse shareholders try to sell their stocks as fast as possible, whereas others hold or even buy more stocks in the hope of a stock price increase after the pandemic.

The classical change detection theory assumes that the considered feature (like the expectation, variance or characteristic function) of an investigated time series stays the same in a (sufficiently large) time period but this feature differs abruptly from one to the next time period (see e. g. [3, Aue and Horváth (2012)], [42, Jirak (2015)], [39, James and Matteson (2014)], [27, Fearnhead and Regaill (2020)] as well as references therein). However, it is well-known that many of the factors (like atmospheric pressure or temperature) which determine the weather may vary smoothly over time and learning effects are often modelled by using continuous learning curves (see e. g. [34, Grosse et al. (2015)]). In addition, the Corona pandemic was accompanied by many often changed regulations of several countries and their federal states (like quarantine duty for infected employees as well as lockdowns), actions carried out by central banks, insolvencies of companies and probably also non-homogeneous reactions of financial market participants with different risk tolerance. Thus, it is expectable for many investment portfolios that this pandemic is not related to a few distribution changes of their returns but to changes of different intensity in (almost) each point in time. In addition, as pointed out in [76, Vogt and Dette (2015), p. 713], changes occur gradually rather than abruptly in a number of applications (like climatology, neuroscience, finance industry) in the sense that underlying properties are (approximately) constant for some time and then slowly start to change. Thus, the classical change detection theory does not seem to be an adequate model for a lot of practical examples (like those mentioned above).

In contrast, the theory of locally stationary processes allows a suitable modelling of many applications which are characterized by gradual distribution changes that may occur in each considered point in time. Hence, this process class seems to describe the climatological, seismological, biological and economic applications mentioned above and many others appropriately.

In the last decades, several methods for investigating distribution changes in locally stationary processes have been proposed, whereby some belonging publications are mentioned in the following. In [1, Adak (1998)], a segmentation procedure is introduced that uses binary trees and windowed spectra to part given data into approximately stationary intervals. A test for detecting whether the local spectral density of a linear locally stationary process depends on the considered point in time can be found in [60, Paparoditis (2009)]. Thereby, the underlying test procedure is based on a comparison between the sample spectral density (calculated locally on a moving window of data) and a global spectral density estimator on the whole stretch of observations. A measure based on an integrated L^2-distance for quantifying the intensity of changes in the local spectral density of a linear locally stationary process $\{X_{t,T} : t \in \{1, \dots, T\}\}_{T=1}^{\infty}$ with i. i. d. centered innovations is introduced in [19, Dette et al. (2011)] and in this publication, it is also explained how this measure can be used to test whether the local spectral density depends on the regarded point in time. In particular, it should be noted that if the local spectral density does not depend on the underlying point in time, the process $(X_{1,T}, \dots, X_{T,T})$ will be approximately (strictly) stationary for large T because covariance stationary linear processes are strictly stationary. In [55, Mallik et al. (2013)], a procedure for estimating the threshold level at which a regression function (that can be handled similarly to a locally stationary process) takes off from its baseline value is proposed. In [76, Vogt and Dette (2015)], a CUSUM-statistic-based method is evolved which allows to detect the first gradual change point $u \in [0, 1]$ in the stationary approximations of locally stationary processes in the sense that $[0, 1] \ni v \mapsto \mathbb{E}\big[f\big(\widetilde{X}_0(v)\big)\big]$ changes at u (smoothly) for the first time for some functions f in a set of functions $\mathbf{F}$ that fulfils some properties (e. g., $\mathbf{F}$ is separable, compact and its elements are measurable as well as real-valued functions). In [20, Dette et al. (2019)], a CUSUM-statistic is used to construct a test which detects whether the correlation function of a locally stationary process changes over time. An approach to test heteroscedasticity of locally stationary time series (that is based on Gini's mean difference of logarithmic local sample variances) is developed in [65, Schmidt et al. (2021)]. In [57, Mies (2021)], a CUSUM-statistic is introduced in the locally stationary framework that allows to reveal changes in parameters which may be expressed as not necessarily linear functions of moments (e. g., expectation, kurtosis, autocorrelation).

In contrast to these researches, as already mentioned in the previous section, characteristic function-based methods are used in the present thesis to quantify for the stationary approximations $\{\widetilde{X}_t(u) : t \in \mathbb{Z}\}$ (with $u \in [0, 1]$) of a locally stationary process how intensive the distribution of $\widetilde{X}_0(u)$ changes for varying u, to test the existence of distribution changes and to estimate the first change point in the distributions of the stationary approximations. Thereby, it is of particular importance that applying the instruments for investigating distribution changes proposed in the present thesis will be justified if $1 + \delta$ moments of $\big|X_{t,T}\big|_1$, of $\sup_{u\in[0,1]} \big|\widetilde{X}_0(u)\big|_1$ and of $\sup_{u\in[0,1]} \big|\partial_u \widetilde{X}_0(u)\big|_1$ with an arbitrary but fixed $\delta \in [0, 1]$ are bounded by a constant $A < \infty$ that does not depend on $t \in \{1, \dots, T\}$. Compared to this, assumptions supposed in [1, Adak (1998)], [60, Paparoditis (2009)], [19, Dette et al. (2011)], [55, Mallik et al. (2013)], [65, Schmidt et al. (2021)] as well as [57, Mies (2021)] demand the existence of more than two finite moments (or even stronger moment conditions). More specifically, to see the benefit of the approaches introduced in the present work, regard for example the locally stationary process, which is defined as $X_{t,T} := (1 + t/T) \cdot (\varepsilon_t - 3)\ \forall\, t \in \{1, \dots, T\}$, $T \in \mathbb{N}$ with i. i. d. innovations $(\varepsilon_t)_{t\in\mathbb{Z}}$, whereby ε_0 is Pareto-distributed with location parameter 1 and shape parameter 1.5. The belonging stationary approximations are given as $\widetilde{X}_t(u) := (1 + u)\,(\varepsilon_t - 3)\ \forall\, t \in \mathbb{Z}$, $u \in [0, 1]$. Since these $X_{t,T}$ and $\widetilde{X}_t(u)$ are centered, a measure based on the expectation does not detect distribution changes. Moreover, methods which uses the variance or spectral density are inappropriate to analyze distribution changes in the present example because the random variables $X_{t,T}$ and $\widetilde{X}_t(u)$ do not own finite second moments. In contrast, the present publication allows to detect and quantify distribution changes of $\widetilde{X}_0(u)$ for varying u.

1.3. Motivation for testing for independence

Testing whether time series depend on each other is very important for several applications. For example, such tests allow to detect for a given time period whether the greenhouse gas concentration in the atmosphere is related to the number of cars on earth, whether a connection exists between the income and the probability of a COVID-19 infection or whether the population numbers of some animal species depend on each other. In addition, these tests play an important role for many economic questions. For instance, they can reveal relationships between the processes that generate share prices of two listed companies, which contributes to the construction of investment portfolios with a broad risk spread because if such relationships are visible, it will be known that factors exist which have an impact on the stock prices of both considered companies.

In the last decades, several tests for independence have been proposed under various assumptions that concern the underlying stochastic processes. Some of them are briefly introduced in the following. A test that investigates based on empirical distribution functions whether two random variables are independent of each other (whereby the underlying distribution functions are assumed to be continuous), is proposed in [36, Hoeffding (1948)] and generalized to the multivariate case in [5, Blum et al. (1960)]. This test is modified in [69, Skaug and Tjøstheim (1993)] to investigate whether a stationary time series consists of independent random variables. Further, the projection covariance, which is a distribution-based dependence measure, is used in [81, Zhu et al. (2017)] for detecting whether two random vectors with not necessarily the same dimension dependent on each other and in [50, Lai et al. (2021)], this approach is extended to functional random variables on separable Hilbert spaces. In [64, Rosenblatt (1975)], it is shown that dependence between the components of a random vector with continuous density function can be tested by using kernel density estimation. Moreover, a density-based method that allows to test whether X_{t-1} and X_t are independent of each other for all $t \in \mathbb{N}\backslash\{1\}$ is constructed in [63, Robinson (1991)], whereby $(X_t)_{t\in\mathbb{N}}$ is supposed to be a real-valued stationary process and X_1 is assumed to be a continuous distributed random variable. Some authors constructed tests for independence based on covariance estimation. However, this is just possible in very restrictive situations. For example, this approach is considered in [38, Horváth et al. (2013)] in order to test whether random functions $X_1, \ldots, X_n$ are i. i. d. against the alternative that $X_1, \ldots, X_n$ form a stationary as well as ergodic sequence, which fulfils that some of these functions are correlated. Further, a test for independence is proposed in [11, Csörgő (1985)], which aims to detect dependences between the components of a random vector by using empirical characteristic functions. However, investigations given in [44, Kankainen and Ushakov (1998)] not only show that the last-mentioned test is inconsistent in the general case but also yield a consistent test, which results from a modification of the approach regarded in [11, Csörgő (1985)] (whereby this modification is based on an L^2-type statistic). Furthermore, a bivariate linear process whose first component is a linear combination of random variables $u_t, u_{t-1}, \ldots$ and whose second component is a linear combination of random variables $v_t, v_{t-1}, \ldots$, whereby $\big(u_t\big)_{t\in\mathbb{Z}}$ and $\big(v_t\big)_{t\in\mathbb{Z}}$ are sequences of i. i. d. random variables, is investigated in [37, Hong (2001)]. In this publication, a method for testing whether u_t depends on v_{t-j} for all $j \in \mathbb{Z}$ is constructed, which is based on empirical characteristic functions. Moreover, in [72, Székely et al. (2007)], the distance covariance and distance correlation are introduced to detect dependences between two random vectors X and Y (which may have different dimensions) under the assumption that i. i. d. copies of $(X', Y')'$ are given. Many other works contain tests for independence which are based on the distance covariance and distance correlation. For instance, the approach of [72, Székely et al. (2007)] is generalized in [18, Davis et al. (2018)] to obtain a test that reveals whether two time series $\big(X_t\big)_{t\in\mathbb{N}}$ and $\big(Y_t\big)_{t\in\mathbb{N}}$ are independent of each other under the assumption that $\big((X_t', Y_t')'\big)_{t\in\mathbb{N}}$ is a stationary α-mixing process.

However, all papers mentioned above investigate stationary processes (or even assume that i. i. d. observations of the underlying random variables are given), which is often a too restrictive assumption. E. g., for obvious reasons, it does not seem to be appropriate to assume that the data generating processes that underlie the greenhouse gas concentration in the atmosphere, the number of cars on earth, household incomes, the probability of a COVID-19 infection, population numbers of animal species or share prices of listed companies and belonging log returns are stationary. Thus, it is necessary to develop tests for independence, which are suitable to be applied in non-stationary frameworks - like locally stationary ones. Such a test for independence is introduced in [4, Beering (2021)], which can be regarded as an adaption of the approach used in [44, Kankainen and Ushakov (1998)] to a locally stationary framework. Concretely, in [4, Beering (2021)], linear multivariate locally stationary time series $\big\{\big({Y_{t,T}}', {Z_{t,T}}'\big)' : t \in \{1, \ldots, T\}\big\}_{T=1}^{\infty}$ with stationary approximations $\big\{\big(\big(\widetilde{Y}_t(u)\big)', \big(\widetilde{Z}_t(u)\big)'\big)' : t \in \mathbb{Z}\big\}$ are considered and it is tested for an arbitrary but fixed $u \in (0,1)$ whether $\widetilde{Y}_0(u)$ is independent of $\widetilde{Z}_0(u)$ against the alternative that $\widetilde{Y}_0(u)$ and $\widetilde{Z}_0(u)$ depend on each other. Further, it is worth mentioning that estimators and belonging convergence results for the distance covariance and distance correlation for

linear locally stationary processes are given in [40, Jentsch et al. (2020a)], whereby it seems possible that these results can also be used to construct an alternative test for the test problem considered in [4, Beering (2021)]. However, Theorem 4.3 in [40, Jentsch et al. (2020a), p. 126] indicates that justifying this test requires stronger assumptions with respect to the existence of finite moments than those supposed in [4, Beering (2021)].
In contrast to the investigations in [4, Beering (2021)] and [40, Jentsch et al. (2020a)], the present test for independence is based on locally stationary Bernoulli shift processes $\{(X_{t,T}^{[1]'}, X_{t,T}^{[2]'})' : t \in \{1, \ldots, T\}\}_{T=1}^{\infty}$, which allows to consider many non-linear processes. In addition, the test problem that underlies this thesis does not just allow to identify dependences between $\tilde{X}_0^{[1]}(u)$ and $\tilde{X}_0^{[2]}(u)$ for a fixed single $u \in (0,1)$ but to detect whether dependences between $(\tilde{X}_{-\mathfrak{d}}^{[1]}(u))_{\mathfrak{d} \in \mathfrak{D}_1}$ and $(\tilde{X}_{-\mathfrak{d}}^{[2]}(u))_{\mathfrak{d} \in \mathfrak{D}_2}$ exist for any $u \in [0,1]$ and arbitrary, non-empty, fixed as well as finite sets $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$. For instance, in order to better understand this difference between the present research and that in [4, Beering (2021)], consider daily log returns $x_{t,T}^{[1]}$ of one stock and daily log returns $x_{t,T}^{[2]}$ of another stock belonging to trading days $t \in \{1, \ldots, T\}$, whereby $(x_{t,T}^{[1]}, x_{t,T}^{[2]})_{t=1}^{T}$ is regarded as a sample path of a sequence of random variables $(X_{t,T}^{[1]}, X_{t,T}^{[2]})_{t=1}^{T}$ which is contained in a locally stationary process. Applying the test proposed in [4, Beering (2021)] to these log returns may detect for sufficiently large T dependences between $X_{t,T}^{[1]}$ and $X_{t,T}^{[2]}$ in a single previously defined day $t = \lfloor uT \rfloor$ (i. e., for a chosen $u \in (0,1)$) because $\mathbb{E}\big[|\tilde{X}_{\lfloor uT \rfloor}^{[k]}(u) - X_{\lfloor uT \rfloor,T}^{[k]}|_1^{\mathfrak{n}}\big]$ vanishes asymptotically for $T \to \infty$ and all $k \in \{1, 2\}$, $u \in (0,1)$ according to the definition of locally stationary processes given above. In contrast, (as resulting from Remark 4.2 (iii)) the test proposed in the present work allows to detect for sufficiently large T whether days, weeks or months exist in which the log returns generating time series depend on each other.

1.4. Outline

In the first section of Chapter 2, the class of locally stationary Bernoulli shift processes as well as several belonging assumptions are introduced and examples for processes that fulfil theses assumptions are given. Motivated by the fact that methods are constructed in the present publication which are based on characteristic functions, the local characteristic function of a locally stationary process as well as of its stationary approximations and a belonging estimator that can be used to estimate both of these local characteristic functions are defined in the second section of this chapter. In addition, this section also contains results concerning the quality of this estimator.
The research presented in Chapter 3 aims to identify and quantify the intensity of distribution changes in the stationary approximations of locally stationary Bernoulli shift processes. Therefor, two measures for quantifying distribution changes (which are based on the local characteristic function of the stationary approximations) and also empirical versions for both measures as well as estimated confidence intervals for these measures are introduced in the first section of Chapter 3. In the second section of Chapter 3, consistent level-alpha tests for the test problem that $\tilde{X}_0(u)$ owns the same distribution as $\tilde{X}_0(v)$ for all $u, v \in [\mathfrak{U}_0, \mathfrak{U}_1]$ with arbitrary but fixed $\mathfrak{U}_0, \mathfrak{U}_1$ which fulfil $0 \leqslant \mathfrak{U}_0 < \mathfrak{U}_1 \leqslant 1$ against the full alternative are constructed, whereby the belonging test statistic is based on the empirical measures for the distribution change intensity constructed in the first section of Chapter 3. Further, the first change point in the distributions of the stationary approximations is estimated in the third section of Chapter 3, i. e., the (rescaled) point in time $\mathbb{V} \in [0,1]$ at which the distribution of $\tilde{X}_0(u)$ changes for the first time in dependence of $u \in [0,1]$ (whereby $\mathbb{V} = 1$ means that $\tilde{X}_0(u)$ owns the same distribution for all $u \in [0,1]$). Simulation studies that illustrate the quality of the methods introduced in Chapter 3 in regard to the finite sample behaviour are contained in the fourth section of Chapter 3.
In Chapter 4, the consistent level-alpha test for independence, which is briefly described in Section 1.1, is constructed. Thereby, in the first section of Chapter 4, the test problem and belonging framework are introduced formally. In the second section of Chapter 4, the mentioned test problem is investigated in the case $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$, i. e., it is tested whether $\tilde{X}_t^{[1]}(u)$ depends on $\tilde{X}_t^{[2]}(u)$ for some $t \in \mathbb{Z}$, $u \in [0,1]$. Therefor, in the second section of Chapter 4, a test statistic for this problem is constructed, its asymptotic behaviour (for $T \to \infty$) under the null hypothesis as well as under the alternative is analyzed and belonging p-values are estimated. In the third section of Chapter 4, the results of the second section are generalized to arbitrary, non-empty, fixed and finite sets $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$. Simulation studies that illustrate the quality of the tests introduced in the second and third section of Chapter 4 in regard to the finite

sample behaviour are contained in the fourth section of Chapter 4.
In Chapter 5, the instruments proposed in the present thesis are applied to log returns of several listed companies and the results are interpreted from an economic perspective.
Appendix A contains some definitions which are often used in the proofs of the results given in the present work. All theorems, propositions and lemmata which are stated in Chapter 2 (Chapter 3 and Chapter 4, respectively) are verified in the first section of Appendix B (Appendix C and Appendix D, respectively). In addition, for all examples which are given in Chapter 2 (Chapter 3 and Chapter 4, respectively), it is shown in Appendix B (Appendix C and Appendix D, respectively) that they fulfil the claimed properties apart from those examples for which is mentioned that this verification is so straightforward that it is omitted. The second sections of the Appendices B, C and D contain auxiliary results which are used in the proofs given in the first sections of these appendices. Thereby, each of these auxiliary results is shown right after it is stated. Additional calculations which support the understanding of the source codes that underlie the simulation studies in Section 3.4 and 4.4 as well as the applications in Chapter 5 are given in Appendix E.

1.5. General notations

Throughout this work, the following notations are used:
The expression $|\cdot|_1$ denotes the ℓ^1-norm that fits to the dimension of its argument. Moreover, define for all $q > 0$ as well as all $\mathbb{C}^r$-valued random vectors X with fixed $r \in \mathbb{N}$ and $\mathbb{E}\left[|X|_1^q\right] < \infty$ the expression $\|X\|_q := \left(\mathbb{E}\left[|X|_1^q\right]\right)^{1/q}$. Further, λ denotes the Lebesgue measure on $\mathbb{R}^r$. Thereby, the dimension r of the spaces $\mathbb{C}^r$ and $\mathbb{R}^r$ that underlie the notations introduced above is repressed in these notations to shorten them. To express for a random variable X which lives on the sample space Ω the realization of X that is associated with an $\omega \in \Omega$, the icon $X(\omega)$ is used, whereas $f(x)\big|_\omega$ denotes the realization of $f(x)$ belonging to ω for a random functions f on Ω and an arbitrary element x of the domain of f. Further, the abbreviated forms $X^c := X - \mathbb{E}[X]$ as well as $f^c(x) := (f(x))^c$ are used for random vectors X with $\mathbb{E}\left[|X|_1\right] < \infty$ and random functions $f\colon D \to \mathbb{C}^r$ as well as $x \in D$ that fulfil $\mathbb{E}\left[|f(x)|_1\right] < \infty$ (with $D \subseteq \mathbb{C}^{\tilde{r}}$ for a $\tilde{r} \in \mathbb{N}$). The notation $X \perp\!\!\!\perp Y$ means for random variables X and Y, which live on the same probability space, that they are (stochastically) independent, whereas $X \not\!\perp\!\!\!\perp Y$ indicates that they are (stochastically) dependent. Moreover, $X \overset{d}{=} Y$ expresses that the distribution of X equals that of Y and the notation $X \overset{d}{\neq} Y$ is used to advert that the distribution of X differs from that of Y. The icons $\mathcal{O}$ and o express Landau's big-O notation and little-o notation, respectively. Further, $f_1 \ll f_2$ as well as $f_2 \gg f_1$ mean $f_1 = o(f_2)$ for real-valued functions f_1 and f_2. The transpose of a vector v is denoted as v'. For an open set $E \subseteq \mathbb{R}^{r_1}$, $x := (x_1, \ldots, x_{r_1})' \in E$, $i \in \{1, \ldots, r_1\}$ and $r_1, r_2, k \in \mathbb{N}$, the k^{th} partial derivative with respect to x_i of a function $g\colon E \to \mathbb{R}^{r_2}$ which is k times partial differentiable with respect to x_i is written as $\partial_{x_i}^k g(x)$. In addition, set $\sum_{n=x}^{y} z_n := 0$ as well as $\prod_{n=x}^{y} z_n := 1$ for all $x, y \in \mathbb{Z}$ with $x > y$ and all $z_y, \ldots, z_x \in \mathbb{C}$. Further, define $\lfloor x \rfloor := \max\{z \in \mathbb{Z} : z \leqslant x\}$ and $\lceil x \rceil := \min\{z \in \mathbb{Z} : z \geqslant x\}$ $\forall\, x \in \mathbb{R}$. Moreover, $\#A$ expresses the cardinality of a finite set A. In addition, $\Re\{z\}$ denotes the real part and $\Im\{z\}$ the imaginary part of a complex number $z \in \mathbb{C}$, whereby $\Re\{z\}^2$ is the abbreviated form of $(\Re\{z\})^2$ and $\Im\{z\}^2$ means $(\Im\{z\})^2$. The icon $\overline{z}$ with $z \in \mathbb{C}$ is defined as the complex conjoint of z. Further, the expression $\{X_{t,T}\}$ is used to shorten the notation of a locally stationary process $\{X_{t,T} : t \in \{1, \ldots, T\}\}_{t=1}^{\infty}$ and $\{\widetilde{X}_t(u)\}$ (with $u \in [0,1]$) is the abbreviated form of the associated stationary approximation $\{\widetilde{X}_t(u) : t \in \mathbb{Z}\}$. Moreover, $C \in (0, \infty)$ denotes an absolute constant that may have different values at different places.

2. The local characteristic functions of locally stationary Bernoulli shift processes

In this chapter, locally stationary Bernoulli shift processes are defined and some belonging assumptions, which underlie the present work, are introduced. Moreover, the local characteristic functions of these processes and their stationary approximations are presented and estimated, whereby asymptotic properties of this estimation are also evolved.

2.1. Locally stationary Bernoulli shift processes

In the following definition, which is based on [16, Dahlhaus et al. (2019), p. 1015 et seq.] (in particular, Assumption 2.3 (M2) in [16, Dahlhaus et al. (2019), p. 1017]), locally stationary Bernoulli shift processes are introduced.

Definition 2.1 (Locally stationary Bernoulli shift processes).
Let $(\varepsilon_k)_{k\in\mathbb{Z}}$ be a sequence of $\mathbb{R}^{\mathfrak{e}}$-valued i. i. d. random variables for an $\mathfrak{e} \in \mathbb{N}$ on a probability space $(\Omega, \Sigma, \mathbb{P})$ and define $\mathcal{F}_t := (\varepsilon_t, \varepsilon_{t-1}, \dots)$ $\forall\, t \in \mathbb{Z}$. Further, assume for a $d \in \mathbb{N}$ that $\mathbf{H}_{t,T}\colon \mathbb{R}^{\mathfrak{e}\times\mathbb{N}} \to \mathbb{R}^d$ with $\mathbb{R}^{\mathfrak{e}\times\mathbb{N}} := \mathbb{R}^{\mathfrak{e}} \times \mathbb{R}^{\mathfrak{e}} \times \dots$ is a deterministic measurable function for each $t \in \{1, \dots, T\}$, $T \in \mathbb{N}$. Moreover, suppose that $\{X_{t,T}\}$ is an $\mathbb{R}^d$-valued locally stationary process (as defined in Section 1.1) which fulfils:

$$X_{t,T} = \mathbf{H}_{t,T}\left(\mathcal{F}_t\right) \;\; \forall\, t \in \{1, \dots, T\},\, T \in \mathbb{N}.$$

Then, the process $\{X_{t,T}\}$ is called ($\mathbb{R}^d$-valued) locally stationary Bernoulli shift process.

The next Assumptions 2.2 $[\mathbf{StAp}]$ and 2.4 $[\mathbf{DM}]$ contain some additional conditions.

Assumption 2.2 [StAp] (Stationary approximations).
Let $\{X_{t,T}\}$ originate from Definition 2.1. The following properties are assumed to hold for $\mathbb{R}^d$-valued (strictly) stationary processes $\{\widetilde{X}_t(u)\}$ (with $u \in [0,1]$), all $T \in \mathbb{N}$ and arbitrary but fixed $\delta \in (0,1]$ as well as $A < \infty$:

(i)

$$\sup_{T\in\mathbb{N}} \sup_{t=1,\dots,T} \|X_{t,T}\|_{1+\delta} \leqslant A, \quad \left\| \sup_{u\in[0,1]} \left|\widetilde{X}_0(u)\right|_1 \right\|_{1+\delta} \leqslant A \quad \text{and} \quad \sup_{t=1,\dots,T} \left\| X_{t,T} - \widetilde{X}_t\left(\frac{t}{T}\right) \right\|_{1+\delta} \leqslant \frac{A}{T}.$$

(ii) Suppose for all $t \in \mathbb{Z}$ that the function $u \mapsto \widetilde{X}_t(u)$ is a. s. continuously differentiable on $(0,1)$ with derivative $u \mapsto \partial_u \widetilde{X}_t(u)$ and the right-hand derivative at 0 as well as the left-hand derivative at 1 of $u \mapsto \widetilde{X}_t(u)$ should be a. s. existent. Moreover, assume:

$$\left\| \sup_{u\in(0,1)} \left|\partial_u \widetilde{X}_0(u)\right|_1 \right\|_{1+\delta} \leqslant A.$$

In addition, it should hold for all non-empty sets $U, V \subseteq (0,1)$:

$$\left\| \sup_{u\in U, v\in V} \left|\partial_u \widetilde{X}_0(u) - \partial_v \widetilde{X}_0(v)\right|_1 \right\|_1 \leqslant A \sup_{u\in U, v\in V} |u - v|^{\delta}. \tag{2.1}$$

(iii) Let for all $u \in [0,1]$ a measurable function $\mathbf{H}(u,\cdot)\colon \mathbb{R}^{\mathfrak{e}\times\mathbb{N}} \to \mathbb{R}^d$ be existent with $\widetilde{X}_t(u) = \mathbf{H}\left(u, \mathcal{F}_t\right)$ $\forall\, t \in \mathbb{Z}$, whereby $\mathcal{F}_t$ originates from Definition 2.1.

The Assumptions 2.2 $[\mathbf{StAp}]$ (i) and (ii) are analog to Assumption 2.1 in [16, Dahlhaus et al. (2019), p. 1015] with $q = 1 + \delta$ as well as $\alpha = 1$ in the notation of this assumption in [16, Dahlhaus et al. (2019), p. 1015]. (The main difference is that (2.1) is not demanded in Assumption 2.1 in [16, Dahlhaus et al. (2019), p. 1015].) Moreover, Assumption 2.2 $[\mathbf{StAp}]$ (iii) is similar to the first statement of Assumption 2.3 (M1) in [16, Dahlhaus et al. (2019), p. 1017]. In the following remark, it is shown that the first equation of (2) in [16, Dahlhaus et al. (2019), p. 1015] is fulfilled in the present situation for $\alpha = 1$ as well as $q = 1 + \delta$, whereas the second equation of (2) in [16, Dahlhaus et al. (2019), p. 1015] holds for $\alpha = 1$ and $q = 1 + \delta$ due to Assumption 2.2 $[\mathbf{StAp}]$ (i).
Further, (2.1) can be interpreted as a Hölder-type condition and one observes that Assumption 2.2 $[\mathbf{StAp}]$ is less restrictive for smaller choices of $\delta \in (0, 1]$ than larger ones.

Remark 2.3. *The mean value theorem and Assumption 2.2* $[\mathbf{StAp}]$ *(ii) show for all non-empty sets* $U, V \subseteq [0, 1]$ *and a fixed* $M < \infty$*:*

$$\sup_{u \in U, v \in V} \sup_{t \in \mathbb{Z}} \left\| \widetilde{X}_t(u) - \widetilde{X}_t(v) \right\|_{1+\delta} \leqslant \left\| \sup_{u \in U, v \in V} \left| \widetilde{X}_0(u) - \widetilde{X}_0(v) \right|_1 \right\|_{1+\delta} \leqslant M \sup_{u \in U, v \in V} |u - v| \,.$$

In the next assumption, dependence measures are introduced that specify the considered locally stationary Bernoulli shift processes, the belonging stationary approximations and their derivatives with respect to u. These dependence measures can be regarded as versions of the functional dependence measure introduced in [80, Wu (2005)].

Assumption 2.4 [DM] (Uniform functional dependence measures).
Suppose that Assumption 2.2 $[\mathbf{StAp}]$ *holds, let* $\left(\varepsilon_k^{\times}\right)_{k \in \mathbb{Z}}$ *be an (stochastically) independent copy of the sequence of random variables* $(\varepsilon_k)_{k \in \mathbb{Z}}$ *which originates from Definition 2.1 and define* $\mathcal{F}_t^{\times(t-0)} := \left(\varepsilon_t^{\times}, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots\right) \forall\, t \in \mathbb{Z}$ *as well as* $\mathcal{F}_t^{\times(t-l)} := \left(\varepsilon_t, \ldots, \varepsilon_{t-l+1}, \varepsilon_{t-l}^{\times}, \varepsilon_{t-l-1}, \ldots\right) \forall\, t \in \mathbb{Z}, l \in \mathbb{N}$. *In addition, the following assumptions should hold for a sequence* $(\Delta_l)_{l \in \mathbb{N}_0}$ *of non-negative numbers:*

(i) Suppose (note that $\mathbf{H}_{t,T}$ *originates from Definition 2.1):*

$$\sup_{T \in \mathbb{N}} \sup_{t=1,\ldots,T} \left\| X_{t,T} - X_{t,T}^{\times(t-l)} \right\|_{1+\delta} \leqslant \Delta_l \;\; \text{for } X_{t,T}^{\times(t-l)} := \mathbf{H}_{t,T}\big(\mathcal{F}_t^{\times(t-l)}\big) \text{ and all } l \in \mathbb{N}_0. \tag{2.2}$$

(ii) Assume (recall that $\mathbf{H}$ *is introduced in Assumption 2.2* $[\mathbf{StAp}]$ *(iii)):*

$$\sup_{u \in [0,1]} \sup_{t \in \mathbb{Z}} \left\| \widetilde{X}_t(u) - \widetilde{X}_t^{\times(t-l)}(u) \right\|_{1+\delta} \leqslant \Delta_l \;\; \text{for } \widetilde{X}_t^{\times(t-l)}(u) := \mathbf{H}\big(u, \mathcal{F}_t^{\times(t-l)}\big) \text{ and all } l \in \mathbb{N}_0. \tag{2.3}$$

In this framework, the following suppositions will be considered:

$\mathbf{2.4\,[DM.1]}$ *Assumption 2.4* $[\mathbf{DM.1}]$ *will hold by definition if and only if Assumption 2.2* $[\mathbf{StAp}]$*, (2.2), (2.3) as well as the following condition are fulfilled:*

$$\Delta_0 + \sum_{l=1}^{\infty} \Delta_l l^2 \leqslant B \;\; \text{for a } B < \infty.$$

$\mathbf{2.4\,[DM.2]}$ *Assumption 2.4* $[\mathbf{DM.2}]$ *will be valid by definition iff Assumption 2.2* $[\mathbf{StAp}]$*, (2.2), (2.3) as well as the following conditions hold for* $\delta \in (0, 1]$ *which originates from Assumption 2.2* $[\mathbf{StAp}]$*:*

$$\begin{aligned} &\Delta_0 + \sum_{l=1}^{\infty} \Delta_l \, l^{2/\delta} \leqslant B \;\; \text{for a } B < \infty \text{ and} \\ &\sup_{u \in (0,1)} \sup_{t \in \mathbb{Z}} \left\| \partial_u \widetilde{X}_t(u) - \partial_u \widetilde{X}_t^{\times(t-l)}(u) \right\|_1 \leqslant \Delta_{\partial,l} \;\; \forall\, l \in \mathbb{N}_0, \;\; \text{whereby } (\Delta_{\partial,l})_{l \in \mathbb{N}_0} \text{ is a sequence of} \\ &\text{non-negative numbers with } \; \Delta_{\partial,0} + \sum_{l=1}^{\infty} \Delta_{\partial,l} \, l \leqslant B. \end{aligned} \tag{2.4}$$

2.4 [DM.3] *Assumption 2.4 [**DM.3**] will be fulfilled by definition if and only if Assumption 2.2 [**StAp**], (2.2), (2.3) as well as the following condition are valid:*

$$\Delta_l \leqslant B\,\rho^l \text{ for all } l \in \mathbb{N}_0, \text{ a } B < \infty \text{ and a } \rho \in (0,1).$$

Intuitively spoken, the Assumptions 2.4 [**DM.1**], 2.4 [**DM.2**] as well as 2.4 [**DM.3**] quantify the importance of the innovations $\varepsilon_t, \varepsilon_{t-1}, \ldots$ on $X_{t,T}$, $\widetilde{X}_t(u)$ and under Assumption 2.4 [**DM.2**], also on $\partial_u \widetilde{X}_t(u)$. Thereby, the impact of an innovation ε_{t-l} with $l \in \mathbb{N}_0$ will be tendentiously bigger if l is closer to zero (i. e., if ε_{t-l} belongs to a more recent past of the point in time t). In addition, it is worth mentioning that the Assumptions 2.4 [**DM.2**] as well as Assumption 2.4 [**DM.3**] are less general than Assumption 2.4 [**DM.1**] and the first condition of (2.4) is more restrictive for smaller $\delta \in (0,1]$ than larger ones, whereby δ determines moment and smoothness conditions (according to Assumption 2.2 [**StAp**]).
Assumption 2.4 [**DM.1**] underlies Section 3.1 in which confidence intervals for two measures for the intensity of distribution changes (in the stationary approximations of locally stationary processes) are estimated, whereas Assumption 2.4 [**DM.2**] is used in Section 3.2 to construct a test for the existence of distribution changes and in Section 3.3 to estimate the first change point in the stationary approximations.
Further, Assumption 2.4 [**DM.3**] is supposed in Chapter 4 to justify the test for independence constructed in this chapter. It is expectable that applying this test will also be appropriate if the right sides of (2.2) and (2.3) decay polynomially with a sufficiently fast decay rate for growing l. However, assuming exponential decay allows to shorten many proofs belonging to Chapter 4 and it should be noted that this supposition is valid for many processes. For instance, if $(\varepsilon_k)_{k\in\mathbb{Z}}$ is a sequence of i. i. d. real-valued random variables, $\mathcal{G}\colon \mathbb{R} \times \mathbb{R}^d \times [0,1] \to \mathbb{R}$ (with $d \in \mathbb{N}$) is a deterministic function and the locally stationary process $\{X_{t,T}\}$ fulfils the recursion (cf. [16, Dahlhaus et al. (2019), p. 1015]):

$$X_{t,T} := \mathcal{G}\left(\varepsilon_t, X_{t-1,T}, \ldots, X_{t-p,T}, \max\left\{\frac{t}{T}, 0\right\}\right) \quad \forall\, t \leqslant T, T \in \mathbb{N},$$

Proposition 4.4 in [16, Dahlhaus et al. (2019), p. 1029] will provide under suitable moment as well as smoothness conditions that the locally stationary process $\{X_{t,T}\}$ and the belonging stationary approximations fulfil Assumption 2.4 [**DM.3**]. Further, it is worth mentioning that exponentially decaying versions of the functional dependence measure are often used in the literature (see e. g. [57, Mies (2021)]).

In the following example, processes are stated for which the Assumptions 2.4 [**DM.1**], 2.4 [**DM.2**] or 2.4 [**DM.3**] (and hence also Assumption 2.2 [**StAp**]) hold.

Example 2.5. *(i) Causal linear locally stationary processes:*
Assume that $B < \infty$, $\delta \in (0,1]$ and that $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a sequence of i. i. d. real-valued random variables with $\|\varepsilon_0\|_{1+\delta} < \infty$. In addition, let $(\Delta_l)_{l\in\mathbb{N}_0}$ and $(\Delta_{\partial,l})_{l\in\mathbb{N}_0}$ be sequences of non-negative real numbers with:

$$\Delta_0 + \sum_{l=1}^{\infty} \Delta_l l^2 \leqslant B \quad \textit{as well as} \quad \sum_{l=0}^{\infty} \Delta_{\partial,l} \leqslant B \tag{2.5}$$

or

$$\Delta_0 + \sum_{l=1}^{\infty} \Delta_l l^{2/\delta} \leqslant B \quad \textit{and} \quad \Delta_{\partial,0} + \sum_{l=1}^{\infty} \Delta_{\partial,l}\, l \leqslant B. \tag{2.6}$$

Further, define deterministic functions $A_{t,T}\colon \mathbb{N}_0 \to \mathbb{R}$ (with $t \in \{1,\ldots,T\}$, $T \in \mathbb{N}$) as well as $\widetilde{A}\colon [0,1] \times \mathbb{N}_0 \to \mathbb{R}$, whereby $(0,1) \ni u \mapsto \widetilde{A}(u,l)$ should be continuously differentiable for all $l \in \mathbb{N}_0$ with existing right-hand derivative at $u = 0$ as well as left-hand derivative at $u = 1$. Moreover, suppose for all $l \in \mathbb{N}_0$, a $D < \infty$ and all non-empty sets $U, V \subseteq (0,1)$:

$$\sup_{T\in\mathbb{N}} \sup_{t=1,\ldots,T} |A_{t,T}(l)| \leqslant \Delta_l, \quad \sup_{u\in[0,1]} \left|\widetilde{A}(u,l)\right| \leqslant \Delta_l, \quad \sup_{T\in\mathbb{N}} \sup_{t=1,\ldots,T} \left(T\left|A_{t,T}(l) - \widetilde{A}\left(\frac{t}{T}, l\right)\right|\right) \leqslant \Delta_l,$$

$$\sup_{u\in(0,1)} \left|\partial_u \widetilde{A}(u,l)\right| \leqslant \Delta_{\partial,l} \quad \textit{as well as} \quad \sup_{u\in U, v\in V} \left|\partial_u \widetilde{A}(u,l) - \partial_v \widetilde{A}(v,l)\right| \leqslant D \sup_{u\in U, v\in V} |u-v|^\delta\, \Delta_{\partial,l}.$$

Define in this framework the locally stationary Bernoulli shift process $\{X_{t,T}\}$ as:

$$X_{t,T} := \sum_{l=0}^{\infty} A_{t,T}(l) \cdot \varepsilon_{t-l} \ \ \forall\, t \in \{1, \ldots, T\},\, T \in \mathbb{N}.$$

Then, $\{X_{t,T}\}$ will satisfy Assumption 2.4 $[\mathbf{DM.1}]$ if (2.5) is fulfilled, Assumption 2.4 $[\mathbf{DM.2}]$ if (2.6) is valid and Assumption 2.4 $[\mathbf{DM.3}]$ if (2.5) with $\Delta_l \leqslant B\rho^l$ for all $l \in \mathbb{N}_0$, a $B < \infty$ and a $\rho \in (0,1)$ holds.

(ii) Time-varying AR(1)-processes with random coefficients:
Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a sequence of i. i. d. real-valued random variables and suppose for all $k \in \{0,1\}$ that $a_k\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a deterministic function. Moreover, assume for all fixed $x \in \mathbb{R}$, $k \in \{0,1\}$ that the function $u \mapsto a_k(x,u)$ is continuously differentiable for all $u \in (0,1)$ and the right-hand derivative at $u = 0$ as well as the left-hand derivative at $u = 1$ of this function should be existent. In addition, let $a_k(x,u) = 0\ \forall\, k \in \{0,1\}$, $x \in \mathbb{R}$, $u < 0$. Furthermore, assume that arbitrary but fixed $\delta \in (0,1]$, $D < \infty$, $\rho \in (0,1)$ and two sequences of i. i. d. positive random variables $(\widetilde{a}_{k,t})_{t\in\mathbb{Z}}$ (with $k \in \{0,1\}$) exist such that:

$$\left\|\widetilde{a}_{0,0}\right\|_{1+\delta} \leqslant D,\ \ \left\|\widetilde{a}_{1,0}\right\|_{1+\delta} \leqslant \rho,\ \ \sup_{u\in[0,1]} \left|a_k\left(\varepsilon_t, u\right)\right| \leqslant \widetilde{a}_{k,t}\ \text{a. s.},\ \ \sup_{u\in(0,1)} \left|\partial_u a_k\left(\varepsilon_t, u\right)\right| \leqslant D\,\widetilde{a}_{k,t}$$
$$\text{a. s. as well as} \sup_{u\in U, v\in V} \left|\partial_u a_k\left(\varepsilon_t, u\right) - \partial_v a_k\left(\varepsilon_t, v\right)\right| \leqslant D \sup_{u\in U, v\in V} |u-v|^{\delta}\,\widetilde{a}_{k,t}\ \text{a. s. for all}$$
$$k \in \{0,1\},\ t \in \mathbb{Z} \text{ and all non-empty sets } U, V \subseteq (0,1). \tag{2.7}$$

Then, the process $\{X_{t,T}\}$, which is for all $T \in \mathbb{N}$ defined as:

$$X_{t,T} := a_1\left(\varepsilon_t, \frac{t}{T}\right) X_{t-1,T} + a_0\left(\varepsilon_t, \frac{t}{T}\right)\ \forall\, t \in \{1, \ldots, T\} \quad \text{with} \quad X_{0,T} := a_0\left(\varepsilon_0, 0\right),$$

is a locally stationary Bernoulli shift process that fulfils the Assumptions 2.4 $[\mathbf{DM.2}]$ and 2.4 $[\mathbf{DM.3}]$.

(iii) Time-varying ARCH(1)-processes:
Suppose that $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a sequence of i. i. d. real-valued random variables with $\left\|\varepsilon_0\right\|_{2+2\delta} < \infty$ for some $\delta \in (0,1/2]$, $\mathbb{E}\left[\varepsilon_0\right] = 0$, $\operatorname{Var}\left(\varepsilon_0\right) = 1$ and $\mathbb{P}\left(\varepsilon_0 = 0\right) = 0$. Moreover, let $a_k\colon \mathbb{R} \to [0,\infty)$ (with $k \in \{0,1\}$) be deterministic non-negative functions which are twice continuously differentiable on $(0,1)$ and assume that the first right-hand derivative at 0 as well as the first left-hand derivative at 1 of these functions exist. In addition, suppose $a_k(u) = 0\ \forall\, k \in \{0,1\}$, $u < 0$. Furthermore, assume for arbitrary but fixed $\rho \in (0,1)$:

$$0 < m_0 := \inf_{u\in[0,1]} a_0(u) \leqslant \sup_{u\in[0,1]} a_0(u) < \infty, \quad \sup_{u\in[0,1]} a_1(u) \leqslant \rho < \left\|\varepsilon_0\right\|_{2+2\delta}^{-2},$$
$$\sup_{u\in(0,1)} \left|\partial_u^i a_k(u)\right| < \infty\ \forall\, i \in \{1,2\},\ k \in \{0,1\} \ \text{and} \sup_{u\in(0,1):a_1(u)\neq 0} \left|\partial_u a_1(u)\right| / \sqrt{a_1(u)} < \infty. \tag{2.8}$$

Then, the process $\{X_{t,T}\}$, which is for all $T \in \mathbb{N}$ defined as:

$$X_{t,T} := \left(a_0\left(\frac{t}{T}\right) + a_1\left(\frac{t}{T}\right) X_{t-1,T}^2\right)^{\frac{1}{2}} \varepsilon_t\ \ \forall\, t \in \{1, \ldots, T\} \quad \text{with} \quad X_{0,T} := \sqrt{a_0\left(0\right)}\,\varepsilon_0,$$

is a locally stationary Bernoulli shift process that fulfils the Assumptions 2.4 $[\mathbf{DM.2}]$ and 2.4 $[\mathbf{DM.3}]$.

The linear locally stationary processes defined in Example 2.5 (i) are causal in the sense that each $X_{t,T}$ just depends on innovations that belong to the present and past (i. e., on innovations ε_{t-l} for which $l \in \mathbb{N}_0$), whereas some researches that consider not necessarily causal linear locally stationary processes are given in, e. g., [40, Jentsch et al. (2020a)] as well as [19, Dette et al. (2011)]. Example 2.5 (ii) is a special case of the class of time-varying (abbr. tv.) autoregressive random coefficient models, which is investigated in [70, Subba Rao (2006)] and also mentioned in Example 2.2 (iv) in [16, Dahlhaus et al., p. 1016]. One observes that these processes have a more general structure than tvAR-processes, which were often considered in the past (see e. g. Chapter 2 in [14, Dahlhaus (2012)] and Example 2 in [33, Giraud et al. (2015), p. 2415]), because they may depend on their innovations in a non-linear way. Example 2.5 (iii) describes some tvARCH(1)-processes, that can be generalized to tvARCH(∞)-processes, which are studied in [17, Dahlhaus and Subba Rao (2006)]. (Some other analyses concerning these time series can be found in, e. g., [30, Fryzlewicz et al. (2008)]). However, Example 2.5 (iii) ensures that the considered tvARCH(1)-processes themselves fulfil Assumption 2.4 $[\mathbf{DM.3}]$, whereas the investigations of [17, Dahlhaus and Subba Rao (2006)] are focused on properties of squared tvARCH-processes. This difference is motivated by the fact that examples of (stochastically) dependent tvARCH(1)-processes can be constructed whose dependences can be eliminated by squaring them, such that squaring may impact the decision resulting from the test for independence constructed in Chapter 4. (E. g., by using Rademacher distributed i. i. d. innovations, a tvARCH(1)-process $\{X_{t,T}\}$ can be constructed whose random variables are not Dirac distributed but which fulfils that $\{X_{t,T}^2\}$ consists of Dirac distributed random variables, that are obviously stochastically independent of all processes.)

2.2. Definition and estimation of the local characteristic functions

In the next definition, the local characteristic functions of locally stationary processes and their stationary approximations are introduced. These functions are defined for linear locally stationary processes in [40, Jentsch et al. (2020a), p. 114], whereas they are presented below for locally stationary Bernoulli shift processes.

Definition 2.6 (Local characteristic functions)**.**
Let Assumption 2.2 $[\mathbf{StAp}]$ *be fulfilled. Then, the local characteristic functions (abbr. LCFs) of the locally stationary Bernoulli shift process* $\{X_{t,T}\}$ *and its stationary approximations* $\{\widetilde{X}_t(u)\}$ *(with* $u \in [0,1]$*) are defined as:*

$$\varphi_{t,T}(s) := \mathbb{E}\left[e^{i\langle s, X_{t,T}\rangle}\right] \quad \textit{and} \quad \varphi(u,s) := \mathbb{E}\left[e^{i\langle s, \widetilde{X}_0(u)\rangle}\right] \quad \forall\, t \in \{1,\ldots,T\},\, T \in \mathbb{N},\, s \in \mathbb{R}^d,\, u \in [0,1].$$

The following remark, which is similar to Lemma 2.2 (i) in [40, Jentsch et al. (2020a), p. 114], shows for all $t \in \{1,\ldots,T\}$ the connection between the characteristic function of $X_{t,T}$ and that of $\widetilde{X}_t\,(t/T)$.

Remark 2.7. *One obtains for* $T \to \infty$ *from Assumption 2.2* $[\mathbf{StAp}]$ *(i):*

$$\sup_{t=1,\ldots,T} \left|\varphi_{t,T}(s) - \varphi\left(\frac{t}{T}, s\right)\right| = \mathcal{O}\left(\frac{1}{T}\right) |s|_1 \quad \forall\, s \in \mathbb{R}^d,$$

whereby the expression $\mathcal{O}\,(1/T)$ *does not depend on* $s \in \mathbb{R}^d$*.*

The distributions of the random variables $X_{t,T}$ (with $t \in \{1,\ldots,T\}$, $T \in \mathbb{N}$) and $\widetilde{X}_t(u)$ (with $t \in \mathbb{Z}$, $u \in [0,1]$) are unknown in most applications. Therefore, an estimator for the LCF φ is constructed in the following, which can also be used to estimate $\varphi_{t,T}$ due to Remark 2.7.
An estimator for φ has to take into account that realizations of $X_{1,T},\ldots,X_{T,T}$ can be observed in practise, whereas realizations of $\widetilde{X}_t(u)$ are commonly unobservable for all $t \in \mathbb{Z}$, $u \in [0,1]$. Moreover, motivated by (1.1), it seems to be useful to estimate $\varphi(u,\cdot)$ for $u \in [0,1]$ in such a manner that a $X_{t,T}$ for which t/T is closer to u has a stronger impact on this estimation than a $X_{t,T}$ for which the distance $|t/T - u|$ is larger. A kernel-bandwidth-based estimator which meets these demands for a suitable choice of the underlying kernel is defined in [40, Jentsch et al. (2020a)] and this work also investigates for fixed $u \in (0,1)$ the asymptotic behaviour of this estimator in regard to linear locally stationary processes. In contrast, the following research aims to obtain results for this estimator that concern locally stationary Bernoulli shift processes and are formulated uniformly with respect to u. Therefor, this estimator is defined below based on a kernel and a bandwidth that fulfil the next assumption.

Assumption 2.8 [K&b.1] (Kernel and bandwidth)**.**

(i) Suppose that $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ *are arbitrary but fixed with* $\mathfrak{U}_0 < \mathfrak{U}_1$ *and* $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$*. The kernel* $K := K_{\mathfrak{U}_{0,1}} : \mathbb{R} \to [0,\infty)$ *is defined as a non-negative, Lipschitz continuous function with Lipschitz constant* $L_K < \infty$ *that fulfils* $K(z) = K(-z)\ \forall\, z \in \mathbb{R}$*,* $K(z) = 0\ \forall\, z \in \mathbb{R}\backslash\left(\mathfrak{U}_0 - \mathfrak{U}_1, \mathfrak{U}_1 - \mathfrak{U}_0\right)$ *and* $\int_{\mathbb{R}} K(z)\, dz = 1$*.*

(ii) *Define for all $T \in \mathbb{N}$ a bandwidth $b := b_T$ as a real number with $b \in (0, 1/2)$. Furthermore, it should hold for $T \to \infty$ that $b \to 0$, $Tb^2 \to \infty$ as well as $Tb^{2+2\delta} \to 0$, whereby $\delta \in (0,1]$ originates from Assumption 2.2* $[\mathbf{StAp}]$.

Remark 2.9. *The parameter $\delta \in (0,1]$ is commonly unknown in practise and it is expectable that it cannot be estimated appropriately in general. Nevertheless, a sequence of bandwidths, for which Assumption 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *(ii) holds, can be constructed without specifying δ $\big($e. g., $b := b_T := 1/4\mathbb{1}_{\{T=1\}} + \min\left\{T^{-1/2}\sqrt{\ln(T)}, 1/4\right\}\mathbb{1}_{\{T \geqslant 2\}}\ \forall\, T \in \mathbb{N}\big)$.*

The properties of the kernel and bandwidth supposed in Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ are very similar to those given in Assumption 2.2 (i) and (ii) in [40, Jentsch et al. (2020a), p. 115]. In contrast to the present publication, the kernel used in [40, Jentsch et al. (2020a)] is just defined in such a manner that it fulfils Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) with $\mathfrak{U}_0 = 0$ as well as $\mathfrak{U}_1 = 1$ and other choices of $\mathfrak{U}_0$ or $\mathfrak{U}_1$ are not considered explicitly but can be regarded as special cases. (Taking various $\mathfrak{U}_0$ and $\mathfrak{U}_1$ into account is of particular importance in Section 3.3, in which the first change point in the stationary approximations is estimated.) Moreover, the conditions $Tb^{2+2\delta} \to 0$ for $T \to \infty$ as well as $b < 1/2$ are not contained in Assumption 2.2 in [40, Jentsch et al. (2020a), p. 115]. However, $Tb^3 \to 0$ is assumed for the central limit theorem (Theorem 3.1 (ii) in [40, Jentsch et al. (2020a), p. 117]), which evolves the asymptotic distribution of the difference (multiplicated with $\sqrt{bT}$) between the estimated and real local characteristic function φ. This is a stronger assumption than $Tb^{2+2\delta} \to 0$ in the case $\delta > 1/2$. In addition, the supposition $b < 1/2$ is useful in the Chapters 3 and 4 since it ensures $\lfloor 1/(2b) \rfloor \geqslant 1$ and $\lambda\left([b, 1-b]\right) > 0$.
Further, in view of Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii), it is worth mentioning that Tb^2 and $Tb^{2+2\delta}$ are closer together for smaller values of $\delta \in (0,1]$ (which determines moment and smoothness conditions according to Assumption 2.2 $[\mathbf{StAp}]$) than larger ones. This suggests that the quality of the kernel-bandwidth-based estimator for the LCF, which is constructed in the following, will fluctuate tendentiously more for different choices of the bandwidth if δ is smaller.

In the next example, kernels are introduced for which Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) holds. Since it is very straightforward to see that Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) is valid for these kernels, the proof of this statement is omitted in the appendix.

Example 2.10. *Suppose that $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ are arbitrary but fixed with $\mathfrak{U}_0 < \mathfrak{U}_1$. Then, the following kernels fulfil Assumption 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *(i).*

(i) *Triangular kernel:*

$$K_{\text{Tri}} : \mathbb{R} \to [0,\infty),\ z \mapsto \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0}\left(1 - \frac{|z|}{\mathfrak{U}_1 - \mathfrak{U}_0}\right)\mathbb{1}_{\{z \in [\mathfrak{U}_0 - \mathfrak{U}_1, \mathfrak{U}_1 - \mathfrak{U}_0]\}}$$

(ii) *Epanechnikov kernel:*

$$K_{\text{Epa}} : \mathbb{R} \to [0,\infty),\ z \mapsto \frac{3}{4\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)}\left(1 - \left(\frac{z}{\mathfrak{U}_1 - \mathfrak{U}_0}\right)^2\right)\mathbb{1}_{\{z \in [\mathfrak{U}_0 - \mathfrak{U}_1, \mathfrak{U}_1 - \mathfrak{U}_0]\}}$$

In the next definition, an estimator for the LCF φ is stated, which is also proposed in [40, Jentsch et al. (2020a), p. 114].

Definition 2.11 (The local empirical characteristic function)**.**
Let the Assumptions 2.2 $[\mathbf{StAp}]$ *and 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *be fulfilled. The kernel estimator $\widehat{\varphi} := \widehat{\varphi}_{T,\mathfrak{U}_{0,1}}$ for the local characteristic function φ, which is defined as:*

$$\widehat{\varphi}(u,s) := \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) e^{i\left\langle s, X_{t,T}\right\rangle} \quad \forall\, u \in [0,1], s \in \mathbb{R}^d \ \textit{with}\ K_b(x) := \frac{1}{b}K\left(\frac{x}{b}\right)\ \forall\, x \in \mathbb{R},$$

is called local empirical characteristic function (abbr. LECF).

The following Proposition 2.12 shows for arbitrary but fixed $s \in \mathbb{R}^d$ that the expectation of the LECF converges for $T \to \infty$ to the LCF φ uniformly with respect to $u \in \left[\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)b, 1 - \left(\mathfrak{U}_1 - \mathfrak{U}_0\right)b\right]$, whereas Proposition 2.13 given below indicates that this convergence result does not hold uniformly with respect to $u \in [0,1]$.

Proposition 2.12 (Bias of the LECF - Part I).
Suppose that the Assumptions 2.2 $[\mathbf{StAp}]$ *and 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *hold. Moreover, define:*

$$\mathfrak{U}_{0,1,b} := \left[(\mathfrak{U}_1 - \mathfrak{U}_0)\, b, 1 - (\mathfrak{U}_1 - \mathfrak{U}_0)\, b\right]. \tag{2.9}$$

Then, one obtains for $T \to \infty$*:*

$$\sup_{u \in \mathfrak{U}_{0,1,b}} \left| \mathbb{E}\left[\widehat{\varphi}(u,s)\right] - \varphi(u,s) \right| = \mathcal{O}\left(b^{1+\delta} + \frac{1}{Tb}\right)\left(|s|_1^{1+\delta} + 1\right) \quad \forall\, s \in \mathbb{R}^d,$$

whereby the expression $\mathcal{O}\left(b^{1+\delta} + 1/(Tb)\right)$ *does not depend on* $s \in \mathbb{R}^d$*.*

The upper bound of the (uniform) bias of the LECF given in Proposition 2.12 depends on $\delta \in (0,1]$, which determines moment and smoothness conditions according to Assumption 2.2 $[\mathbf{StAp}]$. An upper bound of the bias of the LECF in the scale $\mathcal{O}(b^2 + 1/(Tb))$ results from Lemma 3.1 (ii) in [40, Jentsch et al. (2020a), p. 115] for fixed $u \in (0,1)$ as well as $s \in \mathbb{R}^d$ under the assumption that the i. i. d. innovations belonging to the considered linear locally stationary process (and hence also the random variables contained in this linear locally stationary process and its stationary approximations) own finite third-order moments. In contrast, Proposition 2.12 provides for $\delta = 1$ an upper bound in the same scale without demanding that third-order moments of $X_{1,T}, \ldots, X_{T,T}$ or of $\widetilde{X}_0(u)$ are finite. This weakening with respect to the moment conditions is mainly possible due to the assumed Hölder-type property (2.1), which is not demanded in Lemma 3.1 (ii) in [40, Jentsch et al. (2020a), p. 115].

Proposition 2.13 (Bias of the LECF - Part II).
Let the Assumptions 2.2 $[\mathbf{StAp}]$ *and 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *be fulfilled. Then, fixed* $c_K \in (0, \mathfrak{U}_1 - \mathfrak{U}_0)$*,* $\epsilon_K > 0$*,* $S_K > 0$ *and* $T_K \in \mathbb{N}$ *exist for which:*

$$\inf_{[0, c_K b] \cup [1 - c_K b, 1]} \left| \mathbb{E}\left[\widehat{\varphi}(u,s)\right] - \varphi(u,s) \right| \geqslant \epsilon_K \quad \forall\, s \in [-S_K, S_K]^d,\ T \in \mathbb{N} : T \geqslant T_K.$$

The next proposition yields under Assumption 2.4 $[\mathbf{DM}.\mathbf{1}]$ for arbitrary but fixed $s \in \mathbb{R}^d$ that the variances of the real and imaginary part of the LECF converge to zero uniformly with respect to $u \in [0,1]$. Thereby, it should be recalled that the Assumptions 2.4 $[\mathbf{DM}.\mathbf{2}]$ and 2.4 $[\mathbf{DM}.\mathbf{3}]$ are more restrictive than Assumption 2.4 $[\mathbf{DM}.\mathbf{1}]$, such that Proposition 2.14 holds under each of these suppositions. Moreover, this proposition and Proposition 2.12 imply that $\mathbb{E}\left[\left|\widehat{\varphi}(u,s) - \varphi(u,s)\right|^2\right]$ converges for arbitrary but fixed $s \in \mathbb{R}^d$ to zero uniformly with respect to $u \in \mathfrak{U}_{0,1,b}$.

Proposition 2.14 (Variance of the LECF).
Suppose that the Assumptions 2.4 $[\mathbf{DM}.\mathbf{1}]$ *and 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *are fulfilled. Then, it holds for* $T \to \infty$*:*

$$\sup_{u \in [0,1]} \mathbb{E}\left[\left|\widehat{\varphi}(u,s) - \mathbb{E}\left[\widehat{\varphi}(u,s)\right]\right|^2\right] = \mathcal{O}\left(\frac{1}{Tb}\right)\left(|s|_1 + 1\right) \quad \forall\, s \in \mathbb{R}^d,$$

whereby the expression $\mathcal{O}\left(1/(Tb)\right)$ *does not depend on* $s \in \mathbb{R}^d$*.*

Note that Lemma 3.1 (iii) in [40, Jentsch et al. (2020a), p. 116] shows for fixed $u_1, u_2 \in (0,1)$, $s_1, s_2 \in \mathbb{R}^d$ that the (complex) covariance $\operatorname{Cov}\left(\widehat{\varphi}(u_1,s_1), \widehat{\varphi}(u_2,s_2)\right)$ converges to zero with the rate $\mathcal{O}\left(1/(Tb)\right)$, which is the same rate as that obtained in Proposition 2.14.

3. Investigation of distribution changes

In the present chapter, two measures are defined that quantify the intensity of distribution changes of $\widetilde{X}_0(u)$, which originates from Assumption 2.2 [$\mathbf{StAp}$], for varying u and confidence intervals for these measures are estimated. In addition, consistent level-alpha tests are constructed that detect whether the distribution of $\widetilde{X}_0(u)$ changes in dependence of u. Further, the smallest u (i. e., the first rescaled point in time) at which the distribution of $\widetilde{X}_0(u)$ changes is estimated. Moreover, the finite sample behaviour of the methods proposed in this chapter is evaluated by using simulation studies.

3.1. Quantification of distribution changes

3.1.1. Definition of measures for the distribution change intensity

In order to quantify the degree of distribution changes, an approach which is similar to that proposed in [19, Dette et al. (2011)] is used. In [19, Dette et al. (2011)], a measure for the deviation from second order stationarity under local stationarity is defined based on integrated squared minimal distances, which is briefly described in the following. Let $f_{\mathbf{loc.spec}}\colon [0,1] \times [-\pi,\pi] \to \mathbb{R}$ denote the local spectral density of the stationary approximations which belong to a centered linear locally stationary process and $\mathbf{SPEC}$ should be the set of all spectral densities $f_{\mathbf{spec}}\colon [-\pi,\pi] \to \mathbb{R}$ that are associated with centered covariance stationary processes. Then, the measure for the deviation from second order stationarity is defined as (cf. (1.1) in [19, Dette et al. (2011), p. 1114]):

$$\min_{f_{\mathbf{spec}} \in \mathbf{SPEC}} \int_{-\pi}^{\pi} \int_{0}^{1} \left(f_{\mathbf{loc.spec}}(u,x) - f_{\mathbf{spec}}(x)\right)^2 \, du \, dx. \tag{3.1}$$

Since the distribution of $\widetilde{X}_0(u)$ is uniquely defined by the characteristic function of $\widetilde{X}_0(u)$ for each $u \in [0,1]$, the distribution change intensity of $\widetilde{X}_0(u)$ with respect to varying u can be measured based on the local characteristic function $\varphi : [0,1] \times \mathbb{R}^d \to \mathbb{C}$, which is introduced in Definition 2.6. Concretely, motivated by (3.1), it seems to be appropriate to define the following measure (3.2) for the degree of distribution changes in the stationary approximations of locally stationary Bernoulli shift processes:
Let $\mathbf{CHAR}$ denote the set of all characteristic functions $f_{\mathbf{char}}\colon \mathbb{R}^d \to \mathbb{C}$. Then, define the intensity of distribution changes as:

$$\inf_{f_{\mathbf{char}} \in \mathbf{CHAR}} \int_{\mathbb{R}^d} \int_{0}^{1} \left|\varphi(u,s) - f_{\mathbf{char}}(s)\right|^2 \, du \, \mathbf{w}(s) \, ds, \tag{3.2}$$

whereby $\mathbf{w}\colon \mathbb{R}^d \to [0,\infty)$ is a weight function which should ensure that the integral with respect to $s \in \mathbb{R}^d$ contained in (3.2) is well-defined and should allow to evolve the asymptotic behaviour of an adequate estimator for (3.2). For these purposes, it is assumed that $\mathbf{w}$ fulfils Assumption 3.1 [$\mathbf{WEI.1}$], which is given below. Alternatively, as described at the end of the present section (on the Pages 30 and 32), the weight function given in Lemma 1 in [72, Székely et al. (2007), p. 2771], which does not fulfil Assumption 3.1 [$\mathbf{WEI.1}$], can be used to define other measures for quantifying the distribution change intensity. However, it is also explained on the Pages 30 and 32 why the estimation of the resulting measures is accompanied by serious disadvantages in the present locally stationary framework that will not emerge if Assumption 3.1 [$\mathbf{WEI.1}$] holds.

Assumption 3.1 [WEI.1] (Weight function)**.**
Assume that $d \in \mathbb{N}$ originates from Definition 2.1 and $\delta \in (0,1]$ from Assumption 2.2 [$\mathbf{StAp}$]. The weight function $\mathbf{w}\colon \mathbb{R}^d \to [0,\infty)$ is defined as a Riemann integrable function which is Lebesgue almost

everywhere positive and fulfils:

$$\int_{\mathbb{R}^d} \left(1 + |s|_1^{2+2\delta}\right) \mathbf{w}(s)\, ds < \infty. \tag{3.3}$$

According to Assumption 2.2 [$\mathbf{StAp}$], δ is defined by moment and smoothness conditions, such that it is commonly unknown in applications. However, since $\delta \in (0,1]$ is supposed in Assumption 2.2 [$\mathbf{StAp}$], the condition (3.3) will be fulfilled if $\int_{\mathbb{R}^d} \left(1 + |s|_1^4\right) \mathbf{w}(s)\, ds < \infty$. Hence, appropriate weight functions can be selected in practise, whereby some of them are given in the next example. It is very straightforward to see that Assumption 3.1 [$\mathbf{WEI.1}$] holds for the weight functions introduced in Example 3.2, such that a belonging proof is omitted in the appendix of the present work.

Example 3.2. *Suppose that $d \in \mathbb{N}$ originates from Definition 2.1. Then, the weight functions given below fulfil Assumption 3.1* [$\mathbf{WEI.1}$].

(i) Weight functions inspired by Gaussian distributions:

$$\mathbf{w}_{\mathrm{G},y} \colon \mathbb{R}^d \to (0,\infty),\ s \mapsto e^{-|s|_1^2/y} \quad \forall\, y > 0$$

(ii) Weight functions inspired by Laplace distributions:

$$\mathbf{w}_{\mathrm{L},y} \colon \mathbb{R}^d \to (0,\infty),\ s \mapsto e^{-|s|_1/y} \quad \forall\, y > 0$$

The following Definition 3.3 (i) of a measure for the degree of distribution changes is more general than (3.2) in the sense that it allows to quantify the distribution change intensity on different intervals $[\mathfrak{U}_0, \mathfrak{U}_1] \subseteq [0,1]$ of rescaled time periods and not just on the entire rescaled time period $[0,1]$. Due to this modification of (3.2), it is possible to investigate whether the considered distributions change in one rescaled time period more intensively than in another and to estimate the rescaled point in time u in which the distribution of $\widetilde{X}_0(u)$ changes for the first time. Furthermore, in contrast to (3.2), the measure introduced in Definition 3.3 (i) is based on the infimum with respect to the larger set of all continuous functions (instead of all characteristic functions). However, in Remark 3.7 (i) given below, it is pointed out that the unique minimizer of the integrated squared distance considered in Definition 3.3 (i) (and, therefore, also that of (3.2)) is a characteristic function, such that this difference in the underlying sets does not play a role.
Further, in the following Definition 3.3 (ii), another measure for quantifying distribution changes is proposed, which is defined similar to (3.10) in [19, Dette et al. (2011), p. 1117] and discussed in Remark 3.5 (ii) given below.

Definition 3.3 ((Normalized) measure for the distribution change intensity)**.**
Let $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ be arbitrary but fixed with $\mathfrak{U}_0 < \mathfrak{U}_1$ and the Assumptions 2.2 [$\mathbf{StAp}$] *as well as 3.1* [$\mathbf{WEI.1}$] *be fulfilled.*

(i) Define $\mathscr{C}^0\left(\mathbb{R}^d\right) := \left\{f \colon \mathbb{R}^d \to \mathbb{C} \mid f \text{ is continuous}\right\}$ as well as (see Definition 2.6):

$$\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1] \quad \text{and} \quad \mathbb{D} := \mathbb{D}_{\mathfrak{U}_{0,1}} := \inf_{f \in \mathscr{C}^0(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} |\varphi(u,s) - f(s)|^2 \, du\, \mathbf{w}(s)\, ds.$$

The expression $\mathbb{D}$ is called measure for the distribution change intensity under local stationarity (abbr. MDCI).

(ii) Define:

$$\mathbb{D}_1 := \mathbb{D}_{\mathfrak{U}_{0,1},1} := \int_{\mathbb{R}^d} \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} |\varphi(u,s)|^2 \, du\, \mathbf{w}(s)\, ds, \quad \mathbb{D}_2 := \mathbb{D}_{\mathfrak{U}_{0,1},2} := \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int_{\mathbb{R}^d} \left| \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \varphi(u,s)\, du \right|^2 \mathbf{w}(s)\, ds$$

and $\mathbb{D}^{\mathrm{norm}} := \mathbb{D}^{\mathrm{norm}}_{\mathfrak{U}_{0,1}} := 1 - \dfrac{\mathbb{D}_2}{\mathbb{D}_1}$.

Then, $\mathbb{D}^{\mathrm{norm}}$ is called normalized measure for the distribution change intensity under local stationarity (abbr. NMDCI).

As stated in Remark 3.5 (ii) given below, the next lemma is of importance to ensure that $\mathbb{D}^{\mathrm{norm}}$ is well-defined.

Lemma 3.4. *Suppose that $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ are arbitrary but fixed with $\mathfrak{U}_0 < \mathfrak{U}_1$ and that the Assumptions 2.2 $[\mathbf{StAp}]$ as well as 3.1 $[\mathbf{WEI.1}]$ are fulfilled. Then, it holds $\mathbb{D}_1 > 0$.*

Remark 3.5. *(i) Assumption 2.2 $[\mathbf{StAp}]$ and some obvious arguments imply that the function $\mathfrak{U}_{0,1} \times \mathbb{R}^d \ni (u,s) \mapsto |\varphi(u,s) - f(s)|^2$ is continuous for all $f \in \mathscr{C}^0\left(\mathbb{R}^d\right)$. In addition, Assumption 3.1 $[\mathbf{WEI.1}]$ provides:*

$$0 \leqslant \mathbb{D} \leqslant \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} |\varphi(u,s) - 0|^2 \, du \, \mathbf{w}(s) \, ds < \infty. \tag{3.4}$$

Overall, these considerations yield that $\mathbb{D}$ is well-defined and bounded.

(ii) Straightforward arguments show that the integrals contained in $\mathbb{D}_1$ and $\mathbb{D}_2$ are well-defined. Thus, $\mathbb{D}^{\mathrm{norm}}$ is well-defined due to Lemma 3.4.
Further, it follows from $\mathbb{1}_{\{u \in [\mathfrak{U}_0, \mathfrak{U}_1]\}} = \mathbb{1}_{\{u \in [\mathfrak{U}_0, \mathfrak{U}_1]\}} \mathbb{1}_{\{u \in [\mathfrak{U}_0, \mathfrak{U}_1]\}} \; \forall \, u \in \mathbb{R}$ and the Cauchy-Schwarz inequality:

$$\begin{aligned}
\mathbb{D}_2 &= \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int\limits_{\mathbb{R}^d} \left(\left(\int\limits_{\mathbb{R}} \mathbb{1}_{\{u \in [\mathfrak{U}_0, \mathfrak{U}_1]\}} \Re\{\varphi(u,s)\} \, du \right)^2 + \left(\int\limits_{\mathbb{R}} \mathbb{1}_{\{u \in [\mathfrak{U}_0, \mathfrak{U}_1]\}} \Im\{\varphi(u,s)\} \, du \right)^2 \right) \mathbf{w}(s) \, ds \\
&\leqslant \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}} \mathbb{1}^2_{\{u \in [\mathfrak{U}_0, \mathfrak{U}_1]\}} \, du \left(\int\limits_{\mathbb{R}} \mathbb{1}^2_{\{u \in [\mathfrak{U}_0, \mathfrak{U}_1]\}} \Re\{\varphi(u,s)\}^2 \, du + \int\limits_{\mathbb{R}} \mathbb{1}^2_{\{u \in [\mathfrak{U}_0, \mathfrak{U}_1]\}} \Im\{\varphi(u,s)\}^2 \, du \right) \\
&\quad \cdot \mathbf{w}(s) \, ds \\
&= \mathbb{D}_1.
\end{aligned} \tag{3.5}$$

Assumption 3.1 $[\mathbf{WEI.1}]$ (which ensures $\mathbb{D}_2 \geqslant 0$) and (3.5) yield $\mathbb{D}^{\mathrm{norm}} \in [0,1]$, which motivates that $\mathbb{D}^{\mathrm{norm}}$ is called normalized measure (for the distribution change intensity).
Further, it is worth mentioning that $\mathbb{D}$ is monotonically increasing with respect to the underlying time interval in the sense that $\mathbb{D}_{\mathfrak{W}_{0,1}} \leqslant \mathbb{D}_{\mathfrak{U}_{0,1}}$ for $\mathfrak{W}_{0,1} \subseteq \mathfrak{U}_{0,1} \subseteq [0,1]$. Thus, the MDCI may be unuseful in applications which require to compare the distribution change intensity assigned to a certain rescaled time period $\mathfrak{U}_{0,1}$ with the distribution change intensities belonging to subsets of $\mathfrak{U}_{0,1}$ relatively to the length of the considered rescaled time period, whereby such applications are contained in Chapter 5. In contrast, this monotonicity property does not necessarily hold for the NMDCI, such that this measure can be the better choice for some practical examples.

In the parts (i) and (ii) of the following proposition, alternative ways to express the MDCI are introduced, whereas the parts (iii) and (iv) of this proposition indicate that the MDCI and NMDCI are appropriate instruments for identifying distribution changes.

Proposition 3.6 (Alternative representations of the MDCI as well as properties of the MDCI and NMDCI)**.** *Let $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ be arbitrary but fixed with $\mathfrak{U}_0 < \mathfrak{U}_1$ and suppose that the Assumptions 2.2 $[\mathbf{StAp}]$ as well as 3.1 $[\mathbf{WEI.1}]$ are fulfilled. Then, the following statements are valid.*

(i) It holds:

$$\mathbb{D} = \mathbb{D}_1 - \mathbb{D}_2,$$

and:

$$f^{\mathrm{opt}}_{\mathfrak{U}_{0,1}} : \mathbb{R}^d \to \mathbb{C}, \; s \mapsto \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \varphi(w,s) \, dw \tag{3.6}$$

is the unique minimizer of $\mathbb{D}$.

(ii) One obtains:

$$\mathbb{D} = \frac{1}{2\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)} \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} |\varphi(u,s) - \varphi(w,s)|^2 \, du \, dw \, \mathbf{w}(s) \, ds.$$

(iii) It will hold $\mathbb{D} = 0$ *if and only if* $\varphi(u,s) = \varphi(\mathfrak{U}_0, s) \; \forall \, u \in \mathfrak{U}_{0,1}, \; s \in \mathbb{R}^d$ *(recall that* $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ *according to Definition 3.3 (i)).*

(iv) One will obtain $\mathbb{D}^{\text{norm}} = 0$ *if and only if* $\varphi(u,s) = \varphi(\mathfrak{U}_0, s) \; \forall \, u \in \mathfrak{U}_{0,1}, \; s \in \mathbb{R}^d$.

Remark 3.7. *(i) Proposition 3.6 (i) and Bochner's theorem imply that* $f^{\text{opt}}_{\mathfrak{U}_{0,1}}$ *is a characteristic function, such that the value of the MDCI will not change if one defines this measure based on the infimum with respect to all characteristic functions* $f_{\mathbf{char}} \colon \mathbb{R}^d \to \mathbb{C}$ *(as proposed by (3.2) for the case* $\mathfrak{U}_{0,1} = [0,1]$*) instead of all continuous functions (as in Definition 3.3 (i)).*

(ii) Proposition 3.6 (iii) implies that if $\mathbb{D} = 0$ *(which is equivalent to* $\mathbb{D}^{\text{norm}} = 0$ *due to Proposition 3.6 (iv)), it will hold for sufficiently large* T *(i. e.,* $T \geqslant 1/\mathfrak{U}_1$ *and* $\lceil \mathfrak{U}_0 T \rceil \leqslant \lfloor \mathfrak{U}_1 T \rfloor$*) as well as for all* $s \in \mathbb{R}^d$ *(see Definition 2.6):*

$$\begin{aligned} &\sup_{t_1, t_2 \in \{\max\{1, \lceil \mathfrak{U}_0 T \rceil\}, \ldots, \lfloor \mathfrak{U}_1 T \rfloor\}} |\varphi_{t_1,T}(s) - \varphi_{t_2,T}(s)| \\ &= \sup_{t_1, t_2 \in \{\max\{1, \lceil \mathfrak{U}_0 T \rceil\}, \ldots, \lfloor \mathfrak{U}_1 T \rfloor\}} \left| \varphi_{t_1,T}(s) - \varphi\left(\frac{t_1}{T}, s\right) + \varphi\left(\frac{t_2}{T}, s\right) - \varphi_{t_2,T}(s) \right|, \end{aligned} \tag{3.7}$$

whereby the right side of (3.7) converges to zero for $T \to \infty$ *due to Remark 2.7. Hence, the fact that the characteristic function of a random variable uniquely identifies its distribution motivates to regard the locally stationary random variables* $X_{\max\{1, \lceil \mathfrak{U}_0 T \rceil\},T}, \ldots, X_{\lfloor \mathfrak{U}_1 T \rfloor, T}$ *as approximately equally distributed for large* T *in the case* $\mathbb{D} = 0$ *(and* $\mathbb{D}^{\text{norm}} = 0$*, respectively).*
Otherwise, if $\mathbb{D} > 0$ *(which is equivalent to* $\mathbb{D}^{\text{norm}} > 0$*), it will follow from similar arguments, Remark 2.3 and the reverse triangular inequality:*

$$\begin{aligned} &\exists \, u_0, u_1 \in [\mathfrak{U}_0, \mathfrak{U}_1] \text{ with } u_0 < u_1, \; s \in \mathbb{R}^d, \; \epsilon > 0 \text{ and } T_1 \in \left\{ N \in \mathbb{N} : N \geqslant \frac{1}{u_1}, \lceil u_0 N \rceil \leqslant \lfloor u_1 N \rfloor \right\} : \\ &|\varphi(u_0,s) - \varphi(u_1,s)| > 2\epsilon \quad \text{as well as} \quad \left|\varphi_{\max\{1, \lceil u_0 T \rceil\},T}(s) - \varphi_{\lfloor u_1 T \rfloor, T}(s)\right| > \epsilon \quad \forall \, T \geqslant T_1. \end{aligned} \tag{3.8}$$

Hence, one can conclude in the case $\mathbb{D} > 0$ *(and* $\mathbb{D}^{\text{norm}} > 0$*, respectively) that the distribution of* $X_{\max\{1, \lceil u_0 T \rceil\},T}$ *and that of* $X_{\lfloor u_1 T \rfloor, T}$ *are not approximately the same for sufficiently large* T.

(iii) The MDCI and NMDCI are shift-invariant in the sense that they assign the same distribution change intensity to the processes $\{\widetilde{X}_t(u)\}$ *and* $\{x + \widetilde{X}_t(u)\}$ *for all deterministic* $x \in \mathbb{R}^d$*, which can be easily shown by using Proposition 3.6 (i), Definition 3.3 (ii) and* $\mathbb{E}\left[e^{i\langle s, x + \tilde{X}_0(u)\rangle}\right] = e^{i\langle s,x\rangle}\mathbb{E}\left[e^{i\langle s, \tilde{X}_0(u)\rangle}\right] \; \forall \, u \in [0,1], \; s \in \mathbb{R}^d$.
Moreover, both measures for the distribution change intensity are sign-invariant for weight functions $\mathbf{w}$ *that fulfil* $\mathbf{w}(s) = \mathbf{w}(-s) \; \forall \, s \in \mathbb{R}^d$ *in the sense that they do not change their values if* $\{-\widetilde{X}_t(u)\}$ *instead of* $\{\widetilde{X}_t(u)\}$ *is considered, which follows from Proposition 3.6 (i), Definition 3.3 (ii),* $\mathbb{E}\left[e^{i\langle s, -\tilde{X}_0(u)\rangle}\right] = \mathbb{E}\left[e^{i\langle -s, \tilde{X}_0(u)\rangle}\right] \; \forall \, u \in [0,1], \; s \in \mathbb{R}^d$ *and integration by substitution.*
The MDCI and NMDCI are not scale-invariant in the sense that they do not assign the same distribution change intensity to the processes $\{\widetilde{X}_t(u)\}$ *and* $\{y\widetilde{X}_t(u)\}$ *for all deterministic* $y \in \mathbb{R}\backslash\{0\}$*. The latter claim holds because* $\mathbb{D}_1$ *and* $\mathbb{D}_2$ *assigned to* $\{y\widetilde{X}_t(u)\}$ *converge to* $(\mathfrak{U}_1 - \mathfrak{U}_0) \cdot \int_{\mathbb{R}^d} \mathbf{w}(s) \, ds$ *for* $y \to 0$ *due to Lebesgue's dominated convergence theorem in combination with Assumption 3.1* $[\mathbf{WEI.1}]$*, such that* $\mathbb{D}$ *and* $\mathbb{D}^{\text{norm}}$ *converge to zero for* $y \to 0$ *according to Proposition 3.6 (i) as well as Definition 3.3 (ii).*

The fact that the present measures for the distribution change intensity are not scale-invariant is not a disadvantage per se but even a useful property in many practical applications, whereby two of them are described in the following.

Combining financial products with fixed time-independent payments and others with uncertain payments that vary over time is a suitable strategy to control the liquidity risk of an investment portfolio. Concretely, suppose that I_1 is a financial product which generates at each point in time $t \in \{1, \ldots, T\}$ fixed time-independent payments amounting to P_f per invested U.S. dollar. Moreover, another financial product I_2 should exist that leads to time-dependent payments $p_{t,T}$ per invested U.S. dollar at each point in time $t \in \{1, \ldots, T\}$, whereby the last-mentioned payments are modelled as realizations of random variables $P_{t,T}$ contained in a locally stationary process $\{P_{t,T}\}$. Further, consider a person who invests $p \cdot W$ U.S. dollar (with $p \in [0, 1]$) of his monetary wealth W in the product I_1 and $(1 - p) \cdot W$ U.S. dollar in the product I_2. The total payment obtained by this person at each point in time $t \in \{1, \ldots, T\}$ is a realization of the random variable $P_{t,T}^{\text{total}} := p\, W\, P_f + (1 - p)\, W\, P_{t,T}$, whereby $\left\{P_{t,T}^{\text{total}}\right\}$ is a locally stationary process because $\left\{P_{t,T}\right\}$ is assumed to be one. A shift- and scale-invariant measure assigns for all arbitrary but fixed $p \in [0, 1)$ the same distribution change intensity to $\left\{P_{t,T}^{\text{total}}\right\}$ (and the associated stationary approximations, respectively), although increasing p leads to a considerable reduction of the liquidity uncertainty and should reduce the distribution change intensity because $p \cdot W \cdot P_f$ is deterministic and time-independent.

Another application in which a scale-invariant measure is inappropriate arises in seismology. Many cities (like Kagoshima or Naples) are built in areas with seismic activity, such that it is of high relevance to explore these activities, whereby the theory of locally stationary processes may contribute to these investigations. (E. g., in [1, Adak (1998)], the outcome of a seismic data set is modelled as a locally stationary process). Concretely, the measures for the degree of distribution changes introduced in the present work can be used to compare differences in seismic activities which belong to various time periods. Thereby, it should be taken into account that, according to [25, ESSA (1966), p. 16], in a very active seismic area, several hundred or even a thousand microearthquakes may be recorded a day, whereas a destructive earthquake occurs maybe every few tens of years. Thus, for a fixed seismic area, a scale-invariant measure for the degree of distribution changes does not allow to compare changes in the seismic activity assigned to different time intervals $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ appropriately because one considered time period may contain just some (almost) not recognizable microearthquakes, whereas much more powerful and destructive earthquakes can belong to another investigated time period.

3.1.2. Estimation of the measures for the distribution change intensity

In practical applications, the distribution of $\widetilde{X}_0(u)$ is commonly unknown for $u \in [0, 1]$, such that the LCF φ and, therefore, the MDCI and NMDCI cannot be calculated. Thus, in the present section, the LECF $\widehat{\varphi}$, which is stated in Definition 2.11, is used to construct estimators for $\mathbb{D}_1$ as well as $\mathbb{D}_2$ (which originate from Definition 3.3 (ii)) and, subsequently, these estimators are used to obtain empirical versions of the MDCI as well as the NMDCI.

At first glance, motivated by the Propositions 2.12, 2.13 and 2.14, it seems to be appropriate to estimate $\mathbb{D}_1$ by:

$$\widehat{\mathbb{D}}_{1,T}^{[1.\text{Idea}]} := \int\limits_{\mathbb{R}^d} \int\limits_{\max\{(\mathfrak{U}_1 - \mathfrak{U}_0)b, \mathfrak{U}_0\}}^{\min\{1-(\mathfrak{U}_1 - \mathfrak{U}_0)b, \mathfrak{U}_1\}} |\widehat{\varphi}(u, s)|^2 \, du \, \mathbf{w}(s) \, ds.$$

Concretely, the Propositions 2.12 as well as 2.14 indicate that $\widehat{\mathbb{D}}_{1,T}^{[1.\text{Idea}]}$ is an useful estimator for the expression:

$$\mathbb{D}_{1,T}^{[1.\text{Idea}]} := \int\limits_{\mathbb{R}^d} \int\limits_{\max\{(\mathfrak{U}_1 - \mathfrak{U}_0)b, \mathfrak{U}_0\}}^{\min\{1-(\mathfrak{U}_1 - \mathfrak{U}_0)b, \mathfrak{U}_1\}} |\varphi(u, s)|^2 \, du \, \mathbf{w}(s) \, ds \tag{3.9}$$

and $\mathbb{D}_{1,T}^{[1.\text{Idea}]}$ approximates $\mathbb{D}_1$ with an error of order $\mathcal{O}(b)$, which decays to zero for $T \to \infty$ according to Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii). More precisely, in the case that $\mathfrak{U}_0 > 0$ and $\mathfrak{U}_1 < 1$ are arbitrary but fixed, $\mathbb{D}_{1,T}^{[1.\text{Idea}]}$ equals $\mathbb{D}_1$ for sufficiently large T (i. e., sufficiently small b). However, if $\mathfrak{U}_0 = 0$ or

$\mathfrak{U}_1 = 1$, the error bound $\mathcal{O}(b)$ which results from approximating $\mathbb{D}_1$ by $\mathbb{D}_{1,T}^{[1.\text{Idea}]}$ cannot be improved to $o(b)$ in general. In contrast, it is shown below that using a certain Riemann sum (which is inspired by [19, Dette et al. (2011)]) avoids the issue with respect to the bias of $\widehat{\varphi}$ that originates from Proposition 2.13 and approximates $\mathbb{D}_1$ with an error of order $o(b)$ for all $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ (in particular, also for $\mathfrak{U}_0 = 0$ or $\mathfrak{U}_1 = 1$). Hence, the estimator $\widehat{\mathbb{D}}_{1,T}^{[1.\text{Idea}]}$ is not used in the present work. Instead, to construct an approach to estimate $\mathbb{D}_1$, note at first that a Riemann approximation based on the midpoint rule with $\lfloor 1/(2b) \rfloor$ addends to discretize the integral with respect to $u \in [\mathfrak{U}_0, \mathfrak{U}_1]$ contained in $\mathbb{D}_1$ leads to the following expression (whereby $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ according to Definition 3.3 (i) and $\lfloor 1/(2b) \rfloor \geqslant 1$ due to Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii)):

$$\mathbb{D}_{1,T}^{\text{apprx}} := \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} |\varphi(u_k, s)|^2 \, \mathbf{w}(s) \, ds \quad \text{with} \quad u_k := u_{T,\mathfrak{U}_{0,1},k} := \mathfrak{U}_0 + \left(k - \frac{1}{2}\right) \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor}. \tag{3.10}$$

Thereby, $1/\big(2\lfloor 1/(2b) \rfloor\big) \geqslant 1/\big(2/(2b)\big) = b$ implies for all $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ with $\mathfrak{U}_0 < \mathfrak{U}_1$ (recall (2.9)):

$$u_k := u_{T,\mathfrak{U}_{0,1},k} \in \mathfrak{U}_{0,1,b} \ \ \forall\, k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}, \tag{3.11}$$

such that the Propositions 2.12 and 2.14 provide for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$ that $\widehat{\varphi}(u_k, s)$ is an appropriate estimator for $\varphi(u_k, s)$ (in particular, also in the cases $\mathfrak{U}_0 = 0$ or $\mathfrak{U}_1 = 1$). Thus, the issue resulting from Proposition 2.13 that the LECF will be asymptotically biased if u is too close to 0 or 1 (in dependence of b), is avoided by this Riemann approximation.
Moreover, it should be noted that using the Riemann sum contained in $\mathbb{D}_{1,T}^{\text{apprx}}$ to approximate the integral with respect to $u \in [\mathfrak{U}_0, \mathfrak{U}_1]$ in $\mathbb{D}_1$ can be justified by the following considerations:
Assumption 2.2 $[\mathbf{StAp}]$ (ii) and arguments which are similar to ones stated in [41, Jentsch et al. (2020b), p. 3 et seq.] yield for all $u \in (0,1)$, $s \in \mathbb{R}^d$:

$$\partial_u \mathbb{E}\left[\cos\left(\left\langle s, \widetilde{X}_0(u)\right\rangle\right)\right] = \mathbb{E}\left[-\sin\left(\left\langle s, \widetilde{X}_0(u)\right\rangle\right)\left\langle s, \partial_u \widetilde{X}_0(u)\right\rangle\right], \tag{3.12}$$

such that (recall Definition 2.6):

$$\partial_u \left(\Re\{\varphi(u,s)\}^2\right) = \mathbb{E}\left[2\cos\left(\left\langle s, \widetilde{X}_0(u)\right\rangle\right)\right] \mathbb{E}\left[-\sin\left(\left\langle s, \widetilde{X}_0(u)\right\rangle\right)\left\langle s, \partial_u \widetilde{X}_0(u)\right\rangle\right]. \tag{3.13}$$

In addition, it holds for all $x, y \in \mathbb{R}, q \geqslant 1 + \delta$ (whereby $\delta \in (0,1]$ originates from Assumption 2.2 $[\mathbf{StAp}]$) and for $g \colon \mathbb{R} \to \mathbb{R}$ with either $g(z) = \sin(z)\ \forall\, z \in \mathbb{R}$ or $g(z) = \cos(z)\ \forall\, z \in \mathbb{R}$:

$$|g(x) - g(y)|^q \leqslant \min\{2, |x-y|\}^{q-(1+\delta)} \cdot \min\{2, |x-y|\}^{1+\delta} \leqslant 2^{q-(1+\delta)} \cdot |x-y|^{1+\delta}. \tag{3.14}$$

Overall, one obtains for all $v, w \in (0,1)$, $s \in \mathbb{R}^d$ by using (3.13), Remark 2.3, Assumption 2.2 $[\mathbf{StAp}]$ (ii), (3.14) with $q = (1+\delta)/\delta$ and due to $\delta \in (0,1]$, whereby the latter yields $|v-w| \leqslant |v-w|^\delta$ as well as $|s|_1^{1+\delta} \leqslant |s|_1^2 + 1$ (recall that $C \in (0,\infty)$ denotes an absolute constant that may have different values at different places):

$$\begin{aligned}
&\left|\partial_v \left(\Re\{\varphi(v,s)\}^2\right) - \partial_w \left(\Re\{\varphi(w,s)\}^2\right)\right| \\
&= \Big|\mathbb{E}\left[2\cos\left(\left\langle s, \widetilde{X}_0(v)\right\rangle\right) - 2\cos\left(\left\langle s, \widetilde{X}_0(w)\right\rangle\right)\right] \mathbb{E}\left[-\sin\left(\left\langle s, \widetilde{X}_0(v)\right\rangle\right)\left\langle s, \partial_v \widetilde{X}_0(v)\right\rangle\right] \\
&+ \mathbb{E}\left[2\cos\left(\left\langle s, \widetilde{X}_0(w)\right\rangle\right)\right] \mathbb{E}\Big[\left(-\sin\left(\left\langle s, \widetilde{X}_0(v)\right\rangle\right) + \sin\left(\left\langle s, \widetilde{X}_0(w)\right\rangle\right)\right)\left\langle s, \partial_v \widetilde{X}_0(v)\right\rangle \\
&+ \left(-\left\langle s, \partial_v \widetilde{X}_0(v)\right\rangle + \left\langle s, \partial_w \widetilde{X}_0(w)\right\rangle\right)\sin\left(\left\langle s, \widetilde{X}_0(w)\right\rangle\right)\Big]\Big| \\
&\leqslant C|s|_1|v-w||s|_1 \left\|\partial_v \widetilde{X}_0(v)\right\|_1 + C\left\|-\sin\left(\left\langle s, \widetilde{X}_0(v)\right\rangle\right) + \sin\left(\left\langle s, \widetilde{X}_0(w)\right\rangle\right)\right\|_{\frac{1+\delta}{\delta}} \\
&\cdot \left\|\left\langle s, \partial_v \widetilde{X}_0(v)\right\rangle\right\|_{1+\delta} + C|s|_1 \left\|-\partial_v \widetilde{X}_0(v) + \partial_w \widetilde{X}_0(w)\right\|_1 \\
&\leqslant C\left(|s|_1^2 + 1\right)|v-w|^\delta.
\end{aligned} \tag{3.15}$$

Lemma B.2 (iii) in the appendix in combination with (3.15), a similar construction of an upper bound for the distance $\left|\partial_v\left(\Im\left\{\varphi\left(v,s\right)\right\}^2\right)-\partial_w\left(\Im\left\{\varphi\left(w,s\right)\right\}^2\right)\right|$ (with $v,w\in(0,1)$) and Assumption 3.1 $[\mathbf{WEI.1}]$ show that $\mathbb{D}_{1,T}^{\mathrm{apprx}}$ approximates $\mathbb{D}_1$ with an error of order $\mathcal{O}\left(b^{1+\delta}\right)=o(b)$. This indicates that $\mathbb{D}_{1,T}^{\mathrm{apprx}}$ is a tendentiously better approximation of $\mathbb{D}_1$ than $\mathbb{D}_{1,T}^{[1.\mathrm{Idea}]}$ in the cases $\mathfrak{U}_0=0$ or $\mathfrak{U}_1=1$.
Since the Propositions 2.12 and 2.14 provide together with (3.11) that $\widehat{\varphi}(u_k,\cdot)$ is an adequate estimator for $\varphi(u_k,\cdot)$ for all $k\in\{1,\ldots,\lfloor 1/(2b)\rfloor\}$, an appropriate estimator for $\mathbb{D}_1$ results by replacing $\varphi(u_k,s)$ in $\mathbb{D}_{1,T}^{\mathrm{apprx}}$ by $\widehat{\varphi}(u_k,s)$ for all $k\in\{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $s\in\mathbb{R}^d$. The expression $\mathbb{D}_2$ (see Definition 3.3 (ii)) can be estimated in a similar way. These considerations and Proposition 3.6 (i) as well as Definition 3.3 (ii) motivate the estimators for the MDCI and NMDCI which are introduced in the following definition.

Definition 3.8 ((Normalized) empirical measure for the distribution change intensity)**.**
Suppose that the Assumptions 2.2 $[\mathbf{StAp}]$*, 3.1* $[\mathbf{WEI.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *hold.*

(i) Define (recall that $\mathfrak{U}_{0,1}:=[\mathfrak{U}_0,\mathfrak{U}_1]$ *according to Definition 3.3 (i) and see also Definition 2.11):*

$$u_k:=u_{T,\mathfrak{U}_{0,1},k}:=\mathfrak{U}_0+\left(k-\frac{1}{2}\right)\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\qquad\forall\,k\in\{1,\ldots,\lfloor 1/(2b)\rfloor\},$$

$$\widehat{\mathbb{D}}_{T,1}:=\widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},1}:=\int\limits_{\mathbb{R}^d}\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left|\widehat{\varphi}\left(u_k,s\right)\right|^2\mathbf{w}(s)\,ds,$$

$$\widehat{\mathbb{D}}_{T,2}:=\widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},2}:=\frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\int\limits_{\mathbb{R}^d}\left|\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\widehat{\varphi}\left(u_k,s\right)\right|^2\mathbf{w}(s)\,ds\quad\text{and}$$

$$\widehat{\mathbb{D}}_T:=\widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1}}:=\widehat{\mathbb{D}}_{T,1}-\widehat{\mathbb{D}}_{T,2}.$$

Then, $\widehat{\mathbb{D}}_T$ *is called empirical measure for the distribution change intensity under local stationarity (abbr. EMDCI).*

(ii) The expression:

$$\widehat{\mathbb{D}}_T^{\mathrm{norm}}:=\widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1}}^{\mathrm{norm}}:=\begin{cases}1-\widehat{\mathbb{D}}_{T,2}/\widehat{\mathbb{D}}_{T,1}, & \text{for }\widehat{\mathbb{D}}_{T,1}>0\\ 0, & \text{for }\widehat{\mathbb{D}}_{T,1}=0\end{cases}$$

is called normalized empirical measure for the distribution change intensity under local stationarity (abbr. NEMDCI).

The definition of $\widehat{\mathbb{D}}_T^{\mathrm{norm}}$ ensures that this estimator exists for all $T\in\mathbb{N}$. However, in practical applications, the belonging realization of $\widehat{\mathbb{D}}_{T,1}$ should be positive to obtain a useful estimator, whereby the following lemma shows that this will be the case if T is large enough. In particular, $\widehat{\mathbb{D}}_{T,1}$ does not depend on unobservable/unknown expressions, such that $\widehat{\mathbb{D}}_{T,1}>0$ can be easily examined for given realizations of $X_{1,T},\ldots,X_{T,T}$.

Lemma 3.9. *Let the Assumptions 2.2* $[\mathbf{StAp}]$*, 3.1* $[\mathbf{WEI.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *be fulfilled. Then, a fixed* $\breve{T}\in\mathbb{N}$ *exists with:*

$$\widehat{\mathbb{D}}_{T,1}>0\quad\forall\,T\in\mathbb{N}:T\geqslant\breve{T}.$$

Remark 3.10. *(i) According to Assumption 2.8* $[\mathbf{K\&b.1}]$ *(i) and Definition 2.11, the expressions* $\widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},1}$ *and* $\widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},2}$ *take no* $X_{t,T}$ *into account for which* $|t/T-u_{T,\mathfrak{U}_{0,1},k}|\geqslant(\mathfrak{U}_1-\mathfrak{U}_0)\,b$ *holds for all* $k\in\{1,\ldots,\lfloor 1/(2b)\rfloor\}$*. The opposite condition (i. e.,* $|t/T-u_{T,\mathfrak{U}_{0,1},k}|<(\mathfrak{U}_1-\mathfrak{U}_0)\,b$ *for a* $k\in\{1,\ldots,\lfloor 1/(2b)\rfloor\}$*) implies due to* $1/\big(2\lfloor 1/(2b)\rfloor\big)\geqslant 1/\big(2/(2b)\big)=b$ *(recall Definition 3.8 (i)):*

$$\mathfrak{U}_0T\leqslant u_{T,\mathfrak{U}_{0,1},1}T-(\mathfrak{U}_1-\mathfrak{U}_0)\,Tb<t<u_{T,\mathfrak{U}_{0,1},\lfloor 1/(2b)\rfloor}T+(\mathfrak{U}_1-\mathfrak{U}_0)\,Tb\leqslant\mathfrak{U}_1T.$$

Hence, it suffices to observe $X_{\max\{1,\lceil\mathfrak{U}_0T\rceil\},T},\ldots,X_{\lfloor\mathfrak{U}_1T\rfloor,T}$ *in order to calculate realizations of* $\widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},1}$ *and* $\widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},2}$ *(whereby* $X_{\lfloor\mathfrak{U}_1T\rfloor,T}$ *is well-defined for* $T\geqslant 1/\mathfrak{U}_1$*).*

(ii) Assumption 3.1 $[\mathbf{WEI.1}]$ *and the Cauchy-Schwarz inequality imply* $0 \leqslant \widehat{\mathbb{D}}_{T,2} \leqslant \widehat{\mathbb{D}}_{T,1}$*, such that* $\widehat{\mathbb{D}}_T^{\mathrm{norm}} \in [0,1]$.

(iii) The following claims can be verified by arguments which are similar to those stated in Remark 3.7 (iii):
The realizations of the EMDCI and NEMDCI which belong to a sample path $(X_{t,T}(\omega))_{t=1}^{T}$ *(with* $\omega \in \Omega$*) will not change if this sample path is replaced by* $(x + X_{t,T}(\omega))_{t=1}^{T}$ *with arbitrary* $x \in \mathbb{R}^d$*. Moreover, if the weight function* $\mathbf{w}$ *fulfils* $\mathbf{w}(s) = \mathbf{w}(-s)\ \forall\, s \in \mathbb{R}^d$*, the EMDCI as well as NEMDCI will assign the same estimated distribution change intensity to the sample paths* $(X_{t,T}(\omega))_{t=1}^{T}$ *and* $(-X_{t,T}(\omega))_{t=1}^{T}$*. Further, the realizations of the EMDCI and NEMDCI that are associated with a sample path* $(y\, X_{t,T}(\omega))_{t=1}^{T}$*, whereby* $y \in \mathbb{R}$*, may change in dependence of* y*.*

The following proposition provides alternative ways to express the EMDCI.

Proposition 3.11 (Alternative representations of the EMDCI)**.**
Suppose that the Assumptions 2.2 $[\mathbf{StAp}]$*, 3.1* $[\mathbf{WEI.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *are fulfilled. Then, one obtains:*

(i)

$$\widehat{\mathbb{D}}_T = \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k_1=1}^{\lfloor 1/(2b) \rfloor} \left| \widehat{\varphi}\left(u_{k_1}, s\right) - \frac{1}{\lfloor 1/(2b) \rfloor} \sum_{k_2=1}^{\lfloor 1/(2b) \rfloor} \widehat{\varphi}\left(u_{k_2}, s\right) \right|^2 \mathbf{w}(s)\, ds$$

(ii)

$$\widehat{\mathbb{D}}_T = \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{2\lfloor 1/(2b) \rfloor^2} \sum_{k_1,k_2=1}^{\lfloor 1/(2b) \rfloor} \left| \widehat{\varphi}\left(u_{k_1}, s\right) - \widehat{\varphi}\left(u_{k_2}, s\right) \right|^2 \mathbf{w}(s)\, ds$$

Before the asymptotic behaviour of the EMDCI and NEMDCI is presented in Theorem 3.13 given below, the next lemma is stated, which ensures well-definedness of some expressions that are introduced in Theorem 3.13.

Lemma 3.12. *Let Assumption 2.4* $[\mathbf{DM.1}]$ *be fulfilled. Then, it holds for all* $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$*,* $s_1, s_2 \in \mathbb{R}^d$*:*

$$\sum_{t=-\infty}^{\infty} \sup_{u \in [0,1]} \left| \mathrm{Cov}\left(\mathrm{R}_1 \left\{ e^{i\left\langle s_1, \tilde{X}_0(u) \right\rangle} \right\}, \mathrm{R}_2 \left\{ e^{i\left\langle s_2, \tilde{X}_t(u) \right\rangle} \right\} \right) \right| \leqslant C \left(1 + |s_1|_1 + |s_2|_1\right).$$

The following theorem is the fundamental result of this section. It provides the asymptotic distribution of the (with $\sqrt{T}$ normalized) differences between the EMDCI and the MDCI as well as between the NEMDCI and the NMDCI. In particular, it shows that the EMDCI (NEMDCI, respectively) is an appropriate estimator for the MDCI (NMDCI, respectively).

Theorem 3.13 (Asymptotic behaviour of the EMDCI and NEMDCI)**.**
Suppose that the Assumptions 2.4 $[\mathbf{DM.1}]$*, 3.1* $[\mathbf{WEI.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *are fulfilled. Moreover, define for all* $\mathrm{R} \in \{\Re, \Im\}$*,* $\gamma^{[1]}$*,* $\gamma^{[2]} \in \mathbb{R}$*,* $u \in [0,1]$*,* $s \in \mathbb{R}^d$ *(recall that* $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ *according to Definition 3.3 (i) and see also Definition 2.6):*

$$\tau_{\mathfrak{U}_{0,1},\mathrm{R}}\left(\gamma, u, s\right) := \gamma^{[1]} \mathrm{R}\left\{\varphi\left(u, s\right)\right\} + \frac{\gamma^{[2]}}{\mathfrak{U}_1 - \mathfrak{U}_0} \mathrm{R}\left\{ \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \varphi(w, s)\, dw \right\} \quad \text{with} \quad \gamma := \left(\gamma^{[1]}, \gamma^{[2]}\right), \tag{3.16}$$

for all $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$*,* $u \in [0,1]$*,* $s_1, s_2 \in \mathbb{R}^d$*:*

$$\sigma_{\infty,\mathrm{R}_1,\mathrm{R}_2}\left(u, s_1, s_2\right) := \sum_{t=-\infty}^{\infty} \mathrm{Cov}\left(\mathrm{R}_1 \left\{ e^{i\left\langle s_1, \tilde{X}_0(u) \right\rangle} \right\}, \mathrm{R}_2 \left\{ e^{i\left\langle s_2, \tilde{X}_t(u) \right\rangle} \right\} \right), \tag{3.17}$$

for all $\gamma, \widetilde{\gamma} \in \mathbb{R}^{1\times 2}$*:*

$$\sigma_{\mathfrak{U}_{0,1}}(\gamma,\widetilde{\gamma}) := 8\,(\mathfrak{U}_1-\mathfrak{U}_0)\int\limits_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \tau_{\mathfrak{U}_{0,1},\Re}(\gamma,u,s_1)\,\tau_{\mathfrak{U}_{0,1},\Re}(\widetilde{\gamma},u,s_2)$$
$$\cdot\,\sigma_{\infty,\Re,\Re}(u,s_1,s_2) + \tau_{\mathfrak{U}_{0,1},\Im}(\gamma,u,s_1)\,\tau_{\mathfrak{U}_{0,1},\Im}(\widetilde{\gamma},u,s_2)\,\sigma_{\infty,\Im,\Im}(u,s_1,s_2)$$
$$+\,\tau_{\mathfrak{U}_{0,1},\Re}(\gamma,u,s_1)\,\tau_{\mathfrak{U}_{0,1},\Im}(\widetilde{\gamma},u,s_2)\,\sigma_{\infty,\Re,\Im}(u,s_1,s_2)$$
$$+\,\tau_{\mathfrak{U}_{0,1},\Im}(\gamma,u,s_1)\,\tau_{\mathfrak{U}_{0,1},\Re}(\widetilde{\gamma},u,s_2)\,\sigma_{\infty,\Im,\Re}(u,s_1,s_2)\,du\,\mathbf{w}(s_1)\,ds_1\,\mathbf{w}(s_2)\,ds_2 \tag{3.18}$$

and:

$$\Sigma_{\mathfrak{U}_{0,1}} := \begin{pmatrix} \sigma_{\mathfrak{U}_{0,1}}((1,0),(1,0)) & \sigma_{\mathfrak{U}_{0,1}}((1,0),(0,1)) \\ \sigma_{\mathfrak{U}_{0,1}}((0,1),(1,0)) & \sigma_{\mathfrak{U}_{0,1}}((0,1),(0,1)) \end{pmatrix}. \tag{3.19}$$

Then, the following statements hold:

(i)

$$\sqrt{T}\left(\begin{pmatrix} \widehat{\mathbb{D}}_{T,1} \\ \widehat{\mathbb{D}}_{T,2} \end{pmatrix} - \begin{pmatrix} \mathbb{D}_1 \\ \mathbb{D}_2 \end{pmatrix}\right) \xrightarrow{\mathrm{d}} Z^{\mathrm{joint}}_{\mathfrak{U}_{0,1}} \quad \textit{with} \quad Z^{\mathrm{joint}}_{\mathfrak{U}_{0,1}} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_{\mathfrak{U}_{0,1}}\right). \tag{3.20}$$

(ii)

$$\sqrt{T}\left(\widehat{\mathbb{D}}_T - \mathbb{D}\right) \xrightarrow{\mathrm{d}} Z_{\mathfrak{U}_{0,1}} \quad \textit{with} \quad Z_{\mathfrak{U}_{0,1}} \sim \mathcal{N}\Big(0, \sigma_{\mathfrak{U}_{0,1}}((1,-1),(1,-1))\Big). \tag{3.21}$$

(iii) One obtains for:

$$\gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},1} := \frac{\mathbb{D}_2}{\mathbb{D}_1^2} \quad \textit{and} \quad \gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},2} := -\frac{1}{\mathbb{D}_1} \tag{3.22}$$

that:

$$\sqrt{T}\left(\widehat{\mathbb{D}}^{\mathrm{norm}}_T - \mathbb{D}^{\mathrm{norm}}\right) \xrightarrow{\mathrm{d}} Z^{\mathrm{norm}}_{\mathfrak{U}_{0,1}} \quad \textit{with}$$
$$Z^{\mathrm{norm}}_{\mathfrak{U}_{0,1}} \sim \mathcal{N}\Big(0, \sigma_{\mathfrak{U}_{0,1}}\Big(\big(\gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},1}, \gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},2}\big), \big(\gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},1}, \gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},2}\big)\Big)\Big). \tag{3.23}$$

Remark 3.14. *Lemma 3.12 and Assumption 3.1* [**WEI.1**] *provide that* $\sigma_{\infty,\mathrm{R}_1,\mathrm{R}_2}(u,s_1,s_2)$ *as well as* $\sigma_{\mathfrak{U}_{0,1}}(\gamma,\widetilde{\gamma})$ *are well-defined. Moreover, Lemma 3.4 implies that* $\gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},1}$ *and* $\gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},2}$ *are well-defined.*

Theorem 3.13 (ii) allows to construct asymptotic confidence intervals for the MDCI. Therefor, suppose that $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$ and denote for all $\alpha \in (0,1)$ the quantile of the distribution of $Z_{\mathfrak{U}_{0,1}}$ which belongs to the level α as $q_{\alpha,\mathfrak{U}_{0,1}}$. Then, it holds under the assumption $\sigma_{\mathfrak{U}_{0,1}}((1,-1),(1,-1)) > 0$:

$$\lim_{T\to\infty} \mathbb{P}\left(\widehat{\mathbb{D}}_T - \frac{q_{\alpha_2,\mathfrak{U}_{0,1}}}{\sqrt{T}} \leqslant \mathbb{D} \leqslant \widehat{\mathbb{D}}_T - \frac{q_{\alpha_1,\mathfrak{U}_{0,1}}}{\sqrt{T}}\right) = \alpha_2 - \alpha_1. \tag{3.24}$$

Further, asymptotic confidence intervals for the NMDCI can be derived from Theorem 3.13 (iii). Concretely, denote for all $\alpha \in (0,1)$ the quantile of the distribution of $Z^{\mathrm{norm}}_{\mathfrak{U}_{0,1}}$ which belongs to the level α as $q^{\mathrm{norm}}_{\alpha,\mathfrak{U}_{0,1}}$. Then, one obtains under the assumption $\sigma_{\mathfrak{U}_{0,1}}\big((\gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},1}, \gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},2}), (\gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},1}, \gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},2})\big) > 0$ and for $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$:

$$\lim_{T\to\infty} \mathbb{P}\left(\widehat{\mathbb{D}}^{\mathrm{norm}}_T - \frac{q^{\mathrm{norm}}_{\alpha_2,\mathfrak{U}_{0,1}}}{\sqrt{T}} \leqslant \mathbb{D}^{\mathrm{norm}} \leqslant \widehat{\mathbb{D}}^{\mathrm{norm}}_T - \frac{q^{\mathrm{norm}}_{\alpha_1,\mathfrak{U}_{0,1}}}{\sqrt{T}}\right) = \alpha_2 - \alpha_1. \tag{3.25}$$

However, the quantiles $q_{\alpha,\mathfrak{U}_{0,1}}$ and $q^{\mathrm{norm}}_{\alpha,\mathfrak{U}_{0,1}}$ (with $\alpha \in (0,1)$) are unknown in practise. Hence, bootstrap procedures are evolved in the next section that approximate the distributions of $Z_{\mathfrak{U}_{0,1}}$ and $Z^{\mathrm{norm}}_{\mathfrak{U}_{0,1}}$ asymptotically, which allows to estimate confidence intervals for the MDCI as well as the NMDCI.

3.1.3. Estimation of confidence intervals for the measures for the distribution change intensity

The issue that the distributions of $Z_{\mathfrak{U}_{0,1}}$ and $Z_{\mathfrak{U}_{0,1}}^{\mathrm{norm}}$ are unknown is solved in the present section by using dependent wild bootstrap procedures. As mentioned in [67, Shao (2010), p. 218], the dependent wild bootstrap extends the traditional wild bootstrap proposed in [79, Wu (1986)] to the time series setting by allowing the auxiliary variables, which are involved in the wild bootstrap, to be dependent. Therefore, the dependent wild bootstrap is capable of mimicking the dependence in the original time series. Moreover, similar to the dependent wild bootstrap considered in [52, Leucht and Neumann (2013), p. 260 et seq.], the bootstrap approaches given below are based on reusing some parts of the bootstrap statistics in every bootstrap iteration, which is possible by avoiding the direct bootstrapping of the underlying random variables $X_{1,T}, \ldots, X_{T,T}$. In practise, this recycling in every bootstrap iteration provides that estimated confidence intervals for the MDCI and NMDCI can be calculated with quite low computational costs. Further, it is worth mentioning that wild bootstrap approaches have also been used in some other publications that concern locally stationary frameworks, e. g., in [75, Vogt (2015)], to obtain a bootstrap counterpart for a test statistic that investigates whether structural changes in time-varying non-parametric regression models exist and in [45, Karmakar et al. (2022)], to approximate confidence bands for time-varying coefficients of recursively defined locally stationary time series and time-varying GARCH processes.

In order to evolve bootstrap methods for approximating the distributions of $Z_{\mathfrak{U}_{0,1}}$ and $Z_{\mathfrak{U}_{0,1}}^{\mathrm{norm}}$, the following notations are considered under Assumption 2.2 $[\mathbf{StAp}]$:
The icon $\mathbb{P}^*$ denotes the conditional probability conditioned on $X_{1,T}, \ldots, X_{T,T}$, $\mathbb{E}^*$ marks the associated conditional expectation, Var^* symbolizes the matching conditional variance and Cov^* is referred to the belonging conditional covariance. Moreover, if $(Y_T)_{T\in\mathbb{N}}$ is a sequence of complex-valued random variables that live on the probability space $(\Omega, \Sigma, \mathbb{P})$ which originates from Definition 2.1, define:

$$Y_T = o_{\mathbb{P}}^*(1) \quad :\Longleftrightarrow \lim_{T\to\infty} \mathbb{P}\left(\mathbb{P}^*\left(|Y_T| > \epsilon_1\right) > \epsilon_2\right) = 0 \ \forall\, \epsilon_1, \epsilon_2 > 0. \tag{3.26}$$

In the present work, the property $Y_T = o_{\mathbb{P}}^*(1)$ is often verified by using that if $\mathbb{E}\left[|Y_T|\right] < \infty$, Markov's inequality and its conditional version will imply:

$$\mathbb{P}\left(\mathbb{P}^*\left(|Y_T| > \epsilon_1\right) > \epsilon_2\right) \leqslant \frac{1}{\epsilon_1\epsilon_2}\mathbb{E}\left[\mathbb{E}^*\left[|Y_T|\right]\right] = \frac{1}{\epsilon_1\epsilon_2}\mathbb{E}\left[|Y_T|\right] \ \forall\, \epsilon_1, \epsilon_2 > 0. \tag{3.27}$$

Further, straightforward calculations show for all sequences $(Y_{1,T})_{T\in\mathbb{N}}$ and $(Y_{2,T})_{T\in\mathbb{N}}$ of complex-valued random variables that live on the probability space $(\Omega, \Sigma, \mathbb{P})$:

$$Y_{1,T} + Y_{2,T} = o_{\mathbb{P}}^*(1) \ \text{ if } \ Y_{1,T} = o_{\mathbb{P}}^*(1) \ \text{ and } \ Y_{2,T} = o_{\mathbb{P}}^*(1). \tag{3.28}$$

The bootstrap procedures proposed below are based on an arbitrary process $(W_t^*)_{t\in\mathbb{Z}}$ of bootstrap random variables, which fulfils the conditions contained in the next assumption.

Assumption 3.15 $[\mathbf{W}^*]$ (Process of bootstrap random variables)**.**
Let the suppositions of Definition 2.1 and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ *(ii) be fulfilled.*

(i) Define for all $T \in \mathbb{N}$ *a parameter* $\beta := \beta_T > 0$. *Moreover,* $\beta \to \infty$, $\beta b \to 0$ *and* $\beta/\left(Tb^2\right) \to 0$ *should hold for* $T \to \infty$, *whereby* b *is defined in Assumption 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *(ii).*

(ii) Suppose for an $\mathfrak{c}_* \in \mathbb{N}$ *that* $(\varepsilon_t^*)_{t\in\mathbb{Z}}$ *is a sequence of* $\mathbb{R}^{\mathfrak{c}_*}$*-valued i. i. d. random variables which lives on* $(\Omega, \Sigma, \mathbb{P})$ *and is independent of* $(\varepsilon_t)_{t\in\mathbb{Z}}$, *whereby* $(\Omega, \Sigma, \mathbb{P})$ *as well as* $(\varepsilon_t)_{t\in\mathbb{Z}}$ *originate from Definition 2.1. Moreover, set* $\mathcal{F}_t^* := \left(\varepsilon_t^*, \varepsilon_{t-1}^*, \ldots\right) \forall\, t \in \mathbb{Z}$, *assume for all* $T \in \mathbb{N}$ *that* $\mathbf{H}_T^*\colon \mathbb{R}^{\mathfrak{c}_*\times\mathbb{N}} \to \mathbb{R}$ *is a measurable function with* $\mathbb{R}^{\mathfrak{c}_*\times\mathbb{N}} := \mathbb{R}^{\mathfrak{c}_*} \times \mathbb{R}^{\mathfrak{c}_*} \times \ldots$ *and suppose for all* $T \in \mathbb{N}$ *that* $(W_t^*)_{t\in\mathbb{Z}} := \left(W_{t,T}^*\right)_{t\in\mathbb{Z}}$ *is a stationary process for which:*

$$W_{t,T}^* = \mathbf{H}_T^*\left(\mathcal{F}_t^*\right) \quad \forall\, t \in \mathbb{Z},\, T \in \mathbb{N}.$$

(iii) Let $\mathbb{E}\left[W_0^{*4}\right] \leqslant B_*$ *for an arbitrary but fixed* $B_* \in (0, \infty)$ *that does not depend on* $T \in \mathbb{N}$, $\mathbb{E}\left[W_0^*\right] = 0$ *as well as* $\mathrm{Cov}\left(W_{t_1}^*, W_{t_2}^*\right) = K^*\left((t_1 - t_2)/\beta\right) \forall\, t_1, t_2 \in \mathbb{Z}$, *whereby* $K^*\colon \mathbb{R} \to \mathbb{R}$

should be a function that is continuous at zero with $K^*(0) = 1$ and fulfils:

$$\sum_{t=0}^{2\lfloor Tb\rfloor+T} \left| K^*\left(\frac{t}{\beta}\right)\right| = \mathcal{O}(\beta) \quad \textit{for} \quad T \to \infty.$$

(iv) Suppose that $\left(\varepsilon_t^{**}\right)_{t\in\mathbb{Z}}$ is an independent copy of $(\varepsilon_t^*)_{t\in\mathbb{Z}}$ and define $\mathcal{F}_t^{**(t-0)} := \left(\varepsilon_t^{**}, \varepsilon_{t-1}^*, \varepsilon_{t-2}^*, \ldots\right)$ $\forall\, t \in \mathbb{Z}$ as well as $\mathcal{F}_t^{**(t-l)} := \left(\varepsilon_t^*, \ldots, \varepsilon_{t-l+1}^*, \varepsilon_{t-l}^{**}, \varepsilon_{t-l-1}^*, \ldots\right)$ $\forall\, t \in \mathbb{Z}, l \in \mathbb{N}$. In addition, assume for $W_t^{**(t-l)} := W_{t,T}^{**(t-l)} := \mathbf{H}_T^*\left(\mathcal{F}_t^{**(t-l)}\right)$ $\forall\, T \in \mathbb{N}, t \in \mathbb{Z}, l \in \mathbb{N}_0$ and for arbitrary but fixed $C_* \in (0,\infty)$ as well as $\rho_* \in (0,1)$ that do not depend on T and l:

$$\sup_{t\in\mathbb{Z}} \left\| W_t^* - W_t^{**(t-l)} \right\|_2 \leqslant C_* \rho_*^{l/\beta} \quad \forall\, l \in \mathbb{N}_0, T \in \mathbb{N}.$$

It is worth mentioning that Assumption 3.15 $[\mathbf{W}^*]$ can be regarded as an adaption of Assumption (B2) in [52, Leucht and Neumann (2013), p. 262], whereby, analogously to [52, Leucht and Neumann (2013), p. 258], the role of the process $(W_t^*)_{t\in\mathbb{Z}}$ of bootstrap random variables can be explained in the following way: A non-degenerate distribution of the centered random variables W_t^* introduces the necessary randomness and the property Cov $\left(W_{t_1}^*, W_{t_2}^*\right) \to 1$ for $T \to \infty$ and all arbitrary but fixed $t_1, t_2 \in \mathbb{Z}$ should take care that the dependence structure of the time series $(X_{t,T})_{t=1}^T$ is asymptotically captured. However, in contrast to the research presented in [52, Leucht and Neumann (2013)], the present framework considers locally stationary Bernoulli shift processes that obey versions of the uniform functional dependence measure (according to Assumption 2.4 $[\mathbf{DM}]$), whereas τ-dependent stationary processes are investigated in [52, Leucht and Neumann (2013)]. In particular, this difference motivates that Assumption 3.15 $[\mathbf{W}^*]$ depends on the bandwidth b which originates from Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii), whereas the publication [52, Leucht and Neumann (2013)] does not contain kernel-bandwidth-based estimators.

In the following, some processes of bootstrap random variables are proposed for which Assumption 3.15 $[\mathbf{W}^*]$ holds, whereby these processes originate from [24, Doukhan et al. (2015), p. 301].

Example 3.16. *Suppose that $(\varepsilon_t^*)_{t\in\mathbb{Z}}$ is a sequence of i. i. d. random variables with $\varepsilon_0^* \sim \mathcal{N}(0,1)$ which lives on $(\Omega, \Sigma, \mathbb{P})$ and is independent of the sequence $(\varepsilon_t)_{t\in\mathbb{Z}}$ defined in Definition 2.1. Moreover, assume that a sequence of parameters β is given which fulfils Assumption 3.15 $[\mathbf{W}^*]$ (i). Then, Assumption 3.15 $[\mathbf{W}^*]$ holds for the following processes:*

(i) Moving-average process with rectangular weight function:

$$W_t^{*[1]} := W_{t,T}^{*[1]} := \frac{1}{\sqrt{\beta}} \sum_{k=0}^{\beta-1} \varepsilon_{t-k}^* \quad \forall\, t \in \mathbb{Z}, T \in \mathbb{N}, \textit{ whereby, in addition, } \beta \in \mathbb{N} \textit{ should be fulfilled.}$$

(ii) Discretely sampled Ornstein-Uhlenbeck process (autoregressive wild bootstrap):

$$W_t^{*[2]} := W_{t,T}^{*[2]} := e^{-1/\beta}\, W_{t-1,T}^{*[2]} + \sqrt{1 - e^{-2/\beta}}\, \varepsilon_t^* \quad \forall\, t \in \mathbb{Z}, T \in \mathbb{N}.$$

Next, based on an arbitrary process $(W_t^*)_{t\in\mathbb{Z}}$ that fulfils Assumption 3.15 $[\mathbf{W}^*]$, bootstrap procedures are constructed in a heuristic manner that approximate the distributions of $Z_{\mathfrak{U}_{0,1}}$ and $Z_{\mathfrak{U}_{0,1}}^{\mathrm{norm}}$, whereas Theorem 3.19 given below justifies them rigorously.
Since $\sqrt{T}\left(\widehat{\mathbb{D}}_T - \mathbb{D}\right)$ converges in distribution to $Z_{\mathfrak{U}_{0,1}}$ and $\sqrt{T}\left(\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}}\right)$ to $Z_{\mathfrak{U}_{0,1}}^{\mathrm{norm}}$ (according to Theorem 3.13 (ii) as well as (iii)), bootstrap counterparts of $\widehat{\mathbb{D}}_T - \mathbb{D}$ and $\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}}$ are derived in the following. Therefor, observe at first (recall the Definitions 3.8 (i) as well as 3.3 (ii)):

$$\begin{aligned}
\widehat{\mathbb{D}}_{T,1} - \mathbb{D}_1 &= \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \Re\left\{\widehat{\varphi}\left(u_k, s\right)\right\}^2 - \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re\left\{\varphi\left(u, s\right)\right\}^2 \, du\, \mathbf{w}(s)\, ds \\
&\quad + \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \Im\left\{\widehat{\varphi}\left(u_k, s\right)\right\}^2 - \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Im\left\{\varphi\left(u, s\right)\right\}^2 \, du\, \mathbf{w}(s)\, ds \\
&=: \widehat{\mathbb{D}}_{T,1,\Re}^{\mathrm{gap}} + \widehat{\mathbb{D}}_{T,1,\Im}^{\mathrm{gap}} \qquad \text{and}
\end{aligned} \tag{3.29}$$

$$\widehat{\mathbb{D}}_{T,2} - \mathbb{D}_2 = \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int\limits_{\mathbb{R}^d} \Re\left\{\frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \widehat{\varphi}\left(u_k, s\right)\right\}^2 - \Re\left\{\int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \varphi(u,s)\,du\right\}^2 \mathbf{w}(s)\,ds$$
$$+ \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int\limits_{\mathbb{R}^d} \Im\left\{\frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \widehat{\varphi}\left(u_k, s\right)\right\}^2 - \Im\left\{\int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \varphi(u,s)\,du\right\}^2 \mathbf{w}(s)\,ds$$
$$=: \widehat{\mathbb{D}}_{T,2,\Re}^{\text{gap}} + \widehat{\mathbb{D}}_{T,2,\Im}^{\text{gap}}. \tag{3.30}$$

It follows for all $\mathrm{R} \in \{\Re, \Im\}$ by using a Riemann sum which is similar to that contained in (3.10):

$$\widehat{\mathbb{D}}_{T,1,\mathrm{R}}^{\text{gap}} \approx \int\limits_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\}^2 - \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathrm{R}\left\{\varphi\left(u_k, s\right)\right\}^2 \mathbf{w}(s)\,ds$$
$$= \int\limits_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left(\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\} + \mathrm{R}\left\{\varphi\left(u_k, s\right)\right\}\right)\left(\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\} - \mathrm{R}\left\{\varphi\left(u_k, s\right)\right\}\right) \mathbf{w}(s)\,ds \tag{3.31}$$

and, analogously:

$$\widehat{\mathbb{D}}_{T,2,\mathrm{R}}^{\text{gap}}$$
$$\approx \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int\limits_{\mathbb{R}^d} \frac{\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)^2}{\lfloor 1/(2b) \rfloor^2} \sum_{k_1,k_2=1}^{\lfloor 1/(2b) \rfloor} \left(\mathrm{R}\{\widehat{\varphi}(u_{k_1},s)\} + \mathrm{R}\{\varphi(u_{k_1},s)\}\right)\left(\mathrm{R}\{\widehat{\varphi}(u_{k_2},s)\} - \mathrm{R}\{\varphi(u_{k_2},s)\}\right) \mathbf{w}(s)\,ds. \tag{3.32}$$

In practise, the local characteristic function φ cannot be calculated directly but estimated by using the LECF $\widehat{\varphi}$. This motivates to approximate the expression $\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\} + \mathrm{R}\left\{\varphi\left(u_k, s\right)\right\}$ on the right side of (3.31) by $2\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\}$. However, it is obviously a bad idea to replace $\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\} - \mathrm{R}\left\{\varphi\left(u_k, s\right)\right\}$ on the right side of (3.31) in the same manner, i. e., by $\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\} - \mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\}$ $(= 0)$. In contrast, $\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\} - \mathrm{R}\left\{\varphi\left(u_k, s\right)\right\}$ is approximated by the centered expression $\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\} - \mathbb{E}\left[\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\}\right]$, which will turn out to be useful for the present construction of a dependent wild bootstrap procedure. Modifying the right side of (3.31) by these steps yields for all $\mathrm{R} \in \{\Re, \Im\}$:

$$\widehat{\mathbb{D}}_{T,1,\mathrm{R}}^{\text{gap}} \approx \int\limits_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} 2\,\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\}\left(\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\} - \mathbb{E}\left[\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\}\right]\right) \mathbf{w}(s)\,ds. \tag{3.33}$$

Carrying out similar modifications leads for all $\mathrm{R} \in \{\Re, \Im\}$ to the following approximation of the right side of (3.32):

$$\widehat{\mathbb{D}}_{T,2,\mathrm{R}}^{\text{gap}} \approx \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int\limits_{\mathbb{R}^d} \frac{\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)^2}{\lfloor 1/(2b) \rfloor^2} \sum_{k_1,k_2=1}^{\lfloor 1/(2b) \rfloor} 2\,\mathrm{R}\left\{\widehat{\varphi}\left(u_{k_1}, s\right)\right\}\left(\mathrm{R}\left\{\widehat{\varphi}\left(u_{k_2}, s\right)\right\} - \mathbb{E}\left[\mathrm{R}\left\{\widehat{\varphi}\left(u_{k_2}, s\right)\right\}\right]\right) \mathbf{w}(s)\,ds. \tag{3.34}$$

The expectations contained in (3.33) and (3.34) are unknown in practise. However, a bootstrap counterpart of $\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\} - \mathbb{E}\left[\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\}\right]$ can be constructed in the following manner by adapting the dependent wild bootstrap introduced in [67, Shao (2010)] to the present locally stationary framework. Investigations given in [67, Shao (2010)] show under certain assumptions that a bootstrap counterpart of the statistic $\bar{Z}_T - \mathbb{E}\left[Z_1\right]$ for a real-valued, strongly mixing, stationary time series $\left(Z_t\right)_{t \in \mathbb{Z}}$ and $\bar{Z}_T$ as notation for the arithmetic mean of $Z_1, \ldots, Z_T$ with $T \in \mathbb{N}$ can be derived from pseudo-observations, which are defined as $Z_{t,T}^* := \bar{Z}_T + \left(Z_t - \bar{Z}_T\right) W_{t,T}\ \forall\, t \in \{1, \ldots, T\}$, $T \in \mathbb{N}$ (see (1) in [67, Shao (2010), p. 219]) with certain bootstrap random variables $W_{t,T}$. Concretely, using the approach proposed in [67, Shao (2010)] yields the following bootstrap counterpart of the statistic $\bar{Z}_T - \mathbb{E}\left[Z_1\right]$:

$$\mathbf{T}_T^* := \frac{1}{T} \sum_{t=1}^{T} Z_{t,T}^* - \bar{Z}_T = \frac{1}{T} \sum_{t=1}^{T} \left(Z_t - \bar{Z}_T\right) W_{t,T}. \tag{3.35}$$

Thereby, Theorem 3.1 in [67, Shao (2010), p. 221] shows under some assumptions that the conditional distribution of $\sqrt{T}\,\mathbf{T}_T^*$ (conditioned on $Z_1, \ldots, Z_T$) approximates the limiting distribution of $\sqrt{T}\big(\bar{Z}_T - \mathbb{E}[Z_1]\big)$ asymptotically.
Motivated by (3.35) as well as Remark 2.7 and based on an arbitrary process $(W_t^*)_{t\in\mathbb{Z}}$ of bootstrap random variables that fulfils Assumption 3.15 $[\mathbf{W}^*]$, it is an intuitive idea to take the following expression into account as a bootstrap counterpart of $\mathrm{R}\{\widehat{\varphi}(u_k, s)\} - \mathbb{E}[\mathrm{R}\{\widehat{\varphi}(u_k, s)\}]$ with $\mathrm{R} \in \{\Re, \Im\}$ (recall Definition 2.11):

$$\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right)\left(\mathrm{R}\left\{e^{i\langle s, X_{t,T}\rangle}\right\} - \mathrm{R}\left\{\widehat{\varphi}\left(\frac{t}{T}, s\right)\right\}\right) W_t^*. \tag{3.36}$$

However, according to Proposition 2.13, $\widehat{\varphi}(t/T, s)$ will be an unsuitable estimator for $\varphi(t/T, s)$ $\Big($as well as, due to Remark 2.7, also for $\mathbb{E}\Big[e^{i\langle s, X_{t,T}\rangle}\Big]\Big)$ if $t \in [0, c_K T b] \cup [T - c_K T b, T]$. In contrast, Proposition 2.12 together with (3.11) and Proposition 2.14 yield for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^d$ that $\widehat{\varphi}(u_k, s)$ is an appropriate estimator for $\varphi(u_k, s)$ and the Remarks 2.7 as well as 2.3 provide that $\left|\mathbb{E}\left[e^{i\langle s, X_{t,T}\rangle}\right] - \varphi(u_k, s)\right|$ will be small for such u_k for which $K_b(t/T - u_k) > 0$. This leads to the following bootstrap counterpart of $\mathrm{R}\{\widehat{\varphi}(u_k, s)\} - \mathbb{E}[\mathrm{R}\{\widehat{\varphi}(u_k, s)\}]$:

$$\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right)\left(\mathrm{R}\left\{e^{i\langle s, X_{t,T}\rangle}\right\} - \mathrm{R}\{\widehat{\varphi}(u_k, s)\}\right) W_t^*. \tag{3.37}$$

Motivated by the heuristic considerations given above, the expression $\mathrm{R}\{\widehat{\varphi}(u_k, s)\} - \mathbb{E}[\mathrm{R}\{\widehat{\varphi}(u_k, s)\}]$ contained in (3.33) is replaced by (3.37) in order to obtain a bootstrap counterpart of $\widehat{\mathbb{D}}_{T,1,\mathrm{R}}^{\mathrm{gap}}$ and the expression $\mathrm{R}\{\widehat{\varphi}(u_{k_2}, s)\} - \mathbb{E}[\mathrm{R}\{\widehat{\varphi}(u_{k_2}, s)\}]$ contained in (3.34) is replaced by (3.37) (after k in (3.37) is renamed as k_2) to get a bootstrap counterpart of $\widehat{\mathbb{D}}_{T,2,\mathrm{R}}^{\mathrm{gap}}$.
Replacing $\widehat{\mathbb{D}}_{T,1,\mathrm{R}}^{\mathrm{gap}}$ and $\widehat{\mathbb{D}}_{T,2,\mathrm{R}}^{\mathrm{gap}}$ with $\mathrm{R} \in \{\Re, \Im\}$ in (3.29) as well as (3.30) by these bootstrap counterparts and re-sorting some of the resulting terms motivate the expression $\widehat{\mathbb{D}}_T^*(\gamma)$ introduced in the following as a bootstrap counterpart of $\gamma^{[1]}\big(\widehat{\mathbb{D}}_{T,1} - \mathbb{D}_1\big) + \gamma^{[2]}\big(\widehat{\mathbb{D}}_{T,2} - \mathbb{D}_2\big)$, whereby $\gamma := \big(\gamma^{[1]}, \gamma^{[2]}\big)$ is $\mathbb{R}^{1\times 2}$-valued but, for the present purposes, not necessarily deterministic. One defines for all $\mathrm{R} \in \{\Re, \Im\}$, $s \in \mathbb{R}^d$ (note that $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ according to Definition 3.3 (i) and see also Definition 2.11):

$$\begin{aligned}
\widehat{\mathbb{D}}_{T,1,\mathrm{R}}^*(s) &:= \widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},1,\mathrm{R}}^*(s) \\
&:= \frac{2(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathrm{R}\{\widehat{\varphi}(u_k, s)\}\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right)\mathrm{R}\left\{e^{i\langle s, X_{t,T}\rangle} - \widehat{\varphi}(u_k, s)\right\} W_t^*, \\
\widehat{\mathbb{D}}_{T,2,\mathrm{R}}^*(s) &:= \widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},2,\mathrm{R}}^*(s) := \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0}\mathrm{R}\left\{\frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k_1=1}^{\lfloor 1/(2b)\rfloor}\widehat{\varphi}(u_{k_1}, s)\right\}\frac{2(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor} \\
&\qquad\cdot \sum_{k_2=1}^{\lfloor 1/(2b)\rfloor}\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_{k_2}\right)\mathrm{R}\left\{e^{i\langle s, X_{t,T}\rangle} - \widehat{\varphi}(u_{k_2}, s)\right\} W_t^*, \\
\widehat{\mathbb{D}}_{T,\mathrm{R}}^{*\,\gamma}(s) &:= \widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},\mathrm{R}}^{*\,\gamma}(s) := \gamma^{[1]}\widehat{\mathbb{D}}_{T,1,\mathrm{R}}^*(s) + \gamma^{[2]}\widehat{\mathbb{D}}_{T,2,\mathrm{R}}^*(s), \\
\widehat{\mathbb{D}}_{T,\mathrm{R}}^*(\gamma) &:= \widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},\mathrm{R}}^*(\gamma) := \int_{\mathbb{R}^d}\widehat{\mathbb{D}}_{T,\mathrm{R}}^{*\,\gamma}(s)\,\mathbf{w}(s)\,ds \quad \text{and} \quad \widehat{\mathbb{D}}_T^*(\gamma) := \widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1}}^*(\gamma) := \widehat{\mathbb{D}}_{T,\Re}^*(\gamma) + \widehat{\mathbb{D}}_{T,\Im}^*(\gamma).
\end{aligned} \tag{3.38}$$

Next, bootstrap counterparts of $\widehat{\mathbb{D}}_T - \mathbb{D}$ and $\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}}$ are derived from the heuristically obtained bootstrap counterpart $\widehat{\mathbb{D}}_T^*(\gamma)$ of $\gamma^{[1]}\big(\widehat{\mathbb{D}}_{T,1} - \mathbb{D}_1\big) + \gamma^{[2]}\big(\widehat{\mathbb{D}}_{T,2} - \mathbb{D}_2\big)$ by choosing $\gamma := \big(\gamma^{[1]}, \gamma^{[2]}\big)$ suitably. Concretely, Definition 3.8 (i) and Proposition 3.6 (i) provide:

$$\widehat{\mathbb{D}}_T - \mathbb{D} = \gamma^{[1]}\left(\widehat{\mathbb{D}}_{T,1} - \mathbb{D}_1\right) + \gamma^{[2]}\left(\widehat{\mathbb{D}}_{T,2} - \mathbb{D}_2\right) \quad \text{for} \quad \left(\gamma^{[1]}, \gamma^{[2]}\right) := (1, -1),$$

which motivates to use $\widehat{\mathbb{D}}_T^*((1,-1))$ as a bootstrap counterpart of $\widehat{\mathbb{D}}_T - \mathbb{D}$.
Further, in order to construct a bootstrap counterpart of $\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}}$, note at first that if $\widehat{\mathbb{D}}_{T,1} > 0$, one will obtain for the function $(0,\infty) \times \mathbb{R} \ni (y_1, y_2) \mapsto h((y_1,y_2)') := 1 - y_2/y_1$ and for $\xi_{T,1,2} := a_\xi(\widehat{\mathbb{D}}_{T,1}, \widehat{\mathbb{D}}_{T,2})' + (1-a_\xi)(\mathbb{D}_1, \mathbb{D}_2)'$ (with a certain random variable a_ξ that owns realizations in $[0,1]$) due to Definition 3.8 (ii), Definition 3.3 (ii), Lemma 3.4 and the mean value theorem for real functions of multiple variables (whereby ∇ should denote the gradient as a row vector):

$$\begin{aligned}\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}} &= h\left(\left(\widehat{\mathbb{D}}_{T,1}, \widehat{\mathbb{D}}_{T,2}\right)'\right) - h\left((\mathbb{D}_1,\mathbb{D}_2)'\right) = \nabla h(\xi_{T,1,2}) \begin{pmatrix} \widehat{\mathbb{D}}_{T,1} - \mathbb{D}_1 \\ \widehat{\mathbb{D}}_{T,2} - \mathbb{D}_2 \end{pmatrix} \\ &= \gamma^{[1]}\left(\widehat{\mathbb{D}}_{T,1} - \mathbb{D}_1\right) + \gamma^{[2]}\left(\widehat{\mathbb{D}}_{T,2} - \mathbb{D}_2\right) \quad \text{for} \quad \left(\gamma^{[1]}, \gamma^{[2]}\right) := \nabla h(\xi_{T,1,2}). \end{aligned} \tag{3.39}$$

Theorem 3.13 (i) implies that $(\widehat{\mathbb{D}}_{T,1}, \widehat{\mathbb{D}}_{T,2})'$ converges in probability to $(\mathbb{D}_1, \mathbb{D}_2)'$ for $T \to \infty$, which suggests that $\xi_{T,1,2}$ can be approximated by $\left(\widehat{\mathbb{D}}_{T,1}, \widehat{\mathbb{D}}_{T,2}\right)'$ appropriately. This motivates for $\widehat{\mathbb{D}}_{T,1} > 0$ to replace $\nabla h(\xi_{T,1,2})$ in (3.39) by $\nabla h\big((\widehat{\mathbb{D}}_{T,1}, \widehat{\mathbb{D}}_{T,2})'\big) = \big(\widehat{\mathbb{D}}_{T,2}/\widehat{\mathbb{D}}_{T,1}^2, -1/\widehat{\mathbb{D}}_{T,1}\big)$ to obtain the following approximation of $\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}}$:

$$\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}} \approx \gamma^{[1]}\left(\widehat{\mathbb{D}}_{T,1} - \mathbb{D}_1\right) + \gamma^{[2]}\left(\widehat{\mathbb{D}}_{T,2} - \mathbb{D}_2\right) \quad \text{for} \quad \left(\gamma^{[1]}, \gamma^{[2]}\right) := \left(\widehat{\mathbb{D}}_{T,2}/\widehat{\mathbb{D}}_{T,1}^2, -1/\widehat{\mathbb{D}}_{T,1}\right),$$

which leads to the bootstrap counterpart $\widehat{\mathbb{D}}_T^*\big((\widehat{\mathbb{D}}_{T,2}/\widehat{\mathbb{D}}_{T,1}^2, -1/\widehat{\mathbb{D}}_{T,1})\big)$ of $\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}}$ in the case $\widehat{\mathbb{D}}_{T,1} > 0$.
Overall, these considerations result in the following definition.

Definition 3.17 (Bootstrap counterparts of $\widehat{\mathbb{D}}_T - \mathbb{D}$ and $\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}}$).
Let the Assumptions 2.4 $[\mathbf{DM.1}]$*, 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K\&b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled.*

(i) The expression $\widehat{\mathbb{D}}_T^*((1,-1))$ *is called bootstrap counterpart of the EMDCI-based statistic* $\widehat{\mathbb{D}}_T - \mathbb{D}$ *(abbr. B-EMDCI)*

(ii) Define:

$$\widehat{\gamma}_{T,1}^{\mathrm{norm}} := \widehat{\gamma}_{T,\mathfrak{U}_{0,1},1}^{\mathrm{norm}} := \begin{cases} \widehat{\mathbb{D}}_{T,2}/\widehat{\mathbb{D}}_{T,1}^2, & \text{for } \widehat{\mathbb{D}}_{T,1} > 0 \\ 0, & \text{for } \widehat{\mathbb{D}}_{T,1} = 0 \end{cases} \quad \text{and}$$

$$\widehat{\gamma}_{T,2}^{\mathrm{norm}} := \widehat{\gamma}_{T,\mathfrak{U}_{0,1},2}^{\mathrm{norm}} := \begin{cases} -1/\widehat{\mathbb{D}}_{T,1}, & \text{for } \widehat{\mathbb{D}}_{T,1} > 0 \\ 0, & \text{for } \widehat{\mathbb{D}}_{T,1} = 0 \end{cases}.$$

Then, $\widehat{\mathbb{D}}_T^*\big((\widehat{\gamma}_{T,1}^{\mathrm{norm}}, \widehat{\gamma}_{T,2}^{\mathrm{norm}})\big)$ *is called bootstrap counterpart of the NEMDCI-based statistic* $\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}}$ *(abbr. B-NEMDCI).*

Remark 3.18. *(i) The B-NEMDCI is defined in such a manner that it exists for all* $T \in \mathbb{N}$ *but it is just useful in the case* $\widehat{\mathbb{D}}_{T,1} > 0$*, whereby this property will be valid if* T *is large enough (according to Lemma 3.9).*

(ii) It holds for all $x \in \mathbb{R}^d$ *and* $t_1, t_2 \in \{1, \ldots, T\}$*:*

$$\begin{aligned} &\int_{\mathbb{R}^d} \Re\left\{e^{i\langle s, x+X_{t_1,T}\rangle}\right\} \Re\left\{e^{i\langle s, x+X_{t_2,T}\rangle}\right\} + \Im\left\{e^{i\langle s, x+X_{t_1,T}\rangle}\right\} \Im\left\{e^{i\langle s, x+X_{t_2,T}\rangle}\right\} \mathbf{w}(s)\, ds \\ &= \int_{\mathbb{R}^d} \Re\left\{e^{i\langle s, x+X_{t_1,T}\rangle} \overline{e^{i\langle s, x+X_{t_2,T}\rangle}}\right\} \mathbf{w}(s)\, ds \\ &= \int_{\mathbb{R}^d} \Re\left\{e^{i\langle s, X_{t_1,T}-X_{t_2,T}\rangle}\right\} \mathbf{w}(s)\, ds \\ &= \int_{\mathbb{R}^d} \Re\left\{e^{i\langle s, X_{t_1,T}\rangle}\right\} \Re\left\{e^{i\langle s, X_{t_2,T}\rangle}\right\} + \Im\left\{e^{i\langle s, X_{t_1,T}\rangle}\right\} \Im\left\{e^{i\langle s, X_{t_2,T}\rangle}\right\} \mathbf{w}(s)\, ds. \end{aligned}$$

Moreover, one obtains similarly to Remark 3.7 (iii) that the realizations of $\widehat{\gamma}_{T,1}^{\mathrm{norm}}$ *and* $\widehat{\gamma}_{T,2}^{\mathrm{norm}}$ *which belong to a sample path* $(x + X_{t,T}(\omega))_{t=1}^{T}$ *with arbitrary* $\omega \in \Omega$ *and* $x \in \mathbb{R}^d$ *do not change for different choices of* x*. Overall, these arguments provide that realizations of* $\widehat{\mathbb{D}}_T^*((1,-1))$ *and* $\widehat{\mathbb{D}}_T^*\big((\widehat{\gamma}_{T,1}^{\mathrm{norm}}, \widehat{\gamma}_{T,2}^{\mathrm{norm}})\big)$ *will not change if the underlying sample path* $(X_{t,T}(\omega))_{t=1}^{T}$ *is replaced by* $(x + X_{t,T}(\omega))_{t=1}^{T}$ *for arbitrary* $x \in \mathbb{R}^d$*.*
Further, if the weight function $\mathbf{w}$ *fulfils* $\mathbf{w}(s) = \mathbf{w}(-s)\ \forall\, s \in \mathbb{R}^d$*, one will obtain similarly to Remark 3.7 (iii) that the realizations of* $\widehat{\mathbb{D}}_T^*((1,-1))$ *and* $\widehat{\mathbb{D}}_T^*\big((\widehat{\gamma}_{T,1}^{\mathrm{norm}}, \widehat{\gamma}_{T,2}^{\mathrm{norm}})\big)$ *which are associated with the sample path* $(X_{t,T}(\omega))_{t=1}^{T}$ *will stay the same if this sample path is replaced by* $(-X_{t,T}(\omega))_{t=1}^{T}$*.*

In order to obtain bootstrap procedures that approximate the distributions of $Z_{\mathfrak{U}_{0,1}}$ and $Z_{\mathfrak{U}_{0,1}}^{\mathrm{norm}}$, which are the asymptotic distributions resulting from Theorem 3.13 (ii) as well as (iii), the bootstrap counterparts proposed in Definition 3.17 are scaled with $\sqrt{T}$, whereby this scale is motivated by Theorem 3.13 (ii) and (iii). The following theorem provides that this yields consistent bootstraps.

Theorem 3.19 (Asymptotic behaviour of the B-EMDCI and B-NEMDCI)**.**
Let the Assumptions 2.4 $[\mathbf{DM.1}]$*, 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K\&b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled.*

(i) If $\sigma_{\mathfrak{U}_{0,1}}((1,-1),(1,-1)) > 0$ *(recall (3.18)), it will follow for* $T \to \infty$*:*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}^*\left(\sqrt{T}\,\widehat{\mathbb{D}}_T^*((1,-1)) \leqslant x\right) - \mathbb{P}\left(Z_{\mathfrak{U}_{0,1}} \leqslant x\right)\right| \xrightarrow{\mathbb{P}} 0. \tag{3.40}$$

If $\sigma_{\mathfrak{U}_{0,1}}((1,-1),(1,-1)) = 0$*, one will obtain for* $T \to \infty$ *(see (3.26)):*

$$\sqrt{T}\,\widehat{\mathbb{D}}_T^*((1,-1)) = o_{\mathbb{P}}^*(1). \tag{3.41}$$

(ii) If $\sigma_{\mathfrak{U}_{0,1}}\big((\gamma_{\mathfrak{U}_{0,1},1}^{\mathrm{norm}}, \gamma_{\mathfrak{U}_{0,1},2}^{\mathrm{norm}}), (\gamma_{\mathfrak{U}_{0,1},1}^{\mathrm{norm}}, \gamma_{\mathfrak{U}_{0,1},2}^{\mathrm{norm}})\big) > 0$ *(recall (3.18) as well as (3.22)), it will follow for* $T \to \infty$*:*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}^*\left(\sqrt{T}\,\widehat{\mathbb{D}}_T^*\big((\widehat{\gamma}_{T,1}^{\mathrm{norm}}, \widehat{\gamma}_{T,2}^{\mathrm{norm}})\big) \leqslant x\right) - \mathbb{P}\left(Z_{\mathfrak{U}_{0,1}}^{\mathrm{norm}} \leqslant x\right)\right| \xrightarrow{\mathbb{P}} 0. \tag{3.42}$$

If $\sigma_{\mathfrak{U}_{0,1}}\big((\gamma_{\mathfrak{U}_{0,1},1}^{\mathrm{norm}}, \gamma_{\mathfrak{U}_{0,1},2}^{\mathrm{norm}}), (\gamma_{\mathfrak{U}_{0,1},1}^{\mathrm{norm}}, \gamma_{\mathfrak{U}_{0,1},2}^{\mathrm{norm}})\big) = 0$*, one will obtain for* $T \to \infty$*:*

$$\sqrt{T}\,\widehat{\mathbb{D}}_T^*\big((\widehat{\gamma}_{T,1}^{\mathrm{norm}}, \widehat{\gamma}_{T,2}^{\mathrm{norm}})\big) = o_{\mathbb{P}}^*(1). \tag{3.43}$$

Remark 3.20. *Let the Assumptions 2.4* $[\mathbf{DM.1}]$*, 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K\&b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be valid. Then, if* $\sigma_{\mathfrak{U}_{0,1}}((1,-1),(1,-1)) > 0$*, one will obtain from Theorem 3.19 (i) and Theorem 3.13 (ii) together with Polya's theorem:*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}^*\left(\sqrt{T}\,\widehat{\mathbb{D}}_T^*((1,-1)) \leqslant x\right) - \mathbb{P}\left(\sqrt{T}\left(\widehat{\mathbb{D}}_T - \mathbb{D}\right) \leqslant x\right)\right| \xrightarrow{\mathbb{P}} 0.$$

If $\sigma_{\mathfrak{U}_{0,1}}\big((\gamma_{\mathfrak{U}_{0,1},1}^{\mathrm{norm}}, \gamma_{\mathfrak{U}_{0,1},2}^{\mathrm{norm}}), (\gamma_{\mathfrak{U}_{0,1},1}^{\mathrm{norm}}, \gamma_{\mathfrak{U}_{0,1},2}^{\mathrm{norm}})\big) > 0$*, it will hold due to Theorem 3.19 (ii) and Theorem 3.13 (iii) in combination with Polya's theorem:*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}^*\left(\sqrt{T}\,\widehat{\mathbb{D}}_T^*\big((\widehat{\gamma}_{T,1}^{\mathrm{norm}}, \widehat{\gamma}_{T,2}^{\mathrm{norm}})\big) \leqslant x\right) - \mathbb{P}\left(\sqrt{T}\left(\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \mathbb{D}^{\mathrm{norm}}\right) \leqslant x\right)\right| \xrightarrow{\mathbb{P}} 0.$$

The following algorithm, that estimates a confidence interval which covers the value of the MDCI with a probability of $\alpha_2 - \alpha_1$ for chosen $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$, is constructed based on Theorem 3.13 (ii) in combination with (3.24), Theorem 3.19 (i) as well as Remark 3.5 (i) (the latter ensures $\mathbb{D} \in [0,\infty)$).

Algorithm **CONF.MDCI**

Inputs: $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ with $\mathfrak{U}_0 < \mathfrak{U}_1$; $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$; $T, N \in \mathbb{N}$; a sample path $(X_{t,T}(\omega))_{t=1}^{T}$ (for an $\omega \in \Omega$); a kernel K for which Assumption 2.8 $[\mathbf{K\&b.1}]$ (i) holds; a bandwidth $b \in (0, 1/2)$; a weight function $\mathbf{w}$ that fulfils Assumption 3.1 $[\mathbf{WEI.1}]$; a parameter $\beta > 0$ and an associated process $(W_t^*)_{t\in\mathbb{Z}}$ which satisfies Assumption 3.15 $[\mathbf{W}^*]$;

1: Determine the realization of $\widehat{\mathbb{D}}_T$ that belongs to the sample path $(X_{t,T}(\omega))_{t=1}^T$;
2: Independently, for n in $1 : N$ do
3: Generate a sample path of $(W_t^*)_{t=1}^T$;
4: Calculate the associated realization of $\sqrt{T}\,\widehat{\mathbb{D}}_T^*((1,-1))$;
5: end for
6: Estimate for all $\alpha \in \{\alpha_1, \alpha_2\}$ the quantile $q_{\alpha,\mathfrak{U}_{0,1}}$ of the distribution of $Z_{\mathfrak{U}_{0,1}}$ that belongs to the level α by using the computed realizations of $\sqrt{T}\,\widehat{\mathbb{D}}_T^*((1,-1))$ and denote the resulting estimator for $q_{\alpha,\mathfrak{U}_{0,1}}$ as $\widehat{q}_{T,\alpha,\mathfrak{U}_{0,1}}$;
7: Calculate the realization of the following estimated confidence interval for $\mathbb{D}$:

$$\left[\widehat{\mathbb{D}}_T - \frac{\widehat{q}_{T,\alpha_2,\mathfrak{U}_{0,1}}}{\sqrt{T}}, \widehat{\mathbb{D}}_T - \frac{\widehat{q}_{T,\alpha_1,\mathfrak{U}_{0,1}}}{\sqrt{T}}\right] \cap [0,\infty).$$

The following algorithm, that estimates a confidence interval which covers the value of the NMDCI with a probability of $\alpha_2 - \alpha_1$ for chosen $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$, is constructed based on Theorem 3.13 (iii) in combination with (3.25), Theorem 3.19 (ii) as well as Remark 3.5 (ii) (the latter ensures $\mathbb{D}^{\mathrm{norm}} \in [0,1]$). Thereby, it should be noted in regard of the next algorithm that, in contrast to the previous algorithm, $\widehat{\mathbb{D}}_{T,1}(\omega) > 0$ is demanded for the considered sample path $(X_{t,T}(\omega))_{t=1}^T$ (according to the inputs of this algorithm) and $\widehat{\mathbb{D}}_T^{\mathrm{norm}}$ as well as $\sqrt{T}\,\widehat{\mathbb{D}}_T^*\big(\big(\widehat{\gamma}_{T,1}^{\mathrm{norm}}, \widehat{\gamma}_{T,2}^{\mathrm{norm}}\big)\big)$ are used instead of $\widehat{\mathbb{D}}_T$ and $\sqrt{T}\,\widehat{\mathbb{D}}_T^*((1,-1))$, respectively. Moreover, the seventh steps of the algorithm given below takes care of $\mathbb{D}^{\mathrm{norm}} \in [0,1]$, whereas Algorithm **CONF.MDCI** respects $\mathbb{D} \in [0,\infty)$.

Algorithm **CONF.NMDCI**

Inputs: $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ with $\mathfrak{U}_0 < \mathfrak{U}_1$; $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$; $T, N \in \mathbb{N}$; a sample path $(X_{t,T}(\omega))_{t=1}^T$ (for an $\omega \in \Omega$) with $\widehat{\mathbb{D}}_{T,1}(\omega) > 0$; a kernel K for which Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) holds; a bandwidth $b \in (0, 1/2)$; a weight function $\mathbf{w}$ that fulfils Assumption 3.1 $[\mathbf{WEI.1}]$; a parameter $\beta > 0$ and an associated process $(W_t^*)_{t\in\mathbb{Z}}$ which satisfies Assumption 3.15 $[\mathbf{W}^*]$;

1: Determine the realization of $\widehat{\mathbb{D}}_T^{\mathrm{norm}}$ that belongs to the sample path $(X_{t,T}(\omega))_{t=1}^T$;
2: Independently, for n in $1 : N$ do
3: Generate a sample path of $(W_t^*)_{t=1}^T$;
4: Calculate the associated realization of $\sqrt{T}\,\widehat{\mathbb{D}}_T^*\big(\big(\widehat{\gamma}_{T,1}^{\mathrm{norm}}, \widehat{\gamma}_{T,2}^{\mathrm{norm}}\big)\big)$;
5: end for
6: Estimate for all $\alpha \in \{\alpha_1, \alpha_2\}$ the quantile $q_{\alpha,\mathfrak{U}_{0,1}}^{\mathrm{norm}}$ of the distribution of $Z_{\mathfrak{U}_{0,1}}^{\mathrm{norm}}$ that belongs to the level α by using the computed realizations of $\sqrt{T}\,\widehat{\mathbb{D}}_T^*\big(\big(\widehat{\gamma}_{T,1}^{\mathrm{norm}}, \widehat{\gamma}_{T,2}^{\mathrm{norm}}\big)\big)$ and denote the resulting estimator for $q_{\alpha,\mathfrak{U}_{0,1}}^{\mathrm{norm}}$ as $\widehat{q}_{T,\alpha,\mathfrak{U}_{0,1}}^{\mathrm{norm}}$;
7: Calculate the realization of the following estimated confidence interval for $\mathbb{D}^{\mathrm{norm}}$:

$$\left[\widehat{\mathbb{D}}_T^{\mathrm{norm}} - \frac{\widehat{q}_{T,\alpha_2,\mathfrak{U}_{0,1}}^{\mathrm{norm}}}{\sqrt{T}}, \widehat{\mathbb{D}}_T^{\mathrm{norm}} - \frac{\widehat{q}_{T,\alpha_1,\mathfrak{U}_{0,1}}^{\mathrm{norm}}}{\sqrt{T}}\right] \cap [0,1].$$

Remark 3.21. *The asymptotic properties of b and β which are demanded in the Assumptions 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) as well as 3.15 $[\mathbf{W}^*]$ (i) can be regarded as rough guidances on selecting b and β for a given $T \in \mathbb{N}$. Thereby, it is worth mentioning that (according to Remark 2.9) such guidances can be derived although $\delta \in (0,1]$ is (commonly) unknown and not appropriately estimable. In addition, Assumption 3.1 $[\mathbf{WEI.1}]$ depends on δ but this is also not critical because (3.3) (with arbitrary $\delta \in (0,1]$) is a weaker condition than $\int_{\mathbb{R}^d} \big(1 + |s|_1^4\big)\, \mathbf{w}(s)\, ds < \infty$.*

In the following, a measure for the distribution change intensity under local stationarity is constructed which is not just shift- but also scale-invariant and, subsequently, it is explained why estimating this measure is accompanied by issues in the present framework.
Inspired by the measure for differences between multivariate distributions proposed in [56, Matteson and James (2014), p. 335], define for arbitrary but fixed $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ with $\mathfrak{U}_0 < \mathfrak{U}_1$ the following measure for the degree of distribution changes (whereby $\mathfrak{U}_{0,1}$ originates from Definition 3.3 (i), $d \in \mathbb{N}$

from Definition 2.1 and $\delta \in (0,1]$ from Assumption 2.2 [**StAp**]):

$$\mathbb{D}^{[q]}_{\mathfrak{U}_{0,1}} := \frac{1}{2\,(\mathfrak{U}_1 - \mathfrak{U}_0)} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \int\limits_{\mathbb{R}^d} |\varphi(u,s) - \varphi(w,s)|^2 \, \mathbf{w}_{d,q}(s) \, ds \, du \, dw \qquad \text{with}$$

$$\mathbf{w}_{d,q}(s) := \frac{q \cdot 2^q \Gamma((d+q)/2)}{2\pi^{d/2}\Gamma(1-q/2)\,|s|_2^{d+q}} \quad \text{for some } q \in (0, 1+\delta]\backslash\{2\}. \tag{3.44}$$

According to Proposition 3.6 (ii), this measure can be regarded as a modification of the EMDCI based on the weight function $\mathbf{w}_{d,q}$, which does not fulfil Assumption 3.1 [**WEI.1**] (note that $\int_{\mathbb{R}^d} \mathbf{w}_{d,q}(s)\,ds \nless \infty$ - as mentioned in [72, Székely et al. (2007), p. 2771]).
It follows under Assumption 2.2 [**StAp**] similarly to Lemma 1 in [56, Matteson and James (2014), p. 335] (in particular, also regard the proof of Theorem 2 in [71, Székely and Rizzo (2005), p. 178 et seq.]) that $\mathbb{D}^{[q]}_{\mathfrak{U}_{0,1}} = 0$ is equivalent to $\widetilde{X}_0(u) \overset{d}{=} \widetilde{X}_0(w)\ \forall\, u,w \in [\mathfrak{U}_0, \mathfrak{U}_1]$, that $\mathbb{D}^{[q]}_{\mathfrak{U}_{0,1}} \in [0,\infty)$ is valid as well as for $\widetilde{X}_0(u) := \mathbf{H}\left(u, \mathcal{F}_0^{\times}\right)\ \forall\, u \in [0,1]$ with $\mathcal{F}_0^{\times} := \left(\varepsilon_0^{\times}, \varepsilon_{-1}^{\times}, \dots\right)$, whereby $\mathbf{H}$ originates from Assumption 2.2 [**StAp**] (iii) and $\left(\varepsilon_k^{\times}\right)_{k\in\mathbb{Z}}$ from Assumption 2.4 [**DM**], that:

$$\begin{aligned}
\mathbb{D}^{[q]}_{\mathfrak{U}_{0,1}} &= \frac{1}{2\,(\mathfrak{U}_1 - \mathfrak{U}_0)} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} 2\,\mathbb{E}\left[\left|\widetilde{X}_0(u) - \widetilde{X}_0(w)\right|_2^q\right] - \mathbb{E}\left[\left|\widetilde{X}_0(u) - \widetilde{X}_0^{\times}(u)\right|_2^q\right] \\
&\quad - \mathbb{E}\left[\left|\widetilde{X}_0(w) - \widetilde{X}_0^{\times}(w)\right|_2^q\right] du\,dw \\
&= \frac{1}{(\mathfrak{U}_1 - \mathfrak{U}_0)} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \mathbb{E}\left[\left|\widetilde{X}_0(u) - \widetilde{X}_0(w)\right|_2^q\right] du\,dw - \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \mathbb{E}\left[\left|\widetilde{X}_0(u) - \widetilde{X}_0^{\times}(u)\right|_2^q\right] du \\
&=: \mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}} - \mathbb{D}^{[q]}_{2,\mathfrak{U}_{0,1}}.
\end{aligned} \tag{3.45}$$

Thereby, $\mathbb{D}^{[q]}_{\mathfrak{U}_{0,1}} \in [0,\infty)$, which implies $\mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}} \geqslant \mathbb{D}^{[q]}_{2,\mathfrak{U}_{0,1}}$, motivates to consider the following normalized measure for the distribution change intensity:

$$[0,1] \ni \mathbb{D}^{[q],\mathrm{norm}}_{\mathfrak{U}_{0,1}} := \begin{cases} \dfrac{\mathbb{D}^{[q]}_{\mathfrak{U}_{0,1}}}{\mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}}} = 1 - \dfrac{\mathbb{D}^{[q]}_{2,\mathfrak{U}_{0,1}}}{\mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}}}, & \text{for } \mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}} > 0 \\ 0, & \text{for } \mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}} = 0 \end{cases}. \tag{3.46}$$

The measure $\mathbb{D}^{[q],\mathrm{norm}}_{\mathfrak{U}_{0,1}}$ is scale- and shift-invariant in the sense that it assigns the same distribution change intensity to the processes $\{\widetilde{X}_t(u)\}$ and $\{x + y\widetilde{X}_t(u)\}$ for all deterministic $x \in \mathbb{R}^d$ as well as $y \in \mathbb{R}\backslash\{0\}$. However, if Assumption 2.2 [**StAp**] holds, estimating $\mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}}$ and $\mathbb{D}^{[q]}_{2,\mathfrak{U}_{0,1}}$ is accompanied by issues - as explained in the following.
Using kernel-bandwidth-based estimators for $\mathbb{E}\big[\big|\widetilde{X}_0(u) - \widetilde{X}_0(w)\big|_2^q\big]$ and $\mathbb{E}\big[\big|\widetilde{X}_0(u) - \widetilde{X}_0^{\times}(u)\big|_2^q\big]$ seems to be the only possibility to obtain appropriate empirical versions of $\mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}}$ and $\mathbb{D}^{[q]}_{2,\mathfrak{U}_{0,1}}$. However, $\delta \in (0,1]$ is commonly unknown in practical applications, such that $q \leqslant 1$ should be chosen under Assumption 2.2 [**StAp**] to avoid that a q is selected for which $\mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}}$ and $\mathbb{D}^{[q]}_{2,\mathfrak{U}_{0,1}}$ are not well-defined (i. e., for which $q > 1 + \delta$).
So, let $q \leqslant 1$. A kernel-bandwidth-based estimator for $\mathbb{E}\big[\big|\widetilde{X}_0(u) - \widetilde{X}_0(w)\big|_2^q\big]$ with $u, w \in [0,1]$, that is based on Assumption 2.8 [**K&b.1**], is given as:

$$\frac{1}{T^2} \sum_{t_1,t_2=1}^{T} K_b\left(\frac{t_1}{T} - u\right) K_b\left(\frac{t_2}{T} - w\right) |X_{t_1,T} - X_{t_2,T}|_2^q\,. \tag{3.47}$$

This estimator is accompanied by the same issue with respect to its bias than that originating from Proposition 2.13 because $X_{t,T}$ is not well-defined for $t \leqslant 0$ and $t \geqslant T + 1$, which is critical for the estimation of $\mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}}$ in the cases $\mathfrak{U}_0 = 0$ or $\mathfrak{U}_1 = 1$. To avoid this issue, consider in a first step the following approximation of $\mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}}$, that results by using Riemann sums which are similar to that contained in (3.10):

$$\mathbb{D}^{[q],\mathrm{apprx}}_{1,\mathfrak{U}_{0,1}} := \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor^2} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left|\widetilde{X}_0\left(u_{k_1}\right) - \widetilde{X}_0\left(u_{k_2}\right)\right|_2^q\right] \tag{3.48}$$

and, in a second step, replace $\mathbb{E}\left[\left|\widetilde{X}_0\left(u_{k_1}\right) - \widetilde{X}_0\left(u_{k_2}\right)\right|_2^q\right]$ in (3.48) by (3.47) with $u = u_{k_1}$ as well as $w = u_{k_2}$.

However, in contrast to $\mathfrak{U}_{0,1} \ni u \mapsto |\varphi(u,s)|^2$, the functions $\mathfrak{U}_{0,1} \ni u \mapsto \mathbb{E}\big[|\widetilde{X}_0(u) - \widetilde{X}_0(w)|_2^q\big]$ with fixed $w \in \mathfrak{U}_{0,1}$ and $\mathfrak{U}_{0,1} \ni w \mapsto \mathbb{E}\big[|\widetilde{X}_0(u) - \widetilde{X}_0(w)|_2^q\big]$ with fixed $u \in \mathfrak{U}_{0,1}$ do not need to own Hölder continuous derivatives for $q \leqslant 1$ in general. Thus, the approximation $\mathbb{D}^{[q],\mathrm{apprx}}_{1,\mathfrak{U}_{0,1}}$ of $\mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}}$ cannot be justified by Lemma B.2 (iii) in the appendix. Instead, for $q = 1$, Lemma B.2 (ii) shows that this approximation is accompanied by an approximation error of order $\mathcal{O}(b)$, whereby it is expectable that this error cannot be improved to $o(b)$ in general and, for $q \in (0,1)$, it is conjecturable that the error of this approximation is tendentiously even larger than in the case $q = 1$. This indicates under Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) that $\sqrt{T}\big(\mathbb{D}^{[q],\mathrm{apprx}}_{1,\mathfrak{U}_{0,1}} - \mathbb{D}^{[q]}_{1,\mathfrak{U}_{0,1}}\big)$ is not asymptotically bounded for $T \to \infty$. Moreover, similar issues result with respect to the estimation of $\mathbb{D}^{[q]}_{2,\mathfrak{U}_{0,1}}$ for $q \in (0,1]$.

In contrast, recall that $\mathbb{D}^{\mathrm{apprx}}_{1,T}$ (see (3.10)) approximates $\mathbb{D}_1$ with an error of order $\mathcal{O}\left(b^{1+\delta}\right)$ (as shown on the Pages 20 to 21) and $\sqrt{T}b^{1+\delta} \to 0$ holds for $T \to \infty$ according to Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii). These are some of the main reasons that allow to prove Theorem 3.13, which is based on the convergence rate $\sqrt{T}$. In addition, note also that the Algorithms **CONF.MDCI** as well as **CONF.NMDCI** allow to estimate confidence intervals for the MDCI and NMDCI, respectively, without specifying $\delta \in (0,1]$.

3.2. Testing the existence of distribution changes

3.2.1. Introduction of the test problem and construction of a belonging test statistic

Suppose that Assumption 2.2 $[\mathbf{StAp}]$ holds and consider for arbitrary but fixed $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ with $\mathfrak{U}_0 < \mathfrak{U}_1$ the following test problem (note that $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ according to Definition 3.3 (i)):

$$\mathcal{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}} : \widetilde{X}_0(u) \overset{d}{=} \widetilde{X}_0(v)\ \forall\, u,v \in \mathfrak{U}_{0,1} \quad \text{versus} \quad \mathcal{H}^{\mathrm{distr}}_{1,\mathfrak{U}_{0,1}} : \exists\, u,v \in \mathfrak{U}_{0,1} : \widetilde{X}_0(u) \overset{d}{\neq} \widetilde{X}_0(v). \tag{3.49}$$

In the present section, this test problem is investigated based on the EMDCI and it is pointed out that it can also be decided by using the NEMDCI (see Remark 3.38 (ii)).

Remark 3.22. *Note that Remark 3.7 (ii) implies the following claims:*
If the null hypothesis $\mathcal{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}}$ holds, the locally stationary random variables $X_{\max\{1,\lceil \mathfrak{U}_0 T\rceil\},T}, \ldots, X_{\lfloor \mathfrak{U}_1 T\rfloor,T}$ can be regarded as approximately equally distributed for large T. Instead, if the alternative $\mathcal{H}^{\mathrm{distr}}_{1,\mathfrak{U}_{0,1}}$ is fulfilled, $u_0, u_1 \in [\mathfrak{U}_0, \mathfrak{U}_1]$ will exist for which the distributions of $X_{\max\{1,\lceil u_0 T\rceil\},T}$ and $X_{\lfloor u_1 T\rfloor,T}$ are not approximately the same for large T.

In the case that $\mathcal{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}}$ is valid, one obtains $\tau_{\mathfrak{U}_{0,1},\mathrm{R}}\left((1,-1),u,s\right) = 0$ and $\tau_{\mathfrak{U}_{0,1},\mathrm{R}}\big(\big(\gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},1}, \gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},2}\big),u,s\big) = 0\ \forall\, \mathrm{R} \in \{\Re, \Im\}$, $u \in [0,1]$, $s \in \mathbb{R}^d$ (recall (3.16) as well as (3.22)), whereby the first equality holds obviously and the second one results from Proposition 3.6 (i) as well as (iii) (in particular, $\mathbb{D}_1 = \mathbb{D}_2$ under $\mathcal{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}}$). Thus, Theorem 3.13 (ii) and (iii) as well as Proposition 3.6 (iii) and (iv) yield under $\mathcal{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}}$ by using that convergence in distribution to a Dirac distributed random variable implies convergence in probability to this random variable:

$$\sqrt{T}\,\widehat{\mathbb{D}}_T \overset{\mathbb{P}}{\longrightarrow} 0 \quad \text{as well as} \quad \sqrt{T}\,\widehat{\mathbb{D}}^{\mathrm{norm}}_T \overset{\mathbb{P}}{\longrightarrow} 0. \tag{3.50}$$

In practical applications (i. e., for fixed $T \in \mathbb{N}$), realizations of the EMDCI and NEMDCI may be positive under $\mathcal{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}}$, such that it is not useful to reject $\mathcal{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}}$ whenever $\widehat{\mathbb{D}}_T > 0$ ($\widehat{\mathbb{D}}^{\mathrm{norm}}_T > 0$, respectively), which is suggested by (3.50). Thus, Theorem 3.13 (ii) and (iii) are not helpful for constructing an appropriate test for the problem (3.49).

Instead, inspired by the statistic considered in Corollary 1 in [64, Rosenblatt (1975), p. 2], $\widehat{\mathbb{D}}_T$ is scaled with $T\sqrt{b}$ (and not $\sqrt{T}$) as well as a certain deterministic expression of order $\mathcal{O}(1/\sqrt{b})$ is subtracted from $T\sqrt{b}\,\widehat{\mathbb{D}}_T$ to obtain a test statistic that owns a not necessarily degenerate limiting distribution under the null hypothesis and which is consistent under the alternative. The resulting statistic is presented in the next theorem, in which also the asymptotic behaviour of this statistic is stated. Thereby, using the rate of convergence $T\sqrt{b}$, which is asymptotically larger than $\sqrt{T}$ (due to Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii)) motivates that the - compared to Assumption 2.4 $[\mathbf{DM}.\mathbf{1}]$ - stronger Assumption 2.4 $[\mathbf{DM}.\mathbf{2}]$ is demanded in Theorem 3.23, whereas Assumption 2.4 $[\mathbf{DM}.\mathbf{1}]$ underlies Theorem 3.13.

Theorem 3.23 (Construction and asymptotic behaviour of the test statistic)**.**
Let the Assumptions 2.4 $[\mathbf{DM}.\mathbf{2}]$*, 3.1* $[\mathbf{WEI}.\mathbf{1}]$ *and 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *be fulfilled. Moreover, define for all* $\mathrm{R} \in \{\Re, \Im\}$ *(see (3.17)):*

$$\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\mathrm{R}} := \frac{1}{\sqrt{b}} \int\limits_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\, dz \int\limits_{\mathbb{R}^d}\int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \sigma_{\infty,\mathrm{R},\mathrm{R}}\left(u,s,s\right)\, du\, \mathbf{w}(s)\, ds,$$

$$\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}} := \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\Re} + \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\Im} \tag{3.51}$$

and for all $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$*:*

$$\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1},\mathrm{R}_1,\mathrm{R}_2} := 4\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\left(\int\limits_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K\left(z\right)^2\, dz\right)^2 \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \sigma_{\infty,\mathrm{R}_1,\mathrm{R}_2}\left(u,s_1,s_2\right)^2\, du\, \mathbf{w}\left(s_1\right)\, ds_1\, \mathbf{w}\left(s_2\right)\, ds_2$$

as well as $$\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} := \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1},\Re,\Re} + \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1},\Im,\Im} + \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1},\Re,\Im} + \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1},\Im,\Re}. \tag{3.52}$$

(i) Suppose that the null hypothesis $\mathcal{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}}$ *holds. Then, one obtains for* $T \to \infty$*:*

$$T\sqrt{b}\,\widehat{\mathbb{D}}_T - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}} \xrightarrow{\mathrm{d}} Z^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} \quad \textit{with} \quad Z^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} \sim \mathcal{N}\left(0, \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}}\right). \tag{3.53}$$

(ii) Consider an arbitrary sequence $(\tau_T)_{T\in\mathbb{N}}$ *of deterministic real numbers with:*

$$\tau_T > 0\; \forall\, T \in \mathbb{N}, \quad \tau_T \overset{T\to\infty}{\longrightarrow} \infty \quad \textit{as well as} \quad \frac{\tau_T}{T\sqrt{b}} \overset{T\to\infty}{\longrightarrow} 0 \tag{3.54}$$

and assume that the alternative $\mathcal{H}^{\mathrm{distr}}_{1,\mathfrak{U}_{0,1}}$ *is valid. Then, it holds:*

$$\lim_{T\to\infty} \mathbb{P}\left(T\sqrt{b}\,\widehat{\mathbb{D}}_T - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}} > \tau_T\right) = 1.$$

Remark 3.24. *Lemma 3.12 and Assumption 3.1* $[\mathbf{WEI}.\mathbf{1}]$ *yield that* $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}$ *as well as* $\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}}$ *are well-defined.*

Commonly, in practical applications, $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}$ and $\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}}$ cannot be calculated. Thus, a dependent wild bootstrap approach is used in the following to estimate p-values that belong to the considered test statistic. Instead, it is also possible to estimate $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}$ as well as $\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}}$ directly by local Newey-West estimators which are similar to those proposed in Subsection 3.2.3 given below and allow to transform the test statistic in an asymptotically standard-normally distributed one, that can be used to estimate p-values. However, L^2-distance-based test statistics (whereby $\widehat{\mathbb{D}}_T$ is one of them according to Proposition 3.11 (i)) are commonly skewly distributed for a fixed number of observations, such that transforming them into approximately standard-normally distributed random variables works tendentiously less well than using a suitable bootstrap procedure which mimics this skewness.

3.2.2. Bootstrap-based estimation of p-values

In order to obtain a test for the problem (3.49) based on Theorem 3.23 and an appropriate bootstrap which yields reasonable test decisions, it is necessary to ensure that the conditional distribution (conditioned on $X_{1,T}, \ldots, X_{T,T}$) of the bootstrap counterpart (for the moment denoted as) $\mathscr{D}^*_T$ of $T\sqrt{b}\,\widehat{\mathbb{D}}_T - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}$ converges for $T \to \infty$ in probability to the distribution of $Z^{\mathrm{distr}}_{\mathfrak{U}_{0,1}}$ under $\mathcal{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}}$ and that

$\mathbb{P}\left(\mathbb{P}^*\left(\mathscr{D}_T^* > K(\epsilon)\right) < \epsilon\right) \overset{T\to\infty}{\longrightarrow} 1$ holds under $\mathcal{H}_{1,\mathfrak{U}_{0,1}}^{\text{distr}}$ for all $\epsilon > 0$ with suitable, deterministic $K(\epsilon) < \infty$. The bootstrap procedures introduced in Subsection 3.1.3 are inappropriate for this purpose because the distributions of $Z_{\mathfrak{U}_{0,1}}$ (recall (3.21)) as well as of $Z_{\mathfrak{U}_{0,1}}^{\text{norm}}$ (see (3.23)) may differ from that of $Z_{\mathfrak{U}_{0,1}}^{\text{distr}}$ (note (3.53)) and these procedures do not respect the expression $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{distr}}$ contained in the present test statistic. Therefore, a new bootstrap counterpart of $\widehat{\mathbb{D}}_T$ is constructed in the following heuristically and theorems which display the asymptotic behaviour of it are stated below (see the Theorems 3.25 and 3.27). To obtain such a bootstrap counterpart, note at first that if $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ is valid, it will hold $1/\lfloor 1/(2b)\rfloor \sum_{k_2=1}^{\lfloor 1/(2b)\rfloor} \varphi\left(u_{k_2}, s\right) = \varphi\left(u_{k_1}, s\right) \,\forall\, k_1 \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^d$ due to $u_k \in \mathfrak{U}_{0,1} \,\forall\, k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$ (recall Definition 3.8 (i)). Hence, Proposition 3.11 (i) and Proposition 2.12 together with (3.11) motivate the following approximation under $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$:

$$\widehat{\mathbb{D}}_T \approx \int\limits_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \left|\widehat{\varphi}\left(u_k, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_k, s\right)\right]\right|^2 \mathbf{w}(s)\, ds. \tag{3.55}$$

Moreover, Proposition 2.14 provides that $\sup_{k\in\{1,\ldots,\lfloor 1/(2b)\rfloor\}} \mathbb{E}\left[\left|\widehat{\varphi}(u_k, s) - \mathbb{E}\left[\widehat{\varphi}(u_k, s)\right]\right|^2\right]$ vanishes asymptotically not just under $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ but also under $\mathcal{H}_{1,\mathfrak{U}_{0,1}}^{\text{distr}}$, which suggests that (3.55) is an appropriate basis for constructing a bootstrap counterpart of $\widehat{\mathbb{D}}_T$. In order to derive such a bootstrap counterpart, recall that the expression (3.37) has already been used as a bootstrap counterpart of $\mathrm{R}\{\widehat{\varphi}\left(u_k, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_k, s\right)\right]\}$ for $\mathrm{R} \in \{\Re, \Im\}$ in Subsection 3.1.3. Hence, (3.55) and the equation $|x|^2 = \Re\{x\}^2 + \Im\{x\}^2 \,\forall\, x \in \mathbb{C}$ motivate the following bootstrap counterpart of $\widehat{\mathbb{D}}_T$ (note that $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ according to Definition 3.3 (i)):

$$\widehat{\mathbb{D}}_{T,\text{Test}}^* := \widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},\text{Test}}^* := \int\limits_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \left| \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right)\left(e^{i\langle s, X_{t,T}\rangle} - \widehat{\varphi}\left(u_k, s\right)\right) W_t^* \right|^2 \mathbf{w}(s)\, ds, \tag{3.56}$$

whereby the process $(W_t^*)_{t\in\mathbb{Z}}$ should originate from Assumption 3.15 $[\mathbf{W}^*]$.
The following theorem presents the asymptotic behaviour of $\widehat{\mathbb{D}}_{T,\text{Test}}^*$, whereby it should be noted that this theorem holds not just under $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ but also under $\mathcal{H}_{1,\mathfrak{U}_{0,1}}^{\text{distr}}$.

Theorem 3.25 (Asymptotic behaviour of $\widehat{\mathbb{D}}_{T,\text{Test}}^*$ - general case).
Let the Assumptions 2.4 $[\mathbf{DM.2}]$*, 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K\&b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled. Moreover, define for all* $\mathrm{R} \in \{\Re, \Im\}$*,* $u \in [0, 1]$*,* $s \in \mathbb{R}^d$ *(whereby* β *and* K^* *originate from Assumption 3.15* $[\mathbf{W}^*]$*):*

$$\begin{aligned}
\sigma_{T,\infty,\mathrm{R}}^*(u, s) &:= \sum_{t=-\infty}^{\infty} K^*\left(\frac{t}{\beta}\right) \operatorname{Cov}\left(\mathrm{R}\left\{e^{i\langle s, \tilde{X}_0(u)\rangle}\right\}, \mathrm{R}\left\{e^{i\langle s, \tilde{X}_t(u)\rangle}\right\}\right), \\
\mathbf{Bias}_{T,\mathfrak{U}_{0,1},\mathrm{R}}^{\text{distr}*} &:= \frac{1}{\sqrt{b}} \int\limits_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\, dz \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \sigma_{T,\infty,\mathrm{R}}^*(u, s)\, du\, \mathbf{w}(s)\, ds \quad \textit{and} \\
\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{distr}*} &:= \mathbf{Bias}_{T,\mathfrak{U}_{0,1},\Re}^{\text{distr}*} + \mathbf{Bias}_{T,\mathfrak{U}_{0,1},\Im}^{\text{distr}*}.
\end{aligned} \tag{3.57}$$

If $\sigma_{\mathfrak{U}_{0,1}}^{\text{distr}} > 0$ *(see (3.52)), it will hold for* $T \to \infty$ *(recall (3.53)):*

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}^*\left(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,\text{Test}}^* - \mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{distr}*} \leqslant x\right) - \mathbb{P}\left(Z_{\mathfrak{U}_{0,1}}^{\text{distr}} \leqslant x\right)\right| \overset{\mathbb{P}}{\longrightarrow} 0. \tag{3.58}$$

If $\sigma_{\mathfrak{U}_{0,1}}^{\text{distr}} = 0$*, one will obtain for* $T \to \infty$ *(see (3.26)):*

$$T\sqrt{b}\,\widehat{\mathbb{D}}_{T,\text{Test}}^* - \mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{distr}*} = o_{\mathbb{P}}^*(1). \tag{3.59}$$

Remark 3.26. *(i) Assumption 3.15* $[\mathbf{W}^*]$ *(iii), Lemma 3.12 and Assumption 3.1* $[\mathbf{WEI.1}]$ *imply that the expression* $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{distr}*}$ *is well-defined.*

(ii) One obtains the following statements similarly to Remark 3.7 (iii):

Realizations of $\widehat{\mathbb{D}}^{*}_{T,\text{Test}}$ will not change if the underlying sample path $(X_{t,T}(\omega))_{t=1}^{T}$ (with $\omega \in \Omega$) is replaced by $(x + X_{t,T}(\omega))_{t=1}^{T}$ for arbitrary $x \in \mathbb{R}^d$. Further, in the case that the weight function $\mathbf{w}$ fulfils $\mathbf{w}(s) = \mathbf{w}(-s)\ \forall\, s \in \mathbb{R}^d$, realizations of $\widehat{\mathbb{D}}^{*}_{T,\text{Test}}$ which are associated with the sample path $(X_{t,T}(\omega))_{t=1}^{T}$ will stay the same if this sample path is replaced by $(-X_{t,T}(\omega))_{t=1}^{T}$.

The Theorems 3.23 and 3.25 show that $T\sqrt{b}\,\widehat{\mathbb{D}}^{*}_{T,\text{Test}} - \mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1}}$ is an appropriate bootstrap counterpart of $T\sqrt{b}\,\widehat{\mathbb{D}}_{T} - \mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}}$. However, the expression $\mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1}}$ is commonly unknown in practical applications and its asymptotic behaviour may differ from that of $\mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}}$ (see (3.51) as well as (3.17)), which is mainly caused by the divergent factor $1/\sqrt{b}$ contained in $\mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1}}$ and $\mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}}$. More specifically, Assumption 3.15 $[\mathbf{W}^*]$ (i) and (iii) imply $|K^*(h/\beta) - 1| \overset{T\to\infty}{\longrightarrow} 0\ \forall\, h \in \mathbb{Z}$ but not necessarily $1/\sqrt{b}\ |K^*(h/\beta) - 1| \overset{T\to\infty}{\longrightarrow} 0\ \forall\, h \in \mathbb{Z}$, such that $\left|\sqrt{b}\,\mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1}} - \sqrt{b}\,\mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}}\right| \overset{T\to\infty}{\longrightarrow} 0$ is valid (due to Lemma 3.12, Assumption 3.1 $[\mathbf{WEI.1}]$ and Lebesgue's dominated convergence theorem), whereas the asymptotic property $\left|\mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1}} - \mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}}\right| \overset{T\to\infty}{\longrightarrow} 0$ does not need to hold.
Additional assumptions which ensure that $\left|\mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1}} - \mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}}\right| \overset{T\to\infty}{\longrightarrow} 0$ and lead to an usable test for (3.49) are given in the following theorem as well as in Remark 3.28 (iv). In contrast to Theorem 3.25, the slightly more restrictive condition $\delta \in (1/4, 1]$ is demanded in the next theorem (compared to $\delta \in (0, 1]$ - as supposed in Assumption 2.2 $[\mathbf{StAp}]$, which underlies Theorem 3.25), whereby δ determines moment and smoothness conditions according to Assumption 2.2 $[\mathbf{StAp}]$. Moreover, in the following theorem, the constraint (3.60) is supposed to hold for the process $(W_t^*)_{t\in\mathbb{Z}}$ of bootstrap random variables. As explained in Remark 3.28 (iii) given below, the condition $\delta \in (1/4, 1]$ is necessary in the next theorem to ensure the existence of a sequence of parameters β that fulfils not only Assumption 3.15 $[\mathbf{W}^*]$ (i) but also (3.60). Further, under the constraint $\delta \in (1/4, 1]$, Remark 3.28 (iii) also constructs processes $(W_t^*)_{t\in\mathbb{Z}}$ of bootstrap random variables and belonging sequences of parameters β for which the assumptions of Theorem 3.27 hold.

Theorem 3.27 (Asymptotic behaviour of $\widehat{\mathbb{D}}^{*}_{T,\text{Test}}$ - for $\delta \in (1/4, 1]$)**.**
Suppose that the Assumptions 2.4 $[\mathbf{DM.2}]$, 3.1 $[\mathbf{WEI.1}]$, 2.8 $[\mathbf{K\&b.1}]$ and 3.15 $[\mathbf{W}^*]$ hold. In addition, let $\delta \in (1/4, 1]$ be fulfilled, whereby δ originates from Assumption 2.2 $[\mathbf{StAp}]$. Moreover, assume for β and K^* which are introduced in Assumption 3.15 $[\mathbf{W}^*]$:

$$\beta\sqrt{b} \overset{T\to\infty}{\longrightarrow} \infty \qquad \text{and} \qquad \mathscr{S}^* := \sup_{x\in\mathbb{R}\setminus\{0\}} \frac{|1 - K^*(x)|}{|x|} < \infty. \tag{3.60}$$

If $\sigma^{\text{distr}}_{\mathfrak{U}_{0,1}} > 0$ (see (3.52)), one will obtain for $T \to \infty$ (recall (3.56), (3.51) and (3.53)):

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}^*\left(T\sqrt{b}\,\widehat{\mathbb{D}}^{*}_{T,\text{Test}} - \mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}} \leqslant x\right) - \mathbb{P}\left(Z^{\text{distr}}_{\mathfrak{U}_{0,1}} \leqslant x\right)\right| \overset{\mathbb{P}}{\longrightarrow} 0. \tag{3.61}$$

If $\sigma^{\text{distr}}_{\mathfrak{U}_{0,1}} = 0$, it will hold for $T \to \infty$ (see (3.26)):

$$T\sqrt{b}\,\widehat{\mathbb{D}}^{*}_{T,\text{Test}} - \mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}} = o^*_{\mathbb{P}}(1). \tag{3.62}$$

Remark 3.28. *(i) Theorem 3.27 provides under $\mathcal{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ and under $\mathcal{H}^{\text{distr}}_{1,\mathfrak{U}_{0,1}}$ that the conditional distribution (conditioned on $X_{1,T}, \ldots, X_{T,T}$) of $T\sqrt{b}\,\widehat{\mathbb{D}}^{*}_{T,\text{Test}} - \mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}}$ converges in probability to the distribution of $Z^{\text{distr}}_{\mathfrak{U}_{0,1}}$. Hence, a test for the problem (3.49) which is based on the Theorems 3.23 and 3.27 is consistent.*

(ii) If the assumptions of Theorem 3.27 with $\sigma^{\text{distr}}_{\mathfrak{U}_{0,1}} > 0$ hold and $\mathcal{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ is valid, one will obtain for $T \to \infty$ from (3.61), Theorem 3.23 (i) and Polya's theorem:

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}^*\left(T\sqrt{b}\,\widehat{\mathbb{D}}^{*}_{T,\text{Test}} - \mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}} \leqslant x\right) - \mathbb{P}\left(T\sqrt{b}\,\widehat{\mathbb{D}}_{T} - \mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}} \leqslant x\right)\right| \overset{\mathbb{P}}{\longrightarrow} 0.$$

(iii) The condition $\delta > 1/4$ is necessary in Theorem 3.27 to ensure the existence of a sequence of parameters β for which the demanded properties hold because (3.60), Assumption 3.15 $[\mathbf{W}^*]$ (i)

and Assumption 2.8 [$\mathbf{K}\&\mathbf{b.1}$] (ii) yield $b^{-\frac{1}{2}} \ll \beta \ll Tb^2 \ll b^{-2\delta}$, which implies $\delta > 1/4$. This and Assumption 2.2 [$\mathbf{StAp}$] (which demands $\delta \in (0,1]$) explain why $\delta \in (1/4, 1]$ is supposed in Theorem 3.27.

Further, if $\delta \in (1/4, 1]$, it is possible to fulfil the Assumptions 2.8 [$\mathbf{K}\&\mathbf{b.1}$] (ii), 3.15 [$\mathbf{W}^*$] and (3.60), as demonstrated in the following. For instance, Assumption 2.8 [$\mathbf{K}\&\mathbf{b.1}$] (ii) holds for $b := b_T := 1/4 \mathbb{1}_{\{T=1\}} + \min\left\{1/2 \cdot T^{-1/2.5}\sqrt{\ln(T)}, 1/4\right\} \mathbb{1}_{\{T \geqslant 2\}}\ \forall\, T \in \mathbb{N}$ and, for this choice, $\beta := \beta_T := b^{-1/2} \ln(T)\ \forall\, T \in \mathbb{N}$ ensures that the processes of bootstrap random variables which are defined in Example 3.16 fulfil Assumption 3.15 [$\mathbf{W}^*$] as well as (3.60) with $\mathscr{S}^* = 1$ (the latter can be easily shown for Example 3.16 (i) by using (C.43) in the appendix and for Example 3.16 (ii) by considering (C.46)).

(iv) If $\{\widetilde{X}_t(u)\}$ is a sequence of independent random variables for all $u \in [0,1]$, one will obtain $\mathrm{Cov}\left(\mathrm{R}\left\{e^{i\langle s, \widetilde{X}_0(u)\rangle}\right\}, \mathrm{R}\left\{e^{i\langle s, \widetilde{X}_t(u)\rangle}\right\}\right) = 0\ \forall\, \mathrm{R} \in \{\Re, \Im\}$, $s \in \mathbb{R}^d$, $u \in [0,1]$, $t \in \mathbb{Z}\backslash\{0\}$, such that $K^*(0) = 1$ (as supposed in Assumption 3.15 [$\mathbf{W}^*$] (iii)) will imply $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}} = \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}}$ (recall (3.51), (3.17) as well as (3.57)). In this case, the condition (3.60) is omittable in Theorem 3.27 and the statement of Theorem 3.27 is valid for all $\delta \in (0,1]$.

The Theorems 3.23 and 3.27 provide a consistent level-alpha test for the problem (3.49). The following algorithm describes how this test can be implemented in practical applications and Remark 3.29 (i) given below explains why it is justified that this algorithm avoids to calculate or estimate $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}$.

Algorithm **TEST.MDCI.1**

Inputs: $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ with $\mathfrak{U}_0 < \mathfrak{U}_1$; significance level $\alpha \in (0,1)$; $T, N \in \mathbb{N}$; a sample path $(X_{t,T}(\omega))_{t=1}^T$ (for an $\omega \in \Omega$); a kernel K for which Assumption 2.8 [$\mathbf{K}\&\mathbf{b.1}$] (i) holds; a bandwidth $b \in (0, 1/2)$; a weight function $\mathbf{w}$ that fulfils Assumption 3.1 [$\mathbf{WEI.1}$]; a parameter $\beta > 0$ and an associated process $(W_t^*)_{t\in\mathbb{Z}}$ which satisfies Assumption 3.15 [$\mathbf{W}^*$] as well as the condition $\mathscr{S}^* < \infty$ contained in (3.60);

1: Determine the realization of $\widehat{\mathbb{D}}_T$ that belongs to the sample path $(X_{t,T}(\omega))_{t=1}^T$;
2: Independently, for n in $1:N$ do
3: Generate a sample path of $(W_t^*)_{t=1}^T$;
4: Calculate the associated realization of $\widehat{\mathbb{D}}^*_{T,\mathrm{Test}}$;
5: end for
6: Compute a realization of the empirical distribution function of $\widehat{\mathbb{D}}^*_{T,\mathrm{Test}}$ by using the calculated realizations of $\widehat{\mathbb{D}}^*_{T,\mathrm{Test}}$ and call this realization of the empirical distribution function $\widehat{F}^*_{T,N}$;
7: Reject $\mathcal{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}}$ if $1 - \widehat{F}^*_{T,N}\big(\widehat{\mathbb{D}}_T(\omega)\big) < \alpha$;

Remark 3.29. *(i) At a first glance, one might think that the Theorems 3.23 and 3.27 require to replace in Algorithm **TEST.MDCI.1** all $\widehat{\mathbb{D}}_T$ by $T\sqrt{b}\,\widehat{\mathbb{D}}_T - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}$ and all $\widehat{\mathbb{D}}^*_{T,\mathrm{Test}}$ by $T\sqrt{b}\,\widehat{\mathbb{D}}^*_{T,\mathrm{Test}} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}$. However, if $\widehat{\mathbb{D}}^{*[n]}_{T,\mathrm{Test}}$ denotes the realization of $\widehat{\mathbb{D}}^*_{T,\mathrm{Test}}$ that is calculated in the fourth step of Algorithm **TEST.MDCI.1** and belongs to the $n \in \{1, \ldots, N\}$ which was selected before in the second step of this algorithm, it will hold:*

$$\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{\left\{T\sqrt{b}\,\widehat{\mathbb{D}}^{*[n]}_{T,\mathrm{Test}} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}} \leqslant T\sqrt{b}\,\widehat{\mathbb{D}}_T(\omega) - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}\right\}} = \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{\left\{\widehat{\mathbb{D}}^{*[n]}_{T,\mathrm{Test}} \leqslant \widehat{\mathbb{D}}_T(\omega)\right\}}, \tag{3.63}$$

*such that Algorithm **TEST.MDCI.1** is designed appropriately and avoids the issue that $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}$ is commonly unknown in practical applications.*

(ii) The asymptotic properties of b and β which are demanded in the Assumptions 2.8 [$\mathbf{K}\&\mathbf{b.1}$] (ii), (3.60) as well as 3.15 [$\mathbf{W}^*$] (i) can be regarded as rough guidances on selecting b and β for a given $T \in \mathbb{N}$. Thereby, it is worth mentioning that such guidances are obtainable without knowing $\delta \in (1/4, 1]$ (according to Remark 3.28 (iii)).

In the following subsection, Algorithm **TEST.MDCI.1** is manipulated in such a manner that neither assuming $\delta > 1/4$ and (3.60) nor demanding that $\{\widetilde{X}_t(u)\}$ is a sequence of independent random variables for all $u \in [0,1]$ is necessary to justify the resulting test procedure. However, compared to Algorithm **TEST.MDCI.1**, applying the test introduced in the next subsection is accompanied by (slightly) higher computational costs and requires to choose a larger number of tuning parameters.

3.2.3. Modification of the test statistic based on local Newey-West estimation

To obtain an adaption of Algorithm **TEST.MDCI.1** that is suitable for testing (3.49) with arbitrary but fixed $\delta \in (0,1]$, define at first under the Assumptions 2.4 $[\mathbf{DM.2}]$, 3.1 $[\mathbf{WEI.1}]$, 2.8 $[\mathbf{K\&b.1}]$ and 3.15 $[\mathbf{W}^*]$ (recall that $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ according to Definition 3.3 (i) and see also (3.57) as well as (3.51)):

$$\mathbf{Bias}^{\mathrm{error}}_{T,\mathfrak{U}_{0,1}} := \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}, \tag{3.64}$$

which allows to rewrite the test statistic proposed in Theorem 3.23 as follows:

$$T\sqrt{b}\,\widehat{\mathbb{D}}_T - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}} = T\sqrt{b}\left(\widehat{\mathbb{D}}_T + \frac{1}{T\sqrt{b}}\mathbf{Bias}^{\mathrm{error}}_{T,\mathfrak{U}_{0,1}}\right) - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}}. \tag{3.65}$$

Thereby, the factor $T\sqrt{b}$ and the subtrahend $\mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}}$ contained in (3.65) are also used in the bootstrap statistic introduced in Theorem 3.25. Hence, in (3.65), the original test statistic $T\sqrt{b}\,\widehat{\mathbb{D}}_T - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}$ is rewritten in such a manner that it matches to the bootstrap statistic $T\sqrt{b}\,\widehat{\mathbb{D}}^*_{T,\mathrm{Test}} - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}}$. However, $\mathbf{Bias}^{\mathrm{error}}_{T,\mathfrak{U}_{0,1}}$ is commonly unknown in practical applications. Therefore, it is estimated in the present subsection, whereby Newey–West-type estimators are used because $\mathbf{Bias}^{\mathrm{error}}_{T,\mathfrak{U}_{0,1}}$ contains long-run variances. Originally, Newey–West estimators were proposed in [59, Newey and West (1987)] and often investigated after their first introduction (regard e. g. [2, Andrews (1991)] as well as [46, Keener et al. (1991)]). In the following, the classical approach of Newey–West estimation is manipulated in order to handle the present locally stationary framework that underlies $\mathbf{Bias}^{\mathrm{error}}_{T,\mathfrak{U}_{0,1}}$, such that the resulting estimation method is called local Newey-West estimation. For this purpose, bandwidths and kernels which are introduced in the next assumption will be considered. To ensure that they are not confused with those originating from Assumption 2.8 $[\mathbf{K\&b.1}]$, they are called Newey-West-estimation-kernels (abbr. NW-kernels) and Newey-West-estimation-bandwidths (abbr. NW-bandwidths) in the present work.

Assumption 3.30 [NW] (NW-bandwidth and NW-kernel for the estimation of $\mathbf{Bias}^{\mathrm{error}}_{T,\mathfrak{U}_{0,1}}$**).**
Suppose that the Assumptions 2.4 $[\mathbf{DM.2}]$ *as well as 2.8* $[\mathbf{K\&b.1}]$ *(ii) hold and that a fixed* $C_\delta \in [0,\infty)$ *exists for which* $Tb^{2+\delta} \overset{T\to\infty}{\longrightarrow} C_\delta$*, whereby* $\delta \in (0,1]$ *originates from Assumption 2.2* $[\mathbf{StAp}]$*. The sequence of NW-bandwidths* $(\mathbf{B}_T)_{T\in\mathbb{N}} \subseteq \mathbb{N}$ *and the NW-kernel* $\mathbb{K}_{\mathbf{NW}} : \mathbb{R} \to \mathbb{R}$ *should fulfil the following properties:*

$$\frac{\mathbf{B}_T}{\sqrt{Tb^2}} \overset{T\to\infty}{\longrightarrow} 0, \quad \sup_{x\in[-1,1]} |\mathbb{K}_{\mathbf{NW}}(x)| < \infty, \quad \mathbb{K}_{\mathbf{NW}}(0) = 1, \quad \mathbb{K}_{\mathbf{NW}}(x) = 0\ \forall\, x \in \mathbb{R} : |x| > 1$$

$$\textit{and } \exists\, \eta > \frac{1}{\delta} : \sum_{l=1}^{\infty} \Delta_l l^{1+\eta} < \infty, \quad \mathscr{S}_\eta := \sup_{x\in\mathbb{R}\setminus\{0\}} \frac{|1 - \mathbb{K}_{\mathbf{NW}}(x)|}{|x|^\eta} < \infty \ \textit{ as well as }\ \mathbf{B}_T b^{\frac{1}{2\eta}} \overset{T\to\infty}{\longrightarrow} \infty.$$

Remark 3.31. *(i) The conditions* $\mathbf{B}_T/\sqrt{Tb^2} \overset{T\to\infty}{\longrightarrow} 0$ *and* $\mathbf{B}_T b^{1/(2\eta)} \overset{T\to\infty}{\longrightarrow} \infty$ *given in Assumption 3.30* $[\mathbf{NW}]$ *require* $b^{-1/(2\eta)}/\sqrt{Tb^2} \overset{T\to\infty}{\longrightarrow} 0$*, whereby the latter is equivalent to* $Tb^{2+1/\eta} \overset{T\to\infty}{\longrightarrow} \infty$*. In addition, Assumption 2.8* $[\mathbf{K\&b.1}]$ *(ii) demands* $b \overset{T\to\infty}{\longrightarrow} 0$ *and Assumption 3.30* $[\mathbf{NW}]$ *supposes* $Tb^{2+\delta} \to C_\delta$ *for a fixed* $C_\delta \in [0,\infty)$*, such that* $\eta > 1/\delta$ *is necessary to ensure* $Tb^{2+1/\eta} \overset{T\to\infty}{\longrightarrow} \infty$*.*
Moreover, if $C_\delta \in (0,\infty)$ *is chosen in Assumption 3.30* $[\mathbf{NW}]$ *(which is no contradiction to Assumption 2.8* $[\mathbf{K\&b.1}]$ *(ii)), each sequence of NW-bandwidths* $(\mathbf{B}_T)_{T\in\mathbb{N}} \subseteq \mathbb{N}$ *for which* $b^{-1/(2\eta)} \ll \mathbf{B}_T \ll b^{-\delta/2}$ *with* $\eta > 1/\delta$ *holds will fulfil Assumption 3.30* $[\mathbf{NW}]$*, whereby* $\eta > 1/\delta$ *provides that* $b^{-1/(2\eta)} \ll \mathbf{B}_T$ *is no contradiction to* $\mathbf{B}_T \ll b^{-\delta/2}$*. This shows that sequences of parameters* $\mathbf{B}_T$ *that satisfy Assumption 3.30* $[\mathbf{NW}]$ *exist.*
However, $b^{-1/(2\eta)} \ll \mathbf{B}_T \ll b^{-\delta/2}$ *with* $\eta > 1/\delta$ *implies that less suitable sequences of NW-bandwidths exist for smaller* $\delta \in (0,1]$ *than larger ones.*

(ii) In the case $\delta \in (0,1)$*, assuming* $\eta \leqslant 2/\delta - 1$ *provides that* $\sum_{l=1}^{\infty} \Delta_l l^{1+\eta} < \infty$ *is not more restrictive*

than $\sum_{l=1}^{\infty} \Delta_l l^{2/\delta} < \infty$ (as supposed in Assumption 2.4 $[\mathbf{DM.2}]$), whereby $1/\delta < 2/\delta - 1$ holds for all $\delta \in (0,1)$, such that $\eta \leqslant 2/\delta - 1$ is no contradiction to $\eta > 1/\delta$.
Instead, if $\delta = 1$, the assumption $\sum_{l=1}^{\infty} \Delta_l l^{1+\eta} < \infty$ with $\eta > 1/\delta$ will be (slightly) more restrictive than $\sum_{l=1}^{\infty} \Delta_l l^{2/\delta} < \infty$.

(iii) Less NW-kernels fulfil Assumption 3.30 $[\mathbf{NW}]$ for a smaller $\delta \in (0,1]$ than a larger one because $\eta > 1/\delta$ and $\mathscr{S}_\eta < \infty$ require that the NW-kernel has to be smoother at zero for a smaller $\delta \in (0,1]$. (The supposition that $\mathbb{K}_{\mathbf{NW}}$ owns a bounded support can be weakened by assuming that $\mathbb{K}_{\mathbf{NW}}(x)$ approaches zero sufficiently fast for growing absolute values of x. However, this generalization does not mitigate the very restrictive condition $\eta > 1/\delta$ and makes some proofs given in the appendix more difficult to understand, such that it is not considered in the present work.)

Next, some NW-kernels are stated for which Assumption 3.30 $[\mathbf{NW}]$ holds for all $\eta > 0$ (and all $\delta > 0$, respectively), such that they can be used if $\delta \in (0,1]$ is unknown (which is commonly the case in practical applications). Since showing this property of the NW-kernels given below is very straightforward, the belonging proof is omitted in the appendix.

Example 3.32. *The following NW-kernels fulfil Assumption 3.30 $[\mathbf{NW}]$ for all $\eta > 0$:*

(i) Truncated kernel:

$$\mathbb{K}_{\mathbf{NW}}^{\mathbf{Trun}} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1, & \text{for } x \in [-1,1] \\ 0, & \text{otherwise} \end{cases}.$$

(ii) Trapezoid kernel:

$$\mathbb{K}_{\mathbf{NW},a}^{\mathbf{Trap}} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1, & \text{for } x \in [-a,a] \\ \frac{1}{1-a} - \frac{1}{1-a}|x|, & \text{for } x \in [-1,1] \backslash [-a,a] \\ 0, & \text{otherwise} \end{cases} \qquad \forall\, a \in (0,1).$$

Many NW-kernels which have already been used for Newey-West estimation in the last decades, like the Bartlett, Parzen, Tukey-Hanning or quadratic spectral kernel (their definitions can be found in, e. g., [2, Andrews (1991), p. 821]), do not fulfil Assumption 3.30 $[\mathbf{NW}]$ for all $\delta \in (0,1]$.
Further, (in dependence of the number of given observations T) to infinity growing NW-bandwidths of order $o(T^{1/4})$ (as demanded in [59, Newey and West (1987)]) or $o\left(\sqrt{T}\right)$ (as supposed in e. g. [46, Keener et al. (1991)] and [49, Kool (1988)]) or $o(T)$ (as considered in [2, Andrews (1991)]) are common for Newey-West estimation. Compared to this, Assumption 3.30 $[\mathbf{NW}]$ demands more restrictive conditions with respect to the growth rate because $\mathbf{B}_T$ has to increase asymptotically faster than $b^{-1/(2\eta)}$ and slower than $\sqrt{Tb^2}$, whereby Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) yields $\sqrt{Tb^2} = o(T^{1/4})$.
The fact that stronger assumptions with respect to the NW-kernel and NW-bandwidth are supposed to estimate $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}$ can be explained by the following considerations. The publications cited above do not consider a locally stationary setting, which is captured in the following by combining Newey-West estimation with the kernel and bandwidth introduced in Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$, which motivates why the asymptotic behaviour of $\mathbf{B}_T$ supposed in Assumption 3.30 $[\mathbf{NW}]$ also depends on b. In addition, for Newey-West estimation, it is commonly assumed that the underlying random variables are centered, whereas the expressions $\mathrm{R}\left\{e^{i\left\langle s, \widetilde{X}_t(u)\right\rangle}\right\}$ contained in $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}$ (recall (3.64), (3.57), (3.51) as well as (3.17)) are not centered and their expectations have to be estimated. Beside this, $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}$ may increase with the rate $1/\sqrt{b}$ for $T \to \infty$, whereas the above-mentioned publications consider Newey-West estimators for bounded expressions. Moreover, very weak moment conditions (which are determined by $\delta \in (0,1]$) are assumed in the present thesis.
Further, it should be noted that a method for long-run variance estimation in a locally stationary framework is briefly described in Subsection 4.1.1 in [15, Dahlhaus and Richter (2023), p.1145]. However, this method just considers the case that the belonging NW-kernel is the truncated kernel. In contrast, in [76, Vogt and Dette (2015), p. 728 et seq.], a long-run variance estimator is proposed under local stationarity that is adapted to the variance of a certain CUSUM-type statistic, such that it is more similar to the

traditional approach of Newey-West estimation. In particular, in [76, Vogt and Dette (2015)], localizing kernels and bandwidths are just used to estimate expectations of the random variables that are contained in this long-run variance.

In the next definition, two estimators $\widehat{\mathbf{Bias}}_T^{\text{error}}$ and $\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}$ for $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}$ are introduced, which are based on Newey-West estimation in combination with Assumption 3.30 $[\mathbf{NW}]$. Thereby, the construction of $\widehat{\mathbf{Bias}}_T^{\text{error}}$ is motivated by (2.3) in [2, Andrews (1991), p. 820] as well as Subsection 4.1.1 in [15, Dahlhaus and Richter (2023), p.1145] and, therefore, more intuitive than that of $\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}$. However, calculating realizations of $\widehat{\mathbf{Bias}}_T^{\text{error}}$ is computationally more expensive than realizations of $\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}$.

Definition 3.33 (Estimators for $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}$)**.**
Let the Assumptions 3.30 $[\mathbf{NW}]$ *(which includes Assumption 2.4* $[\mathbf{DM.2}]$*), 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K\&b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled. Moreover, define for all* $s \in \mathbb{R}^d$*,* $t \in \{1, \ldots, T\}$*,* $T \in \mathbb{N}\backslash\{1\}$*,* $u \in [0,1]$*,* $h \in \{-\lfloor T/2 \rfloor + 1, \ldots, \lfloor T/2 \rfloor - 1\}$*,* $\mathrm{R} \in \{\Re, \Im\}$ *(note that* $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ *according to Definition 3.3 (i)):*

$$\left(e^{i\left\langle s,X_{t,T}\right\rangle}\right)^{\widehat{c}(u)} := e^{i\left\langle s,X_{t,T}\right\rangle} - \frac{1}{T}\sum_{j=1}^{T} K_b\left(\frac{j}{T} - u\right) e^{i\left\langle s,X_{j,T}\right\rangle},$$

$$\widehat{\sigma}_{h,T,\mathrm{R}}\,(u,s) := \widehat{\sigma}_{h,T,\mathfrak{U}_{0,1},\mathrm{R}}\,(u,s)$$

$$:= \begin{cases} \frac{1}{T}\sum\limits_{t=1}^{T-h} K_b\left(\frac{t}{T} - u\right) \mathrm{R}\left\{e^{i\left\langle s,X_{t,T}\right\rangle}\right\}^{\widehat{c}(u)} \mathrm{R}\left\{e^{i\left\langle s,X_{t+h,T}\right\rangle}\right\}^{\widehat{c}(u)}, & \text{for } h \geqslant 0 \\ \frac{1}{T}\sum\limits_{t=1-h}^{T} K_b\left(\frac{t}{T} - u\right) \mathrm{R}\left\{e^{i\left\langle s,X_{t,T}\right\rangle}\right\}^{\widehat{c}(u)} \mathrm{R}\left\{e^{i\left\langle s,X_{t+h,T}\right\rangle}\right\}^{\widehat{c}(u)}, & \text{for } h < 0 \end{cases},$$

$$\widehat{\sigma}_{T,\mathrm{R}}^{\text{error}}\,(u,s) := \widehat{\sigma}_{T,\mathfrak{U}_{0,1},\mathrm{R}}^{\text{error}}\,(u,s) := \sum_{h=-\lfloor T/2 \rfloor+1}^{\lfloor T/2 \rfloor-1} \left(K^*\left(\frac{h}{\beta}\right) - 1\right) \mathbb{K}_{\mathbf{NW}}\left(\frac{h}{\mathbf{B}_T}\right) \widehat{\sigma}_{h,T,\mathrm{R}}\,(u,s) \tag{3.66}$$

and (recall that u_k *originates from Definition 3.8 (i)):*

$$\widehat{\mathbf{Bias}}_{T,\mathrm{R}}^{\text{error}} := \widehat{\mathbf{Bias}}_{T,\mathfrak{U}_{0,1},\mathrm{R}}^{\text{error}} := \frac{1}{\sqrt{b}} \int\limits_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz \int\limits_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \widehat{\sigma}_{T,\mathrm{R}}^{\text{error}}\,(u_k, s)\ \mathbf{w}(s)\,ds$$

$$\text{as well as}\quad \widehat{\mathbf{Bias}}_T^{\text{error}} := \widehat{\mathbf{Bias}}_{T,\mathfrak{U}_{0,1}}^{\text{error}} := \widehat{\mathbf{Bias}}_{T,\Re}^{\text{error}} + \widehat{\mathbf{Bias}}_{T,\Im}^{\text{error}}. \tag{3.67}$$

Then, $\widehat{\mathbf{Bias}}_T^{\text{error}}$ *is called local Newey-West estimator for* $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}$ *(see (3.64)). Further, if in addition* $T \geqslant 2\mathbf{B}_T + 1$*, define the computationally efficient local Newey-West estimator for* $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}$ *as:*

$$\begin{aligned} \widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}} &:= \widehat{\widehat{\mathbf{Bias}}}_{T,\mathfrak{U}_{0,1}}^{\text{error}} \\ &:= \frac{1}{\sqrt{b}} \int\limits_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz \cdot \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \frac{1}{T - 2\mathbf{B}_T} \sum_{t=1+\mathbf{B}_T}^{T-\mathbf{B}_T} K_b\left(\frac{t}{T} - u_k\right) \\ &\quad \cdot \sum_{h=-\mathbf{B}_T}^{\mathbf{B}_T} \left(K^*\left(\frac{h}{\beta}\right) - 1\right) \mathbb{K}_{\mathbf{NW}}\left(\frac{h}{\mathbf{B}_T}\right) \Re\left\{ \int\limits_{\mathbb{R}^d} \left(e^{i\left\langle s,X_{t,T}\right\rangle}\right)^{\widehat{c}(u_k)} \overline{\left(e^{i\left\langle s,X_{t+h,T}\right\rangle}\right)^{\widehat{c}(u_k)}}\, \mathbf{w}(s)\,ds \right\}. \end{aligned} \tag{3.68}$$

Remark 3.34. *Since* $\left(e^{i\langle s,x+X_{t,T}\rangle}\right)^{\widehat{c}(u_k)} = e^{i\langle s,x\rangle}\left(e^{i\langle s,X_{t,T}\rangle}\right)^{\widehat{c}(u_k)}$*,* $\Re\left\{e^{i\langle s,x\rangle}\right\}\Re\left\{e^{i\langle s,x\rangle}\right\} + \Im\left\{e^{i\langle s,x\rangle}\right\} \cdot \Im\left\{e^{i\langle s,x\rangle}\right\} = 1$ *as well as* $e^{i\langle s,x\rangle}\overline{e^{i\langle s,x\rangle}} = 1$ *hold for all* $s \in \mathbb{R}^d$*,* $t \in \{1, \ldots, T\}$*,* $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$*,* $x \in \mathbb{R}^d$*, the realizations of* $\widehat{\mathbf{Bias}}_T^{\text{error}}$ *and* $\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}$ *which are associated with a sample path* $(X_{t,T}(\omega))_{t=1}^{T}$ *(with* $\omega \in \Omega$*) will not change if this sample path is replaced by* $(x + X_{t,T}(\omega))_{t=1}^{T}$ *with arbitrary* $x \in \mathbb{R}^d$*. Moreover, in the case that the weight function* $\mathbf{w}$ *fulfils* $\mathbf{w}(s) = \mathbf{w}(-s)\ \forall\, s \in \mathbb{R}^d$*, it follows from* $\left(e^{i\langle s,-X_{t,T}\rangle}\right)^{\widehat{c}(u_k)} = \left(e^{i\langle -s,X_{t,T}\rangle}\right)^{\widehat{c}(u_k)}\ \forall\, s \in \mathbb{R}^d$*,* $t \in \{1, \ldots, T\}$*,* $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$ *and integration* *by substitution that realizations of* $\widehat{\mathbf{Bias}_T}^{\text{error}}$ *as well as* $\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}$ *will stay the same if the underlying process* $(X_{t,T})_{t=1}^T$ *is replaced by* $(-X_{t,T})_{t=1}^T$.

The next lemma shows that $\widehat{\mathbf{Bias}_T}^{\text{error}}$ and $\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}$ are useful estimators for $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}$.

Lemma 3.35 (L^1-convergence of the estimators for $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}$)**.**
Suppose that the Assumptions 3.30 $[\mathbf{NW}]$*, 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *hold. Then, one obtains for* $T \to \infty$*:*

(i) $\mathbb{E}\left[\left|\widehat{\mathbf{Bias}_T}^{\text{error}} - \mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}\right|\right] = o(1)$.

(ii) $\mathbb{E}\left[\left|\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}} - \mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}\right|\right] = o(1)$.

The following theorem can be used together with Theorem 3.25 to investigate the test problem (3.49) under the assumption that $\delta \in (0,1]$ is arbitrary but fixed.

Theorem 3.36 (Asymptotic behaviour of the Newey-West-estimation-based test statistic)**.**
Let the Assumptions 3.30 $[\mathbf{NW}]$*, 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled.*

(i) If $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ *(recall (3.49)) is valid, it will hold for* $T \to \infty$ *and all* $\mathbf{Y}_T \in \left\{\widehat{\mathbf{Bias}_T}^{\text{error}}, \widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}\right\}$ *(see Definition 3.8 (i), (3.57) as well as (3.53)):*

$$T\sqrt{b}\left(\widehat{\mathbb{D}}_T + \frac{1}{T\sqrt{b}}\mathbf{Y}_T\right) - \mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{distr}*} \xrightarrow{\text{d}} Z_{\mathfrak{U}_{0,1}}^{\text{distr}}.$$

(ii) Suppose that $(\tau_T)_{T\in\mathbb{N}}$ *is an arbitrary sequence of deterministic real numbers that fulfils (3.54). If* $\mathcal{H}_{1,\mathfrak{U}_{0,1}}^{\text{distr}}$ *(recall (3.49)) holds, one will obtain for all* $\mathbf{Y}_T \in \left\{\widehat{\mathbf{Bias}_T}^{\text{error}}, \widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}\right\}$*:*

$$\lim_{T\to\infty} \mathbb{P}\left(T\sqrt{b}\left(\widehat{\mathbb{D}}_T + \frac{1}{T\sqrt{b}}\mathbf{Y}_T\right) - \mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{distr}*} > \tau_T\right) = 1.$$

Remark 3.37. *If the assumptions of Theorem 3.36 with* $\sigma_{\mathfrak{U}_{0,1}}^{\text{distr}} > 0$ *(see (3.52)) are valid and* $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ *holds, one will obtain for* $T \to \infty$ *as well as all* $\mathbf{Y}_T \in \left\{\widehat{\mathbf{Bias}_T}^{\text{error}}, \widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}\right\}$ *from (3.58), Theorem 3.36 (i) and Polya's theorem:*

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}^*\left(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,\text{Test}}^* - \mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{distr}*} \leqslant x\right) - \mathbb{P}\left(T\sqrt{b}\left(\widehat{\mathbb{D}}_T + \frac{1}{T\sqrt{b}}\mathbf{Y}_T\right) - \mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{distr}*} \leqslant x\right)\right| \xrightarrow{\mathbb{P}} 0.$$

The Theorems 3.36 and 3.25 provide a consistent level-alpha test for (3.49) in the case that $\delta \in (0,1]$ is arbitrary but fixed, whereby δ originates from Assumption 2.2 $[\mathbf{StAp}]$. The following algorithm describes how this test can be implemented. To shorten the notation, this algorithm is just formulated based on the estimator $\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}$. The alternative test procedure that takes $\widehat{\mathbf{Bias}_T}^{\text{error}}$ into account can be obtained by replacing $\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}$ by $\widehat{\mathbf{Bias}_T}^{\text{error}}$ in the algorithm given below. However, this alternative test is accompanied by higher computational costs, such that belonging numerical examples are omitted in the present publication, whereas the next algorithm is used in simulation studies and practical examples contained in Section 3.4 as well as Chapter 5. Further, calculating or estimating $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{distr}*}$ is avoided in this algorithm, which can be justified similarly to Remark 3.29 (i) by using the Theorems 3.36 and 3.25. (In particular, note also that Algorithm **TEST.MDCI.1** and the following one are based on slightly different inputs and the algorithm given below demands to calculate a realization of $\widehat{\mathbb{D}}_T + 1/(T\sqrt{b})\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}$ instead of $\widehat{\mathbb{D}}_T$ - as in Algorithm **TEST.MDCI.1**.)

Algorithm **TEST.MDCI.2**

Inputs: $\mathfrak{U}_0, \mathfrak{U}_1 \in [0,1]$ with $\mathfrak{U}_0 < \mathfrak{U}_1$; significance level $\alpha \in (0,1)$; $T, N \in \mathbb{N}$; a sample path $(X_{t,T}(\omega))_{t=1}^T$ (for an $\omega \in \Omega$); a kernel K for which Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) holds; a bandwidth $b \in (0, 1/2)$; a weight function $\mathbf{w}$ that fulfils Assumption 3.1 $[\mathbf{WEI.1}]$; a parameter $\beta > 0$ and an associated process $(W_t^*)_{t\in\mathbb{Z}}$ which satisfies Assumption 3.15 $[\mathbf{W}^*]$; an NW-bandwidth $\mathbf{B}_T \in \mathbb{N}$; an NW-kernel $\mathbb{K}_{\mathbf{NW}}$ for which Assumption 3.30 $[\mathbf{NW}]$ holds;

Determine the realization of $\widehat{\mathbb{D}}_T + 1/(T\sqrt{b})\widehat{\mathbf{Bias}}_T^{\mathrm{error}}$ that belongs to the sample path $(X_{t,T}(\omega))_{t=1}^T$;
Independently, for n in $1 : N$ do
    Generate a sample path of $(W_t^*)_{t=1}^T$;
    Calculate the associated realization of $\widehat{\mathbb{D}}_{T,\mathrm{Test}}^*$;
end for
Compute a realization of the empirical distribution function of $\widehat{\mathbb{D}}_{T,\mathrm{Test}}^*$ by using the calculated realizations of $\widehat{\mathbb{D}}_{T,\mathrm{Test}}^*$ and call this realization of the empirical distribution function $\widehat{F}_{T,N}^*$;
Reject $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\mathrm{distr}}$ if $1 - \widehat{F}_{T,N}^*\big(\widehat{\mathbb{D}}_T(\omega) + 1/(T\sqrt{b})\widehat{\mathbf{Bias}}_T^{\mathrm{error}}(\omega)\big) < \alpha$;

Remark 3.38. *(i) Remark 3.31 (i) ensures that sequences of parameters b and $\mathbf{B}_T$ which fulfil Assumption 3.30 $[\mathbf{NW}]$ exist, but constructing them without knowing $\delta \in (0,1]$ is impossible, such that no useful guidances on selecting b and $\mathbf{B}_T$ can be derived from Assumption 3.30 $[\mathbf{NW}]$. The belonging simulation studies and practical examples given in Section 3.4 as well as Chapter 5 consider several choices of b, β (whose asymptotic behaviour is influenced by b according to Assumption 3.15 $[\mathbf{W}^*]$) and $\mathbf{B}_T$ in order to analyze how much changes of these tuning parameters impact the test decision that results from Algorithm **TEST.MDCI.2** for fixed $T \in \mathbb{N}$.*

*(ii) If $\widehat{\mathbb{D}}_{T,1}(\omega) > 0$, a test for the test problem (3.49) can be constructed that is based on the NEMDCI $\widehat{\mathbb{D}}_T^{\mathrm{norm}}$ $\big(= \widehat{\mathbb{D}}_T/\widehat{\mathbb{D}}_{T,1}$ according to Definition 3.8$\big)$ instead of $\widehat{\mathbb{D}}_T$. Concretely, just the realizations of $\widehat{\mathbb{D}}_T + 1/(T\sqrt{b})\widehat{\mathbf{Bias}}_T^{\mathrm{error}}$ and $\widehat{\mathbb{D}}_{T,\mathrm{Test}}^*$ calculated in Algorithm **TEST.MDCI.2** need to be divided by $\widehat{\mathbb{D}}_{T,1}(\omega) > 0$. However, this modification of Algorithm **TEST.MDCI.2** leads to the same test decision as Algorithm **TEST.MDCI.2**, such that it is redundant and not discussed anymore in the present publication. Moreover, in the case $\mathbf{B}_T = 0$, the property $K^*(0) = 1$ (as supposed in Assumption 3.15 $[\mathbf{W}^*]$ (iii)) shows $\widehat{\mathbf{Bias}}_T^{\mathrm{error}} = 0$ (recall (3.68)), such that Algorithm **TEST.MDCI.2** equals Algorithm **TEST.MDCI.1** in this case. Thereby, demand the convention $0/0 = 1$ since $\mathbb{K}_{\mathbf{NW}}(0/0)$ is contained in $\widehat{\mathbf{Bias}}_T^{\mathrm{error}}$ for $\mathbf{B}_T = 0$.*

3.3. Estimation of the first change point in the distributions of the stationary approximations

In this section, methods for estimating the first rescaled point in time $\mathbb{V} \in [0,1]$ at which the distribution of $\widetilde{X}_0(u)$ changes (smoothly) are introduced, whereby $\mathbb{V} \in [0,1]$ is defined in the following formally. The present investigations are inspired by the publication [76, Vogt and Dette (2015)] and some differences between them and those in [76, Vogt and Dette (2015)] are described in Remark 3.44 (ii) given below.
For $w \in (0,1]$, let $\mathbb{D}_{[0,w]}$ originate from Definition 3.3 (i) with $\mathfrak{U}_{0,1} = [0,w]$ and call $\mathbb{V} \in [0,1]$ the first change point (in the distributions of the stationary approximations) if:

$$\Big(\mathbb{V} < 1 \ \wedge \ \mathbb{D}_{[0,w]} = 0 \ \forall\, w \in [0,\mathbb{V}] \text{ with } \mathbb{D}_{[0,0]} := 0 \ \wedge \ C_{\mathfrak{A}} := \inf_{w \in (\mathbb{V},1]} \frac{\mathbb{D}_{[0,w]}}{(w-\mathbb{V})^{\mathfrak{A}}} > 0 \text{ for a } \mathfrak{A} > 0\Big)$$
$$\vee \ \Big(\mathbb{V} = 1 \ \wedge \ \mathbb{D}_{[0,w]} = 0 \ \forall\, w \in [0,\mathbb{V}] \text{ with } \mathbb{D}_{[0,0]} := 0\Big). \tag{3.69}$$

Remark 3.39. *(i) Note that $\mathbb{D}_{[0,w]} = 0 \ \forall\, w \in [0,\mathbb{V}]$ is equivalent to $\mathbb{D}_{[0,\mathbb{V}]} = 0$. Further, the case $\mathbb{V} = 1 \wedge \mathbb{D}_{[0,w]} = 0 \ \forall\, w \in [0,\mathbb{V}]$ means that no distribution change exists in the sense that $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\mathrm{distr}}$ holds for $\mathfrak{U}_{0,1} = [0,1]$.*

(ii) If $\mathbb{V} < 1$, the expression $C_{\mathfrak{A}}$ can be rewritten as:

$$C_{\mathfrak{A}} = \inf_{\epsilon \in (0,1-\mathbb{V}]} \frac{\mathbb{D}_{[0,\mathbb{V}+\epsilon]} - \mathbb{D}_{[0,\mathbb{V}]}}{\epsilon^{\mathfrak{A}}}. \tag{3.70}$$

Thus, the smoother the function $[0,1] \ni w \mapsto \mathbb{D}_{[0,w]}$ changes at $\mathbb{V}$, the larger $\mathfrak{A}$ has to be to ensure $C_{\mathfrak{A}} > 0$. In this sense, the (in most practical applications unknown) smallest parameter $\mathfrak{A}$ for which $C_{\mathfrak{A}} > 0$ holds determines how smooth the distribution changes at $\mathbb{V}$.

(iii) Assume that $\mathbb{V} < 1$. Then, Proposition 3.6 (ii) and (iii) imply for all $w \in (\mathbb{V}, 1]$:

$$\mathbb{D}_{[0,w]} = \frac{1}{2w} \int_{\mathbb{R}^d} \int_0^{\mathbb{V}} \int_0^{\mathbb{V}} |\varphi(u,s) - \varphi(v,s)|^2 \, du \, dv + \int_0^{\mathbb{V}} \int_{\mathbb{V}}^{w} |\varphi(u,s) - \varphi(v,s)|^2 \, du \, dv$$
$$+ \int_{\mathbb{V}}^{w} \int_0^{\mathbb{V}} |\varphi(u,s) - \varphi(v,s)|^2 \, du \, dv + \int_{\mathbb{V}}^{w} \int_{\mathbb{V}}^{w} |\varphi(u,s) - \varphi(v,s)|^2 \, du \, dv \, \mathbf{w}(s) \, ds$$
$$= \frac{1}{2w} \int_{\mathbb{R}^d} 2\mathbb{V} \int_{\mathbb{V}}^{w} |\varphi(u,s) - \varphi(\mathbb{V},s)|^2 \, du + \int_{\mathbb{V}}^{w} \int_{\mathbb{V}}^{w} |\varphi(u,s) - \varphi(v,s)|^2 \, du \, dv \, \mathbf{w}(s) \, ds,$$

such that Remark 2.3 and Assumption 3.1 $[\mathbf{WEI.1}]$ yield (recall that $C \in (0,\infty)$ denotes an absolute constant which may have different values at different places):

$$\mathbb{D}_{[0,w]} \leqslant C \left(\frac{\mathbb{V}}{w} (w - \mathbb{V})^3 + \frac{1}{2w} (w - \mathbb{V})^4 \right).$$

Hence, $\mathfrak{A} \geqslant 3$ needs to hold in order to fulfil $C_{\mathfrak{A}} > 0$.

In the following, estimators for $\mathbb{V}$ are constructed. Therefor, let $(\tau_T)_{T \in \mathbb{N}}$ be a sequence of deterministic real numbers that fulfils (3.54). Then, Theorem 3.36 implies under its assumptions (see (3.49) and set $\mathfrak{U}_{0,1} = [0,w]$ in Definition 3.8 (i) as well as Definition 3.33):

$$\begin{cases} \lim\limits_{T\to\infty} \mathbb{E}\Big[\mathbb{1}_{\left\{ T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T \right\}} \Big] = 1 \ \text{ for fixed } w \in (0, \mathbb{V}] \\ \lim\limits_{T\to\infty} \mathbb{E}\Big[\mathbb{1}_{\left\{ T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T \right\}} \Big] = 0 \ \text{ for fixed } w \in (\mathbb{V}, 1]. \end{cases}$$

This consideration motivates the following (non-deterministic) approximation $\widehat{\mathbb{V}}_T$ of $\mathbb{V}$:

$$\begin{aligned} \widehat{\mathbb{V}}_T &:= \int_0^1 \mathbb{1}_{\left\{ T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T \right\}} \, d\lambda(w) \\ &= \lambda\left(\left\{ w \in [0,1] : T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T \right\} \right) \\ &\quad \text{with } \ \widehat{\mathbb{D}}_{T,[0,0]} := 0, \quad \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,0]} := 0 \ \text{ and } \ \mathbf{Bias}^{\text{distr}*}_{T,[0,0]} := 0, \end{aligned} \tag{3.71}$$

whereby τ_T can be interpreted as a threshold level for rejecting $\mathcal{H}^{\text{distr}}_{0,[0,w]}$.
The next lemma provides that $\widehat{\mathbb{V}}_T$ converges in mean square to $\mathbb{V}$. However, $\widehat{\mathbb{V}}_T$ is not an useful estimator for $\mathbb{V}$ because $\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}$ is commonly unknown in practise but, as shown below, estimators for $\mathbb{V}$ can be derived from $\widehat{\mathbb{V}}_T$.

Lemma 3.40. *Let Assumption 3.30 $[\mathbf{NW}]$ (which includes Assumption 2.4 $[\mathbf{DM.2}]$ and 2.8 $[\mathbf{K\&b.1}]$ (ii)), 3.1 $[\mathbf{WEI.1}]$ as well as 3.15 $[\mathbf{W}^*]$ be valid. Moreover, suppose that $\mathbb{V} \in [0,1]$ fulfils (3.69) and that $(\tau_T)_{T\in\mathbb{N}}$ is a sequence of deterministic real numbers for which (3.54) holds. In addition, assume that K is a kernel that suffices Assumption 2.8 $[\mathbf{K\&b.1}]$ (i) with $\mathfrak{U}_{0,1} = [0,1]$ and define:*

$$K_{[0,w]}(x) := \frac{1}{w} K\left(\frac{x}{w}\right) \quad \forall\, w \in (0,1],\ x \in \mathbb{R}. \tag{3.72}$$

Further, let for all $w \in (0,1]$ the expression $\widehat{\mathbb{D}}_{T,[0,w]}$ originate from Definition 3.8 (i), $\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]}$ from Definition 3.33 and $\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}$ from (3.57) with $\mathfrak{U}_{0,1} = [0,w]$, whereby $K_{[0,w]}$ is the underlying kernel. Then, one obtains:

$$\lim_{T\to\infty} \left\| \widehat{\mathbb{V}}_T - \mathbb{V} \right\|_2 = 0.$$

Remark 3.41. *(i) By construction, $K_{[0,w]}$ suffices Assumption 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *(i) with $\mathfrak{U}_{0,1} = [0, w]$.*

(ii) It holds for all $t \in \{1, \ldots, T\}$, $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$ as well as $w \in (0, 1/(2T)]$ (recall Definition 3.8 (i) and note that $1/b \geqslant 2$ according to Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ *(ii)):*

$$\frac{t/T - u_{T,[0,w],k}}{wb} = \frac{t/T}{wb} - \frac{k - 1/2}{\lfloor 1/(2b) \rfloor b} \geqslant \frac{1}{wTb} - \frac{1}{b} \geqslant \frac{1}{b} > 1,$$

such that $K_{[0,w]}\big(\left(t/T - u_{T,[0,w],k}\right)/b\big) = 0$ (see (3.72)) due to Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ *(i). Hence, $\widehat{\mathbb{D}}_{T,[0,w]} = \widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_{T,[0,w]} = 0 \;\forall\, w \in (0, 1/(2T)]$. This, the last line of (3.71) and some obvious arguments show that $[0,1] \ni w \mapsto \widehat{\mathbb{D}}_{T,[0,w]}$, $[0,1] \ni w \mapsto \widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_{T,[0,w]}$ as well as $[0,1] \ni w \mapsto \mathbf{Bias}^{\text{distr}*}_{T,[0,w]}$ (recall (3.57)) are continuous for fixed $T \in \mathbb{N}$ with one-sided continuity at 0 and 1. These considerations provide that $\widehat{\mathbb{V}}_T$ is well-defined.*

(iii) Theorem 3.36 and Lemma 3.35 (i) indicate that $\widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_{T,[0,w]}$ can be replaced by $\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]}$ for all $w \in [0,1]$ in $\widehat{\mathbb{V}}_T$. However, to reduce the number of notations introduced in the present section, this alternative approach is omitted. In particular, also note that $\widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_{T,[0,w]}$ can be computed with less computational costs than $\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]}$.

One possibility to obtain an estimator for $\mathbb{V}$ from $\widehat{\mathbb{V}}_T$ is to replace $\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}$ in $\widehat{\mathbb{V}}_T$ by a suitable Newey-West estimator. Such an estimator can be obtained by replacing the expression $K^*(h/\beta) - 1$ contained in $\widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_{T,[0,w]}$ by $K^*(h/\beta)$. It can be proved very similarly to Lemma 3.40 and Lemma 3.35 (ii) that this yields a consistent estimator for $\mathbb{V}$.
However, another approach is used in the following, which is based on the next proposition and described in detail below.

Proposition 3.42 (Bootstrap-supported estimation of the first change point in the distributions of the stationary approximations)**.**
Suppose that the assumptions of Lemma 3.40 are fulfilled and that the following property holds for some $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$ (recall (3.17)):

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_{\infty,\mathrm{R}_1,\mathrm{R}_2}(0, s_1, s_2)^2\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 > 0. \tag{3.73}$$

Moreover, for all $w \in (0,1]$, the expression $\widehat{\mathbb{D}}^*_{T,[0,w],\text{Test}}$ should originate from (3.56) with $\mathfrak{U}_{0,1} = [0,w]$, whereby $K_{[0,w]}$ (see (3.72)) is the underlying kernel. Further, define for all $\alpha \in (0,1)$, $w \in (0,1]$ (recall (3.57), the last line of (3.71) as well as Definition 3.33):

$$\begin{aligned} &q^{\text{distr}*}_{T,1-\alpha}(0) := 0, \quad q^{\text{distr}*}_{T,1-\alpha}(w) := \inf\left\{x \in \mathbb{R} : \mathbb{P}^*\left(T\sqrt{b}\,\widehat{\mathbb{D}}^*_{T,[0,w],\text{Test}} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant x\right) \geqslant 1-\alpha\right\} \\ &\text{and} \quad \widehat{\mathbb{V}}_{T,1-\alpha} := \lambda\left(\left\{w \in [0,1] : T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant q^{\text{distr}*}_{T,1-\alpha}(w)\right\}\right). \end{aligned} \tag{3.74}$$

(i) Then, it holds for all fixed $\alpha \in (0,1)$:

$$\limsup_{T\to\infty} \left\|\widehat{\mathbb{V}}_{T,1-\alpha} - \mathbb{V}\right\|_2 \leqslant \sqrt{\alpha}\,\mathbb{V}.$$

(ii) In addition to the assumptions given above, let $(\tau_T)_{T\in\mathbb{N}}$ be a sequence of deterministic real numbers that fulfils (3.54) and:

$$\frac{\tau_T}{\sqrt{T}} \overset{T\to\infty}{\longrightarrow} \infty. \tag{3.75}$$

Moreover, suppose that $(\alpha_T)_{T\in\mathbb{N}}$ is a sequence of deterministic real numbers for which the follow-

ing properties hold:

$$\alpha_T \in (0,1)\ \forall T \in \mathbb{N}, \quad \alpha_T \overset{T\to\infty}{\longrightarrow} 0 \quad \textit{and} \quad \alpha_T \frac{\tau_T}{\sqrt{T}} \overset{T\to\infty}{\longrightarrow} \mathfrak{C} \quad \textit{for a } \mathfrak{C} > 0. \tag{3.76}$$

Then, one obtains:

$$\lim_{T\to\infty} \left\| \widehat{\mathbb{V}}_{T,1-\alpha_T} - \mathbb{V} \right\|_2 = 0.$$

Remark 3.43. *Assumption 2.8* $[\mathbf{K\&b.1}]$ *(ii) provides* $\sqrt{T} = o\big(T\sqrt{b}\big)$*, such that (3.54) is not a contradiction to (3.75). In addition, (3.75) ensures that a sequence* $(\alpha_T)_{T\in\mathbb{N}}$ *which suffices (3.76) exists.*

To obtain an useful procedure for estimating $\mathbb{V}$ that results from Proposition 3.42, $\widehat{\mathbb{V}}_{T,1-\alpha}$ is approximated at first in such a manner that the contained Lebesgue measure vanishes, which simplifies the implementation of such a procedure. Concretely, choose a (large) $M \in \mathbb{N}$ and approximate $\widehat{\mathbb{V}}_{T,1-\alpha}$ by the following expression:

$$\begin{aligned}
\widehat{\mathbb{V}}^{[M]}_{T,1-\alpha} &:= \lambda\Bigg(\Bigg\{ w \in \bigcup_{j=1}^{M} \left(\frac{j-1}{M}, \frac{j}{M}\right] : T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,\lceil wM\rceil/M]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,\lceil wM\rceil/M]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,\lceil wM\rceil/M]} \\
&\qquad \leqslant q^{\text{distr}*}_{T,1-\alpha}(\lceil wM\rceil/M) \Bigg\}\Bigg) \\
&= \frac{1}{M}\sum_{j=1}^{M} \mathbb{1}_{\left\{\widehat{\mathbb{D}}_{T,[0,j/M]} + 1/\left(T\sqrt{b}\right)\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,j/M]} \leqslant \left(q^{\text{distr}*}_{T,1-\alpha}(j/M) + \mathbf{Bias}^{\text{distr}*}_{T,[0,j/M]}\right)/\left(T\sqrt{b}\right)\right\}}.
\end{aligned}$$

Thereby, Theorem 3.25 provides that $\big(q^{\text{distr}*}_{T,1-\alpha}(j/M) + \mathbf{Bias}^{\text{distr}*}_{T,[0,j/M]}\big)/\big(T\sqrt{b}\big)$ can be estimated by using (conditionally on $X_{1,T},\ldots,X_{T,T}$) sampled realizations of $\widehat{\mathbb{D}}^*_{T,[0,j/M],\text{Test}}$ (note that $\mathbf{Bias}^{\text{distr}*}_{T,[0,j/M]}$ is deterministic), which leads to the next algorithm that aims to estimate $\mathbb{V}$.

Algorithm **DETECT.MDCI**

Inputs: $\alpha \in (0,1)$; $T, N, M \in \mathbb{N}$; a sample path $(X_{t,T}(\omega))_{t=1}^T$ (for an $\omega \in \Omega$); the kernel $K_{[0,w]}$ which originates from (3.72); a bandwidth $b \in (0, 1/2)$; a weight function $\mathbf{w}$ that fulfils Assumption 3.1 $[\mathbf{WEI.1}]$; a parameter $\beta > 0$ and an associated process $(W^*_t)_{t\in\mathbb{Z}}$ which satisfies Assumption 3.15 $[\mathbf{W}^*]$; an NW-bandwidth $\mathbf{B}_T \in \mathbb{N}$; an NW-kernel $\mathbb{K}_{\mathbf{NW}}$ for which Assumption 3.30 $[\mathbf{NW}]$ holds;

1: Set $\mathbb{V}_M \leftarrow 0$;
2: For j in $1:M$ do
3: Set $j_M \leftarrow j/M$;
4: Compute the realization of $\widehat{\mathbb{D}}_{T,[0,j_M]} + 1/\big(T\sqrt{b}\big)\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,j_M]}$ that belongs to the sample path $(X_{t,T}(\omega))_{t=1}^T$;
5: Independently, for n in $1:N$ do
6: Generate a sample path of $(W^*_t)_{t=1}^T$;
7: Calculate the associated realization of $\widehat{\mathbb{D}}^*_{T,[0,j_M],\text{Test}}$;
8: end for
9: Estimate $1/\big(T\sqrt{b}\big)\big(q^{\text{distr}*}_{T,1-\alpha}(j_M) + \mathbf{Bias}^{\text{distr}*}_{T,[0,j_M]}\big)$ by using the computed realizations of $\widehat{\mathbb{D}}^*_{T,[0,j_M],\text{Test}}$ and denote the resulting estimator as $\widehat{q}^{\,\text{change}}_{T,1-\alpha}(j_M)$;
10: If $\widehat{\mathbb{D}}_{T,[0,j_M]} + 1/(T\sqrt{b})\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,j_M]} \leqslant \widehat{q}^{\,\text{change}}_{T,1-\alpha}(j_M)$ holds for the calculated realizations of the expressions that are contained in this inequality, set $\mathbb{V}_M \leftarrow \mathbb{V}_M + 1$;
11: end for;
12: Estimate $\mathbb{V}$ by $\mathbb{V}_M/M$;

Remark 3.44. *(i) If the suppositions of Proposition 3.42 are valid except Assumption 3.30* $[\mathbf{NW}]$*, which does not need to hold in all its entirety but the therein contained Assumptions 2.4* $[\mathbf{DM.2}]$ *as well as 2.8* $[\mathbf{K\&b.1}]$ *(ii) are satisfied and either* $\delta \in (1/4, 1]$ *as well as (3.60) are fulfilled (whereby* δ *originates from Assumption 2.2* $[\mathbf{StAp}]$*) or* $\{\widetilde{X}_t(u)\}$ *is a sequence of independent* *random variables for all* $u \in [0,1]$*, replacing* $\widehat{\mathbb{D}}_{T,[0,j_M]} + 1/(T\sqrt{b})\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,j_M]}$ *by* $\widehat{\mathbb{D}}_{T,[0,j_M]}$ *in the fourth and tenth step of Algorithm* ***DETECT.MDCI*** *will also lead to a suitable algorithm for estimating* $\mathbb{V}$ *due to Theorem 3.27 as well as Remark 3.28 (iv).*

(ii) The Definition (3.69) of the change point $\mathbb{V}$ *can be regarded as a modification of the kinds of change points considered in [76, Vogt and Dette (2015)]. In the latter mentioned work, the first change point* $u_0 \in [0,1]$ *is estimated, at which* $[0,1] \ni v \mapsto \mathbb{E}\big[f\big(\widetilde{X}_0(v)\big)\big]$ *changes (smoothly) for the first time for some functions* f *in a set* $\mathbf{F}$ *of functions that fulfils some properties (e. g.,* $\mathbf{F}$ *is separable, compact and its elements are measurable as well as real-valued functions). It seems that the method for estimating* u_0 *which is proposed in this publication can also be used to detect changes in the entire distribution of* $\widetilde{X}_0(u)$ *for varying* u*. Therefor, set* $\mathbf{F} := \big\{\mathbb{R}^d \ni x \mapsto \mathrm{R}\left\{e^{i\langle s,x\rangle}\right\} \cdot \mathbf{W}(s) : s \in \mathbb{R}^d, \mathrm{R} \in \{\Re, \Im\}\big\}$ *with a weight function* $\mathbf{W}\colon \mathbb{R}^d \to (0,\infty)$ *which decays to zero sufficiently fast for* $|s|_1 \to \infty$ *to ensure that Assumption 5.1 in [76, Vogt and Dette (2015), p. 721 et seq.] is fulfilled. In contrast to the present thesis, a CUSUM-type statistic (instead of an* L^2*-distance-based one) is used in [76, Vogt and Dette (2015)], which does not depend on a bandwidth* b *but owns a weaker rate of convergence, such that it is tendentiously less powerful in rejecting* $\mathcal{H}^{\text{distr}}_{0,[0,w]}$ *for some* $w \in (0,1]$ *(in regard of fixed alternatives). This property provides that the threshold level that underlies the work [76, Vogt and Dette (2015)] is assumed to grow to infinity with the speed* $o\big(\sqrt{T}\big)$*, whereas the present sequence* $(\tau_T)_{T\in\mathbb{N}}$ *suffices the (according to Assumption 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *(ii) weaker) condition* $\tau_T = o\big(T\sqrt{b}\big)$*. In addition, an approach that is based on a given level* $\alpha \in (0,1)$ *and Newey-West-type estimation is proposed in [76, Vogt and Dette (2015)] to choose the threshold level for practical applications, whereby, similarly to the present work, this method requires to select a bandwidth for the Newey-West estimation approach and another one for the Nadaraya-Watson estimator for* $\mathbb{E}\big[f\big(X_{t,T}\big)\big]$ *with* $t \in \{1,\ldots,T\}$ *(details can be found in [76, Vogt and Dette (2015), p. 727 et. seq.]). Let* $\widehat{u}_{0,T,\alpha}$ *denote this estimator for the first change point* $u_0 < 1$*. Then, Corollary 5.5 in [76, Vogt and Dette (2015), p. 727] provides for a chosen* $\alpha \in (0,1)$*:*

$$\mathbb{P}\left(\widehat{u}_{0,T,\alpha} < u_0\right) \leqslant \alpha + o(1) \quad \textit{and} \quad \mathbb{P}\left(\widehat{u}_{0,T,\alpha} > u_0 + o(1)\right) = o(1), \tag{3.77}$$

i. e., the probability that the estimated and true change point differ from each other is small for large T *and small* α*. However, this result does not provide an asymptotic upper bound for the difference between* $\widehat{u}_{0,T,\alpha}$ *and* u_0 *in dependence of* α *(as Proposition 3.42 (i) does), whereby such an upper bound can be regarded as an approximative error bound for large values of* T*.*

(iii) In some papers (e. g., [39, James and Matteson (2014)]) that belong to the classical change point theory, which considers abrupt (instead of smooth) distribution changes, estimators not just for the first but also subsequent change points are constructed. However, the continuity of the local characteristic function $[0,1] \times \mathbb{R}^d \ni (w,s) \mapsto \varphi(w,s)$ *with respect to* w *(which holds due to Assumption 2.2* $[\mathbf{StAp}]$*) implies that* $[0,1] \ni w \mapsto \widetilde{X}_0(w)$ *contains either no or uncountable many distribution changes directly after the first change point. Moreover,* $\widetilde{X}_0(w)$ *may have a different distribution for each* $w \in [0,1]$*. Thus, in general, it is expectable that it is impossible to detect all of these change points in practical applications because commonly just a sample path of* $(X_{t,T})_{t=1}^T$ *is observable. So, the present research aims to estimate the first change point (like [76, Vogt and Dette (2015)]).*

3.4. Simulation studies

In the present section, the finite sample behaviour of the Algorithms **CONF.MDCI**, **CONF.NMDCI**, **TEST.MDCI.1**, **TEST.MDCI.2** and **DETECT.MDCI** is investigated by simulation studies. Therefor, these algorithms are implemented in the programming language $\mathbf{R}$ and applied to sample paths of several locally stationary processes with various selections of the rescaled time interval $\mathfrak{U}_{0,1}$ (which is introduced in Definition 3.3 (i)), whereby the next setting describes all versions of these algorithms that are used in this section. In contrast, the considered locally stationary processes and the investigated rescaled time intervals $\mathfrak{U}_{0,1}$ are specified in each simulation study.

Setting 3.45. *All of the algorithms mentioned above are applied to sample paths which contain* $T = 1000$ *observations that originate from a real-valued locally stationary process. Thereby, the weight function* $\mathbb{R} \ni s \mapsto \mathbf{w}(s) := \mathbf{w}_{\mathrm{L},1}(s)$ *(see Example 3.2 (ii)) and the kernel* K_{Epa} *originating from Example 2.10 (ii) are used for each applied algorithm. Moreover, all algorithms are carried out with the bandwidths* $b \in \{1/10, 1/14\}$*, such that the Riemann sums contained in the EMDCI as well as NEMDCI (recall Definition 3.8) are based on* $\lfloor 1/(2b) \rfloor = 5$ *addends for* $b = 1/10$ *and* $\lfloor 1/(2b) \rfloor = 7$ *summands for* $b = 1/14$*. Since* $\lfloor 1/(2b) \rfloor = 1/(2b)$ *holds for these choices of* b*, no observations of the investigated processes are excluded from the EMDCI and NEMDCI because of the fact that* $1/(2b)$ *is rounded in these empirical measures. Moreover, the process of bootstrap random variables introduced in Example 3.16 (ii) with* $\beta \in \{0.3\,Tb^2, 0.7\,Tb^2\}$ *is used for each algorithm (which, in particular, fulfils* $\mathscr{S}^* = 1$ *(see (3.60)), as mentioned in Remark 3.28 (iii)). Further,* $\alpha_1 = 1 - \alpha_2$ *with* $\alpha_2 \in \{0.975, 0.95\}$ *and* $N = 500$ *bootstrap iterations are selected for the applied versions of the Algorithms* ***CONF.MDCI*** *as well as* ***CONF.NMDCI****. The considered versions of the Algorithms* ***TEST.MDCI.1*** *and* ***TEST.MDCI.2*** *are based on* $\alpha \in \{0.05, 0.1\}$ *as well as* $N = 500$*, whereas* $\alpha = 0.05$*,* $N = 200$ *and* $M = 100$ *underlie all used versions of Algorithm* ***DETECT.MDCI****. (A compared to the other algorithms smaller choice of the number of bootstrap iterations* N *is taken for Algorithm* ***DETECT.MDCI*** *because this algorithm is much more computationally expensive than the other ones.) Further, Example 3.32 (ii) with* $a = 0.25$ *is used as the Newey-West-estimation-kernel* $\mathbb{K}_{\mathbf{NW}}$ *and the Newey-West-estimation-bandwidths* $\mathbf{B}_T \in \{0, 3, 5\}$ *are chosen for the Algorithms* ***TEST.MDCI.2*** *as well as* ***DETECT.MDCI****. Thereby, recall that, in the case* $\mathbf{B}_T = 0$*, Algorithm* ***TEST.MDCI.2*** *equals Algorithm* ***TEST.MDCI.1*** *(according to Remark 3.38 (ii)) and applying Algorithm* ***DETECT.MDCI*** *with* $\mathbf{B}_T = 0$ *is justified under the conditions mentioned in Remark 3.44 (i).*

3.4.1. Estimation of confidence intervals for the measures for the distribution change intensity

In this subsection, the versions of the Algorithms **CONF.MDCI** and **CONF.NMDCI** that are described in Setting 3.45 are evaluated based on the locally stationary processes $\{\mathrm{A}_{t,T}\}$ as well as $\{\mathrm{B}_{t,T}\}$, which are introduced in the following.

Suppose that $\big(\varepsilon_t^{[\mathrm{A}]}\big)_{t\in\mathbb{Z}}$ is a sequence of i. i. d. random variables with $\varepsilon_0^{[\mathrm{A}]} \sim \mathcal{N}(0.1, 0.1)$ and define:

$$\mathrm{A}_{t,T} := \begin{cases} 0.2 \cdot \left(\varepsilon_t^{[\mathrm{A}]} + \varepsilon_{t-1}^{[\mathrm{A}]}\right), & \text{for } t \leqslant T/2 \\ \left(\sin\left(2\pi\left(\frac{t}{T} - 0.5\right)\right)^2 + 0.2\right)\left(\varepsilon_t^{[\mathrm{A}]} + \varepsilon_{t-1}^{[\mathrm{A}]}\right), & \text{for } t > T/2 \end{cases} \quad \forall\, t \in \{1, \ldots, T\},\, T \in \mathbb{N}. \tag{3.78}$$

Further, let $\mathcal{S}^0_{\bar{\alpha}}\left(\bar{\beta}, \bar{\gamma}, \bar{\delta}\right)$ denote the $\mathcal{S}^0$-parametrization of the stable distribution (which is contained in the **R**-package 'stabledist') with index parameter $\bar{\alpha}$, skewness parameter $\bar{\beta}$, scale parameter $\bar{\gamma}$ as well as location parameter $\bar{\delta}$ and take a sequence $\big(\varepsilon_t^{[\mathrm{B}]}\big)_{t\in\mathbb{Z}}$ of i. i. d. random variables with $\varepsilon_0^{[\mathrm{B}]} \sim \mathcal{S}^0_{1.5}(0, 1, 0)$. (3.79)

Define:

$$\mathrm{B}_{t,T} := 16 \cdot \left(\frac{t}{T} - 0.5\right)^3 \sum_{j=1}^{100} \frac{1}{j^{3.2}} \varepsilon_{t-j}^{[\mathrm{B}]} \quad \forall\, t \in \{1, \ldots, T\},\, T \in \mathbb{N}. \tag{3.80}$$

The locally stationary processes $\{\mathrm{A}_{t,T}\}$ and $\{\mathrm{B}_{t,T}\}$ fulfil Assumption 2.4 $[\mathbf{DM.1}]$ (in particular, note that $(0,1) \ni u \mapsto 0.2 \cdot \mathbb{1}_{\{u\in[0,0.5]\}} + \big(\sin(2\pi(u - 0.5))^2 + 0.2\big) \cdot \mathbb{1}_{\{u\in(0.5,1]\}}$ is one-time differentiable with Lipschitz continuous derivative). Hence, applying the Algorithms **CONF.MDCI** and **CONF.NMDCI** to these locally stationary processes is justified. In addition, it is worth mentioning that $(\mathrm{A}_{t,T})_{t=1}^T$ is m-dependent with $m = 1$ and the impact of $\varepsilon_{t-j}^{[\mathrm{B}]}$ on $\mathrm{B}_{t,T}$ decays polynomially for growing $j \in \{1, \ldots, 100\}$. Furthermore $\{\mathrm{A}_{t,T}\}$ and $\{\mathrm{B}_{t,T}\}$ do not fulfil the assumptions which are supposed in [19, Dette et al. (2011)] because $(0,1) \ni u \mapsto 0.2 \cdot \mathbb{1}_{\{u\in[0,0.5]\}} + \big(\sin(2\pi(u - 0.5))^2 + 0.2\big) \cdot \mathbb{1}_{\{u\in(0.5,1]\}}$ is not twice continuously differentiable (in $u = 0.5$), as demanded in [19, Dette et al. (2011), p. 1114], and $\varepsilon_0^{[\mathrm{B}]}$ does not own 1.5 (or more) finite moments, such that the local spectral density $[0,1] \times [-\pi, \pi] \ni (u, x) \mapsto$

$f_{\mathbf{loc.spec}}(u, x)$ of the stationary approximations of $\{\mathrm{B}_{t,T}\}$ is not well-defined for all $u \in [0,1]$. Hence, it is not reasoned to use the in this publication proposed estimated confidence bands for the measure (3.1), which quantifies the intensity of second-order stationarity in locally stationary processes (whereby second-order stationarity is equivalent to strict stationarity under the assumptions supposed in [19, Dette et al. (2011), p. 1114]). It should also be noted that many approaches which are considered in the change point theory (see e. g. [3, Aue and Horváth (2012)]) are focused on detecting changes in the expectation, such that they do not detect changes in the stationary approximations of $\{\mathrm{B}_{t,T}\}$ because $\mathbb{E}\big[\varepsilon_0^{[B]}\big] = 0$. In contrast, the MDCI as well as NMDCI (recall Definition 3.3) are based on the local characteristic function, such that they measure changes in the entire distribution and not just in the expectation.

The measures $\mathbb{D}_{\mathfrak{U}_{0,1}}$ as well as $\mathbb{D}^{\text{norm}}_{\mathfrak{U}_{0,1}}$ (see Definition 3.3) that belong to $\{\mathrm{A}_{t,T}\}$ and $\{\mathrm{B}_{t,T}\}$ are approximated for the following purposes because calculating them for these locally stationary processes explicitly is very complicated - except in the case that $\{\mathrm{A}_{t,T}\}$ with $\mathfrak{U}_{0,v} := [0, v]$ for some $v \leqslant 0.5$ is considered, in which the MDCI as well as NMDCI equal zero according to Proposition 3.6 (iii) and (iv), such that they are not approximated but calculated exactly in this case. In a first step, to approximate $\mathbb{D}_{\mathfrak{U}_{0,1}}$ and $\mathbb{D}^{\text{norm}}_{\mathfrak{U}_{0,1}}$, the integral with respect to $s \in \mathbb{R}^d$ (with $d = 1$ in the present framework) contained in $\mathbb{D}_{\mathfrak{U}_{0,1},1}$ as well as $\mathbb{D}_{\mathfrak{U}_{0,1},2}$ is approximated by an integral with respect to $s \in (-10, 10)$ and, in a second step, the resulting integral with respect to $s \in (-10, 10)$ is replaced by a belonging Riemann sum with 1000 addends that is based on the midpoint rule. Moreover, the integral with respect to $u \in [\mathfrak{U}_0, \mathfrak{U}_1]$ which is contained in $\mathbb{D}_{\mathfrak{U}_{0,1},1}$ and $\mathbb{D}_{\mathfrak{U}_{0,1},2}$ is also approximated by a Riemann sum based on the midpoint rule with 1000 addends. Next, denote the resulting approximations of $\mathbb{D}_{\mathfrak{U}_{0,1},1}$ as well as of $\mathbb{D}_{\mathfrak{U}_{0,1},2}$ as $\mathbb{D}^{[\text{apprx}]}_{\mathfrak{U}_{0,1},1}$ and $\mathbb{D}^{[\text{apprx}]}_{\mathfrak{U}_{0,1},2}$, respectively. Then, approximate $\mathbb{D}_{\mathfrak{U}_{0,1}}$ by $\mathbb{D}^{[\text{apprx}]}_{\mathfrak{U}_{0,1}} := \mathbb{D}^{[\text{apprx}]}_{\mathfrak{U}_{0,1},1} - \mathbb{D}^{[\text{apprx}]}_{\mathfrak{U}_{0,1},2}$ and $\mathbb{D}^{\text{norm}}_{\mathfrak{U}_{0,1}}$ by $\mathbb{D}^{[\text{norm.apprx}]}_{\mathfrak{U}_{0,1}} := 1 - \mathbb{D}^{[\text{apprx}]}_{\mathfrak{U}_{0,1},2} / \mathbb{D}^{[\text{apprx}]}_{\mathfrak{U}_{0,1},1}$. These approximations can be justified by Lemma B.2 (iii) in the appendix together with (3.15) as well as similar arguments, Proposition 3.6 (i) and Definition 3.3 (ii).

In the following, 500 sample paths of $\big(\mathrm{A}_{t,T}\big)_{t=1}^{T}$ with $T = 1000$ are generated independently of each other and confidence intervals for the MDCI as well as NMDCI with $\mathfrak{U}_{0,1} \in \{[0,1], [0,0.5], [0.25,0.75], [0.5,1]\}$ are estimated by applying the versions of the Algorithms **CONF.MDCI** and **CONF.NMDCI** given in Setting 3.45 to each of these sample paths. Thereby, for each of the chosen $\mathfrak{U}_{0,1}$, Table 3.1 shows the relative frequencies that $\mathbb{D}^{[\text{apprx}]}_{\mathfrak{U}_{0,1}}$ belonging to $\{\mathrm{A}_{t,T}\}$ is contained in the 500 calculated estimated confidence intervals which result from the used versions of Algorithm **CONF.MDCI**, whereas Table 3.2 displays the relative frequencies that $\mathbb{D}^{[\text{norm.apprx}]}_{\mathfrak{U}_{0,1}}$ associated with $\{\mathrm{A}_{t,T}\}$ is covered by the computed estimated confidence intervals which originate from the applied versions of Algorithm **CONF.NMDCI**. The relative frequencies belonging to $\alpha_2 = 0.975$ that are the closest to 0.95 and the relative frequencies associated with $\alpha_2 = 0.95$ which are the closest to 0.9 are printed in bold for each considered interval $\mathfrak{U}_{0,1}$ in both tables, which is motivated by the fact that, according to (3.24) ((3.25), respectively), the versions of Algorithm **CONF.MDCI** (Algorithm **CONF.NMDCI**, respectively) defined in Setting 3.45 aim to generate estimated confidence intervals that contain the true value of the MDCI (NMDCI, respectively) with a probability of $\alpha_2 - \alpha_1 = 2\alpha_2 - 1 = 0.95$ for $\alpha_2 = 0.975$ and $\alpha_2 - \alpha_1 = 2\alpha_2 - 1 = 0.9$ for $\alpha_2 = 0.95$.

Table 3.1.: Relative frequencies that $\mathbb{D}^{[\text{apprx}]}_{\mathfrak{U}_{0,1}}$ belonging to $\{\mathrm{A}_{t,T}\}$ is covered by the estimated confidence intervals which originate from the versions of Algorithm **CONF.MDCI** introduced in Setting 3.45

b	β	α_2 \ $\mathfrak{U}_{0,1}$	$[0,1]$	$[0,0.5]$	$[0.25,0.75]$	$[0.5,1]$
1/10	$0.3\,Tb^2$	0.975	0.822	**0.96**	**0.916**	**0.93**
		0.95	0.756	**0.89**	0.862	0.888
	$0.7\,Tb^2$	0.975	0.828	0.93	0.892	0.926
		0.95	0.77	0.858	0.838	**0.89**
1/14	$0.3\,Tb^2$	0.975	0.902	0.846	**0.916**	0.916
		0.95	0.862	0.698	0.858	0.846

	$0.7\,Tb^2$	0.975	**0.926**	0.868	0.912	0.916
		0.95	**0.872**	0.706	**0.864**	0.852

Table 3.2.: Relative frequencies that $\mathbb{D}_{\mathfrak{U}_{0,1}}^{[\text{norm.apprx}]}$ associated with $\{\mathrm{A}_{t,T}\}$ is contained in the estimated confidence intervals which result from the versions of Algorithm **CONF.NMDCI** introduced in Setting 3.45

b	β	α_2 \ $\mathfrak{U}_{0,1}$	$[0,1]$	$[0,0.5]$	$[0.25,0.75]$	$[0.5,1]$
1/10	$0.3\,Tb^2$	0.975	0.834	**0.96**	**0.918**	**0.928**
		0.95	0.764	**0.888**	0.858	**0.886**
	$0.7\,Tb^2$	0.975	0.838	0.93	0.894	0.926
		0.95	0.776	0.86	0.838	**0.886**
1/14	$0.3\,Tb^2$	0.975	0.904	0.848	0.914	0.916
		0.95	0.866	0.698	0.86	0.846
	$0.7\,Tb^2$	0.975	**0.926**	0.868	0.914	0.92
		0.95	**0.874**	0.706	**0.866**	0.848

In the following, 500 sample paths of $\left(\mathrm{B}_{t,T}\right)_{t=1}^{T}$ with $T = 1000$ are generated independently of each other and confidence intervals for the MDCI as well as NMDCI with $\mathfrak{U}_{0,1} \in \{[0,1], [0,0.5], [0.25,0.75], [0.5,1]\}$ are estimated by applying the versions of the Algorithms **CONF.MDCI** and **CONF.NMDCI** given in Setting 3.45 to each of these sample paths. Thereby, for each of the chosen $\mathfrak{U}_{0,1}$, Table 3.3 shows the relative frequencies that $\mathbb{D}_{\mathfrak{U}_{0,1}}^{[\text{apprx}]}$ belonging to $\{\mathrm{B}_{t,T}\}$ is contained in the 500 calculated estimated confidence intervals which result from the used versions of Algorithm **CONF.MDCI**, whereas Table 3.4 displays the relative frequencies that $\mathbb{D}_{\mathfrak{U}_{0,1}}^{[\text{norm.apprx}]}$ associated with $\{\mathrm{B}_{t,T}\}$ is covered by the computed estimated confidence intervals which originate from the applied versions of Algorithm **CONF.NMDCI**. As in the previous tables, the relative frequencies belonging to $\alpha_2 = 0.975$ that are the closest to 0.95 and the relative frequencies associated with $\alpha_2 = 0.95$ which are the closest to 0.9 are printed in bold for each considered interval $\mathfrak{U}_{0,1}$ in the next two tables.

Table 3.3.: Relative frequencies that $\mathbb{D}_{\mathfrak{U}_{0,1}}^{[\text{apprx}]}$ belonging to $\{\mathrm{B}_{t,T}\}$ is covered by the estimated confidence intervals which originate from the versions of Algorithm **CONF.MDCI** introduced in Setting 3.45

b	β	α_2 \ $\mathfrak{U}_{0,1}$	$[0,1]$	$[0,0.5]$	$[0.25,0.75]$	$[0.5,1]$
1/10	$0.3\,Tb^2$	0.975	0.752	**0.934**	0.77	**0.918**
		0.95	0.63	**0.884**	0.694	**0.86**
	$0.7\,Tb^2$	0.975	0.802	0.912	0.772	0.916
		0.95	0.72	0.87	0.7	0.854
1/14	$0.3\,Tb^2$	0.975	0.91	0.918	**0.948**	0.906
		0.95	0.858	0.854	**0.918**	0.832
	$0.7\,Tb^2$	0.975	**0.918**	0.912	0.946	0.89
		0.95	**0.866**	0.848	0.922	0.816

Table 3.4.: Relative frequencies that $\mathbb{D}_{\mathfrak{U}_{0,1}}^{[\text{norm.apprx}]}$ associated with $\{\mathrm{B}_{t,T}\}$ is contained in the estimated confidence intervals which result from the versions of Algorithm **CONF.NMDCI** introduced in Setting 3.45

b	β	α_2 \ $\mathfrak{U}_{0,1}$	$[0,1]$	$[0,0.5]$	$[0.25,0.75]$	$[0.5,1]$
1/10	$0.3\,Tb^2$	0.975	0.79	**0.93**	0.764	**0.918**
		0.95	0.682	**0.874**	0.688	**0.868**
	$0.7\,Tb^2$	0.975	0.856	0.918	0.772	0.91
		0.95	0.764	0.862	0.698	0.864

1/14	$0.3\,Tb^2$	0.975	0.91	0.92	**0.948**	0.912
		0.95	0.852	0.872	0.918	0.846
	$0.7\,Tb^2$	0.975	**0.92**	0.914	0.944	0.9
		0.95	**0.864**	0.858	**0.916**	0.83

Overall, the Tables 3.1 to 3.4 indicate that many of the computed estimated confidence intervals own reliable coverage ratios. Thereby, it is worth mentioning that the stationary approximation of $\{\mathrm{A}_{t,T}\}$ (see (3.78)) fulfil $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ (recall (3.49)) for $\mathfrak{U}_{0,1} := [0, 0.5]$, such that (3.50) holds and $\sigma_{\mathfrak{U}_{0,1}}((1,-1),(1,-1)) = 0$ as well as $\sigma_{\mathfrak{U}_{0,1}}\left(\left(\gamma_{\mathfrak{U}_{0,1},1}^{\text{norm}}, \gamma_{\mathfrak{U}_{0,1},2}^{\text{norm}}\right), \left(\gamma_{\mathfrak{U}_{0,1},1}^{\text{norm}}, \gamma_{\mathfrak{U}_{0,1},2}^{\text{norm}}\right)\right) = 0$ (see (3.21) and (3.23)). In this case, (3.41) and (3.43) are valid (recall (3.26)), which may explain together with (3.50) the good results concerning $\mathfrak{U}_{0,1} = [0, 0.5]$ which are displayed in the Tables 3.1 as well as 3.2. However, confidence intervals that belong to degenerated distributions and their estimators should be seen critically from a theoretical point of view.

3.4.2. Testing the existence of distribution changes

In this subsection, the versions of the Algorithms **TEST.MDCI.1** and **TEST.MDCI.2** that are described in Setting 3.45 are evaluated based on the locally stationary processes $\{\mathrm{A}_{t,T}\}$ (see (3.78)) as well as $\{\mathrm{C}_{t,T,\Lambda}\}$, whereby the latter is introduced in the following and depends on a parameter Λ that is chosen as element of the set $\{0, 0.25, 0.5, 0.75, 1\}$ in the simulations given below.

Let $\left(\varepsilon_t^{[\mathrm{C},1]}\right)_{t\in\mathbb{Z}}$ be a sequence of i. i. d. random variables with $\varepsilon_0^{[\mathrm{C},1]} \sim \mathcal{N}(0,1)$, define

$$c_0(u) := 0.4 \cdot (1.1 - u)^2\, \mathbb{1}_{\{u\in[0,1]\}}, \quad c_1(u) := 0.4 \cdot (1 + \cos(2\pi u))\, \mathbb{1}_{\{u\in[0,1]\}} \quad \forall\, u \in \mathbb{R},$$

$$\mathrm{C}_{0,T}^{[1]} := \sqrt{c_0(0)}\, \varepsilon_0^{[\mathrm{C},1]}, \quad \mathrm{C}_{t,T}^{[1]} := \left(c_0\left(\frac{t}{T}\right) + c_1\left(\frac{t}{T}\right) \cdot \left(C_{t-1,T}^{[1]}\right)^2\right)^{\frac{1}{2}} \varepsilon_t^{[\mathrm{C},1]} \quad \forall\, t \in \{1, \ldots, T\},\, T \in \mathbb{N},$$

suppose that $\mathrm{Pareto}(x_{\min}, \mathfrak{S})$ denotes the Pareto distribution with location parameter $x_{\min} > 0$ as well as shape parameter $\mathfrak{S} > 0$, $\left(\varepsilon_t^{[\mathrm{C},2]}\right)_{t\in\mathbb{Z}}$ should be a sequence of i. i. d. random variables with $\varepsilon_0^{[\mathrm{C},2]} + 11 \sim \mathrm{Pareto}(1, 1.1)$, which is stochastically independent of $\left(\varepsilon_t^{[\mathrm{C},1]}\right)_{t\in\mathbb{Z}}$ and set:

$$\mathrm{C}_t^{[2]} := \varepsilon_t^{[\mathrm{C},2]} + \varepsilon_{t-1}^{[\mathrm{C},2]} + \varepsilon_{t-2}^{[\mathrm{C},2]} + \varepsilon_{t-3}^{[\mathrm{C},2]} + \varepsilon_{t-4}^{[\mathrm{C},2]} + \varepsilon_{t-5}^{[\mathrm{C},2]} \quad \forall\, t \in \mathbb{Z} \quad \text{as well as}$$

$$\mathrm{C}_{t,T,\Lambda} := \Lambda \cdot 20\, \mathrm{C}_{t,T}^{[1]} + (1 - \Lambda) \cdot \mathrm{C}_t^{[2]} \quad \forall\, t \in \{1, \ldots, T\},\, T \in \mathbb{N}. \tag{3.81}$$

Assumption 2.4 $[\mathbf{DM.2}]$ holds for the locally stationary processes $\{\mathrm{A}_{t,T}\}$ and $\{\mathrm{C}_{t,T,\Lambda}\}$ (for all fixed $\Lambda \in [0,1]$), which can be verified for $\{\mathrm{A}_{t,T}\}$ directly by its definition (recall (3.78)) as well as for $\{\mathrm{C}_{t,T,\Lambda}\}$ by using straightforward calculations and that $\left\{\mathrm{C}_{t,T}^{[1]}\right\}$ fulfils the conditions of Example 2.5 (iii), (in particular, note $\sup_{u\in[0,1]} c_1(u) \leqslant 0.8 < \left\|\varepsilon_0^{[\mathrm{C},1]}\right\|_{2+2\delta}^{-2}$ for $\delta = 0.26$). Hence, applying Algorithm **TEST.MDCI.2** to test the problem (3.49) for the locally stationary processes $\{\mathrm{A}_{t,T}\}$ and $\{\mathrm{C}_{t,T,\Lambda}\}$ (with $\Lambda \in [0,1]$) is appropriate. In contrast, Algorithm **TEST.MDCI.1** is suitable to investigate (3.49) for $\{\mathrm{A}_{t,T}\}$ and for $\{\mathrm{C}_{t,T,1}\}$ but using it is not justified by Theorem 3.27 or Remark 3.28 (iv) for $\{\mathrm{C}_{t,T,\Lambda}\}$ with $\Lambda \in [0,1)$.

Further, since the random variables contained in the process $\left(\mathrm{C}_t^{[2]}\right)_{t\in\mathbb{Z}}$ own less than 1.1 finite moments, the assumptions supposed in [60, Paparoditis (2009)], [19, Dette et al. (2011)], [65, Schmidt et al. (2021)] and [57, Mies (2021)] do not hold for $\{\mathrm{C}_{t,T,\Lambda}\}$ with $\Lambda \in [0,1)$, such that applying the tests proposed in these publications to $\{\mathrm{C}_{t,T,\Lambda}\}$ with $\Lambda \in [0,1)$ is not justified. (Recall that these tests are briefly described in Section 1.2 and also aim to detect the existence of changes with respect to some kinds of distribution properties of locally stationary time series). In addition, the stationary approximations of $\{\mathrm{C}_{t,T,\Lambda}\}$ are centered for all $\Lambda \in [0,1]$, such that tests which investigate whether the expectations of the belonging stationary approximations change in dependence of the rescaled time would not detect distribution changes.

In the following, 500 sample paths of $\left(\mathrm{A}_{t,T}\right)_{t=1}^{T}$ with $T = 1000$ are generated independently of each other and the versions of the Algorithms **TEST.MDCI.1** as well as **TEST.MDCI.2** given in Setting 3.45 are applied to all of these sample paths. Thereby, it should be recalled that, as explained in Remark 3.38

(ii), Algorithm **TEST.MDCI.1** equals Algorithm **TEST.MDCI.2** in the case $\mathbf{B}_T = 0$. The belonging relative frequencies of rejecting $\mathcal{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ (see (3.49)) for the intervals $\mathfrak{U}_{0,1} \in \{[0,1], [0,0.5], [0,0.6], [0,0.65], [0.25,0.75], [0.5,1]\}$ are given in Table 3.5.

Table 3.5.: Relative frequencies of rejecting $\mathcal{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ that result from applying the versions of the Algorithms **TEST.MDCI.1** and **TEST.MDCI.2** which are introduced in Setting 3.45 to sample paths of $\left(\mathrm{A}_{t,T}\right)_{t=1}^{T}$

b	β	$\mathbf{B}_T$	α \ $\mathfrak{U}_{0,1}$	$[0,1]$	$[0,0.5]$	$[0,0.6]$	$[0,0.65]$	$[0.25,0.75]$	$[0.5,1]$
1/10	$0.3\,Tb^2$	0	0.05	1	0.082	0.156	0.874	1	1
			0.1	1	0.148	0.254	0.968	1	1
		3	0.05	1	0.066	0.132	0.85	1	1
			0.1	1	0.112	0.208	0.952	1	1
		5	0.05	1	0.064	0.13	0.844	1	1
			0.1	1	0.116	0.21	0.954	1	1
	$0.7\,Tb^2$	0	0.05	1	0.092	0.152	0.828	1	1
			0.1	1	0.172	0.252	0.952	1	1
		3	0.05	1	0.088	0.134	0.818	1	1
			0.1	1	0.152	0.222	0.94	1	1
		5	0.05	1	0.088	0.136	0.818	1	1
			0.1	1	0.162	0.224	0.94	1	1
1/14	$0.3\,Tb^2$	0	0.05	1	0.136	0.334	0.966	1	1
			0.1	1	0.222	0.434	0.998	1	1
		3	0.05	1	0.102	0.264	0.932	1	1
			0.1	1	0.142	0.362	0.986	1	1
		5	0.05	1	0.106	0.256	0.926	1	1
			0.1	1	0.14	0.36	0.982	1	1
	$0.7\,Tb^2$	0	0.05	1	0.122	0.278	0.916	1	1
			0.1	1	0.192	0.4	0.982	1	1
		3	0.05	1	0.106	0.246	0.9	1	1
			0.1	1	0.154	0.36	0.966	1	1
		5	0.05	1	0.112	0.244	0.898	1	1
			0.1	1	0.16	0.37	0.968	1	1

The locally stationary process $\{\mathrm{A}_{t,T}\}$ (see (3.78)) fulfils $\mathcal{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ for $\mathfrak{U}_{0,1} = [0,0.5]$ but not for the other considered choices of $\mathfrak{U}_{0,1}$. In the case $\mathfrak{U}_{0,1} = [0,0.5]$, the test versions with $b = 1/10$, $\beta = 0.3\,Tb^2$ and $\mathbf{B}_T \in \{3,5\}$ can be regarded as the most suitable ones among the applied tests in the present simulation study because they reject $\mathcal{H}^{\text{distr}}_{0,[0,0.5]}$ very satisfactory rarely for the significance levels $\alpha \in \{0.05, 0.1\}$. Further, if $[0,0.5]$ is a large proper subset of $\mathfrak{U}_{0,1}$, identifying the validity of $\mathcal{H}^{\text{distr}}_{1,\mathfrak{U}_{0,1}}$ will be tendentiously challenging for the Algorithms **TEST.MDCI.1** and **TEST.MDCI.2**. Concretely, the applied versions of these algorithms reject $\mathcal{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ with $\mathfrak{U}_{0,1} = [0,0.6]$ not often $\big($based on the significance levels $\alpha \in \{0.05, 0.1\}\big)$. However, if the interval $\mathfrak{U}_{0,1} = [0,0.6]$ is slightly enlarged, which means that $\mathfrak{U}_{0,1} = [0,0.65]$ is considered, all of the used test versions will reject $\mathcal{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ very reliably. Moreover, for $\mathfrak{U}_{0,1} \in \{[0,1], [0.25,0.75], [0.5,1]\}$, each applied version of the Algorithms **TEST.MDCI.1** and **TEST.MDCI.2** rejects $\mathcal{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ for all generated sample paths of $\{\mathrm{A}_{t,T}\}$.

In the following, for all $\Lambda \in \{0, 0.25, 0.5, 0.75, 1\}$, 500 sample paths of $\left(\mathrm{C}_{t,T,\Lambda}\right)_{t=1}^{T}$ with $T = 1000$ are generated independently of each other and the versions of the Algorithms **TEST.MDCI.1** as well as **TEST.MDCI.2** given in Setting 3.45 are applied to all of these sample paths. Thereby, it should be recalled again that Algorithm **TEST.MDCI.1** equals Algorithm **TEST.MDCI.2** in the case $\mathbf{B}_T = 0$. The belonging relative frequencies of rejecting $\mathcal{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ for the interval $\mathfrak{U}_{0,1} = [0,1]$ and each $\Lambda \in \{0, 0.25, 0.5, 0.75, 1\}$ are given in Table 3.6.

Table 3.6.: Relative frequencies of rejecting $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ with $\mathfrak{U}_{0,1} := [0,1]$ that result from applying the versions of the Algorithms **TEST.MDCI.1** and **TEST.MDCI.2** which are introduced in Setting 3.45 to sample paths of $\left(\mathrm{C}_{t,T,\Lambda}\right)_{t=1}^{T}$ with $\Lambda \in \{0, 0.25, 0.5, 0.75, 1\}$

b	β	$\mathbf{B}_T$	α \ Λ	0	0.25	0.5	0.75	1
1/10	$0.3\,Tb^2$	0	0.05	0.16	0.268	0.816	1	1
			0.1	0.26	0.358	0.892	1	1
		3	0.05	0.068	0.178	0.754	1	1
			0.1	0.112	0.266	0.846	1	1
		5	0.05	0.042	0.126	0.712	1	1
			0.1	0.072	0.2	0.802	1	1
	$0.7\,Tb^2$	0	0.05	0.044	0.144	0.722	1	1
			0.1	0.1	0.232	0.83	1	1
		3	0.05	0.03	0.106	0.688	1	1
			0.1	0.066	0.2	0.8	1	1
		5	0.05	0.026	0.086	0.652	1	1
			0.1	0.05	0.174	0.772	1	1
1/14	$0.3\,Tb^2$	0	0.05	0.744	0.672	0.944	1	1
			0.1	0.86	0.768	0.978	1	1
		3	0.05	0.28	0.378	0.87	0.998	1
			0.1	0.388	0.518	0.928	1	1
		5	0.05	0.14	0.24	0.808	0.998	1
			0.1	0.194	0.334	0.87	1	1
	$0.7\,Tb^2$	0	0.05	0.236	0.308	0.828	0.996	1
			0.1	0.396	0.458	0.912	1	1
		3	0.05	0.112	0.204	0.758	0.996	1
			0.1	0.19	0.326	0.858	1	1
		5	0.05	0.064	0.148	0.698	0.996	1
			0.1	0.132	0.246	0.822	1	1

The locally stationary process $\{\mathrm{C}_{t,T,\Lambda}\}$ (recall (3.81)) fulfils $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ with $\mathfrak{U}_{0,1} = [0,1]$ just in the case $\Lambda = 0$. In this case, the tuning parameters $b = 1/10$ and either $\beta = 0.3\,Tb^2$ as well as $\mathbf{B}_T \in \{3,5\}$ or $\beta = 0.7\,Tb^2$ and $\mathbf{B}_T \in \{0,3,5\}$ can be regarded as suitable choices because the belonging test versions reject $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ very satisfactorily rarely based on the significance levels $\alpha \in \{0.05, 0.1\}$. In contrast, some other choices of the tuning parameters (in particular, $b = 1/14$, $b = 0.3\,Tb^2$ and $\mathbf{B}_T = 0$) lead to inappropriate high rejection frequencies for $\Lambda = 0$. One explanation therefor may be that the conditions $Tb^{2+2\delta} \overset{T\to\infty}{\longrightarrow} 0$ and $Tb^2 \overset{T\to\infty}{\longrightarrow} \infty$ (as supposed in Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii)) are more restrictive for smaller values of δ than larger ones, whereby $\{\mathrm{C}_{t,T,\Lambda}\}$ with $\Lambda = 0$ fulfils Assumption 2.2 $[\mathbf{StAp}]$ just for $\delta \in (0, 0.1)$. In this context, note also that, according to the Assumptions 3.15 $[\mathbf{W}^*]$ (i) as well as 3.30 $[\mathbf{NW}]$, b impacts the boundaries for the asymptotic growing rates of β and $\mathbf{B}_T$. In addition, $\mathrm{C}_t^{[2]}$ depends on $\mathrm{C}_{t-j}^{[2]}$ for all $j \in \{1, \ldots, 5\}$ quite strongly. Thus, local Newey-West estimation of $\mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\text{error}}$ (recall (3.64)) may have a large impact on the test decision belonging to $\{\mathrm{C}_{t,T,\Lambda}\}$ for $\Lambda \in [0,1)$, whereby, as mentioned above, applying Algorithm **TEST.MDCI.1** to $\{\mathrm{C}_{t,T,\Lambda}\}$ with $\Lambda \in [0,1)$ is not justified by Theorem 3.27 or Remark 3.28 (iv).

Further, if $\Lambda \in [0,1]$ is closer to one, the influence of $\{\mathrm{C}_{t,T}^{[1]}\}$ on $\{\mathrm{C}_{t,T,\Lambda}\}$ is higher, whereas the impact of the stationary process $\left(\mathrm{C}_t^{[2]}\right)_{t\in\mathbb{Z}}$ on $\{\mathrm{C}_{t,T,\Lambda}\}$ is lower, which may be the reason why the relative frequencies of rejecting $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ which are displayed in Table 3.6 increase with growing Λ for fixed b, β, $\mathbf{B}_T$ and α tendentiously. In particular, rejecting $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ reliably is more challenging for smaller $\Lambda \in [0,1]$ than larger ones. Concretely, most of the applied test versions reject $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ with $\mathfrak{U}_{0,1} = [0,1]$ based on the significance levels $\alpha \in \{0.05, 0.1\}$ not reliable for $\Lambda = 0.25$ but very satisfactory often for $\Lambda = 0.5$ and $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ is rejected by the used test versions for (almost) each generated sample path of $\left(\mathrm{C}_{t,T,\Lambda}\right)_{t=1}^{T}$ in the cases $\Lambda \in \{0.75, 1\}$.

Overall, the applied versions of the Algorithms **TEST.MDCI.1** and **TEST.MDCI.2** are quite powerful in both considered simulation studies under the alternative and the (by the relative rejection frequencies) estimated type I error rate is appropriately small for some investigated choices of the tuning parameters b, β as well as $\mathbf{B}_T$ but large for other selections of them.

3.4.3. Estimation of the first change point in the distributions of the stationary approximations

In this subsection, the versions of Algorithm **DETECT.MDCI** that are described in Setting 3.45 are evaluated based on the locally stationary processes $\{\mathrm{A}_{t,T}\}$ (see (3.78)) as well as $\{D_{t,T}\}$, whereby the latter is introduced in the following.

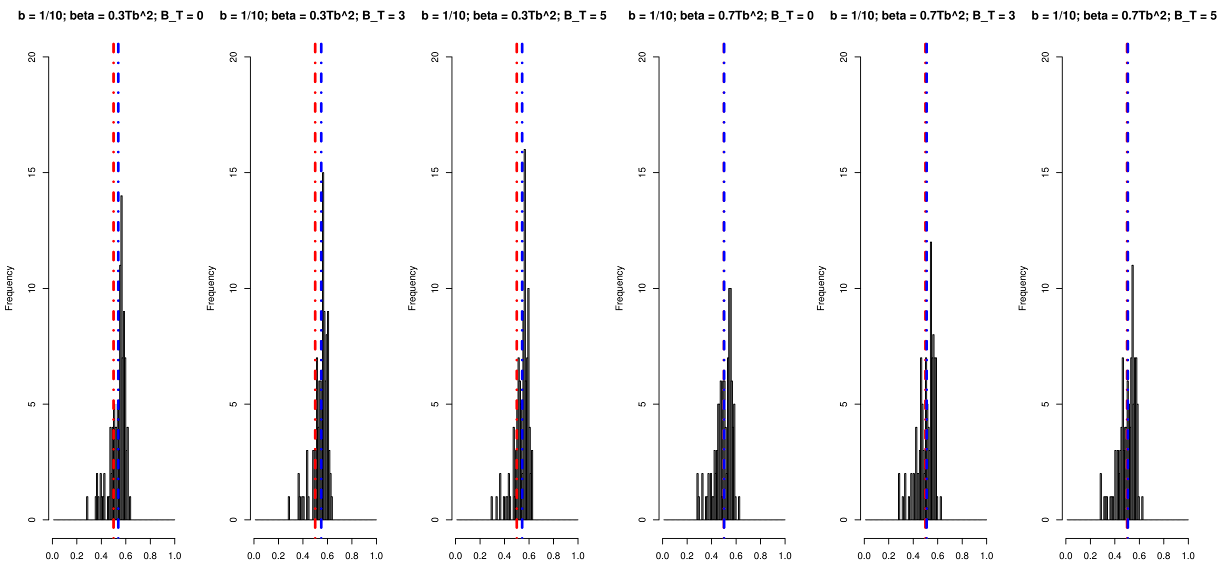
Suppose that the sequence $\left(\varepsilon_t^{[\mathrm{B}]}\right)_{t \in \mathbb{Z}}$ originates from (3.79) and define:

$$\mathrm{D}_{t,T} := \begin{cases} \sum_{j=1}^{100} \frac{1}{j^{3.2}} \varepsilon_{t-j}^{[\mathrm{B}]}, & \text{for } t \leqslant 0.3\,T \\ \left(\frac{200}{3} \cdot \left(\frac{t}{T} - 0.3\right)^{3/2} + 1\right) \sum_{j=1}^{100} \frac{1}{j^{3.2}} \varepsilon_{t-j}^{[\mathrm{B}]}, & \text{for } t > 0.3\,T \end{cases} \quad \forall\, t \in \{1, \ldots, T\},\, T \in \mathbb{N}. \tag{3.82}$$

The locally stationary processes $\{\mathrm{A}_{t,T}\}$ and $\{\mathrm{D}_{t,T}\}$ fulfil Assumption 2.4 $[\mathbf{DM.2}]$ with $\delta \in (1/4, 1]$ (in particular, note that $(0,1) \ni u \mapsto 0.2 \cdot \mathbb{1}_{\{u \in [0,0.5]\}} + \left(\sin\left(2\pi\left(u - 0.5\right)\right)^2 + 0.2\right) \cdot \mathbb{1}_{\{u \in (0.5,1]\}}$ is one-time differentiable with Lipschitz continuous derivative, whereas $(0,1) \ni u \mapsto \mathbb{1}_{\{u \in [0,0.3]\}} + \big(200/3 \cdot (u - 0.3)^{3/2} + 1\big)\mathbb{1}_{\{u \in (0.3,1]\}}$ is one-time differentiable with Hölder continuous derivative - with Hölder exponent $1/2$). Hence, applying Algorithm **DETECT.MDCI** (even with $\mathbf{B}_T = 0$) to these processes is justified.

In the following, 100 sample paths of $\left(\mathrm{A}_{t,T}\right)_{t=1}^T$ and of $\left(\mathrm{D}_{t,T}\right)_{t=1}^T$ with $T = 1000$ are generated independently of each other and the versions of Algorithm **DETECT.MDCI** given in Setting 3.45 are applied to each of these sample paths. Figure 3.1, which is based on the computed sample paths of $\left(\mathrm{A}_{t,T}\right)_{t=1}^T$, and Figure 3.2, that is based on the generated sample paths of $\left(\mathrm{D}_{t,T}\right)_{t=1}^T$, show belonging histograms of the estimated first change point in the distributions of the stationary approximations (see (3.69) for a formal definition of this change point). In each histogram, the true first change point $\big(\mathbb{V} = 0.5$ for $\left(\mathrm{A}_{t,T}\right)_{t=1}^T$ as well as $\mathbb{V} = 0.3$ for $\left(\mathrm{D}_{t,T}\right)_{t=1}^T\big)$ is indicated by a red dashed line and the arithmetic mean associated with this histogram by a blue dashed line.
Thereby, all of these arithmetic means that are marked in the histograms contained in the Figures 3.1 and 3.2 are close or very close to the belonging true first change point $\mathbb{V}$. In addition, the given histograms are quite narrow (especially those histograms that belong to the tuning parameters $b = 1/10$, $\beta = 0.3\,Tb^2$ and $\mathbf{B}_T \in \{0, 3, 5\}$ lie tightly around $\mathbb{V}$ in both figures), which indicates that these first change points are estimated adequately.

Figure 3.1.: Histograms of the estimated first change point belonging to $\left(\mathrm{A}_{t,T}\right)_{t=1}^T$



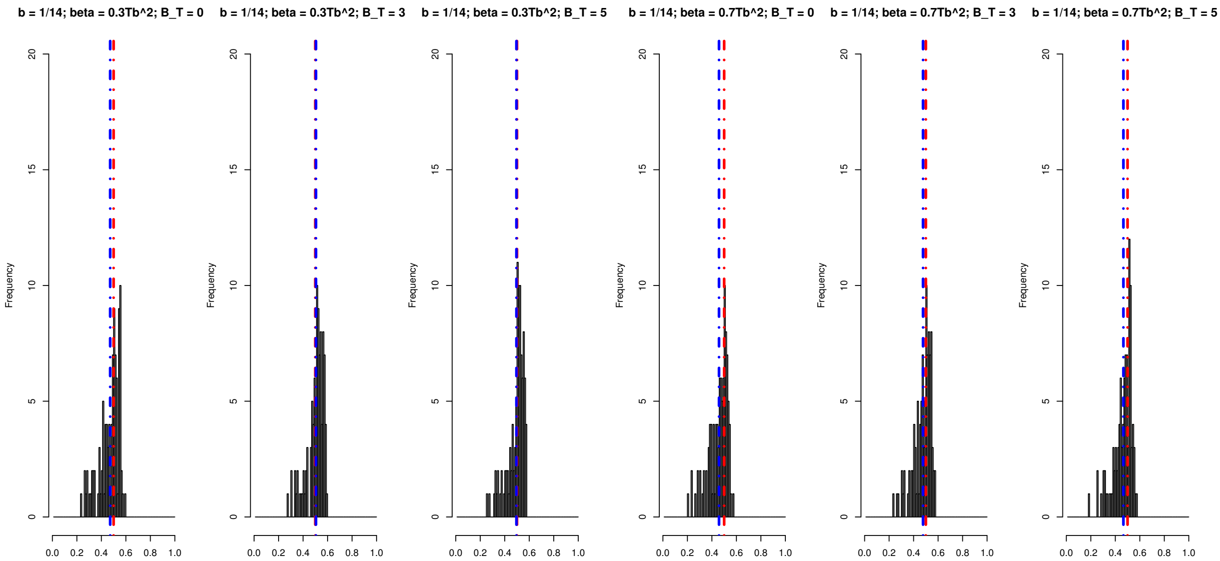
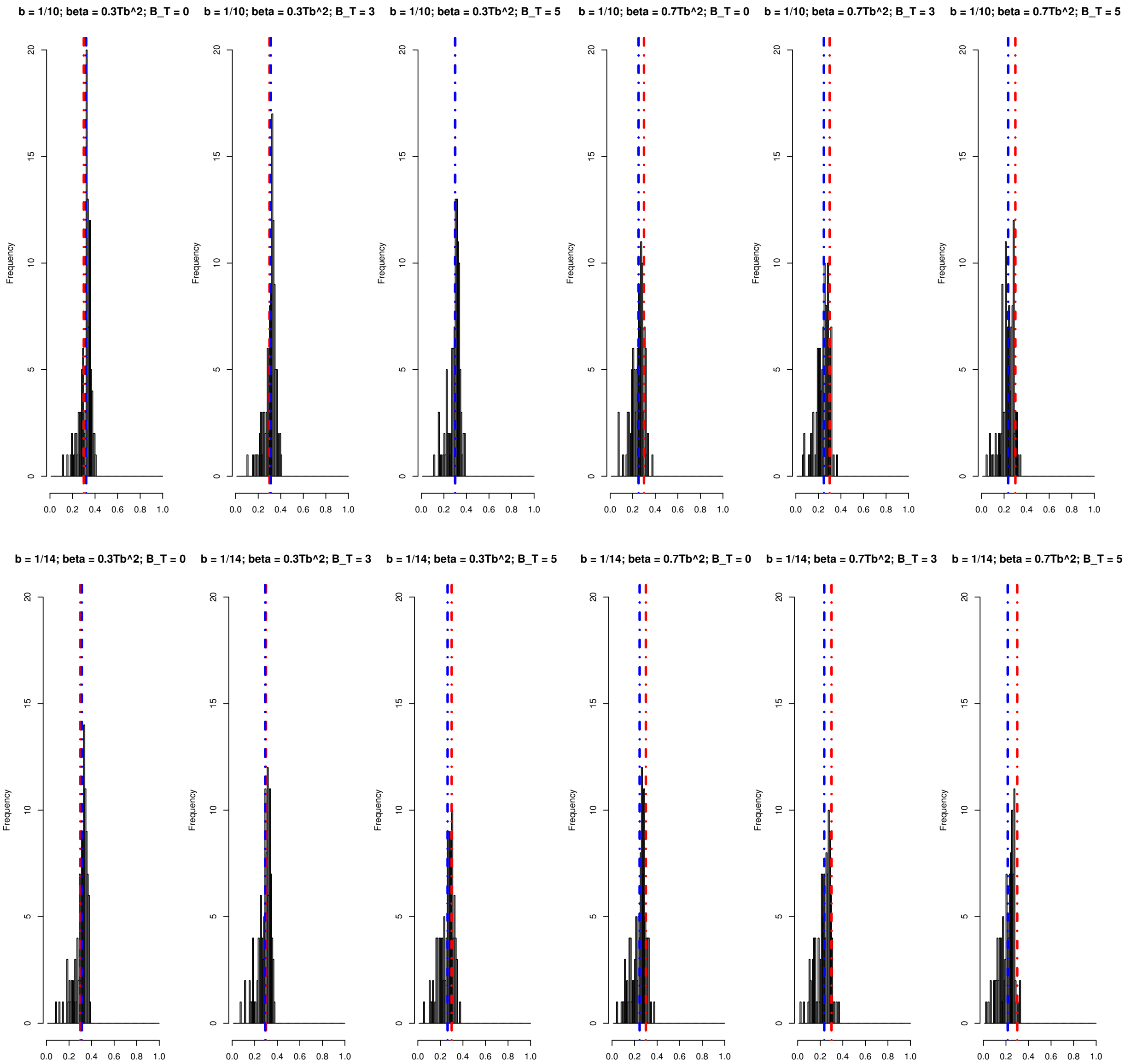


Figure 3.2.: Histograms of the estimated first change point belonging to $\left(\mathrm{D}_{t,T}\right)_{t=1}^{T}$



4. Testing for independence

In the present chapter, it is tested whether dependences between two locally stationary Bernoulli shift processes exist, whereby the concrete test problem is introduced in (4.1) given below. In order to improve the readability of the investigations presented in this chapter, a consistent level-alpha test for a special case of this test problem is constructed at first and, thereafter, this test is modified in such a manner that it allows to handle the original test problem. Further, at the end of this chapter, the finite sample behaviour of the proposed tests is investigated by using simulation studies.

4.1. Preliminaries

At first, the central assumption which underlies this chapter is stated in the following.

Assumption 4.1 [INDEP]. *Let $d := d_1 + d_2$ for fixed $d_1, d_2 \in \mathbb{N}$. Moreover, assume that $\{X_{t,T}\}$ is an $\mathbb{R}^d$-valued locally stationary Bernoulli shift process which fulfils Assumption 2.4 $[\mathbf{DM.3}]$ (especially also Assumption 2.2 $[\mathbf{StAp}]$) and owns the following representation:*

$$X_{t,T} = \mathbf{H}_{t,T}\left(\mathcal{F}_t\right) = \left(X_{t,T}^{[1]\prime}, X_{t,T}^{[2]\prime}\right)' \quad \forall\, t \in \{1, \ldots, T\},\ T \in \mathbb{N},$$

whereby $\mathbf{H}_{t,T}$ (with $t \in \{1, \ldots, T\}$, $T \in \mathbb{N}$) as well as $\mathcal{F}_t$ (with $t \in \mathbb{Z}$) are defined according to Definition 2.1 and $X_{t,T}^{[k]}$ (with $k \in \{1, 2\}$, $t \in \{1, \ldots, T\}$, $T \in \mathbb{N}$) is an $\mathbb{R}^{d_k}$-valued random variable.
Furthermore, introduce for the stationary approximations $\{\widetilde{X}_t(u)\}$ of $\{X_{t,T}\}$ the following notation:

$$\widetilde{X}_t(u) = \mathbf{H}\left(u, \mathcal{F}_t\right) = \left(\left(\widetilde{X}_t^{[1]}(u)\right)', \left(\widetilde{X}_t^{[2]}(u)\right)'\right)' \quad \forall\, t \in \mathbb{Z},\ u \in [0, 1],$$

whereby $\mathbf{H}(u, \cdot)$ originates from Assumption 2.2 $[\mathbf{StAp}]$ (iii) and $\widetilde{X}_t^{[k]}(u)$ is an $\mathbb{R}^{d_k}$-valued random variable.

In this chapter, the following test problem is investigated under Assumption 4.1 $[\mathbf{INDEP}]$ for arbitrary, non-empty, fixed as well as finite sets $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$ (note that $\perp\!\!\!\perp$ denotes (stochastic) independence and $\not\perp\!\!\!\perp$ (stochastic) dependence):

$$\begin{aligned} &\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}:\ \left(\widetilde{X}_{-\mathfrak{d}}^{[1]}(u)\right)_{\mathfrak{d}\in\mathfrak{D}_1} \perp\!\!\!\perp \left(\widetilde{X}_{-\mathfrak{d}}^{[2]}(u)\right)_{\mathfrak{d}\in\mathfrak{D}_2} \ \forall\, u \in [0, 1] \quad \text{versus} \\ &\mathcal{H}_{1,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}:\ \exists u \in [0, 1] : \left(\widetilde{X}_{-\mathfrak{d}}^{[1]}(u)\right)_{\mathfrak{d}\in\mathfrak{D}_1} \not\perp\!\!\!\perp \left(\widetilde{X}_{-\mathfrak{d}}^{[2]}(u)\right)_{\mathfrak{d}\in\mathfrak{D}_2}. \end{aligned} \tag{4.1}$$

This test problem is formulated based on the negative points in time $-\mathfrak{d}$ (with $\mathfrak{d} \in \mathfrak{D}_1$ and $\mathfrak{d} \in \mathfrak{D}_2$, respectively), which is motivated by the fact that, for $k \in \{1, 2\}$, $u \in [0, 1]$ and $j, t \in \{1, \ldots, T\}$ with $j \leqslant t$, the random variables $X_{j,T}^{[k]}$ and $\widetilde{X}_j^{[k]}(u)$ are measurable with respect to the sigma-algebra generated by $\mathcal{F}_t := (\varepsilon_t, \varepsilon_{t-1}, \ldots)$. This property turns out to be useful in Section 4.3.

Remark 4.2. *(i) Straightforward arguments show that if Assumption 4.1 $[\mathbf{INDEP}]$ is valid, the triangular array $\{X_{t,T}^{[k]}\} := \{X_{t,T}^{[k]} : t \in \{1, \ldots, T\}\}_{T=1}^{\infty}$ (with $k \in \{1, 2\}$) will be an $\mathbb{R}^{d_k}$-valued locally stationary process for which Assumption 2.4 $[\mathbf{DM.3}]$ holds with the belonging stationary approximations $\{\widetilde{X}_t^{[k]}(u)\} := \{\widetilde{X}_t^{[k]}(u) : t \in \mathbb{Z}\}$ (with $u \in [0, 1]$). Hence, (4.1) can be regarded as a test problem that concerns dependences between the stationary approximations of two locally stationary processes.*

(ii) Denote for $k \in \{1, 2\}$ the elements of the set $\mathfrak{D}_k$ as $\mathfrak{d}_{k,1}, \ldots, \mathfrak{d}_{k,\#\mathfrak{D}_k}$ and define for all $t \in \mathbb{Z}$,

$u \in [0,1]$, $s^{[k]} \in \mathbb{R}^{d_k \cdot \#\mathfrak{D}_k}$, $s := \left(s^{[1]'}, s^{[2]'}\right)'$:

$$\widetilde{X}^{[k]}_{\mathfrak{D}_k,t}(u) := \left(\left(\widetilde{X}^{[k]}_{t-\mathfrak{d}_{k,1}}(u)\right)', \ldots, \left(\widetilde{X}^{[k]}_{t-\mathfrak{d}_{k,\#\mathfrak{D}_k}}(u)\right)'\right)', \quad \widetilde{X}_{\mathfrak{D}_1,\mathfrak{D}_2,t}(u) := \left(\left(\widetilde{X}^{[1]}_{\mathfrak{D}_1,t}(u)\right)', \left(\widetilde{X}^{[2]}_{\mathfrak{D}_2,t}(u)\right)'\right)'$$

$$\textit{and} \quad \mathbb{Q}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) := \mathbb{E}\left[e^{i\left\langle s, \widetilde{X}_{\mathfrak{D}_1,\mathfrak{D}_2,0}(u)\right\rangle}\right] - \mathbb{E}\left[e^{i\left\langle s^{[1]}, \widetilde{X}^{[1]}_{\mathfrak{D}_1,0}(u)\right\rangle}\right]\mathbb{E}\left[e^{i\left\langle s^{[2]}, \widetilde{X}^{[2]}_{\mathfrak{D}_2,0}(u)\right\rangle}\right]. \tag{4.2}$$

Since two random variables X (with characteristic function φ_X) and Y (with characteristic function φ_Y) that live on the same probability space but do not need to own the same dimension will be independent of each other if and only if the characteristic function of $(X', Y')'$ (denoted as $\varphi_{X,Y}$) fulfils $\varphi_{X,Y} = \varphi_X \cdot \varphi_Y$ (cf. [43, Kankainen (1995), p. 23]), the test problem (4.1) can be rewritten as:

$$\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}: \ \mathbb{Q}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) = 0 \ \ \forall\, u \in [0,1],\ s \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2} \quad \textit{versus}$$
$$\mathcal{H}^{\text{indep}}_{1,\mathfrak{D}_1,\mathfrak{D}_2}: \ \exists u \in [0,1],\ s \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2} : \mathbb{Q}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) \neq 0.$$

(iii) For many practical applications, it may be more useful to formulate statements about dependences between the locally stationary processes $\left\{X^{[1]}_{t,T}\right\}$ and $\left\{X^{[2]}_{t,T}\right\}$ than about dependences between the belonging stationary approximations. Such statements can be derived in the following manner under the null hypothesis and under the alternative.
Therefor, note at first that $\left(\widetilde{X}_{\mathfrak{D}_1,\mathfrak{D}_2,t}(u)\right)_{t\in\mathbb{Z}}$ is stationary for all $u \in [0,1]$ due to Theorem 3.35 in [78, White (2001), p. 44], such that it holds for all $u \in [0,1]$, $s^{[k]} \in \mathbb{R}^{d_k \cdot \#\mathfrak{D}_k}$, $s := \left(s^{[1]'}, s^{[2]'}\right)'$, $t \in \mathbb{Z}$:

$$\mathbb{Q}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) := \mathbb{E}\left[e^{i\left\langle s^{[1]}, \widetilde{X}^{[1]}_{\mathfrak{D}_1,t}(u)\right\rangle} e^{i\left\langle s^{[2]}, \widetilde{X}^{[2]}_{\mathfrak{D}_2,t}(u)\right\rangle}\right] - \mathbb{E}\left[e^{i\left\langle s^{[1]}, \widetilde{X}^{[1]}_{\mathfrak{D}_1,t}(u)\right\rangle}\right]\mathbb{E}\left[e^{i\left\langle s^{[2]}, \widetilde{X}^{[2]}_{\mathfrak{D}_2,t}(u)\right\rangle}\right]. \tag{4.3}$$

Moreover, assume for the remaining part of this remark that $T \geqslant 1 + \max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2} \mathfrak{d}$ and define for all $k \in \{1,2\}$, $t \in \{1 + \max_{\mathfrak{d}\in\mathfrak{D}_k} \mathfrak{d}, \ldots, T\}$ (recall that $\mathfrak{d}_{k,1}, \ldots, \mathfrak{d}_{k,\#\mathfrak{D}_k}$ denote the elements of the set $\mathfrak{D}_k$):

$$X^{[k]}_{\mathfrak{D}_k,t,T} := \left(X^{[k]'}_{t-\mathfrak{d}_{k,1},T}, \ldots, X^{[k]'}_{t-\mathfrak{d}_{k,\#\mathfrak{D}_k},T}\right)'. \tag{4.4}$$

If $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ is valid, which implies $\mathbb{Q}_{\mathfrak{D}_1,\mathfrak{D}_2}(t/T, s) = 0 \ \forall\, t \in \{1,\ldots,T\}$, $T \in \mathbb{N}$, $s \in \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1 + d_2\cdot\#\mathfrak{D}_2}$ due to Remark 4.2 (ii), one will obtain from (4.3), Assumption 2.2 [**StAp**] *(ii) and Remark 2.3:*

$$\sup_{t=1+\max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2}\mathfrak{d},\ldots,T}\left|\mathbb{E}\left[e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,t,T}\right\rangle} e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,t,T}\right\rangle}\right] - \mathbb{E}\left[e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,t,T}\right\rangle}\right]\mathbb{E}\left[e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,t,T}\right\rangle}\right]\right|$$
$$\overset{T\to\infty}{\longrightarrow} 0 \ \ \forall\, s^{[k]} \in \mathbb{R}^{d_k\cdot\#\mathfrak{D}_k},\ k \in \{1,2\}. \tag{4.5}$$

Hence, under $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$, the (sub-)processes $\left(X^{[1]}_{t-\mathfrak{d},T}\right)_{\mathfrak{d}\in\mathfrak{D}_1}$ and $\left(X^{[2]}_{t-\mathfrak{d},T}\right)_{\mathfrak{d}\in\mathfrak{D}_2}$ can be interpreted as approximately totally independent of each other for all $t \in \{1 + \max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2}\mathfrak{d}, \ldots, T\}$ and large $T \in \mathbb{N}$.
Instead, if $\mathcal{H}^{\text{indep}}_{1,\mathfrak{D}_1,\mathfrak{D}_2}$ holds, it will follow similarly to (4.5):

$$\exists\, u \in [0,1],\ s := \left(s^{[1]'}, s^{[2]'}\right)' \in \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1} \times \mathbb{R}^{d_2\cdot\#\mathfrak{D}_2},\ \epsilon > 0 \textit{ and } \overline{T} \in \mathbb{N}:$$
$$\left|\mathbb{E}\left[e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,\overline{t_u},T}\right\rangle} e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,\overline{t_u},T}\right\rangle}\right] - \mathbb{E}\left[e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,\overline{t_u},T}\right\rangle}\right]\mathbb{E}\left[e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,\overline{t_u},T}\right\rangle}\right]\right| > \epsilon \quad \forall\, T \geqslant \overline{T}$$
$$\textit{with} \quad \overline{t_u} := \max\left\{1 + \max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2}\mathfrak{d}, \lceil uT\rceil\right\}.$$

Therefore, under $\mathcal{H}^{\text{indep}}_{1,\mathfrak{D}_1,\mathfrak{D}_2}$, the (sub-)processes $\left(X^{[1]}_{\overline{t_u}-\mathfrak{d},T}\right)_{\mathfrak{d}\in\mathfrak{D}_1}$ and $\left(X^{[2]}_{\overline{t_u}-\mathfrak{d},T}\right)_{\mathfrak{d}\in\mathfrak{D}_2}$ can be re-

garded as approximately dependent of each other for large $T \in \mathbb{N}$.
These considerations show for instance that, as mentioned in Section 1.3, a test for (4.1) (with suitable choices of $\mathfrak{D}_1, \mathfrak{D}_2$) allows to investigate under appropriate assumptions whether days, weeks or months exist in which the data generating processes that belong to daily log returns of two stocks depend on each other.

In the following section, a procedure for deciding the test problem (4.1) in the case $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ is evolved, i. e., one that investigates whether the stationary approximations are pairwise independent of each other in the same points in time. Thereafter, the belonging results are generalized in the subsequent section to obtain a test for total blockwise independence, i. e., one that belongs to the problem (4.1) with arbitrary, non-empty, fixed and finite sets $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$.

4.2. Testing for pairwise independence in the same points in time

4.2.1. Construction of a test statistic and derivation of its asymptotic behaviour

To investigate the test problem (4.1) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$, note at first that Assumption 2.2 $[\mathbf{StAp}]$ (ii) (which is contained in Assumption 4.1 $[\mathbf{INDEP}]$) and Remark 4.2 (ii) show that $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ will be valid if and only if (recall Definition 2.6 as well as $d = d_1 + d_2$ according to Assumption 4.1 $[\mathbf{INDEP}]$):

$$\mathfrak{Q} := \int_0^1 \int_{\mathbb{R}^d} |\mathbb{Q}(u,s)|^2 \, \mathbf{w}(s) \, ds \, du = 0 \quad \text{with}$$

$$\mathbb{Q}(u,s) := \varphi(u,s) - \varphi^{[1]}\left(u, s^{[1]}\right)\varphi^{[2]}\left(u, s^{[2]}\right), \quad \varphi^{[k]}\left(u, s^{[k]}\right) := \mathbb{E}\left[e^{i\left\langle s^{[k]}, \tilde{X}_0^{[k]}(u)\right\rangle}\right]$$
$$\forall\, u \in [0,1],\ s^{[k]} \in \mathbb{R}^{d_k},\ s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \tag{4.6}$$

and a weight function $\mathbf{w}$ that fulfils the next assumption.

Assumption 4.3 [WEI.2] (Weight function - Part 2)**.**
Let $d \in \mathbb{N}$ originate from Assumption 4.1 $[\mathbf{INDEP}]$ and $\delta \in (0,1]$ from Assumption 2.2 $[\mathbf{StAp}]$ (which is contained in Assumption 4.1 $[\mathbf{INDEP}]$). The weight function $\mathbf{w}\colon \mathbb{R}^d \to [0,\infty)$ is defined as a Riemann integrable function which is Lebesgue almost everywhere positive and fulfils:

$$\int_{\mathbb{R}^d} \left(1 + |s|_1^{2+2\delta} + |s|_1^3\right) \mathbf{w}(s) \, ds < \infty. \tag{4.7}$$

Remark 4.4. *The Assumption 4.3 $[\mathbf{WEI.2}]$ is stronger than necessary to ensure that $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ is equivalent to $\mathfrak{Q} = 0$ because this assumption is also used below to construct and justify a test statistic for the present test problem that is based on an estimator for $\mathfrak{Q}$. In addition, it is worth mentioning that Assumption 4.3 $[\mathbf{WEI.2}]$ is more restrictive for $\delta \in (0, 1/2)$ than Assumption 3.1 $[\mathbf{WEI.1}]$, which underlies Chapter 3. However, $\delta \in (0,1]$ is (commonly) unknown in applications, such that it is necessary in practise to take a Riemann integrable and Lebesgue almost everywhere positive function $\mathbf{w}\colon \mathbb{R}^d \to [0,\infty)$ for which $\int_{\mathbb{R}^d} \left(1+|s|_1^4\right) \mathbf{w}(s) \, ds < \infty$ holds to ensure that Assumption 3.1 $[\mathbf{WEI.1}]$ or 4.3 $[\mathbf{WEI.2}]$ is fulfilled, whereby some of these weight functions are given in Example 3.2. Hence, the difference between Assumption 3.1 $[\mathbf{WEI.1}]$ and 4.3 $[\mathbf{WEI.2}]$ is (often) not of practical interest.*

In order to derive a test statistic for the problem (4.1) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ from $\mathfrak{Q}$, consider at first under Assumption 4.1 $[\mathbf{INDEP}]$ the following estimator for $\varphi^{[k]}\left(u, s^{[k]}\right)$ with $k \in \{1,2\}$, $u \in [0,1]$, $s^{[k]} \in \mathbb{R}^{d_k}$:

$$\hat{\varphi}^{[k]}\left(u, s^{[k]}\right) := \hat{\varphi}_T^{[k]}\left(u, s^{[k]}\right) := \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) e^{i\left\langle s^{[k]}, X_{t,T}^{[k]}\right\rangle}, \tag{4.8}$$

whereby K_b originates from Definition 2.11 and the underlying kernel K as well as bandwidth b should fulfil the next assumption.

Assumption 4.5 [K&b.2] (Kernel and bandwidth - Part 2)**.**

(i) Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ *(i) with* $\mathfrak{U}_0 = 0$ *and* $\mathfrak{U}_1 = 1$ *should hold for the kernel* K.

(ii) Let the sequence of the bandwidths $b := b_T$ *suffice Assumption 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *(ii) and, in addition,* $Tb^2/\ln(T)^3 \to \infty$ *for* $T \to \infty$.

Remark 4.6. *(i) Although the parameter* $\delta \in (0,1]$ *is commonly unknown in practise and it is expectable that it cannot be estimated appropriately in general, a sequence of bandwidths that fulfils Assumption 4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *(ii) can be selected* $\big($*e. g.,* $b := b_T := 1/4\mathbb{1}_{\{T=1\}} + \min\left\{T^{-1/2}\ln(T)^{1.6}, 1/4\right\}\mathbb{1}_{\{T\geqslant 2\}}\ \forall\, T \in \mathbb{N}\big)$. *Note also that the supposition* $Tb^2/\ln(T)^3 \overset{T\to\infty}{\longrightarrow} \infty$ *(contained in Assumption 4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *(ii)) is slightly stronger than* $Tb^2 \overset{T\to\infty}{\longrightarrow} \infty$ *(as demanded in Assumption 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *(ii)). However, all asymptotic results given in the present chapter remain valid under weaker assumptions on* b *but demanding* $Tb^2/\ln(T)^3 \overset{T\to\infty}{\longrightarrow} \infty$ *allows to shorten some belonging proofs significantly.*

The Propositions 2.12 (with $\mathfrak{U}_{0,1,b} = [b, 1-b]$ under Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$) and 2.14 motivate the following estimator for $\mathfrak{Q}$, which can be used to test (4.1) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ (recall (4.6), Definition 2.11 as well as (4.8)):

$$\widehat{\mathfrak{Q}}_T := \int_b^{1-b}\int_{\mathbb{R}^d} \left|\widehat{\mathbb{Q}}_T(u,s)\right|^2 \mathbf{w}(s)\,ds\,du \quad \text{with} \quad \widehat{\mathbb{Q}}_T(u,s) := \widehat{\varphi}\,(u,s) - \widehat{\varphi}^{[1]}\left(u, s^{[1]}\right)\cdot \widehat{\varphi}^{[2]}\left(u, s^{[2]}\right)$$

$$\forall\, u \in [0,1],\ s^{[k]} \in \mathbb{R}^{d_k},\ s := \left(s^{[1]\prime}, s^{[2]\prime}\right)'. \qquad (4.9)$$

Thereby, it is worth mentioning that $\widehat{\mathfrak{Q}}_T$ equals the integral (with respect to $u \in [b, 1-b]$) of the statistic (5.2) considered in [4, Beering (2021), p. 72]. (Note that the belonging test problem investigated in [4, Beering (2021)] is briefly described in Section 1.3.)

Remark 4.7. *(i) In view of the research presented in Chapter 3, it is expectable that an alternative useful estimator for* $\mathfrak{Q}$ *results by replacing the integral with respect to* $u \in [b, 1-b]$ *contained in* $\widehat{\mathfrak{Q}}_T$ *by a Riemann sum based on the midpoint rule with the evolution points* $u_{T,\mathfrak{U}_{0,1},k}$ *with* $\mathfrak{U}_{0,1} = [0,1]$ *that originate from Definition 3.8 (i). This estimator for* $\mathfrak{Q}$ *may also be suitable for testing (4.1) with* $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ *but it is expectable for practical applications, i. e., for fixed* $T \in \mathbb{N}$*, that this approach is tendentiously less powerful in detecting dependences between* $\widetilde{X}_0^{[1]}(u)$ *and* $\widetilde{X}_0^{[2]}(u)$ *for* $u \in [b, 1-b]$ *than a test which is based on* $\widehat{\mathfrak{Q}}_T$.
In contrast, such Riemann approximations have been used in Chapter 3 because the EMDCI as well as NEMDCI should not just allow to detect whether distribution changes exist but also to estimate confidence intervals for the MDCI as well as NMDCI on the entire rescaled time period $[0,1]$ *and to estimate the first change point, whereby, for the latter, a good estimation of the MDCI belonging to the rescaled time intervals* $[0, w]$ *with* $w \in (0,1]$ *is useful.*

(ii) It is also conceivable to test (4.1) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ *based on integrated distance covariances, i. e., based on the weight function* $\mathbf{w}_{d,q}$ *(see (3.44)). However, the following considerations explain that this approach is accompanied by some issues in the present framework.*
If the weight function $\mathbf{w}_{d,q}$ *is used* $\big($*which is non-integrable according to [72, Székely et al. (2007), p. 2771], i. e.,* $\int_{\mathbb{R}^d}\mathbf{w}_{d,q}(s)ds \nless \infty\big)$*, the Propositions 2.12 and 2.14 are not suitable for justifying the estimator* $\widehat{\mathfrak{Q}}_T$ *for* $\mathfrak{Q}$*, whereby this estimation is essential for analyzing the behaviour of the present test statistic under the considered null hypothesis and alternative. To overcome this issue, define at first* $\widetilde{X}_0^{[k]\times}(u)$ *and* $\widetilde{X}_0^{[k]\times\times}(u)$ *(with* $k \in \{1,2\}$*) in such a manner that* $\big(\widetilde{X}_0^{[k]}(u), \widetilde{X}_0^{[k]\times}(u), \widetilde{X}_0^{[k]\times\times}(u)\big)$ *is a sequence of i. i. d. random variables for all* $u \in [0,1]$ *(whereby constructing them is possible by using independent copies of the sequence* $(\varepsilon_t)_{t\in\mathbb{Z}}$ *that originates from Definition 2.1). Remark 3 in [72, Székely et al. (2007), p. 2783] motivates to test*

(4.1) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ by using an kernel-bandwidth-based estimator for the following expression (with $q \in (0,2)$):

$$\int_0^1 \mathbb{E}\left[\left|\widetilde{X}_0^{[1]}(u) - \widetilde{X}_0^{[1]\times}(u)\right|_2^q \left|\widetilde{X}_0^{[2]}(u) - \widetilde{X}_0^{[2]\times}(u)\right|_2^q\right] + \mathbb{E}\left[\left|\widetilde{X}_0^{[1]}(u) - \widetilde{X}_0^{[1]\times}(u)\right|_2^q\right]$$
$$\cdot\,\mathbb{E}\left[\left|\widetilde{X}_0^{[2]}(u) - \widetilde{X}_0^{[2]\times}(u)\right|_2^q\right] - 2\mathbb{E}\left[\left|\widetilde{X}_0^{[1]}(u) - \widetilde{X}_0^{[1]\times}(u)\right|_2^q \left|\widetilde{X}_0^{[2]\times}(u) - \widetilde{X}_0^{[2]\times\times}(u)\right|_2^q\right] du. \tag{4.10}$$

However, if $\mathcal{H}^{\text{indep}}_{1,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ is fulfilled, (4.10) will not be well-defined in general for $q > (1+\delta)/2$ under Assumption 2.2 $[\mathbf{StAp}]$ (which underlies Assumption 4.1 $[\mathbf{INDEP}]$). This causes issues in regard to power considerations of a test for (4.1) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ that is based on an estimator for (4.10). Further, for $q \leqslant (1+\delta)/2$, (4.10) is well-defined under Assumption 2.2 $[\mathbf{StAp}]$ but the contained expectations are just Hölder continuous (with exponent q) with respect to u, which is problematic (especially for small values of the commonly unknown parameter δ) for the estimation of (4.10) by kernel-bandwidth-based estimators.
In particular, note also that Theorem 4.3 in [40, Jentsch et al. (2020a), p. 126], which states for linear locally stationary processes the asymptotic behaviour of a kernel-bandwidth-based estimator for (4.10) in the case $q = 1$ with fixed $u \in (0,1)$ (instead of the integral with respect to u), demands that the $|\cdot|_1$-norm of the innovations that underlie the investigated linear locally stationary processes own more than four finite moments.
Overall, these considerations motivate that the present test is not evolved based on distance covariances but on weight functions for which Assumption 4.3 $[\mathbf{WEI.2}]$ holds to handle the quite weak assumed moment conditions. Thereby, it is worth mentioning that the statistic that underlies the test for independence proposed in [4, Beering (2021)] is also not based on distance covariances but on integrable weight functions in order to consider linear locally stationary processes for which the same (weak) moment conditions as demanded in Assumption 2.2 $[\mathbf{StAp}]$ hold.

Before the asymptotic behaviour of $\widehat{\mathfrak{Q}}_T$ is presented in Theorem 4.9 given below, the following lemma is stated, which ensures well-definedness of some expressions that are introduced in Theorem 4.9.

Lemma 4.8. *Suppose that Assumption 4.1 $[\mathbf{INDEP}]$ holds. Define for all $u \in [0,1]$, $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $x := \left(x^{[1]\prime}, x^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $t \in \mathbb{Z}$, $\mathrm{R} \in \{\Re, \Im\}$ (recall (4.6)):*

$$\mathbf{g}_{u,s}(x) := e^{i\langle s,x\rangle} - \varphi^{[1]}\left(u, s^{[1]}\right) \cdot e^{i\left\langle s^{[2]},x^{[2]}\right\rangle} - \varphi^{[2]}\left(u, s^{[2]}\right) \cdot e^{i\left\langle s^{[1]},x^{[1]}\right\rangle},$$
$$\widetilde{\mathbf{G}}_t(u,s) := \mathbf{g}_{u,s}\left(\widetilde{X}_t(u)\right) \quad \text{and} \quad \widetilde{\mathbf{G}}_{t,\mathrm{R}}(u,s) := \mathrm{R}\left\{\widetilde{\mathbf{G}}_t(u,s)\right\}. \tag{4.11}$$

Then, one obtains for all $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$, $s_1, s_2 \in \mathbb{R}^d$:

$$\sum_{t\in\mathbb{Z}} \sup_{u\in[0,1]} \left|\mathrm{Cov}\left(\widetilde{\mathbf{G}}_{0,\mathrm{R}_1}(u,s_1), \widetilde{\mathbf{G}}_{t,\mathrm{R}_2}(u,s_2)\right)\right| \leqslant C\left(|s_1|_1 + |s_2|_1 + 1\right).$$

The following theorem describes the asymptotic behaviour of $\widehat{\mathfrak{Q}}_T$ under the null hypothesis and the alternative of the test problem (4.1) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$.

Theorem 4.9 (Asymptotic behaviour of $\widehat{\mathfrak{Q}}_T$).
Let the Assumptions 4.1 $[\mathbf{INDEP}]$, 4.3 $[\mathbf{WEI.2}]$ and 4.5 $[\mathbf{K\&b.2}]$ be fulfilled. Moreover, define:

$$\gamma_{\widetilde{\mathbf{G}}}(u,s,t) := \mathrm{Cov}\left(\widetilde{\mathbf{G}}_{0,\Re}(u,s), \widetilde{\mathbf{G}}_{t,\Re}(u,s)\right) + \mathrm{Cov}\left(\widetilde{\mathbf{G}}_{0,\Im}(u,s), \widetilde{\mathbf{G}}_{t,\Im}(u,s)\right) \quad \forall\, u \in [0,1],\ s \in \mathbb{R}^d,\ t \in \mathbb{Z},$$
$$\mathbf{Bias}^{\text{indep}}_T := \frac{1}{\sqrt{b}} \int_{-1}^{1} K(z)^2\,dz \cdot \int_{\mathbb{R}^d} \int_0^1 \sum_{t\in\mathbb{Z}} \gamma_{\widetilde{\mathbf{G}}}(u,s,t)\,du\,\mathbf{w}(s)\,ds \tag{4.12}$$

as well as:

$$\mathbf{Cov}^{\text{indep}}_{\mathrm{R}_1,\mathrm{R}_2}(u,s_1,s_2) := \sum_{t\in\mathbb{Z}} \mathrm{Cov}\left(\widetilde{\mathbf{G}}_{0,\mathrm{R}_1}(u,s_1), \widetilde{\mathbf{G}}_{t,\mathrm{R}_2}(u,s_2)\right)$$
$$\forall\, \mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\},\ u \in [0,1],\ s_1, s_2 \in \mathbb{R}^d \quad \text{and}$$

$$\sigma^{\text{indep}} := 2 \int_{-2}^{2} \left(\int_{-1}^{1} K(q) K(q+v)\, dq \right)^2 dv \cdot \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 \mathbf{Cov}^{\text{indep}}_{\Re,\Re}(u, s_1, s_2)^2 + \mathbf{Cov}^{\text{indep}}_{\Im,\Im}(u, s_1, s_2)^2 + 2\, \mathbf{Cov}^{\text{indep}}_{\Re,\Im}(u, s_1, s_2)^2 \, du\, \mathbf{w}(s_2)\, ds_2\, \mathbf{w}(s_1)\, ds_1. \tag{4.13}$$

(i) If $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ holds, one will obtain for $T \to \infty$:

$$T\sqrt{b}\, \widehat{\mathfrak{Q}}_T - \mathbf{Bias}^{\text{indep}}_T \xrightarrow{\text{d}} Z^{\text{indep}} \quad \textit{with} \quad Z^{\text{indep}} \sim \mathcal{N}\left(0, \sigma^{\text{indep}}\right). \tag{4.14}$$

(ii) Assume that $\mathcal{H}^{\text{indep}}_{1,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ is valid and consider an arbitrary sequence $(\tau_T)_{T\in\mathbb{N}}$ of deterministic real numbers which fulfils (3.54). Then, it holds:

$$\lim_{T\to\infty} \mathbb{P}\left(T\sqrt{b}\, \widehat{\mathfrak{Q}}_T - \mathbf{Bias}^{\text{indep}}_T > \tau_T\right) = 1.$$

Remark 4.10. *Lemma 4.8 and Assumption 4.3 $[\mathbf{WEI.2}]$ ensure that $\mathbf{Bias}^{\text{indep}}_T$, $\mathbf{Cov}^{\text{indep}}_{\mathrm{R}_1,\mathrm{R}_2}(u, s_1, s_2)$ as well as σ^{indep} are well-defined.*

Commonly, in practical applications, $\mathbf{Bias}^{\text{indep}}_T$ and σ^{indep} cannot be calculated. Thus, a dependent wild bootstrap approach is used in the following to estimate p-values that belong to the considered test statistic. Instead, it is also possible to estimate $\mathbf{Bias}^{\text{indep}}_T$ as well as σ^{indep} directly by local Newey-West estimators which are similar to those proposed in Subsection 3.2.3 and allow to transform the test statistic in an asymptotically standard-normally distributed one, that can be used to estimate p-values. However, L^2-distance-based test statistics (whereby $\widehat{\mathfrak{Q}}_T$ is one of them) are commonly skewly distributed for a fixed number of observations, such that transforming them into approximately standard-normally distributed random variables works tendentiously less well than using a suitable bootstrap procedure which mimics this skewness.

4.2.2. Bootstrap-based estimation of p-values

In order to obtain a test for the problem (4.1) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ based on Theorem 4.9 and an appropriate bootstrap which yields reasonable test decisions, it is necessary to ensure that the conditional distribution (conditioned on $X_{1,T}, \ldots, X_{T,T}$) of the bootstrap counterpart (for the moment denoted as) $\mathscr{Q}^*_T$ of $T\sqrt{b}\, \widehat{\mathfrak{Q}}_T - \mathbf{Bias}^{\text{indep}}_T$ converges for $T \to \infty$ in probability to the distribution of Z^{indep} under $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ and that $\mathbb{P}(\mathbb{P}^*(\mathscr{Q}^*_T > K(\epsilon)) < \epsilon) \overset{T\to\infty}{\longrightarrow} 1$ holds under $\mathcal{H}^{\text{indep}}_{1,\mathfrak{D}_1,\mathfrak{D}_2}$ for all $\epsilon > 0$ with suitable, deterministic $K(\epsilon) < \infty$.
A bootstrap approach which owns these properties is constructed in the following heuristically and theorems which display the asymptotic behaviour of the proposed bootstrap statistic are stated below (see the Theorems 4.11 and 4.13).
To obtain such an approach, note at first that if $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ is valid, Remark 4.2 (ii) will provide that $\widehat{\mathfrak{Q}}_T$ can be rewritten in the following manner (see (4.9) and (4.6)):

$$\widehat{\mathfrak{Q}}_T := \int_b^{1-b} \int_{\mathbb{R}^d} \left| \widehat{\mathbb{Q}}_T(u,s) - \mathbb{Q}(u,s) \right|^2 \mathbf{w}(s)\, ds\, du. \tag{4.15}$$

In addition, since Assumption 4.5 $[\mathbf{K\&b.2}]$ underlies $\widehat{\mathfrak{Q}}_T$, the Propositions 2.12 and 2.14 indicate that $\widehat{\mathbb{Q}}_T(u,s)$ is a reasonable estimator for $\mathbb{Q}(u,s)$ (with $u \in [b, 1-b], s \in \mathbb{R}^d$). This motivates under the null hypothesis the following approximation:

$$\widehat{\mathfrak{Q}}_T \approx \int_b^{1-b} \int_{\mathbb{R}^d} \left| \widehat{\mathbb{Q}}_T(u,s) - \mathbb{E}\left[\widehat{\mathbb{Q}}_T(u,s)\right] \right|^2 \mathbf{w}(s)\, ds\, du. \tag{4.16}$$

Further, $\widehat{\mathbb{Q}}_T$ can be rewritten as (see (4.9), Definition 2.11 as well as (4.8)):

$$\widehat{\mathbb{Q}}_T(u,s) = \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right)\widehat{q}_{t,T}(u,s) \quad \text{with} \quad \widehat{q}_{t,T}(u,s) := e^{i\left\langle s, X_{t,T}\right\rangle} - \widehat{\varphi}^{[1]}\left(u, s^{[1]}\right) e^{i\left\langle s^{[2]}, X_{t,T}^{[2]}\right\rangle}$$

$$\forall\, u \in [0,1],\ s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2},\ t \in \{1,\ldots,T\}.$$

Thus, considerations which are similar to those that lead to the bootstrap complement (3.37) of $\mathrm{R}\{\widehat{\varphi}(u_k, s)\} - \mathbb{E}\left[\mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\}\right]$ (with $\mathrm{R} \in \{\Re, \Im\}$), the equation $|x|^2 = \Re\{x\}^2 + \Im\{x\}^2\ \forall\, x \in \mathbb{C}$ and (4.16) motivate the following bootstrap counterpart of $\widehat{\mathfrak{Q}}_T$ (whereby $(W_t^*)_{t\in\mathbb{Z}}$ should be a process of bootstrap random variables for which Assumption 3.15 $[\mathbf{W}^*]$ holds):

$$\widehat{\mathfrak{Q}}_T^{[1.\text{Idea}]*} := \int_b^{1-b}\int_{\mathbb{R}^d}\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right)\left(\widehat{q}_{t,T}(u,s) - \widehat{\mathbb{Q}}_T(u,s)\right) W_t^*\right|^2 \mathbf{w}(s)\, ds\, du. \tag{4.17}$$

As explained in the following, this bootstrap approach is accompanied by a serious issue, such that $\widehat{\mathfrak{Q}}_T^{[1.\text{Idea}]*}$ is modified below to solve it:
Alternatively to $\{X_{t,T}\}$, one may also consider the locally stationary process $\left\{\left(X_{t,T}^{[2]\prime}, X_{t,T}^{[1]\prime}\right)'\right\}$, whereby $X_{t,T}^{[1]}$ and $X_{t,T}^{[2]}$ originate from Assumption 4.1 $[\mathbf{INDEP}]$. However, the test for independence which results from $\widehat{\mathfrak{Q}}_T$ and its bootstrap counterpart $\widehat{\mathfrak{Q}}_T^{[1.\text{Idea}]*}$ may revise its test decision in practise (i. e., for fixed $T \in \mathbb{N}$) if its test decision is not longer based on the sample path $\left(X_{t,T}(\omega)\right)_{t=1}^T$ (with arbitrary $\omega \in \Omega$) but on the sample path $\left(\left(X_{t,T}^{[2]}(\omega)', X_{t,T}^{[1]}(\omega)'\right)'\right)_{t=1}^T$. This property is commonly not appropriate because interchanging the components of the process does not effect whether they depend on each other. This issue can be solved in the following way:
At first, use a weight function $\mathbf{w}$ that fulfils not only Assumption 4.3 $[\mathbf{WEI.2}]$ but also for a Riemann integrable function $w\colon \mathbb{R} \to (0,\infty)$:

$$\mathbf{w}(s) = \prod_{n=1}^{d} w\left({}_n s\right)\ \forall\, s := \left({}_1 s, \ldots, {}_d s\right)' \in \mathbb{R}^d, \tag{4.18}$$

which ensures that interchanging $\left\langle s^{[1]}, X_{t,T}^{[1]}\right\rangle$ and $\left\langle s^{[2]}, X_{t,T}^{[2]}\right\rangle$ for all $s^{[k]} \in \mathbb{R}^{d_k}$ with $k \in \{1,2\}$ as well as all $t \in \{1,\ldots,T\}$ does not change realizations of $\widehat{\mathfrak{Q}}_T$ (recall (4.9)). However, even if (4.18) holds, this interchange may have an impact on realizations of $\widehat{\mathfrak{Q}}_T^{[1.\text{Idea}]*}$. To avoid the latter, consider at first the following modification of $\widehat{\mathbb{Q}}_T$:

$$\widehat{\mathbb{Q}}_T^{\text{mod}}(u,s) := \widehat{\mathbb{Q}}_T(u,s) + \left(\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) - 1\right)\widehat{\varphi}^{[1]}\left(u, s^{[1]}\right)\widehat{\varphi}^{[2]}\left(u, s^{[2]}\right)$$

$$\forall\, u \in [0,1],\ s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}.$$

Thereby, (B.40) and (B.41) in the appendix with $\mathfrak{U}_0 = 0$ as well as $\mathfrak{U}_1 = 1$ show under Assumption 4.5 $[\mathbf{K\&b.2}]$ for $T \to \infty$:

$$\sup_{u\in[b,1-b]}\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) - 1\right| = \mathcal{O}\left(\frac{1}{Tb}\right), \tag{4.19}$$

such that $\widehat{\mathbb{Q}}_T$ can be approximated by $\widehat{\mathbb{Q}}_T^{\text{mod}}$. Moreover, straightforward calculations show:

$$\widehat{\mathbb{Q}}_T^{\text{mod}}(u,s) = \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right)\widehat{q}_{t,T}^{\text{mod}}(u,s) \quad \text{with}$$

$$\widehat{q}_{t,T}^{\text{mod}}(u,s) := e^{i\left\langle s, X_{t,T}\right\rangle} - \widehat{\varphi}^{[1]}\left(u, s^{[1]}\right) e^{i\left\langle s^{[2]}, X_{t,T}^{[2]}\right\rangle} + \left(\widehat{\varphi}^{[1]}\left(u, s^{[1]}\right) - e^{i\left\langle s^{[1]}, X_{t,T}^{[1]}\right\rangle}\right)\widehat{\varphi}^{[2]}\left(u, s^{[2]}\right)$$

$$\forall\, u \in [0,1],\ s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}.$$

Replacing $\widehat{q}_{t,T}$ by $\widehat{q}_{t,T}^{\text{mod}}$ in (4.17) leads to the following expression (recall (4.9)):

$$\widehat{\mathfrak{Q}}_T^* := \int_b^{1-b} \int_{\mathbb{R}^d} \left| \widehat{\mathbb{Q}}_T^*(u,s) \right|^2 \mathbf{w}(s)\, ds\, du \quad \text{with}$$

$$\begin{aligned} \widehat{\mathbb{Q}}_T^*(u,s) &:= \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) \left(\widehat{q}_{t,T}^{\text{mod}}(u,s) - \widehat{\mathbb{Q}}_T(u,s)\right) W_t^* \\ &= \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) \Bigg(e^{i\langle s, X_{t,T}\rangle} - \widehat{\varphi}\,(u,s) + \widehat{\varphi}^{[1]}\left(u, s^{[1]}\right) \\ &\quad \cdot \left(\widehat{\varphi}^{[2]}\left(u, s^{[2]}\right) - e^{i\left\langle s^{[2]}, X_{t,T}^{[2]}\right\rangle}\right) + \left(\widehat{\varphi}^{[1]}\left(u, s^{[1]}\right) - e^{i\left\langle s^{[1]}, X_{t,T}^{[1]}\right\rangle}\right) \cdot \widehat{\varphi}^{[2]}\left(u, s^{[2]}\right) \Bigg) W_t^* \\ &\qquad \forall\, u \in [0,1],\, s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}. \end{aligned} \tag{4.20}$$

If (4.18) holds, realizations of $\widehat{\mathfrak{Q}}_T^*$ will be unaffected by interchanging $\langle s^{[1]}, X_{t,T}^{[1]}\rangle$ and $\langle s^{[2]}, X_{t,T}^{[2]}\rangle$ for all $s^{[k]} \in \mathbb{R}^{d_k}$ with $k \in \{1,2\}$ as well as all $t \in \{1,\ldots,T\}$, which avoids the issue described above. Hence, $\widehat{\mathfrak{Q}}_T^*$ seems to be an useful bootstrap counterpart of $\widehat{\mathfrak{Q}}_T$. Further, note that the random function $\widehat{\mathbb{Q}}_T$ contained in (4.17) is not replaced by $\widehat{\mathbb{Q}}_T^{\text{mod}}$ to obtain $\widehat{\mathfrak{Q}}_T^*$ because this would increase the computational costs for calculating realizations of $\widehat{\mathfrak{Q}}_T^*$.

The following theorem presents the asymptotic behaviour of the heuristically motivated bootstrap counterpart $\widehat{\mathfrak{Q}}_T^*$. In particular, it should be noted that this theorem holds not just under $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ but also under $\mathcal{H}_{1,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$. Further, in this theorem, it is not demanded that (4.18) holds because, in some applications, if additional informations that concern properties of the distributions of $X_{t,T}^{[1]}$ and $X_{t,T}^{[2]}$ with $t \in \{1,\ldots,T\}$ are available, it might be appropriate to differ from using a weight function $\mathbf{w}$ that fulfils (4.18). (However, in this case, the test decision may differ if $\{(X_{t,T}^{[2]\prime}, X_{t,T}^{[1]\prime})'\}$ instead of $\{X_{t,T}\}$ is considered.)

Theorem 4.11 (Asymptotic behaviour of $\widehat{\mathfrak{Q}}_T^*$ - general case)**.**
Let the Assumptions 4.1 $[\mathbf{INDEP}]$, *4.3* $[\mathbf{WEI.2}]$, *4.5* $[\mathbf{K\&b.2}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled. Moreover, define (recall (4.12)):*

$$\mathbf{Bias}_T^{\text{indep}*} := \frac{1}{\sqrt{b}} \int_{-1}^{1} K(z)^2\, dz \cdot \int_{\mathbb{R}^d} \int_0^1 \sum_{t\in\mathbb{Z}} K^*\left(\frac{t}{\beta}\right) \gamma_{\tilde{\mathbf{G}}}(u,s,t)\, du\, \mathbf{w}(s)\, ds. \tag{4.21}$$

If $\sigma^{\text{indep}} > 0$ *(see (4.13)), it will hold for* $T \to \infty$ *(recall (4.14)):*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}^*\left(T\sqrt{b}\,\widehat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}*} \leqslant x\right) - \mathbb{P}\left(Z^{\text{indep}} \leqslant x\right) \right| \xrightarrow{\mathbb{P}} 0. \tag{4.22}$$

If $\sigma^{\text{indep}} = 0$, *one will obtain for* $T \to \infty$ *(see (3.26)):*

$$T\sqrt{b}\,\widehat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}*} = o_{\mathbb{P}}^*(1). \tag{4.23}$$

Remark 4.12. *Assumption 3.15* $[\mathbf{W}^*]$ *(iii), Lemma 4.8 and Assumption 4.3* $[\mathbf{WEI.2}]$ *ensure that* $\mathbf{Bias}_T^{\text{indep}*}$ *is well-defined.*

The latter theorem shows that the conditional distribution (conditioned on $X_{1,T},\ldots,X_{T,T}$) of the bootstrap statistic $T\sqrt{b}\,\widehat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}*}$ approximates for large values of T the distribution of Z^{indep}, which is the asymptotic distribution of $T\sqrt{b}\,\widehat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}}$ under $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ according to Theorem 4.9 (i). Thereby, analog to Theorem 3.25 and the subsequent considerations, the asymptotic behaviour of $\mathbf{Bias}_T^{\text{indep}*}$ may differ from that of $\mathbf{Bias}_T^{\text{indep}}$ (recall (4.12)). However, as stated in the next theorem, this issue can be solved similarly to Theorem 3.27 under the additional assumptions $\delta \in (1/4, 1]$ and (3.60).

Theorem 4.13 (Asymptotic behaviour of $\widehat{\mathfrak{Q}}_T^*$ - for $\delta \in (1/4, 1]$).
Suppose that the Assumptions 4.1 $[\mathbf{INDEP}]$*, 4.3* $[\mathbf{WEI.2}]$*, 4.5* $[\mathbf{K\&b.2}]$ *and 3.15* $[\mathbf{W}^*]$ *hold. In addition, let* $\delta \in (1/4, 1]$ *be fulfilled, whereby* δ *originates from Assumption 2.2* $[\mathbf{StAp}]$ *(which is contained in Assumption 4.1* $[\mathbf{INDEP}]$*). Moreover, assume for* β *and* K^**, which are introduced in Assumption 3.15* $[\mathbf{W}^*]$*, that (3.60) is valid.*
If $\sigma^{\text{indep}} > 0$ *(see (4.13)), it will hold for* $T \to \infty$ *(recall (4.12) and (4.14)):*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}^* \left(T\sqrt{b}\, \widehat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}} \leqslant x \right) - \mathbb{P}\left(Z^{\text{indep}} \leqslant x \right) \right| \xrightarrow{\mathbb{P}} 0. \tag{4.24}$$

If $\sigma^{\text{indep}} = 0$*, one will obtain for* $T \to \infty$ *(see (3.26)):*

$$T\sqrt{b}\, \widehat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}} = o_{\mathbb{P}}^*(1). \tag{4.25}$$

Remark 4.14. *(i) Theorem 4.13 provides under* $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ *and under* $\mathcal{H}_{1,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ *with* $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ *that the conditional distribution (conditioned on* $X_{1,T}, \ldots, X_{T,T}$*) of* $T\sqrt{b}\, \widehat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}}$ *converges in probability to the distribution of* Z^{indep}*. Hence, a test which is based on the Theorems 4.9 and 4.13 is consistent under the assumptions supposed in Theorem 4.13.*

(ii) If the assumptions of Theorem 4.13 with $\sigma^{\text{indep}} > 0$ *hold and* $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ *is valid, one will obtain for* $T \to \infty$ *from (4.24), Theorem 4.9 (i) and Polya's theorem:*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}^* \left(T\sqrt{b}\, \widehat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}} \leqslant x \right) - \mathbb{P}\left(T\sqrt{b}\, \widehat{\mathfrak{Q}}_T - \mathbf{Bias}_T^{\text{indep}} \leqslant x \right) \right| \xrightarrow{\mathbb{P}} 0.$$

(iii) It can be argued similarly to Remark 3.28 (iii) that the condition $\delta \in (1/4, 1]$ *is necessary under the Assumptions 4.5* $[\mathbf{K\&b.2}]$ *(ii) and 3.15* $[\mathbf{W}^*]$ *(i) to ensure that parameters* β *exist for which (3.60) holds. Moreover, if* $\delta \in (1/4, 1]$*, (e. g.) the bandwidths* $b := b_T := 1/4\mathbb{1}_{\{T=1\}} + \min\left\{1/10 \cdot T^{-1/2.5}\ln(T)^{1.6}, 1/4\right\}\mathbb{1}_{\{T \geqslant 2\}}\ \forall\, T \in \mathbb{N}$ *suffice Assumption 4.5* $[\mathbf{K\&b.2}]$ *(ii) and, for this choice of the sequence of bandwidths,* $\beta := \beta_T := b^{-1/2}\ln(T)\ \forall\, T \in \mathbb{N}$ *can be selected to ensure that the processes which are defined in Example 3.16 fulfil Assumption 3.15* $[\mathbf{W}^*]$ *as well as (3.60) with* $\mathscr{S}^* = 1$ *(recall that the property* $\mathscr{S}^* = 1$ *originates from Remark 3.28 (iii)).*
Further, if $\{\widetilde{X}_t(u)\}$ *is a sequence of independent random variables for all* $u \in [0, 1]$*, one will obtain* $\gamma_{\widetilde{\mathbf{G}}}(u, s, h) = 0\ \forall\, u \in [0, 1],\ s \in \mathbb{R}^d,\ h \in \mathbb{Z}\backslash\{0\}$ *(see (4.12) and (4.11)). In this case, it follows analogously to Remark 3.28 (iv) that the condition (3.60) is omittable in Theorem 4.13 and the statement of Theorem 4.13 is valid for all* $\delta \in (0, 1]$.

The Theorems 4.9 and 4.13 provide a consistent level-alpha test for the problem (4.1) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$. The following algorithm describes how this test can be implemented, whereby one obtains similarly to Remark 3.29 (i) why it is justified that this algorithm avoids to calculate or estimate $\mathbf{Bias}_T^{\text{indep}}$.

Algorithm **TEST.INDEP.1**

Inputs: significance level $\alpha \in (0, 1)$; $T, N \in \mathbb{N}$; a sample path $(X_{t,T}(\omega))_{t=1}^T$ (for an $\omega \in \Omega$); a kernel K for which Assumption 4.5 $[\mathbf{K\&b.2}]$ (i) holds; a bandwidth $b \in (0, 1/2)$; a weight function $\mathbf{w}$ that fulfils Assumption 4.3 $[\mathbf{WEI.2}]$ (and possibly also (4.18)); a parameter $\beta > 0$ and an associated process $(W_t^*)_{t \in \mathbb{Z}}$ which satisfies Assumption 3.15 $[\mathbf{W}^*]$ as well as the condition $\mathscr{S}^* < \infty$ contained in (3.60);

1: Determine the realization of $\widehat{\mathfrak{Q}}_T$ that belongs to the sample path $(X_{t,T}(\omega))_{t=1}^T$;
2: Independently, for n in $1 : N$ do
3: Generate a sample path of $(W_t^*)_{t=1}^T$;
4: Calculate the associated realization of $\widehat{\mathfrak{Q}}_T^*$;
5: end for
6: Compute a realization of the empirical distribution function of $\widehat{\mathfrak{Q}}_T^*$ by using the calculated realizations of $\widehat{\mathfrak{Q}}_T^*$ and call this realization of the empirical distribution function $\widehat{F}_{T,N}^{*\text{indep}}$;
7: Reject $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ if $1 - \widehat{F}_{T,N}^{*\text{indep}}\big(\widehat{\mathfrak{Q}}_T(\omega)\big) < \alpha$;

Remark 4.15. *(i) The asymptotic properties of* b *and* β *which are demanded in the Assumptions 4.5* $[\mathbf{K\&b.2}]$ *(ii), 3.15* $[\mathbf{W}^*]$ *(i) as well as (3.60) can be regarded as rough guidances on selecting* b *and* β *for a given* $T \in \mathbb{N}$*. Thereby, Remark 4.14 (iii) provides that such guidances are* *obtainable without knowing $\delta \in (1/4, 1]$. In addition, Assumption 4.3* $[\mathbf{WEI.2}]$ *depends on δ but this is also not an issue because (4.7) is a weaker condition than $\int_{\mathbb{R}^d} \left(1 + |s|_1^4\right) \mathbf{w}(s)\, ds < \infty$.*

*(ii) Algorithm **TEST.INDEP.1** can be modified in such a manner that its application to locally stationary processes $\{X_{t,T}\}$ for which Assumption 4.1* $[\mathbf{INDEP}]$ *holds for arbitrary but fixed $\delta \in (0,1]$ is justified $\big($without assuming that $\{\widetilde{X}_t(u)\}$ is a sequence of independent random variables for all $u \in [0,1]\big)$. This target can be meet similarly to the research given in Subsection 3.2.3, i. e., by omitting the condition (3.60) and replacing $\widehat{\mathfrak{Q}}_T$ by $\widehat{\mathfrak{Q}}_T + 1/(T\sqrt{b})\widehat{\mathbf{Bias}}_T^{\text{indep.error}}$ in the first and seventh step of Algorithm **TEST.INDEP.1**, whereby $\widehat{\mathbf{Bias}}_T^{\text{indep.error}}$ should be a local Newey-West estimator for $\mathbf{Bias}_T^{\text{indep}*} - \mathbf{Bias}_T^{\text{indep}}$, which is constructed similarly to one of those proposed in Definition 3.33. This modification of Algorithm **TEST.INDEP.1** is omitted in the present thesis to shorten it.*

*(iii) The following two claims can be verified by arguments which are similar to those stated in Remark 3.7 (iii): The test decision that results from applying Algorithm **TEST.INDEP.1** to a sample path $(X_{t,T}(\omega))_{t=1}^T$ will not change if this underlying sample path is replaced by $(x + X_{t,T}(\omega))_{t=1}^T$ with arbitrary $x \in \mathbb{R}^d$. Moreover, if the weight function $\mathbf{w}$ fulfils $\mathbf{w}(s) = \mathbf{w}(-s)\ \forall\, s \in \mathbb{R}^d$, Algorithm **TEST.INDEP.1** assigns the same test decision to the sample path $(X_{t,T}(\omega))_{t=1}^T$ and $(-X_{t,T}(\omega))_{t=1}^T$.*
*However, Algorithm **TEST.INDEP.1** is not scale-invariant in the sense that it does not inevitably yield the same test decision for the sample path $(X_{t,T}(\omega))_{t=1}^T$ and $(y\, X_{t,T}(\omega))_{t=1}^T$ with arbitrary $y \in \mathbb{R}\backslash\{0\}$. A distance covariance (or distance correlation) based approach may provide this kind of scale-invariance but Remark 4.7 (ii) indicates that using such an approach is problematic under the weak moment conditions that underlie the present framework.*

In the following, the results evolved in the present section are generalized in such a manner that they yield a test for total blockwise independence, i. e., a procedure which investigates (4.1) for arbitrary, non-empty, fixed and finite sets $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$.

4.3. Testing for total blockwise independence

In order to investigate under Assumption 4.1 $[\mathbf{INDEP}]$ the test problem (4.1) with arbitrary, non-empty, fixed and finite sets $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$, define at first the triangular array $\{X^+_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\} := \{X^+_{\mathfrak{D}_1,\mathfrak{D}_2,t,T} : t \in \{1,\ldots,T\}\}_{T=1}^\infty$ in the following way (recall (4.2) and (4.4)):

$$X^+_{\mathfrak{D}_1,\mathfrak{D}_2,t,T} := \left(X^{[1]+\,\prime}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}, X^{[2]+\,\prime}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\right)' \quad \text{with}$$
$$X^{[k]+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T} := \widetilde{X}^{[k]}_{\mathfrak{D}_k,t}\left(\frac{t}{T}\right)\mathbb{1}_{\left\{t\in\left\{1,\ldots,\max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2}\mathfrak{d}\right\}\right\}} + X^{[k]}_{\mathfrak{D}_k,t,T}\mathbb{1}_{\left\{t\geqslant 1+\max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2}\mathfrak{d}\right\}}$$
$$\forall\, k \in \{1,2\},\ t \in \{1,\ldots,T\},\ T \in \mathbb{N}. \qquad (4.26)$$

Remark 4.16. *If Assumption 4.1* $[\mathbf{INDEP}]$ *is valid, $\{X^+_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\}$ will be an $\mathbb{R}^{d_1\cdot\#\mathfrak{D}_1+d_2\cdot\#\mathfrak{D}_2}$-valued locally stationary Bernoulli shift process (as defined in Definition 2.1) that fulfils Assumption 2.4* $[\mathbf{DM.3}]$ *(especially also Assumption 2.2* $[\mathbf{StAp}]$*) for the same $\delta \in (0,1]$ as the locally stationary Bernoulli shift process $\{X_{t,T}\}$, which originates from Assumption 4.1* $[\mathbf{INDEP}]$*, whereby $\{\widetilde{X}_{\mathfrak{D}_1,\mathfrak{D}_2,t}(u)\}$ with $u \in [0,1]$ (see (4.2)) are the stationary approximations belonging to $\{X^+_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\}$.*
This claim follows from the fact that Assumption 2.2 $[\mathbf{StAp}]$ *(i) as well as Remark 2.3 imply for all $k \in \{1,2\}$:*

$$\sup_{t=1,\ldots,T}\left\|X^+_{\mathfrak{D}_1,\mathfrak{D}_2,t,T} - \widetilde{X}_{\mathfrak{D}_1,\mathfrak{D}_2,t}\left(\frac{t}{T}\right)\right\|_{1+\delta}$$
$$\leqslant \mathbb{1}_{\left\{T\leqslant\max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2}\mathfrak{d}\right\}}0 + \mathbb{1}_{\left\{T\geqslant 1+\max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2}\mathfrak{d}\right\}}\cdot\sup_{t=1+\max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2}\mathfrak{d},\ldots,T}\sum_{k=1}^2\sum_{\widetilde{\mathfrak{d}}\in\mathfrak{D}_k}\Bigg\|X^{[k]}_{t-\widetilde{\mathfrak{d}},T}$$
$$- \widetilde{X}^{[k]}_{t-\widetilde{\mathfrak{d}}}\left(\frac{t-\widetilde{\mathfrak{d}}}{T}\right) + \widetilde{X}^{[k]}_{t-\widetilde{\mathfrak{d}}}\left(\frac{t-\widetilde{\mathfrak{d}}}{T}\right) - \widetilde{X}^{[k]}_{t-\widetilde{\mathfrak{d}}}\left(\frac{t}{T}\right)\Bigg\|_{1+\delta}$$
$$\leqslant \frac{C}{T}$$

and from some other more straightforward arguments.

The test problem (4.1) with arbitrary, non-empty, fixed and finite sets $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$ is equivalent to considering the null hypothesis $\widetilde{X}^{[1]}_{\mathfrak{D}_1,0}(u) \perp\!\!\!\perp \widetilde{X}^{[2]}_{\mathfrak{D}_2,0}(u)\ \forall\, u \in [0,1]$ and the alternative $\exists\, u \in [0,1]$: $\widetilde{X}^{[1]}_{\mathfrak{D}_1,0}(u) \not\perp\!\!\!\perp \widetilde{X}^{[2]}_{\mathfrak{D}_2,0}(u)$ (recall (4.2)), whereby Remark 4.16 indicates that a test for this problem can be constructed by using the results of the previous section.
Concretely, to obtain such a test, adapt at first the domain of the weight function $\mathbf{w}$ (which is introduced in Assumption 4.3 $[\mathbf{WEI.2}]$) to the dimension of the random variables $X^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}$ (with $t \in \{1,\ldots,T\}$, $T \in \mathbb{N}$), i. .e., consider the following assumption:

Assumption 4.17 [WEI.3] (Weight function - Part 3)**.**
Suppose that $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$ are the sets which underlie the considered test problem (4.1), let $d_1, d_2 \in \mathbb{N}$ originate from Assumption 4.1 $[\mathbf{INDEP}]$ *and $\delta \in (0,1]$ from Assumption 2.2* $[\mathbf{StAp}]$*. The weight function $\mathbf{w}\colon \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1 + d_2\cdot\#\mathfrak{D}_2} \to [0,\infty)$ is defined as a Riemann integrable function that is Lebesgue almost everywhere positive and fulfils:*

$$\int\limits_{\mathbb{R}^{d_1\cdot\#\mathfrak{D}_1 + d_2\cdot\#\mathfrak{D}_2}} \left(1 + |s|_1^{2+2\delta} + |s|_1^3\right) \mathbf{w}(s)\, ds < \infty. \tag{4.27}$$

Defining the test statistic $\widehat{\mathfrak{Q}}_T$ (recall (4.9), Definition 2.11 as well as (4.8)) and its bootstrap counterpart $\widehat{\mathfrak{Q}}^*_T$ (see (4.20)) based on $\{X^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\}$ instead of $\{X_{t,T}\}$ does not yield a feasible approach for testing (4.1) because the random variables $X^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}$ are commonly unobservable in practical applications for $t \leqslant \max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2} \mathfrak{d}$ due to the contained stationary approximations of $\{X_{t,T}\}$. However, removing these unobservable random variables from $\widehat{\mathfrak{Q}}_T$ and $\widehat{\mathfrak{Q}}^*_T$ is asymptotically uncritical because just a finite number of them exist since $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$ are finite sets. Moreover, by definition, each $X^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}$ contains random variables that originate from different points in time, which motivates to modify the localizing terms $K_b(t/T-u)$ in $\widehat{\mathfrak{Q}}_T$ and $\widehat{\mathfrak{Q}}^*_T$ similarly to (4.18) in [40, Jentsch et al. (2020a), p. 124 et. seq.]. Overall, these considerations lead to the statistic $\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ and its bootstrap counterpart $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$, which are introduced in the following.
Let $T \geqslant 1 + \max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2} \mathfrak{d}$ and the Assumptions 4.5 $[\mathbf{K\&b.2}]$ as well as 4.17 $[\mathbf{WEI.3}]$ be fulfilled. Define for all $k \in \{1,2\}$, $u \in [0,1]$, $s^{[k]} \in \mathbb{R}^{d_k\cdot\#\mathfrak{D}_k}$, $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)'$ (recall Definition 2.11 as well as (4.4)):

$$\mathfrak{D}_{\max} := \max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2} \mathfrak{d}, \qquad \mathfrak{D}_{\text{mean}} := \frac{1}{\#\mathfrak{D}_1 + \#\mathfrak{D}_2} \sum_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2} \mathfrak{d},$$

$$\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) := \widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s) := \frac{1}{T}\sum_{t=1+\mathfrak{D}_{\max}}^{T} K_b\left(\frac{t-\mathfrak{D}_{\text{mean}}}{T} - u\right) e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,t,T}\right\rangle} e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,t,T}\right\rangle},$$

$$\widehat{\varphi}^{[k]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[k]}\right) := \widehat{\varphi}^{[k]}_{\mathfrak{D}_1,\mathfrak{D}_2,T}\left(u,s^{[k]}\right) := \frac{1}{T}\sum_{t=1+\mathfrak{D}_{\max}}^{T} K_b\left(\frac{t-\mathfrak{D}_{\text{mean}}}{T} - u\right) e^{i\left\langle s^{[k]}, X^{[k]}_{\mathfrak{D}_k,t,T}\right\rangle},$$

$$\widehat{\mathbb{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s) := \widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) - \widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[1]}\right)\cdot\widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[2]}\right),$$

$$\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} := \int\limits_b^{1-b} \int\limits_{\mathbb{R}^{d_1\cdot\#\mathfrak{D}_1 + d_2\cdot\#\mathfrak{D}_2}} \left|\widehat{\mathbb{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s)\right|^2 \mathbf{w}(s)\, ds\, du \tag{4.28}$$

and for a process of bootstrap random variables $(W^*_t)_{t\in\mathbb{Z}}$ which fulfils Assumption 3.15 $[\mathbf{W}^*]$:

$$\begin{aligned}\widehat{\mathbb{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s) := \frac{1}{T}\sum_{t=1+\mathfrak{D}_{\max}}^{T} K_b\left(\frac{t-\mathfrak{D}_{\text{mean}}}{T} - u\right) &\Bigg(e^{i\left\langle s, \left(X^{[1]\prime}_{\mathfrak{D}_1,t,T}, X^{[2]\prime}_{\mathfrak{D}_2,t,T}\right)'\right\rangle} \\ &- \widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) + \widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[1]}\right)\cdot\left(\widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[2]}\right) - e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,t,T}\right\rangle}\right) \\ &+ \left(\widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[1]}\right) - e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,t,T}\right\rangle}\right)\cdot\widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[2]}\right)\Bigg) W^*_t \quad \text{as well as}\end{aligned}$$

$$\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T} := \int_b^{1-b} \int_{\mathbb{R}^{d_1\cdot\#\mathfrak{D}_1+d_2\cdot\#\mathfrak{D}_2}} \left|\mathbb{Q}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s)\right|^2 \mathbf{w}(s)\,ds\,du. \tag{4.29}$$

The following theorem justifies that the test statistic $\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ and its bootstrap counterpart $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ can be used to investigate (4.1) for arbitrary, non-empty, fixed and finite sets $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$. To shorten the present work, $\delta \in (1/4, 1]$ is assumed in this theorem to ensure that it yields a practicable test procedure. Instead, if $\delta \in (0, 1]$ is arbitrary, one can proceed similarly to Subsection 3.2.3 and construct a local Newey-West estimator which allows to modify the test statistic accordingly.

Theorem 4.18 (Testing for total blockwise independence - for $\delta \in (1/4, 1]$).
Suppose that $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$ are arbitrary, non-empty, fixed and finite sets. Moreover, let the Assumptions 4.1 $[\mathbf{INDEP}]$, 4.17 $[\mathbf{WEI.3}]$, 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ as well as 3.15 $[\mathbf{W}^*]$ be fulfilled and $\delta \in (1/4, 1]$, whereby δ originates from Assumption 2.2 $[\mathbf{StAp}]$. In addition, (3.60) should be valid. Further, define for all $u \in [0,1]$, $s^{[k]} \in \mathbb{R}^{d_k\cdot\#\mathfrak{D}_k}$ with $k \in \{1,2\}$, $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)'$, $s_1, s_2 \in \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1} \times \mathbb{R}^{d_2\cdot\#\mathfrak{D}_2}$, $t \in \mathbb{Z}$, $\mathrm{R}, \mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$ (see (4.2)):

$$\begin{aligned}
\widetilde{\mathbf{G}}_{\mathfrak{D}_1,\mathfrak{D}_2,t,\mathrm{R}}(u,s) &:= \mathrm{R}\Bigg\{ e^{i\left\langle s, \tilde{X}_{\mathfrak{D}_1,\mathfrak{D}_2,t}(u)\right\rangle} - \mathbb{E}\left[e^{i\left\langle s^{[1]}, \tilde{X}^{[1]}_{\mathfrak{D}_1,0}(u)\right\rangle}\right] \cdot e^{i\left\langle s^{[2]}, \tilde{X}^{[2]}_{\mathfrak{D}_2,t}(u)\right\rangle} \\
&\qquad - \mathbb{E}\left[e^{i\left\langle s^{[2]}, \tilde{X}^{[2]}_{\mathfrak{D}_2,0}(u)\right\rangle}\right] \cdot e^{i\left\langle s^{[1]}, \tilde{X}^{[1]}_{\mathfrak{D}_1,t}(u)\right\rangle} \Bigg\}, \\
\gamma_{\mathfrak{D}_1,\mathfrak{D}_2,\widetilde{\mathbf{G}}}(u,s,t) &:= \mathrm{Cov}\left(\widetilde{\mathbf{G}}_{\mathfrak{D}_1,\mathfrak{D}_2,0,\Re}(u,s), \widetilde{\mathbf{G}}_{\mathfrak{D}_1,\mathfrak{D}_2,t,\Re}(u,s)\right) \\
&\qquad + \mathrm{Cov}\left(\widetilde{\mathbf{G}}_{\mathfrak{D}_1,\mathfrak{D}_2,0,\Im}(u,s), \widetilde{\mathbf{G}}_{\mathfrak{D}_1,\mathfrak{D}_2,t,\Im}(u,s)\right), \\
\mathbf{Bias}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} &:= \frac{1}{\sqrt{b}} \int_{-1}^{1} K(z)^2\,dz \cdot \int_{\mathbb{R}^{d_1\cdot\#\mathfrak{D}_1+d_2\cdot\#\mathfrak{D}_2}} \int_0^1 \sum_{t\in\mathbb{Z}} \gamma_{\mathfrak{D}_1,\mathfrak{D}_2,\widetilde{\mathbf{G}}}(u,s,t)\,du\,\mathbf{w}(s)\,ds, \\
\mathbf{Cov}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,\mathrm{R}_1,\mathrm{R}_2}(u,s_1,s_2) &:= \sum_{t\in\mathbb{Z}} \mathrm{Cov}\left(\widetilde{\mathbf{G}}_{\mathfrak{D}_1,\mathfrak{D}_2,0,\mathrm{R}_1}(u,s_1), \widetilde{\mathbf{G}}_{\mathfrak{D}_1,\mathfrak{D}_2,t,\mathrm{R}_2}(u,s_2)\right) \quad \textit{and} \\
\sigma^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2} &:= 2\int_{-2}^{2} \left(\int_{-1}^{1} K(q)K(q+v)\,dq\right)^2 dv \cdot \int_{\mathbb{R}^{d_1\cdot\#\mathfrak{D}_1+d_2\cdot\#\mathfrak{D}_2}} \int_{\mathbb{R}^{d_1\cdot\#\mathfrak{D}_1+d_2\cdot\#\mathfrak{D}_2}} \int_0^1 \mathbf{Cov}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,\Re,\Re}(u,s_1,s_2)^2 \\
&\quad + \mathbf{Cov}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,\Im,\Im}(u,s_1,s_2)^2 + 2\,\mathbf{Cov}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,\Re,\Im}(u,s_1,s_2)^2\,du\,\mathbf{w}(s_2)\,ds_2\,\mathbf{w}(s_1)\,ds_1.
\end{aligned} \tag{4.30}$$

(i) If $\mathcal{H}^{\mathrm{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ is fulfilled, it will hold for $T \to \infty$:

$$T\sqrt{b}\,\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \mathbf{Bias}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} \xrightarrow{\mathrm{d}} Z^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2} \quad \textit{with} \quad Z^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2} \sim \mathcal{N}\left(0, \sigma^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2}\right). \tag{4.31}$$

(ii) Assume that $\mathcal{H}^{\mathrm{indep}}_{1,\mathfrak{D}_1,\mathfrak{D}_2}$ is valid and consider an arbitrary sequence $(\tau_T)_{T\in\mathbb{N}}$ of deterministic real numbers which fulfils (3.54). Then, one obtains:

$$\lim_{T\to\infty} \mathbb{P}\left(T\sqrt{b}\,\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \mathbf{Bias}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} > \tau_T\right) = 1.$$

(iii) If $\sigma^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2} > 0$, the following statement will hold for $T \to \infty$:

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}^*\left(T\sqrt{b}\,\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \mathbf{Bias}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} \leqslant x\right) - \mathbb{P}\left(Z^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2} \leqslant x\right)\right| \xrightarrow{\mathbb{P}} 0. \tag{4.32}$$

If $\sigma^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2} = 0$, one will obtain for $T \to \infty$ (see (3.26)):

$$T\sqrt{b}\,\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \mathbf{Bias}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} = o^*_{\mathbb{P}}(1). \tag{4.33}$$

Remark 4.19. *Applying Lemma 4.8 to the stationary approximations* $\{\widetilde{X}_{\mathfrak{D}_1,\mathfrak{D}_2,t}(u)\}$ *(which is justified due to Remark 4.16) and Assumption 4.17* $[\mathbf{WEI.3}]$ *yield that* $\mathbf{Bias}^{\text{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$, $\mathbf{Cov}^{\text{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,\mathrm{R}_1,\mathrm{R}_2}(u,s_1,s_2)$ *as well as* $\sigma^{\text{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2}$ *are well-defined.*

Theorem 4.18 provides a consistent level-alpha test for the problem (4.1) with arbitrary, non-empty, fixed and finite sets $\mathfrak{D}_1,\mathfrak{D}_2 \subset \mathbb{N}_0$. The following algorithm describes how this test can be implemented, whereby one obtains similarly to Remark 3.29 (i) why it is justified that this algorithm avoids to calculate or estimate $\mathbf{Bias}^{\text{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$. Note also, that the following algorithm is very similar to Algorithm **TEST.INDEP.1** with the only differences that $\mathbf{w}$ fulfils Assumption 4.17 $[\mathbf{WEI.3}]$ instead of 4.3 $[\mathbf{WEI.2}]$ and $\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ as well as $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ instead of $\widehat{\mathfrak{Q}}_T$ and $\widehat{\mathfrak{Q}}^*_T$, respectively, have to be computed.

Algorithm **TEST.INDEP.2**

Inputs: arbitrary, non-empty, fixed and finite sets $\mathfrak{D}_1,\mathfrak{D}_2 \subset \mathbb{N}_0$; significance level $\alpha \in (0,1)$; $T, N \in \mathbb{N}$; a sample path $(X_{t,T}(\omega))_{t=1}^T$ (for an $\omega \in \Omega$); a kernel K for which Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) holds; a bandwidth $b \in (0,1/2)$; a weight function $\mathbf{w}$ that fulfils Assumption 4.17 $[\mathbf{WEI.3}]$; a parameter $\beta > 0$ and an associated process $(W_t^*)_{t\in\mathbb{Z}}$ which satisfies Assumption 3.15 $[\mathbf{W}^*]$ as well as the condition $\mathscr{S}^* < \infty$ contained in (3.60);

1: Determine the realization of $\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ that belongs to the sample path $(X_{t,T}(\omega))_{t=1}^T$;
2: Independently, for n in $1:N$ do
3: Generate a sample path of $(W_t^*)_{t=1}^T$;
4: Calculate the associated realization of $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$;
5: end for
6: Compute a realization of the empirical distribution function of $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ by using the calculated realizations of $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ and call this realization of the empirical distribution function $\widehat{F}^{*\text{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T,N}$;
7: Reject $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ if $1 - \widehat{F}^{*\text{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T,N}\big(\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(\omega)\big) < \alpha$;

Remark 4.20. *Note that Algorithm* ***TEST.INDEP.2*** *equals Algorithm* ***TEST.INDEP.1*** *in the case* $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ *and that the considerations mentioned in Remark 4.15 can be easily adapted to Algorithm* ***TEST.INDEP.2****.*

4.4. Simulation studies

In this section, the finite sample behaviour of the Algorithms **TEST.INDEP.1** and **TEST.INDEP.2** is investigated by simulation studies. Therefor, these algorithms are implemented in the programming language $\mathbf{R}$ and applied to sample paths of several locally stationary processes. The next setting describes all versions of Algorithm **TEST.INDEP.2** that are used in the present section, whereas the considered versions of Algorithm **TEST.INDEP.1** equal those of Algorithm **TEST.INDEP.2** with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$. In contrast, the regarded locally stationary processes and sets $\mathfrak{D}_1$ as well as $\mathfrak{D}_2$ are specified in each simulation study.

Setting 4.21. *All of the algorithms mentioned above are applied to sample paths which contain* $T = 1000$ *observations that originate from* $\mathbb{R}^2$*-valued locally stationary processes. Thereby, the weight function* $\mathbb{R}^{\#\mathfrak{D}_1+\#\mathfrak{D}_2} \ni s \mapsto \mathbf{w}_{\mathfrak{D}_1,\mathfrak{D}_2}(s) := e^{-|s|_1}$ *and the kernel* K_{Epa} *with* $\mathfrak{U}_0 = 0$ *as well as* $\mathfrak{U}_1 = 1$ *which originates from Example 2.10 (ii) are used. Moreover, the bandwidths* $b \in \{0.1, 0.15\}$*, the bootstrap random variables from Example 3.16 (ii) (which, in particular, fulfil* $\mathscr{S}^* = 1$*) with* $\beta \in \{0.3\,Tb^2, 0.7\,Tb^2\}$ *and the significance levels* $\alpha \in \{0.05, 0.1\}$ *are considered. In addition,* $N = 500$ *iterations of the underlying bootstrap procedure are taken into account. Further, the integrals with respect to* $u \in [b, 1-b]$ *contained in the realizations of* $\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ *and of* $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ *are not calculated exactly but approximated by right Riemann sums. Therefor,* $d_T := \lfloor T/(2b) \rfloor$ *summands are used for the Riemann approximation of* $\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ *and* $e_T = \lfloor T/4 \rfloor$ *addends for that of* $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ *(especially, note that* d_T *varies in dependence of b). These approximations of* $\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ *and* $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ *are justified by Proposition E.4 given in the appendix.* $\big($*In the course of the proof of the latter named proposition, it is presented that* $\widehat{Q}_{T,\text{apprx}}$ *as well as* $\widehat{Q}^*_{T,\text{apprx}}$ *introduced in this proposition are these Riemann approximations of* $\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ *and* $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$*, respectively.*$\big)$

4.4.1. Testing for pairwise independence in the same points in time

In this subsection, the versions of Algorithm **TEST.INDEP.2** that are described in Setting 4.21 are evaluated based on $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ (i. e., Algorithm **TEST.INDEP.1** is actually considered) and the locally stationary processes $\{\mathrm{E}_{t,T,\Lambda}\}$ as well as $\{\mathrm{F}_{t,T,\Lambda}\}$, which are introduced in the following. These locally stationary processes depend on a parameter $\Lambda \in [0,1]$, whereby several choices of Λ are considered below.

Suppose that $\big(\varepsilon_t^{[\mathrm{E},1]}\big)_{t\in\mathbb{Z}}$ as well as $\big(\varepsilon_t^{[\mathrm{E},2]}\big)_{t\in\mathbb{Z}}$ are sequences of i. i. d. random variables with $\varepsilon_0^{[\mathrm{E},1]} \sim \mathcal{S}^0_{1.3}(0,1,2)$ and $\varepsilon_0^{[\mathrm{E},2]} \sim \mathcal{S}^0_{1.3}(0,1,1)$ that are independent of each other. (Recall that $\mathcal{S}^0_{\bar{\alpha}}\big(\bar{\beta},\bar{\gamma},\bar{\delta}\big)$ denotes the $\mathcal{S}^0$-parametrization of the stable distribution - for details see (3.79).) Moreover, define:

$$\Big(\mathrm{E}^{[1]}_{0,T}, \mathrm{E}^{[2]}_{0,T}\Big)' := \Big(0.8\,\varepsilon_0^{[\mathrm{E},1]}, 0.3\,\varepsilon_0^{[\mathrm{E},2]}\Big)',$$

$$\mathrm{e}_1(u) := 0.6\sin(2\pi u)\,\mathbb{1}_{\{u\in[0,1]\}},\quad \mathrm{e}_2(u) := 0.6\sin(4\pi u)\,\mathbb{1}_{\{u\in[0,1]\}}\quad \forall\, u\in\mathbb{R}$$

$$\begin{pmatrix}\mathrm{E}^{[1]}_{t,T}\\ \mathrm{E}^{[2]}_{t,T}\end{pmatrix} := \begin{pmatrix}\mathrm{e}_1\big(\frac{t}{T}\big) & 0\\ 0 & \mathrm{e}_2\big(\frac{t}{T}\big)\end{pmatrix}\begin{pmatrix}\mathrm{E}^{[1]}_{t-1,T}\\ \mathrm{E}^{[2]}_{t-1,T}\end{pmatrix} + \begin{pmatrix}0.8\,\varepsilon_t^{[\mathrm{E},1]}\\ 0.3\,\varepsilon_t^{[\mathrm{E},2]}\end{pmatrix}\quad \forall\, t\in\{1,\dots,T\},\, T\in\mathbb{N}$$

$$\text{and}\quad \mathrm{E}_{t,T,\Lambda} := \begin{pmatrix}1 & 0\\ \Lambda & 1-\Lambda\end{pmatrix}\cdot\begin{pmatrix}\mathrm{E}^{[1]}_{t,T}\\ \mathrm{E}^{[2]}_{t,T}\end{pmatrix}\quad \forall\, t\in\{1,\dots,T\}, T\in\mathbb{N}, \Lambda\in[0,1].$$

Further, let $\big(\varepsilon_t^{[\mathrm{C}]}\big)_{t\in\mathbb{Z}}$ as well as $\big\{\mathrm{C}^{[1]}_{t,T}\big\}$ originate from (3.81), assume that $\big(\varepsilon_t^{[\mathrm{F}]}\big)_{t\in\mathbb{Z}}$ is a sequence of i. i. d. random variables with $\varepsilon_0^{[\mathrm{F}]} \sim \mathcal{U}\big[-\sqrt{3},\sqrt{3}\big]$ which is independent of $\big(\varepsilon_t^{[\mathrm{C}]}\big)_{t\in\mathbb{Z}}$ and define:

$$\mathrm{f}_0(u) := 0.2\cdot(1.1-u)^2\,\mathbb{1}_{\{u\in[0,1]\}},\quad \mathrm{f}_1(u) := 0.4\cdot(1+\sin(2\pi u))\,\mathbb{1}_{\{u\in[0,1]\}}\quad \forall\, u\in\mathbb{R}$$

$$\mathrm{F}_{0,T} := \sqrt{\mathrm{f}_0(0)}\,\varepsilon_0^{[\mathrm{F}]},\quad \mathrm{F}_{t,T} := \left(\mathrm{f}_0\left(\frac{t}{T}\right) + \mathrm{f}_1\left(\frac{t}{T}\right)\cdot\big(\mathrm{F}_{t-1,T}\big)^2\right)^{\frac{1}{2}}\varepsilon_t^{[\mathrm{F}]}\quad \forall\, t\in\{1,\dots,T\},\, T\in\mathbb{N}$$

$$\text{and}\quad \mathrm{F}_{t,T,\Lambda} := \begin{pmatrix}1 & 0\\ \Lambda & 1-\Lambda\end{pmatrix}\cdot\begin{pmatrix}\mathrm{C}^{[1]}_{t,T}\\ \mathrm{F}_{t,T}\end{pmatrix}\quad \forall\, t\in\{1,\dots,T\}, T\in\mathbb{N}, \Lambda\in[0,1].$$

Since $\big\{\mathrm{E}^{[1]}_{t,T}\big\}$ as well as $\big\{\mathrm{E}^{[2]}_{t,T}\big\}$ fulfil the conditions of Example 2.5 (ii) with $\delta = 0.26$ and $\big\{\mathrm{C}^{[1]}_{t,T}\big\}$ as well as $\big\{\mathrm{F}_{t,T}\big\}$ satisfy the conditions of Example 2.5 (iii) with $\delta = 0.26$, it is easy to verify that Assumption 4.1 $[\textbf{INDEP}]$ with $\delta = 0.26$ holds for the locally stationary processes $\{\mathrm{E}_{t,T,\Lambda}\}$ and $\{\mathrm{F}_{t,T,\Lambda}\}$ (for all fixed $\Lambda\in[0,1]$). Hence, it is justified for all $\Lambda\in[0,1]$ to apply Algorithm **TEST.INDEP.1** in order to investigate whether the stationary approximations of $\big\{\mathrm{E}^{[1]}_{t,T}\big\}$ depend on those of $\big\{\Lambda\,\mathrm{E}^{[1]}_{t,T} + (1-\Lambda)\,\mathrm{E}^{[2]}_{t,T}\big\}$ and whether the stationary approximations of $\big\{C^{[1]}_{t,T}\big\}$ depend on those of $\big\{\Lambda\,\mathrm{C}^{[1]}_{t,T} + (1-\Lambda)\,\mathrm{F}_{t,T}\big\}$ (in the sense that $\mathcal{H}^{\text{indep}}_{1,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ holds).

In the following, for all $\Lambda \in \{0, 0.025, 0.05, 0.075, 0.1\}$, 200 sample paths of $\big(\mathrm{E}_{t,T,\Lambda}\big)_{t=1}^{T}$ with $T = 1000$ are generated independently of each other and the versions of Algorithm **TEST.INDEP.2** given in Setting 4.21 with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ are applied to all of these sample paths. The belonging relative frequencies of rejecting $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ are given in Table 4.1.

Table 4.1.: Relative frequencies of rejecting $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ that result from applying the versions of Algorithm **TEST.INDEP.2** which are introduced in Setting 4.21 to sample paths of $\big(\mathrm{E}_{t,T,\Lambda}\big)_{t=1}^{T}$ with $\Lambda \in \{0, 0.025, 0.05, 0.075, 0.1\}$

b	β	α \ Λ	0	0.025	0.05	0.075	0.1
0.1	$0.3\,Tb^2$	0.05	0.07	0.125	0.66	0.98	1
		0.1	0.135	0.29	0.8	0.995	1
	$0.7\,Tb^2$	0.05	0.075	0.165	0.675	0.98	1
		0.1	0.195	0.355	0.845	0.995	1
0.15	$0.3\,Tb^2$	0.05	0.06	0.135	0.605	0.98	1

		0.1	0.135	0.25	0.8	0.99	1
	$0.7\,Tb^2$	0.05	0.07	0.145	0.665	0.99	1
		0.1	0.22	0.36	0.885	0.99	1

The locally stationary process $\{\mathrm{E}_{t,T,\Lambda}\}$ fulfils $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ in the case $\Lambda = 0$ but not for the other considered choices of Λ. For $\Lambda = 0$ and $\alpha \in \{0.05, 0.1\}$, the test version with $b = 0.15$ as well as $\beta = 0.3\,Tb^2$ can be regarded as the most suitable one among the applied ones. However, the other considered test versions also own adequate rejection frequencies for the significance level $\alpha = 0.05$ and that with $b = 0.1$ as well as $\beta = 0.3\,Tb^2$ also for $\alpha = 0.1$.
Further, if $\Lambda \in [0,1]$ is closer to one, $\left(E^{[1]}_{t,T}\right)_{t=1}^{T}$ and $\left(\Lambda\,\mathrm{E}^{[1]}_{t,T} + (1-\Lambda)\,\mathrm{E}^{[2]}_{t,T}\right)_{t=1}^{T}$ depend stronger on each other, which may explain why the relative frequencies of rejecting $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ which are displayed in Table 4.1 increase with growing Λ for fixed b, β as well as α. In particular, rejecting $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ reliably is more challenging for smaller values of $\Lambda \in [0,1]$ than larger ones. Concretely, the applied test versions reject $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ based on the significance levels $\alpha \in \{0.05, 0.1\}$ with not satisfactory frequency for $\Lambda = 0.025$ but much more often for $\Lambda = 0.05$ and this null hypothesis is rejected by the used test versions for (almost) each generated sample path of $\left(\mathrm{E}_{t,T,\Lambda}\right)_{t=1}^{T}$ in the cases $\Lambda \in \{0.075, 0.1\}$.

In the following, for all $\Lambda \in \{0, 0.05, 0.1, 0.15, 0.2\}$, 200 sample paths of $\left(\mathrm{F}_{t,T,\Lambda}\right)_{t=1}^{T}$ with $T = 1000$ are generated independently of each other and the versions of Algorithm **TEST.INDEP.2** given in Setting 4.21 with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ are applied to all of these sample paths. The belonging relative frequencies of rejecting $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ are given in Table 4.2.

Table 4.2.: Relative frequencies of rejecting $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ that result from applying the versions of Algorithm **TEST.INDEP.2** which are introduced in Setting 4.21 to sample paths of $\left(\mathrm{F}_{t,T,\Lambda}\right)_{t=1}^{T}$ with $\Lambda \in \{0, 0.05, 0.1, 0.15, 0.2\}$

b	β	α \ Λ	0	0.05	0.1	0.15	0.2
0.1	$0.3\,Tb^2$	0.05	0.055	0.115	0.435	0.93	1
		0.1	0.105	0.205	0.605	0.975	1
	$0.7\,Tb^2$	0.05	0.065	0.135	0.485	0.95	1
		0.1	0.13	0.23	0.655	0.99	1
0.15	$0.3\,Tb^2$	0.05	0.05	0.12	0.545	0.97	1
		0.1	0.105	0.23	0.675	0.99	1
	$0.7\,Tb^2$	0.05	0.065	0.16	0.56	0.965	1
		0.1	0.12	0.265	0.76	0.99	1

Table 4.2 indicates for each significance level $\alpha \in \{0.05, 0.1\}$ quite or very satisfactory rejection frequencies of all considered test versions under the null hypothesis $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$, which holds just for $\Lambda = 0$. However, compared to the previous simulation study, Λ has to be chosen noticeably larger (i. e., $\Lambda \in \{0.15, 0.2\}$) to ensure that the applied test versions detect the validity of $\mathcal{H}^{\text{indep}}_{1,\mathfrak{D}_1,\mathfrak{D}_2}$ reliably.

4.4.2. Testing for total blockwise independence

In this subsection, the versions of Algorithm **TEST.INDEP.2** that are described in Setting 4.21 are evaluated based on $\mathfrak{D}_1 = \{0,1\}$, $\mathfrak{D}_2 \in \{\mathfrak{D}_{2,0}, \dots, \mathfrak{D}_{2,4}\}$ with $\mathfrak{D}_{2,j} := \{j, j+1\}\ \forall\, j \in \{0,\dots,4\}$ and the locally stationary process $\{\mathrm{G}_{t,T}\}$, which is introduced in the following.
Suppose that $\left(\varepsilon^{[\mathrm{G},1]}_t\right)_{t\in\mathbb{Z}}$ as well as $\left(\varepsilon^{[\mathrm{G},2]}_t\right)_{t\in\mathbb{Z}}$ are sequences of i. i. d. random variables with $\varepsilon^{[\mathrm{G},1]}_0 \sim \mathcal{S}^0_{1.7}(-0.5, 1, 0)$ and $\varepsilon^{[\mathrm{G},2]}_t \sim \mathcal{N}(-2, 1)$ which are independent of each other. (Recall that $\mathcal{S}^0_{\bar\alpha}\left(\bar\beta, \bar\gamma, \bar\delta\right)$ denotes the $\mathcal{S}^0$-parametrization of the stable distribution - for details see (3.79).) Moreover, define:

$$\mathrm{G}_{t,T} := \begin{pmatrix} \mathrm{G}^{[1]}_{t,T} \\ \mathrm{G}^{[2]}_{t,T} \end{pmatrix} := \frac{1}{10} \begin{pmatrix} \varepsilon^{[\mathrm{G},1]}_t + \cos\left(2\pi \frac{t}{T}\right) \varepsilon^{[\mathrm{G},1]}_{t-1} + \sin\left(2\pi \frac{t}{T}\right) \varepsilon^{[\mathrm{G},1]}_{t-2} \\ \varepsilon^{[\mathrm{G},1]}_t + \cos\left(2\pi \frac{t}{T}\right) \varepsilon^{[\mathrm{G},1]}_{t-1} \varepsilon^{[\mathrm{G},2]}_t \end{pmatrix} \quad \forall\, t \in \{1,\dots,T\},\ T \in \mathbb{N}.$$

The locally stationary process $\{G_{t,T}\}$ fulfils Assumption 4.1 $[\mathbf{INDEP}]$ with $\delta = 0.5$. Hence, it is justified to apply Algorithm **TEST.INDEP.2** in order to investigate whether the stationary approximations of $\{G_{t,T}^{[1]}\}$ depend on those of $\{G_{t,T}^{[2]}\}$ (in the sense that $\mathcal{H}_{1,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ holds).

In the following, 200 sample paths of $(G_{t,T})_{t=1}^{T}$ with $T = 1000$ are generated independently of each other and the versions of Algorithm **TEST.INDEP.2** given in Setting 4.21 with the above-mentioned choices of $\mathfrak{D}_1$ as well as $\mathfrak{D}_2$ are applied to all of these sample paths. The belonging relative frequencies of rejecting $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ are given in Table 4.3.

Table 4.3.: Relative frequencies of rejecting $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ with $\mathfrak{D}_1 = \{0,1\}$ and $\mathfrak{D}_2 \in \{\mathfrak{D}_{2,0}, \ldots, \mathfrak{D}_{2,4}\}$ that result from applying the versions of Algorithm **TEST.INDEP.2** which are introduced in Setting 4.21 to sample paths of $(G_{t,T})_{t=1}^{T}$

b	β	α \ $\mathfrak{D}_2$	$\mathfrak{D}_{2,0}$	$\mathfrak{D}_{2,1}$	$\mathfrak{D}_{2,2}$	$\mathfrak{D}_{2,3}$	$\mathfrak{D}_{2,4}$
0.1	$0.3\,Tb^2$	0.05	1	1	1	1	0.02
		0.1	1	1	1	1	0.07
	$0.7\,Tb^2$	0.05	1	1	1	1	0.045
		0.1	1	1	1	1	0.09
0.15	$0.3\,Tb^2$	0.05	1	1	1	1	0.045
		0.1	1	1	1	1	0.09
	$0.7\,Tb^2$	0.05	1	1	1	1	0.06
		0.1	1	1	1	1	0.145

The locally stationary process $\{G_{t,T}\}$ fulfils $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ with $\mathfrak{D}_1 = \{0,1\}$ for $\mathfrak{D}_2 = \mathfrak{D}_{2,4} = \{4,5\}$ but not for $\mathfrak{D}_2 \in \{\mathfrak{D}_{2,0}, \ldots, \mathfrak{D}_{2,3}\}$. In the case $\mathfrak{D}_2 = \mathfrak{D}_{2,4}$, just $(b, \beta, \alpha) = (0.15, 0.7\,Tb^2, 0.1)$ leads to a noticeable too large frequency of rejecting $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$. In contrast, the rejection frequencies that belong to the other choices of (b, β, α) are quite or very satisfactory for $\mathfrak{D}_2 = \mathfrak{D}_{2,4}$. Furthermore, for $\mathfrak{D}_2 \in \{\mathfrak{D}_{2,0}, \ldots, \mathfrak{D}_{2,3}\}$, the validity of $\mathcal{H}_{1,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ is detected by all applied versions of Algorithm **TEST.INDEP.2** for each generated sample path of $(G_{t,T})_{t=1}^{T}$.

5. Applications: Financial time series

In this chapter, the Algorithms **CONF.NMDCI**, **TEST.MDCI.2**, **DETECT.MDCI** as well as **TEST.INDEP.2** are applied to log returns of several listed companies and the outputs of these algorithms are interpreted from an economic perspective.

5.1. Preliminaries

The log returns which are considered in the present chapter are based on opening and adjusted closing stock prices (adjusted by splits) from $T = 1000$ trading days of GameStop Corporation (share price: GME) and of Wirecard AG (share price: WRCDF) from July 3, 2017, to June 22, 2021, as well as from $T = 1000$ trading days of Deutsche Bank AG (share price: DBK.DE) and of Deutsche Lufthansa AG (share price: LHA.DE) from January 2, 2018, to December 13, 2021, whereby these stock prices originate from yahoo! finance. Concretely, the regarded log returns of the stocks GameStop (abbr. GS), Wirecard (abbr. WC), Deutsche Bank (abbr. DB) and Deutsche Lufthansa (abbr. LH) are defined as follows.

Suppose that $o_{t,T}^{[S]}$ denotes the opening price and $c_{t,T}^{[S]}$ the adjusted closing price on trading day $t \in \{1, \dots, T = 1000\}$ of stock $S \in \{\mathrm{GS}, \mathrm{WC}, \mathrm{DB}, \mathrm{LH}\}$. Then, the log return on day t of stock S is given as:

$$x_{t,T}^{[S]} := \ln\left(c_{t,T}^{[S]}\right) - \ln\left(o_{t,T}^{[S]}\right) \quad \forall\, t \in \{1, \dots, T = 1000\},\ S \in \{\mathrm{GS}, \mathrm{WC}, \mathrm{DB}, \mathrm{LH}\}. \tag{5.1}$$

Remark 5.1. *(i) Note that the date which corresponds with specific t is not the same for all considered stocks because the investigated $T = 1000$ log returns of GameStop as well as Wirecard originate from another period of time than those of Deutsche Bank and Deutsche Lufthansa.*

(ii) Defining log returns based on closing prices of two consecutive trading days is quite common in the literature (e. g., as in [28, Fryzlewicz (2005)]). However, the present stock price data are characterized by the fact that the actual time gap between two consecutive trading days may vary sizeably since the considered stock markets stay closed on Saturday, Sunday and some official holidays (like December 25), whereby, in this time gap between two consecutive trading days, events may happen which have an impact on the investigated log returns. In this context, it is worth mentioning that Lufthansa offers flights on these non-trading days, GameStop's stores are open on Saturday and Deutsche Bank as well as Wirecard operate offices in some countries (like China) in which Saturday and December 25 are working days.
To avoid that the regarded log returns represent considerably differently long periods of time, they are defined in this chapter based on opening and (adjusted) closing prices that originate from the same trading day.

*(iii) The Remarks 3.10 (iii), 3.18 (ii), 3.26 (ii), 3.34 and 4.15 (iii) (whereby it is easy to see that the claims of the latter remark also hold for Algorithm **TEST.INDEP.2**) provide together with the equation $\ln(y\,z) = \ln(y) + \ln(z)\ \forall\, y, z > 0$ that applying the Algorithms **CONF.NMDCI**, **TEST.MDCI.2**, **DETECT.MDCI** as well as **TEST.INDEP.2** to log returns allows to investigate distribution changes and the existence of dependences, respectively, without distorting these explorations by the average scale of the simple returns which belong to each stock.*

The following macroeconomic events are contained in the considered time periods from July 3, 2017, to June 22, 2021 and from January 2, 2018, to December 13, 2021: On February 1, 2020, Great Britain withdrew from the European Union and the European Atomic Energy Community (cf. [9, BMI (2020)]). On March 11, 2020, the World Health Organization classified the COVID-19 spread as a pandemic (cf. [12, Cucinotta and Vanelli (2020)]) and first Coronavirus vaccines were authorized in the EU on December 21, 2020 (cf. [10, Cavaleri et al. (2021)]).

Furthermore, among others, the following events, which might have influenced the regarded sequences of stock prices of GameStop and Wirecard, happened in the time period from July 3, 2017, to June 22, 2021: On March 22, 2020, GameStop temporarily stopped customer access to US storefronts due to the COVID-19 pandemic (cf. [32, GameStop Corporation (2020)]). In addition, the GameStop short squeeze occurred at the end of January 2021. Moreover, Germany's Federal Financial Supervisory Authority (named BaFin) prohibited establishing and increasing of Wirecard net short positions from February 18, 2019, to April 18, 2019 (cf. [7, BaFin (2019)]) and leading representatives of Wirecard were arrested for fraud by Munich Public Prosecution Office I on July 22, 2020 (cf. [51, Leiding (2020)]).
Among others, the time period from January 2, 2018, to December 13, 2021, contains the following events, which might have influenced the above-mentioned stock price sequences of Deutsche Bank and Deutsche Lufthansa: On November 29, 2018, police conducted an investigation at some Deutsche Bank offices in Germany, which was related to the "Panama Papers" (cf. [21, Deutsche Bank (2018)]). Moreover, on July 7, 2019, Deutsche Bank outlined significant strategic transformation and restructuring plans (cf. [23, Deutsche Bank (2019b)]). In particular, these plans envisaged to exit the Equities Sales & Trading business and to reduce the costs in order to obtain a cost income ratio of 70% in 2022. Further, on March 6, 2020, Lufthansa Group announced to reduce its flight programme by up to 50% due to the COVID-19 spread (cf. [53, Lufthansa Group (2020)]) and on November 3, 2021, Lufthansa Group released that the company's third quarter adjusted EBIT was positive again for the first time since the beginning of the pandemic (cf. [54, Lufthansa Group (2021)]).

5.2. Investigation of distribution changes

In the present section, the Algorithms **TEST.MDCI.2**, **CONF.NMDCI** as well as **DETECT.MDCI** are applied to the sequences $\big(x_{t,T}^{[S]}\big)_{t=1}^{T}$ (with $T = 1000$ and $S \in \{\mathrm{GS}, \mathrm{WC}, \mathrm{DB}, \mathrm{LH}\}$), which are defined in the previous section. Thereby, the used versions of these algorithms are almost the same as those described in Setting 3.45 with the only differences that the weight function $\mathbb{R} \ni s \mapsto \mathbf{w}(s) := \mathbf{w}_{\mathrm{L},20}(s)$ (recall Example 3.2 (ii)) instead of $\mathbb{R} \ni s \mapsto \mathbf{w}(s) := \mathbf{w}_{\mathrm{L},1}(s)$ is used for all of these algorithms and that $M = 1000$ is selected for Algorithm **DETECT.MDCI**. This choice of M ensures that each day can be identified as the day of the first change point, whereas $M = 100$ is selected in Setting 3.45 to reduce the computational costs of Algorithm **DETECT.MDCI**. Further, note that both of the weight functions $\mathbf{w}_{\mathrm{L},1}$ and $\mathbf{w}_{\mathrm{L},20}$ fulfil Assumption 3.1 $[\mathbf{WEI.1}]$ but the following considerations indicate that $\mathbf{w}_{\mathrm{L},20}$ is a more suitable choice for the present investigations than $\mathbf{w}_{\mathrm{L},1}$.
To motivate the weight function $\mathbf{w}_{\mathrm{L},20}$, define at first for each function $\mathbf{w}$ for which Assumption 3.1 $[\mathbf{WEI.1}]$ holds the belonging non-unitary Fourier transform (with angular frequency $x \in \mathbb{R}^d$):

$$\mathfrak{F}\mathbf{w}(x) := \int_{\mathbb{R}^d} e^{-i\langle s,x\rangle}\mathbf{w}(s)\,ds. \tag{5.2}$$

The equality $|z|^2 = z\overline{z}\;\forall\, z \in \mathbb{C}$ provides (recall the Definitions 3.8 (i) as well as 2.11):

$$\widehat{\mathbb{D}}_{T,1} = \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \frac{1}{T^2} \sum_{t_1,t_2=1}^{T} K_b\left(\frac{t_1}{T} - u_k\right) K_b\left(\frac{t_2}{T} - u_k\right) \mathfrak{F}\mathbf{w}\left(X_{t_2,T} - X_{t_1,T}\right) \quad \text{and}$$

$$\widehat{\mathbb{D}}_{T,2} = \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor^2} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \frac{1}{T^2} \sum_{t_1,t_2=1}^{T} K_b\left(\frac{t_1}{T} - u_{k_1}\right) K_b\left(\frac{t_2}{T} - u_{k_2}\right) \mathfrak{F}\mathbf{w}\left(X_{t_2,T} - X_{t_1,T}\right). \tag{5.3}$$

Thereby, it holds $\mathfrak{F}\mathbf{w}_{\mathrm{L},1}(x) = 2/(1 + x^2)\;\forall\, x \in \mathbb{R}$ and $\mathfrak{F}\mathbf{w}_{\mathrm{L},20}(x) = 40/(1 + 400\,x^2)\;\forall\, x \in \mathbb{R}$. Thus, for choices of x with small absolute value, the additive 1 in the denominator of $\mathfrak{F}\mathbf{w}_{\mathrm{L},1}(x)$ is very large compared to x^2. Hence, just due to the fact that the investigated log returns oscillate on a small level, the EMDCI and NEMDCI which belong to the present log returns are very close to zero in the case that the weight function $\mathbf{w}_{\mathrm{L},1}$ is used. Compared to $\mathfrak{F}\mathbf{w}_{\mathrm{L},1}$, the effect of the additive 1 in the denominator of $\mathfrak{F}\mathbf{w}_{\mathrm{L},20}$ is much smaller. (Note also that the factors 2 as well as 40 in $\mathfrak{F}\mathbf{w}_{\mathrm{L},1}$ and $\mathfrak{F}\mathbf{w}_{\mathrm{L},20}$, respectively, can be replaced by any $y > 0$ without changing the outputs of the Algorithms **TEST.MDCI.2**, **CONF.NMDCI** as well as **DETECT.MDCI**, which can be shown similarly to Remark 3.29 (i) since the same weight function is contained in the underlying statistics and their bootstrap counterparts.)

In the following, the versions of the Algorithms **TEST.MDCI.2**, **CONF.NMDCI** as well as **DETECT.MDCI** which originate from Setting 3.45 (apart from the weight function that is selected as $\mathbf{w} = \mathbf{w}_{\mathrm{L},20}$ and M which is chosen as $M = 1000$) are applied to the log returns of GameStop as well as Wirecard that belong to the time period from July 3, 2017, to June 22, 2021, which is taken as the interval $\mathfrak{U}_{0,1} = [0, 1]$. In addition, these versions of the Algorithms **TEST.MDCI.2** and **CONF.NMDCI** are also applied to the log returns of GameStop for $\mathfrak{U}_{0,1} = [0, 0.684]$, which is the interval that corresponds with the time period from July 3, 2017, to March 20, 2020 (in particular, the pandemic induced closing of GameStop's US storefronts is not contained in this time period) as well as for the rescaled time period $\mathfrak{U}_{0,1} = [0.685, 1]$ (at whose beginning GameStop's US storefronts were closed and which also includes the GameStop short squeeze). Furthermore, the mentioned versions of the Algorithms **TEST.MDCI.2** and **CONF.NMDCI** are applied to the log returns of Wirecard for the interval $\mathfrak{U}_{0,1} = [0, 0.409]$, which belongs to the time period from July 3, 2017, to February 15, 2019 (in particular, BaFin's prohibition of establishing and increasing of Wirecard net short positions is not contained in this time period) as well as for the rescaled time period $\mathfrak{U}_{0,1} = [0.41, 1]$ (at whose beginning this prohibition applied and in which leading representatives of Wirecard were arrested).
Table 5.1 (which is based on the log returns of GameStop) and Table 5.2 (that is based on the log returns of Wirecard) contain the estimated p-values $1 - \widehat{F}^{*}_{T,N}\big(\mathbb{D}_T(\omega) + 1/(T\sqrt{b})\widehat{\mathbf{Bias}}^{\mathrm{error}}_T(\omega)\big)$ $\Big($resulting from the used versions of Algorithm **TEST.MDCI.2**, whereby, as explained in Remark 3.38 (ii), $\widehat{\mathbf{Bias}}^{\mathrm{error}}_T = 0$ for $\mathbf{B}_T = 0\Big)$, the estimated 95%-confidence intervals for the NMDCI (obtained by the applied versions of Algorithm **CONF.NMDCI**) and the dates (notated in US date format) that correspond with the estimated points in time of the first gradual distribution change (which originate from the considered versions of Algorithm **DETECT.MDCI**). Thereby, the lower and upper bounds of the estimated confidence intervals displayed in the Tables 5.1 and 5.2 are values in percentage, whereby the lower bounds in percentage of these estimated intervals are rounded off to two decimals and their upper bounds are rounded up to two decimals in order to ensure that rounding does not reduce the estimated confidence intervals.

Table 5.1.: Estimated p-values, estimated 95%-confidence intervals for the NMDCI and estimated points in time of the first gradual distribution change that belong to the investigated log returns of GameStop

b	β	$\mathbf{B}_T$	**TEST.MDCI.2**			**CONF.NMDCI**			**DETECT.MDCI**
			$\mathfrak{U}_{0,1}$			$\mathfrak{U}_{0,1}$			
			$[0,1]$	$[0,0.684]$	$[0.685,1]$	$[0,1]$	$[0,0.684]$	$[0.685,1]$	
1/10	$0.3\,Tb^2$	0	0	0.002	0				07/24/2019
		3	0	0.002	0	$[2.94, 5.73]$	$[0.36, 1.42]$	$[1.74, 8.81]$	07/17/2019
		5	0	0.002	0				07/02/2019
	$0.7\,Tb^2$	0	0	0.002	0.002				04/22/2019
		3	0	0.002	0.002	$[2.64, 6.03]$	$[0.34, 1.39]$	$[1.63, 8.95]$	04/10/2019
		5	0	0.002	0.002				04/15/2019
1/14	$0.3\,Tb^2$	0	0	0	0.028				06/21/2019
		3	0	0	0.024	$[2.58, 5.03]$	$[0.60, 1.80]$	$[1.98, 7.03]$	05/20/2019
		5	0	0	0.024				05/03/2019
	$0.7\,Tb^2$	0	0	0	0.028				03/21/2019
		3	0	0	0.026	$[2.35, 5.10]$	$[0.56, 1.88]$	$[1.61, 7.05]$	03/28/2019
		5	0	0	0.026				03/20/2019

Table 5.1 indicates that all applied versions of Algorithm **TEST.MDCI.2** reject $\mathbf{H}^{\mathrm{distr}}_{0,\mathfrak{U}_{0,1}}$ (see (3.49)) for $\mathfrak{U}_{0,1} \in \{[0,1], [0,0.684], [0.685,1]\}$ based on the significance level $\alpha = 0.05$ (or even lower levels - like $\alpha = 0.01$). Moreover, the estimated 95%-confidence intervals for the NMDCI suggest that the distribution change intensity is probably higher in the considered time period in which US storefronts of GameStop were closed due to the pandemic and the GameStop short squeeze occurred than in that before. Further, the estimated dates of the first gradual distribution change generated by the regarded versions of Algorithm **DETECT.MDCI** vary considerably in dependence of the underlying tuning parameters (but all of these dates are contained in a time period which is quite narrow compared to the total number of analyzed trading days which belong to several years). Thereby, it should be noted that Proposition 3.42 (i) implies that the present method for estimating the first point in time $\mathbb{V}$ of a gradual distribution change is accompanied by the asymptotic error $\sqrt{\alpha}\,\mathbb{V}$. The two very different simulation studies given in Subsection 3.4.3 suggest based on $T = 1000$ as well as $\alpha = 0.05$ that $b = 1/10$ and $\beta = 0.3\,Tb^2$ are recommended choices for Algorithm **DETECT.MDCI**. For these selections of b and β

as well as each $\mathbf{B}_T \in \{0, 3, 5\}$, the estimated point in time of the first gradual distribution change is close to (or contained in) the time period from July 7-20, 2019, in which GameStop offered deep discounts on its products (cf. [31, GameStop Corporation (2019)]).

Table 5.2.: Estimated p-values, estimated 95%-confidence intervals for the NMDCI and estimated points in time of the first gradual distribution change that belong to the investigated log returns of Wirecard

b	β	$\mathbf{B}_T$	**TEST.MDCI.2**			**CONF.NMDCI**			**DETECT.MDCI**
			$\mathfrak{U}_{0,1}$			$\mathfrak{U}_{0,1}$			
			$[0,1]$	$[0,0.409]$	$[0.41,1]$	$[0,1]$	$[0,0.409]$	$[0.41,1]$	
1/10	$0.3\,Tb^2$	0	0	0	0				05/24/2018
		3	0	0	0	$[4.96, 8.60]$	$[0.16, 0.62]$	$[4.52, 9.39]$	05/03/2018
		5	0	0	0				05/10/2018
	$0.7\,Tb^2$	0	0	0	0				02/21/2018
		3	0	0	0	$[4.68, 8.93]$	$[0.18, 0.61]$	$[4.35, 9.57]$	02/12/2018
		5	0	0	0				02/15/2018
1/14	$0.3\,Tb^2$	0	0	0	0				04/19/2018
		3	0	0	0	$[5.78, 9.23]$	$[0.23, 0.91]$	$[4.56, 8.55]$	03/12/2018
		5	0	0	0				03/09/2018
	$0.7\,Tb^2$	0	0	0	0				02/13/2018
		3	0	0	0	$[5.75, 9.41]$	$[0.24, 0.91]$	$[4.92, 8.63]$	01/22/2018
		5	0	0	0				01/25/2018

Table 5.2 indicates that the applied versions of Algorithm **TEST.MDCI.2** reject $\mathbf{H}_{0,\mathfrak{U}_{0,1}}^{\text{distr}}$ highly significantly for $\mathfrak{U}_{0,1} \in \{[0,1], [0, 0.409], [0.41, 1]\}$. In addition, the estimated 95%-confidence intervals for the NMDCI suggest that the distribution change intensity is probably higher in the considered time period in which BaFin prohibited establishing and increasing of Wirecard net short positions as well as leading representatives of Wirecard were arrested than in that before. Further, the outputs generated by Algorithm **DETECT.MDCI** vary considerably in dependence of the underlying tuning parameters. For the above-motivated selections $b = 1/10$, $\beta = 0.3\,Tb^2$ and $\mathbf{B}_T = \{0, 3, 5\}$, the point in time of the first gradual distribution change lies closely to the date May 8, 2018, at which Wirecard Bank applied for becoming a subsidiary company of Wirecard AG from BaFin (cf. [8, BMF (2020), p. 13]).
Further, from July 3, 2017 to June 22, 2021 (i. e., for $\mathfrak{U}_{0,1} = [0,1]$), it seems that stronger distribution changes are associated with Wirecard than with GameStop. However, this statement is doubtful because many of the used versions of Algorithm **CONF.NMDCI** yield estimated confidence intervals of GameStop and of Wirecard that overlap each other for $\mathfrak{U}_{0,1} = [0,1]$.

In the following, the versions of the Algorithms **TEST.MDCI.2**, **CONF.NMDCI** as well as **DETECT.MDCI** which originate from Setting 3.45 (apart from the weight function that is selected as $\mathbf{w} = \mathbf{w}_{\text{L},20}$ and M which is chosen as $M = 1000$) are applied to the log returns of Deutsche Bank as well as Deutsche Lufthansa that belong to the time period from January 2, 2018, to December 13, 2021, which is taken as the interval $\mathfrak{U}_{0,1} = [0,1]$. In addition, these versions of the Algorithms **TEST.MDCI.2** and **CONF.NMDCI** are also applied to the log returns of Deutsche Bank for $\mathfrak{U}_{0,1} = [0, 0.525]$, which is the interval that corresponds with the time period from January 2, 2018, to January 31, 2020 (in particular, Great Britain's withdrawal from the European Union and the European Atomic Energy Community is not contained in this time period) as well as for the rescaled time period $\mathfrak{U}_{0,1} = [0.526, 1]$ (at whose beginning this withdrawal was effective). It is worth mentioning that in the time period from January 2, 2018, to January 31, 2020, official investigations at some Deutsche Bank offices in Germany were carried out (due to the "Panama Papers") and Deutsche Bank outlined the strategic transformation as well as restructuring plans described in the previous section. Furthermore, the considered versions of the Algorithms **TEST.MDCI.2** and **CONF.NMDCI** are applied to the log returns of Deutsche Lufthansa for the interval $\mathfrak{U}_{0,1} = [0, 0.549]$, which belongs to the time period from January 2, 2018 to March 5, 2020 (in particular, Lufthansa Group's announcement to reduce its flight programme by up to 50% is not included in this time period) as well as for the rescaled time period $\mathfrak{U}_{0,1} = [0.55, 1]$ (at whose beginning this announcement was published).
Table 5.3 (which is based on the log returns of Deutsche Bank) and Table 5.4 (that is based on the log returns of Deutsche Lufthansa) contain the estimated p-values $1 - \widehat{F}^{*}_{T,N}\big(\widehat{\mathbb{D}}_T(\omega) + 1/(T\sqrt{b})\widehat{\widehat{\mathbf{Bias}}}{}_T^{\text{error}}(\omega)\big)$ (resulting from the used versions of Algorithm **TEST.MDCI.2**), the estimated 95%-confidence intervals for the NMDCI (obtained by the applied versions of Algorithm **CONF.NMDCI**) and the dates (notated in US date format) that correspond with the estimated points in time of the first gradual distribution

change (which originate from the considered versions of Algorithm **DETECT.MDCI**). Thereby, the lower and upper bounds of the estimated confidence intervals displayed in the Tables 5.3 and 5.4 are values in percentage, whereby the lower bounds in percentage of these estimated intervals are rounded off to two decimals and their upper bounds are rounded up to two decimals in order to ensure that rounding does not reduce the estimated confidence intervals.

Table 5.3.: Estimated p-values, estimated 95%-confidence intervals for the NMDCI and estimated points in time of the first gradual distribution change that belong to the investigated log returns of Deutsche Bank

b	β	$\mathbf{B}_T$	**TEST.MDCI.2**			**CONF.NMDCI**			**DETECT.MDCI**
			$\mathfrak{U}_{0,1}$			$\mathfrak{U}_{0,1}$			
			$[0,1]$	$[0,0.525]$	$[0.526,1]$	$[0,1]$	$[0,0.525]$	$[0.526,1]$	
1/10	$0.3\,Tb^2$	0	0.036	0.058	0.002				01/02/2020
		3	0.036	0.056	0.002	$[0.07,0.45]$	$[0.11,0.69]$	$[0.44,1.53]$	12/09/2019
		5	0.038	0.05	0.002				11/19/2019
	$0.7\,Tb^2$	0	0.042	0.046	0				08/14/2019
		3	0.042	0.046	0	$[0.07,0.45]$	$[0.13,0.70]$	$[0.47,1.49]$	08/14/2019
		5	0.044	0.046	0				08/07/2019
1/14	$0.3\,Tb^2$	0	0.012	0.032	0.028				09/27/2019
		3	0.012	0.028	0.028	$[0.12,0.65]$	$[0.24,0.92]$	$[0.41,1.38]$	08/26/2019
		5	0.012	0.032	0.024				08/13/2019
	$0.7\,Tb^2$	0	0.014	0.034	0.016				06/11/2019
		3	0.014	0.034	0.016	$[0.13,0.65]$	$[0.25,0.90]$	$[0.39,1.40]$	06/06/2019
		5	0.014	0.034	0.016				05/30/2019

Table 5.3 indicates that most of the applied versions of Algorithm **TEST.MDCI.2** reject $\mathbf{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ for $\mathfrak{U}_{0,1} \in \{[0,1],[0,0.525],[0.526,1]\}$ based on the significance level $\alpha = 0.05$ (and partially also for $\alpha = 0.01$). Moreover, the estimated confidence intervals for the NMDCI may suggest that the distribution change intensity is probably higher in the considered period of time since Great Britain's withdrawal from the EU than in that before. However, it should be noted that all of the applied versions of Algorithm **CONF.NMDCI** yield estimated 95%-confidence intervals for $\mathfrak{U}_{0,1} = [0,0.525]$ and $\mathfrak{U}_{0,1} = [0.526,1]$ that overlap each other. Further, the outputs generated by Algorithm **DETECT.MDCI** vary considerably in dependence of the underlying tuning parameters. For the above-recommended selections $b = 1/10$, $\beta = 0.3\,Tb^2$ and $\mathbf{B}_T = \{0,3,5\}$, the point in time of the first gradual distribution change is close to the date December 6, 2019, at which Deutsche Bank announced that Frankfurt public prosecutor's office dropped certain allegations of aiding and abetting tax evasion and of money laundering that it made against Deutsche Bank more than one year before (cf. [22, Deutsche Bank (2019a)]).

Table 5.4.: Estimated p-values, estimated 95%-confidence intervals for the NMDCI and estimated points in time of the first gradual distribution change that belong to the investigated log returns of Deutsche Lufthansa

b	β	$\mathbf{B}_T$	**TEST.MDCI.2**			**CONF.NMDCI**			**DETECT.MDCI**
			$\mathfrak{U}_{0,1}$			$\mathfrak{U}_{0,1}$			
			$[0,1]$	$[0,0.549]$	$[0.55,1]$	$[0,1]$	$[0,0.549]$	$[0.55,1]$	
1/10	$0.3\,Tb^2$	0	0	0.918	0				03/31/2020
		3	0	0.92	0	$[0.23,1.05]$	$[0,0.18]$	$[0.77,2.55]$	03/13/2020
		5	0	0.928	0				03/02/2020
	$0.7\,Tb^2$	0	0.002	0.9	0.002				02/19/2020
		3	0.002	0.906	0.002	$[0.21,1.07]$	$[0,0.18]$	$[0.71,2.49]$	01/14/2020
		5	0.002	0.906	0.002				12/06/2019
1/14	$0.3\,Tb^2$	0	0	0.496	0				03/17/2020
		3	0	0.484	0	$[0.28,1.08]$	$[0,0.47]$	$[1.19,3.58]$	03/06/2020
		5	0	0.474	0				02/27/2020
	$0.7\,Tb^2$	0	0	0.49	0				02/19/2020
		3	0	0.482	0	$[0.26,1.08]$	$[0,0.48]$	$[1.21,3.41]$	01/21/2020
		5	0	0.476	0				11/20/2019

Table 5.4 indicates that all applied versions of Algorithm **TEST.MDCI.2** reject $\mathbf{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ highly significantly for $\mathfrak{U}_{0,1} \in \{[0,1],[0.55,1]\}$. However, $\mathbf{H}^{\text{distr}}_{0,\mathfrak{U}_{0,1}}$ is not rejected for any statistically relevant significance level in the case $\mathfrak{U}_{0,1} = [0,0.549]$, i. e., before Lufthansa announced to reduce its flight programme by up to 50%. In particular, note that all of the generated estimated 95%-confidence intervals for the NMDCI with $\mathfrak{U}_{0,1} = [0,0.549]$ contain 0. Further, the outputs generated by Algorithm **DETECT.MDCI** vary considerably in dependence of the underlying tuning parameters. For the above-motivated selections $b = 1/10$, $\beta = 0.3\,Tb^2$ and $\mathbf{B}_T = \{0,3,5\}$, the point in time of the first gradual distribution change lies closely to the date March 6, 2020, at which Lufthansa Group announced to reduce its flight programme.

Further, from January 2, 2018, to December 13, 2021 (i. e., for $\mathfrak{U}_{0,1} = [0,1]$), it seems that stronger distribution changes are associated with Deutsche Lufthansa than with Deutsche Bank. However, this statement is doubtful because all of the used versions of Algorithm **CONF.NMDCI** yield estimated confidence intervals for GameStop and for Wirecard that overlap each other in the case $\mathfrak{U}_{0,1} = [0,1]$.

5.3. Testing for independence

In the present section, Algorithm **TEST.INDEP.2** is applied for different choices of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ to the sequences $\big((x_{t,T}^{[\mathrm{GS}]}, x_{t,T}^{[\mathrm{WC}]})'\big)_{t=1}^{T}$ as well as $\big((x_{t,T}^{[\mathrm{DB}]}, x_{t,T}^{[\mathrm{LH}]})'\big)_{t=1}^{T}$ (with $T = 1000$), which are defined in Section 5.1. It is worth mentioning that the results of the previous section indicate that applying tests for independence which are evolved for stationary frameworks (like those mentioned in Section 1.3) to these log returns is not appropriate.
The versions of Algorithm **TEST.INDEP.2** that are used below are almost the same as those described in Setting 4.21 with the only difference that the weight function $\mathbb{R}^{\#\mathfrak{D}_1+\#\mathfrak{D}_2} \ni s \mapsto \tilde{\mathbf{w}}_{\mathfrak{D}_1,\mathfrak{D}_2}(s) := e^{-|s|_1/20}$ instead of $\mathbb{R}^{\#\mathfrak{D}_1+\#\mathfrak{D}_2} \ni s \mapsto \mathbf{w}_{\mathfrak{D}_1,\mathfrak{D}_2}(s) := e^{-|s|_1}$ is regarded. Both of these weight functions fulfil Assumption 4.17 $[\mathbf{WEI.3}]$ but the following considerations motivate why $\tilde{\mathbf{w}}_{\mathfrak{D}_1,\mathfrak{D}_2}$ is a more suitable choice for the present investigations than $\mathbf{w}_{\mathfrak{D}_1,\mathfrak{D}_2}$.
Define $\mathbb{R} \ni s \mapsto w_y(s) := e^{-|s|/y}\ \forall\, y > 0$. If the weight function $\mathbf{w}_{\mathfrak{D}_1,\mathfrak{D}_2}$ is used, set $y = 1$, whereas $y = 20$ is taken for the weight function $\tilde{\mathbf{w}}_{\mathfrak{D}_1,\mathfrak{D}_2}$. Then, the equality $|z|^2 = z\overline{z}\ \forall\, z \in \mathbb{C}$ provides in the case $d_1 = d_2 = 1$ for the weight functions $\mathbf{w}_{\mathfrak{D}_1,\mathfrak{D}_2}$ and $\tilde{\mathbf{w}}_{\mathfrak{D}_1,\mathfrak{D}_2}$, respectively (see (4.28) as well as (5.2)):

$$
\begin{aligned}
\widehat{\mathfrak{Q}}_{T,\mathfrak{D}_1,\mathfrak{D}_2} = {} & \frac{1}{T^2} \sum_{t_1,t_4=1+\mathfrak{D}_{\max}}^{T} K_b\Big(\frac{t_1-\mathfrak{D}_{\mathrm{mean}}}{T}-u\Big)\, K_b\Big(\frac{t_4-\mathfrak{D}_{\mathrm{mean}}}{T}-u\Big) \\
& \cdot \prod_{d_1\in\mathfrak{D}_1} \mathfrak{F}w_y\Big(X^{[1]}_{t_4-\mathfrak{d}_1,T}-X^{[1]}_{t_1-\mathfrak{d}_1,T}\Big) \prod_{d_2\in\mathfrak{D}_2} \mathfrak{F}w_y\Big(X^{[2]}_{t_4-\mathfrak{d}_2,T}-X^{[2]}_{t_1-\mathfrak{d}_2,T}\Big) \\
& - \frac{2}{T^3} \sum_{t_2,t_3,t_4=1+\mathfrak{D}_{\max}}^{T} K_b\Big(\frac{t_2-\mathfrak{D}_{\mathrm{mean}}}{T}-u\Big)\, K_b\Big(\frac{t_3-\mathfrak{D}_{\mathrm{mean}}}{T}-u\Big)\, K_b\Big(\frac{t_4-\mathfrak{D}_{\mathrm{mean}}}{T}-u\Big) \\
& \cdot \prod_{d_1\in\mathfrak{D}_1} \mathfrak{F}w_y\Big(X^{[1]}_{t_4-\mathfrak{d}_1,T}-X^{[1]}_{t_2-\mathfrak{d}_1,T}\Big) \prod_{d_2\in\mathfrak{D}_2} \mathfrak{F}w_y\Big(X^{[2]}_{t_4-\mathfrak{d}_2,T}-X^{[2]}_{t_3-\mathfrak{d}_2,T}\Big) \\
& + \frac{1}{T^4} \sum_{t_2,t_5=1+\mathfrak{D}_{\max}}^{T} K_b\Big(\frac{t_2-\mathfrak{D}_{\mathrm{mean}}}{T}-u\Big)\, K_b\Big(\frac{t_5-\mathfrak{D}_{\mathrm{mean}}}{T}-u\Big) \prod_{d_1\in\mathfrak{D}_1} \mathfrak{F}w_y\Big(X^{[1]}_{t_5-\mathfrak{d}_1,T}-X^{[1]}_{t_2-\mathfrak{d}_1,T}\Big) \\
& \cdot \sum_{t_3,t_6=1+\mathfrak{D}_{\max}}^{T} K_b\Big(\frac{t_3-\mathfrak{D}_{\mathrm{mean}}}{T}-u\Big)\, K_b\Big(\frac{t_6-\mathfrak{D}_{\mathrm{mean}}}{T}-u\Big) \prod_{d_2\in\mathfrak{D}_2} \mathfrak{F}w_y\Big(X^{[2]}_{t_6-\mathfrak{d}_2,T}-X^{[2]}_{t_3-\mathfrak{d}_2,T}\Big)\,.
\end{aligned}
$$

Thus, it can be argued similarly to the previous section that $\tilde{\mathbf{w}}_{\mathfrak{D}_1,\mathfrak{D}_2}$ is more suitable for investigating the present log returns than $\mathbf{w}_{\mathfrak{D}_1,\mathfrak{D}_2}$.

In the following, the versions of Algorithm **TEST.INDEP.2** which originate from Setting 4.21 (apart from the weight function that is chosen as $\tilde{\mathbf{w}}_{\mathfrak{D}_1,\mathfrak{D}_2}$) are applied to $\big((x_{t,T}^{[\mathrm{GS}]}, x_{t,T}^{[\mathrm{WC}]})'\big)_{t=1}^{T}$, i. e., to the log returns of GameStop and Wirecard that belong to the time period from July 3, 2017, to June 22, 2021. Thereby, $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0,\ldots,4\}$ is selected (which means that time periods of the duration of a common trading week are regarded because the considered market place is closed on weekends) to investigate whether the random variables that underlie these log returns depend on each other approximately (in the sense described in Remark 4.2 (iii)).
Table 5.5 contains the estimated p-values $1 - \widehat{F}^{*\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T,N}\big(\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(\omega)\big)$, which result from the used versions of Algorithm **TEST.INDEP.2**.

Table 5.5.: Estimated p-values that belong to the investigated log returns of GameStop and Wirecard

TEST.INDEP.2		
b \ β	$0.3\,Tb^2$	$0.7\,Tb^2$
0.1	0.328	0.196
0.15	0.306	0.202

The applied versions of Algorithm **TEST.INDEP.2** cannot reject the null hypothesis $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ (recall (4.1)) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0,\dots,4\}$ for commonly used significance levels α (i. e., $\alpha \in \{0.1, 0.05, 0.01\}$).

In the following, the versions of Algorithm **TEST.INDEP.2** which originate from Setting 4.21 $\big($apart from the weight function that is chosen as $\widetilde{\mathbf{w}}_{\mathfrak{D}_1,\mathfrak{D}_2}\big)$ are applied to $\big((x_{t,T}^{[\mathrm{DB}]}, x_{t,T}^{[\mathrm{LH}]})'\big)_{t=1}^{T}$, i. e., to the log returns of Deutsche Bank and Deutsche Lufthansa that belong to the time period from January 2, 2018, to December 13, 2021. Thereby, $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ is selected to investigate whether trading days exist for which the random variables that underlie these log returns depend on each other (in the sense described in Remark 4.2 (iii)).
Table 5.6 contains the estimated p-values $1 - \widehat{F}_{\mathfrak{D}_1,\mathfrak{D}_2,T,N}^{*\text{indep}}\big(\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(\omega)\big)$, which result from the used versions of Algorithm **TEST.INDEP.2**.

Table 5.6.: Estimated p-values that belong to the investigated log returns of Deutsche Bank and Deutsche Lufthansa

TEST.INDEP.2		
b \ β	$0.3\,Tb^2$	$0.7\,Tb^2$
0.1	0	0
0.15	0	0

Each of the applied versions of Algorithm **TEST.INDEP.2** rejects the null hypothesis $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ highly significantly. This suggests that days between January 2, 2018, and December 13, 2021, exist for which the random variables that generate these log returns of Deutsche Lufthansa and Deutsche Bank depend on each other approximately.

6. Conclusions and Outlook

In the present thesis, local empirical characteristic functions are used to evolve L^2-distance-based instruments that allow to investigate distribution changes and to test for independence in a quite general locally stationary framework, which is characterized by weak moment conditions.
In detail, in Chapter 3, based on the local characteristic function φ, two measures $\mathbb{D}$ and $\mathbb{D}^{\mathrm{norm}}$ are defined which quantify the intensity of distribution changes of the stationary approximations within choosable rescaled time periods, whereby $\mathbb{D}^{\mathrm{norm}}$ can be regarded as a normalized version of $\mathbb{D}$. Subsequently, since the local characteristic function is commonly unknown in practise, estimators $\widehat{\mathbb{D}}_T$ and $\widehat{\mathbb{D}}_T^{\mathrm{norm}}$, respectively, for both measures are introduced to quantify distribution changes in applications. These estimators are constructed by using the local empirical characteristic function in combination with a Riemann sum, which allows to handle that the local empirical characteristic function is asymptotically biased for rescaled points in time that are (in dependence of the chosen bandwidth) too close to zero or one. Next, $\widehat{\mathbb{D}}_T$ as well as $\widehat{\mathbb{D}}_T^{\mathrm{norm}}$ together with appropriate dependent wild bootstrap procedures are used to estimate confidence intervals for $\mathbb{D}$ and $\mathbb{D}^{\mathrm{norm}}$, respectively. In addition, two tests that aim to detect distribution changes under local stationarity are derived from $\widehat{\mathbb{D}}_T$. Thereby, applying the secondly constructed test for this purpose is justified under the weaker condition $\delta \in (0,1]$ (compared to $\delta \in (1/4,1]$ for the first test), whereby δ determines moment and smoothness conditions according to Assumption 2.2 $[\mathbf{StAp}]$. Handling this weaker condition is achieved by using local Newey-West estimators, which modify the test statistic that underlies the firstly proposed test accordingly. Furthermore, based on these tests for detecting distribution changes, the first change point in the distributions of the stationary approximations is estimated. At the end of Chapter 3, the proposed tools for investigating distribution changes are applied in several simulation studies, whereby all of these methods provide very reasonable results for appropriate selections of the underlying tuning parameters.
The investigations presented in Chapter 4 aim to test for independence under local stationarity. For a better understanding of these investigations, a method for testing for pairwise independence in the same points in time is evolved at first, whereby the belonging L^2-distance-based test statistic is constructed by local empirical characteristic functions and belonging p-values are estimated by a suitable dependent wild bootstrap procedure. Next, this method is modified in such a manner that it is capable of rejecting the null hypothesis of total blockwise independence. Further, the simulation studies given at the end of Chapter 4 show very reliable rejection frequencies of the proposed tests for suitable choices of the underlying tuning parameters.
In Chapter 5, the tools introduced in the Chapters 3 and 4 are applied to some log returns of GameStop, Wirecard, Deutsche Bank as well as Deutsche Lufthansa. Thereby, the obtained results indicate that it is inappropriate to model these log returns in the considered time periods by stationary processes. In addition, the applied tests for independence detect with high significance that days exist between January 2, 2018 and December 13, 2021, for which the random variables that generate the log returns associated with Deutsche Bank and Deutsche Lufthansa depend on each other. In contrast, it cannot be rejected based on statistically relevant significance levels that the random variables underlying the analyzed daily log returns of GameStop and Wirecard are totally independent of each other between July 3, 2017 and June 22, 2021 within five days ongoing periods (i. e., within periods with the duration of a common trading week).

As a recommendation for future research, developing methods for selecting the tuning parameters b, β and $\mathbf{B}_T$ (that belong to the instruments evolved in the present thesis) optimally for a given sample path of $(X_{t,T})_{t=1}^T$ would be useful. However, this is very challenging due to the following reasons. The Assumptions 2.8 $[\mathbf{K\&b.1}]$ (ii) and 4.5 $[\mathbf{K\&b.2}]$ (ii) depend on δ, which originates from Assumption 2.2 $[\mathbf{StAp}]$ and does not seem to be appropriately estimable in practise. Moreover, the asymptotic conditions on β as well as $\mathbf{B}_T$ are influenced by b. In addition, the approach for selecting b that is introduced in [15, Dahlhaus and Richter (2023)] is not appropriate under Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii)

or 4.5 $[\mathbf{K\&b.2}]$ (ii) because it proposes to choose $b = c_{\mathrm{opt}} T^{-1/5}$ for a certain positive constant c_{opt}.
Constructing optimal weight functions, kernels and Newey-West-estimation-kernels for the tools presented in this thesis is also a topic for future researches that is of practical interest. Furthermore, using the non-integrable weight function $\mathbf{w}_{d,q}$ (see (3.44)) for investigating distribution changes and testing for independence under local stationarity should also be investigated in further details. However, the considerations given below Remark 3.21 and those in Remark 4.7 (ii) suggest that the latter requires to demand assumptions which are stronger than the suppositions that underlie the present work.
In addition, motivated by many applications, it is of importance to extent the present theory of local Newey-West estimation to many other contexts. (E. g., Remark 4.15 (ii) mentions that the proposed test for independence can be generalized from the case $\delta \in (1/4, 1]$ to the case $\delta \in (0, 1]$ by using local Newey-West estimators which are similar to those constructed in Subsection 3.2.3.)
Further, combining the approaches given in Chapter 3 and Chapter 4 suggests that replacing the Riemann sums contained in $\widehat{\mathbb{D}}_T$ by belonging integrals with respect to $u \in \big[\max\big\{(\mathfrak{U}_1 - \mathfrak{U}_0)b, \mathfrak{U}_0\big\}, \min\big\{1 - \big(\mathfrak{U}_1 - \mathfrak{U}_0\big)b, \mathfrak{U}_1\big\}\big]$ also leads to an appropriate test statistic for (3.49), which owns the rate of convergence $T\sqrt{b}$ (like the EMDCI based one derived in Section 3.2). However, as explained at the beginning of Subsection 3.1.2, it is expectable that this approach is less suitable for detecting distribution changes that belong to (rescaled) time periods which are close or even include the points in time 0 and 1. Thereby, it should be noted that the choice $\mathfrak{U}_{0,1} = [0, w]$ with $w \in (0, 1]$ plays a considerable role for the in Section 3.3 evolved estimation of the first change point in the distributions of the stationary approximations.
Further topics of future researches may be to define measures for quantifying deviations in the mean or variance which can be constructed similarly to those introduced in Chapter 3. Note also that, according to Remark 4.7 (i), it is expectable that an empirical measure for quantifying how strong locally stationary processes depend on each other can also be obtained analogously.
Under Assumption 4.1 $[\mathbf{INDEP}]$, one could contemplate to test the null hypothesis that the random variable $\tilde{X}^{[1]}_{t_1}(u)$ is independent of $\tilde{X}^{[2]}_{t_2}(w)$ for all $u, w \in [0, 1]$, $t_1, t_2 \in \mathbb{N}_0$ against the alternative that $\tilde{X}^{[1]}_{t_1}(u)$ and $\tilde{X}^{[2]}_{t_2}(w)$ depend on each other for some $u, w \in [0, 1]$, $t_1, t_2 \in \mathbb{N}_0$. It follows similarly to Remark 4.2 (iii) that this test problem allows to investigate whether $\big(X^{[1]}_{t,T}\big)_{t=1}^{T}$ and $\big(X^{[2]}_{t,T}\big)_{t=1}^{T}$ are asymptotically pairwise independent in the sense that:

$$\sup_{t_1,t_2=1,\dots,T} \left| \mathbb{E}\left[e^{i\left\langle s^{[1]}, X^{[1]}_{t_1,T} \right\rangle} e^{i\left\langle s^{[2]}, X^{[2]}_{t_2,T} \right\rangle} \right] - \mathbb{E}\left[e^{i\left\langle s^{[1]}, X^{[1]}_{t_1,T} \right\rangle} \right] \mathbb{E}\left[e^{i\left\langle s^{[2]}, X^{[2]}_{t_2,T} \right\rangle} \right] \right| \overset{T\to\infty}{\longrightarrow} 0 \quad \forall\, s^{[k]} \in \mathbb{R}^{d_k}, \quad k \in \{1, 2\}.$$

To investigate this test problem in a similar manner to that in Chapter 4, it is necessary to construct a consistent estimator for the following expression:

$$\int_0^1 \int_0^1 \int_{\mathbb{R}^{d_1+d_2}} \left| \mathbb{E}\left[e^{i\left\langle s^{[1]}, \tilde{X}^{[1]}_{t_1}(u) \right\rangle} e^{i\left\langle s^{[2]}, \tilde{X}^{[2]}_{t_2}(w) \right\rangle} \right] - \mathbb{E}\left[e^{i\left\langle s^{[1]}, \tilde{X}^{[1]}_{t_1}(u) \right\rangle} \right] \mathbb{E}\left[e^{i\left\langle s^{[2]}, \tilde{X}^{[2]}_{t_2}(w) \right\rangle} \right] \right|^2 \mathbf{w}(s)\, ds\, du\, dw$$

with $\quad s := \left(s^{[1]\prime}, s^{[2]\prime} \right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2},\ t_1, t_2 \in \mathbb{N}_0,$

which is very challenging (maybe even impossible) due to the contained expressions $\mathbb{E}\Big[e^{i\left\langle s^{[1]}, \tilde{X}^{[1]}_{t_1}(u) \right\rangle} \cdot e^{i\left\langle s^{[2]}, \tilde{X}^{[2]}_{t_2}(w) \right\rangle} \Big]$. Furthermore, analog issues result from trying to evolve such a test statistic based on other tools for detecting dependences (like the distribution function or density). In addition, it is expectable that arguments which are similar to those used in Chapter 4 allow to investigate the test problem (4.1) not just for fixed sets $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$ but also in the case that the numbers of elements of these sets grow to infinity for $T \to \infty$ sufficiently slowly.

Appendices

A. Auxiliary definitions

The next definitions are often used in the proofs that are given in the subsequent appendices.

Definition A.1. *(i) Let the sequence of random variables* $(\varepsilon_t)_{t\in\mathbb{Z}}$ *originate from Definition 2.1 and Assumption 2.2* $[\mathbf{StAp}]$ *be fulfilled. Define for all* $\mathrm{n} \in \mathbb{N}_0$ *and all measurable functions* $g\colon \mathbb{R}^d \to \mathbb{C}$ *with* $\|g(X_{t,T})\|_1 < \infty\ \forall\, t \in \{1,\ldots,T\}$, $T \in \mathbb{N}$ *as well as* $\|g(\widetilde{X}_r(u))\|_1 < \infty\ \forall\, r \in \mathbb{Z}$, $u \in [0,1]$*:*

$$\begin{aligned}
&\mathcal{F}_{r_1,r_2} := (\varepsilon_{r_1},\ldots,\varepsilon_{r_2}) \quad \forall\, r_1, r_2 \in \mathbb{Z} \text{ with } r_1 \geqslant r_2,\\
&X_{t,T,\{\mathrm{n}\}} := \mathbb{E}\left[X_{t,T}\middle|\mathcal{F}_{t,t-\mathrm{n}}\right] \ \forall\, t \in \{1,\ldots,T\},\ T \in \mathbb{N},\\
&\widetilde{X}_{r,\{\mathrm{n}\}}(u) := \mathbb{E}\left[\widetilde{X}_r(u)\middle|\mathcal{F}_{r,r-\mathrm{n}}\right] \ \forall\, r \in \mathbb{Z}, u \in [0,1],\\
&g(X_{t,T})_{\mathrm{n}} := \mathbb{E}\left[g(X_{t,T})\middle|\mathcal{F}_{t,t-\mathrm{n}}\right] \ \forall\, t \in \{1,\ldots,T\},\ T \in \mathbb{N} \quad \text{and}\\
&g\left(\widetilde{X}_r(u)\right)_{\mathrm{n}} := \mathbb{E}\left[g\left(\widetilde{X}_r(u)\right)\middle|\mathcal{F}_{r,r-\mathrm{n}}\right] \ \forall\, r \in \mathbb{Z}, u \in [0,1].
\end{aligned}$$

Further, if Assumption 3.15 $[\mathbf{W}^*]$ *is valid, introduce the following notations:*

$$\mathcal{F}^*_{r_1,r_2} := \left(\varepsilon^*_{r_1},\ldots,\varepsilon^*_{r_2}\right) \ \forall\, r_1, r_2 \in \mathbb{Z} \text{ with } r_1 \geqslant r_2 \text{ and } W^*_{r,\{\mathrm{n}\}} := \mathbb{E}\left[W^*_r\middle|\mathcal{F}^*_{r,r-\mathrm{n}}\right] \ \forall\, r \in \mathbb{Z},\ \mathrm{n} \in \mathbb{N}_0.$$

(ii) Suppose that the Assumptions 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ *(ii) and 2.4* $[\mathbf{DM.1}]$ *hold. Define for each* $T \in \mathbb{N}$ *parameters* $\widetilde{m} := \widetilde{m}_T \in \mathbb{N}$ *as well as* $m := m_T \in \mathbb{N}$ *which fulfil:*

$$\frac{\widetilde{m}}{\ln(Tb^2)} \overset{T\to\infty}{\longrightarrow} \infty, \quad \frac{\widetilde{m}}{\sqrt{T}b} \overset{T\to\infty}{\longrightarrow} 0 \quad \text{and} \quad m := \max\left\{\left\lceil \sqrt{T}b\left(\sum_{l=\widetilde{m}}^{\infty} \Delta_l l^2\right)^{1/4}\right\rceil, \widetilde{m}\right\}.$$

(iii) Let the Assumptions 3.15 $[\mathbf{W}^*]$ *(which includes Assumption 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *(ii)) and 2.4* $[\mathbf{DM.1}]$ *be valid. Define for each* $T \in \mathbb{N}$ *the parameters* $m_* := m_{*,T} > 0$ *and* $m_\beta := m_{\beta,T} \in \mathbb{N}$ *in the following manner (recall that* $\beta > 0$, $\rho_* \in (0,1)$ *as well as that* e *denotes Euler's number - as always in the present thesis):*

$$\beta^{\mathrm{inv}}_{\sup} := 1 + \sup_{T\in\mathbb{N}} \frac{e}{\beta}, \quad m_* := \max\left\{-\frac{\ln(Tb^2)}{\ln(\rho_*)}, m\beta^{\mathrm{inv}}_{\sup}\right\} \quad \text{and} \quad m_\beta := \left\lceil \beta\left(m_* - \frac{\ln(\beta^{\mathrm{inv}}_{\sup}\beta)}{\ln(\rho_*)}\right)\right\rceil.$$

(iv) Suppose that the Assumptions 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ *(ii) and 2.4* $[\mathbf{DM.2}]$ *hold. Define for an arbitrary but fixed constant* $\mathcal{N} \in \mathbb{N}$ *and each* $T \in \mathbb{N}$ *the parameters* $\widetilde{n} := \widetilde{n}_T \in \mathbb{N}$ *as well as* $n := n_T \in \mathbb{N}$ *as follows:*

$$\widetilde{n} := \mathcal{N}\left\lceil \frac{1}{\sqrt{b}}\right\rceil \quad \text{and} \quad n := \max\left\{\left\lceil \sqrt{T}b\left(\sum_{l=\widetilde{n}}^{\infty} \Delta_l l^{2/\delta}\right)^{\delta/4}\right\rceil, \widetilde{n}\right\}.$$

(v) Let the Assumptions 3.15 $[\mathbf{W}^*]$ *and 2.4* $[\mathbf{DM.2}]$ *be fulfilled. Define for each* $T \in \mathbb{N}$ *the parameter* $n_\beta := n_{\beta,T} \in \mathbb{N}$ *in the following manner (note that* $\beta^{\mathrm{inv}}_{\sup}$ *is introduced in Definition A.1 (iii)):*

$$n_\beta := \left\lceil \frac{\beta}{\ln(\rho_*)}\left(-\ln(e+Tb) - \ln(\beta^{\mathrm{inv}}_{\sup}\beta)\right)\right\rceil + n.$$

(vi) Suppose that $\rho \in (0,1)$ *originates from Assumption 2.4* $[\mathbf{DM.3}]$*. Define for arbitrary but fixed constants* $C_1 > 1$, $C_2 \geqslant \max\{1, -1/\ln(\rho)\}$ *as well as* $C_3 \geqslant \max\{e, C_1\}$ *and each* $T \in \mathbb{N}$ *a*

parameter $\mathscr{a} := \mathscr{a}_T \in \mathbb{N}$ *that fulfils:*

$$-\frac{\ln(C_1 T)}{\ln(\rho)} \leqslant \mathscr{a} \leqslant C_2 \ln(C_3 T)\,.$$

(vii) Let C_2 *as well as* C_3 *originate from Definition A.1 (vi) and* $\beta^{\text{inv}}_{\text{sup}}$ *from Definition A.1 (iii). In addition, suppose that the Assumptions 3.15* $[\mathbf{W}^*]$ *and 4.5* $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ *(ii) are valid. Define for each* $T \in \mathbb{N}$ *the parameters* $\mathscr{a}_* := \mathscr{a}_{*,T} > 0$ *as well as* $\mathscr{a}_\beta := \mathscr{a}_{\beta,T} \in \mathbb{N}$ *as follows:*

$$\mathscr{a}_* := \max\left\{-\frac{\ln(C_3 T)}{\ln(\rho_*)}, 8C_2\beta^{\text{inv}}_{\text{sup}} \ln(C_3 T)\right\} \quad \textit{and} \quad \mathscr{a}_\beta := \left\lceil \beta\left(\mathscr{a}_* - \frac{\ln(\beta^{\text{inv}}_{\text{sup}}\beta)}{\ln(\rho_*)}\right)\right\rceil.$$

Remark A.2. *(i) Assumption 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *(ii) provides* $\sqrt{T}b \overset{T\to\infty}{\longrightarrow} \infty$ *and* $\big($*together with* $\widetilde{\mathscr{m}}/\ln(Tb^2) \overset{T\to\infty}{\longrightarrow} \infty\big)$ *that* $\widetilde{\mathscr{m}} \overset{T\to\infty}{\longrightarrow} \infty$*. Hence, Assumption 2.4* $[\mathbf{DM}.\mathbf{1}]$ *implies* $\mathscr{m}/\big(\sqrt{T}b\big) \overset{T\to\infty}{\longrightarrow} 0$.

(ii) Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *(ii) ensures* $\widetilde{\mathscr{n}} \overset{T\to\infty}{\longrightarrow} \infty$ *and* $\widetilde{\mathscr{n}}/\sqrt{Tb} \overset{T\to\infty}{\longrightarrow} 0$*. Thus, Assumption 2.4* $[\mathbf{DM}.\mathbf{2}]$ *shows* $\mathscr{n}/\sqrt{Tb} \overset{T\to\infty}{\longrightarrow} 0$.

(iii) Assumption 4.5 $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ *(ii) yields* $T \ll b^{-(2+2\delta)}$*, such that* $\ln(T) \leqslant C\ln\big(b^{-1}\big) \ll Cb^{-\epsilon}$ *for all fixed* $\epsilon > 0$ *(note that, according to Section 1.5,* $C \in (0,\infty)$ *denotes an absolute constant which may have different values at different places). Moreover, Assumption 4.5* $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ *(ii) also provides* $\ln(T) \ll \big(Tb^2\big)^{1/3}$*. Thus, one obtains for* $\delta \in (0,1]$ *which originates from Assumption 2.2* $[\mathbf{StAp}]$ *(that is contained in Assumption 4.1* $[\mathbf{INDEP}]$*):*

$$\mathscr{a} \ll \min\left\{b^{-1/12}, \big(Tb^2\big)^{1/3}, b^{-\delta/2}\right\}. \tag{A.1}$$

(iv) Assumption 3.15 $[\mathbf{W}^*]$ *(i) implies* $\beta^{\text{inv}}_{\text{sup}} < \infty$.

B. Appendix to Chapter 2

B.1. Proofs of the statements given in Chapter 2

Verification of Example 2.5. (i) Straightforward calculations show that Example 2.5 (i) will satisfy Assumption 2.4 $[\mathbf{DM}.\mathbf{1}]$ if (2.5) is fulfilled, Assumption 2.4 $[\mathbf{DM}.\mathbf{2}]$ if (2.6) is valid and Assumption 2.4 $[\mathbf{DM}.\mathbf{3}]$ if (2.5) with $\Delta_l \leqslant B\rho^l$ for all $l \in \mathbb{N}_0$, a $B < \infty$ and a $\rho \in (0,1)$ holds.

(ii) At first, it is proved that Example 2.5 (ii) fulfils Assumption 2.2 $[\mathbf{StAp}]$. Therefor, one defines:

$$\widetilde{X}_t(u) := \sum_{k=0}^{\infty}\left(\prod_{r=0}^{k-1} a_1(\varepsilon_{t-r}, u)\right) a_0(\varepsilon_{t-k}, u) \quad \forall\, t \in \mathbb{Z}, u \in [0,1]. \tag{B.1}$$

The stationary process $\big(\widetilde{X}_t(u)\big)_{t\in\mathbb{Z}}$ is for all $u \in [0,1]$ the a. s. well-defined unique causal solution of (1.2) in [70, Subba Rao (2006), p. 1156] with $\mathbf{A}_t(u) = a_1(\varepsilon_t, u)$ and $\mathbf{b}_t(u) = a_0(\varepsilon_t, u)\ \forall\, t \in \mathbb{Z}, u \in [0,1]$, which can be shown by using Theorem 1.1 in [6, Bougerol and Picard (1992), p. 1715] together with (2.7) as well as $\mathbb{E}\left[\ln^+(|Z|)\right] := \mathbb{E}\left[\max\{0, \ln(|Z|)\}\right] \leqslant \mathbb{E}\left[\ln(1+|Z|)\right] \leqslant \ln(1+\mathbb{E}[|Z|]) < \infty$ which holds for all real-valued random variables Z with finite first moment. In addition, it follows from (2.7):

$$\left\|\sup_{u\in[0,1]}\left|\widetilde{X}_0(u)\right|\right\|_{1+\delta} \leqslant C\sum_{k=0}^{\infty}\rho^k \leqslant C. \tag{B.2}$$

Further, one observes that Example 2.5 (ii) fulfils Assumption 2.1[1] in [70, Subba Rao (2006), p. 1156 et seq.] with $\mathbf{A}_t(u) = a_1(\varepsilon_t, u)$, $\mathbf{b}_t(u) = a_0(\varepsilon_t, u) \; \forall \, t \in \mathbb{Z}, u \in [0, 1]$, $M = 1$, $\epsilon = 1 + \delta$ and $\beta = 1$ (whereby Part (ii) of this assumption holds due to the mean value theorem). Hence, Proposition 2.1 in [70, Subba Rao (2006), p. 1158] with $n = 1+\delta$ and (2.7) prove (2.3) in [70, Subba Rao (2006), p. 1157] with $\epsilon = 1 + \delta$. Thus, (2.11) and (2.4) in [70, Subba Rao (2006), p. 1157 et seq.] as well as (B.2) yield that Assumption 2.2 $[\mathbf{StAp}]$ (i) holds.

Moreover, Example 2.5 (ii) fulfils Assumption 3.1 in [70, Subba Rao (2006), p. 1159 et seq.] with $\beta' = \delta$. Therefore, Proposition 3.1 in [70, Subba Rao (2006), p. 1163] with $n = 1 + \delta$ shows the validity of the assumptions of Corollary 3.1 in [70, Subba Rao (2006), p. 1161] with $\epsilon = 1 + \delta$, such that one obtains from (3.11)[2] in [70, Subba Rao (2006), p. 1162] as well as (2.7):

$$\left\| \sup_{u \in (0,1)} \left| \partial_u \widetilde{X}_0(u) \right| \right\|_{1+\delta} \leqslant \sum_{k=0}^{\infty} \sum_{r=0}^{k-1} C \left\| \widetilde{a}_{1,0} \right\|_{1+\delta}^k \left\| \widetilde{a}_{0,0} \right\|_{1+\delta} + \sum_{k=0}^{\infty} \left\| \widetilde{a}_{1,0} \right\|_{1+\delta}^k C \left\| \widetilde{a}_{0,0} \right\|_{1+\delta} \leqslant C \sum_{k=0}^{\infty} \left(k\rho^k + \rho^k \right) \leqslant C. \tag{B.3}$$

The validity of Assumption 2.2 $[\mathbf{StAp}]$ (ii) follows from (B.3), (3.12)[2] in [70, Subba Rao (2006), p. 1162] with $\beta' = \delta$ and Proposition 3.1 in [70, Subba Rao (2006), p. 1163] with $n = 1 + \delta$ (the a. s. existence of the right-hand derivative at 0 as well as of the left-hand derivative at 1 of $u \mapsto \widetilde{X}_t(u)$ follows similarly to the a. s. existence of the derivative for $u \in (0, 1)$). In addition, (2.7) yields that Assumption 2.2 $[\mathbf{StAp}]$ (iii) is fulfilled (recall (B.1)).

In the following, it is proved that Example 2.5 (ii) also satisfies the other conditions of Assumption 2.4 $[\mathbf{DM.3}]$ and the last property stated in (2.4), which verifies that Assumption 2.4 $[\mathbf{DM.2}]$ is fulfilled, too. To meet these targets, one defines $(\chi_{k,t-l})_{k \in \mathbb{Z}}$ as $\chi_{k,t-l} := \varepsilon_k \mathbb{1}_{\{k \neq t-l\}} + \varepsilon_{t-l}^{\times} \mathbb{1}_{\{k=t-l\}} \; \forall \, k, t \in \mathbb{Z}, \; l \in \mathbb{N}_0$, whereby $\varepsilon_{t-l}^{\times}$ is introduced in Assumption 2.4 $[\mathbf{DM}]$. It follows for all $l \in \mathbb{N}_0$ by using (2.1)[1] in [70, Subba Rao (2006), p. 1157], (2.7) as well as $\sum_{k=l}^{\infty} \rho^k = \sum_{k=0}^{\infty} \rho^{l+k} \leqslant C\rho^l$ (see (2.2)):

$$\begin{aligned} \sup_{T \in \mathbb{N}} \sup_{t=1,\ldots,T} \left\| X_{t,T} - X_{t,T}^{\times(t-l)} \right\|_{1+\delta} &= \sup_{T \in \mathbb{N}} \sup_{t=1,\ldots,T} \Bigg\| \sum_{k=l}^{\infty} \Bigg[\left(\prod_{r=0}^{k-1} a_1 \left(\varepsilon_{t-r}, \frac{t-r}{T} \right) \right) a_0 \left(\varepsilon_{t-k}, \frac{t-k}{T} \right) \\ &\quad - \left(\prod_{r=0}^{k-1} a_1 \left(\chi_{t-r,t-l}, \frac{t-r}{T} \right) \right) a_0 \left(\chi_{t-k,t-l}, \frac{t-k}{T} \right) \Bigg] \Bigg\|_{1+\delta} \\ &\leqslant C\rho^l, \end{aligned} \tag{B.4}$$

such that (2.2) with $\Delta_l := C\rho^l$ for a $\rho \in (0, 1)$ and all $l \in \mathbb{N}_0$ is fulfilled. The validity of (2.3) with $\Delta_l := C\rho^l \; \forall \, l \in \mathbb{N}_0$ can be proved similarly (recall (B.1)). In summary, it is shown that Example 2.5 (ii) satisfies Assumption 2.4 $[\mathbf{DM.3}]$.

[1]For convenience, in [70, Subba Rao (2006), p. 1157], $\mathbf{A}_t(u) = 0 \; \forall \, u \leqslant 0$ is setted, which is avoided in the present Example 2.5 (ii) for $u = 0$ by setting $a_0(x, u) = 0 \; \forall \, x \in \mathbb{R}, u < 0$. In particular, this does not change the validity of (2.1) in [70, Subba Rao (2006), p. 1157] because one obtains in both of these cases $X_{t,T} = \sum_{k=0}^{t} \left(\prod_{r=0}^{k-1} \mathbf{A}_{t-r}((t-r)/T) \right) \mathbf{b}_{t-k}((t-k)/T) = \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k-1} \mathbf{A}_{t-r}((t-r)/T) \right) \mathbf{b}_{t-k}((t-k)/T)$.

[2]In Corollary 3.2 in [70, Subba Rao (2006), p. 1162], just Assumption 2.1 in [70, Subba Rao (2006), p. 1156 et seq.] is demanded, which does not suffice to obtain the statement of this corollary because this assumption does not even ensure the a. s. differentiability of $u \mapsto \widetilde{X}_t(u)$. However, it follows from the proof of this corollary (which is stated below this corollary) that it is possible to show its assertions under the assumptions of Corollary 3.1 in [70, Subba Rao (2006), p. 1161]. In addition, the last sum in (3.11) in [70, Subba Rao (2006), p. 1162] should sum over all $k \in \{0, \ldots, \infty\}$ and not just $k \in \{1, \ldots, \infty\}$ to handle that the summand contained in $\widetilde{X}_t(u)$ which belongs to $k = 0$ equals $a_0(\varepsilon_t, u) \; \forall \, t \in \mathbb{Z}$, $u \in [0, 1]$ (recall that, according to Section 1.5, $\sum_{n=x}^{y} z_n := 0$ as well as $\prod_{n=x}^{y} z_n := 1$ for all $x, y \in \mathbb{Z}$ with $x > y$ and all $z_y, \ldots, z_x \in \mathbb{C}$). Further, the right-hand side of the inequality (3.12) in [70, Subba Rao (2006), p. 1162] has to be corrected to $\sup_{u,v} |u - v|^{\beta'} \mathbf{W}_t(2)$.

In order to prove the last property stated in (2.4), note that (3.11) in [70, Subba Rao (2006), p. 1162] and (2.7) yield for all $l \in \mathbb{N}_0$:

$$
\begin{aligned}
&\sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \partial_u \widetilde{X}_t(u) - \partial_u \widetilde{X}_t^{\times(t-l)}(u) \right\|_{1+\delta} \\
&\leqslant \sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \sum_{k=l}^{\infty} \sum_{r=0}^{k-1} \left[\left(\prod_{\substack{i=0\\ i\neq r}}^{k-1} a_1\left(\varepsilon_{t-i}, u\right) \right) \left(\partial_u a_1\left(\varepsilon_{t-r}, u\right)\right) a_0\left(\varepsilon_{t-k}, u\right) \right.\right. \\
&\left.\left. - \left(\prod_{\substack{i=0\\ i\neq r}}^{k-1} a_1\left(\chi_{t-i,t-l}, u\right) \right) \left(\partial_u a_1\left(\chi_{t-r,t-l}, u\right)\right) a_0\left(\chi_{t-k,t-l}, u\right) \right] \right\|_{1+\delta} \\
&+ \sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \sum_{k=l}^{\infty} \left[\left(\prod_{i=0}^{k-1} a_1\left(\varepsilon_{t-i}, u\right) \right) \left(\partial_u a_0\left(\varepsilon_{t-k}, u\right)\right) - \left(\prod_{i=0}^{k-1} a_1\left(\chi_{t-i,t-l}, u\right) \right) \left(\partial_u a_0\left(\chi_{t-k,t-l}, u\right)\right) \right] \right\|_{1+\delta} \\
&\leqslant C \sum_{k=l}^{\infty} \left(k\rho^k + \rho^k \right) \\
&= C\sqrt{\rho}^l \sum_{k=0}^{\infty} \left((k+l)\sqrt{\rho}^l \rho^k + \sqrt{\rho}^l \rho^k \right) \\
&\leqslant C\sqrt{\rho}^l,
\end{aligned}
\tag{B.5}
$$

such that $\Delta_{\partial,l} := C\sqrt{\rho}^l \;\forall\, l \in \mathbb{N}_0$ can be setted, which fulfils $\Delta_{\partial,0} + \sum_{l=1}^{\infty} \Delta_{\partial,l}\, l \leqslant B$ for a $B < \infty$. Overall, it is verified that Example 2.5 (ii) satisfies also Assumption 2.4 [$\mathbf{DM.2}$] (and not just Assumption 2.4 [$\mathbf{DM.3}$]).

(iii) At first, it is proved that Example 2.5 (iii) satisfies Assumption 2.2 [$\mathbf{StAp}$]. Therefor, one defines for all $t \in \mathbb{Z}$, $u \in [0,1]$:

$$
\begin{aligned}
&\widetilde{X}_{1,t}(u) := \sum_{k=1}^{\infty} a_1^k(u) \prod_{r=0}^{k} \varepsilon_{t-r}^2, \quad \widetilde{X}_{2,t}(u) := \sum_{k=1}^{\infty} k \left(\partial_u a_1(u)\right) a_1^{k-1}(u) \prod_{r=0}^{k} \varepsilon_{t-r}^2 \quad \text{and} \\
&\widetilde{X}_t(u) := \operatorname{sign}\left(\varepsilon_t\right) \sqrt{a_0(u)\varepsilon_t^2 + a_0(u)\widetilde{X}_{1,t}(u)}.
\end{aligned}
\tag{B.6}
$$

Arguments from the proof of Theorem 2 in [58, Nelson (1990), p. 331] and (2.8) show for all $t \in \mathbb{Z}, u \in [0,1]$ that $\widetilde{X}_t^2(u) = f_u\left(\varepsilon_t, \varepsilon_{t-1}, \dots\right)$ for a non-negative measurable function f_u (whereby the measurability of $x \mapsto \operatorname{sign}(x)$ holds due to Corollary 1.89 in [48, Klenke (2013), p. 38]). Hence, Theorem 3.35 in [78, White (2001), p. 44] implies that $\left(\widetilde{X}_t(u)\right)_{t\in\mathbb{Z}}$ is stationary for all $u \in [0,1]$.
One obtains from (2.8), $\rho \in (0,1)$, $\|\varepsilon_0\|_{2+2\delta}^{-2} \leqslant 1$ (which is an implication of $\|\varepsilon_0\|_2 = 1$), $\sum_{j=2}^{\infty} \left(j^4\left(\pi^4/90 - 1\right)\right)^{-1} = 1$ and the mean value theorem that Example 2.5 (iii) fulfils Assumption 1 in [17, Dahlhaus and Subba Rao (2006), p. 1077] with:

$$
\begin{aligned}
&Z_t = \varepsilon_t \;\forall\, t \in \mathbb{Z}, \quad a_j(u) = 0 \;\forall\, u \in \mathbb{R}, j \geqslant 2, \quad Q = 1, \quad \nu = 1 - \frac{1}{2}\left(\rho + \|\varepsilon_0\|_{2+2\delta}^{-2}\right), \\
&l(1) = \rho^{-1}, \quad l(j) = j^4 \left(\frac{\pi^4}{90} - 1\right) (1 - \nu - \rho)^{-1} \;\forall\, j \geqslant 2 \quad \text{as well as} \quad M = \sup_{u\in(0,1)} \left|\partial_u a_1(u)\right| / \rho.
\end{aligned}
\tag{B.7}
$$

It follows from (2.8) and $\|\varepsilon_0\|_2 = 1$ (see (B.6)):

$$
\left\| \sup_{u\in[0,1]} \left| \widetilde{X}_0(u) \right| \right\|_2 \leqslant C.
\tag{B.8}
$$

Furthermore, Proposition 1 and (45) in [17, Dahlhaus and Subba Rao (2006), p. 1078 as well as p. 1091] imply due to $a_j(u) = 0 \;\forall\, j \in \{0,1\}$, $u < 0$ and (B.7) that:

$$X_{t,T} = \operatorname{sign}(\varepsilon_t)\sqrt{X_{t,T}^2} \quad \text{with} \quad X_{t,T}^2 = a_0\left(\frac{t}{T}\right)\varepsilon_t^2 + \sum_{k=1}^{\infty} a_0\left(\frac{t-k}{T}\right)\left(\prod_{r=1}^{k} a_1\left(\frac{t-r+1}{T}\right)\right)\prod_{r=0}^{k}\varepsilon_{t-r}^2$$
$$\forall\, t \in \{1,\ldots,T\},\, T \in \mathbb{N} \qquad \text{(B.9)}$$

is the a. s. well-defined unique causal solution of Example 2.5 (iii). In addition, (44) in [17, Dahlhaus and Subba Rao (2006), p. 1091] as well as (B.7) yield that $\left(\widetilde{X}_t^2(u)\right)_{t\in\mathbb{Z}}$, $u \in [0,1]$ (see (B.6)) is the a. s. well-defined unique solution of the square of (6)[3] in [17, Dahlhaus and Subba Rao (2006), p. 1078]. Hence, one obtains from $x - y = \left(x^2 - y^2\right)/\left(x+y\right) \;\forall\, x, y \in \mathbb{R}$ with $x \neq -y$ in combination with $\operatorname{sign}\left(\widetilde{X}_t(u)\right) = \operatorname{sign}\left(X_{t,T}\right)$ (recall (B.6) as well as (B.9)), Theorem 1 in [17, Dahlhaus and Subba Rao (2006), p. 1078], (B.9), (2.8) and $\mathbb{P}(\varepsilon_0 = 0) = 0$ (note that U_t is defined in (48) in [17, Dahlhaus and Subba Rao (2006), p. 1093]):

$$\sup_{t=1,\ldots,T}\left\|X_{t,T} - \widetilde{X}_t\left(\frac{t}{T}\right)\right\|_{1+\delta} \leqslant \frac{C}{T}\sup_{t=1,\ldots,T}\left\|U_t\left|X_{t,T} + \widetilde{X}_t\left(\frac{t}{T}\right)\right|^{-1}\right\|_{1+\delta} \leqslant \frac{C}{T}\sup_{t=1,\ldots,T}\left\|\frac{U_t}{\varepsilon_t}\right\|_{1+\delta}. \qquad \text{(B.10)}$$

It follows from $a_j(u) = 0 \;\forall\, j \geqslant 2$, $u \in \mathbb{R}$ (see (B.7)) for the expressions $\widetilde{m}_t(u,k)$ and $m_{t,N}(k)$ defined in [17, Dahlhaus and Subba Rao (2006), p. 1091] that $\widetilde{m}_t(u,k) = \mathbb{1}_{\{j_k = t-k < \cdots < j_1 = t-1 < j_0 = t\}} \cdot \widetilde{g}_u\left(k, j_0, j_1, \ldots, j_k\right)\prod_{i=0}^{k} Z_{j_i}^2$ as well as $m_{t,N}(k) = \mathbb{1}_{\{j_k = t-k < \cdots < j_1 = t-1 < j_0 = t\}} g_{t,N}\left(k, j_0, j_1, \ldots, j_k\right) \cdot \prod_{i=0}^{k} Z_{j_i}^2$ ($\widetilde{g}_u$ and $g_{t,N}$ are also introduced in [17, Dahlhaus and Subba Rao (2006), p. 1091]). Therefore, arguments which are similar to those used in the proof of Theorem 1[4] given in [17, Dahlhaus and Subba Rao (2006), p. 1093 et seq.] provide (recall that U_t is defined in (48) in [17, Dahlhaus and Subba Rao (2006), p. 1093]):

$$U_t = Z_t^2 + \sum_{k=1}^{\infty} Q^{k-1}\mathbb{1}_{\{j_k = t-k < \cdots < j_1 = t-1 < j_0 = t\}}\frac{k\left|j_0 - j_k\right|}{\prod_{i=1}^{k}\ell\left(j_{i-1} - j_i\right)}\prod_{i=0}^{k} Z_{j_i}^2, \qquad \text{(B.11)}$$

such that (B.7), (2.8) and $\mathbb{P}\left(\varepsilon_0 = 0\right) = 0$ imply:

$$\sup_{t=1,\ldots,T}\left\|\frac{U_t}{\varepsilon_t}\right\|_{1+\delta} \leqslant \left\|\varepsilon_0\right\|_{1+\delta} + \sum_{k=1}^{\infty} k^2\rho^k\left\|\varepsilon_0\right\|_{1+\delta}\left\|\varepsilon_0\right\|_{2+2\delta}^{2k} \leqslant C. \qquad \text{(B.12)}$$

The statements $\sup_{T\in\mathbb{N}}\sup_{t=1,\ldots,T}\mathbb{E}\left[X_{t,T}^2\right] \leqslant C$ (which follows from (46)[4] in [17, Dahlhaus and Subba Rao (2006), p. 1092] by using (2.8) as well as (B.7)), (B.8), (B.10) and (B.12) yield the validity of Assumption 2.2 $[\mathbf{StAp}]$ (i).
In order to verify that Assumption 2.2 $[\mathbf{StAp}]$ (ii) holds, one firstly observes for all $t \in \mathbb{Z}$, $u \in (0,1)$ that (51) in [17, Dahlhaus and Subba Rao (2006), p. 1095] together with (B.7) provides (see (B.6)):

$$\partial_u\widetilde{X}_t^2(u) = \partial_u a_0(u)\varepsilon_t^2 + \left(\partial_u a_0(u)\right)\widetilde{X}_{1,t}(u) + a_0(u)\widetilde{X}_{2,t}(u) \quad \text{a. s.} \qquad \text{(B.13)}$$

In the following, some auxiliary investigations are carried out, which help to prove the validity of Assumption 2.2 $[\mathbf{StAp}]$ (ii). It follows from $a_1(u) \geqslant 0 \;\forall\, u \in [0,1]$ and the fact that a_1 is differentiable on $(0,1)$:

$$\sup_{u\in(0,1):a_1(u)=0}\left|\partial_u a_1(u)\right| = 0. \qquad \text{(B.14)}$$

[3]In contrast to [17, Dahlhaus and Subba Rao (2006)], it is assumed here that (6) in [17, Dahlhaus and Subba Rao (2006), p. 1078] is defined for all $u_0 \in [0,1]$ and not just for $u_0 \in (0,1]$.

[4]The proof of Theorem 1 and (46) in [17, Dahlhaus and Subba Rao (2006), p. 1092 et seq.] contain typos: All expressions $\ell\left(j_i - j_{i-1}\right)$ in this equation and this proof have to be replaced with $\ell\left(j_{i-1} - j_i\right)$ according to the definitions of $g_{t,N}$ and $\widetilde{g}_u$ as well as (48) in [17, Dahlhaus and Subba Rao (2006), p. 1091 as well as 1093]. In particular, the mentioned expressions $\ell\left(j_i - j_{i-1}\right)$ are not defined because $j_i - j_{i-1} < 0$, whereas Assumption 1 in [17, Dahlhaus and Subba Rao (2006), p. 1077] just considers arguments of ℓ which are natural numbers.

Let $\sup_{u\in\varnothing} x(u) := 0\ \forall\, x\colon \mathbb{R}\to[0,\infty)$. Then, one obtains due to (2.8), $a_1(u)\geqslant 0\ \forall\, u\in\mathbb{R}$, $\widetilde{X}_0^2(u)\geqslant m_0\max\left\{\varepsilon_0^2, \widetilde{X}_{1,0}(u)\right\}$ a. s. $\forall\, u\in[0,1]$, $\mathbb{P}\left(\varepsilon_0=0\right)=0$, $\left\|\varepsilon_0\right\|_{2+2\delta}<\infty$, $\rho\left\|\varepsilon_0\right\|_{2+2\delta}\leqslant\rho^{1/2}\left\|\varepsilon_0\right\|_{2+2\delta}<1$ as well as (B.14) (recall (B.6)):

$$\begin{aligned}
\left\|\sup_{u\in[0,1]}\left|\frac{\varepsilon_0^2}{\sqrt{\widetilde{X}_0^2(u)}}\right|\right\|_{2+2\delta} &\leqslant C\left\|\varepsilon_0\right\|_{2+2\delta}\leqslant C,\\
\left\|\sup_{u\in[0,1]}\left|\frac{\widetilde{X}_{1,0}(u)}{\sqrt{\widetilde{X}_0^2(u)}}\right|\right\|_{2+2\delta} &\leqslant C\left\|\sup_{u\in[0,1]}\left|\sqrt{\widetilde{X}_{1,0}(u)}\right|\mathbb{1}_{\{a_1(u)\neq 0\}}+\left|\frac{\widetilde{X}_{1,0}(u)}{\sqrt{\varepsilon_0^2}}\right|\mathbb{1}_{\{a_1(u)=0\}}\right\|_{2+2\delta}\leqslant C \quad\text{and}\\
\left\|\sup_{u\in(0,1)}\left|\frac{\widetilde{X}_{2,0}(u)}{\sqrt{\widetilde{X}_0^2(u)}}\right|\right\|_{2+2\delta} &\leqslant C\Bigg\|\sup_{u\in(0,1):a_1(u)=0}\left|\partial_u a_1(u)\right|\frac{\varepsilon_0^2}{\sqrt{\varepsilon_0^2}}\varepsilon_{-1}^2+\sup_{u\in(0,1):a_1(u)\neq 0}\frac{\left|\partial_u a_1(u)\right|}{\sqrt{a_1(u)}}\prod_{r=0}^{1}\frac{\varepsilon_{-r}^2}{\sqrt{\varepsilon_{-r}^2}}\\
&\quad+\sup_{u\in(0,1)}\sum_{k=2}^{\infty}k\left|\partial_u a_1(u)\right|\rho^{\frac{k}{2}-1}\prod_{r=0}^{k}\frac{\varepsilon_{-r}^2}{\sqrt{\varepsilon_{-r}^2}}\Bigg\|_{2+2\delta}\\
&\leqslant C.
\end{aligned}\tag{B.15}$$

Now, the auxiliary work is finished. Next, the validity of Assumption 2.2 $[\mathbf{StAp}]$ (ii) is shown. At first, (B.13), (2.8), $\mathbb{P}\left(\varepsilon_0=0\right)=0$ and (B.15) yield (see (B.6)):

$$\left\|\sup_{u\in(0,1)}\left|\partial_u\widetilde{X}_0(u)\right|\right\|_{2+2\delta}=\left\|\sup_{u\in(0,1)}\left|\operatorname{sign}\left(\varepsilon_0\right)\frac{\partial_u\widetilde{X}_0^2(u)}{2\sqrt{\widetilde{X}_0^2(u)}}\right|\right\|_{2+2\delta}\leqslant C.\tag{B.16}$$

Moreover, assume that $U,V\subseteq(0,1)$ are arbitrary non-empty sets. In order to prove (2.1), note that it follows similarly to the first equation in (B.16):

$$\begin{aligned}
&\left\|\sup_{u\in U,v\in V}\left|\partial_u\widetilde{X}_0(u)-\partial_v\widetilde{X}_0(v)\right|\right\|_1\\
&\leqslant\left\|\sup_{u\in U,v\in V}\frac{\left|\partial_u\widetilde{X}_0^2\left(u\right)-\partial_v\widetilde{X}_0^2\left(v\right)\right|}{2\sqrt{\widetilde{X}_0^2\left(u\right)}}\right\|_1+\left\|\sup_{u\in U,v\in V}\left|\frac{1}{2\sqrt{\widetilde{X}_0^2\left(u\right)}}-\frac{1}{2\sqrt{\widetilde{X}_0^2\left(v\right)}}\right|\left|\partial_v\widetilde{X}_0^2\left(v\right)\right|\right\|_1\\
&=:\mathrm{I}+\mathrm{II}.
\end{aligned}\tag{B.17}$$

One obtains from (2.8) as well as $\left\|\varepsilon_0^2\right\|_1=1$ that $\sum_{k=1}^{\infty}\sup_{u\in[0,1]}\left|a_1^k(u)\right|\prod_{r=0}^{k}\varepsilon_{-r}^2$ owns a finite first moment and, obviously, this expression is non-negative, such that it is a. s. finite. Hence, (2.8), $\widetilde{X}_0^2(u)\geqslant m_0\varepsilon_0^2$ a. s. $\forall\, u\in[0,1]$, $\mathbb{P}\left(\varepsilon_0=0\right)=0$, $\left\|\varepsilon_0^2\right\|_1=1$ and the mean value theorem $\big($the latter provides together with (2.8) that $\sup_{u\in U,v\in V}\left|a_1^k\left(u\right)-a_1^k\left(v\right)\right|\leqslant k\rho^{k-1}\sup_{w\in(0,1)}\left|\partial_w a_1(w)\right|\left|u-v\right|\big)$ imply (recall (B.6)):

$$\left\|\sup_{u\in U,v\in V}\frac{\left|\widetilde{X}_{1,0}\left(u\right)-\widetilde{X}_{1,0}\left(v\right)\right|}{\sqrt{\widetilde{X}_0^2\left(u\right)}}\right\|_1\leqslant\sum_{k=1}^{\infty}\sup_{u\in U,v\in V}\left|a_1^k\left(u\right)-a_1^k\left(v\right)\right|\frac{\left\|\varepsilon_0\right\|_1}{\sqrt{m_0}}\prod_{r=1}^{k}\left\|\varepsilon_{-r}^2\right\|_1\leqslant C\sup_{u\in U,v\in V}\left|u-v\right|.\tag{B.18}$$

Arguments which are similar to those that show (B.18) prove (see (B.6)):

$$\left\|\sup_{u\in U,v\in V}\frac{\left|\widetilde{X}_{2,0}\left(u\right)-\widetilde{X}_{2,0}\left(v\right)\right|}{\sqrt{\widetilde{X}_0^2\left(u\right)}}\right\|_1\leqslant C\sup_{u\in U,v\in V}\left|u-v\right|\tag{B.19}$$

and:

$$\left\| \sup_{u\in U, v\in V} \frac{\left|\widetilde{X}_0^2(u) - \widetilde{X}_0^2(v)\right|}{m_0 \varepsilon_0^2} \right\|_1 \leqslant C \sup_{u\in U, v\in V} |u-v|\,. \tag{B.20}$$

It follows from (B.13), (2.8), the mean value theorem, (B.15), (B.18), (B.19), $U, V \subseteq (0,1)$ and $\delta \in (0, 1/2]$ (recall (B.17)):

$$\mathrm{I} \leqslant C \sup_{u\in U, v\in V} |u-v| \leqslant C \sup_{u\in U, v\in V} |u-v|^{\delta}\,. \tag{B.21}$$

The function $f\colon [0,\infty) \to \mathbb{R}, z \mapsto \sqrt{z}$ is Hölder continuous with exponent $1/2$, such that one obtains from $\widetilde{X}_0^2(u) \geqslant m_0\varepsilon_0^2$ a. s. $\forall\, u \in [0,1]$, (B.20), arguments which are similar to those that show (B.16), $U, V \subseteq (0,1)$ and $\delta \in (0,1/2]$ (see (B.17)):

$$\begin{aligned}
\mathrm{II} &= \left\| \sup_{u\in U, v\in V} \left| \frac{\sqrt{\widetilde{X}_0^2(v)} - \sqrt{\widetilde{X}_0^2(u)}}{\sqrt{\widetilde{X}_0^2(u)}} \right| \left| \frac{\partial_v \widetilde{X}_0^2(v)}{2\sqrt{\widetilde{X}_0^2(v)}} \right| \right\|_1 \\
&\leqslant C \left\| \sup_{u\in U, v\in V} \frac{\left|\widetilde{X}_0^2(v) - \widetilde{X}_0^2(u)\right|}{\widetilde{X}_0^2(u)} \right\|_1^{\frac{1}{2}} \left\| \sup_{v\in V} \left(\frac{\partial_v \widetilde{X}_0^2(v)}{2\sqrt{\widetilde{X}_0^2(v)}} \right)^2 \right\|_1^{\frac{1}{2}} \\
&\leqslant C \sup_{u\in U, v\in V} |u-v|^{\delta}\,.
\end{aligned} \tag{B.22}$$

The validity of Assumption 2.2 $[\mathbf{StAp}]$ (ii) holds due to (B.16), (B.17), (B.21), (B.22) and the fact that the a. s. existence of the right-hand derivative at 0 as well as of the left-hand derivative at 1 of $u \mapsto \widetilde{X}_t(u)$ follows similarly to the a. s. existence of the derivative for $u \in (0,1)$. In addition, the same arguments from the proof of Theorem 2 in [58, Nelson (1990), p. 331] which have already been used above show that Assumption 2.2 $[\mathbf{StAp}]$ (iii) holds (recall (B.6)).
In the following, it is verified that Example 2.5 (iii) satisfies also the other conditions of Assumption 2.4 $[\mathbf{DM.3}]$ and the last property stated in (2.4), which provides that Assumption 2.4 $[\mathbf{DM.2}]$ is fulfilled, too. Therefor, one firstly observes that (B.9), $\|\varepsilon_0\|_{2+2\delta} < \infty$ and (2.8) $\big($note thereby $\rho\left\|\varepsilon_0^2\right\|_{1+\delta} = \rho\left\|\varepsilon_0\right\|_{2+2\delta}^2 \in (0,1)\big)$ imply $\sup_{T\in\mathbb{N}}\sup_{t=1,\ldots,T}\|X_{t,T}\|_{2+2\delta} = \sup_{T\in\mathbb{N}}\sup_{t=1,\ldots,T}\left\|X_{t,T}^2\right\|_{1+\delta}^{1/2} \leqslant C$. Thus, (B.9), the fact that $f\colon [0,\infty) \to \mathbb{R}, z \mapsto \sqrt{z}$ is Hölder continuous with exponent $1/2$, iterative applying of the inequality $\sqrt{x+y} \leqslant \sqrt{x} + \sqrt{y}\ \forall\, x, y \geqslant 0$, $\|\varepsilon_0\|_{2+2\delta} < \infty$, (2.8) and shifting the index of a sum yield for all $l \in \mathbb{N}_0$ (see (2.2)):

$$\begin{aligned}
&\sup_{T\in\mathbb{N}}\sup_{t=1,\ldots,T} \left\| X_{t,T} - X_{t,T}^{\times(t-l)} \right\|_{2+2\delta} \\
&\leqslant \sup_{T\in\mathbb{N}}\sup_{t=1,\ldots,T} \mathbf{1}_{\{l=0\}} \left(\left\| \left(\operatorname{sign}(\varepsilon_t) - \operatorname{sign}(\varepsilon_t^{\times})\right)\sqrt{X_{t,T}^2} \right\|_{2+2\delta} + \left\| \sqrt{a_0\left(\frac{t}{T}\right)\left|\varepsilon_t^2 - \left(\varepsilon_t^{\times}\right)^2\right|} \right\|_{2+2\delta} \right) \\
&+ \sup_{T\in\mathbb{N}}\sup_{t=1,\ldots,T} \left\| \sum_{k=\max\{1,l\}}^{\infty} \sqrt{a_0\left(\frac{t-k}{T}\right)} \sqrt{\prod_{r=1}^{k} a_1\left(\frac{t-r+1}{T}\right)} \sqrt{\left|\varepsilon_{t-l}^2 - \left(\varepsilon_{t-l}^{\times}\right)^2\right|} \sqrt{\prod_{\substack{r=0\\ r\neq l}}^{k} \varepsilon_{t-r}^2} \right\|_{2+2\delta} \\
&\leqslant C\mathbf{1}_{\{l=0\}} \left(1 + \sum_{k=1}^{\infty} \rho^{k/2} \|\varepsilon_0\|_{2+2\delta}^k \right) + C\mathbf{1}_{\{l\geqslant 1\}} \sum_{k=l}^{\infty} \rho^{k/2} \|\varepsilon_0\|_{2+2\delta}^k \\
&\leqslant C \left(\sqrt{\rho}\,\|\varepsilon_0\|_{2+2\delta}\right)^l.
\end{aligned} \tag{B.23}$$

This proves that (2.2) with $\Delta_l := C\big(\sqrt{\rho}\,\|\varepsilon_0\|_{2+2\delta}\big)^l\ \forall\, l \in \mathbb{N}_0$ is fulfilled, whereby (2.8) ensures $\sqrt{\rho}\,\|\varepsilon_0\|_{2+2\delta} \in (0,1)$. The validity of (2.3) with $\Delta_l := C\big(\sqrt{\rho}\,\|\varepsilon_0\|_{2+2\delta}\big)^l\ \forall\, l \in \mathbb{N}_0$ follows similarly (recall (B.6)). In summary, it is shown that Example 2.5 (iii) satisfies Assumption 2.4 $[\mathbf{DM.3}]$.
In order to verify that the last property stated in (2.4) is fulfilled, one defines $(\chi_{k,t-l})_{k\in\mathbb{Z}}$ by $\chi_{k,t-l} :=$

$\varepsilon_k \mathbb{1}_{\{k \neq t-l\}} + \varepsilon^{\times}_{t-l} \mathbb{1}_{\{k=t-l\}}$ $\forall\, k, t \in \mathbb{Z}$, $l \in \mathbb{N}_0$, whereby $\varepsilon^{\times}_{t-l}$ originates from Assumption 2.4 $[\mathbf{DM}]$.
It follows for all $l \in \mathbb{N}_0$ from $\mathbb{P}(\varepsilon_0 = 0) = 0$ and $\partial_u \widetilde{X}_t(u) = \operatorname{sign}(\varepsilon_t)\, \partial_u \widetilde{X}^2_t(u) / \big(2\sqrt{\widetilde{X}^2_t(u)}\big)$ a. s. $\forall\, t \in \mathbb{Z}, u \in (0,1)$ (see (B.6) as well as (B.13)):

$$\begin{aligned}
&\sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \left|\partial_u \widetilde{X}_t(u) - \partial_u \widetilde{X}^{\times(t-l)}_t(u)\right| \mathbb{1}_{\{|\chi_{t-l,t-l}| \geqslant |\varepsilon_{t-l}|\}} \right\|_{1+\delta} \\
&\leqslant \sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \left|\operatorname{sign}(\varepsilon_t) - \operatorname{sign}(\chi_{t,t-l})\right| \left| \frac{\partial_u \widetilde{X}^2_t(u)}{2\sqrt{\widetilde{X}^2_t(u)}} \right| \right\|_{1+\delta} \\
&+ \sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \left|\operatorname{sign}(\chi_{t,t-l})\right| \left| \frac{1}{2\sqrt{\widetilde{X}^2_t(u)}} - \frac{1}{2\sqrt{\left(\widetilde{X}^{\times(t-l)}_t(u)\right)^2}} \right| \left|\partial_u \widetilde{X}^2_t(u)\right| \mathbb{1}_{\{|\chi_{t-l,t-l}| \geqslant |\varepsilon_{t-l}|\}} \right\|_{1+\delta} \\
&+ \sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \left|\operatorname{sign}(\chi_{t,t-l})\right| \left|\partial_u \widetilde{X}^2_t(u) - \partial_u \left(\widetilde{X}^{\times(t-l)}_t(u)\right)^2\right| \left| \frac{1}{2\sqrt{\left(\widetilde{X}^{\times(t-l)}_t(u)\right)^2}} \right| \mathbb{1}_{\{|\chi_{t-l,t-l}| \geqslant |\varepsilon_{t-l}|\}} \right\|_{1+\delta} \\
&=: \mathrm{III}(l) + \mathrm{IV}(l) + \mathrm{V}(l).
\end{aligned} \tag{B.24}$$

One obtains for all $l \in \mathbb{N}_0$ analogously to (B.16):

$$\mathrm{III}(l) \leqslant C \mathbb{1}_{\{l=0\}}. \tag{B.25}$$

Applying arguments which are similar to those that show the first inequality of (B.22) (recall thereby also the definition of II given in (B.17)) as well as those that yield (B.16) and using (2.8) together with $\big(\widetilde{X}^{\times(t-l)}_t(u)\big)^2 \geqslant m_0 \chi^2_{t,t-l}$ a. s. $\forall\, t \in \mathbb{Z}, l \in \mathbb{N}_0, u \in [0,1]$, $\mathbb{P}(\varepsilon_0 = 0) = \mathbb{P}\big(\varepsilon^{\times}_0 = 0\big) = 0$ as well as shifting the index of a sum provide for all $l \in \mathbb{N}_0$ (see (B.6)):

$$\begin{aligned}
\mathrm{IV}(l) &\leqslant C \sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \frac{\left|\widetilde{X}^2_t(u) - \left(\widetilde{X}^{\times(t-l)}_t(u)\right)^2\right|}{\chi^2_{t,t-l}} \mathbb{1}_{\{|\chi_{t-l,t-l}| \geqslant |\varepsilon_{t-l}|\}} \right\|^{\frac{1}{2}}_{1+\delta} \left\| \left(\frac{\partial_u \widetilde{X}^2_t(u)}{2\sqrt{\widetilde{X}^2_t(u)}} \right)^2 \right\|^{\frac{1}{2}}_{1+\delta} \\
&\leqslant C \sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| a_0(u) \mathbb{1}_{\{|\chi_{t-l,t-l}| \geqslant |\varepsilon_{t-l}|\}} \left(\frac{\varepsilon^2_t - \chi^2_{t,t-l}}{\chi^2_{t,t-l}} + \sum_{k=\max\{1,l\}}^{\infty} a^k_1(u) \frac{\left(\varepsilon^2_{t-l} - \chi^2_{t-l,t-l}\right) \prod_{\substack{r=0\\ r\neq l}}^{k} \varepsilon^2_{t-r}}{\chi^2_{t,t-l}} \right) \right\|^{\frac{1}{2}}_{1+\delta} \\
&\leqslant C \left(\mathbb{1}_{\{l=0\}} \left(1 + \sum_{k=1}^{\infty} \rho^k \left\|\varepsilon_0\right\|^{2k}_{2+2\delta} \right) + C \mathbb{1}_{\{l\geqslant 1\}} \sum_{k=0}^{\infty} \rho^{k+l} \left\|\varepsilon_0\right\|^{2\cdot(k+l)}_{2+2\delta} \right)^{1/2} \\
&\leqslant C \left(\rho \left\|\varepsilon_0\right\|^2_{2+2\delta} \right)^{\frac{l}{2}}.
\end{aligned} \tag{B.26}$$

One denotes for all $t \in \mathbb{Z}$, $l \in \mathbb{N}_0$, $u \in [0,1]$ the term $\widetilde{X}^{\times(t-l)}_{1,t}(u)$ as the expression that results from replacing ε_{t-l} in $\widetilde{X}_{1,t}(u)$ by $\varepsilon^{\times}_{t-l}$ (recall (B.6)). Then, (B.13), (2.8), $\big(\widetilde{X}^{\times(t-l)}_t(u)\big)^2 \geqslant m_0 \max\big\{\chi^2_{t,t-l}, \widetilde{X}^{\times(t-l)}_{1,t}(u)\big\}$ a. s. $\forall\, t \in \mathbb{Z}, l \in \mathbb{N}_0, u \in [0,1]$, $\mathbb{P}(\varepsilon_0 = 0) = \mathbb{P}\big(\varepsilon^{\times}_0 = 0\big) = 0$, (B.14), $\left\|\varepsilon_0\right\|_{1+\delta} \leqslant \left\|\varepsilon_0\right\|_2 = 1$ and shifting the index of a sum imply for all $l \in \mathbb{N}_0$:

$$\mathrm{V}(l) \leqslant C \sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \mathbb{1}_{\{l=0\}} \left\| \frac{\left|\varepsilon^2_t - \chi^2_{t,t-l}\right|}{|\chi_{t,t-l}|} \mathbb{1}_{\{|\chi_{t-l,t-l}| \geqslant |\varepsilon_{t-l}|\}} \right\|_{1+\delta}$$

$$
\begin{aligned}
&+ C \sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \sum_{k=\max\{1,l\}}^{\infty} a_1^{\frac{k}{2}}(u) \frac{\left|\varepsilon_{t-l}^2 - \chi_{t-l,t-l}^2\right|}{|\chi_{t-l,t-l}|} \mathbb{1}_{\{|\chi_{t-l,t-l}| \geqslant |\varepsilon_{t-l}|\}} \prod_{\substack{r=0\\r\neq l}}^{k} |\varepsilon_{t-r}| \right\|_{1+\delta} \\
&+ C\mathbb{1}_{\{l=0,1\}} \sup_{t\in\mathbb{Z}} \left\| \sum_{k=1}^{1} k \sup_{u\in(0,1):a_1(u)=0} |\partial_u a_1(u)| \frac{\left|\varepsilon_{t-l}^2 - \chi_{t-l,t-l}^2\right|}{|\chi_{t,t-l}|} \mathbb{1}_{\{|\chi_{t-l,t-l}| \geqslant |\varepsilon_{t-l}|\}} \prod_{\substack{r=0\\r\neq l}}^{k} \varepsilon_{t-r}^2 \right\|_{1+\delta} \\
&+ C\mathbb{1}_{\{l=0,1\}} \sup_{t\in\mathbb{Z}} \left\| \sum_{k=1}^{1} k \sup_{u\in(0,1):a_1(u)\neq 0} \frac{|\partial_u a_1(u)|}{\sqrt{a_1(u)}} \frac{\left|\varepsilon_{t-l}^2 - \chi_{t-l,t-l}^2\right|}{|\chi_{t-l,t-l}|} \mathbb{1}_{\{|\chi_{t-l,t-l}| \geqslant |\varepsilon_{t-l}|\}} \prod_{\substack{r=0\\r\neq l}}^{k} |\varepsilon_{t-r}| \right\|_{1+\delta} \\
&+ C \sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \sum_{k=\max\{2,l\}}^{\infty} k \left|\partial_u a_1(u)\right| a_1^{\frac{k}{2}-1}(u) \frac{\left|\varepsilon_{t-l}^2 - \chi_{t-l,t-l}^2\right|}{|\chi_{t-l,t-l}|} \mathbb{1}_{\{|\chi_{t-l,t-l}| \geqslant |\varepsilon_{t-l}|\}} \prod_{\substack{r=0\\r\neq l}}^{k} |\varepsilon_{t-r}| \right\|_{1+\delta} \\
&\leqslant C\mathbb{1}_{\{l=0,1\}} + C\mathbb{1}_{\{l\geqslant 2\}} \sum_{k=l}^{\infty} \left(\rho^{\frac{k}{2}} + k\rho^{\frac{k}{2}-1}\right) \\
&= C\mathbb{1}_{\{l=0,1\}} + C\mathbb{1}_{\{l\geqslant 2\}} \rho^{\frac{l}{4}} \sum_{k=0}^{\infty} \rho^{\frac{k}{4}} \left(\rho^{\frac{k+l}{4}} + (k+l)\rho^{\frac{k+l}{4}-1}\right) \\
&\leqslant C\rho^{\frac{l}{4}}. \qquad\qquad \text{(B.27)}
\end{aligned}
$$

Arguments which are similar to those that show (B.24), (B.25), (B.26) and (B.27) yield for all $l \in \mathbb{N}_0$:

$$
\sup_{u\in(0,1)} \sup_{t\in\mathbb{Z}} \left\| \left| \partial_u \widetilde{X}_t^{\times(t-l)}(u) - \partial_u \widetilde{X}_t(u) \right| \mathbb{1}_{\{|\chi_{t-l,t-l}| \leqslant |\varepsilon_{t-l}|\}} \right\|_{1+\delta} \leqslant C \left(\rho \left\|\varepsilon_0\right\|_{2+2\delta}^2\right)^{\frac{l}{2}} + C\rho^{\frac{l}{4}}. \qquad \text{(B.28)}
$$

One obtains from (B.24), (B.25), (B.26), (B.27), (B.28) and $\rho \left\|\varepsilon_0\right\|_{2+2\delta}^2 \in (0,1)$ (which holds due to (2.8)) that Example 2.5 (iii) fulfils the last property stated in (2.4). Overall, it is verified that Example 2.5 (iii) satisfies Assumption 2.4 $[\mathbf{DM.2}]$ (and not just Assumption 2.4 $[\mathbf{DM.3}]$). ☐

Proof of Proposition 2.12. At first, define for all $u \in [0,1]$, $s \in \mathbb{R}^d$ (cf. (5.5) in [4, Beering (2021), p. 78]):

$$
\widetilde{\varphi}(u,s) := \widetilde{\varphi}_{T,\mathfrak{U}_{0,1}}(u,s) := \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) e^{i\langle s, \tilde{X}_t(u)\rangle} \qquad \text{(B.29)}
$$

and observe for all $s \in \mathbb{R}^d$ (see (2.9) as well as the Definitions 2.11 and 2.6):[5]

$$
\begin{aligned}
\sup_{u\in\mathfrak{U}_{0,1,b}} \left|\mathbb{E}\left[\widehat{\varphi}(u,s)\right] - \mathbb{E}\left[\widetilde{\varphi}(u,s)\right]\right| &\leqslant \sup_{u\in\mathfrak{U}_{0,1,b}} \left| \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) \cdot \left(\varphi_{t,T}(s) - \varphi\left(\frac{t}{T}, s\right)\right) \right| \\
&+ \sup_{u\in\mathfrak{U}_{0,1,b}} \left| \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) \cdot \left(\varphi\left(\frac{t}{T}, s\right) - \varphi(u,s)\right) \right| \\
&=: \mathrm{I}(s) + \mathrm{II}(s). \qquad\qquad \text{(B.30)}
\end{aligned}
$$

Lemma B.1 with $\kappa_1 = 1$ and Remark 2.7 imply for all $s \in \mathbb{R}^d$:[5]

$$
\mathrm{I}(s) \leqslant \frac{C}{T}|s|_1. \qquad \text{(B.31)}
$$

Next, $\mathrm{II}(s)$ is bounded. Therefor, note at first that it follows for all $s \in \mathbb{R}^d$ as well as for a not necessarily deterministic value $\xi_{t,T,u}$ between t/T and u from the mean value theorem together with Assumption

[5]The inequalities (B.30) and (B.31) are similar to (6.3), (6.4) as well as (6.6) in [41, Jentsch et al. (2020b), p. 4].

2.2 [$\mathbf{StAp}$] (ii) (recall Definition 2.11):

$$\begin{aligned}
&\sup_{u\in\mathfrak{U}_{0,1,b}}\left|\frac{1}{T}\sum_{t=1}^{T}K_b\left(\frac{t}{T}-u\right)\mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_0\left(\frac{t}{T}\right)\right\rangle\right)-\cos\left(\left\langle s,\widetilde{X}_0\left(u\right)\right\rangle\right)\right]\right|\\
&\leqslant \sup_{u\in\mathfrak{U}_{0,1,b}}\left|\frac{1}{Tb}\sum_{t=1}^{T}K\left(\frac{\frac{t}{T}-u}{b}\right)\left(\frac{t}{T}-u\right)\mathbb{E}\left[\partial_{\widetilde{u}}\cos\left(\left\langle s,\widetilde{X}_0\left(\widetilde{u}\right)\right\rangle\right)\Big|_{\widetilde{u}=\xi_{t,T,u}}-\partial_u\cos\left(\left\langle s,\widetilde{X}_0\left(u\right)\right\rangle\right)\right]\right|\\
&+\sup_{u\in\mathfrak{U}_{0,1,b}}\left|\frac{1}{T}\sum_{t=1}^{T}K\left(\frac{\frac{t}{T}-u}{b}\right)\frac{\frac{t}{T}-u}{b}\mathbb{E}\left[\partial_u\cos\left(\left\langle s,\widetilde{X}_0\left(u\right)\right\rangle\right)\right]\right|\\
&=:\mathrm{II}_1(s)+\mathrm{II}_2(s).
\end{aligned}\tag{B.32}$$

Lemma B.1 with $\kappa_1=\kappa_2=1$ and Assumption 2.8 [$\mathbf{K\&b.1}$] (i) (which provides $K\left((t/T-u)/b\right)=K\left((t/T-u)/b\right)\mathbb{1}_{\{|t/T-u|\leqslant b\}}$) yield for all $s\in\mathbb{R}^d$:

$$\begin{aligned}
&\mathrm{II}_1(s)\\
&\leqslant Cb\sup_{u\in\mathfrak{U}_{0,1,b}}\sup_{t=1,\ldots,T:\left|\frac{t}{T}-u\right|\leqslant b}\mathbb{E}\left[\left|\sin\left(\left\langle s,\widetilde{X}_0\left(\xi_{t,T,u}\right)\right\rangle\right)\left(-\left\langle s,\partial_{\widetilde{u}}\widetilde{X}_0\left(\widetilde{u}\right)|_{\widetilde{u}=\xi_{t,T,u}}\right\rangle+\left\langle s,\partial_u\widetilde{X}_0\left(u\right)\right\rangle\right)\right|\right]\\
&+Cb\sup_{u\in\mathfrak{U}_{0,1,b}}\sup_{t=1,\ldots,T:\left|\frac{t}{T}-u\right|\leqslant b}\mathbb{E}\left[\left|\left(-\sin\left(\left\langle s,\widetilde{X}_0\left(\xi_{t,T,u}\right)\right\rangle\right)+\sin\left(\left\langle s,\widetilde{X}_0\left(u\right)\right\rangle\right)\right)\left\langle s,\partial_u\widetilde{X}_0\left(u\right)\right\rangle\right|\right]\\
&=:\mathrm{II}_{1.1}(s)+\mathrm{II}_{1.2}(s).
\end{aligned}\tag{B.33}$$

One obtains for all $s\in\mathbb{R}^d$ from Assumption 2.2 [$\mathbf{StAp}$] (ii):

$$\begin{aligned}
\mathrm{II}_{1.1}(s)&\leqslant Cb|s|_1\cdot\mathbb{E}\left[\sup_{u\in\mathfrak{U}_{0,1,b}}\sup_{t=1,\ldots,T:\left|\frac{t}{T}-u\right|\leqslant b}\left|-\partial_{\widetilde{u}}\widetilde{X}_0\left(\widetilde{u}\right)|_{\widetilde{u}=\xi_{t,T,u}}+\partial_u\widetilde{X}_0\left(u\right)\right|_1\right]\\
&\leqslant Cb^{1+\delta}|s|_1.
\end{aligned}\tag{B.34}$$

It follows for all $s\in\mathbb{R}^d$ from (3.14) with $q=(1+\delta)/\delta$, Remark 2.3 and Assumption 2.2 [$\mathbf{StAp}$] (ii):

$$\begin{aligned}
\mathrm{II}_{1.2}(s)&\leqslant Cb\,\mathbb{E}\left[\sup_{u\in\mathfrak{U}_{0,1,b}}\sup_{t=1,\ldots,T:\left|\frac{t}{T}-u\right|\leqslant b}\left|-\sin\left(\left\langle s,\widetilde{X}_0\left(\xi_{t,T,u}\right)\right\rangle\right)+\sin\left(\left\langle s,\widetilde{X}_0\left(u\right)\right\rangle\right)\right|^{\frac{1+\delta}{\delta}}\right]^{\frac{\delta}{1+\delta}}\\
&\quad\cdot\left\|\sup_{u\in(0,1)}\left|\left\langle s,\partial_u\widetilde{X}_0\left(u\right)\right\rangle\right|\right\|_{1+\delta}\\
&\leqslant Cb^{1+\delta}|s|_1^{1+\delta}.
\end{aligned}\tag{B.35}$$

Furthermore, Assumption 2.8 [$\mathbf{K\&b.1}$] (i) yields for all $z_1,z_2\in\mathbb{R}$:

$$\begin{aligned}
&\sup_{u\in[0,1]}\left|K\left(\frac{z_1-u}{b}\right)\frac{z_1-u}{b}-K\left(\frac{z_2-u}{b}\right)\frac{z_2-u}{b}\right|\\
&=\sup_{u\in[0,1]}\left|K\left(\frac{z_1-u}{b}\right)\frac{z_1-u}{b}-K\left(\frac{z_2-u}{b}\right)\frac{z_2-u}{b}\right|\mathbb{1}_{\{|z_1-u|\leqslant b\vee|z_2-u|\leqslant b\}}\\
&\leqslant\sup_{u\in[0,1]}\min\left\{\left|K\left(\frac{z_1-u}{b}\right)-K\left(\frac{z_2-u}{b}\right)\right|\left|\frac{z_1-u}{b}\right|+\left|\frac{z_1-u}{b}-\frac{z_2-u}{b}\right|\left|K\left(\frac{z_2-u}{b}\right)\right|,\right.\\
&\left.\left|K\left(\frac{z_1-u}{b}\right)\right|\left|\frac{z_1-u}{b}-\frac{z_2-u}{b}\right|+\left|K\left(\frac{z_1-u}{b}\right)-K\left(\frac{z_2-u}{b}\right)\right|\left|\frac{z_2-u}{b}\right|\right\}\mathbb{1}_{\{|z_1-u|\leqslant b\vee|z_2-u|\leqslant b\}}\\
&\leqslant\frac{C}{b}\left|z_1-z_2\right|.
\end{aligned}\tag{B.36}$$

In addition, it follows for all $u\in\mathfrak{U}_{0,1,b}$ (see (2.9)) from the substitution $y:=(z-u)/b$, the inequalities $-u/b\leqslant\mathfrak{U}_0-\mathfrak{U}_1$ as well as $(1-u)/b\geqslant\mathfrak{U}_1-\mathfrak{U}_0$ (note that Assumption 2.8 [$\mathbf{K\&b.1}$] (ii) demands

$b \in (0, 1/2)$, which ensures $(\mathfrak{U}_1 - \mathfrak{U}_0)b \leqslant 1 - (\mathfrak{U}_1 - \mathfrak{U}_0)b)$ and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i):

$$\int_0^1 K\left(\frac{z-u}{b}\right)\frac{z-u}{b}\,dz = b\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(y)\,y\,dy = 0. \tag{B.37}$$

One obtains for all $s \in \mathbb{R}^d$ from Lemma B.2 (i) together with (B.36), Assumption 2.2 $[\mathbf{StAp}]$ (ii) and (B.37) (recall (B.32)):

$$\mathrm{II}_2(s) \leqslant C\left(\sup_{u\in\mathfrak{U}_{0,1,b}}\left|\int_0^1 K\left(\frac{z-u}{b}\right)\frac{z-u}{b}\,dz\right| + \frac{C}{Tb}\right)|s|_1 = \frac{C}{Tb}|s|_1. \tag{B.38}$$

It follows for all $s \in \mathbb{R}^d$ from (B.32), (B.33), (B.34), (B.35) and (B.38) as well as similar arguments (see (B.30)):

$$\mathrm{II}(s) \leqslant Cb^{1+\delta}\cdot\left(|s|_1 + |s|_1^{1+\delta}\right) + \frac{C}{Tb}|s|_1. \tag{B.39}$$

Proposition 2.12 is an implication of (B.30), (B.31), (B.39) and Lemma B.3. □

Proof of Proposition 2.13. At first, one obtains from Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) and the fact that $z \in [(t-1)/T, t/T]$ implies $zT \leqslant t \leqslant zT + 1\ \forall\, t \in \{1, \ldots, T\}$ (recall Definition 2.11):[6]

$$\begin{aligned}
&\sup_{u\in[0,1]}\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) - \int_0^1 \frac{1}{b}K\left(\frac{z-u}{b}\right)dz\right| \\
&\leqslant \sup_{u\in[0,1]}\frac{1}{b}\sum_{t=1}^{T}\int_{\frac{t-1}{T}}^{\frac{t}{T}}\left|K\left(\frac{\frac{t}{T}-u}{b}\right) - K\left(\frac{z-u}{b}\right)\right|\cdot\left(\mathbb{1}_{\{uT-Tb\leqslant t\leqslant uT+Tb\}} + \mathbb{1}_{\{uT-Tb\leqslant zT\leqslant uT+Tb\}}\right)dz \\
&\leqslant \sup_{u\in[0,1]}\frac{C}{b^2}\sum_{t=1}^{T}\int_{\frac{t-1}{T}}^{\frac{t}{T}}\left|\frac{t}{T}-z\right|\cdot\left(\mathbb{1}_{\{uT-Tb\leqslant t\leqslant uT+Tb\}} + \mathbb{1}_{\{uT-Tb\leqslant zT\leqslant zT+1\leqslant uT+Tb+1\}}\mathbb{1}_{\{zT\leqslant t\leqslant zT+1\}}\right)dz \\
&\leqslant \sup_{u\in[0,1]}\frac{C}{T^2b^2}\sum_{t=1}^{T}\mathbb{1}_{\{uT-Tb\leqslant t\leqslant uT+Tb+1\}} \\
&\leqslant \frac{C}{Tb}.
\end{aligned} \tag{B.40}$$

Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) as well as (ii) (the latter demands $b \in (0, 1/2)$, which ensures $(\mathfrak{U}_1 - \mathfrak{U}_0)b \leqslant 1 - (\mathfrak{U}_1 - \mathfrak{U}_0)b)$, the substitution $y := (z-u)/b$ and the inequalities $-v/b \leqslant \mathfrak{U}_0 - \mathfrak{U}_1$ as well as $(1-v)/b \geqslant \mathfrak{U}_1 - \mathfrak{U}_0\ \forall\, v \in [(\mathfrak{U}_1 - \mathfrak{U}_0)\,b, 1 - (\mathfrak{U}_1 - \mathfrak{U}_0)\,b]$ provide for all $u \in [0, 1]$:[6]

$$\begin{aligned}
1 - \int_0^1 \frac{1}{b}K\left(\frac{z-u}{b}\right)dz &= \int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(y)\,dy - \int_{-\frac{u}{b}}^{\frac{1-u}{b}} K(y)\,dy \\
&= \int_{\mathfrak{U}_0-\mathfrak{U}_1}^{-\frac{u}{b}} K(y)\,dy\,\mathbb{1}_{\{u\in[0,(\mathfrak{U}_1-\mathfrak{U}_0)b)\}} + \int_{\frac{1-u}{b}}^{\mathfrak{U}_1-\mathfrak{U}_0} K(y)\,dy\,\mathbb{1}_{\{u\in(1-(\mathfrak{U}_1-\mathfrak{U}_0)b,1]\}}.
\end{aligned} \tag{B.41}$$

In addition, for each kernel which satisfies Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) exists a $c_K \in (0, \mathfrak{U}_1 - \mathfrak{U}_0)$ with

[6]Note that (B.40) and (B.41) are similar to the statement $\left|1/T\sum_{t=1}^{T} K_b(t/T-u) - 1\right| = \mathcal{O}(1/(Tb))$ for arbitrary but fixed $u \in (0, 1)$, which is stated in [41, Jentsch et al. (2020b), p. 4].

$\int_{c_K}^{\mathfrak{U}_1-\mathfrak{U}_0} K(y)\,dy > 0$. It follows from the reverse triangle inequality, (B.40), (B.41), $\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{-c_K} K(y)\,dy = \int_{c_K}^{\mathfrak{U}_1-\mathfrak{U}_0} K(y)\,dy$ and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) that a $T_K \in \mathbb{N}$ exists which fulfils (note the Definitions 2.11 as well as 2.6):

$$\begin{aligned}
&\inf_{[0,c_K b]\cup[1-c_K b,1]} \left|\mathbb{E}\left[\widehat{\varphi}(u,0)\right] - \varphi(u,0)\right| \\
&\geqslant \inf_{[0,c_K b]\cup[1-c_K b,1]} \left(- \left|\mathbb{E}\left[\widehat{\varphi}(u,0)\right] - \int_0^1 \frac{1}{b} K\left(\frac{z-u}{b}\right) dz\right| + \left|\int_0^1 \frac{1}{b} K\left(\frac{z-u}{b}\right) dz - \varphi(u,0)\right|\right) \\
&\geqslant -\frac{C}{Tb} + \int_{c_K}^{\mathfrak{U}_1-\mathfrak{U}_0} K(y)\,dy \geqslant \frac{1}{2}\int_{c_K}^{\mathfrak{U}_1-\mathfrak{U}_0} K(y)\,dy > 0 \quad \forall\, T \in \mathbb{N} : T \geqslant T_K. \qquad \text{(B.42)}
\end{aligned}$$

Moreover, Assumption 2.2 $[\mathbf{StAp}]$ (i) provides that an absolute constant $\widetilde{C} \in (0,\infty)$ exists for which the following inequality holds:

$$\sup_{u\in[0,1]} \left|\varphi(u,r) - \varphi(u,s)\right| \leqslant \widetilde{C}\,|r-s|_1 \quad \forall\, r,s \in \mathbb{R}^d \qquad \text{(B.43)}$$

and, according to Lemma B.1 with $\kappa_1 = 1$ as well as Assumption 2.2 $[\mathbf{StAp}]$ (i), this constant $\widetilde{C}$ can be chosen in such a manner that it also fulfils:

$$\sup_{u\in[0,1]} \left\|\widehat{\varphi}(u,r) - \widehat{\varphi}(u,s)\right\|_1 \leqslant \sup_{u\in[0,1]} \frac{C}{Tb}\sum_{t=1}^{T} K\left(\frac{\frac{t}{T}-u}{b}\right)|r-s|_1 \leqslant \widetilde{C}\,|r-s|_1 \quad \forall\, r,s \in \mathbb{R}^d. \qquad \text{(B.44)}$$

Since the reverse triangle inequality implies for all $u \in [0,1]$, $s \in \mathbb{R}^d$:

$$\left|\mathbb{E}\left[\widehat{\varphi}(u,s)\right] - \varphi(u,s)\right| \geqslant -\left|\mathbb{E}\left[\widehat{\varphi}(u,0)\right] - \mathbb{E}\left[\widehat{\varphi}(u,s)\right]\right| + \left|\mathbb{E}\left[\widehat{\varphi}(u,0)\right] - \varphi(u,0)\right| - \left|\varphi(u,s) - \varphi(u,0)\right|,$$

Proposition 2.13 follows from (B.44), (B.42) and (B.43) with $S_K := \left(8d\widetilde{C}\right)^{-1}\int_{c_K}^{\mathfrak{U}_1-\mathfrak{U}_0} K(y)\,dy$ as well as $\epsilon_K := 1/4\int_{c_K}^{\mathfrak{U}_1-\mathfrak{U}_0} K(y)\,dy$. □

Proof of Proposition 2.14. At first, one obtains from shifting the index of a sum, Tonelli's theorem for infinite series ($\Delta_l \geqslant 0\ \forall\, l \in \mathbb{N}$ according to Assumption 2.4 $[\mathbf{DM}]$) and Assumption 2.4 $[\mathbf{DM.1}]$:

$$\sup_{\widetilde{t}_1\in\mathbb{N}_0} \sum_{t_2=\widetilde{t}_1+1}^{\infty} \sum_{l=t_2-\widetilde{t}_1}^{\infty} \Delta_l = \sum_{t_2=1}^{\infty}\sum_{l=t_2}^{\infty} \Delta_l = \sum_{t_2=1}^{\infty}\sum_{l=1}^{\infty} \mathbb{1}_{\{t_2\leqslant l\}}\Delta_l = \sum_{l=1}^{\infty}\sum_{t_2=1}^{\infty} \mathbb{1}_{\{t_2\leqslant l\}}\Delta_l = \sum_{l=1}^{\infty}\Delta_l l \leqslant C. \qquad \text{(B.45)}$$

Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) (which ensures $K(z) = 0\ \forall\, z \in \mathbb{R} : |z| > 1$), Lemma B.4 (v), Lemma B.1 with $\kappa_1 = 1$ and (B.45) provide for all $s \in \mathbb{R}^d$ (see Definition 2.11):

$$\begin{aligned}
&\sup_{u\in[0,1]} \mathbb{E}\left[\Re\left\{\widehat{\varphi}(u,s) - \mathbb{E}\left[\widehat{\varphi}(u,s)\right]\right\}^2\right] \\
&\leqslant \sup_{u\in[0,1]} \frac{1}{(Tb)^2} \sum_{t=\max\{1,\lfloor uT-Tb\rfloor\}}^{\min\{T,\lceil uT+Tb\rceil\}} K\left(\frac{\frac{t}{T}-u}{b}\right)^2 \mathrm{Var}\left(\cos\left(\langle s, X_{t,T}\rangle\right)\right) \\
&+ \sup_{u\in[0,1]} \frac{2}{(Tb)^2} \sum_{t_1=1}^{T-1} K\left(\frac{\frac{t_1}{T}-u}{b}\right) \sum_{t_2=t_1+1}^{T} K\left(\frac{\frac{t_2}{T}-u}{b}\right) \left|\mathrm{Cov}\left(\cos\left(\langle s, X_{t_1,T}\rangle\right), \cos\left(\langle s, X_{t_2,T}\rangle\right)\right)\right| \\
&\leqslant \frac{C}{Tb} + \sup_{u\in[0,1]} \frac{C}{(Tb)^2}\sum_{t_1=1}^{T-1} K\left(\frac{\frac{t_1}{T}-u}{b}\right)\left(\sup_{\widetilde{t}_1=1,\ldots,T-1} \sum_{t_2=\widetilde{t}_1+1}^{T}\sum_{l=t_2-\widetilde{t}_1}^{\infty} \Delta_l |s|_1\right) \\
&\leqslant \frac{C}{Tb} + \frac{C}{Tb}|s|_1. \qquad \text{(B.46)}
\end{aligned}$$

This and similar arguments show Proposition 2.14. □

B.2. Auxiliary results belonging to Chapter 2 and their proofs

Lemma B.1. *Let Assumption 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *be fulfilled. Then, it holds for all* $T \in \mathbb{N}$, $\kappa_1 > 0$, $\kappa_2 > 0$*:*

$$\sup_{u\in[0,1]} \frac{1}{Tb}\sum_{t=1}^{T} K\left(\frac{\frac{t}{T}-u}{b}\right)^{\kappa_1} \leqslant C \quad \textit{and} \quad \sup_{u\in[0,1]} \frac{1}{Tb}\sum_{t=1}^{T} K\left(\frac{\frac{t}{T}-u}{b}\right)^{\kappa_1} \left|\frac{t}{T}-u\right|^{\kappa_2} \leqslant Cb^{\kappa_2}.$$

Proof. In the following, just the second statement of Lemma B.1 is proved because the first one results from similar arguments. One obtains $K\left((t/T-u)/b\right) = 0$ for $|t/T-u| > b$ from Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i), such that:

$$\sup_{u\in[0,1]} \frac{1}{Tb}\sum_{t=1}^{T} K\left(\frac{\frac{t}{T}-u}{b}\right)^{\kappa_1} \left|\frac{t}{T}-u\right|^{\kappa_2} \leqslant \sup_{u\in[0,1]} \frac{1}{Tb}\sum_{t=\lfloor uT-Tb\rfloor}^{\lceil uT+Tb\rceil} K\left(\frac{\frac{t}{T}-u}{b}\right)^{\kappa_1} \mathbb{1}_{\{|\frac{t}{T}-u|\leqslant b\}} \left|\frac{t}{T}-u\right|^{\kappa_2},$$

which implies the second statement of Lemma B.1. □

Lemma B.2. *Assume that* $N \in \mathbb{N}$, $x, y \in \mathbb{R}$ *with* $x < y$ *and that* $f\colon [x,y] \to \mathbb{R}$ *is a function which fulfils for all* $q_1, q_2 \in [x,y]$ *as well as a* $L_f \in [0,\infty)$*:*

$$|f(q_1) - f(q_2)| \leqslant L_f\,|q_1-q_2|\,. \tag{B.47}$$

Moreover, set:

$$\Delta z := \frac{y-x}{N},\quad z_k := x + k\Delta z \ \forall\, k \in \{0,\dots,N\} \ \textit{and} \ c_k := x + \left(k-\frac{1}{2}\right)\Delta z \ \forall\, k \in \{1,\dots,N\}. \tag{B.48}$$

Then, it holds:

(i)

$$\left|\int_x^y f(z)\,dz - \Delta z\sum_{k=1}^{N} f(z_k)\right| \leqslant \frac{L_f|y-x|^2}{N}.$$

(ii)

$$\left|\int_x^y f(z)\,dz - \Delta z\sum_{k=1}^{N} f(c_k)\right| \leqslant \frac{3L_f|y-x|^2}{2N}.$$

(iii) [7] *If in addition* f *is differentiable on* (x,y) *and* $L_{f'} \in [0,\infty)$ *as well as* $\kappa \in (0,1]$ *exist which fulfil:*

$$|\partial_{q_1} f(q_1) - \partial_{q_2} f(q_2)| \leqslant L_{f'}\,|q_1-q_2|^{\kappa} \quad \forall\, q_1, q_2 \in (x,y), \tag{B.49}$$

one will obtain:

$$\left|\int_x^y f(z)\,dz - \Delta z\sum_{k=1}^{N} f(c_k)\right| \leqslant \frac{L_{f'}|y-x|^{2+\kappa}}{2^{\kappa}N^{1+\kappa}}.$$

Proof. (i) Lemma B.2 (i) follows from the Qth Example in [77, Walter (2007), p. 234] by using the equidistant partition which is defined in [77, Walter (2007), p. 235].

[7] Proofs of error bounds which belong to numerical integrals that are based on the midpoint rule are well-known for functions with bounded second derivative (see e. g. [26, Fazekas and Mercer (2009)]). In contrast, Lemma B.2 (iii) provides an error bound under weaker assumptions, which is shown by using some similar arguments, among other things.

(ii) Lemma B.2 (i) and (B.47) provide (recall (B.48)):

$$\left|\int_x^y f(z)\,dz - \Delta z \sum_{k=1}^N f(c_k)\right| \leqslant \left|\int_x^y f(z)\,dz - \Delta z \sum_{k=1}^N f(z_k)\right|$$
$$+ \Delta z \sum_{k=1}^N \left| f(x + k\Delta z) - f\left(x + \left(k - \frac{1}{2}\right)\Delta z\right)\right|$$
$$\leqslant \frac{L_f |y-x|^2}{N} + \Delta z \cdot N \cdot L_f \cdot \frac{\Delta z}{2},$$

which shows Lemma B.2 (ii) (note $\Delta z := (y-x)/N$).

(iii) At first, define for all $k \in \{1, \ldots, N\}$, $z \in (x, y)$:

$$g_k(z) := f(c_k) + \partial_v f(v)\big|_{v=c_k} (z - c_k). \tag{B.50}$$

It holds for all $k \in \{1, \ldots, N\}$ due to $c_k = 1/2(z_k + z_{k-1})$ (see (B.48)):

$$\int_{z_{k-1}}^{z_k} f(c_k)\,dz = \int_{z_{k-1}}^{z_k} f(c_k)\,dz + \partial_v f(v)\big|_{v=c_k} \left[\frac{z^2}{2} - z c_k\right]\Big|_{z_{k-1}}^{z_k} = \int_{z_{k-1}}^{z_k} g_k(z)\,dz,$$

such that:

$$\left|\int_x^y f(z)\,dz - \Delta z \sum_{k=1}^N f(c_k)\right| \leqslant \sum_{k=1}^N \left|\int_{z_{k-1}}^{z_k} f(z) - f(c_k)\,dz\right| = \sum_{k=1}^N \left|\int_{z_{k-1}}^{z_k} f(z) - g_k(z)\,dz\right|. \tag{B.51}$$

By assumption, one obtains for all $k \in \{1, \ldots, N\}$, $z \in [c_k, z_k]$ that the function $f\colon\ [c_k, z] \to \mathbb{R}$ is continuous on $[c_k, z]$ and differentiable on (c_k, z). (These properties should hold by convention for $z = c_k$.) Hence, the mean value theorem implies for all $k \in \{1, \ldots, N\}$, $z \in [c_k, z_k]$ the existence of a value $\xi(c_k, z)$ between c_k and z which fulfils $f(z) = f(c_k) + \partial_v f(v)\big|_{v=\xi(c_k,z)} (z - c_k)$, whereby $\xi(c_k, z) := c_k$ for $z = c_k$. Similar arguments show for all $k \in \{1, \ldots, N\}$, $\widetilde{z} \in [z_{k-1}, c_k]$ as well as a value $\xi(\widetilde{z}, c_k)$ between $\widetilde{z}$ and c_k that $f(c_k) = f(\widetilde{z}) + \partial_v f(v)\big|_{v=\xi(\widetilde{z},c_k)} (c_k - \widetilde{z})$, which is equivalent to $f(\widetilde{z}) = f(c_k) + \partial_v f(v)\big|_{v=\xi(\widetilde{z},c_k)} (\widetilde{z} - c_k)$, whereby $\xi(\widetilde{z}, c_k) := c_k$ for $\widetilde{z} = c_k$. Thus, (B.49) provides (note (B.50) and (B.48)):

$$\sup_{k=1,\ldots,N} \sup_{z \in [z_{k-1}, z_k]} |f(z) - g_k(z)|$$
$$\leqslant L_{f'} \sup_{k=1,\ldots,N} \left(\sup_{\widetilde{z} \in [z_{k-1}, c_k]} |\xi(\widetilde{z}, c_k) - c_k|^\kappa \cdot |\widetilde{z} - c_k| + \sup_{z \in [c_k, z_k]} |\xi(c_k, z) - c_k|^\kappa \cdot |z - c_k| \right)$$
$$\leqslant L_{f'} 2 \left(\frac{1}{2}\Delta z\right)^\kappa \frac{1}{2}\Delta z. \tag{B.52}$$

Overall, (B.51), (B.52), $z_k - z_{k-1} = \Delta z \;\forall\, k \in \{1, \ldots, N\}$ and $\Delta z := (y-x)/N$ prove Lemma B.2 (iii).

□

Lemma B.3. *Let the Assumptions 2.2* $[\mathbf{StAp}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *be valid. Then, the following statement holds for* $T \to \infty$ *(recall (2.9), (B.29) and Definition 2.6):*

$$\sup_{u \in \mathfrak{U}_{0,1,b}} |\mathbb{E}[\widetilde{\varphi}(u, s)] - \varphi(u, s)| = \mathcal{O}\left(\frac{1}{Tb}\right) \quad \forall\, s \in \mathbb{R}^d,$$

whereby the expression $\mathcal{O}(1/(Tb))$ *does not depend on* $s \in \mathbb{R}^d$*.*

Proof. The right side of (B.41) is zero for all $u \in \mathfrak{U}_{0,1,b}$ (see (2.9)). Thus, $\mathbb{E}\big[e^{i\langle s, \tilde{X}_t(u)\rangle}\big] = \mathbb{E}\big[e^{i\langle s, \tilde{X}_0(u)\rangle}\big]$ $\forall\, s \in \mathbb{R}^d$, $t \in \mathbb{Z}$, $u \in [0,1]$, (B.40) and (B.41) verify Lemma B.3. □

Lemma B.4. *Suppose for some $d, d_1, d_2 \in \mathbb{N}$ and all $k \in \{1,2\}$ that the d-variate triangular array $\{X_{t,T}\} := \{X_{t,T} : t \in \{1,\dots,T\}\}_{T=1}^{\infty}$ and the d_k-variate triangular array $\{Y_{t,T}^{[k]}\} := \{Y_{t,T}^{[k]} : t \in \{1,\dots,T\}\}_{T=1}^{\infty}$ are locally stationary Bernoulli shift processes (in the sense of Definition 2.1) which fulfil Assumption 2.4 $[\mathbf{DM.1}]$ or 2.4 $[\mathbf{DM.2}]$ or 2.4 $[\mathbf{DM.3}]$ (and, therefore, also Assumption 2.2 $[\mathbf{StAp}]$). In addition, assume that $\{Y_{t,T}^{[1]}\}$ and $\{Y_{t,T}^{[2]}\}$ are based on the same i.i.d. innovations $(\varepsilon_t)_{t\in\mathbb{Z}}$. Moreover, the stationary approximations of $\{X_{t,T}\}$ are called $\{\tilde{X}_t(u) : t \in \mathbb{Z}\}$ (with $u \in [0,1]$) and those of $\{Y_{t,T}^{[k]}\}$ are denoted as $\{\tilde{Y}_t^{[k]}(u) : t \in \mathbb{Z}\}$ (for $k \in \{1,2\}$). For all $l \in \mathbb{N}_0$, let $X_{t,T}^{\times(t-l)}$ (with $t \in \{1,\dots,T\}$, $T \in \mathbb{N}$) as well as $\tilde{X}_t^{\times(t-l)}(u)$ (with $t \in \mathbb{Z}$, $u \in [0,1]$) be defined as in Assumption 2.4 $[\mathbf{DM}]$ and $\mathcal{F}_{r_1,r_2}$ (with $r_1, r_2 \in \mathbb{Z} : r_1 \geqslant r_2$) originate from Definition A.1 (i). Further, $g\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ should be an arbitrary measurable function and suppose that $g_j\colon \mathbb{R} \to \mathbb{R}$ (with $j \in \{1,2\}$) fulfils either $g_j(x) = \sin(x)\ \forall\, x \in \mathbb{R}$ or $g_j(x) = \cos(x)\ \forall\, x \in \mathbb{R}$ (whereby $g_1 \neq g_2$ is allowed). Then, it holds:*

(i)

$$\left\|\mathbb{E}\left[X_{t,T}\middle|\mathcal{F}_{t,t-l}\right] - \mathbb{E}\left[X_{t,T}\middle|\mathcal{F}_{t,t-l-1}\right]\right\|_{1+\delta} \leqslant \Delta_{l+1} \quad \forall\, t \in \{1,\dots,T\},\ T \in \mathbb{N},\ l \in \mathbb{N}_0.$$

(ii) If $\left\|g\big(\tilde{X}_t(u), \partial_u \tilde{X}_t(u)\big)\right\|_1 < \infty\ \forall\, u \in [0,1], t \in \mathbb{Z}$, one will obtain:

$$\begin{aligned}&\left\|\mathbb{E}\left[g\left(\tilde{X}_t(u), \partial_u \tilde{X}_t(u)\right)\middle|\mathcal{F}_{t,t-l}\right] - \mathbb{E}\left[g\left(\tilde{X}_t(u), \partial_u \tilde{X}_t(u)\right)\middle|\mathcal{F}_{t,t-l-1}\right]\right\|_1 \\ &\leqslant \left\|g\left(\tilde{X}_t^{\times(t-l-1)}(u), \partial_u \tilde{X}_t^{\times(t-l-1)}(u)\right) - g\left(\tilde{X}_t(u), \partial_u \tilde{X}_t(u)\right)\right\|_1 \quad \forall\, t \in \mathbb{Z},\ u \in [0,1],\ l \in \mathbb{N}_0.\end{aligned}$$

(iii)

$$\begin{aligned}&\left\|\mathbb{E}\left[g_1(\langle s, X_{t,T}\rangle)\middle|\mathcal{F}_{t,t-l}\right] - \mathbb{E}\left[g_1(\langle s, X_{t,T}\rangle)\middle|\mathcal{F}_{t,t-l-1}\right]\right\|_q \\ &\leqslant \left\|g_1\left(\left\langle s, X_{t,T}^{\times(t-l-1)}\right\rangle\right) - g_1(\langle s, X_{t,T}\rangle)\right\|_q \\ &\leqslant C\Delta_{l+1}^{\frac{1+\delta}{q}} |s|_1^{\frac{1+\delta}{q}} \qquad \forall\, s \in \mathbb{R}^d,\ t \in \{1,\dots,T\},\ T \in \mathbb{N},\ l \in \mathbb{N}_0,\ q \geqslant 1+\delta.\end{aligned}$$

(iv)

$$\begin{aligned}&\left\|\mathbb{E}\left[g_1\left(\left\langle s, \tilde{X}_t(u)\right\rangle\right)\middle|\mathcal{F}_{t,t-l}\right] - \mathbb{E}\left[g_1\left(\left\langle s, \tilde{X}_t(u)\right\rangle\right)\middle|\mathcal{F}_{t,t-l-1}\right]\right\|_q \\ &\leqslant \left\|g_1\left(\left\langle s, \tilde{X}_t^{\times(t-l-1)}(u)\right\rangle\right) - g_1\left(\left\langle s, \tilde{X}_t(u)\right\rangle\right)\right\|_q \\ &\leqslant C\Delta_{l+1}^{\frac{1+\delta}{q}} |s|_1^{\frac{1+\delta}{q}} \qquad \forall\, s \in \mathbb{R}^d,\ t \in \mathbb{Z},\ u \in [0,1],\ l \in \mathbb{N}_0,\ q \geqslant 1+\delta.\end{aligned}$$

(v)

$$\begin{aligned}&\left|\operatorname{Cov}\left(g_1\left(\left\langle s^{[1]}, Y_{t_1,T}^{[1]}\right\rangle\right), g_2\left(\left\langle s^{[2]}, Y_{t_2,T}^{[2]}\right\rangle\right)\right)\right| \\ &\leqslant C \sum_{l=|t_1-t_2|}^{\infty} \Delta_l \left(\left|s^{[1]}\right|_1 \mathbb{1}_{\{t_1>t_2\}} + \left|s^{[2]}\right|_1 \mathbb{1}_{\{t_2>t_1\}}\right) + C\mathbb{1}_{\{t_1=t_2\}} \\ &\qquad\qquad \forall\, s^{[1]} \in \mathbb{R}^{d_1},\ s^{[2]} \in \mathbb{R}^{d_2},\ t_1, t_2 \in \{1,\dots,T\},\ T \in \mathbb{N}.\end{aligned}$$

(vi)

$$\sup_{u,w\in[0,1]} \left|\operatorname{Cov}\left(g_1\left(\left\langle s^{[1]}, \tilde{Y}_{t_1}^{[1]}(u)\right\rangle\right), g_2\left(\left\langle s^{[2]}, \tilde{Y}_{t_2}^{[2]}(w)\right\rangle\right)\right)\right|$$

$$\leqslant C \sum_{l=|t_1-t_2|}^{\infty} \Delta_l \left(\left| s^{[1]} \right|_1 \mathbb{1}_{\{t_1>t_2\}} + \left| s^{[2]} \right|_1 \mathbb{1}_{\{t_2>t_1\}} \right) + C \mathbb{1}_{\{t_1=t_2\}}$$
$$\forall\, s^{[1]} \in \mathbb{R}^{d_1},\ s^{[2]} \in \mathbb{R}^{d_2},\ t_1, t_2 \in \mathbb{Z}.$$

(vii) (note Definition A.1 (i))

$$\left| \operatorname{Cov} \left(g_1 \left(\left\langle s^{[1]}, Y_{t_1,T}^{[1]} \right\rangle \right)_{\mathrm{n}}, g_2 \left(\left\langle s^{[2]}, Y_{t_2,T}^{[2]} \right\rangle \right)_{\mathrm{n}} \right) \right|$$
$$\leqslant C \sum_{l=|t_1-t_2|}^{\infty} \Delta_l \left(\left| s^{[1]} \right|_1 \mathbb{1}_{\{t_1>t_2\}} + \left| s^{[2]} \right|_1 \mathbb{1}_{\{t_2>t_1\}} \right) + C \mathbb{1}_{\{t_1=t_2\}}$$
$$\forall\, s^{[1]} \in \mathbb{R}^{d_1},\ s^{[2]} \in \mathbb{R}^{d_2},\ t_1, t_2 \in \{1, \ldots, T\},\ T \in \mathbb{N},\ \mathrm{n} \in \mathbb{N}_0.$$

(viii) (see Definition A.1 (i))

$$\sup_{u,w \in [0,1]} \left| \operatorname{Cov} \left(g_1 \left(\left\langle s^{[1]}, \tilde{Y}_{t_1}^{[1]}(u) \right\rangle \right)_{\mathrm{n}}, g_2 \left(\left\langle s^{[2]}, \tilde{Y}_{t_2}^{[2]}(w) \right\rangle \right)_{\mathrm{n}} \right) \right|$$
$$\leqslant C \sum_{l=|t_1-t_2|}^{\infty} \Delta_l \left(\left| s^{[1]} \right|_1 \mathbb{1}_{\{t_1>t_2\}} + \left| s^{[2]} \right|_1 \mathbb{1}_{\{t_2>t_1\}} \right) + C \mathbb{1}_{\{t_1=t_2\}}$$
$$\forall\, s^{[1]} \in \mathbb{R}^{d_1},\ s^{[2]} \in \mathbb{R}^{d_2},\ t_1, t_2 \in \mathbb{Z},\ \mathrm{n} \in \mathbb{N}_0.$$

Proof. (i) One defines $\mathcal{F}_{t,t-0}^{\times(t-0)} := \varepsilon_t^{\times}$ and $\mathcal{F}_{t,t-l}^{\times(t-l)} := \left(\varepsilon_t, \ldots, \varepsilon_{t-l+1}, \varepsilon_{t-l}^{\times} \right)$ $\forall\, t \in \mathbb{Z}$, $l \in \mathbb{N}$, whereby $\left(\varepsilon_k^{\times} \right)_{k \in \mathbb{Z}}$ originates from Assumption 2.4 [$\mathbf{DM}$]. Then, Lemma B.4 (i) follows similarly to the proofs of Theorem 1 (ii) and (i) in [80, Wu (2005), p. 14151] (note Definition A.1 (i) as well as (2.2)):

$$\begin{aligned}
&\left\| \mathbb{E}\left[X_{t,T} \middle| \mathcal{F}_{t,t-l} \right] - \mathbb{E}\left[X_{t,T} \middle| \mathcal{F}_{t,t-l-1} \right] \right\|_{1+\delta} \\
&= \left\| \mathbb{E}\left[\mathbb{E}\left[X_{t,T}^{\times(t-l-1)} \middle| \mathcal{F}_{t,t-l-1}^{\times(t-l-1)} \right] - \mathbb{E}\left[X_{t,T} \middle| \mathcal{F}_{t,t-l-1} \right] \middle| \mathcal{F}_{t,t-l-1} \right] \right\|_{1+\delta} \\
&\leqslant \left\| \mathbb{E}\left[X_{t,T}^{\times(t-l-1)} - X_{t,T} \middle| \mathcal{F}_{t,t-l}, \varepsilon_{t-l-1}^{\times}, \varepsilon_{t-l-1} \right] \right\|_{1+\delta} \\
&\leqslant \left\| X_{t,T}^{\times(t-l-1)} - X_{t,T} \right\|_{1+\delta} \\
&\leqslant \Delta_{l+1} \quad \forall\, t \in \{1, \ldots, T\},\ T \in \mathbb{N},\ l \in \mathbb{N}_0.
\end{aligned} \tag{B.53}$$

(ii) Lemma B.4 (ii) can be verified analogously to Lemma B.4 (i).

(iii) It follows for all $s \in \mathbb{R}^d$, $t \in \{1, \ldots, T\}$, $T \in \mathbb{N}$, $l \in \mathbb{N}_0$, $q \geqslant 1 + \delta$ from arguments which are similar to those that show (B.53) and by using (3.14):

$$\begin{aligned}
&\left\| \mathbb{E}\left[g_1\left(\langle s, X_{t,T} \rangle \right) \middle| \mathcal{F}_{t,t-l} \right] - \mathbb{E}\left[g_1\left(\langle s, X_{t,T} \rangle \right) \middle| \mathcal{F}_{t,t-l-1} \right] \right\|_q \\
&\leqslant \left\| g_1\left(\left\langle s, X_{t,T}^{\times(t-l-1)} \right\rangle \right) - g_1\left(\langle s, X_{t,T} \rangle \right) \right\|_q \\
&\leqslant C\, \mathbb{E}\left[\left| \left\langle s, X_{t,T}^{\times(t-l-1)} - X_{t,T} \right\rangle \right|^{1+\delta} \right]^{\frac{1+\delta}{q(1+\delta)}} \\
&\leqslant C \left\| X_{t,T}^{\times(t-l-1)} - X_{t,T} \right\|_{1+\delta}^{\frac{1+\delta}{q}} |s|_1^{\frac{1+\delta}{q}}.
\end{aligned} \tag{B.54}$$

This proves Lemma B.4 (iii) (recall (2.2)).

(iv) Lemma B.4 (iv) can be verified analogously to Lemma B.4 (iii).

(v) At first, consider $t_1, t_2 \in \{1, \ldots, T\}$ with $t_1 > t_2$. Lemma B.4 (iii) with $X_{t,T} = Y_{t,T}^{[1]}$ as well as $q = 1 + \delta$ implies for all $s^{[1]} \in \mathbb{R}^{d_1}$, $s^{[2]} \in \mathbb{R}^{d_2}$ (note that $\mathcal{F}_t$ originates from Definition 2.1 and

$\mathcal{F}_{t,t-l}$ from Definition A.1 (i)):

$$\begin{aligned}
&\left|\mathrm{Cov}\left(g_1\left(\left\langle s^{[1]}, Y^{[1]}_{t_1,T}\right\rangle\right), g_2\left(\left\langle s^{[2]}, Y^{[2]}_{t_2,T}\right\rangle\right)\right)\right| \\
&= \left|\mathbb{E}\left[\left(\mathbb{E}\left[g_1\left(\left\langle s^{[1]}, Y^{[1]}_{t_1,T}\right\rangle\right)\middle|\mathcal{F}_{t_1}\right] - \mathbb{E}\left[g_1\left(\left\langle s^{[1]}, Y^{[1]}_{t_1,T}\right\rangle\right)\middle|\mathcal{F}_{t_1,t_2+1}\right]\right) g_2\left(\left\langle s^{[2]}, Y^{[2]}_{t_2,T}\right\rangle\right)\right]\right| \\
&\leqslant \sum_{l=t_1-t_2-1}^{\infty} \left\|\mathbb{E}\left[g_1\left(\left\langle s^{[1]}, Y^{[1]}_{t_1,T}\right\rangle\right)|\mathcal{F}_{t_1,t_1-l}\right] - \mathbb{E}\left[g_1\left(\left\langle s^{[1]}, Y^{[1]}_{t_1,T}\right\rangle\right)|\mathcal{F}_{t_1,t_1-l-1}\right]\right\|_1 \\
&\leqslant C \sum_{l=t_1-t_2-1}^{\infty} \Delta_{l+1} \left|s^{[1]}\right|_1 . \qquad \text{(B.55)}
\end{aligned}$$

For $t_1 \neq t_2$, Lemma B.4 (v) follows from (B.55), shifting the index of a sum and similar arguments in the case $t_2 > t_1$. For $t_1 = t_2$, the proof of Lemma B.4 (v) is trivial.

(vi) Lemma B.4 (vi) can be shown analogously to Lemma B.4 (v).

(vii) At first, consider $t_1, t_2 \in \{1, \ldots, T\}$ with $t_1 > t_2$. It follows for all $s^{[1]} \in \mathbb{R}^{d_1}$, $s^{[2]} \in \mathbb{R}^{d_2}$ similarly to (B.55) by shifting the index of a sum (see Definition A.1 (i)):

$$\begin{aligned}
&\left|\mathrm{Cov}\left(g_1\left(\left\langle s^{[1]}, Y^{[1]}_{t_1,T}\right\rangle\right)_{\mathrm{n}}, g_2\left(\left\langle s^{[2]}, Y^{[2]}_{t_2,T}\right\rangle\right)_{\mathrm{n}}\right)\right| \\
&\leqslant \sum_{l=t_1-t_2-1}^{\infty} \left\|\mathbb{E}\left[g_1\left(\left\langle s^{[1]}, Y^{[1]}_{t_1,T}\right\rangle\right)_{\mathrm{n}}|\mathcal{F}_{t_1,t_1-l}\right] - \mathbb{E}\left[g_1\left(\left\langle s^{[1]}, Y^{[1]}_{t_1,T}\right\rangle\right)_{\mathrm{n}}|\mathcal{F}_{t_1,t_1-l-1}\right]\right\|_1 \\
&= \sum_{l=t_1-t_2-1}^{\infty} \left\|\mathbb{E}\left[\mathbb{E}\left[g_1\left(\left\langle s^{[1]}, Y^{[1]}_{t_1,T}\right\rangle\right)|\mathcal{F}_{t_1,t_1-l}\right] - \mathbb{E}\left[g_1\left(\left\langle s^{[1]}, Y^{[1]}_{t_1,T}\right\rangle\right)|\mathcal{F}_{t_1,t_1-l-1}\right]|\mathcal{F}_{t_1,t_1-\mathrm{n}}\right]\right\|_1 \\
&\leqslant C \sum_{l=t_1-t_2}^{\infty} \Delta_l \left|s^{[1]}\right|_1 .
\end{aligned}$$

This and analog arguments yield Lemma B.4 (vii) in the case $t_1 \neq t_2$. For $t_1 = t_2$, the proof of Lemma B.4 (vii) is trivial.

(viii) Lemma B.4 (viii) follows similarly to Lemma B.4 (vii).

□

C. Appendix to Chapter 3

C.1. Proofs of the statements given in Chapter 3

Proof of Lemma 3.4. At first, one obtains for $\breve{S} := 1/\big(2d\widetilde{C}\big)$ $\big($whereby $\widetilde{C} \in (0,\infty)$ originates from (B.43) and, according to Definition 2.1, d denotes the dimension of $\widetilde{X}_0(u)\big)$ by using (B.43) as well as $\varphi(u,0) = 1\ \forall\, u \in [0,1]$:

$$\inf_{s\in\left[-\breve{S},\breve{S}\right]^d} \inf_{u\in[0,1]} |\varphi(u,s)| \geqslant \inf_{s\in\left[-\breve{S},\breve{S}\right]^d} \inf_{u\in[0,1]} \left(-|\varphi(u,0)-\varphi(u,s)| + |\varphi(u,0)|\right) \geqslant \frac{1}{2}. \qquad \text{(C.1)}$$

It follows from (C.1) and Assumption 3.1 $[\mathbf{WEI.1}]$ (note Definition 3.3 (ii)):

$$\mathbb{D}_1 \geqslant \int_{\left[-\breve{S},\breve{S}\right]^d} \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} |\varphi(u,s)|^2\, du\, \mathbf{w}(s)\, ds \geqslant \left(\inf_{s\in\left[-\breve{S},\breve{S}\right]^d} \inf_{u\in[0,1]} |\varphi(u,s)|\right)^2 (\mathfrak{U}_1 - \mathfrak{U}_0) \int_{\left[-\breve{S},\breve{S}\right]^d} \mathbf{w}(s)\, ds > 0,$$

which proves Lemma 3.4. □

Proof of Proposition 3.6. (i) At first, one defines for all $\mathrm{R} \in \{\Re, \Im\}$:

$$f^{\mathrm{opt}}_{\mathfrak{U}_{0,1},\mathrm{R}} \colon \mathbb{R}^d \to \mathbb{C},\ s \mapsto \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \mathrm{R}\left\{\varphi(w,s)\right\}\, dw \tag{C.2}$$

and observes:

$$\int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \mathrm{R}\left\{\varphi(u,s)\right\}\, du = \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} f^{\mathrm{opt}}_{\mathfrak{U}_{0,1},\mathrm{R}}(s)\, du \quad \forall\, \mathrm{R} \in \{\Re, \Im\},\ s \in \mathbb{R}^d. \tag{C.3}$$

It follows for all $f \in \mathscr{C}^0\big(\mathbb{R}^d\big) := \big\{f \colon \mathbb{R}^d \to \mathbb{C} \mid f \text{ is continuous}\big\}$ from (C.3) (recall (C.2) and (3.6)):

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re\left\{\varphi(u,s) - f(s)\right\}^2 du\, \mathbf{w}(s)\, ds \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re\left\{\varphi(u,s)\right\}^2 du\, \mathbf{w}(s)\, ds + \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re\left\{f(s)\right\}^2 du\, \mathbf{w}(s)\, ds - 2 \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} f^{\mathrm{opt}}_{\mathfrak{U}_{0,1},\Re}(s)\, du\, \Re\left\{f(s)\right\}\, \mathbf{w}(s)\, ds \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re\left\{\varphi(u,s)\right\}^2 + \Re\left\{f(s)\right\}^2 - 2 f^{\mathrm{opt}}_{\mathfrak{U}_{0,1},\Re}(s) \Re\left\{f(s)\right\} du\, \mathbf{w}(s)\, ds \\
&+ \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Big(f^{\mathrm{opt}}_{\mathfrak{U}_{0,1},\Re}(s)\Big)^2 + \Big(f^{\mathrm{opt}}_{\mathfrak{U}_{0,1},\Re}(s)\Big)^2 du\, \mathbf{w}(s)\, ds - 2 \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re\left\{\varphi(u,s)\right\}\, du\, f^{\mathrm{opt}}_{\mathfrak{U}_{0,1},\Re}(s)\, \mathbf{w}(s)\, ds \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re\Big\{\varphi(u,s) - f^{\mathrm{opt}}_{\mathfrak{U}_{0,1}}(s)\Big\}^2 du\, \mathbf{w}(s)\, ds + \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re\Big\{f(s) - f^{\mathrm{opt}}_{\mathfrak{U}_{0,1}}(s)\Big\}^2 du\, \mathbf{w}(s)\, ds.
\end{aligned} \tag{C.4}$$

One obtains for all $f \in \mathscr{C}^0\big(\mathbb{R}^d\big)$ from (C.4) and similar arguments:

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \left|\varphi(u,s) - f(s)\right|^2 du\, \mathbf{w}(s)\, ds \\
&\quad = \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Big|\varphi(u,s) - f^{\mathrm{opt}}_{\mathfrak{U}_{0,1}}(s)\Big|^2 du\, \mathbf{w}(s)\, ds + \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Big|f(s) - f^{\mathrm{opt}}_{\mathfrak{U}_{0,1}}(s)\Big|^2 du\, \mathbf{w}(s)\, ds.
\end{aligned} \tag{C.5}$$

Since the first addend on the right side of (C.5) does not depend on $f \in \mathscr{C}^0\big(\mathbb{R}^d\big)$, it is easy to see that the unique minimizer of (C.5) with respect to $f \in \mathscr{C}^0\big(\mathbb{R}^d\big)$ is $f = f^{\mathrm{opt}}_{\mathfrak{U}_{0,1}}$ (note that, according to Assumption 3.1 $[\mathbf{WEI}.\mathbf{1}]$, $\mathbf{w} > 0$ λ-a.e.). In addition, one obtains for all $\mathrm{R} \in \{\Re, \Im\}$ (see (3.6) as well as (C.2)):

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \mathrm{R}\Big\{\varphi(u,s) - f^{\mathrm{opt}}_{\mathfrak{U}_{0,1}}(s)\Big\}^2 du\, \mathbf{w}(s)\, ds \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \mathrm{R}\left\{\varphi(u,s)\right\}^2 du + \left(\mathfrak{U}_1 - \mathfrak{U}_0\right) \Big(f^{\mathrm{opt}}_{\mathfrak{U}_{0,1},\mathrm{R}}(s)\Big)^2 - 2\left(\mathfrak{U}_1 - \mathfrak{U}_0\right) \Big(f^{\mathrm{opt}}_{\mathfrak{U}_{0,1},\mathrm{R}}(s)\Big)^2\, \mathbf{w}(s)\, ds \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \mathrm{R}\left\{\varphi(u,s)\right\}^2 du - \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \mathrm{R}\left\{\int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \varphi(w,s)\, dw\right\}^2 \mathbf{w}(s)\, ds,
\end{aligned} \tag{C.6}$$

which finishes the proof of Proposition 3.6 (i) due to $|x|^2 = \Re\{x\}^2 + \Im\{x\}^2 \;\forall\, x \in \mathbb{C}$ (note Definition 3.3 (i) and (ii)).

(ii) Proposition 3.6 (i) and the equation $|x|^2 = x\overline{x} \;\forall\, x \in \mathbb{C}$ imply (recall Definition 3.3 (ii)):

$$\begin{aligned}
\mathbb{D} &= \mathbb{D}_1 - \mathbb{D}_2 \\
&= \int\limits_{\mathbb{R}^d} \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \frac{\varphi(u,s)\,\overline{\varphi(u,s)} + \varphi(w,s)\,\overline{\varphi(w,s)}}{2}\, du\, dw \\
&\quad - \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \frac{\varphi(u,s)\,\overline{\varphi(w,s)} + \overline{\varphi(u,s)}\varphi(w,s)}{2}\, du\, dw\, \mathbf{w}(s)\, ds \\
&= \frac{1}{2\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)} \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Big(\varphi(u,s)\,\overline{\varphi(u,s)} - \varphi(u,s)\,\overline{\varphi(w,s)} - \overline{\varphi(u,s)}\varphi(w,s) + \varphi(w,s)\,\overline{\varphi(w,s)}\Big)\, du\, dw \\
&\quad \cdot \mathbf{w}(s)\, ds,
\end{aligned}$$

which shows Proposition 3.6 (ii) due to $x\overline{x} - x\overline{y} - \overline{x}y + y\overline{y} = |x-y|^2 \;\forall\, x, y \in \mathbb{C}$.

(iii) Proposition 3.6 (i) yields (see Definition 3.3 (i)):

$$\mathbb{D} = \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \left|\varphi(u,s) - f^{\text{opt}}_{\mathfrak{U}_{0,1}}(s)\right|^2 du\, \mathbf{w}(s)\, ds$$

with $f^{\text{opt}}_{\mathfrak{U}_{0,1}} \in \mathscr{C}^0\big(\mathbb{R}^d\big)$, such that Assumption 3.1 $[\mathbf{WEI.1}]$ $\big($in particular, $\mathbf{w} > 0$ λ-a.e.$\big)$ implies that $\mathbb{D} = 0$ iff $\varphi(u,s) = f^{\text{opt}}_{\mathfrak{U}_{0,1}}(s) \;\forall\, u \in [\mathfrak{U}_0, \mathfrak{U}_1]$, $s \in \mathbb{R}^d$. This proves Proposition 3.6 (iii).

(iv) Lemma 3.4 ensures $1/\mathbb{D}_1 > 0$ and Proposition 3.6 (i) provides $\mathbb{D}^{\text{norm}} = \mathbb{D}/\mathbb{D}_1$ (recall Definition 3.3 (ii)). Therefore, $\mathbb{D}^{\text{norm}} = 0$ will be valid iff $\mathbb{D} = 0$. Thus, Proposition 3.6 (iv) follows from Proposition 3.6 (iii). □

Proof of Lemma 3.9. At first, it will be shown that absolute constants $\check{K} > 0$ and $\check{T} \in \mathbb{N}$ exist that fulfil (note that u_1 originates from Definition 3.8 (i)):

$$\text{For all } T \in \mathbb{N} \text{ with } T \geqslant \check{T} \text{ exists } \check{t}_T \in \{1, \ldots, T\} \text{ for which } K\left(\frac{\frac{\check{t}_T}{T} - u_1}{b}\right) \geqslant \check{K} > 0. \tag{C.7}$$

Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) provides that fixed $\check{z} \in (0, \mathfrak{U}_1 - \mathfrak{U}_0)$ and $\check{K} > 0$ exist which fulfil:

$$K\left(\check{z}\right) \geqslant 2\check{K} > 0. \tag{C.8}$$

It follows from Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) that $Tb \to \infty$ for $T \to \infty$. In addition, $u_1 T \geqslant 0$ holds obviously. Hence, $\check{z} > 0$ ensures $1 \leqslant \lfloor u_1 T + \check{z} Tb \rfloor$ for sufficiently large values of T. Moreover, Definition 3.8 (i) and $1/\left(2\lfloor 1/(2b)\rfloor\right) \geqslant b$ imply $u_1 T \leqslant u_{\lfloor 1/(2b)\rfloor} T \leqslant \mathfrak{U}_0 T + (\mathfrak{U}_1 - \mathfrak{U}_0)(T - Tb)$. Thus, $\check{z} < \mathfrak{U}_1 - \mathfrak{U}_0$ shows $\lfloor u_1 T + \check{z} Tb \rfloor \leqslant \lfloor \mathfrak{U}_1 T \rfloor \leqslant T$. Further, $Tb \to \infty$ provides $-L_K/(Tb) + 2\check{K} \geqslant \check{K}$ for sufficiently large values of T, whereby (according to Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i)) L_K denotes the Lipschitz constant of K. Overall, one obtains that a $\check{T} \in \mathbb{N}$ exists for which the following statement is valid:

$$\text{For all } T \in \mathbb{N} \text{ with } T \geqslant \check{T} \text{ holds: } \check{t}_T := \lfloor u_1 T + \check{z} Tb \rfloor \text{ fulfils } 1 \leqslant \check{t}_T \leqslant T \text{ and } -\frac{L_K}{Tb} + 2\check{K} \geqslant \check{K}. \tag{C.9}$$

In conclusion, Assumption 2.8 [$\mathbf{K\&b.1}$] (i), (C.8) and (C.9) imply for all $T \geqslant \breve{T}$:

$$K\left(\frac{\frac{\breve{t}_T}{T} - u_1}{b}\right) \geqslant -\left|K\left(\frac{\breve{z}Tb}{Tb}\right) - K\left(\frac{\breve{t}_T - u_1 T}{Tb}\right)\right| + K\left(\breve{z}\right)$$
$$\geqslant -L_K\left(\frac{(u_1T + \breve{z}Tb) - \lfloor u_1 T + \breve{z}Tb\rfloor}{Tb}\right) + K\left(\breve{z}\right)$$
$$\geqslant -\frac{L_K}{Tb} + 2\breve{K} \geqslant \breve{K} > 0,$$

such that (C.7) is proved.

In order to derive Lemma 3.9 from (C.7), one selects an arbitrary $\omega \in \Omega$, whereby Ω is the sample space from Definition 2.1. It follows for all $s \in \mathbb{R}^d$ and for an absolute constant $\breve{C} \in (0, \infty)$, which does not depend on ω and s, by using $|x|^2 = x\overline{x}\ \forall\, x \in \mathbb{C}$ (recall Definition 2.11):

$$\left|\left|\widehat{\varphi}\left(u_1, 0\right)\big|_\omega\right|^2 - \left|\widehat{\varphi}\left(u_1, s\right)\big|_\omega\right|^2\right|$$
$$\leqslant \frac{1}{T^2}\sum_{t_1,t_2=1}^{T} K_b\left(\frac{t_1}{T} - u_1\right) K_b\left(\frac{t_2}{T} - u_1\right)\left|e^{i\left\langle 0, X_{t_1,T}(\omega) - X_{t_2,T}(\omega)\right\rangle} - e^{i\left\langle s, X_{t_1,T}(\omega) - X_{t_2,T}(\omega)\right\rangle}\right|$$
$$\leqslant \frac{\breve{C}}{T^2}\sum_{t_1,t_2=1}^{T} K_b\left(\frac{t_1}{T} - u_1\right) K_b\left(\frac{t_2}{T} - u_1\right)\left|X_{t_1,T}(\omega) - X_{t_2,T}(\omega)\right|_1 |s|_1 . \tag{C.10}$$

Moreover, one defines for the selected $\omega \in \Omega$:

$$\mathscr{S}_T(\omega) := \left\{ s \in \mathbb{R}^d : |s|_1 \leqslant \frac{\left|\widehat{\varphi}\left(u_1, 0\right)\big|_\omega\right|^2}{2\breve{C}}\left(1 + \frac{1}{T^2}\sum_{t_1,t_2=1}^{T} K_b\left(\frac{t_1}{T} - u_1\right) K_b\left(\frac{t_2}{T} - u_1\right) \cdot \left|X_{t_1,T}(\omega) - X_{t_2,T}(\omega)\right|_1\right)^{-1}\right\}. \tag{C.11}$$

It follows from (C.7) that $\left|\widehat{\varphi}\left(u_1, 0\right)\big|_\omega\right| > 0\ \forall\, T \geqslant \breve{T}$. Thus, $\lambda\left(\mathscr{S}_T(\omega)\right) > 0$ holds for all $T \geqslant \breve{T}$. Assumption 3.1 [$\mathbf{WEI.1}$], (C.10), $\lambda\left(\mathscr{S}_T(\omega)\right) > 0$ and (C.7) provide for all $T \geqslant \breve{T}$:

$$\int_{\mathbb{R}^d} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \left|\widehat{\varphi}\left(u_k, s\right)\big|_\omega\right|^2 \mathbf{w}(s)\, ds \geqslant \int_{\mathscr{S}_T(\omega)} \left|\widehat{\varphi}\left(u_1, s\right)\big|_\omega\right|^2 \mathbf{w}(s)\, ds$$
$$\geqslant -\int_{\mathscr{S}_T(\omega)} \left|\left|\widehat{\varphi}\left(u_1, 0\right)\big|_\omega\right|^2 - \left|\widehat{\varphi}\left(u_1, s\right)\big|_\omega\right|^2\right| \mathbf{w}(s)\, ds + \int_{\mathscr{S}_T(\omega)} \left|\widehat{\varphi}\left(u_1, 0\right)\big|_\omega\right|^2 \mathbf{w}(s)\, ds$$
$$\geqslant \frac{1}{2}\int_{\mathscr{S}_T(\omega)} \left|\widehat{\varphi}\left(u_1, 0\right)\big|_\omega\right|^2 \mathbf{w}(s)\, ds \geqslant \frac{1}{2T^2b^2} K\left(\frac{\frac{\breve{t}_T}{T} - u_1}{b}\right)^2 \int_{\mathscr{S}_T(\omega)} \mathbf{w}(s)\, ds > 0, \tag{C.12}$$

which proves Lemma 3.9 (see Definition 3.8 (i)). □

Proof of Proposition 3.11. (i) It follows from $|x|^2 = x\overline{x}\ \forall\, x \in \mathbb{C}$ (note Definition 3.8 (i)):

$$\widehat{\mathbb{D}}_T = \widehat{\mathbb{D}}_{T,1} - \widehat{\mathbb{D}}_{T,2}$$
$$= \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\widehat{\varphi}\left(u_k, s\right)\overline{\widehat{\varphi}\left(u_k, s\right)} - \widehat{\varphi}\left(u_k, s\right)\frac{1}{\lfloor 1/(2b)\rfloor}\sum_{k_2=1}^{\lfloor 1/(2b)\rfloor}\overline{\widehat{\varphi}\left(u_{k_2}, s\right)}\right)\mathbf{w}(s)\, ds$$
$$= \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\widehat{\varphi}\left(u_k, s\right)\overline{\widehat{\varphi}\left(u_k, s\right)} - \widehat{\varphi}\left(u_k, s\right)\frac{1}{\lfloor 1/(2b)\rfloor}\sum_{k_2=1}^{\lfloor 1/(2b)\rfloor}\overline{\widehat{\varphi}\left(u_{k_2}, s\right)}\right.$$

$$- \overline{\widehat{\varphi}\left(u_k, s\right)} \frac{1}{\lfloor 1/(2b) \rfloor} \sum_{k_2=1}^{\lfloor 1/(2b) \rfloor} \widehat{\varphi}\left(u_{k_2}, s\right) + \frac{1}{\lfloor 1/(2b) \rfloor^2} \sum_{k_1,k_2=1}^{\lfloor 1/(2b) \rfloor} \widehat{\varphi}\left(u_{k_1}, s\right) \overline{\widehat{\varphi}\left(u_{k_2}, s\right)} \Bigg) \mathbf{w}(s)\, ds,$$

which implies Proposition 3.11 (i) due to $x\overline{x} - x\overline{y} - \overline{x}y + y\overline{y} = |x-y|^2 \; \forall\, x, y \in \mathbb{C}$.

(ii) Proposition 3.11 (ii) can be shown similarly to the proof of Proposition 3.6 (ii).

□

Proof of Lemma 3.12. In the following, Lemma 3.12 with $\mathrm{R}_1 = \Re$ and $\mathrm{R}_2 = \Im$ will be verified. Obviously, one obtains for all $s_1, s_2 \in \mathbb{R}^d$:

$$\sup_{u \in [0,1]} \left| \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u) \right\rangle \right), \sin\left(\left\langle s_2, \widetilde{X}_0(u) \right\rangle \right) \right) \right| \leqslant C. \tag{C.13}$$

Moreover, it holds for all $s_1, s_2 \in \mathbb{R}^d$ due to Lemma B.4 (vi) and (B.45):

$$\sum_{t=1}^{\infty} \sup_{u \in [0,1]} \left| \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u) \right\rangle \right), \sin\left(\left\langle s_2, \widetilde{X}_t(u) \right\rangle \right) \right) \right| \leqslant C \left| s_2 \right|_1. \tag{C.14}$$

Further, one obtains similarly to (B.45):

$$\sum_{t=-\infty}^{-1} \sum_{l=-t}^{\infty} \Delta_l = \sum_{t=-\infty}^{-1} \sum_{l=1}^{\infty} \mathbb{1}_{\{l \geqslant -t\}} \Delta_l = \sum_{l=1}^{\infty} \sum_{t=-\infty}^{-1} \mathbb{1}_{\{-l \leqslant t\}} \Delta_l = \sum_{l=1}^{\infty} \Delta_l l \leqslant C. \tag{C.15}$$

Lemma B.4 (vi) and (C.15) imply for all $s_1, s_2 \in \mathbb{R}^d$:

$$\sum_{t=-\infty}^{-1} \sup_{u \in [0,1]} \left| \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u) \right\rangle \right), \sin\left(\left\langle s_2, \widetilde{X}_t(u) \right\rangle \right) \right) \right| \leqslant C \left| s_1 \right|_1. \tag{C.16}$$

Lemma 3.12 with $\mathrm{R}_1 = \Re$ and $\mathrm{R}_2 = \Im$ follows from (C.13), (C.14) as well as (C.16). The other versions of Lemma 3.12 (with the other choices of R_1 and R_2) can be proved similarly. □

Proof of Theorem 3.13. (i) Throughout this proof, it is supposed that T is large enough to ensure $2\lfloor T(\mathfrak{U}_1 - \mathfrak{U}_0)b \rfloor - 1 - \mathscr{m} \geqslant 1 + \mathscr{m}$ (see Definition A.1 (ii)), which holds for sufficiently large T due to Remark A.2 (i).
Further, one defines for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $t \in \mathbb{Z}$, $s \in \mathbb{R}^d$, $\mathrm{R} \in \{\Re, \Im\}$, $j \in \{1, 2\}$ (recall the Definitions 3.3 (i), 3.8 (i), A.1 (ii) as well as (i), (3.16) and that $X^c := X - \mathbb{E}[X]$ for all random variables X with finite first moment):

$$T_{\mathfrak{U}} := T_{\mathfrak{U}_{0,1},\mathfrak{U}} := T\left(\mathfrak{U}_1 - \mathfrak{U}_0\right), \quad \widetilde{u}_{k,t} := \widetilde{u}_{T,\mathfrak{U}_{0,1},k,t} := u_k + \frac{t - \lfloor T_{\mathfrak{U}} b \rfloor}{T}, \quad \gamma_1 := (1,0), \quad \gamma_2 := (0,1),$$

$$\varphi^{\circ}_{\mathscr{m}}(u_k, s) := \varphi^{\circ}_{T,\mathfrak{U}_{0,1},\mathscr{m}}(u_k, s) := \frac{1}{\lfloor Tb \rfloor} \sum_{t=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \mathscr{m}} K\left(\frac{t - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} \left(\mathfrak{U}_1 - \mathfrak{U}_0\right) \right) \left(e^{i \left\langle s, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t}\left(\widetilde{u}_{k,t}\right) \right\rangle} \right)_{\mathscr{m}},$$

$$\mathbb{D}^{\circ}_{T,k,j,\mathrm{R}} := \mathbb{D}^{\circ}_{T,\mathfrak{U}_{0,1},k,j,\mathrm{R}} := \frac{2\sqrt{T}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)}{\lfloor 1/(2b) \rfloor} \int_{\mathbb{R}^d} \mathrm{R}\left\{ \varphi^{\circ}_{\mathscr{m}}(u_k, s) \right\}^c \cdot \tau_{\mathfrak{U}_{0,1},\mathrm{R}}\left(\gamma_j, u_k, s\right) \mathbf{w}(s)\, ds,$$

$$\mathbb{D}^{\circ}_{T,k,j} := \mathbb{D}^{\circ}_{T,\mathfrak{U}_{0,1},k,j} := \mathbb{D}^{\circ}_{T,k,j,\Re} + \mathbb{D}^{\circ}_{T,k,j,\Im} \quad \text{and} \quad \mathbb{D}^{\circ}_{T,j} := \mathbb{D}^{\circ}_{T,\mathfrak{U}_{0,1},j} := \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{D}^{\circ}_{T,k,j}. \tag{C.17}$$

Thereby, it holds for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $t \in \{1, \ldots, 2\lfloor T_{\mathfrak{U}} b \rfloor\}$ (see Definition 3.8 (i)):

$$\mathfrak{U}_0 \leqslant \mathfrak{U}_0 + \left(1 - \frac{1}{2}\right) \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{1/(2b)} + \frac{1 - T_{\mathfrak{U}} b}{T} \leqslant \widetilde{u}_{k,t} \quad \text{and}$$

$$\widetilde{u}_{k,t} \leqslant \mathfrak{U}_0 + \left(\left\lfloor \frac{1}{2b} \right\rfloor - \frac{1}{2}\right) \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} + \frac{2\lfloor T_{\mathfrak{U}} b \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor}{T} \leqslant \mathfrak{U}_0 + \left(1 - \frac{1/2}{1/(2b)}\right)(\mathfrak{U}_1 - \mathfrak{U}_0) + \frac{T_{\mathfrak{U}} b}{T} = \mathfrak{U}_1, \tag{C.18}$$

such that $\varphi^{\circ}_{\mathscr{m}}(u_k, s)$ just takes $\widetilde{X}_r(u)$ with $u \in [0,1]$ (and, obviously, $r \in \mathbb{Z}$) into account. Therefore, $\varphi^{\circ}_{\mathscr{m}}(u_k, s)$ is well-defined for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $s \in \mathbb{R}^d$.
The Lemmata C.1 (regard that $\mathbb{D}_j = \mathbb{D}_{j,\Re} + \mathbb{D}_{j,\Im}$ and $\widehat{\mathbb{D}}_{T,j} = \widehat{\mathbb{D}}_{T,j,\Re} + \widehat{\mathbb{D}}_{T,j,\Im}\ \forall\, j \in \{1,2\}$ follow from the Definitions 3.3 (ii), 3.8 (i) and (C.187)), C.2 as well as C.5 provide for $T \to \infty$ (recall (C.17)):

$$\left\| \sqrt{T} \begin{pmatrix} \widehat{\mathbb{D}}_{T,1} - \mathbb{D}_1 \\ \widehat{\mathbb{D}}_{T,2} - \mathbb{D}_2 \end{pmatrix} - \begin{pmatrix} \mathbb{D}^{\circ}_{T,1} \\ \mathbb{D}^{\circ}_{T,2} \end{pmatrix} \right\|_1 = o(1). \tag{C.19}$$

Hence, Theorem 3.13 (i) will be proved if the following statement holds for $T \to \infty$:

$$\begin{pmatrix} \mathbb{D}^{\circ}_{T,1} \\ \mathbb{D}^{\circ}_{T,2} \end{pmatrix} \xrightarrow{\mathrm{d}} Z^{\text{joint}}_{\mathfrak{U}_{0,1}}. \tag{C.20}$$

From now on, the validity of the assumptions demanded in Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274 et seq.] will be examined in order to use this theorem to verify (C.20), whereby the expression $\mathbb{D}^{\circ}_{T,k,j}$ takes the role of $X_{n,k,j}$ which originates from Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274 et seq.].

Obviously, $\mathbb{E}\big[\mathbb{D}^{\circ}_{T,k,j}\big] = 0\ \forall\, k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $j \in \{1,2\}$, such that the first assumption of Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274] holds.
Further, $\lfloor x \rfloor \lfloor y \rfloor \lfloor z \rfloor \leqslant \lfloor xyz \rfloor\ \forall\, x,y,z \geqslant 0$ implies (see (C.17)):

$$\left\lfloor \frac{T_{\mathfrak{U}}}{\lfloor 1/(2b) \rfloor} \right\rfloor - \lfloor T_{\mathfrak{U}} b \rfloor \geqslant \left\lfloor \frac{\lfloor 2 \rfloor \lfloor T_{\mathfrak{U}} b \rfloor \lfloor 1/(2b) \rfloor}{\lfloor 1/(2b) \rfloor} \right\rfloor - \lfloor T_{\mathfrak{U}} b \rfloor = \lfloor T_{\mathfrak{U}} b \rfloor. \tag{C.21}$$

One obtains for all $k_1, k_2 \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $t_1, t_2 \in \{1 + \mathscr{m}, \ldots, 2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \mathscr{m}\}$ with $k_1 \geqslant k_2 + 1$ due to $\lfloor x \rfloor + \lfloor y \rfloor \leqslant \lfloor x + y \rfloor\ \forall\, x, y \geqslant 0$ and (C.21) (recall Definition 3.8 (i)):

$$\begin{aligned} \lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1 &\geqslant \left\lfloor \mathfrak{U}_0 T + \frac{k_1 - \frac{3}{2}}{\lfloor 1/(2b) \rfloor} T_{\mathfrak{U}} \right\rfloor + \left\lfloor \frac{T_{\mathfrak{U}}}{\lfloor 1/(2b) \rfloor} \right\rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + 1 + \mathscr{m} \\ &\geqslant \lfloor u_{k_2} T \rfloor + \lfloor T_{\mathfrak{U}} b \rfloor + 1 + \mathscr{m} \\ &\geqslant \lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2 + 2 + 2\mathscr{m}. \end{aligned} \tag{C.22}$$

Moreover, $\mathbb{D}^{\circ}_{T,k,j}$ (with $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $j \in \{1,2\}$) is measurable with respect to the sigma algebra generated by $\mathcal{F}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + 2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \mathscr{m}, \lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + 1}$ (see (C.17), Definition A.1 (i) and (3.16)). Thus, it follows from (C.22) with $t_1 = 1 + \mathscr{m}$ and $t_2 = 2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \mathscr{m}$:

$$\left(\left(\mathbb{D}^{\circ}_{T,k,1}, \mathbb{D}^{\circ}_{T,k,2}\right)'\right)_{k=1}^{\lfloor 1/(2b) \rfloor} \text{ is a sequence of independent random variables,} \tag{C.23}$$

which implies for all $j \in \{1,2\}$ by using $\mathbb{E}\big[\mathbb{D}^{\circ}_{T,k,j}\big] = 0\ \forall\, k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$:

$$\operatorname{Var}\left(\mathbb{D}^{\circ}_{T,j}\right) = \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left(\mathbb{D}^{\circ}_{T,k,j}\right)^2\right].$$

Hence, one obtains for all $j \in \{1,2\}$ from Corollary C.7, Lemma 3.12 and Assumption 3.1 [**WEI.1**], whereby the latter two imply $\sigma_{\mathfrak{U}_{0,1}}(\gamma_j, \gamma_j) < \infty$ (recall (3.18), (3.16) as well as (3.17)) that the sequence $\left(\sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{E}\big[\left(\mathbb{D}^{\circ}_{T,k,j}\right)^2\big]\right)_{T \in \mathbb{N}}$ converges for $T \to \infty$, such that it is bounded from above by an absolute constant $\nu_0 < \infty$. Thus, the second assumption of Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274] holds.
Moreover, Corollary C.7 implies (6.25) in [52, Leucht and Neumann (2013), p. 274] with $\Sigma = \Sigma_{\mathfrak{U}_{0,1}}$, whereby Σ originates from [52, Leucht and Neumann (2013), p. 274] (see (C.17) and (3.19)).

Next, the validity of (6.26) in [52, Leucht and Neumann (2013), p. 275] is proved. Therefor, note at first that the Cauchy–Schwarz inequality and Markov's inequality provide for each real-valued random variable X with finite fourth moments and all $\epsilon > 0$:

$$\mathbb{E}\left[X^2 \mathbb{1}_{\{|X|>\epsilon\}}\right] = \mathbb{E}\left[|X|^2 \mathbb{1}_{\{|X|^4>\epsilon^4\}}\right] \leqslant \mathbb{E}\left[|X|^4\right]^{\frac{1}{2}} \mathbb{P}\left(|X|^4 > \epsilon^4\right)^{\frac{1}{2}} \leqslant \frac{1}{\epsilon^2}\mathbb{E}\left[|X|^4\right]. \tag{C.24}$$

In addition, the Cauchy-Schwarz inequality for sums yields:

$$|z_1 + \cdots + z_M|^2 \leqslant M |z_1|^2 + \cdots + M |z_M|^2 \quad \forall\, z_1, \ldots, z_M \in \mathbb{C},\ M \in \mathbb{N}, \tag{C.25}$$

which implies (in the case $M = 2$):

$$\left(x_1 + x_2\right)^4 \leqslant \left(2x_1^2 + 2x_2^2\right)^2 \leqslant 8x_1^4 + 8x_2^4 \quad \forall\, x_1, x_2 \in \mathbb{R}. \tag{C.26}$$

One obtains for all $j \in \{1, 2\}$, $\epsilon > 0$ from (C.24) and (C.26) (recall (C.17)):

$$\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left(\mathbb{D}^{\circ}_{T,k,j}\right)^2 \mathbb{1}_{\{|\mathbb{D}^{\circ}_{T,k,j}|>\epsilon\}}\right] \leqslant \frac{C}{\epsilon^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left(\mathbb{D}^{\circ}_{T,k,j,\Re}\right)^4\right] + \frac{C}{\epsilon^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left(\mathbb{D}^{\circ}_{T,k,j,\Im}\right)^4\right]. \tag{C.27}$$

In the following, it is shown that the first sum on the right side of (C.27) converges to zero. This and similar arguments together with (C.27) yield that (6.26) in [52, Leucht and Neumann (2013), p. 275] holds. Therefor, one defines for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $\underline{s} := (s_1, s_2, s_3, s_4)$, $\underline{t} := (t_1, t_2, t_3, t_4)$ with $s_1, \ldots, s_4 \in \mathbb{R}^d$ as well as $t_1, \ldots, t_4 \in \{1, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor\}$:

$$\begin{aligned}\Gamma_{T,k}\left(\underline{s}, \underline{t}\right) := \mathbb{E}\Big[&\cos\left(\left\langle s_1, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1}\left(\widetilde{u}_{k,t_1}\right)\right\rangle\right)^c_{\mathscr{m}} \cos\left(\left\langle s_2, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2}\left(\widetilde{u}_{k,t_2}\right)\right\rangle\right)^c_{\mathscr{m}} \\ &\cdot \cos\left(\left\langle s_3, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_3}\left(\widetilde{u}_{k,t_3}\right)\right\rangle\right)^c_{\mathscr{m}} \cos\left(\left\langle s_4, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_4}\left(\widetilde{u}_{k,t_4}\right)\right\rangle\right)^c_{\mathscr{m}}\Big].\end{aligned} \tag{C.28}$$

If a $p_1 \in \{1, \ldots, 4\}$ exists with $|t_{p_1} - t_{p_2}| > \mathscr{m}\, \forall\, p_2 \in \{1, \ldots, 4\} \setminus \{p_1\}$, it will hold (recall Definition A.1 (i)):

$$\Gamma_{T,k}\left(\underline{s}, \underline{t}\right) = 0. \tag{C.29}$$

One obtains for all $j \in \{1, 2\}$ due to (C.29) (see (C.17) and (3.16)):

$$\begin{aligned}\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left(\mathbb{D}^{\circ}_{T,k,j,\Re}\right)^4\right] \leqslant \frac{C}{T^2} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2,t_3,t_4=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}} \left|\Gamma_{T,k}\left(\underline{s}, \underline{t}\right)\right| \\ &\cdot \mathbb{1}_{\{\forall\, p_1 \in \{1,\ldots,4\} \exists p_2 \in \{1,\ldots,4\}\setminus\{p_1\}: \left|t_{p_1} - t_{p_2}\right| \leqslant \mathscr{m}\}}\, \mathbf{w}\left(s_1\right)\, ds_1\, \mathbf{w}\left(s_2\right)\, ds_2\, \mathbf{w}\left(s_3\right)\, ds_3 \\ &\cdot\ \mathbf{w}\left(s_4\right)\, ds_4.\end{aligned} \tag{C.30}$$

It holds for all $t_1, \ldots, t_4 \in \{1 + \mathscr{m}, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}\}$:

$$\begin{aligned}&\mathbb{1}_{\{\forall\, p_1 \in \{1,\ldots,4\}\, \exists p_2 \in \{1,\ldots,4\}\setminus\{p_1\}: \left|t_{p_1} - t_{p_2}\right| \leqslant \mathscr{m}\}} \\ &\leqslant \mathbb{1}_{\{|t_1 - t_2| \leqslant \mathscr{m}\}} + \mathbb{1}_{\{|t_1 - t_3| \leqslant \mathscr{m}\}} + \mathbb{1}_{\{|t_1 - t_4| \leqslant \mathscr{m}\}} + \mathbb{1}_{\{|t_2 - t_3| \leqslant \mathscr{m}\}} + \mathbb{1}_{\{|t_2 - t_4| \leqslant \mathscr{m}\}} + \mathbb{1}_{\{|t_3 - t_4| \leqslant \mathscr{m}\}}.\end{aligned} \tag{C.31}$$

Further, one observes for all $\underline{s} := (s_1, \ldots, s_4) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{aligned}&\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2,t_3,t_4=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}} \left|\Gamma_{T,k}\left(\underline{s}, \underline{t}\right)\right| \mathbb{1}_{\{|t_1 - t_2| \leqslant \mathscr{m}\}} \\ &\leqslant \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2,t_3,t_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left|\Gamma_{T,k}\left(\underline{s}, \underline{t}\right)\right| \mathbb{1}_{\{|t_1 - t_2| \leqslant \mathscr{m}\}} \mathbb{1}_{\{|t_3 - t_1| \leqslant 2\mathscr{m}\}}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2,t_3,t_4=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\left|\Gamma_{T,k}\left(\underline{s},\underline{t}\right)\right|\mathbb{1}_{\{|t_1-t_2|\leqslant \mathscr{m}\}}\mathbb{1}_{\{|t_4-t_1|\leqslant 2\mathscr{m}\}}\\
&+\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2,t_3,t_4=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\left|\Gamma_{T,k}\left(\underline{s},\underline{t}\right)\right|\mathbb{1}_{\{|t_1-t_2|\leqslant \mathscr{m}\}}\mathbb{1}_{\{|t_3-t_1|\geqslant 2\mathscr{m}+1\}}\mathbb{1}_{\{|t_4-t_1|\geqslant 2\mathscr{m}+1\}}\\
&=: \mathrm{L}_{T,1}\left(\underline{s}\right)+\mathrm{L}_{T,2}\left(\underline{s}\right)+\mathrm{L}_{T,3}\left(\underline{s}\right), \qquad \text{(C.32)}
\end{aligned}
$$

whereby (C.29) provides for all $\underline{s} := (s_1,\ldots,s_4) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ (recall (C.28)):

$$
\begin{aligned}
\mathrm{L}_{T,1}\left(\underline{s}\right) &\leqslant \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2,t_3,t_4=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\left|\Gamma_{T,k}\left(\underline{s},\underline{t}\right)\right|\mathbb{1}_{\{|t_1-t_2|\leqslant \mathscr{m}\}}\mathbb{1}_{\{|t_3-t_1|\leqslant 2\mathscr{m}\}}\mathbb{1}_{\left\{\exists p_2\in\{1,2,3\}:\left|t_4-t_{p_2}\right|\leqslant \mathscr{m}\right\}}\\
&\leqslant C\left\lfloor 1/(2b)\right\rfloor\left\lfloor Tb\right\rfloor \mathscr{m}^3 \qquad \text{(C.33)}
\end{aligned}
$$

and similar arguments show for all $\underline{s} := (s_1,\ldots,s_4) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$:

$$
\mathrm{L}_{T,2}\left(\underline{s}\right) \leqslant C\left\lfloor 1/(2b)\right\rfloor\left\lfloor Tb\right\rfloor \mathscr{m}^3. \qquad \text{(C.34)}
$$

If $|t_1-t_2| \leqslant \mathscr{m}$, $|t_3-t_1| \geqslant 2\mathscr{m}+1$ as well as $|t_4-t_1| \geqslant 2\mathscr{m}+1$ are valid (with $t_1,\ldots,t_4 \in \{1,\ldots,2\lfloor T_{\mathfrak{u}}b\rfloor\}$), it will follow from the reverse triangle inequality that $|t_2-t_3| \geqslant ||t_2-t_1|-|t_3-t_1||$ $\geqslant \mathscr{m}+1$ and $|t_4-t_2| \geqslant ||t_4-t_1|-|t_2-t_1|| \geqslant \mathscr{m}+1$. In conclusion, all of these conditions with respect to $t_1,\ldots,t_4$ imply $([t_1-\mathscr{m},t_1]\cup[t_2-\mathscr{m},t_2])\cap([t_3-\mathscr{m},t_3]\cup[t_4-\mathscr{m},t_4])=\varnothing$. Hence, one obtains for all $\underline{s} := (s_1,\ldots,s_4) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ from Lemma B.4 (viii) and (B.45) (see (C.28) as well as Definition A.1 (i)):

$$
\begin{aligned}
&\mathrm{L}_{T,3}\left(\underline{s}\right)\\
&\leqslant \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\sum_{t_1,t_2=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\sup_{u,v\in[0,1]}\left|\mathbb{E}\left[\cos\left(\left\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{u}}b\rfloor+t_1}(u)\right\rangle\right)^c_{\mathscr{m}}\cos\left(\left\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{u}}b\rfloor+t_2}(v)\right\rangle\right)^c_{\mathscr{m}}\right]\right|\right.\\
&\left.\cdot\sum_{t_3,t_4=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\sup_{u,v\in[0,1]}\left|\mathbb{E}\left[\cos\left(\left\langle s_3,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{u}}b\rfloor+t_3}(u)\right\rangle\right)^c_{\mathscr{m}}\cos\left(\left\langle s_4,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{u}}b\rfloor+t_4}(v)\right\rangle\right)^c_{\mathscr{m}}\right]\right|\right)\\
&\leqslant C\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left[\sum_{t_1,t_2=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\mathbb{1}_{\{t_1=t_2\}}+\left(\sum_{t_2=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\sum_{t_1=t_2+1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\sum_{l=t_1-t_2}^{\infty}\Delta_l\left|s_1\right|_1+\sum_{t_1=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\sum_{t_2=t_1+1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\sum_{l=t_2-t_1}^{\infty}\Delta_l\left|s_2\right|_1\right)\right]\\
&\cdot\left[\sum_{t_3,t_4=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\mathbb{1}_{\{t_3=t_4\}}+\left(\sum_{t_4=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\sum_{t_3=t_4+1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\sum_{l=t_3-t_4}^{\infty}\Delta_l\left|s_3\right|_1+\sum_{t_3=1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\sum_{t_4=t_3+1}^{2\lfloor T_{\mathfrak{u}}b\rfloor}\sum_{l=t_4-t_3}^{\infty}\Delta_l\left|s_4\right|_1\right)\right]\\
&\leqslant C\left\lfloor 1/(2b)\right\rfloor\left\lfloor Tb\right\rfloor^2\left(1+\left|s_1\right|_1+\left|s_2\right|_1\right)\left(1+\left|s_3\right|_1+\left|s_4\right|_1\right). \qquad \text{(C.35)}
\end{aligned}
$$

Overall, it follows for all $\underline{s} := (s_1,\ldots,s_4) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ from (C.32) to (C.35), Remark A.2 (i) as well as Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) $\big($the last two and $\delta \leqslant 1$ provide $\mathscr{m}^3 = o\left(Tb\right)\big)$:

$$
\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2,t_3,t_4=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{u}}b\rfloor-1-\mathscr{m}}\left|\Gamma_{T,k}\left(\underline{s},\underline{t}\right)\right|\mathbb{1}_{\{|t_1-t_2|\leqslant \mathscr{m}\}} \leqslant CT^2b\left(1+\left|s_1\right|_1+\left|s_2\right|_1\right)\left(1+\left|s_3\right|_1+\left|s_4\right|_1\right). \qquad \text{(C.36)}
$$

One obtains for all $j \in \{1,2\}$ from (C.30), (C.31), (C.36) and similar arguments as well as the Assumptions 3.1 $[\mathbf{WEI.1}]$ and 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii):

$$
\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{E}\left[\left(\mathbb{D}^{\circ}_{T,k,j,\mathfrak{R}}\right)^4\right] = o(1). \qquad \text{(C.37)}
$$

In conclusion, (C.27), (C.37) and analog arguments prove that (6.26) in [52, Leucht and Neumann (2013), p. 275] is fulfilled.

Further, (C.23) implies that (6.27) and (6.28) in [52, Leucht and Neumann (2013), p. 275] are valid with

$\theta_r = 0 \; \forall \, r \in \mathbb{N}$, whereby θ_r is defined in [52, Leucht and Neumann (2013), p. 275].
Overall, Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274 et seq.] shows (C.20) (note (C.17) and (3.20)).
Theorem 3.13 (i) is an implication of (C.19) and (C.20).

(ii) In order to prove Theorem 3.13 (ii), observe at first for all $x_1, x_2 \in \mathbb{R}$ (recall (3.19) and (3.18)):

$$
\begin{aligned}
&(x_1, x_2)\, \Sigma_{\mathfrak{U}_{0,1}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= 8\,(\mathfrak{U}_1 - \mathfrak{U}_0) \int\limits_{\mathfrak{U}_0 - \mathfrak{U}_1}^{\mathfrak{U}_1 - \mathfrak{U}_0} K(z)^2\, dz \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Big[x_1^2 \tau_{\mathfrak{U}_{0,1},\Re}((1,0),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Re}((1,0),u,s_2) + x_1 x_2 \\
&\cdot \tau_{\mathfrak{U}_{0,1},\Re}((1,0),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Re}((0,1),u,s_2) + x_2 x_1 \tau_{\mathfrak{U}_{0,1},\Re}((0,1),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Re}((1,0),u,s_2) \\
&+ x_2^2 \tau_{\mathfrak{U}_{0,1},\Re}((0,1),u,s_2)\, \tau_{\mathfrak{U}_{0,1},\Re}((0,1),u,s_2) \Big] \sigma_{\infty,\Re,\Re}(u,s_1,s_2) + \Big[x_1^2 \tau_{\mathfrak{U}_{0,1},\Im}((1,0),u,s_1) \\
&\cdot \tau_{\mathfrak{U}_{0,1},\Im}((1,0),u,s_2) + x_1 x_2 \tau_{\mathfrak{U}_{0,1},\Im}((1,0),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Im}((0,1),u,s_2) + x_2 x_1 \tau_{\mathfrak{U}_{0,1},\Im}((0,1),u,s_1) \\
&\cdot \tau_{\mathfrak{U}_{0,1},\Im}((1,0),u,s_2) + x_2^2 \tau_{\mathfrak{U}_{0,1},\Im}((0,1),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Im}((0,1),u,s_2) \Big] \sigma_{\infty,\Im,\Im}(u,s_1,s_2) \\
&+ \Big[x_1^2 \tau_{\mathfrak{U}_{0,1},\Re}((1,0),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Im}((1,0),u,s_2) + x_1 x_2 \tau_{\mathfrak{U}_{0,1},\Re}((1,0),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Im}((0,1),u,s_2) \\
&+ x_2 x_1 \tau_{\mathfrak{U}_{0,1},\Re}((0,1),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Im}((1,0),u,s_2) + x_2^2 \tau_{\mathfrak{U}_{0,1},\Re}((0,1),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Im}((0,1),u,s_2) \Big] \\
&\cdot \sigma_{\infty,\Re,\Im}(u,s_1,s_2) + \Big[x_1^2 \tau_{\mathfrak{U}_{0,1},\Im}((1,0),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Re}((1,0),u,s_2) + x_1 x_2 \tau_{\mathfrak{U}_{0,1},\Im}((1,0),u,s_1) \\
&\cdot \tau_{\mathfrak{U}_{0,1},\Re}((0,1),u,s_2) + x_2 x_1 \tau_{\mathfrak{U}_{0,1},\Im}((0,1),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Re}((1,0),u,s_2) + x_2^2 \tau_{\mathfrak{U}_{0,1},\Im}((0,1),u,s_1) \\
&\cdot \tau_{\mathfrak{U}_{0,1},\Re}((0,1),u,s_2) \Big] \sigma_{\infty,\Im,\Re}(u,s_1,s_2)\; du \cdot \mathbf{w}(s_1)\; ds_1\, \mathbf{w}(s_2)\; ds_2. \qquad \text{(C.38)}
\end{aligned}
$$

Moreover, it holds for all $x_1, x_2 \in \mathbb{R}$, $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$, $u \in [0,1]$, $s_1, s_2 \in \mathbb{R}^d$ (see (3.16)):

$$
\begin{aligned}
&x_1^2 \tau_{\mathfrak{U}_{0,1},\mathrm{R}_1}((1,0),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\mathrm{R}_2}((1,0),u,s_2) + x_1 x_2 \tau_{\mathfrak{U}_{0,1},\mathrm{R}_1}((1,0),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\mathrm{R}_2}((0,1),u,s_2) \\
&+ x_2 x_1 \tau_{\mathfrak{U}_{0,1},\mathrm{R}_1}((0,1),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\mathrm{R}_2}((1,0),u,s_2) + x_2^2 \tau_{\mathfrak{U}_{0,1},\mathrm{R}_1}((0,1),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\mathrm{R}_2}((0,1),u,s_2) \\
&= \tau_{\mathfrak{U}_{0,1},\mathrm{R}_1}((x_1,x_2),u,s_1)\, \tau_{\mathfrak{U}_{0,1},\mathrm{R}_2}((x_1,x_2),u,s_2). \qquad \text{(C.39)}
\end{aligned}
$$

In conclusion, (C.38) and (C.39) provide for all $x_1, x_2 \in \mathbb{R}$ (note (3.18)):

$$
(x_1, x_2)\, \Sigma_{\mathfrak{U}_{0,1}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sigma_{\mathfrak{U}_{0,1}}((x_1,x_2),(x_1,x_2)). \qquad \text{(C.40)}
$$

Proposition 3.6 (i), Theorem 3.13 (i) together with the delta method and (C.40) with $(x_1, x_2) = (1, -1)$ imply for $h_1((y_1,y_2)') := y_1 - y_2 \; \forall \, y_1, y_2 \in \mathbb{R}$ (recall Definition 3.8 (i) as well as (3.21)):

$$
\sqrt{T}\left(\widehat{\mathbb{D}}_T - \mathbb{D}\right) = \sqrt{T}\left(h_1 \begin{pmatrix} \widehat{\mathbb{D}}_{T,1} \\ \widehat{\mathbb{D}}_{T,2} \end{pmatrix} - h_1 \begin{pmatrix} \mathbb{D}_1 \\ \mathbb{D}_2 \end{pmatrix} \right) \xrightarrow{\mathrm{d}} Z_{\mathfrak{U}_{0,1}},
$$

which shows Theorem 3.13 (ii).

(iii) Lemma 3.4 yields that the function $h_2((y_1,y_2)') := 1 - y_2/y_1 \; \forall \, y_1 > 0,\ y_2 \in \mathbb{R}$ is partially differentiable at $(y_1,y_2)' = (\mathbb{D}_1, \mathbb{D}_2)'$. Moreover, Lemma 3.9 provides $\widehat{\mathbb{D}}_T^{\mathrm{norm}} = h_2\big((\widehat{\mathbb{D}}_{T,1}, \widehat{\mathbb{D}}_{T,2})'\big)$ for sufficiently large T (note Definition 3.8 (ii)) and it holds $\mathbb{D}^{\mathrm{norm}} = h_2((\mathbb{D}_1, \mathbb{D}_2)')$ (see Definition 3.3 (ii)). Hence, Theorem 3.13 (i) together with the delta method and (C.40) with $(x_1, x_2) = \big(\gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},1}, \gamma^{\mathrm{norm}}_{\mathfrak{U}_{0,1},2}\big)$ (recall (3.22)) prove Theorem 3.13 (iii). □

Verification of Example 3.16. (i) The Assumptions 3.15 $[\mathbf{W}^*]$ (i) as well as (ii) are obviously fulfilled and:

$$\mathbb{E}\left[\left(W_0^{*[1]}\right)^4\right] \leqslant \frac{1}{\beta^2}\sum_{k_1,k_2=0}^{\beta-1}\mathbb{E}\left[{\varepsilon_{-k_1}^*}^2\right]\mathbb{E}\left[{\varepsilon_{-k_2}^*}^2\right] + \frac{1}{\beta^2}\sum_{k=0}^{\beta-1}\mathbb{E}\left[{\varepsilon_{-k}^*}^4\right] \leqslant C. \tag{C.41}$$

Moreover, it holds for all $t_1, t_2 \in \mathbb{Z}$ with $t_1 \geqslant t_2$:

$$\begin{aligned}\mathbb{E}\left[W_{t_1}^{*[1]}W_{t_2}^{*[1]}\right] = \frac{1}{\beta}\sum_{k_1,k_2=0}^{\beta-1}\mathbb{1}_{\{t_1-t_2+k_2=k_1\}} &= \frac{1}{\beta}\sum_{k_2=t_1-t_2}^{\beta-1+t_1-t_2}\sum_{k_1=0}^{\beta-1}\mathbb{1}_{\{k_2=k_1\}}\\ &= \frac{\beta-(t_1-t_2)}{\beta}\mathbb{1}_{\{t_1-t_2<\beta\}}.\end{aligned} \tag{C.42}$$

The validity of Assumption 3.15 $[\mathbf{W}^*]$ (iii) is an implication of (C.41) and (C.42) as well as obvious arguments, whereby:

$$K^*\colon \mathbb{R}\to\mathbb{R},\ x\mapsto \max\{1-|x|,0\}. \tag{C.43}$$

Furthermore, Assumption 3.15 $[\mathbf{W}^*]$ (iv) holds with $\rho_* := e^{-1}$ due to $1/\sqrt{\beta}\,\mathbb{1}_{\{l\leqslant\beta-1\}} \leqslant C\,e^{-l/\beta}$ $\forall\, l\in\mathbb{N}_0$.

(ii) Assumption 3.15 $[\mathbf{W}^*]$ (i) is obviously fulfilled and Assumption 3.15 $[\mathbf{W}^*]$ (ii) is valid due to:

$$W_t^{*[2]} = \sum_{k=0}^{\infty} e^{-\frac{k}{\beta}}\sqrt{1-e^{-\frac{2}{\beta}}}\,\varepsilon_{t-k}^* \text{ a.s. } \ \forall\, t\in\mathbb{Z}. \tag{C.44}$$

It follows from (C.44) and $\lim_{T\to\infty}\beta\left(1-e^{-2/\beta}\right) = 2$:

$$\begin{aligned}\mathbb{E}\left[\left(W_0^{*[2]}\right)^4\right] &\leqslant \frac{1}{\beta^2}\left(\beta\left(1-e^{-\frac{2}{\beta}}\right)\right)^2\left(\sum_{k_1,k_2=0}^{\infty}e^{-\frac{2k_1}{\beta}}e^{-\frac{2k_2}{\beta}}\mathbb{E}\left[{\varepsilon_{-k_1}^*}^2\right]\mathbb{E}\left[{\varepsilon_{-k_2}^*}^2\right] + \sum_{k=0}^{\infty}e^{-\frac{4k}{\beta}}\mathbb{E}\left[{\varepsilon_{-k}^*}^4\right]\right)\\ &\leqslant C.\end{aligned} \tag{C.45}$$

Moreover, one observes:

$$\operatorname{Cov}\left(W_{t_1}^{*[2]}, W_{t_2}^{*[2]}\right) = K^*\left(\frac{t_1-t_2}{\beta}\right)\ \forall\, t_1,t_2\in\mathbb{Z} \quad \text{with} \quad K^*\colon\mathbb{R}\to\mathbb{R},\ x\mapsto e^{-|x|} \tag{C.46}$$

and $\lim_{T\to\infty}\beta\left(1-e^{-1/\beta}\right) = 1$ yields:

$$\sum_{t=0}^{2\lfloor Tb\rfloor+T}\left|K^*\left(\frac{t}{\beta}\right)\right| \leqslant \sum_{t=0}^{\infty}e^{-\frac{t}{\beta}} = \frac{e^{\frac{1}{\beta}}}{e^{\frac{1}{\beta}}-1} = \frac{\beta}{\beta\left(1-e^{-\frac{1}{\beta}}\right)} \leqslant C\beta. \tag{C.47}$$

The validity of Assumption 3.15 $[\mathbf{W}^*]$ (iii) is an implication of (C.45), (C.46) and (C.47) as well as obvious arguments. Further, it follows from (C.44) that Assumption 3.15 $[\mathbf{W}^*]$ (iv) is fulfilled with $\rho_* := e^{-1}$.

□

Proof of Theorem 3.19. Throughout this proof, suppose that T is large enough to ensure (see (C.17) as well as Definition A.1 (iii)):

$$2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - m_\beta \geqslant 1 + m_\beta, \tag{C.48}$$

which holds for sufficiently large T due to Lemma C.8 (ii), Assumption 3.15 $[\mathbf{W}^*]$ (i), Remark A.2 (i) and Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii).
To prove Theorem 3.19, it will be shown at first for $T\to\infty$ and all $\gamma := \left(\gamma^{[1]},\gamma^{[2]}\right)\in\mathbb{R}^{1\times 2}$ (recall

(3.18)):

$$\sqrt{T}\,\widehat{\mathbb{D}}^{*}_{T}(\gamma) \xrightarrow{\mathrm{d}} Z_{\mathfrak{U}_{0,1},\gamma} \text{ in probability with } Z_{\mathfrak{U}_{0,1},\gamma} \sim \mathcal{N}\left(0, \sigma_{\mathfrak{U}_{0,1}}(\gamma,\gamma)\right), \tag{C.49}$$

which means (as described in [52, Leucht and Neumann (2013), p. 262]) that the distance (quantified by the Prokhorov metric) between the conditional distribution of $\sqrt{T}\,\widehat{\mathbb{D}}^{*}_{T}(\gamma)$ (conditioned on $X_{1,T},\ldots,$ $X_{T,T}$) and the distribution of $Z_{\mathfrak{U}_{0,1},\gamma}$ converges to zero in probability.
In order to verify (C.49), observe (see (3.38)):

$$\sqrt{T}\,\widehat{\mathbb{D}}^{*}_{T}(\gamma) = \sqrt{T}\int_{\mathbb{R}^d} \gamma^{[1]}\widehat{\mathbb{D}}^{*}_{T,1,\Re}(s) + \gamma^{[2]}\widehat{\mathbb{D}}^{*}_{T,2,\Re}(s) + \gamma^{[1]}\widehat{\mathbb{D}}^{*}_{T,1,\Im}(s) + \gamma^{[2]}\widehat{\mathbb{D}}^{*}_{T,2,\Im}(s)\,\mathbf{w}(s)\,ds \tag{C.50}$$

and define for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^d$ (note the Definitions A.1 (i) and (iii), 3.8 (i), 3.3 (i), (C.17) as well as (3.16)):

$$\begin{aligned}
&\varphi^{\circ *}_{\mathscr{m}_\beta}(u_k,s) := \varphi^{\circ *}_{T,\mathfrak{U}_{0,1}\mathscr{m}_\beta}(u_k,s) \\
&:= \frac{1}{\lfloor Tb\rfloor}\sum_{t=1+\mathscr{m}_\beta}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}_\beta} K\left(\frac{t-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)\left(e^{i\left\langle s, X_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t,T}\right\rangle}\right)^c W^{*}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t,\{\mathscr{m}_\beta\}}, \\
&\mathbb{D}^{\circ *}_{T,k,\gamma,\mathrm{R}} := \mathbb{D}^{\circ *}_{T,\mathfrak{U}_{0,1},k,\gamma,\mathrm{R}} := \frac{2\sqrt{T}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor}\int_{\mathbb{R}^d}\mathrm{R}\left\{\varphi^{\circ *}_{\mathscr{m}_\beta}(u_k,s)\right\}\cdot\tau_{\mathfrak{U}_{0,1},\mathrm{R}}(\gamma,u_k,s)\;\mathbf{w}(s)\,ds, \\
&\mathbb{D}^{\circ *}_{T,k,\gamma} := \mathbb{D}^{\circ *}_{T,\mathfrak{U}_{0,1},k,\gamma} := \mathbb{D}^{\circ *}_{T,k,\gamma,\Re} + \mathbb{D}^{\circ *}_{T,k,\gamma,\Im} \text{ and } \mathbb{D}^{\circ *}_{T,\gamma} := \mathbb{D}^{\circ *}_{T,\mathfrak{U}_{0,1},\gamma} := \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{D}^{\circ *}_{T,k,\gamma}.
\end{aligned} \tag{C.51}$$

Thereby, it holds for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $t \in \{1,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor\}$ due to $\lfloor x\rfloor + \lfloor y\rfloor \leqslant \lfloor x+y\rfloor\ \forall\, x,y>0$, $b \leqslant (1/2)/\lfloor 1/(2b)\rfloor$ and $\lfloor x-y\rfloor \leqslant x - \lfloor y\rfloor\ \forall\, x,y>0$ (recall Definition 3.8 (i) as well as (C.17)):

$$\begin{aligned}
&1 \leqslant \lfloor \mathfrak{U}_0T\rfloor + \lfloor T_{\mathfrak{U}}b\rfloor - \lfloor T_{\mathfrak{U}}b\rfloor + 1 \leqslant \left\lfloor \mathfrak{U}_0T + \frac{1/2}{\lfloor 1/(2b)\rfloor}T_{\mathfrak{U}}\right\rfloor - \lfloor T_{\mathfrak{U}}b\rfloor + 1 \leqslant \lfloor u_kT\rfloor - \lfloor T_{\mathfrak{U}}b\rfloor + t \text{ and} \\
&\lfloor u_kT\rfloor - \lfloor T_{\mathfrak{U}}b\rfloor + t \leqslant \left\lfloor \mathfrak{U}_0T + T_{\mathfrak{U}} - \frac{1/2}{\lfloor 1/(2b)\rfloor}T_{\mathfrak{U}}\right\rfloor - \lfloor T_{\mathfrak{U}}b\rfloor + 2\lfloor T_{\mathfrak{U}}b\rfloor \leqslant \mathfrak{U}_0T + T_{\mathfrak{U}} \leqslant \mathfrak{U}_1T,
\end{aligned} \tag{C.52}$$

such that $\varphi^{\circ *}_{\mathscr{m}_\beta}(u_k,s)$ just takes $X_{r,T}$ with $r \in \{1,\ldots,T\}$ into account. Therefore, $\varphi^{\circ *}_{\mathscr{m}}(u_k,s)$ is well-defined for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^d$.

Moreover, one observes for all $\mathrm{R} \in \{\Re,\Im\}$, $\gamma := \left(\gamma^{[1]},\gamma^{[2]}\right) \in \mathbb{R}^{1\times 2}$, $u \in [0,1]$, $s \in \mathbb{R}^d$ (recall (3.16) and regard that γ_1 as well as γ_2 originate from (C.17)):

$$\tau_{\mathfrak{U}_{0,1},\mathrm{R}}(\gamma,u,s) = \gamma^{[1]}\tau_{\mathfrak{U}_{0,1},\mathrm{R}}(\gamma_1,u,s) + \gamma^{[2]}\tau_{\mathfrak{U}_{0,1},\mathrm{R}}(\gamma_2,u,s),$$

such that (see (C.51)):

$$\mathbb{D}^{\circ *}_{T,\gamma} = \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\gamma^{[1]}\mathbb{D}^{\circ *}_{T,k,\gamma_1,\Re} + \gamma^{[2]}\mathbb{D}^{\circ *}_{T,k,\gamma_2,\Re} + \gamma^{[1]}\mathbb{D}^{\circ *}_{T,k,\gamma_1,\Im} + \gamma^{[2]}\mathbb{D}^{\circ *}_{T,k,\gamma_2,\Im}\right). \tag{C.53}$$

Overall, (C.50), (C.53), the Lemmata C.10 as well as C.12 and (3.27) provide (recall (3.26)):

$$\sqrt{T}\,\widehat{\mathbb{D}}^{*}_{T}(\gamma) - \mathbb{D}^{\circ *}_{T,\gamma} = o^{*}_{\mathbb{P}}(1). \tag{C.54}$$

Thus, in order to prove (C.49), it suffices to show for $T \to \infty$ that:

$$\mathbb{D}^{\circ *}_{T,\gamma} \xrightarrow{d} Z_{\mathfrak{U}_{0,1},\gamma} \text{ in probability.} \tag{C.55}$$

From now on, the validity of the assumptions demanded in Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275 et seq.] will be examined in order to use this corollary to verify (C.55), whereby the expression $\mathbb{D}^{\circ *}_{T,k,\gamma}$ takes the role of $X^{*}_{n,k}$ which originates from Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275 et seq.].

The Assumptions 3.15 $[\mathbf{W}^{*}]$ (ii) and (iii) imply (see (C.51), (3.16) as well as Definition A.1 (i)):

$$\mathbb{E}^{*}\left[\mathbb{D}^{\circ *}_{T,k,\gamma}\right] = 0 \quad \forall\, k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}, \tag{C.56}$$

such that the first assumption of Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275] holds.
It follows for all $k_1, k_2 \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $t_1, t_2 \in \{1 + \mathscr{m}_{\beta}, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}_{\beta}\}$ with $k_1 \geqslant k_2 + 1$ similarly to (C.22):

$$\lfloor u_{k_1} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1 \geqslant \lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2 + 2 + 2\mathscr{m}_{\beta}. \tag{C.57}$$

One obtains from Definition A.1 (i), Assumption 3.15 $[\mathbf{W}^{*}]$ (ii) and (C.57) with $t_1 = 1 + \mathscr{m}_{\beta}$ as well as $t_2 = 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}_{\beta}$ that $\left(\mathbb{D}^{\circ *}_{T,k,\gamma}\right)_{k=1}^{\lfloor 1/(2b)\rfloor}$ is a sequence of conditioned on $(X_{t,T})_{t=1}^{T}$ independent random variables, such that (C.56) provides:

$$\operatorname{Var}^{*}\left(\mathbb{D}^{\circ *}_{T,\gamma}\right) = \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}^{*}\left[\left(\mathbb{D}^{\circ *}_{T,k,\gamma}\right)^{2}\right] \quad \text{a. s.} \tag{C.58}$$

Markov's inequality, (C.58) and Lemma C.13 show:

$$\mathbb{P}\left(\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}^{*}\left[\left(\mathbb{D}^{\circ *}_{T,k,\gamma}\right)^{2}\right] > \sigma_{\mathfrak{U}_{0,1}}(\gamma,\gamma) + 1\right) \leqslant \mathbb{E}\left[\left|\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}^{*}\left[\left(\mathbb{D}^{\circ *}_{T,k,\gamma}\right)^{2}\right] - \sigma_{\mathfrak{U}_{0,1}}(\gamma,\gamma)\right|\right] \overset{T\to\infty}{\longrightarrow} 0. \tag{C.59}$$

The second assumption of Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275] holds due to (C.59), Lemma 3.12 and Assumption 3.1 $[\mathbf{WEI.1}]$, whereby the latter two provide $\sigma_{\mathfrak{U}_{0,1}}(\gamma,\gamma) < \infty$ (recall (3.18), (3.16) as well as (3.17)).
Lemma C.13 (note also (C.51)) shows the validity of (6.29) in [52, Leucht and Neumann (2013), p. 275] with $\Sigma = \sigma_{\mathfrak{U}_{0,1}}(\gamma,\gamma)$, whereby Σ originates from (6.29) in [52, Leucht and Neumann (2013), p. 275].
Next, it is proved that the condition (6.30) in [52, Leucht and Neumann (2013), p. 275] is fulfilled. Therefor, one defines for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^{d}$ (observe that the following expressions result from those in (C.51) by replacing the contained terms $e^{i\langle s, X_{\bullet,T}\rangle}$ by terms of the form $\left(e^{i\langle s, \tilde{X}_{\bullet}(\times)\rangle}\right)_{\mathscr{m}}$ and that according to Definition A.1 (i), $\left(X^{c}\right)_{\mathscr{m}} = \left(\left(X\right)_{\mathscr{m}}\right)^{c}$ for all random variables X with finite first moment, such that the order of the operators $\bullet^{c}$ and $(\bullet)_{\mathscr{m}}$ does not play a role in the notation $(X)^{c}_{\mathscr{m}}$):

$$\begin{aligned}
&\tilde{\varphi}^{\circ *}_{\mathscr{m},\mathscr{m}_{\beta}}(u_k, s) := \tilde{\varphi}^{\circ *}_{T,\mathfrak{U}_{0,1},\mathscr{m},\mathscr{m}_{\beta}}(u_k, s)\\
&:= \frac{1}{\lfloor Tb\rfloor} \sum_{t=1+\mathscr{m}_{\beta}}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}_{\beta}} K\left(\frac{t - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right)\left(e^{i\left\langle s, \tilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t}\left(\tilde{u}_{k,t}\right)\right\rangle}\right)^{c}_{\mathscr{m}} W^{*}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t, \{\mathscr{m}_{\beta}\}},\\
&\widetilde{\mathbb{D}}^{\circ *}_{T,k,\gamma,\mathrm{R}} := \widetilde{\mathbb{D}}^{\circ *}_{T,\mathfrak{U}_{0,1},k,\gamma,\mathrm{R}} := \frac{2\sqrt{T}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor} \int_{\mathbb{R}^{d}} \mathrm{R}\left\{\tilde{\varphi}^{\circ *}_{\mathscr{m},\mathscr{m}_{\beta}}(u_k, s)\right\} \cdot \tau_{\mathfrak{U}_{0,1},\mathrm{R}}(\gamma, u_k, s)\, \mathbf{w}(s)\, ds,\\
&\widetilde{\mathbb{D}}^{\circ *}_{T,k,\gamma} := \widetilde{\mathbb{D}}^{\circ *}_{T,\mathfrak{U}_{0,1},k,\gamma} := \widetilde{\mathbb{D}}^{\circ *}_{T,k,\gamma,\Re} + \widetilde{\mathbb{D}}^{\circ *}_{T,k,\gamma,\Im} \quad \text{and} \quad \widetilde{\mathbb{D}}^{\circ *}_{T,\gamma} := \widetilde{\mathbb{D}}^{\circ *}_{T,\mathfrak{U}_{0,1},\gamma} := \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \widetilde{\mathbb{D}}^{\circ *}_{T,k,\gamma},
\end{aligned} \tag{C.60}$$

whereby (C.18) ensures that $\tilde{\varphi}^{\circ *}_{\mathscr{m},\mathscr{m}_{\beta}}(u_k, s)$ is well-defined for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^{d}$.
In the following, it is verified that (note (C.51)):

$$\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \left|\mathbb{E}\left[\left(\mathbb{D}^{\circ *}_{T,k,\gamma,\Re}\right)^{4} - \left(\widetilde{\mathbb{D}}^{\circ *}_{T,k,\gamma,\Re}\right)^{4}\right]\right| = o(1) \quad \text{and} \tag{C.61}$$

$$\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left(\widetilde{\mathbb{D}}^{\circ *}_{T,k,\gamma,\Re}\right)^{4}\right] = o(1). \tag{C.62}$$

Subsequently, the validity of (6.30) in [52, Leucht and Neumann (2013), p. 275] will be proved by using (C.61), (C.62) and similar arguments.
In order to verify (C.61), it will be just shown that the following expression converges to zero because this and very similar arguments imply (C.61) (see (C.51) as well as (C.60) and note in particular the condition $t_1 \geqslant t_2 \geqslant \max\{t_3, t_4\}$ which restricts the fourfold sum with respect to $t_1, \ldots, t_4$ in the expression given below):

$$\mathrm{R}_T^* := \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Bigg| \frac{2^4 \cdot T^2 (\mathfrak{U}_1 - \mathfrak{U}_0)^4}{\lfloor 1/(2b) \rfloor^4 \lfloor Tb \rfloor^4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{\substack{t_1, t_2, t_3, t_4 = 1 + m_\beta \\ t_1 \geqslant t_2 \geqslant \max\{t_3, t_4\}}}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - m_\beta} \tau_{\mathfrak{U}_{0,1}, \Re}(\gamma, u_k, s_1)\, \tau_{\mathfrak{U}_{0,1}, \Re}(\gamma, u_k, s_2)$$
$$\cdot \tau_{\mathfrak{U}_{0,1}, \Re}(\gamma, u_k, s_3)\, \tau_{\mathfrak{U}_{0,1}, \Re}(\gamma, u_k, s_4)\, K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0)\right) K\left(\frac{t_2 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0)\right)$$
$$\cdot K\left(\frac{t_3 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0)\right) K\left(\frac{t_4 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0)\right) \mathbb{E}\Bigg[\Bigg\{ \cos\left(\left\langle s_1, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, T} \right\rangle\right)^c$$
$$\cdot \cos\left(\left\langle s_2, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, T} \right\rangle\right)^c \cos\left(\left\langle s_3, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_3, T} \right\rangle\right)^c \cos\left(\left\langle s_4, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_4, T} \right\rangle\right)^c$$
$$- \cos\left(\left\langle s_1, \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1} (\tilde{u}_{k, t_1}) \right\rangle\right)_m^c \cos\left(\left\langle s_2, \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2} (\tilde{u}_{k, t_2}) \right\rangle\right)_m^c$$
$$\cdot \cos\left(\left\langle s_3, \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_3} (\tilde{u}_{k, t_3}) \right\rangle\right)_m^c \cos\left(\left\langle s_4, \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_4} (\tilde{u}_{k, t_4}) \right\rangle\right)_m^c \Bigg\}$$
$$\cdot W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, \{m_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, \{m_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_3, \{m_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_4, \{m_\beta\}} \Bigg]$$
$$\cdot \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2\, \mathbf{w}(s_3)\, ds_3\, \mathbf{w}(s_4)\, ds_4 \Bigg|. \tag{C.63}$$

One observes for all $x_1, \ldots, x_4, y_1, \ldots, y_4 \in \mathbb{R}$:

$$x_1 x_2 x_3 x_4 - y_1 y_2 y_3 y_4 = (x_1 - y_1) x_2 x_3 x_4 + (x_2 - y_2) y_1 x_3 x_4 + (x_3 - y_3) y_1 y_2 x_4 + (x_4 - y_4) y_1 y_2 y_3. \tag{C.64}$$

Further, (3.14), Assumption 2.2 $[\mathbf{StAp}]$ (i) as well as Remark 2.3 imply for all $s \in \mathbb{R}^d$, $q \geqslant 1 + \delta$ (recall (C.17) and regard that the following expressions are well-defined due to (C.52) as well as (C.18)):

$$\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t=1,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor} \left\| e^{i\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t, T} \right\rangle} - e^{i\left\langle s, \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t} (\tilde{u}_{k,t}) \right\rangle} \right\|_q$$
$$\leqslant C \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t=1,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor} \left(\left\| X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t, T} - \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t} \left(\frac{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t}{T} \right) \right\|_{1+\delta} \right.$$
$$\left. + \left\| \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t} \left(\frac{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t}{T} \right) - \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t} \left(u_k + \frac{t - \lfloor T_{\mathfrak{U}} b \rfloor}{T} \right) \right\|_{1+\delta} \right)^{\frac{1+\delta}{q}} |s|_1^{\frac{1+\delta}{q}}$$
$$\leqslant \frac{C}{T^{\frac{1+\delta}{q}}} |s|_1^{\frac{1+\delta}{q}}. \tag{C.65}$$

Lemma C.4 (i) with $q = 1 + \delta$, (C.65) with $q = 1 + \delta$ and Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii) provide for all $s \in \mathbb{R}^d$ (see Definition A.1 (i)):

$$\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t=1,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor} \mathbb{E}\left[\left| \mathrm{R}\left\{ \left(e^{i\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t, T} \right\rangle} \right)^c - \left(e^{i\left\langle s, \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t} (\tilde{u}_{k,t}) \right\rangle} \right)_m^c \right\} \right| \right]$$
$$\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t=1,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor} \left(\mathbb{E}\left[\left| \left(e^{i\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t, T} \right\rangle} - \left(e^{i\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t, T} \right\rangle} \right)_m \right)^c \right| \right] \right.$$
$$\left. + \mathbb{E}\left[\left| \left(e^{i\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t, T} \right\rangle} - e^{i\left\langle s, \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t} (\tilde{u}_{k,t}) \right\rangle} \right)_m^c \right| \right] \right)$$
$$\leqslant \frac{C}{T b^2} |s|_1. \tag{C.66}$$

Moreover, Assumption 3.15 [$\mathbf{W}^*$] (iii) implies (recall Definition A.1 (i)):

$$\sup_{r_1,\ldots,r_4\in\mathbb{Z}} \left|\mathbb{E}\left[W^*_{r_1,\{m_\beta\}} W^*_{r_2,\{m_\beta\}} W^*_{r_3,\{m_\beta\}} W^*_{r_4,\{m_\beta\}}\right]\right| \leqslant \sup_{r\in\mathbb{Z}} \left\|W^*_{r,\{m_\beta\}}\right\|_4^4 \leqslant \|W_0^*\|_4^4 \leqslant C. \quad \text{(C.67)}$$

It follows from Assumption 3.15 [$\mathbf{W}^*$] (ii), (C.64) and (C.66) (see (C.63), (3.16), note that the following expressions $\mathrm{R}^*_{T,1}$, $\mathrm{R}^*_{T,2}$ as well as $\mathrm{R}^*_{T,3}$ are well-defined due to (C.67) and regard the different conditions on $t_1, \ldots, t_4$ which underlie these expressions):

$$\begin{aligned}
\mathrm{R}^*_T \leqslant{} & \frac{C}{T^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{\substack{t_1,t_2,t_3,t_4=1+m_\beta\\ t_1>t_2\geqslant\max\{t_3,t_4\}}}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-m_\beta} \bigg|\mathbb{E}\bigg[\cos\left(\left\langle s_1, X_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1,T}\right\rangle\right)^c \\
& \cdot \cos\left(\left\langle s_2, X_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_2,T}\right\rangle\right)^c \cos\left(\left\langle s_3, X_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_3,T}\right\rangle\right)^c \cos\left(\left\langle s_4, X_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_4,T}\right\rangle\right)^c \\
& - \cos\left(\left\langle s_1, \widetilde{X}_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1}\left(\widetilde{u}_{k,t_1}\right)\right\rangle\right)^c_{m} \cos\left(\left\langle s_2, \widetilde{X}_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_2}\left(\widetilde{u}_{k,t_2}\right)\right\rangle\right)^c_{m} \\
& \cdot \cos\left(\left\langle s_3, \widetilde{X}_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_3}\left(\widetilde{u}_{k,t_3}\right)\right\rangle\right)^c_{m} \cos\left(\left\langle s_4, \widetilde{X}_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_4}\left(\widetilde{u}_{k,t_4}\right)\right\rangle\right)^c_{m}\bigg]\bigg| \\
& \cdot \left|\mathbb{E}\left[W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1,\{m_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_2,\{m_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_3,\{m_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_4,\{m_\beta\}}\right]\right| \\
& \cdot \mathbb{1}_{\{|t_1-t_h|\leqslant 3m_\beta\ \forall h\in\{3,4\}\}} \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2\, \mathbf{w}(s_3)\, ds_3\, \mathbf{w}(s_4)\, ds_4 \\
& + \frac{C}{T^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{\substack{t_1,t_2,t_3,t_4=1+m_\beta\\ t_1=t_2\geqslant\max\{t_3,t_4\}}}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-m_\beta} \frac{C}{Tb^2}\cdot\left(|s_1|_1+|s_2|_1+|s_3|_1+|s_4|_1\right) \\
& \cdot \left|\mathbb{E}\left[W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1,\{m_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_2,\{m_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_3,\{m_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_4,\{m_\beta\}}\right]\right| \\
& \cdot \mathbb{1}_{\{|t_1-t_h|\leqslant 3m_\beta\ \forall h\in\{3,4\}\}} \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2\, \mathbf{w}(s_3)\, ds_3\, \mathbf{w}(s_4)\, ds_4 \\
& + \frac{C}{T^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{\substack{t_1,t_2,t_3,t_4=1+m_\beta\\ t_1\geqslant t_2\geqslant\max\{t_3,t_4\}}}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-m_\beta} \frac{C}{Tb^2}\cdot\left(|s_1|_1+|s_2|_1+|s_3|_1+|s_4|_1\right) \\
& \cdot \left|\mathbb{E}\left[W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1,\{m_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_2,\{m_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_3,\{m_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_4,\{m_\beta\}}\right]\right| \\
& \cdot \left(\mathbb{1}_{\{|t_1-t_3|>3m_\beta\}} + \mathbb{1}_{\{|t_1-t_4|>3m_\beta\}}\right) \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2\, \mathbf{w}(s_3)\, ds_3\, \mathbf{w}(s_4)\, ds_4 \\
& =: \mathrm{R}^*_{T,1} + \mathrm{R}^*_{T,2} + \mathrm{R}^*_{T,3}.
\end{aligned} \quad \text{(C.68)}$$

To shorten the notation used in the following, set $Y_{s,r,T} := \cos\left(\left\langle s, X_{r,T}\right\rangle\right)$ as well as $\widetilde{Y}_{s,r,T}\left(v_{r,T}\right) := \cos\left(\left\langle s, \widetilde{X}_r\left(v_{r,T}\right)\right\rangle\right)_{m}$ for all $s \in \mathbb{R}^d$, $r \in \{1, \ldots, T\}$ and arbitrary, deterministic parameters $v_{r,T} \in [0,1]$ that may depend on r as well as T. Assumption 2.2 [$\mathbf{StAp}$] (iii), Lemma B.4 (iii) and (iv) with $q = 1+\delta$ as well as shifting the index of a sum provide for all $s_1, \ldots, s_4 \in \mathbb{R}^d$, $r_1, \ldots, r_4 \in \{1, \ldots, T\}$ with $r_1 > r_2 \geqslant \max\{r_3, r_4\}$ (recall that $\mathcal{F}_r$ originates from Definition 2.1 and Definition A.1 (i)):

$$\begin{aligned}
& \left|\mathbb{E}\left[Y^c_{s_1,r_1,T} Y^c_{s_2,r_2,T} Y^c_{s_3,r_3,T} Y^c_{s_4,r_4,T} - \widetilde{Y}_{s_1,r_1,T}(v_{r_1,T})^c\, \widetilde{Y}_{s_2,r_2,T}(v_{r_2,T})^c\, \widetilde{Y}_{s_3,r_3,T}(v_{r_3,T})^c\, \widetilde{Y}_{s_4,r_4,T}(v_{r_4,T})^c\right]\right| \\
& \leqslant \left|\mathbb{E}\left[\left(Y_{s_1,r_1,T} - \widetilde{Y}_{s_1,r_1,T}\left(v_{r_1,T}\right)\right)^c Y^c_{s_2,r_2,T} Y^c_{s_3,r_3,T} Y^c_{s_4,r_4,T}\right]\right| \\
& + \left|\mathbb{E}\left[\widetilde{Y}_{s_1,r_1,T}\left(v_{r_1,T}\right)^c \left(Y^c_{s_2,r_2,T} Y^c_{s_3,r_3,T} Y^c_{s_4,r_4,T} - \widetilde{Y}_{s_2,r_2,T}\left(v_{r_2,T}\right)^c \widetilde{Y}_{s_3,r_3,T}\left(v_{r_3,T}\right)^c \widetilde{Y}_{s_4,r_4,T}\left(v_{r_4,T}\right)^c\right)\right]\right| \\
& \leqslant \bigg|\mathbb{E}\bigg[\left(\mathbb{E}\left[Y_{s_1,r_1,T} - \widetilde{Y}_{s_1,r_1,T}\left(v_{r_1,T}\right)\Big|\mathcal{F}_{r_1}\right] - \mathbb{E}\left[Y_{s_1,r_1,T} - \widetilde{Y}_{s_1,r_1,T}\left(v_{r_1,T}\right)\Big|\mathcal{F}_{r_1,r_2+1}\right]\right) \\
& \cdot Y^c_{s_2,r_2,T} Y^c_{s_3,r_3,T} Y^c_{s_4,r_4,T}\bigg]\bigg| + \bigg|\mathbb{E}\bigg[\left(\mathbb{E}\left[\widetilde{Y}_{s_1,r_1,T}\left(v_{r_1,T}\right)\Big|\mathcal{F}_{r_1}\right] - \mathbb{E}\left[\widetilde{Y}_{s_1,r_1,T}\left(v_{r_1,T}\right)\Big|\mathcal{F}_{r_1,r_2+1}\right]\right) \\
& \cdot \left(Y^c_{s_2,r_2,T} Y^c_{s_3,r_3,T} Y^c_{s_4,r_4,T} - \widetilde{Y}_{s_2,r_2,T}\left(v_{r_2,T}\right)^c \widetilde{Y}_{s_3,r_3,T}\left(v_{r_3,T}\right)^c \widetilde{Y}_{s_4,r_4,T}\left(v_{r_4,T}\right)^c\right)\bigg]\bigg|
\end{aligned}$$

$$\leqslant C \sum_{l=r_1-r_2-1}^{\infty} \left\| \mathbb{E}\left[Y_{s_1,r_1,T} \middle| \mathcal{F}_{r_1,r_1-l} \right] - \mathbb{E}\left[Y_{s_1,r_1,T} \middle| \mathcal{F}_{r_1,r_1-l-1} \right] \right\|_1$$

$$+ C \sum_{l=r_1-r_2-1}^{\infty} \left\| \mathbb{E}\left[\widetilde{Y}_{s_1,r_1,T}\left(v_{r_1,T}\right) \middle| \mathcal{F}_{r_1,r_1-l} \right] - \mathbb{E}\left[\widetilde{Y}_{s_1,r_1,T}\left(v_{r_1,T}\right) \middle| \mathcal{F}_{r_1,r_1-l-1} \right] \right\|_1$$

$$\leqslant C \sum_{l=r_1-r_2}^{\infty} \Delta_l |s_1|_1. \tag{C.69}$$

Overall, (C.69) with $r_j := \lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + t_j$ as well as $v_{r_j,T} := \tilde{u}_{k,t_j}$ $\forall\, j \in \{1, \ldots, 4\}$, Assumption 3.1 $[\mathbf{WEI.1}]$, (C.67), (B.45), Lemma C.8 (ii), Assumption 3.15 $[\mathbf{W}^*]$ (i) $\big($which ensures $\beta = o(1/b)\big)$ and Remark A.2 (i) show (see (C.68) as well as (C.17)):

$$\mathrm{R}^*_{T,1} \leqslant \frac{C}{T^2} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \sum_{t_2=1}^{2\lfloor T_{\amalg} b \rfloor - 1} \sum_{t_1=t_2+1}^{2\lfloor T_{\amalg} b \rfloor} \sum_{l=t_1-t_2}^{\infty} \Delta_l \left(\sup_{\tilde{t}_1=1,\ldots,2\lfloor T_{\amalg} b \rfloor} \sum_{t_3,t_4=1}^{2\lfloor T_{\amalg} b \rfloor} \mathbb{1}_{\left\{ \left| \tilde{t}_1 - t_h \right| \leqslant 3 m_\beta \,\forall\, h \in \{3,4\} \right\}} \right) \leqslant C \frac{m_\beta^2}{T} = o(1). \tag{C.70}$$

It follows from Assumption 3.1 $[\mathbf{WEI.1}]$, (C.67), Lemma C.8 (ii), Assumption 3.15 $[\mathbf{W}^*]$ (i) $\big($which ensures $\beta = o(1/b)\big)$, Remark A.2 (i) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (note (C.68) as well as (C.17)):

$$\mathrm{R}^*_{T,2} \leqslant \frac{C \lfloor 1/(2b) \rfloor \lfloor T_{\amalg} b \rfloor m_\beta^2}{T^2 T b^2} \leqslant \frac{C m_\beta^2}{(Tb)^2} = o(1). \tag{C.71}$$

Below, $\mathrm{R}^*_{T,3}$ will be bounded from above. Therefor, one observes at first that Assumption 3.15 $[\mathbf{W}^*]$ (iii), shifting the index of a sum, Lemma C.8 (ii) and (iii), Remark A.2 (i) as well as Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) imply (recall (C.17) and Definition A.1 (i)):

$$\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t=1,\ldots,2\lfloor T_{\amalg} b \rfloor} \sum_{r=1}^{2\lfloor T_{\amalg} b \rfloor} \left| \mathbb{E}\left[W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + t, \{m_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + r, \{m_\beta\}} \right] \right|$$

$$= \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t=1,\ldots,2\lfloor T_{\amalg} b \rfloor} \sum_{\substack{r=1 \\ |t-r| \leqslant m_\beta}}^{2\lfloor T_{\amalg} b \rfloor} \left| \mathbb{E}\left[W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + t, \{m_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + r, \{m_\beta\}} \right] \right|$$

$$\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \left(\sup_{t=1,\ldots,2\lfloor T_{\amalg} b \rfloor} \sum_{\substack{r=1 \\ |t-r| \leqslant m_\beta}}^{2\lfloor T_{\amalg} b \rfloor} \left| \mathbb{E}\left[\left(W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + t, \{m_\beta\}} - W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + t} \right) W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + r, \{m_\beta\}} \right] \right. \right.$$

$$\left. \left. + \mathbb{E}\left[W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + t} \left(W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + r, \{m_\beta\}} - W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + r} \right) \right] \right| + \sup_{t=1,\ldots,2\lfloor T_{\amalg} b \rfloor} \sum_{r=1}^{2\lfloor T_{\amalg} b \rfloor} \left| K^*\left(\frac{r-t}{\beta} \right) \right| \right)$$

$$\leqslant C m_\beta \sup_{\tilde{t} \in \mathbb{Z}} \left\| W^*_{\tilde{t},\{m_\beta\}} - W^*_{\tilde{t}} \right\|_2 \left(\sup_{\tilde{r} \in \mathbb{Z}} \left\| W^*_{\tilde{r},\{m_\beta\}} \right\|_2 + \|W^*_0\|_2 \right) + \sup_{t=1,\ldots,2\lfloor T_{\amalg} b \rfloor} \sum_{r=1-t}^{2\lfloor T_{\amalg} b \rfloor - t} \left| K^*\left(\frac{r}{\beta} \right) \right|$$

$$\leqslant C \frac{\beta m}{Tb^2} + \sum_{r=-2\lfloor T_{\amalg} b \rfloor}^{2\lfloor T_{\amalg} b \rfloor} \left| K^*\left(\frac{r}{\beta} \right) \right|$$

$$\leqslant C\beta. \tag{C.72}$$

Further, it holds (see Definition A.1 (i)):

$$\frac{C}{T^2 T b^2} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \sum_{\substack{t_1,t_2,t_3,t_4 = 1 + m_\beta \\ t_1 \geqslant t_2 \geqslant \max\{t_3,t_4\}}}^{2\lfloor T_{\amalg} b \rfloor - 1 - m_\beta} \left| \mathbb{E}\left[W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + t_1, \{m_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + t_2, \{m_\beta\}} \right. \right.$$

$$\left. \left. \cdot W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + t_3, \{m_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\amalg} b \rfloor + t_4, \{m_\beta\}} \right] \right| \mathbb{1}_{\left\{ |t_1 - t_3| > 3 m_\beta \right\}} \mathbb{1}_{\left\{ \forall l_1 \in \{1,\ldots,4\}\, \exists l_2 \in \{1,\ldots,4\} \setminus \{l_1\} : \left| t_{l_1} - t_{l_2} \right| \leqslant m_\beta \right\}}$$

$$
\begin{aligned}
&\leqslant \frac{C}{T^2Tb^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{\substack{t_1,t_2,t_3,t_4=1+\mathscr{m}_\beta \\ t_1\geqslant t_2 \geqslant \max\{t_3,t_4\}}}^{2\lfloor T_{\mathfrak{U}} b\rfloor -1-\mathscr{m}_\beta} \Big| \mathbb{E}\Big[W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\mathscr{m}_\beta\}} W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\mathscr{m}_\beta\}} \\
&\cdot W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_3,\{\mathscr{m}_\beta\}} W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_4,\{\mathscr{m}_\beta\}} \Big]\Big| \mathbb{1}_{\{|t_1-t_3|>3\mathscr{m}_\beta\}} \mathbb{1}_{\{|t_1-t_2|\leqslant \mathscr{m}_\beta\}} \mathbb{1}_{\{|t_3-t_4|\leqslant \mathscr{m}_\beta\}} \\
&=: \mathrm{R}^*_{T,3,1}. \qquad\qquad (C.73)
\end{aligned}
$$

If the conditions $|t_1 - t_3| > 3\mathscr{m}_\beta$, $|t_1 - t_2| \leqslant \mathscr{m}_\beta$ and $|t_3 - t_4| \leqslant \mathscr{m}_\beta$ hold for some $t_1, \ldots, t_4 \in \mathbb{Z}$, one will obtain from the reverse triangle inequality:

$$
|t_1 - t_4| > 2\mathscr{m}_\beta, \quad |t_3 - t_2| > 2\mathscr{m}_\beta \quad \text{and} \quad |t_2 - t_4| > \mathscr{m}_\beta. \qquad (C.74)
$$

The conditions $|t_1 - t_3| > 3\mathscr{m}_\beta$, $|t_1 - t_2| \leqslant \mathscr{m}_\beta$ as well as $|t_3 - t_4| \leqslant \mathscr{m}_\beta$ contained in $\mathrm{R}^*_{T,3,1}$, (C.74) and Definition A.1 (i) provide that $W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\mathscr{m}_\beta\}} \cdot W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\mathscr{m}_\beta\}}$ is independent of $W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_3,\{\mathscr{m}_\beta\}} \cdot W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_4,\{\mathscr{m}_\beta\}}$. Thus, one obtains from (C.72) and Assumption 3.15 $[\mathbf{W}^*]$ (i) (see (C.17)):

$$
\begin{aligned}
\mathrm{R}^*_{T,3,1} &\leqslant \frac{C}{T^2Tb^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big| \mathbb{E}\Big[W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\mathscr{m}_\beta\}} W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\mathscr{m}_\beta\}} \Big]\Big| \\
&\quad \cdot \sum_{t_3,t_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big| \mathbb{E}\Big[W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_3,\{\mathscr{m}_\beta\}} W^*_{\lceil u_k T\rceil - \lfloor T_{\mathfrak{U}} b\rfloor + t_4,\{\mathscr{m}_\beta\}} \Big]\Big| \\
&\leqslant \frac{C \lfloor 1/(2b)\rfloor \lfloor T_{\mathfrak{U}} b\rfloor^2 \beta^2}{T^2Tb^2} \\
&= \frac{o\left(\frac{1}{b}\right) o\left(Tb^2\right)}{Tb} \\
&= o(1). \qquad\qquad (C.75)
\end{aligned}
$$

The indicator function on the left side of (C.73) which contains the condition $\forall l_1 \in \{1, \ldots, 4\}\, \exists l_2 \in \{1, \ldots, 4\}\backslash\{l_1\} : |t_{l_1} - t_{l_2}| \leqslant \mathscr{m}_\beta$ can be omitted because $\mathbb{E}\big[W^*_{t,\{\mathscr{m}_\beta\}}\big] = 0\ \forall\, t \in \mathbb{Z}$ (which follows from Assumption 3.15 $[\mathbf{W}^*]$ (iii)) implies that replacing this condition with the opposite one generates addends which equal zero. Hence, it follows from Assumption 3.1 $[\mathbf{WEI.1}]$, (C.73), (C.75) and similar arguments (recall (C.68)):

$$
\mathrm{R}^*_{T,3} = o(1). \qquad (C.76)
$$

In conclusion, (C.61) is an implication of (C.68), (C.70), (C.71), (C.76) and analog arguments (see (C.51) as well as (C.60)).
Further, one obtains similarly to (C.37) by using Assumption 3.15 $[\mathbf{W}^*]$ (ii), (C.67) and Lemma C.8 (ii) that (C.62) holds (note (C.60), (C.17) as well as (3.16)).
Overall, analogously to (C.27), it follows for all $\epsilon > 0$ from (C.61), (C.62) and similar arguments (recall (C.51)):

$$
\mathbb{E}\left[\left| \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}^*\left[\left(\mathbb{D}^{\circ *}_{T,k,\gamma}\right)^2 \mathbb{1}_{\{|\mathbb{D}^{\circ *}_{T,k,\gamma}|>\epsilon\}} \right] - 0 \right|\right] = \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left(\mathbb{D}^{\circ *}_{T,k,\gamma}\right)^2 \mathbb{1}_{\{|\mathbb{D}^{\circ *}_{T,k,\gamma}|>\epsilon\}} \right] = o(1),
$$

such that (6.30) in [52, Leucht and Neumann (2013), p. 275] holds.

The validity of (6.31) as well as (6.32)[8] in [52, Leucht and Neumann (2013), p. 275] with $\theta_r = 0$ $\forall\, r \in \mathbb{N}$ can be verified by using that $\left(\mathbb{D}^{\circ *}_{T,k,\gamma}\right)_{k=1}^{\lfloor 1/(2b)\rfloor}$ is a sequence of conditioned on $(X_{t,T})_{t=1}^T$ independent random variables (as explained above (C.58)), whereby θ_r is defined in [52, Leucht and Neumann (2013), p. 275].

[8]Since Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275 et seq.] is a direct implication of Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274 et seq.], it is easy to see that the comma between X^*_{n,t_1,j_1} and X^*_{n,t_2,j_2} in (6.32) in [52, Leucht and Neumann (2013), p. 275] is a typo.

Overall, Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275 et seq.] shows (C.55) (see (C.51)). It follows from (C.54) and (C.55) that (C.49) holds.

Next, (C.49) is used to prove Theorem 3.19 (i) and (ii).

(i) In the case $\sigma_{\mathfrak{U}_{0,1}}\left((1,-1),(1,-1)\right) > 0$, (C.49) with $\gamma = (1,-1)$ implies (3.40) by using the following chaining argument (recall also (3.21)):
One assumes that $J \in \mathbb{N}$ is arbitrary but fixed. It follows from the continuity of the distribution function $x \mapsto \mathbb{P}\left(Z_{\mathfrak{U}_{0,1}} \leqslant x\right)$ (which holds due to $\sigma_{\mathfrak{U}_{0,1}}\left((1,-1),(1,-1)\right) > 0$) that some points $-\infty = x_0 < x_1 < \cdots < x_J = \infty$ exist with $\mathbb{P}\left(Z_{\mathfrak{U}_{0,1}} \leqslant x_j\right) = j/J \; \forall j = 0,\ldots,J$. Since $x \mapsto \mathbb{P}^*\left(\sqrt{T}\,\widehat{\mathbb{D}}_T^*\left((1,-1)\right) \leqslant x\right)$ is monotonically increasing, one obtains for all $x \in \mathbb{R}$ similarly to the proof of Lemma 2.11 in [73, van der Vaart (2000), p. 12]:

$$\begin{aligned}&\left|\mathbb{P}^*\left(\sqrt{T}\,\widehat{\mathbb{D}}_T^*\left((1,-1)\right) \leqslant x\right) - \mathbb{P}\left(Z_{\mathfrak{U}_{0,1}} \leqslant x\right)\right| \\ &\leqslant \sup_{j=0,\ldots,J}\left|\mathbb{P}^*\left(\sqrt{T}\,\widehat{\mathbb{D}}_T^*\left((1,-1)\right) \leqslant x_j\right) - \mathbb{P}\left(Z_{\mathfrak{U}_{0,1}} \leqslant x_j\right)\right| + \frac{1}{J} \quad \text{a. s.}\end{aligned} \tag{C.77}$$

The right side of (C.77) converges in probability to $1/J$ due to (C.49) with $\gamma = (1,-1)$, which proves (3.40) because J is an arbitrary natural number.

Next, (3.41) is shown. If $\sigma_{\mathfrak{U}_{0,1}}\left((1,-1),(1,-1)\right) = 0$, one will obtain for $\gamma = (1,-1)$ from (C.56), (C.58) and the convergence in mean stated on the right side of (C.59):

$$\mathbb{E}\left[{\mathbb{D}^{\circ *}_{T,(1,-1)}}^2\right] = \mathbb{E}\left[\mathbb{E}^*\left[{\mathbb{D}^{\circ *}_{T,(1,-1)}}^2\right]\right] = \mathbb{E}\left[\mathrm{Var}^*\left(\mathbb{D}^{\circ *}_{T,(1,-1)}\right)\right] = o(1). \tag{C.78}$$

In conclusion, (3.41) is an implication of (C.54), (C.78), (3.27) and (3.28).

(ii) In the case $\sigma_{\mathfrak{U}_{0,1}}\left(\left(\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1},\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},2}\right),\left(\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1},\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},2}\right)\right) > 0$, one obtains from Lemma C.9, (C.49) with $\gamma = \left(\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1},\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},2}\right)$ and a chaining argument which is similar to (C.77) that (3.42) holds (see (3.23)).

Further, if $\sigma_{\mathfrak{U}_{0,1}}\left(\left(\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1},\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},2}\right),\left(\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1},\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},2}\right)\right) = 0$, one will obtain similarly to (C.78):

$$\mathbb{E}\left[{\mathbb{D}^{\circ *}_{T,\left(\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1},\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},2}\right)}}^2\right] = o(1). \tag{C.79}$$

In conclusion, (3.43) is an implication of Lemma C.9, (C.54) with $\gamma = \left(\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1},\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},2}\right)$, (C.79), (3.27) and (3.28).

□

Proof of Theorem 3.23. (i) Throughout this proof, it is supposed that T is large enough to ensure $2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{n} \geqslant 9\mathscr{n} + 2$ (see (C.17) as well as Definition A.1 (iv)), which holds for sufficiently large T due to Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii).
Further, one defines for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $\mathrm{R} \in \{\Re,\Im\}$, $t, j \in \{1,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor\}$, $s \in \mathbb{R}^d$ (recall the Definitions 3.3 (i), 3.8 (i), (C.17), A.1 (i) and (iv) as well as $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\begin{aligned}\widetilde{\mathbb{X}}_{T,k,\mathrm{R}}\left(t,s\right) &:= \widetilde{\mathbb{X}}_{T,\mathfrak{U}_{0,1},k,\mathrm{R}}\left(t,s\right) := \mathrm{R}\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}}b\rfloor + t}\left(\tilde{u}_{k,t}\right)\right\rangle}\right\}_{\mathscr{n}}, \\ \widetilde{\mathbb{K}}_{T,k,\mathrm{R}}\left(t,s\right) &:= \widetilde{\mathbb{K}}_{T,\mathfrak{U}_{0,1},k,\mathrm{R}}\left(t,s\right) := K\left(\frac{t - \lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right)\widetilde{\mathbb{X}}_{T,k,\mathrm{R}}\left(t,s\right), \\ \widetilde{\mathbb{I}}_{T,k,\mathrm{R}}\left(t,j\right) &:= \widetilde{\mathbb{I}}_{T,\mathfrak{U}_{0,1},k,\mathrm{R}}\left(t,j\right) := \int_{\mathbb{R}^d} \widetilde{\mathbb{K}}^c_{T,k,\mathrm{R}}\left(t,s\right)\widetilde{\mathbb{K}}^c_{T,k,\mathrm{R}}\left(j,s\right)\,\mathbf{w}(s)\,ds,\end{aligned}$$

$$\widetilde{\mathbb{S}}_{T,k,\mathrm{R}} := \widetilde{\mathbb{S}}_{T,\mathfrak{U}_{0,1},k,\mathrm{R}} := \frac{2T\sqrt{b}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \sum_{t=9\varkappa+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa} \sum_{j=1+2\varkappa}^{t-7\varkappa-1} \widetilde{\mathbb{I}}_{T,k,\mathrm{R}}\,(t,j)\,,$$

$$\widetilde{\mathbb{S}}_{T,k} := \widetilde{\mathbb{S}}_{T,\mathfrak{U}_{0,1},k} := \widetilde{\mathbb{S}}_{T,k,\Re} + \widetilde{\mathbb{S}}_{T,k,\Im}, \quad \widetilde{\mathbb{S}}_{T,\mathrm{R}} := \widetilde{\mathbb{S}}_{T,\mathfrak{U}_{0,1},\mathrm{R}} := \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \widetilde{\mathbb{S}}_{T,k,\mathrm{R}} \quad \text{and}$$

$$\widetilde{\mathbb{S}}_{T} := \widetilde{\mathbb{S}}_{T,\mathfrak{U}_{0,1}} := \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \widetilde{\mathbb{S}}_{T,k}, \tag{C.80}$$

whereby $\widetilde{\mathbb{X}}_{T,k,\mathrm{R}}\,(t,s)$ is well-defined for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $\mathrm{R} \in \{\Re,\Im\}$, $t \in \{1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor\}$, $s \in \mathbb{R}^d$ due to (C.18).
Since the null hypothesis $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\mathrm{distr}}$ holds by assumption, the Lemmata C.14, C.15, C.17 and C.18 as well as (C.399) provide (see (3.51)):

$$\left\| T\sqrt{b}\,\widehat{\mathbb{D}}_T - \mathbf{Bias}_{T,\mathfrak{U}_{0,1}}^{\mathrm{distr}} - \widetilde{\mathbb{S}}_T \right\|_1 = o(1). \tag{C.81}$$

Hence, Theorem 3.23 (i) will be proved if the following statement holds for $T \to \infty$:

$$\widetilde{\mathbb{S}}_T \xrightarrow{\mathrm{d}} Z_{\mathfrak{U}_{0,1}}^{\mathrm{distr}}. \tag{C.82}$$

From now on, the validity of the assumptions demanded in Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274 et seq.] will be examined in order to use this theorem to verify (C.82), whereby the expression $\widetilde{\mathbb{S}}_{T,k}$ takes the role of $X_{n,k}$ which originates from Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274 et seq.].

It holds for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $\mathrm{R} \in \{\Re,\Im\}$, $t,j \in \{1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor\}$ with $|t-j| > \varkappa$ (note (C.80) and Definition A.1 (i)):

$$\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k,\mathrm{R}}\,(t,j)\right] = 0, \tag{C.83}$$

which implies (recall (C.80)):

$$\mathbb{E}\left[\widetilde{\mathbb{S}}_{T,k,\mathrm{R}}\right] = 0 \quad \forall\, \mathrm{R} \in \{\Re,\Im\}\,,\, k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}, \tag{C.84}$$

such that the first assumption of Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274] is valid.
Further, one obtains for all $k_1,k_2 \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $t_1,t_2 \in \{1+\varkappa,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa\}$ with $k_1 \geqslant k_2+1$ similarly to (C.22) (see Definition 3.8 (i) and (C.17)):

$$\lfloor u_{k_1} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1 \geqslant \lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2 + 2 + 2\varkappa. \tag{C.85}$$

The random variable $\widetilde{\mathbb{S}}_{T,k,\mathrm{R}}$ with $\mathrm{R} \in \{\Re,\Im\}$ is measurable with respect to the sigma algebra generated by $\mathcal{F}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa, \lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + 1 + \varkappa}$ (recall Definition A.1 (i)), such that (C.85) with $t_1 = 1+\varkappa$ and $t_2 = 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa$ shows:

$$\left(\left(\widetilde{\mathbb{S}}_{T,k,\Re}, \widetilde{\mathbb{S}}_{T,k,\Im} \right)' \right)_{k=1}^{\lfloor 1/(2b)\rfloor} \text{ is a sequence of independent random variables.} \tag{C.86}$$

Hence, (C.84) implies:

$$\mathrm{Var}\left(\widetilde{\mathbb{S}}_T\right) = \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\widetilde{\mathbb{S}}_{T,k}^2\right]. \tag{C.87}$$

In addition, (C.400) and Lemma C.23 provide (note (3.52)):

$$\mathrm{Var}\left(\widetilde{\mathbb{S}}_T\right) = \sigma_{\mathfrak{U}_{0,1}}^{\mathrm{distr}} + o(1). \tag{C.88}$$

One obtains from (C.87), (C.88), Lemma 3.12 and Assumption 3.1 $[\mathbf{WEI.1}]$ (the latter two show

$\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} < \infty$ (see (3.52) as well as (3.17)) that the sequence $\left(\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\big[\widetilde{\mathbb{S}}^2_{T,k}\big]\right)_{T\in\mathbb{N}}$ converges for $T \to \infty$, such that it is bounded from above by an absolute constant $\nu_0 < \infty$. Thus, the second assumption of Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274] holds.

Moreover, (C.88) provides (6.25) in [52, Leucht and Neumann (2013), p. 274] with $\Sigma = \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}}$, whereby Σ originates from [52, Leucht and Neumann (2013), p. 274].

In the following, the validity of (6.26) in [52, Leucht and Neumann (2013), p. 275] is proved. Therefor, note at first that one obtains for all $\epsilon > 0$ similarly to (C.27) (recall (C.80)):

$$\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left(\widetilde{\mathbb{S}}_{T,k}\right)^2 \mathbb{1}_{\{|\widetilde{\mathbb{S}}_{T,k}|>\epsilon\}}\right] \leqslant \frac{C}{\epsilon^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left(\widetilde{\mathbb{S}}_{T,k,\Re}\right)^4\right] + \frac{C}{\epsilon^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left(\widetilde{\mathbb{S}}_{T,k,\Im}\right)^4\right]. \tag{C.89}$$

Next, it is shown that the first sum on the right side of (C.89) converges to zero. This and similar arguments together with (C.89) yield that (6.26) in [52, Leucht and Neumann (2013), p. 275] holds.

To meet this target, define for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $\underline{t} := (t_1,\ldots,t_4)$, $\underline{j} := (j_1,\ldots,j_4)$ and $\underline{s} := (s_1,\ldots,s_4)$ with $t_1,\ldots,t_4, j_1,\ldots,j_4 \in \{1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor\}$ as well as $s_1,\ldots,s_4 \in \mathbb{R}^d$ (see (C.80)):

$$\begin{aligned}\Theta_{T,k,\Re}\left(\underline{t},\underline{j},\underline{s}\right) := \mathbb{E}\Big[&\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_2)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_2,s_2)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_3,s_3)\\ &\cdot\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_3,s_3)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_4,s_4)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_4,s_4)\Big].\end{aligned} \tag{C.90}$$

Moreover, assume $|t_q - j_q| \geqslant 7\varkappa + 1\ \forall\, q \in \{1,\ldots,4\}$.

In the following, the process $\left(\widetilde{\mathbb{K}}^c_{T,k,\Re}(r,s)\right)_{r\in\{t_1,\ldots,t_4,j_1,\ldots,j_4\},s\in\{s_1,\ldots,s_4\}}$ will be disassembled into $P \leqslant 8$ sub-processes $\left(\widetilde{\mathbb{K}}^c_{T,k,\Re}(r,s)\right)_{r\in M_p,s\in\{s_1,\ldots,s_4\}}$ with $M_p \subseteq \mathcal{M} := \{t_1,\ldots,t_4,j_1,\ldots,j_4\}$, $p \in \{1,\ldots,P\}$ and $\left(\widetilde{\mathbb{K}}^c_{T,k,\Re}(r,s)\right)_{r\in M_p,s\in\{s_1,\ldots,s_4\}} \perp\!\!\!\perp \left(\widetilde{\mathbb{K}}^c_{T,k,\Re}(r,s)\right)_{r\in\mathcal{M}\setminus M_p,s\in\{s_1,\ldots,s_4\}}\ \forall\, p \in \{1,\ldots,P\}$, whereby the last property should hold by convention for $P = 1$ (in this case, the original process is the only sub-process). In addition, this decomposition has to be carried out in such a manner that the sets $M_1,\ldots,M_P$ are disjoint and P is as large as possible (the latter ensures that not more from each other independent sub-processes can be separated). These properties imply $\sup_{o_1,o_2\in M_p}|o_1 - o_2| \leqslant (\#M_p - 1)\,\varkappa$ $\forall\, p \in \{1,\ldots,P\}$ due to the definition of $\widetilde{\mathbb{K}}^c_{T,k,\Re}(r,s)$ (note (C.80) and Definition A.1 (i)). This, $(\#M_p - 1) \leqslant 7\ \forall\, p \in \{1,\ldots,P\}$ and the fact that $|t_q - j_q| \geqslant 7\varkappa + 1\ \forall\, q \in \{1,\ldots,4\}$ is assumed provide that no M_p exists which contains t_q and j_q for a $q \in \{1,\ldots,4\}$. Therefore, $\max_{p=1,\ldots,P}\#M_p \leqslant 4$.

Hence, the expectation contained in (C.90) can be splitted by using the following analysis:

If $\max_{p=1,\ldots,P}\#M_p = 4$ and $\Theta_{T,k,\Re}\left(\underline{t},\underline{j},\underline{s}\right) \neq 0$, the property $\mathbb{E}\big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r,s)\big] = 0\ \forall\, r \in \mathcal{M},\ s \in \mathbb{R}^d$ will imply that either $\big(\#M_p = 4$ for $p \in \{1,2\} \wedge P = 2\big)$ or $\big(\exists!\, p \in \{1,\ldots,4\} : \big(\#M_p = 4 \wedge \#M_{p'} = 2\ \forall\, p' \in \{1,\ldots,P\}\backslash\{p\}\big) \wedge P = 3\big)$.

If $\max_{p=1,\ldots,P}\#M_p = 3$ and $\Theta_{T,k,\Re}\left(\underline{t},\underline{j},\underline{s}\right) \neq 0$, the property $\mathbb{E}\big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r,s)\big] = 0\ \forall\, r \in \mathcal{M},\ s \in \mathbb{R}^d$ will ensure that $\#M_{p_1} = 3$, $\#M_{p_2} = 3$ and $\#M_{p_3} = 2$ for differing $p_1,p_2,p_3 \in \{1,\ldots,P\}$, whereby $P = 3$ will hold.

If $\max_{p=1,\ldots,P}\#M_p = 2$ as well as $\Theta_{T,k,\Re}\left(\underline{t},\underline{j},\underline{s}\right) \neq 0$, the fact that $\mathbb{E}\big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r,s)\big] = 0\ \forall\, r \in \mathcal{M}$, $s \in \mathbb{R}^d$ will provide $\#M_p = 2\ \forall\, p \in \{1,\ldots,P\}$ and $P = 4$.

If $\max_{p=1,\ldots,P}\#M_p = 1$, the property $\mathbb{E}\big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r,s)\big] = 0\ \forall\, r \in \mathcal{M},\ s \in \mathbb{R}^d$ will yield $\Theta_{T,k,\Re}\left(\underline{t},\underline{j},\underline{s}\right) = 0$.

Overall, it holds for all $\underline{s} := (s_1,\ldots,s_4) \in \mathbb{R}^{d\times 4}$ $\big($recall (C.90) and $\sup_{o_1,o_2\in M_p}|o_1 - o_2| \leqslant (\#M_p - 1)\,\varkappa\,\forall\, p = 1,\ldots,P\big)$:

$$\begin{aligned}&\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\ \sum_{t_1,\ldots,t_4=9\varkappa+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa}\ \sum_{j_1=1+2\varkappa}^{t_1-7\varkappa-1}\ \sum_{j_2=1+2\varkappa}^{t_2-7\varkappa-1}\ \sum_{j_3=1+2\varkappa}^{t_3-7\varkappa-1}\ \sum_{j_4=1+2\varkappa}^{t_4-7\varkappa-1}\left|\Theta_{T,k,\Re}\left(\underline{t},\underline{j},\underline{s}\right)\right|\\ \leqslant &\sum_{\substack{q_1,\ldots,q_8\in\{1,\ldots,4\}:\\ \nexists\, x\in\{1,\ldots,8\},y\in\{1,\ldots,8\}\backslash\{x\},z\in\{1,\ldots,8\}\backslash\{x,y\}:\\ q_x=q_y=q_z}}\ \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left\{\sum_{r_1,\ldots,r_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor}\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1,s_{q_1})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2,s_{q_2})\right.\right.\right.\end{aligned}$$

$$\cdot \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3, s_{q_3})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4, s_{q_4})\Big]\Big| \cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_1,\ldots,r_4\}}|o_1-o_2|\leqslant 3\varkappa\}} \cdot \Bigg(\sum_{r_5,\ldots,r_8=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_5, s_{q_5})$$

$$\cdot \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_6, s_{q_6})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_7, s_{q_7})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_8, s_{q_8})\Big]\Big| \cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_5,\ldots,r_8\}}|o_1-o_2|\leqslant 3\varkappa\}}$$

$$+ \sum_{r_5,r_6=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_5, s_{q_5})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_6, s_{q_6})\Big]\Big| \sum_{r_7,r_8=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_7, s_{q_7})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_8, s_{q_8})\Big]\Big| \Bigg)$$

$$+ \sum_{r_1,\ldots,r_3=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1, s_{q_1})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2, s_{q_2})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3, s_{q_3})\Big]\Big| \cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_1,\ldots,r_3\}}|o_1-o_2|\leqslant 2\varkappa\}}$$

$$\cdot \sum_{r_4,\ldots,r_6=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4, s_{q_4})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_5, s_{q_5})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_6, s_{q_6})\Big]\Big| \cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_4,\ldots,r_6\}}|o_1-o_2|\leqslant 2\varkappa\}}$$

$$\cdot \sum_{r_7,r_8=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_7, s_{q_7})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_8, s_{q_8})\Big]\Big| + \sum_{r_1,r_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1, s_{q_1})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2, s_{q_2})\Big]\Big|$$

$$\cdot \sum_{r_3,r_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3, s_{q_3})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4, s_{q_4})\Big]\Big| \cdot \sum_{r_5,r_6=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_5, s_{q_5})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_6, s_{q_6})\Big]\Big|$$

$$\cdot \sum_{r_7,r_8=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_7, s_{q_7})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_8, s_{q_8})\Big]\Big| \Bigg\}. \tag{C.91}$$

Further, one obtains for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $r_1, \ldots, r_4 \in \{1, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor\}$, $s_{q_1}, \ldots, s_{q_4} \in \mathbb{R}^d$ with $r_1 > r_2 \geqslant \max\{r_3, r_4\}$ by using Lemma B.4 (iv) with $q = 1 + \delta$ and shifting the index of a sum (see (C.80) as well as Definition A.1 (i)):

$$\Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1, s_{q_1})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2, s_{q_2})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3, s_{q_3})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4, s_{q_4})\Big]\Big|$$

$$= \Big|\mathbb{E}\Big[\Big(\mathbb{E}\Big[\widetilde{\mathbb{K}}_{T,k,\Re}(r_1, s_{q_1}) \,\Big|\mathcal{F}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1}\Big] - \mathbb{E}\Big[\widetilde{\mathbb{K}}_{T,k,\Re}(r_1, s_{q_1}) \,\Big|\mathcal{F}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1, \lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_2 + 1}\Big]\Big)$$

$$\cdot \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2, s_{q_2})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3, s_{q_3})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4, s_{q_4})\Big]\Big|$$

$$\leqslant C \sup_{u\in[0,1]} \Big\|\mathbb{E}\Big[\mathbb{E}\Big[\cos\Big(\Big\langle s_{q_1}, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1}(u)\Big\rangle\Big) \,\Big|\mathcal{F}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1}\Big] - \mathbb{E}\Big[\cos\Big(\Big\langle s_{q_1}, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1}(u)\Big\rangle\Big)$$

$$\Big|\mathcal{F}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1, \lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_2 + 1}\Big] \,\Big|\mathcal{F}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1, \lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1 - \varkappa}\Big]\Big\|_{1+\delta}$$

$$\leqslant C \sup_{u\in[0,1]} \sum_{l=r_1-r_2-1}^{\infty} \Big\|\mathbb{E}\Big[\cos\Big(\Big\langle s_{q_1}, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1}(u)\Big\rangle\Big) \,\Big|\mathcal{F}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1, \lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1 - l}\Big]$$

$$- \mathbb{E}\Big[\cos\Big(\Big\langle s_{q_1}, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1}(u)\Big\rangle\Big) \,\Big|\mathcal{F}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1, \lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + r_1 - l - 1}\Big]\Big\|_{1+\delta}$$

$$\leqslant C \sum_{l=r_1-r_2}^{\infty} \Delta_l \left|s_{q_1}\right|_1. \tag{C.92}$$

It follows for all $s_{q_1}, s_{q_2}, s_{q_3}, s_{q_4} \in \mathbb{R}^d$ from (C.92) and (B.45):

$$\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{\substack{r_1,\ldots,r_4=1 \\ r_1\geqslant r_2\geqslant \max\{r_3,r_4\}}}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1, s_{q_1})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2, s_{q_2})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3, s_{q_3})\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4, s_{q_4})\Big]\Big|$$

$$\cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_1,\ldots,r_4\}}|o_1-o_2|\leqslant 3\varkappa\}}$$

$$\leqslant C \sum_{r_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left(\sum_{r_1=r_2+1}^{\infty} \sum_{l=r_1-r_2}^{\infty} \Delta_l \left|s_{q_1}\right|_1 + \sum_{r_1=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \mathbb{1}_{\{r_1=r_2\}} \right) \sup_{\tilde{r}_2=1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{r_3,r_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \mathbb{1}_{\{|r_3-\tilde{r}_2|\leqslant 3\varkappa\}} \mathbb{1}_{\{|r_4-\tilde{r}_2|\leqslant 3\varkappa\}}$$

$$\leqslant C \lfloor T_{\mathfrak{U}} b\rfloor \varkappa^2 \left(\left|s_{q_1}\right|_1 + 1\right). \tag{C.93}$$

This and similar arguments show for all $s_{q_1}, s_{q_2}, s_{q_3}, s_{q_4} \in \mathbb{R}^d$:

$$\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{r_1,\ldots,r_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\mathfrak{R}}(r_1, s_{q_1})\, \widetilde{\mathbb{K}}^c_{T,k,\mathfrak{R}}(r_2, s_{q_2})\, \widetilde{\mathbb{K}}^c_{T,k,\mathfrak{R}}(r_3, s_{q_3})\, \widetilde{\mathbb{K}}^c_{T,k,\mathfrak{R}}(r_4, s_{q_4})\right]\right|$$
$$\cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_1,\ldots,r_4\}}|o_1-o_2|\leqslant 3\varkappa\}}$$
$$\leqslant C\,\lfloor T_{\mathfrak{U}} b\rfloor\, \varkappa^2 \left(1 + |s_{q_1}|_1 + |s_{q_2}|_1 + |s_{q_3}|_1 + |s_{q_4}|_1\right). \tag{C.94}$$

One obtains for all $\underline{s} := (s_1, \ldots, s_4) \in \mathbb{R}^{d\times 4}$ from (C.91), (C.94) and analogous arguments:

$$\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,\ldots,t_4=9\varkappa+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa} \sum_{j_1=1+2\varkappa}^{t_1-7\varkappa-1} \sum_{j_2=1+2\varkappa}^{t_2-7\varkappa-1} \sum_{j_3=1+2\varkappa}^{t_3-7\varkappa-1} \sum_{j_4=1+2\varkappa}^{t_4-7\varkappa-1} \left|\Theta_{T,k,\mathfrak{R}}\left(\underline{t}, \underline{j}, \underline{s}\right)\right|$$
$$\leqslant \sum_{\substack{q_1,\ldots,q_8\in\{1,\ldots,4\}:\\ \nexists\,(x\in\{1,\ldots,8\}\wedge y\in\{1,\ldots,8\}\setminus\{x\}\wedge z\in\{1,\ldots,8\}\setminus\{x,y\}:\\ q_x=q_y=q_z)}} C\,\lfloor 1/(2b)\rfloor \cdot \Big\{ \lfloor T_{\mathfrak{U}} b\rfloor\, \varkappa^2 \left(1 + |s_{q_1}|_1 + |s_{q_2}|_1 + |s_{q_3}|_1 + |s_{q_4}|_1\right)$$
$$\cdot \Big(\lfloor T_{\mathfrak{U}} b\rfloor\, \varkappa^2 \left(1 + |s_{q_5}|_1 + |s_{q_6}|_1 + |s_{q_7}|_1 + |s_{q_8}|_1\right) + \lfloor T_{\mathfrak{U}} b\rfloor \left(1 + |s_{q_5}|_1 + |s_{q_6}|_1\right) \lfloor T_{\mathfrak{U}} b\rfloor$$
$$\cdot \left(1 + |s_{q_7}|_1 + |s_{q_8}|_1\right) \Big) + \lfloor T_{\mathfrak{U}} b\rfloor\, \varkappa \left(1 + |s_{q_1}|_1 + |s_{q_2}|_1 + |s_{q_3}|_1\right) \cdot \lfloor T_{\mathfrak{U}} b\rfloor\, \varkappa \left(1 + |s_{q_4}|_1 + |s_{q_5}|_1 + |s_{q_6}|_1\right)$$
$$\cdot \lfloor T_{\mathfrak{U}} b\rfloor \left(1 + |s_{q_7}|_1 + |s_{q_8}|_1\right) + \lfloor T_{\mathfrak{U}} b\rfloor \left(1 + |s_{q_1}|_1 + |s_{q_2}|_1\right) \cdot \lfloor T_{\mathfrak{U}} b\rfloor \cdot \left(1 + |s_{q_3}|_1 + |s_{q_4}|_1\right)$$
$$\cdot \lfloor T_{\mathfrak{U}} b\rfloor \left(1 + |s_{q_5}|_1 + |s_{q_6}|_1\right) \cdot \lfloor T_{\mathfrak{U}} b\rfloor \left(1 + |s_{q_7}|_1 + |s_{q_8}|_1\right) \Big\}. \tag{C.95}$$

Thus, Assumption 3.1 $[\mathbf{WEI.1}]$ and Remark A.2 (ii) show (recall (C.80), (C.90) as well as (C.17)):

$$\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left(\widetilde{\mathbb{S}}_{T,k,\mathfrak{R}}\right)^4\right] \leqslant \frac{C}{T^4 b^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,\ldots,t_4=9\varkappa+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa} \sum_{j_1=1+2\varkappa}^{t_1-7\varkappa-1} \sum_{j_2=1+2\varkappa}^{t_2-7\varkappa-1} \sum_{j_3=1+2\varkappa}^{t_3-7\varkappa-1} \sum_{j_4=1+2\varkappa}^{t_4-7\varkappa-1}$$
$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\Theta_{T,k,\mathfrak{R}}\left(\underline{t}, \underline{j}, \underline{s}\right)\right| \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2\, \mathbf{w}(s_3)\, ds_3\, \mathbf{w}(s_4)\, ds_4$$
$$\leqslant \frac{C}{T^4 b^2} \lfloor 1/(2b)\rfloor \left(\lfloor T_{\mathfrak{U}} b\rfloor^2 \varkappa^4 + \lfloor T_{\mathfrak{U}} b\rfloor^3 \varkappa^2 + \lfloor T_{\mathfrak{U}} b\rfloor^3 \varkappa^2 + \lfloor T_{\mathfrak{U}} b\rfloor^4\right)$$
$$= o(1). \tag{C.96}$$

The validity of (6.26) in [52, Leucht and Neumann (2013), p. 275] is an implication of (C.89), (C.96) and similar arguments.
The conditions (6.27) and (6.28) in [52, Leucht and Neumann (2013), p. 275] are valid with $\theta_r = 0$ $\forall\, r \in \mathbb{N}$ due to (C.86), whereby θ_r is defined in [52, Leucht and Neumann (2013), p. 275].
Overall, Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274 et seq.] shows (C.82). Theorem 3.23 (i) is an implication of (C.81) and (C.82).

(ii) To prove Theorem 3.23 (ii), note at first that it holds due to Lemma 3.12 and Assumption 3.1 $[\mathbf{WEI.1}]$ (see (3.51) as well as (3.17)):

$$\left|\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}\right| \leqslant \frac{C}{\sqrt{b}}. \tag{C.97}$$

Further, the inclusion-exclusion principle provides for all measurable sets $\mathcal{A}$ and $\mathcal{B}$:

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geqslant \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{B}) - 1. \tag{C.98}$$

One obtains from (C.97) and (C.98) (recall (3.54) as well as Definition 3.3 (i)):

$$\lim_{T\to\infty} \mathbb{P}\left(T\sqrt{b}\,\widehat{\mathbb{D}}_T - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}} > \tau_T\right)$$
$$= \lim_{T\to\infty} \mathbb{P}\left(T\sqrt{b}\left(\widehat{\mathbb{D}}_T - \mathbb{D}\right) - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}} + T\sqrt{b}\,\mathbb{D} > \tau_T\right)$$

$$\geqslant \lim_{T\to\infty} \mathbb{P}\left(-\left|\widehat{\mathbb{D}}_T - \mathbb{D}\right| - \frac{\left|\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}\right|}{T\sqrt{b}} + \mathbb{D} > \frac{\tau_T}{T\sqrt{b}}, \left|\widehat{\mathbb{D}}_T - \mathbb{D}\right| \leqslant T^{-1/3} \right)$$

$$\geqslant \lim_{T\to\infty} \mathbb{P}\left(-T^{-1/3} - \frac{C}{Tb} + \mathbb{D} > \frac{\tau_T}{T\sqrt{b}} \right) + \lim_{T\to\infty} \mathbb{P}\left(\left|\widehat{\mathbb{D}}_T - \mathbb{D}\right| \leqslant T^{-1/3} \right) - 1. \tag{C.99}$$

Since $\mathbf{H}^{\mathrm{distr}}_{1,\mathfrak{U}_{0,1}}$ (see (3.49)) holds by assumption, Proposition 3.6 (iii) provides $\mathbb{D} > 0$ (recall Definition 3.3 (i)). Hence, (3.54) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) imply that the first limit on the right side of (C.99) is one. Moreover, Theorem 3.13 (ii) shows that the second limit on the right side of (C.99) is one. Thus, Theorem 3.23 (ii) follows from (C.99). □

Proof of Theorem 3.25. Throughout this proof, it is supposed that T is large enough to ensure $2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta \geqslant 9\varkappa_\beta + 2$ (see (C.17) as well as Definition A.1 (v)), which holds for sufficiently large T due to Lemma C.25 (i) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii).
To verify Theorem 3.25, it will be shown at first for $T \to \infty$ that:

$$T\sqrt{b}\,\widehat{\mathbb{D}}^{*}_{T,\mathrm{Test}} - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}} \xrightarrow{\mathrm{d}} Z^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} \quad \text{in probability,} \tag{C.100}$$

which means that the distance (quantified by the Prokhorov metric) between the conditional distribution of $T\sqrt{b}\,\widehat{\mathbb{D}}^{*}_{T,\mathrm{Test}} - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}}$ (conditioned on $X_{1,T},\ldots,X_{T,T}$) and the distribution of $Z^{\mathrm{distr}}_{\mathfrak{U}_{0,1}}$ converges to zero in probability.
To prove (C.100), one defines for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $\mathrm{R} \in \{\Re,\Im\}$, $t, j \in \{1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor\}$, $s \in \mathbb{R}^d$ (recall Definition 3.3 (i), 3.8 (i), (C.17), A.1 (i) and (v) as well as $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\begin{aligned}
&\mathbb{X}_{T,k,\mathrm{R}}(t,s) := \mathbb{X}_{T,\mathfrak{U}_{0,1},k,\mathrm{R}}(t,s) := \mathrm{R}\left\{ e^{i\left\langle s, X_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t, T}\right\rangle} \right\},\\
&\mathbb{K}_{T,k,\mathrm{R}}(t,s) := \mathbb{K}_{T,\mathfrak{U}_{0,1},k,\mathrm{R}}(t,s) := K\left(\frac{t - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0) \right) \mathbb{X}_{T,k,\mathrm{R}}(t,s),\\
&\mathbb{I}_{T,k,\mathrm{R}}(t,j) := \mathbb{I}_{T,\mathfrak{U}_{0,1},k,\mathrm{R}}(t,j) := \int_{\mathbb{R}^d} \mathbb{K}^c_{T,k,\mathrm{R}}(t,s)\, \mathbb{K}^c_{T,k,\mathrm{R}}(j,s)\, \mathbf{w}(s)\, ds,\\
&\mathbb{I}^*_{T,k,\mathrm{R}}(t,j) := \mathbb{I}^*_{T,\mathfrak{U}_{0,1},k,\mathrm{R}}(t,j) := \mathbb{I}_{T,k,\mathrm{R}}(t,j)\, W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t, \{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j, \{\varkappa_\beta\}},\\
&\mathbb{S}^*_{T,k,\mathrm{R}} := \mathbb{S}^*_{T,\mathfrak{U}_{0,1},k,\mathrm{R}} := \frac{2T\sqrt{b}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \sum_{t=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta} \sum_{j=1+2\varkappa_\beta}^{t-7\varkappa_\beta - 1} \mathbb{I}^*_{T,k,\mathrm{R}}(t,j),\\
&\mathbb{S}^*_{T,k} := \mathbb{S}^*_{T,\mathfrak{U}_{0,1},k} := \mathbb{S}^*_{T,k,\Re} + \mathbb{S}^*_{T,k,\Im}, \quad \mathbb{S}^*_{T,\mathrm{R}} := \mathbb{S}^*_{T,\mathfrak{U}_{0,1},\mathrm{R}} := \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{S}^*_{T,k,\mathrm{R}} \quad \text{and}\\
&\mathbb{S}^*_T := \mathbb{S}^*_{T,\mathfrak{U}_{0,1}} := \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{S}^*_{T,k},
\end{aligned} \tag{C.101}$$

whereby $\mathbb{X}_{T,k,\mathrm{R}}(t,s)$ is well-defined for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $\mathrm{R} \in \{\Re,\Im\}$, $t \in \{1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor\}$, $s \in \mathbb{R}^d$ because (C.52) ensures that this expression just takes $X_{r,T}$ with $r \in \{1,\ldots,T\}$ into account.

One obtains from the Lemmata C.26 as well as C.31 and (3.27) (see (3.26)):

$$T\sqrt{b}\,\widehat{\mathbb{D}}^{*}_{T,\mathrm{Test}} - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}} - \mathbb{S}^*_T = o^*_{\mathbb{P}}(1). \tag{C.102}$$

Thus, to prove (C.100), it suffices to show for $T \to \infty$ that:

$$\mathbb{S}^*_T \xrightarrow{\mathrm{d}} Z^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} \quad \text{in probability.} \tag{C.103}$$

From now on, the validity of the assumptions demanded in Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275 et seq.] will be examined in order to use this corollary to verify (C.103), whereby the expression $\mathbb{S}^*_{T,k}$ takes the role of $X^*_{n,k}$ which originates from Corollary 6.1 in [52, Leucht and Neumann

(2013), p. 275 et seq.].

The Assumptions 3.15 $[\mathbf{W}^*]$ (ii) and (iii) imply (recall (C.101) as well as Definition A.1 (i)):

$$\mathbb{E}^*\left[\mathbb{S}^*_{T,k}\right] = 0 \quad \forall\, k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}, \tag{C.104}$$

such that the first assumption of Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275] is valid. Further, it follows for all $k_1, k_2 \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $t_1, t_2 \in \{1 + \varkappa_\beta, \ldots, 2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa_\beta\}$ with $k_1 \geqslant k_2 + 1$ similarly to (C.22):

$$\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1 \geqslant \lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2 + 2 + 2\varkappa_\beta. \tag{C.105}$$

One obtains from Definition A.1 (i), Assumption 3.15 $[\mathbf{W}^*]$ (ii) and (C.105) with $t_1 = 1 + \varkappa_\beta$ as well as $t_2 = 2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa_\beta$ that $\left(\mathbb{S}^*_{T,k}\right)_{k=1}^{\lfloor 1/(2b) \rfloor}$ is a sequence of conditioned on $(X_{t,T})_{t=1}^{T}$ independent random variables, such that (C.104) provides:

$$\mathrm{Var}^*\left(\mathbb{S}^*_T\right) = \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{E}^*\left[\left(\mathbb{S}^*_{T,k}\right)^2\right] \quad \text{a. s.} \tag{C.106}$$

Markov's inequality, (C.106) and Lemma C.32 show:

$$\mathbb{P}\left(\sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{E}^*\left[\left(\mathbb{S}^*_{T,k}\right)^2\right] > \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} + 1\right) \leqslant \mathbb{E}\left[\left|\sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{E}^*\left[\left(\mathbb{S}^*_{T,k}\right)^2\right] - \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}}\right|\right] \overset{T\to\infty}{\longrightarrow} 0. \tag{C.107}$$

The second assumption of Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275] holds due to (C.107), Lemma 3.12 and Assumption 3.1 $[\mathbf{WEI.1}]$, whereby the latter two provide $\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} < \infty$ (see (3.52) as well as (3.17)).
Lemma C.32 (recall also (C.101)) shows the validity of (6.29) in [52, Leucht and Neumann (2013), p. 275] with $\Sigma = \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}}$ (note that Σ originates from (6.29) in [52, Leucht and Neumann (2013), p. 275]).
Next, it is proved that the condition (6.30) in [52, Leucht and Neumann (2013), p. 275] is fulfilled. Therefor, define for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $\mathrm{R} \in \{\Re, \Im\}$, $t, j \in \{1, \ldots, 2\lfloor T_{\mathfrak{U}} b \rfloor\}$ (recall the Definitions 3.3 (i), (C.80), 3.8 (i), (C.17), A.1 (i) as well as (v) and that, as mentioned at the beginning of this proof, T is assumed to be large enough to ensure $2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa_\beta \geqslant 9\varkappa_\beta + 2$):

$$\begin{aligned}
&\widetilde{\mathbb{I}}^*_{T,k,\mathrm{R}}(t,j) := \widetilde{\mathbb{I}}^*_{T,\mathfrak{U}_{0,1},k,\mathrm{R}}(t,j) := \widetilde{\mathbb{I}}_{T,k,\mathrm{R}}(t,j)\, W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t, \{\varkappa_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + j, \{\varkappa_\beta\}}, \\
&\widetilde{\mathbb{S}}^*_{T,k,\mathrm{R}} := \widetilde{\mathbb{S}}^*_{T,\mathfrak{U}_{0,1},k,\mathrm{R}} := \frac{2T\sqrt{b}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b) \rfloor \lfloor Tb \rfloor^2} \sum_{t=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa_\beta} \; \sum_{j=1+2\varkappa_\beta}^{t-7\varkappa_\beta-1} \widetilde{\mathbb{I}}^*_{T,k,\mathrm{R}}(t,j), \\
&\widetilde{\mathbb{S}}^*_{T,k} := \widetilde{\mathbb{S}}^*_{T,\mathfrak{U}_{0,1},k} := \widetilde{\mathbb{S}}^*_{T,k,\Re} + \widetilde{\mathbb{S}}^*_{T,k,\Im}, \quad \widetilde{\mathbb{S}}^*_{T,\mathrm{R}} := \widetilde{\mathbb{S}}^*_{T,\mathfrak{U}_{0,1},\mathrm{R}} := \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \widetilde{\mathbb{S}}^*_{T,k,\mathrm{R}} \quad \text{and} \\
&\widetilde{\mathbb{S}}^*_T := \widetilde{\mathbb{S}}^*_{T,\mathfrak{U}_{0,1}} := \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \widetilde{\mathbb{S}}^*_{T,k}.
\end{aligned} \tag{C.108}$$

In the following, it is verified that (see (C.101)):

$$\sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left(\mathbb{S}^*_{T,k,\Re} - \widetilde{\mathbb{S}}^*_{T,k,\Re}\right)^4\right] = o(1) \tag{C.109}$$

and:

$$\sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left|\mathbb{E}\left[\left(\widetilde{\mathbb{S}}^*_{T,k,\Re}\right)^4\right]\right| = o(1). \tag{C.110}$$

Subsequently, the validity of (6.30) in [52, Leucht and Neumann (2013), p. 275] will be concluded from (C.109), (C.110) and similar arguments.

Lemma C.16 (i) and (C.65) provide for all $s \in \mathbb{R}^d$, $q \geqslant 1+\delta$ (recall (C.101), (C.80) as well as Definition A.1 (i)):

$$\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor}\sup_{t=1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor}\left\|\mathbb{X}^c_{T,k,\mathfrak{R}}(t,s)-\widetilde{\mathbb{X}}^c_{T,k,\mathfrak{R}}(t,s)\right\|_q$$
$$\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor}\sup_{t=1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor}\left(\left\|\left(e^{i\left\langle s,X_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t,T}\right\rangle}-\left(e^{i\left\langle s,X_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t,T}\right\rangle}\right)_{\varkappa}\right)^c\right\|_q\right.$$
$$\left.+\left\|\left(e^{i\left\langle s,X_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t,T}\right\rangle}-e^{i\left\langle s,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t}\left(\widetilde{u}_{k,t}\right)\right\rangle}\right)^c_{\varkappa}\right\|_q\right)$$
$$\leqslant C\left(\frac{1}{(Tb)^{(1+\delta)/(\delta q)}}+\frac{1}{T^{(1+\delta)/q}}\right)|s|_1^{\frac{1+\delta}{q}}. \tag{C.111}$$

The equation:

$$x_1x_2-y_1y_2=(x_1-y_1)\,x_2+(x_2-y_2)\,y_1\ \ \forall\, x_1,x_2,y_1,y_2\in\mathbb{R}, \tag{C.112}$$

(C.111), $\left|\mathbb{K}^c_{T,k,\mathfrak{R}}(t,s)\right|\leqslant C$ and $\left|\widetilde{\mathbb{K}}^c_{T,k,\mathfrak{R}}(t,s)\right|\leqslant C$ a. s. (see (C.101) and (C.80)) for all $k\in\{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $t\in\{1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor\}$, $s\in\mathbb{R}^d$, Assumption 3.1 $[\mathbf{WEI.1}]$, $(1+\delta)/\delta\geqslant 2$ (which holds because $\delta\in(0,1]$ according to Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$) as well as Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) show for all $q\geqslant 1+\delta$ (recall (C.101) and (C.80)):

$$\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor}\sup_{t,j=1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor}\left\|\mathbb{I}_{T,k,\mathrm{R}}(t,j)-\widetilde{\mathbb{I}}_{T,k,\mathrm{R}}(t,j)\right\|_q\leqslant C\left(\frac{1}{(Tb)^{(1+\delta)/(\delta q)}}+\frac{1}{T^{(1+\delta)/q}}\right)=o\left(\frac{1}{T^{1/q}}\right). \tag{C.113}$$

In addition, one obtains similarly to (C.72) by using Lemma C.25 (ii) and (i) as well as Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (see (C.17) as well as Definition A.1 (i)):

$$\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor}\sup_{t=1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor}\sum_{r=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor}\left|\mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+r,\{\varkappa_\beta\}}\right]\right|\leqslant C\left(\frac{\varkappa_\beta}{Tb}+\beta\right)\leqslant C\beta. \tag{C.114}$$

Moreover, it holds for all $t_1,\ldots,t_4\in\{1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor\}$ with $t_1\geqslant\cdots\geqslant t_4$ that either $|t-t'|\leqslant 3\varkappa_\beta$ for all $t,t'\in\{t_1,\ldots,t_4\}$ or $t,t'\in\{t_1,\ldots,t_4\}$ exist with $|t-t'|>3\varkappa_\beta$. In the latter case, one obtains that $t_1-t_2>\varkappa_\beta$ or $t_2-t_3>\varkappa_\beta$ or $t_3-t_4>\varkappa_\beta$. If $t_1-t_2>\varkappa_\beta$, the random variable $W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1,\{\varkappa_\beta\}}$ is independent of $(W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t,\{\varkappa_\beta\}})_{t\in\{t_2,t_3,t_4\}}$ (recall Definition A.1 (i)). If $t_3-t_4>\varkappa_\beta$, the random variable $W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_4,\{\varkappa_\beta\}}$ is independent of $(W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t,\{\varkappa_\beta\}})_{t\in\{t_1,t_2,t_3\}}$. Thus, Assumption 3.15 $[\mathbf{W}^*]$ (iii) $\big($which ensures $\mathbb{E}\big[W^*_{t,\{\varkappa_\beta\}}\big]=\mathbb{E}\big[W^*_t\big]=0\ \forall\, t\in\mathbb{Z}\big)$ and (C.114) provide:

$$\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,t_2,t_3,t_4=1\\ t_1\geqslant\cdots\geqslant t_4}}^{2\lfloor T_{\mathfrak{U}} b\rfloor}\left|\mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_2,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_3,\{\varkappa_\beta\}}\right.\right.$$
$$\left.\left.\cdot W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_4,\{\varkappa_\beta\}}\right]\right|$$
$$\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,t_2,t_3,t_4=1\\ t_1\geqslant\cdots\geqslant t_4}}^{2\lfloor T_{\mathfrak{U}} b\rfloor}\left|\mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_2,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_3,\{\varkappa_\beta\}}\right.\right.$$
$$\left.\left.\cdot W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_4,\{\varkappa_\beta\}}\right]\right|\mathbb{1}_{\{\forall\, t,t'\in\{t_1,\ldots,t_4\}:|t-t'|\leqslant 3\varkappa_\beta\}}$$
$$+\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,t_2,t_3,t_4=1\\ t_1\geqslant\cdots\geqslant t_4}}^{2\lfloor T_{\mathfrak{U}} b\rfloor}\left|\mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_2,\{\varkappa_\beta\}}\right]\right|\left|\mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_3,\{\varkappa_\beta\}}\right.\right.$$
$$\left.\left.\cdot W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_4,\{\varkappa_\beta\}}\right]\right|\mathbb{1}_{\{\exists\, t,t'\in\{t_1,\ldots,t_4\}:|t-t'|>3\varkappa_\beta\}}$$
$$\leqslant C\left(\lfloor T_{\mathfrak{U}} b\rfloor\,\varkappa_\beta^3+\lfloor T_{\mathfrak{U}} b\rfloor^2\beta^2\right). \tag{C.115}$$

This and analog arguments show:

$$\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2,t_3,t_4=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor} \Big|\mathbb{E}\Big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_3,\{\varkappa_\beta\}} \cdot W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_4,\{\varkappa_\beta\}}\Big]\Big| \leqslant C\left(\lfloor T_{\mathfrak{U}}b\rfloor \varkappa_\beta^3 + \lfloor T_{\mathfrak{U}}b\rfloor^2\beta^2\right). \tag{C.116}$$

Moreover, it follows similarly to (C.115):

$$\begin{aligned}&\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2,t_3,t_4=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor} \Big|\mathbb{E}\Big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_3,\{\varkappa_\beta\}}\Big]\Big| \\ &= \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2,t_3=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor} \Big|\mathbb{E}\Big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_3,\{\varkappa_\beta\}}\Big]\Big| \\ &\cdot \mathbb{1}_{\{\forall\, t,t'\in\{t_1,\ldots,t_3\}:|t-t'|\leqslant 2\varkappa_\beta\}} + 0 \\ &\leqslant C\lfloor T_{\mathfrak{U}}b\rfloor \varkappa_\beta^2.\end{aligned} \tag{C.117}$$

Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii), Definition A.1 (iii) and Assumption 3.15 $[\mathbf{W}^*]$ (i) $\big($the latter ensures $\beta > 0\big)$ provide $\ln(e+Tb) \geqslant 1$ as well as $\ln\left(\beta^{\text{inv}}_{\text{sup}}\beta\right) \geqslant 1$ and Assumption 3.15 $[\mathbf{W}^*]$ (iv) demands $\rho_* \in (0,1)$, such that $\beta \leqslant -\ln(\rho_*)\lceil \beta/\ln(\rho_*)\cdot(-1)\rceil \leqslant C\varkappa_\beta$ (see Definition A.1 (v)). One obtains analogously to (C.91) (regard also the text between (C.90) and (C.91)) by using (C.116), (C.114), (C.117), $\beta \leqslant C\varkappa_\beta$ (as just shown), the second inequality of Lemma C.25 (i) together with $(x_1+x_2)^n \leqslant 2^n\left(x_1^n+x_2^n\right)\,\forall\, x_1,x_2 \geqslant 0$, $n\in\mathbb{N}$ and Remark A.2 (ii) (recall (C.17)):

$$\begin{aligned}&\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\ \sum_{t_1,\ldots,t_4=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\ \sum_{j_1=1+2\varkappa_\beta}^{t_1-7\varkappa_\beta-1}\ \sum_{j_2=1+2\varkappa_\beta}^{t_2-7\varkappa_\beta-1}\ \sum_{j_3=1+2\varkappa_\beta}^{t_3-7\varkappa_\beta-1}\ \sum_{j_4=1+2\varkappa_\beta}^{t_4-7\varkappa_\beta-1} \\ &\Big|\mathbb{E}\Big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\varkappa_\beta\}} \\ &\cdot W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_3,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_3,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_4,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_4,\{\varkappa_\beta\}}\Big]\Big| \\ &\leqslant C\lfloor 1/(2b)\rfloor\Big\{\left(\lfloor T_{\mathfrak{U}}b\rfloor\varkappa_\beta^3+\lfloor T_{\mathfrak{U}}b\rfloor^2\beta^2\right)\Big[\left(\lfloor T_{\mathfrak{U}}b\rfloor\varkappa_\beta^3+\lfloor T_{\mathfrak{U}}b\rfloor^2\beta^2\right)+\lfloor T_{\mathfrak{U}}b\rfloor\beta\lfloor T_{\mathfrak{U}}b\rfloor\beta\Big] \\ &+C\lfloor T_{\mathfrak{U}}b\rfloor\varkappa_\beta^2\lfloor T_{\mathfrak{U}}b\rfloor\varkappa_\beta^2\lfloor T_{\mathfrak{U}}b\rfloor\beta+C\lfloor T_{\mathfrak{U}}b\rfloor\beta\lfloor T_{\mathfrak{U}}b\rfloor\beta\lfloor T_{\mathfrak{U}}b\rfloor\beta\lfloor T_{\mathfrak{U}}b\rfloor\beta\Big\} \\ &\leqslant C\lfloor 1/(2b)\rfloor\Big\{\lfloor T_{\mathfrak{U}}b\rfloor^2\varkappa_\beta^6+\lfloor T_{\mathfrak{U}}b\rfloor^3\varkappa_\beta^4\beta+\lfloor T_{\mathfrak{U}}b\rfloor^4\beta^4\Big\} \\ &\leqslant C\lfloor 1/(2b)\rfloor\Big\{T^2b^2\beta^6\ln(e+Tb)^6+T^2b^2o(Tb)^3+T^3b^3\beta^5\ln(e+Tb)^4+T^3b^3o(Tb)^2\beta+T^4b^4\beta^4\Big\}.\end{aligned} \tag{C.118}$$

It holds for all arbitrary but fixed $p\in\mathbb{N}$ due to Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii):

$$b\ln(e+Tb)^p \leqslant CT^{-1/(2+2\delta)}\ln(e+T)^p \longrightarrow 0 \ \text{ for } \ T\to\infty. \tag{C.119}$$

In conclusion, Assumption 3.15 $[\mathbf{W}^*]$ (ii), the fact that $\|X_1X_2X_3X_4\|_1 \leqslant \|X_1\|_4\|X_2\|_4\|X_3\|_4\|X_4\|_4$ for all real-valued random variables $X_1,\ldots,X_4$ which live on the same probability space and own finite fourth moments, (C.113) with $q=4$, (C.118), Assumption 3.15 $[\mathbf{W}^*]$ (i) and (C.119) show (see (C.101) as well as (C.108)):

$$\begin{aligned}&\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\Big|\mathbb{E}\Big[\Big(\mathbb{S}^*_{T,k,\Re}-\widetilde{\mathbb{S}}^*_{T,k,\Re}\Big)^4\Big]\Big| \\ &\leqslant \frac{C}{T^4b^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\ \sum_{t_1,\ldots,t_4=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\ \sum_{j_1=1+2\varkappa_\beta}^{t_1-7\varkappa_\beta-1}\ \sum_{j_2=1+2\varkappa_\beta}^{t_2-7\varkappa_\beta-1}\ \sum_{j_3=1+2\varkappa_\beta}^{t_3-7\varkappa_\beta-1}\ \sum_{j_4=1+2\varkappa_\beta}^{t_4-7\varkappa_\beta-1}\end{aligned}$$

$$
\begin{aligned}
&\cdot \Big|\mathbb{E}\Big[\Big(\mathbb{I}_{T,k,\Re}(t_1,j_1)-\widetilde{\mathbb{I}}_{T,k,\Re}(t_1,j_1)\Big)\Big(\mathbb{I}_{T,k,\Re}(t_2,j_2)-\widetilde{\mathbb{I}}_{T,k,\Re}(t_2,j_2)\Big)\Big(\mathbb{I}_{T,k,\Re}(t_3,j_3)\\
&-\widetilde{\mathbb{I}}_{T,k,\Re}(t_3,j_3)\Big)\Big(\mathbb{I}_{T,k,\Re}(t_4,j_4)-\widetilde{\mathbb{I}}_{T,k,\Re}(t_4,j_4)\Big)\Big]\Big|\\
&\cdot \Big|\mathbb{E}\Big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\mathscr{n}_\beta\}}\\
&\cdot W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_3,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_3,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_4,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_4,\{\mathscr{n}_\beta\}}\Big]\Big|\\
&\leqslant \frac{C}{T^4b^2}o\left(\frac{1}{T}\right)\lfloor 1/(2b)\rfloor\left(T^2b^2o\left(\frac{T^3b^6}{b^3}\right)\ln(e+Tb)^6+o\left(T^5b^5\right)+T^3b^3o\left(\frac{T^2b^4}{b^3}\right)\ln(e+Tb)^4\right.\\
&\left.+o\left(T^5b^5\right)o\left(\frac{1}{b}\right)+T^4b^4o\left(\frac{Tb^2}{b^3}\right)\right)\\
&=o(1),
\end{aligned}
$$

such that (C.109) is valid. In order to verify (C.110), note at first that one obtains from disassembling the set of the random variables contained in the expectation in (C.120) similarly to the decomposition described below (C.90) (which is possible according to Definition A.1 (i)) and from Assumption 3.15 $[\mathbf{W}^*]$ (iii) $\big($the latter provides $\mathbb{E}\big[\big|W^*_{r_1,\{\mathscr{n}_\beta\}}\dots W^*_{r_l,\{\mathscr{n}_\beta\}}\big|\big]\leqslant\|W^*_0\|_4^l\ \forall\, r_1,\dots,r_l\in\mathbb{Z},\, l\in\{1,\dots,4\}\big)$:

$$
\begin{aligned}
&\sup_{k=1,\dots,\lfloor 1/(2b)\rfloor}\ \sup_{t_1,\dots,t_4=9\mathscr{n}_\beta+2,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{n}_\beta}\ \sup_{j_1=1,\dots,t_1-7\mathscr{n}_\beta-1}\ \sup_{j_2=1,\dots,t_2-7\mathscr{n}_\beta-1}\ \sup_{j_3=1,\dots,t_3-7\mathscr{n}_\beta-1}\\
&\sup_{j_4=1,\dots,t_4-7\mathscr{n}_\beta-1}\Big|\mathbb{E}\Big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\mathscr{n}_\beta\}}\\
&\cdot W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_3,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_3,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_4,\{\mathscr{n}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_4,\{\mathscr{n}_\beta\}}\Big]\Big|\\
&\leqslant C. \qquad\qquad (C.120)
\end{aligned}
$$

It follows from Assumption 3.15 $[\mathbf{W}^*]$ (ii), (C.120), Lemma C.25 (i) (which ensures $\mathscr{n}_\beta\geqslant\mathscr{n}$) as well as the second inequality and last equality of (C.96) (see (C.108), (C.90) as well as (C.80)):

$$
\begin{aligned}
\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left|\mathbb{E}\left[\left(\widetilde{\mathbb{S}}^*_{T,k,\Re}\right)^4\right]\right| &\leqslant \frac{C}{T^4b^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\ \sum_{t_1,\dots,t_4=9\mathscr{n}_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{n}_\beta}\ \sum_{j_1=1+2\mathscr{n}_\beta}^{t_1-7\mathscr{n}_\beta-1}\ \sum_{j_2=1+2\mathscr{n}_\beta}^{t_2-7\mathscr{n}_\beta-1}\ \sum_{j_3=1+2\mathscr{n}_\beta}^{t_3-7\mathscr{n}_\beta-1}\ \sum_{j_4=1+2\mathscr{n}_\beta}^{t_4-7\mathscr{n}_\beta-1}\\
&\quad\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\Theta_{T,k,\Re}\left(\underline{t},\underline{j},\underline{s}\right)\right|\mathbf{w}(s_1)\,ds_1\,\mathbf{w}(s_2)\,ds_2\,\mathbf{w}(s_3)\,ds_3\,\mathbf{w}(s_4)\,ds_4\\
&= o(1),
\end{aligned}
$$

which proves (C.110). Moreover, (C.26), (C.109) and (C.110) provide:

$$
\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left|\mathbb{E}\left[\left(\mathbb{S}^*_{T,k,\Re}\right)^4\right]\right| = \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left|\mathbb{E}\left[\left(\mathbb{S}^*_{T,k,\Re}-\widetilde{\mathbb{S}}^*_{T,k,\Re}+\widetilde{\mathbb{S}}^*_{T,k,\Re}\right)^4\right]\right| = o(1). \tag{C.121}
$$

Analogously to (C.27), one obtains for all $\epsilon>0$ from (C.121) and similar arguments (recall (C.101)):

$$
\mathbb{E}\left[\left|\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{E}^*\left[\left(\mathbb{S}^*_{T,k}\right)^2\mathbb{1}_{\{|\mathbb{S}^*_{T,k}|>\epsilon\}}\right]-0\right|\right] = \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{E}\left[\left(\mathbb{S}^*_{T,k}\right)^2\mathbb{1}_{\{|\mathbb{S}^*_{T,k}|>\epsilon\}}\right] = o(1),
$$

such that (6.30) in [52, Leucht and Neumann (2013), p. 275] holds.

The validity of (6.31) as well as (6.32)[8] in [52, Leucht and Neumann (2013), p. 275] with $\theta_r = 0$ $\forall\, r\in\mathbb{N}$ can be verified by using that $\left(\mathbb{S}^*_{T,k}\right)_{k=1}^{\lfloor 1/(2b)\rfloor}$ is a sequence of conditioned on $(X_{t,T})_{t=1}^T$ independent random variables (as explained above (C.106)), whereby θ_r is defined in [52, Leucht and Neumann (2013), p. 275].
Overall, Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275 et seq.] shows (C.103) (see (C.101) as well as (3.53)). It follows from (C.102) and (C.103) that (C.100) holds.

In the case $\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} > 0$, (3.58) follows from (C.100) and a chaining argument that is similar to (C.77).
Instead, if $\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} = 0$, one will obtain from (C.104), (C.106) and the convergence in mean stated on the right side of (C.107):

$$\mathbb{E}\left[\mathbb{S}_T^{*\,2}\right] = \mathbb{E}\left[\mathbb{E}^*\left[\mathbb{S}_T^{*\,2}\right]\right] = \mathbb{E}\left[\mathrm{Var}^*\left(\mathbb{S}_T^*\right)\right] = o(1). \tag{C.122}$$

In conclusion, (3.59) is an implication of (3.28), (C.102), (C.122) and (3.27) (recall (3.26)). □

Proof of Theorem 3.27. At first, note that arguments which are similar to those that show (B.45) and (C.15) as well as Assumption 2.4 $[\mathbf{DM.2}]$ imply:

$$\sum_{t=1}^{\infty}\sum_{l=t}^{\infty} l\Delta_l \leqslant \sum_{l=1}^{\infty} l^2\Delta_l \leqslant C \quad \text{and} \quad \sum_{t=-\infty}^{-1}\sum_{l=-t}^{\infty} l\Delta_l \leqslant \sum_{l=1}^{\infty} l^2\Delta_l \leqslant C. \tag{C.123}$$

One obtains from Assumption 3.15 $[\mathbf{W}^*]$ (iii) (which ensures $K^*(0) = 1$), Lemma B.4 (vi), Assumption 3.1 $[\mathbf{WEI.1}]$, (3.60) and (C.123) (see (3.51), (3.17) as well as (3.57)):

$$\begin{aligned}
&\left|\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\Re} - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1},\Re}\right| \\
&\leqslant \frac{1}{\sqrt{b}} \int\limits_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz \int\limits_{\mathbb{R}^d}\int\limits_{\mathfrak{U}_0}^{\mathfrak{U}_1} \sum_{t\in\mathbb{Z}\backslash\{0\}} \left|1 - K^*\left(\frac{t}{\beta}\right)\right| \left|\mathrm{Cov}\left(\Re\left\{e^{i\left\langle s,\tilde{X}_0(u)\right\rangle}\right\}, \Re\left\{e^{i\left\langle s,\tilde{X}_t(u)\right\rangle}\right\}\right)\right| du\,\mathbf{w}(s)\,ds \\
&\leqslant \frac{C}{\sqrt{b}\beta} \sum_{t\in\mathbb{Z}\backslash\{0\}} \frac{\left|1 - K^*\left(\frac{t}{\beta}\right)\right|}{|t|/\beta} |t| \sum_{l=|t|}^{\infty} \Delta_l \\
&\leqslant \frac{C}{\sqrt{b}\beta} \sum_{t\in\mathbb{Z}\backslash\{0\}} \sup_{x\in\mathbb{R}\backslash\{0\}} \frac{|1 - K^*(x)|}{|x|} \sum_{l=|t|}^{\infty} l\Delta_l \\
&= o(1).
\end{aligned}$$

This and analog arguments show (note (3.56), (3.51), (3.57) as well as (3.26)):

$$\left|T\sqrt{b}\,\widehat{\mathbb{D}}^*_{T,\mathrm{Test}} - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}} - \left(T\sqrt{b}\,\widehat{\mathbb{D}}^*_{T,\mathrm{Test}} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}}\right)\right| = o^*_{\mathbb{P}}(1), \tag{C.124}$$

such that if $\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} > 0$, (3.58) will provide for $T \to \infty$:

$$T\sqrt{b}\,\widehat{\mathbb{D}}^*_{T,\mathrm{Test}} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1}} \stackrel{\mathrm{d}}{\longrightarrow} Z^{\mathrm{distr}}_{\mathfrak{U}_{0,1}} \quad \text{in probability.} \tag{C.125}$$

One obtains (3.61) from (C.125) and a chaining argument which is similar to (C.77). Further, (3.28), (C.124) and (3.59) show (3.62). □

Proof of Lemma 3.35. Throughout this proof, it is assumed that T is large enough to ensure:

$$T - \mathbf{B}_T \geqslant 1 + \mathbf{B}_T \quad \text{and} \quad \mathbf{B}_T \leqslant \lfloor T/2 \rfloor - 1, \tag{C.126}$$

which holds for sufficiently large T due to Assumption 3.30 $[\mathbf{NW}]$.
(i) In order to prove Lemma 3.35 (i), observe at first that Assumption 3.30 $[\mathbf{NW}]$ (which ensures $\mathbb{K}_{\mathbf{NW}}(x) = 0\ \forall\, x \in \mathbb{R} : |x| > 1$, $\sup_{x\in[-1,1]} |\mathbb{K}_{\mathbf{NW}}(x)| < \infty$, $\mathbf{B}_T/\sqrt{Tb^2} \stackrel{T\to\infty}{\longrightarrow} 0$ as well as $Tb^{2+\delta} \to C_\delta < \infty$ for a $\delta \in (0,1]$), (C.126), Lemma C.33 together with (3.11), Assumption 3.15 $[\mathbf{W}^*]$ (iii) (that provides $|K^*(h/\beta)| \leqslant C\ \forall\, h \in \mathbb{Z}$) and Assumption 3.1 $[\mathbf{WEI.1}]$ imply (recall (3.66)):

$$\frac{1}{\sqrt{b}} \int\limits_{\mathbb{R}^d} \mathbb{E}\left[\left|\frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \hat{\sigma}^{\mathrm{error}}_{T,\Re}(u_k, s) - \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \sum_{h=-\infty}^{\infty} \left(K^*\left(\frac{h}{\beta}\right) - 1\right)\right.\right.$$

$$\cdot \mathbb{K}_{\mathbf{NW}}\left(\frac{h}{\mathbf{B}_T}\right) \operatorname{Cov}\left(\Re\left\{e^{i\left\langle s,\tilde{X}_0(u_k)\right\rangle}\right\}, \Re\left\{e^{i\left\langle s,\tilde{X}_h(u_k)\right\rangle}\right\}\right)\Bigg|\Bigg] \mathbf{w}(s)\, ds$$

$$\leqslant \frac{C}{\sqrt{b}} \sum_{h=-\mathbf{B}_T}^{\mathbf{B}_T} \left|K^*\left(\frac{h}{\beta}\right) - 1\right| \left|\mathbb{K}_{\mathbf{NW}}\left(\frac{h}{\mathbf{B}_T}\right)\right| \left(\frac{C}{Tb}\left(|h|+1\right) + C\left(b + \frac{1}{\sqrt{Tb}} + \frac{|h|}{T}\right)\right)$$

$$= o(1). \tag{C.127}$$

Further, one defines for all $u \in [0,1]$, $s \in \mathbb{R}^d$:

$$\sigma^{\text{error}}_{T,\infty,\Re}(u,s) := \sum_{h=-\infty}^{\infty} \left(K^*\left(\frac{h}{\beta}\right) - 1\right) \operatorname{Cov}\left(\Re\left\{e^{i\left\langle s,\tilde{X}_0(u)\right\rangle}\right\}, \Re\left\{e^{i\left\langle s,\tilde{X}_h(u)\right\rangle}\right\}\right). \tag{C.128}$$

Arguments which are similar to those that show (B.45) and (C.15) provide due to Assumption 3.30 $[\mathbf{NW}]$:

$$\sum_{h\in\mathbb{Z}\backslash\{0\}} \sum_{l=|h|}^{\infty} l^{\eta}\Delta_l = \sum_{h=1}^{\infty}\sum_{l=h}^{\infty} l^{\eta}\Delta_l + \sum_{h=-\infty}^{-1}\sum_{l=-h}^{\infty} l^{\eta}\Delta_l = 2\sum_{l=1}^{\infty} l^{1+\eta}\Delta_l \leqslant C. \tag{C.129}$$

One obtains by using Lemma B.4 (vi), Assumption 3.1 $[\mathbf{WEI.1}]$, Assumption 3.30 $[\mathbf{NW}]$ (which ensures $\sup_{x\in\mathbb{R}\backslash\{0\}} |1 - \mathbb{K}_{\mathbf{NW}}(x)|/|x|^{\eta} \leqslant C$ as well as $\mathbf{B}_T b^{1/(2\eta)} \overset{T\to\infty}{\longrightarrow} \infty$), Assumption 3.15 $[\mathbf{W}^*]$ (iii) (that implies $|K^*(h/\beta)| \leqslant C \;\forall\, h \in \mathbb{Z}$) and (C.129) (see (3.17) as well as (C.128)):

$$\frac{1}{\sqrt{b}} \int_{\mathbb{R}^d} \Bigg| \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{h=-\infty}^{\infty} \left(K^*\left(\frac{h}{\beta}\right) - 1\right) \mathbb{K}_{\mathbf{NW}}\left(\frac{h}{\mathbf{B}_T}\right) \operatorname{Cov}\left(\Re\left\{e^{i\left\langle s,\tilde{X}_0(u_k)\right\rangle}\right\}, \Re\left\{e^{i\left\langle s,\tilde{X}_h(u_k)\right\rangle}\right\}\right)$$

$$- \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sigma^{\text{error}}_{T,\infty,\Re}(u_k,s) \Bigg| \mathbf{w}(s)\, ds$$

$$\leqslant \frac{C}{\sqrt{b}} \left(|K^*(0) - 1|\, |\mathbb{K}_{\mathbf{NW}}(0) - 1| + \frac{1}{\mathbf{B}_T^{\eta}} \sum_{h\in\mathbb{Z}\backslash\{0\}} \left|K^*\left(\frac{h}{\beta}\right) - 1\right| \frac{\left|\mathbb{K}_{\mathbf{NW}}\left(\frac{h}{\mathbf{B}_T}\right) - 1\right|}{|h/\mathbf{B}_T|^{\eta}} |h|^{\eta} \sum_{l=|h|}^{\infty} \Delta_l \right)$$

$$\leqslant \frac{C}{\sqrt{b}\mathbf{B}_T^{\eta}} \sup_{x\in\mathbb{R}\backslash\{0\}} \frac{|\mathbb{K}_{\mathbf{NW}}(x) - 1|}{|x|^{\eta}} \sum_{h\in\mathbb{Z}\backslash\{0\}} \sum_{l=|h|}^{\infty} l^{\eta}\Delta_l$$

$$= o(1). \tag{C.130}$$

In the following, it is proved:

$$\frac{1}{\sqrt{b}} \int_{\mathbb{R}^d} \left| \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sigma^{\text{error}}_{T,\infty,\Re}(u_k,s) - \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \sigma^{\text{error}}_{T,\infty,\Re}(u,s)\, du \right| \mathbf{w}(s)\, ds = o(1). \tag{C.131}$$

Hölder's inequality for series together with $2/(2-\delta) > 1$ as well as $2/\delta > 1$ (the latter two follow from $\delta \in (0,1]$), $\Delta_l \leqslant Cl^{-2/\delta} \;\forall\, l \in \mathbb{N}$ (which holds because Assumption 2.4 $[\mathbf{DM.2}]$ implies $\Delta_l l^{2/\delta} \to 0$ for $l \to \infty$), the fact that the Riemann zeta function $x \mapsto \zeta(x)$ converges for $x = 2/(2-\delta) > 1$ and Assumption 2.4 $[\mathbf{DM.2}]$ show:

$$\sum_{l=1}^{\infty} \Delta_l^{\delta} l \leqslant \left(\sum_{l=1}^{\infty} \left(\Delta_l^{\delta/2}\right)^{\frac{2}{2-\delta}}\right)^{\frac{2-\delta}{2}} \left(\sum_{l=1}^{\infty} \left(\Delta_l^{\delta/2} l\right)^{\frac{2}{\delta}}\right)^{\frac{\delta}{2}} \leqslant \left(C\sum_{l=1}^{\infty} l^{-\frac{2}{2-\delta}}\right)^{\frac{2-\delta}{2}} \left(\sum_{l=1}^{\infty} \Delta_l l^{2/\delta}\right)^{\frac{\delta}{2}} \leqslant C, \tag{C.132}$$

such that one obtains analogously to (B.45):

$$\sum_{h=1}^{\infty}\sum_{l=h}^{\infty}\Delta_l^{\delta}\leqslant C. \tag{C.133}$$

Moreover, Assumption 2.4 $[\mathbf{DM.2}]$ implies similarly to (B.45):

$$\sum_{h=1}^{\infty}\sum_{l=h}^{\infty}\Delta_{\partial,l}\leqslant C. \tag{C.134}$$

Further, it follows for all $s\in\mathbb{R}^d$, $u\in(0,1)$, $r_1,r_2\in\mathbb{Z}$ from Assumption 2.2 $[\mathbf{StAp}]$ (ii) by using arguments which are analog to ones stated in [41, Jentsch et al. (2020b), p. 3 et seq.]:

$$\begin{aligned}
&\partial_u\mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\cos\left(\left\langle s,\widetilde{X}_{r_2}(u)\right\rangle\right)\right]\\
&=\mathbb{E}\left[-\sin\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_1}(u)\right\rangle\cos\left(\left\langle s,\widetilde{X}_{r_2}(u)\right\rangle\right)\right]\\
&+\mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\left(-\sin\left(\left\langle s,\widetilde{X}_{r_2}(u)\right\rangle\right)\right)\left\langle s,\partial_u\widetilde{X}_{r_2}(u)\right\rangle\right]
\end{aligned} \tag{C.135}$$

and:

$$\begin{aligned}
&\partial_u\left(\mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\right]\mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_{r_2}(u)\right\rangle\right)\right]\right)\\
&=\mathbb{E}\left[-\sin\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_1}(u)\right\rangle\right]\mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_{r_2}(u)\right\rangle\right)\right]\\
&+\mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\right]\mathbb{E}\left[-\sin\left(\left\langle s,\widetilde{X}_{r_2}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_2}(u)\right\rangle\right].
\end{aligned} \tag{C.136}$$

For $r_1\geqslant r_2+1$, one obtains from (C.135), (C.136), Lemma B.4 (ii) as well as (iv) with $q=(1+\delta)/\delta$, Assumption 2.2 $[\mathbf{StAp}]$ (ii), Assumption 2.4 $[\mathbf{DM.2}]$ and shifting the index of a sum (recall Definition A.1 (i)):

$$\begin{aligned}
&\sup_{u\in(0,1)}\left|\partial_u\mathrm{Cov}\left(\cos\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right),\cos\left(\left\langle s,\widetilde{X}_{r_2}(u)\right\rangle\right)\right)\right|\\
&=\sup_{u\in(0,1)}\Bigg|\mathbb{E}\Big[\Big(\mathbb{E}\Big[-\sin\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_1}(u)\right\rangle\Big|\mathcal{F}_{r_1}\Big]\\
&-\mathbb{E}\Big[-\sin\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_1}(u)\right\rangle\Big|\mathcal{F}_{r_1,r_2+1}\Big]\Big)\cos\left(\left\langle s,\widetilde{X}_{r_2}(u)\right\rangle\right)\Big]\\
&+\mathbb{E}\Big[\Big(\mathbb{E}\Big[\cos\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\Big|\mathcal{F}_{r_1}\Big]-\mathbb{E}\Big[\cos\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\Big|\mathcal{F}_{r_1,r_2+1}\Big]\Big)\\
&\cdot\left(-\sin\left(\left\langle s,\widetilde{X}_{r_2}(u)\right\rangle\right)\right)\left\langle s,\partial_u\widetilde{X}_{r_2}(u)\right\rangle\Big]\Bigg|\\
&\leqslant\sup_{u\in(0,1)}\Bigg(\Big\|\mathbb{E}\Big[-\sin\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_1}(u)\right\rangle\Big|\mathcal{F}_{r_1}\Big]-\mathbb{E}\Big[-\sin\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_1}(u)\right\rangle\Big|\\
&\mathcal{F}_{r_1,r_2+1}\Big]\Big\|_1+\Big\|\mathbb{E}\Big[\cos\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\Big|\mathcal{F}_{r_1}\Big]-\mathbb{E}\Big[\cos\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\Big|\mathcal{F}_{r_1,r_2+1}\Big]\Big\|_{\frac{1+\delta}{\delta}}\Big\|\left\langle s,\partial_u\widetilde{X}_{r_2}(u)\right\rangle\Big\|_{1+\delta}\Bigg)\\
&\leqslant\sup_{u\in(0,1)}\sum_{l=r_1-r_2-1}^{\infty}\Bigg(\Big\|-\sin\left(\left\langle s,\widetilde{X}_{r_1}^{\times(r_1-l-1)}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_1}^{\times(r_1-l-1)}(u)\right\rangle\\
&+\sin\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_1}^{\times(r_1-l-1)}(u)\right\rangle-\sin\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_1}^{\times(r_1-l-1)}(u)\right\rangle\\
&+\sin\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\left\langle s,\partial_u\widetilde{X}_{r_1}(u)\right\rangle\Big\|_1+\Big\|\cos\left(\left\langle s,\widetilde{X}_{r_1}^{\times(r_1-l-1)}(u)\right\rangle\right)-\cos\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\Big\|_{\frac{1+\delta}{\delta}}C|s|_1\Bigg)\\
&\leqslant\sum_{l=r_1-r_2-1}^{\infty}\sup_{u\in(0,1)}\Bigg(\Big\|-\sin\left(\left\langle s,\widetilde{X}_{r_1}^{\times(r_1-l-1)}(u)\right\rangle\right)+\sin\left(\left\langle s,\widetilde{X}_{r_1}(u)\right\rangle\right)\Big\|_{\frac{1+\delta}{\delta}}\\
&\cdot\Big\|\left\langle s,\partial_u\widetilde{X}_{r_1}^{\times(r_1-l-1)}(u)\right\rangle\Big\|_{1+\delta}+\Big\|\left\langle s,-\partial_u\widetilde{X}_{r_1}^{\times(r_1-l-1)}(u)+\partial_u\widetilde{X}_{r_1}(u)\right\rangle\Big\|_1
\end{aligned}$$

$$+ \left\| \cos\left(\left\langle s, \widetilde{X}_{r_1}^{\times(r_1-l-1)}(u)\right\rangle\right) - \cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right)\right\|_{\frac{1+\delta}{\delta}} C|s|_1 \Bigg)$$

$$\leqslant C \sum_{l=r_1-r_2-1}^{\infty} \left(\Delta_{l+1}^{\delta}|s|_1^{1+\delta} + \Delta_{\partial,l+1}|s|_1 + \Delta_{l+1}^{\delta}|s|_1^{1+\delta}\right)$$

$$= C \sum_{l=r_1-r_2}^{\infty} \left(\Delta_{l}^{\delta}|s|_1^{1+\delta} + \Delta_{\partial,l}|s|_1 + \Delta_{l}^{\delta}|s|_1^{1+\delta}\right). \tag{C.137}$$

Overall, (C.137) (with $r_1 = h$ and $r_2 = 0$), (C.133) as well as (C.134) provide for all $s \in \mathbb{R}^d$:

$$\sum_{h=1}^{\infty} \sup_{u \in (0,1)} \left|\partial_u \mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_0(u)\right\rangle\right), \cos\left(\left\langle s, \widetilde{X}_h(u)\right\rangle\right)\right)\right| \leqslant C\left(|s|_1^{1+\delta} + |s|_1\right). \tag{C.138}$$

Further, it follows for all $s \in \mathbb{R}^d$ from (C.135), (C.136) and Assumption 2.2 $[\mathbf{StAp}]$ (ii):

$$\sup_{u \in (0,1)} \left|\partial_u \mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_0(u)\right\rangle\right), \cos\left(\left\langle s, \widetilde{X}_0(u)\right\rangle\right)\right)\right| \leqslant C|s|_1. \tag{C.139}$$

One obtains for all $s \in \mathbb{R}^d$ due to (C.138) and similar arguments as well as (C.139):

$$\sum_{h=-\infty}^{\infty} \sup_{u \in (0,1)} \left|\partial_u \mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_0(u)\right\rangle\right), \cos\left(\left\langle s, \widetilde{X}_h(u)\right\rangle\right)\right)\right| \leqslant C\left(|s|_1^{1+\delta} + |s|_1\right). \tag{C.140}$$

Assumption 3.15 $[\mathbf{W}^*]$ (iii), Lemma 3.12, the mean value theorem and (C.140) show for all $v, w \in [0,1]$, $s \in \mathbb{R}^d$ (see (C.128)):

$$\left|\sigma_{T,\infty,\Re}^{\text{error}}(v,s) - \sigma_{T,\infty,\Re}^{\text{error}}(w,s)\right|$$

$$\leqslant C \sum_{h=-\infty}^{\infty} \left|\mathrm{Cov}\left(\Re\left\{e^{i\left\langle s, \widetilde{X}_0(v)\right\rangle}\right\}, \Re\left\{e^{i\left\langle s, \widetilde{X}_h(v)\right\rangle}\right\}\right) - \mathrm{Cov}\left(\Re\left\{e^{i\left\langle s, \widetilde{X}_0(w)\right\rangle}\right\}, \Re\left\{e^{i\left\langle s, \widetilde{X}_h(w)\right\rangle}\right\}\right)\right|$$

$$\leqslant C\left(|s|_1^{1+\delta} + |s|_1\right)|v - w|. \tag{C.141}$$

Lemma B.2 (ii) in combination with (C.141) and Assumption 3.1 $[\mathbf{WEI.1}]$ imply (C.131).
Lemma 3.35 (i) follows from (C.127), (C.130), (C.131) and similar arguments.

(ii) In order to prove Lemma 3.35 (ii), define at first (recall (3.68)):

$$\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error.new}} := \frac{T - 2\mathbf{B}_T}{T} \widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}}. \tag{C.142}$$

It holds due to Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) (see Definition 2.11):

$$K_b(t/T - u_k) = 0 \quad \forall\, t \in \mathbb{N}\backslash[\lfloor u_k T - Tb\rfloor, \lceil u_k T + Tb\rceil],\ k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}. \tag{C.143}$$

Moreover, Lemma B.1 with $\kappa_1 = 1$ provides for all $\mathrm{R} \in \{\Re, \Im\}$ (recall (3.66)):

$$\left|\mathrm{R}\left\{e^{i\left\langle s, X_{t,T}\right\rangle}\right\}^{\widehat{c}(u)}\right| \leqslant C \quad \forall\, s \in \mathbb{R}^d,\ t \in \{1, \ldots, T\},\ u \in [0,1]. \tag{C.144}$$

Overall, (C.126), (C.143), (C.144), Assumption 3.1 $[\mathbf{WEI.1}]$ and Assumption 3.30 $[\mathbf{NW}]$ imply (see (3.68), (C.142), Definition 2.11 as well as (3.66)):

$$\mathbb{E}\left[\left|\widehat{\widehat{\mathbf{Bias}}}_T^{\text{error}} - \widehat{\widehat{\mathbf{Bias}}}_T^{\text{error.new}}\right|\right] \leqslant \frac{CTb\mathbf{B}_T}{\sqrt{b}b}\left|\frac{1}{T - 2\mathbf{B}_T} - \frac{1}{T}\right| \leqslant \frac{C}{\sqrt{b}} \frac{\mathbf{B}_T^2}{T - 2\mathbf{B}_T} = o(1). \tag{C.145}$$

Further, one defines for all $h \in \{-\mathbf{B}_T, \ldots, \mathbf{B}_T\}$, $\mathrm{R} \in \{\Re, \Im\}$, $u \in [0,1]$, $s \in \mathbb{R}^d$ (recall (C.126), Assumption 3.30 [**NW**], (3.66) as well as Assumption 3.15 [$\mathbf{W}^*$] (iii)):

$$\widehat{\widehat{\sigma}}_{h,T,\mathrm{R}}(u,s) := \frac{1}{T}\sum_{t=1+\mathbf{B}_T}^{T-\mathbf{B}_T} K_b\left(\frac{t}{T}-u\right)\cdot \mathrm{R}\left\{e^{i\langle s,X_{t,T}\rangle}\right\}^{\widehat{c}(u)}\cdot \mathrm{R}\left\{e^{i\langle s,X_{t+h,T}\rangle}\right\}^{\widehat{c}(u)} \quad \text{and}$$

$$\widehat{\widehat{\sigma}}^{\text{error}}_{T,\mathrm{R}}(u,s) := \sum_{h=-\mathbf{B}_T}^{\mathbf{B}_T}\left(K^*\left(\frac{h}{\beta}\right)-1\right)\mathbb{K}_{\mathbf{NW}}\left(\frac{h}{\mathbf{B}_T}\right)\widehat{\widehat{\sigma}}_{h,T,\mathrm{R}}(u,s). \tag{C.146}$$

Since:

$$\Re\{x\overline{y}\} = \Re\{x\}\Re\{y\} + \Im\{x\}\Im\{y\} \quad \forall\, x,y\in\mathbb{C}, \tag{C.147}$$

(C.126) provides (see (C.142), (3.68), (3.66) and (C.146)):

$$\begin{aligned}
\widehat{\widehat{\mathbf{Bias}}}^{\text{error.new}}_T &= \frac{1}{\sqrt{b}}\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz \int_{\mathbb{R}^d}\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{h=-\mathbf{B}_T}^{\mathbf{B}_T}\left(K^*\left(\frac{h}{\beta}\right)-1\right)\mathbb{K}_{\mathbf{NW}}\left(\frac{h}{\mathbf{B}_T}\right)\\
&\quad\cdot\frac{1}{T}\sum_{t=1+\mathbf{B}_T}^{T-\mathbf{B}_T} K_b\left(\frac{t}{T}-u_k\right)\Re\left\{\left(e^{i\langle s,X_{t,T}\rangle}\right)^{\widehat{c}(u_k)}\overline{\left(e^{i\langle s,X_{t+h,T}\rangle}\right)^{\widehat{c}(u_k)}}\right\}\mathbf{w}(s)\,ds\\
&= \frac{1}{\sqrt{b}}\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz \int_{\mathbb{R}^d}\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\widehat{\widehat{\sigma}}^{\text{error}}_{T,\Re}(u_k,s)+\widehat{\widehat{\sigma}}^{\text{error}}_{T,\Im}(u_k,s)\;\mathbf{w}(s)\,ds.
\end{aligned} \tag{C.148}$$

Moreover, one obtains for all $\mathrm{R}\in\{\Re,\Im\}$ from (C.126) and (C.144) (recall (C.146), (3.66) as well as Definition 2.11):

$$\sup_{s\in\mathbb{R}^d}\ \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor}\ \sup_{h=-\mathbf{B}_T,\ldots,\mathbf{B}_T}\mathbb{E}\left[\left|\widehat{\widehat{\sigma}}_{h,T,\mathrm{R}}(u_k,s)-\widehat{\sigma}_{h,T,\mathrm{R}}(u_k,s)\right|\right]\leqslant\frac{C}{Tb}\mathbf{B}_T. \tag{C.149}$$

It follows from (C.148), Assumption 3.30 [**NW**] together with (C.126), Assumption 3.15 [$\mathbf{W}^*$] (iii), (C.149) and Assumption 3.1 [**WEI.1**] (see (C.146), (3.67) as well as (3.66)):

$$\mathbb{E}\left[\left|\widehat{\widehat{\mathbf{Bias}}}^{\text{error.new}}_T-\widehat{\mathbf{Bias}}^{\text{error}}_T\right|\right]\leqslant\frac{C\mathbf{B}_T^2}{\sqrt{b}Tb}=o(1). \tag{C.150}$$

Overall, (C.145), (C.150) and Lemma 3.35 (i) show Lemma 3.35 (ii). □

***Proof of Theorem 3.36*.** (i) Lemma 3.35 implies for all $\mathbf{Y}_T\in\left\{\widehat{\mathbf{Bias}}^{\text{error}}_T,\widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_T\right\}$ (recall (3.64)):

$$\mathbb{E}\left[\left|T\sqrt{b}\left(\widehat{\mathbb{D}}_T+\frac{1}{T\sqrt{b}}\mathbf{Y}_T\right)-\mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1}}-\left(T\sqrt{b}\,\widehat{\mathbb{D}}_T-\mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}}\right)\right|\right]=o(1),$$

such that Theorem 3.23 (i) proves Theorem 3.36 (i).

(ii) One obtains under $\mathcal{H}^{\text{distr}}_{1,\mathfrak{U}_{0,1}}$ (see (3.49)) for all $\mathbf{Y}_T\in\left\{\widehat{\mathbf{Bias}}^{\text{error}}_T,\widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_T\right\}$ from (C.98), Theorem 3.23 (ii) (observe that (3.54) and Assumption 2.8 [**K**&**b.1**] (ii) yield that $(\tau_T+1)_{T\in\mathbb{N}}$ is a sequence of positive numbers which grows to infinity for $T\to\infty$ slower than $T\sqrt{b}$) as well as Lemma 3.35 together with Markov's inequality (note (3.64)):

$$\begin{aligned}
&\lim_{T\to\infty}\mathbb{P}\left(T\sqrt{b}\left(\widehat{\mathbb{D}}_T+\frac{1}{T\sqrt{b}}\mathbf{Y}_T\right)-\mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1}}>\tau_T\right)\\
&\geqslant\lim_{T\to\infty}\mathbb{P}\left(T\sqrt{b}\,\widehat{\mathbb{D}}_T+\mathbf{Y}_T-\mathbf{Bias}^{\text{error}}_{T,\mathfrak{U}_{0,1}}+\mathbf{Bias}^{\text{error}}_{T,\mathfrak{U}_{0,1}}-\mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1}}>\tau_T,\left|\mathbf{Y}_T-\mathbf{Bias}^{\text{error}}_{T,\mathfrak{U}_{0,1}}\right|\leqslant 1\right)\\
&\geqslant\lim_{T\to\infty}\mathbb{P}\left(T\sqrt{b}\,\widehat{\mathbb{D}}_T-1-\mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1}}>\tau_T,\left|\mathbf{Y}_T-\mathbf{Bias}^{\text{error}}_{T,\mathfrak{U}_{0,1}}\right|\leqslant 1\right)\\
&=1,
\end{aligned}$$

which verifies Theorem 3.36 (ii). □

Proof of Lemma 3.40. The Fatou–Lebesgue theorem and Theorem 3.36 (i) imply (recall (3.69), that $\widehat{\mathbb{D}}_{T,[0,w]}$, $\widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_{T,[0,w]}$ and $\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}$ are defined as in Definition 3.8 (i), (3.68) as well as (3.57), whereby $\mathfrak{U}_{0,1} = [0,w]$ is chosen, (3.54) and the conventions given in the last line of (3.71)):

$$\begin{aligned}\mathbb{V} &\leqslant \liminf_{T\to\infty} \int_0^{\mathbb{V}} \mathbb{P}\left(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T\right) d\lambda(w)\\ &\leqslant \limsup_{T\to\infty} \int_0^{\mathbb{V}} \mathbb{P}\left(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T\right) d\lambda(w)\\ &\leqslant \mathbb{V}, \end{aligned} \tag{C.151}$$

such that $\mathbb{P}(\mathcal{A}) = \mathbb{E}[\mathbb{1}_{\mathcal{A}}]$ and $\int \mathbb{1}_{\mathcal{A}} d\lambda = \lambda(\mathcal{A})$ (which hold for each measurable set $\mathcal{A}$) yield:

$$\lim_{T\to\infty} \mathbb{E}\left[\lambda\left(\left\{w \in [0,\mathbb{V}] : T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\widehat{\mathbf{Bias}}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T\right\}\right)\right] = \mathbb{V}. \tag{C.152}$$

Further, one obtains for all measurable sets $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \Omega$ (whereby Ω originates from Definition 2.1, $\mathcal{B}'$ denotes the complement of $\mathcal{B}$ and $\mathcal{C}'$ is the complement of $\mathcal{C}$):

$$\mathbb{P}(\mathcal{A}) \leqslant \mathbb{P}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) + \mathbb{P}(\mathcal{B}') + \mathbb{P}(\mathcal{C}'). \tag{C.153}$$

Moreover, assume that $\mathbf{Bias}^{\text{error}}_{T,[0,w]}$ as well as $\mathbf{Bias}^{\text{distr}}_{T,[0,w]}$ with $w \in (0,1]$ are defined as in (3.64) and (3.51), respectively, whereby $K_{[0,w]}$ (see (3.72)) is the underlying kernel. In addition, recall that (3.64) and (3.51) are based on Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i), which demands $\mathfrak{U}_0 < \mathfrak{U}_1$. To expand these definitions to the case $\mathfrak{U}_0 = \mathfrak{U}_1 = 0$, set:

$$\mathbf{Bias}^{\text{error}}_{T,[0,0]} := 0 \quad \text{and} \quad \mathbf{Bias}^{\text{distr}}_{T,[0,0]} := 0. \tag{C.154}$$

Lemma 3.12 and Assumption 3.1 $[\mathbf{WEI.1}]$ yield (note (3.17)):

$$\begin{aligned}\sup_{w\in[0,1]} \left|\mathbf{Bias}^{\text{distr}}_{T,[0,w]}\right| &\leqslant 0 + \sup_{w\in(0,1]} \left(\frac{1}{\sqrt{b}} \int_{-w}^{w} \frac{1}{w^2} K\left(\frac{z}{w}\right)^2 dz \cdot Cw\right)\\ &\leqslant \frac{C_{\mathbf{Bias}}}{\sqrt{b}} \quad \text{for an absolute constant } C_{\mathbf{Bias}} \in (0,\infty).\end{aligned} \tag{C.155}$$

Since convergence in distribution to a Dirac distributed random variable implies convergence in probability to this random variable, Theorem 3.13 (ii) shows for $T \to \infty$ (note in the case $\mathbb{V} = 0$ that $\widehat{\mathbb{D}}_{T,[0,0]} - \mathbb{D}_{[0,0]} = 0$ holds according to the last line of (3.71) and (3.69)):

$$\mathfrak{f}_T\left(\widehat{\mathbb{D}}_{T,[0,w]} - \mathbb{D}_{[0,w]}\right) \xrightarrow{\mathbb{P}} 0 \text{ for all } w \in [\mathbb{V},1] \text{ and an arbitrary deterministic sequence } (\mathfrak{f}_T)_{T\in\mathbb{N}} \text{ with}$$
$$\mathfrak{f}_T > 0\,\forall\, T \in \mathbb{N},\ \mathfrak{f}_T \overset{T\to\infty}{\longrightarrow} \infty \text{ and } \frac{\mathfrak{f}_T}{\sqrt{T}} \overset{T\to\infty}{\longrightarrow} 0. \tag{C.156}$$

From now on up to and including (C.160), suppose $\mathbb{V} < 1$. One defines for $\mathfrak{A}$ and $C_{\mathfrak{A}}$, which originate from (3.69) (see (3.54) as well as (C.155)):

$$E_{\mathbb{V},T} := \left(\frac{T\sqrt{b}/\mathfrak{f}_T + 1 + \tau_T + C_{\mathbf{Bias}}/\sqrt{b}}{T\sqrt{b}\,C_{\mathfrak{A}}}\right)^{\frac{1}{\mathfrak{A}}}, \tag{C.157}$$

whereby (C.156), (3.54) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) imply:

$$E_{\mathbb{V},T} = o(1). \tag{C.158}$$

If T is large enough to ensure $\mathbb{V} + E_{\mathbb{V},T} < 1$, which holds for sufficiently large T due to (C.158) and $\mathbb{V} < 1$ (as supposed above), one will obtain from (3.69) as well as (C.155):

$$-\frac{T\sqrt{b}}{\mathfrak{f}_T} - 1 + T\sqrt{b}\,\mathbb{D}_{[0,w]} - \mathbf{Bias}^{\text{distr}}_{T,[0,w]} \geqslant -\frac{T\sqrt{b}}{\mathfrak{f}_T} - 1 + T\sqrt{b}\,(w-\mathbb{V})^{\mathfrak{A}}\,C_{\mathfrak{A}} - \frac{C_{\mathbf{Bias}}}{\sqrt{b}} > \tau_T \quad \forall\, w \in (\mathbb{V} + E_{\mathbb{V},T}, 1]. \tag{C.159}$$

It follows from (C.153), the Fatou–Lebesgue theorem, $\mathbf{Bias}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} = -\mathbf{Bias}^{\text{distr}}_{T,[0,w]}\ \forall\, w \in [0,1]$ (which holds due to (C.154), the last line of (3.71) as well as (3.64)), (C.156), Lemma 3.35 (ii), (C.158) and (C.159):

$$\begin{aligned}
&\limsup_{T\to\infty} \mathbb{E}\left[\lambda\left(\left\{w \in [\mathbb{V},1] : T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T\right\}\right)\right] \\
&= \limsup_{T\to\infty} \int_{\mathbb{V}}^{1} \mathbb{P}\left(T\sqrt{b}\left(\widehat{\mathbb{D}}_{T,[0,w]} - \mathbb{D}_{[0,w]}\right) + \left(\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{error}}_{T,[0,w]}\right) + T\sqrt{b}\,\mathbb{D}_{[0,w]}\right. \\
&\quad \left. + \mathbf{Bias}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T\right) d\lambda(w) \\
&\leqslant \limsup_{T\to\infty} \int_{\mathbb{V}}^{1} \mathbb{P}\left(T\sqrt{b}\left(\widehat{\mathbb{D}}_{T,[0,w]} - \mathbb{D}_{[0,w]}\right) + \left(\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{error}}_{T,[0,w]}\right) + T\sqrt{b}\,\mathbb{D}_{[0,w]}\right. \\
&\quad \left. + \mathbf{Bias}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T, \left|\widehat{\mathbb{D}}_{T,[0,w]} - \mathbb{D}_{[0,w]}\right| \leqslant \frac{1}{\mathfrak{f}_T}, \left|\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{error}}_{T,[0,w]}\right| \leqslant 1\right) d\lambda(w) \\
&\quad + \int_{\mathbb{V}}^{1} \limsup_{T\to\infty} \left(\mathbb{P}\left(\left|\widehat{\mathbb{D}}_{T,[0,w]} - \mathbb{D}_{[0,w]}\right| \geqslant \frac{1}{\mathfrak{f}_T}\right) + \mathbb{P}\left(\left|\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{error}}_{T,[0,w]}\right| \geqslant 1\right)\right) d\lambda(w) \\
&\leqslant \limsup_{T\to\infty} \int_{\mathbb{V}}^{\mathbb{V}+E_{\mathbb{V},T}} \mathbb{P}\left(-\frac{T\sqrt{b}}{\mathfrak{f}_T} - 1 + T\sqrt{b}\,\mathbb{D}_{[0,w]} - \mathbf{Bias}^{\text{distr}}_{T,[0,w]} \leqslant \tau_T\right) d\lambda(w) \\
&\quad + \limsup_{T\to\infty} \int_{\mathbb{V}+E_{\mathbb{V},T}}^{1} \mathbb{P}\left(-\frac{T\sqrt{b}}{\mathfrak{f}_T} - 1 + T\sqrt{b}\,\mathbb{D}_{[0,w]} - \mathbf{Bias}^{\text{distr}}_{T,[0,w]} \leqslant \tau_T\right) d\lambda(w) \\
&= 0.
\end{aligned} \tag{C.160}$$

One obtains in the cases $\mathbb{V} \in [0,1)$ and $\mathbb{V} = 1$ from (C.152) as well as (C.160) (see (3.71)):

$$\mathbb{E}\left[\widehat{\mathbb{V}}_T\right] = \mathbb{V} + o(1). \tag{C.161}$$

Further, it follows similarly to (C.151) and (C.160) by using that the function $[0,1] \ni x \mapsto \sqrt{x}$ is continuous as well as monotonically increasing and $\sqrt{x+y+z} \leqslant \sqrt{x} + \sqrt{y} + \sqrt{z}\ \forall\, x,y,z \geqslant 0$:

$$\lim_{T\to\infty} \int_0^1 \sqrt{\mathbb{P}\left(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant \tau_T\right)}\, d\lambda(w) = \mathbb{V}.$$

Thus, the Cauchy-Schwarz inequality provides (note (3.71)):

$$\begin{aligned}
\limsup_{T\to\infty} \mathbb{E}\left[\widehat{\mathbb{V}}_T^2\right] &= \limsup_{T\to\infty} \int_0^1 \int_0^1 \mathbb{E}\left[\mathbb{1}_{\left\{T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w_1]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w_1]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w_1]} \leqslant \tau_T\right\}}\right. \\
&\quad \left. \cdot \mathbb{1}_{\left\{T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w_2]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w_2]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w_2]} \leqslant \tau_T\right\}}\right] d\lambda(w_1)\ d\lambda(w_2) \\
&\leqslant \mathbb{V}^2.
\end{aligned} \tag{C.162}$$

In conclusion, (C.162) and (C.161) imply:

$$\limsup_{T\to\infty} \operatorname{Var}\left(\widehat{\mathbb{V}}_T\right) \leqslant \limsup_{T\to\infty} \mathbb{E}\left[\widehat{\mathbb{V}}_T^2\right] - \liminf_{T\to\infty} \left(\mathbb{E}\left[\widehat{\mathbb{V}}_T\right]\right)^2 \leqslant 0. \tag{C.163}$$

The statement of Lemma 3.40 follows from (C.161) and (C.163). □

Proof of Proposition 3.42. (i) At first, suppose for all $\alpha \in (0,1)$ and $w \in (0,1]$ that $q_{1-\alpha}^{\text{distr}}(w)$ denotes the $(1-\alpha)$-quantile of the distribution of $Z_{[0,w]}^{\text{distr}}$ (recall (3.53)), i. e.:

$$q_{1-\alpha}^{\text{distr}}(w) := \inf\left\{x \in \mathbb{R} : \mathbb{P}\left(Z_{[0,w]}^{\text{distr}} \leqslant x\right) \geqslant 1-\alpha\right\}. \tag{C.164}$$

In the following, it is shown (see (3.74)):

$$\lim_{T\to\infty} \mathbb{P}\left(\left|q_{T,1-\alpha}^{\text{distr}*}(w) - q_{1-\alpha}^{\text{distr}}(w)\right| > \epsilon\right) = 0 \quad \forall\, \alpha \in (0,1),\ w \in (0,1],\ \epsilon > 0. \tag{C.165}$$

Therefor, one observes at first that Lemma 3.12, Assumption 3.1 $[\mathbf{WEI}.1]$ and straightforward arguments imply for all $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$ that (note (3.17)):

$$[0,1] \ni u \mapsto \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_{\infty,\mathrm{R}_1,\mathrm{R}_2}(u, s_1, s_2)^2\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 \tag{C.166}$$

is a continuous function. Thus, (3.73) provides $\operatorname{Var}\big(Z_{[0,w]}^{\text{distr}}\big) = \sigma_{[0,w]}^{\text{distr}} > 0$ (recall (3.53) and (3.52)) for all $w \in (0,1]$. Moreover, introduce for all $w \in (0,1]$ and $x \in \mathbb{R}$ the shorthands $F_w^{[Test]}(x) := \mathbb{P}\big(Z_{[0,w]}^{\text{distr}} \leqslant x\big)$ as well as $F_{w,T}^{[Test]*}(x) := \mathbb{P}^*\big(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w],\text{Test}}^* - \mathbf{Bias}_{T,[0,w]}^{\text{distr}*} \leqslant x\big)$. Since $\operatorname{Var}\big(Z_{[0,w]}^{\text{distr}}\big) > 0$, the (Gaussian) distribution function $F_w^{[Test]}$ is strictly increasing for all $w \in (0,1]$ and an $\epsilon_{w,\alpha} > 0$ exists for each $\epsilon > 0$, $w \in (0,1]$, $\alpha \in (0,1)$ that fulfils $F_w^{[Test]}\big(q_{1-\alpha}^{\text{distr}}(w) - \epsilon/2\big) \leqslant 1 - \alpha - \epsilon_{w,\alpha}$. These considerations, (C.98) and (3.58) imply for all $\alpha \in (0,1)$, $w \in (0,1]$, $\epsilon > 0$ (see (3.74) as well as (C.164)):

$$\begin{aligned}
&\mathbb{P}\left(q_{T,1-\alpha}^{\text{distr}*}(w) + \epsilon/2 \geqslant q_{1-\alpha}^{\text{distr}}(w) - \epsilon/2\right)\\
&= \mathbb{P}\left(F_w^{[Test]}\left(q_{T,1-\alpha}^{\text{distr}*}(w) + \epsilon/2\right) \geqslant F_w^{[Test]}\left(q_{1-\alpha}^{\text{distr}}(w) - \epsilon/2\right)\right)\\
&\geqslant \mathbb{P}\Bigg(-\left|-F_w^{[Test]}\left(q_{T,1-\alpha}^{\text{distr}*}(w) + \epsilon/2\right) + F_{w,T}^{[Test]*}\left(q_{T,1-\alpha}^{\text{distr}*}(w) + \epsilon/2\right)\right| + F_{w,T}^{[Test]*}\left(q_{T,1-\alpha}^{\text{distr}*}(w) + \epsilon/2\right)\\
&\geqslant 1 - \alpha - \epsilon_{w,\alpha}, \left|-F_w^{[Test]}\left(q_{T,1-\alpha}^{\text{distr}*}(w) + \epsilon/2\right) + F_{w,T}^{[Test]*}\left(q_{T,1-\alpha}^{\text{distr}*}(w) + \epsilon/2\right)\right| \leqslant \frac{\epsilon_{w,\alpha}}{2}\Bigg)\\
&\geqslant \mathbb{P}\Bigg(-\frac{\epsilon_{w,\alpha}}{2} + 1 - \alpha \geqslant 1 - \alpha - \epsilon_{w,\alpha}, \left|F_w^{[Test]}\left(q_{T,1-\alpha}^{\text{distr}*}(w) + \epsilon/2\right) - F_{w,T}^{[Test]*}\left(q_{T,1-\alpha}^{\text{distr}*}(w) + \epsilon/2\right)\right|\\
&\leqslant \frac{\epsilon_{w,\alpha}}{2}\Bigg)\\
&\overset{T\to\infty}{\longrightarrow} 1
\end{aligned} \tag{C.167}$$

and analog arguments show:

$$\mathbb{P}\left(q_{T,1-\alpha}^{\text{distr}*}(w) - \epsilon/2 \leqslant q_{1-\alpha}^{\text{distr}}(w) + \epsilon/2\right) \overset{T\to\infty}{\longrightarrow} 1. \tag{C.168}$$

Overall, (C.167) and (C.168) yield (C.165).
Further, one observes for all measurable sets $\mathcal{A}, \mathcal{B} \subseteq \Omega$ (whereby Ω originates from Definition 2.1 and $\mathcal{B}'$ denotes the complement of $\mathcal{B}$):

$$\mathbb{P}(\mathcal{A}) \leqslant \mathbb{P}(\mathcal{A} \cap \mathcal{B}) + \mathbb{P}(\mathcal{B}'). \tag{C.169}$$

It follows for all $\alpha \in (0,1)$, $\epsilon > 0$ from (C.169), the Fatou–Lebesgue theorem, (C.165) and Theorem 3.36 (i) in combination with $\operatorname{Var}\big(Z_{[0,w]}^{\text{distr}}\big) > 0$ for all $w \in (0,1]$ - as shown above (recall (3.69), that

$\widehat{\mathbb{D}}_{T,[0,w]}$, $\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]}$ and $\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}$ are defined as in Definition 3.8 (i), (3.68) as well as (3.57), whereby $\mathfrak{U}_{0,1} = [0, w]$ is chosen and see also the last line of (3.71)):

$$\limsup_{T\to\infty} \mathbb{E}\Bigg[\lambda\Bigg(\Bigg\{w \in [0,\mathbb{V}] : T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant q^{\text{distr}*}_{T,1-\alpha}(w)\Bigg\}\Bigg)\Bigg]$$

$$\leqslant \limsup_{T\to\infty} \int_0^{\mathbb{V}} \mathbb{P}\Bigg(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant q^{\text{distr}*}_{T,1-\alpha}(w),$$

$$\Big|q^{\text{distr}*}_{T,1-\alpha}(w) - q^{\text{distr}}_{1-\alpha}(w)\Big| \leqslant \epsilon\Bigg) + \mathbb{P}\Bigg(\Big|q^{\text{distr}*}_{T,1-\alpha}(w) - q^{\text{distr}}_{1-\alpha}(w)\Big| > \epsilon\Bigg)\, d\lambda(w)$$

$$\leqslant \int_0^{\mathbb{V}} \limsup_{T\to\infty} \mathbb{P}\Bigg(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant q^{\text{distr}}_{1-\alpha}(w) + \epsilon\Bigg)\, d\lambda(w)$$

$$= \int_0^{\mathbb{V}} \mathbb{P}\Bigg(Z^{\text{distr}}_{[0,w]} \leqslant q^{\text{distr}}_{1-\alpha}(w) + \epsilon\Bigg)\, d\lambda(w). \tag{C.170}$$

Moreover, the Fatou–Lebesgue theorem, (C.98), Theorem 3.36 (i) in combination with $\operatorname{Var}\big(Z^{\text{distr}}_{[0,w]}\big) > 0$ for all $w \in (0,1]$ and (C.165) imply for all $\alpha \in (0,1)$, $\epsilon > 0$:

$$\liminf_{T\to\infty} \mathbb{E}\Bigg[\lambda\Bigg(\Bigg\{w \in [0,\mathbb{V}] : T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant q^{\text{distr}*}_{T,1-\alpha}(w)\Bigg\}\Bigg)\Bigg]$$

$$\geqslant \liminf_{T\to\infty} \int_0^{\mathbb{V}} \mathbb{P}\Bigg(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant q^{\text{distr}*}_{T,1-\alpha}(w),$$

$$\Big|q^{\text{distr}*}_{T,1-\alpha}(w) - q^{\text{distr}}_{1-\alpha}(w)\Big| \leqslant \epsilon\Bigg)\, d\lambda(w)$$

$$\geqslant \int_0^{\mathbb{V}} \liminf_{T\to\infty} \mathbb{P}\Bigg(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant q^{\text{distr}}_{1-\alpha}(w) - \epsilon\Bigg)$$

$$+ \liminf_{T\to\infty} \mathbb{P}\left(\Big|q^{\text{distr}*}_{T,1-\alpha}(w) - q^{\text{distr}}_{1-\alpha}(w)\Big| \leqslant \epsilon\right) - 1\, d\lambda(w)$$

$$= \int_0^{\mathbb{V}} \mathbb{P}\Bigg(Z^{\text{distr}}_{[0,w]} \leqslant q^{\text{distr}}_{1-\alpha}(w) - \epsilon\Bigg)\, d\lambda(w). \tag{C.171}$$

Lebesgue's dominated convergence theorem and $\operatorname{Var}\big(Z^{\text{distr}}_{[0,w]}\big) > 0$ for all $w \in (0,1]$ show for all $\alpha \in (0,1)$ (see (C.164)):

$$(1-\alpha)\mathbb{V} = \lim_{\epsilon\to 0} \int_0^{\mathbb{V}} \mathbb{P}\Bigg(Z^{\text{distr}}_{[0,w]} \leqslant q^{\text{distr}}_{1-\alpha}(w) + \epsilon\Bigg)\, d\lambda(w) \geqslant \lim_{\epsilon\to 0} \int_0^{\mathbb{V}} \mathbb{P}\Bigg(Z^{\text{distr}}_{[0,w]} \leqslant q^{\text{distr}}_{1-\alpha}(w) - \epsilon\Bigg)\, d\lambda(w)$$

$$= (1-\alpha)\mathbb{V}. \tag{C.172}$$

Overall, (C.170), (C.171) and (C.172) provide for all $\alpha \in (0,1)$:

$$\lim_{T\to\infty} \mathbb{E}\left[\lambda\Bigg(\Bigg\{w \in [0,\mathbb{V}] : T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]} + \widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]} - \mathbf{Bias}^{\text{distr}*}_{T,[0,w]} \leqslant q^{\text{distr}*}_{T,1-\alpha}(w)\Bigg\}\Bigg)\right] = (1-\alpha)\mathbb{V}. \tag{C.173}$$

Since $\sup_{w\in(0,1]} \operatorname{Var}\big(Z^{\text{distr}}_{[0,w]}\big) = \sup_{w\in(0,1]} \sigma^{\text{distr}}_{[0,w]} \leqslant C$ (which follows from Lemma 3.12 as well as Assumption 3.1 $[\mathbf{WEI.1}]$ by recalling (3.52) and that the kernel $K_{[0,w]}$ defined in (3.72) is considered) as well as (3.54) yield $1 + q^{\text{distr}}_{1-\alpha}(w) \leqslant \tau_T$ for all $\alpha \in (0,1)$, $w \in (0,1]$ and sufficiently large T (which depends on α but not on w), one obtains from (C.169), the Fatou–Lebesgue theorem, (C.165) as well as

(C.160):

$$\limsup_{T\to\infty}\mathbb{E}\Big[\lambda\Big(\Big\{w\in[\mathbb{V},1]:T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\widehat{\mathbf{Bias}}}{}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant q^{\text{distr}*}_{T,1-\alpha}(w)\Big\}\Big)\Big]$$

$$\leqslant \limsup_{T\to\infty}\int_{\mathbb{V}}^{1}\mathbb{P}\Big(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\widehat{\mathbf{Bias}}}{}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant q^{\text{distr}*}_{T,1-\alpha}(w),$$

$$\Big|q^{\text{distr}*}_{T,1-\alpha}(w)-q^{\text{distr}}_{1-\alpha}(w)\Big|\leqslant 1\Big)\,d\lambda(w)+\int_{\mathbb{V}}^{1}\limsup_{T\to\infty}\mathbb{P}\left(\Big|q^{\text{distr}*}_{T,1-\alpha}(w)-q^{\text{distr}}_{1-\alpha}(w)\Big|\geqslant 1\right)\,d\lambda(w)$$

$$\leqslant \limsup_{T\to\infty}\int_{\mathbb{V}}^{1}\mathbb{P}\Big(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\widehat{\mathbf{Bias}}}{}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant 1+q^{\text{distr}}_{1-\alpha}(w)\Big)\,d\lambda(w)$$

$$=0. \tag{C.174}$$

Further, it follows for all $\alpha\in(0,1)$ similarly to (C.170), (C.171), (C.172) and (C.174) by using that the function $[0,1]\ni x\mapsto\sqrt{x}$ is continuous as well as monotonically increasing and $\sqrt{x+y}\leqslant\sqrt{x}+\sqrt{y}$ $\forall\,x,y\geqslant 0$:

$$\lim_{T\to\infty}\int_{0}^{1}\sqrt{\mathbb{P}\left(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\widehat{\mathbf{Bias}}}{}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant q^{\text{distr}*}_{T,1-\alpha}(w)\right)}\,d\lambda(w)=\sqrt{1-\alpha}\,\mathbb{V}. \tag{C.175}$$

One obtains for all $\alpha\in(0,1)$ analogously to (C.162) and (C.163) by using (C.175), (C.173) as well as (C.174):

$$\limsup_{T\to\infty}\mathrm{Var}\Big(\lambda\Big(\Big\{w\in[0,1]:T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\widehat{\mathbf{Bias}}}{}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant q^{\text{distr}*}_{T,1-\alpha}(w)\Big\}\Big)\Big)$$
$$\leqslant(1-\alpha)\mathbb{V}^2-(1-\alpha)^2\mathbb{V}^2. \tag{C.176}$$

Proposition 3.42 (i) is an implication of $\Big\|\widehat{\mathbb{V}}_{T,1-\alpha}-\mathbb{V}\Big\|_2\leqslant\sqrt{\big(\mathbb{E}\big[\widehat{\mathbb{V}}_{T,1-\alpha}\big]-\mathbb{V}\big)^2+\mathrm{Var}\big(\widehat{\mathbb{V}}_{T,1-\alpha}\big)}$, (C.173), (C.174), (C.176) and $(-\alpha)^2+1-\alpha-(1-\alpha)^2=\alpha$.

(ii) In order to prove Proposition 3.42 (ii), observe at first for all $w\in(0,1]$ that (as already explained in Part (i) of the present proof) $\mathbb{R}\ni x\mapsto F^{[Test]}_{w}(x):=\mathbb{P}\big(Z^{\text{distr}}_{[0,w]}\leqslant x\big)$ is strictly increasing and $\mathrm{Var}\big(Z^{\text{distr}}_{[0,w]}\big)=\sigma^{\text{distr}}_{[0,w]}>0$ holds. Thus, it follows for all $\epsilon\in(0,1)$ from the Fatou–Lebesgue theorem, (C.98), (3.58), Theorem 3.36 (i) together with Slutzky's lemma as well as Polya's theorem and by using (3.76) $\big($note $F^{[Test]*}_{w,T}(x):=\mathbb{P}^*\big(T\sqrt{b}\,\widehat{\mathbb{D}}^*_{T,[0,w],\text{Test}}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant x\big)$ as well as $F^{[Test]*}_{w,T}\big(q^{\text{distr}*}_{T,1-\alpha_T}(w)+1/T\big)\geqslant 1-\alpha_T$ according to (3.74) and $1-\alpha_T-\epsilon\in(0,1)$ for sufficiently large $T\big)$:

$$\liminf_{T\to\infty}\mathbb{E}\left[\lambda\left(\Big\{w\in[0,\mathbb{V}]:T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\widehat{\mathbf{Bias}}}{}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant q^{\text{distr}*}_{T,1-\alpha_T}(w)\Big\}\right)\right]$$

$$=\liminf_{T\to\infty}\int_{0}^{\mathbb{V}}\mathbb{P}\Big(F^{[Test]}_{w}\Big(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\widehat{\mathbf{Bias}}}{}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}+\frac{1}{T}\Big)\leqslant F^{[Test]}_{w}\Big(q^{\text{distr}*}_{T,1-\alpha_T}(w)$$

$$+\frac{1}{T}\Big)\Big)\,d\lambda(w)$$

$$\geqslant\liminf_{T\to\infty}\int_{0}^{\mathbb{V}}\mathbb{P}\Big(F^{[Test]}_{w}\Big(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\widehat{\mathbf{Bias}}}{}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}+\frac{1}{T}\Big)\leqslant F^{[Test]*}_{w,T}\Big(q^{\text{distr}*}_{T,1-\alpha_T}(w)$$

$$+\frac{1}{T}\Bigg)-\epsilon,\left|F_w^{[Test]}\left(q^{\text{distr}*}_{T,1-\alpha_T}(w)+\frac{1}{T}\right)-F_{w,T}^{[Test]*}\left(q^{\text{distr}*}_{T,1-\alpha_T}(w)+\frac{1}{T}\right)\right|\leqslant\epsilon\Bigg)\,d\lambda(w)$$

$$\geqslant\int_0^{\mathbb{V}}\liminf_{T\to\infty}\mathbb{P}\Bigg(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}+\frac{1}{T}\leqslant F_w^{[Test]-1}\left(1-\alpha_T-\epsilon\right)\Bigg)\,d\lambda(w)$$

$$=\mathbb{V}\left(1-\epsilon\right). \tag{C.177}$$

Further, let $(w_T)_{T\in\mathbb{N}}$ be an arbitrary sequence of deterministic real numbers that fulfils:

$$w_T\in(0,1)\;\forall\,T\in\mathbb{N},\quad w_T\overset{T\to\infty}{\longrightarrow}0,\quad w_TTb\overset{T\to\infty}{\longrightarrow}\infty\quad\text{and}\quad\frac{\beta}{w_T}=o\left(\sqrt{Tb}\right). \tag{C.178}$$

A sequence $(w_T)_{T\in\mathbb{N}}$ which converges slowly enough to zero that it fulfils (C.178) exists because Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) provides $Tb\overset{T\to\infty}{\longrightarrow}\infty$ and Assumption 3.15 $[\mathbf{W}^*]$ (i) ensures $\beta=o\big(\sqrt{Tb^2}\sqrt{1/b}\big)$.
Next, it is shown for sufficiently large T:

$$q^{\text{distr}*}_{T,1-\alpha_T}(w)\leqslant\tau_T\quad\forall\,w\in[w_T,1]. \tag{C.179}$$

To verify (C.179), let for $w\in(0,1]$ the expression $\widehat{\varphi}_{T,[0,w]}$ be defined as in Definition 2.11 with $\mathfrak{U}_{0,1}=[0,w]$, whereby $K_{[0,w]}$ introduced in (3.72) should be the underlying kernel. One obtains for all $w\in[w_T,1]$, $s\in\mathbb{R}^d$ from applying Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) to the kernel K contained in (3.72) and by using (C.178):

$$\begin{aligned}\sup_{u\in[0,1]}\left|\widehat{\varphi}_{T,[0,w]}\left(u,s\right)\right|&\leqslant\sup_{u\in[0,1]}\frac{1}{Twb}\sum_{t=\lfloor uT-Twb\rfloor}^{\lceil uT+Twb\rceil}K\left(\frac{\frac{t}{T}-u}{wb}\right)\\&\leqslant\frac{C}{Twb}\sup_{u\in[0,1]}\left(uT+Twb-\left(uT-Twb\right)+2\right)\leqslant C+\frac{C}{Twb}\leqslant C+\frac{C}{Tw_Tb}\leqslant C.\end{aligned} \tag{C.180}$$

It follows for all $w\in[w_T,1]$ from Assumption 3.15 $[\mathbf{W}^*]$ (ii), (C.180), Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) applied to the kernel K contained in $K_{[0,w]}$, shifting the indices of sums and Assumption 3.15 $[\mathbf{W}^*]$ (iii) together with $2\lfloor Tb\rfloor+3\leqslant2\lfloor Tb\rfloor+T$ for sufficiently large T $\big($recall that $\widehat{\mathbb{D}}^*_{T,[0,w],\text{Test}}$ originates from (3.56) with $\mathfrak{U}_{0,1}=[0,w]$, whereby $K_{[0,w]}$ is the underlying kernel$\big)$:

$$\begin{aligned}\mathbb{E}^*\left[\widehat{\mathbb{D}}^*_{T,[0,w],\text{Test}}\right]&\leqslant\frac{Cw}{T^2b^2w^2}\sup_{u\in[0,1]}\sum_{t_1,t_2=\lfloor uT\rfloor-\lfloor Twb\rfloor-1}^{\lfloor uT\rfloor+\lfloor Twb\rfloor+2}K\left(\frac{\frac{t_1}{T}-u}{wb}\right)K\left(\frac{\frac{t_2}{T}-u}{wb}\right)\left|\mathbb{E}\left[W^*_{t_1}W^*_{t_2}\right]\right|\\&\leqslant\frac{Cw}{T^2b^2w^2}\sum_{t_1=-\lfloor Twb\rfloor-1}^{\lfloor Twb\rfloor+2}\;\sum_{t_2=-\lfloor Twb\rfloor-1-t_1}^{\lfloor Twb\rfloor+2-t_1}\left|K^*\left(\frac{t_2}{\beta}\right)\right|\\&\leqslant\frac{Cw}{T^2b^2w^2}\sum_{t_1=-\lfloor Tb\rfloor-1}^{\lfloor Tb\rfloor+2}\;\sum_{t_2=-2\lfloor Tb\rfloor-3}^{2\lfloor Tb\rfloor+3}\left|K^*\left(\frac{t_2}{\beta}\right)\right|\\&\leqslant\frac{C}{Tb}\frac{\beta}{w_T}.\end{aligned} \tag{C.181}$$

The conditional version of Markov's inequality, (C.181) and arguments which are similar to those that provide (C.155) together with Assumption 3.15 $[\mathbf{W}^*]$ (iii) yield for all $w\in[w_T,1]$ (see (3.57)):

$$\mathbb{P}^*\Big(T\sqrt{b}\,\widehat{\mathbb{D}}^*_{T,[0,w],\text{Test}}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant\tau_T\Big)\geqslant1-\frac{\frac{C}{\sqrt{b}}\frac{\beta}{w_T}+\frac{C}{\sqrt{b}}}{\tau_T}, \tag{C.182}$$

whereby (C.178) as well as Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) imply:

$$\frac{C}{\sqrt{b}}\frac{\beta}{w_T}+\frac{C}{\sqrt{b}}=o\left(\sqrt{T}\right).$$

Thus, one obtains from (3.76) for sufficiently large T:

$$1-\frac{\frac{C}{\sqrt{b}}\frac{\beta}{w_T}+\frac{C}{\sqrt{b}}}{\tau_T}\geqslant 1-\alpha_T,$$

such that (C.182) yields (C.179) (recall (3.74)).
It follows from (C.179), (C.178) and (C.160) (note that $\mathbb{V}<1$ is assumed for (C.160), however, (C.183) holds obviously in the case $\mathbb{V}=1$):

$$\begin{aligned}
&\limsup_{T\to\infty}\mathbb{E}\left[\lambda\left(\left\{w\in[\mathbb{V},1]:T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant q^{\text{distr}*}_{T,1-\alpha_T}(w)\right\}\right)\right]\\
&\leqslant\limsup_{T\to\infty}\lambda\left([0,w_T]\right)+\limsup_{T\to\infty}\mathbb{E}\Bigg[\lambda\Bigg(\Bigg\{w\in\left[\max\left\{w_T,\mathbb{V}\right\},1\right]:T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]}\\
&-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant\tau_T\Bigg\}\Bigg)\Bigg]\\
&=0.
\end{aligned}\tag{C.183}$$

Since $\epsilon\in(0,1)$ is arbitrary in (C.177) and the expectation in the first line of (C.177) is smaller or equals $\lambda\left([0,\mathbb{V}]\right)=\mathbb{V}$ for all $T\in\mathbb{N}$, one obtains from (C.177) as well as (C.183):

$$\lim_{T\to\infty}\mathbb{E}\left[\lambda\left(\left\{w\in[0,1]:T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant q^{\text{distr}*}_{T,1-\alpha_T}(w)\right\}\right)\right]=\mathbb{V}\tag{C.184}$$

and similar arguments show:

$$\lim_{T\to\infty}\int_0^1\sqrt{\mathbb{P}\left(T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant q^{\text{distr}*}_{T,1-\alpha_T}(w)\right)}\,d\lambda(w)=\mathbb{V}.\tag{C.185}$$

It follows analogously to (C.162) and (C.163) by using (C.185) as well as (C.184):

$$\limsup_{T\to\infty}\operatorname{Var}\left(\lambda\left(\left\{w\in[0,1]:T\sqrt{b}\,\widehat{\mathbb{D}}_{T,[0,w]}+\widehat{\mathbf{Bias}}^{\text{error}}_{T,[0,w]}-\mathbf{Bias}^{\text{distr}*}_{T,[0,w]}\leqslant q^{\text{distr}*}_{T,1-\alpha_T}(w)\right\}\right)\right)=0.\tag{C.186}$$

Overall, (C.184) and (C.186) prove Proposition 3.42 (ii) (see (3.74)). □

C.2. Auxiliary results belonging to Chapter 3 and their proofs

Lemma C.1. *Let the Assumptions 2.4* $[\mathbf{DM}.\mathbf{1}]$*, 3.1* $[\mathbf{WEI}.\mathbf{1}]$ *and 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *be fulfilled. Moreover, define for all* $\mathrm{R}\in\{\Re,\Im\}$ *(recall that* $X^c:=X-\mathbb{E}[X]$ *for each random variable* X *with finite first moment):*

$$\widehat{\mathbb{D}}_{T,1,\mathrm{R}}:=\widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},1,\mathrm{R}}:=\int_{\mathbb{R}^d}\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathrm{R}\left\{\widehat{\varphi}\left(u_k,s\right)\right\}^2\,\mathbf{w}(s)\,ds,$$

$$\mathbb{D}_{1,\mathrm{R}}:=\mathbb{D}_{\mathfrak{U}_{0,1},1,\mathrm{R}}:=\int_{\mathbb{R}^d}\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\mathrm{R}\left\{\varphi(u,s)\right\}^2\,du\,\mathbf{w}(s)\,ds,$$

$$\widehat{\mathbb{D}}^{[1]}_{T,1,\mathrm{R}}:=\widehat{\mathbb{D}}^{[1]}_{T,\mathfrak{U}_{0,1},1,\mathrm{R}}:=\int_{\mathbb{R}^d}\frac{2\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathrm{R}\left\{\widehat{\varphi}\left(u_k,s\right)\right\}^c\mathrm{R}\left\{\varphi\left(u_k,s\right)\right\}\,\mathbf{w}(s)\,ds,$$

$$\widehat{\mathbb{D}}_{T,2,\mathrm{R}}:=\widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},2,\mathrm{R}}:=\int_{\mathbb{R}^d}\frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\mathrm{R}\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\widehat{\varphi}\left(u_k,s\right)\right\}^2\,\mathbf{w}(s)\,ds,$$

$$\mathbb{D}_{2,\mathrm{R}} := \mathbb{D}_{\mathfrak{U}_{0,1},2,\mathrm{R}} := \int_{\mathbb{R}^d} \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \mathrm{R} \left\{ \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \varphi(u,s)\, du \right\}^2 \mathbf{w}(s)\, ds \quad \textit{and}$$

$$\widehat{\mathbb{D}}_{T,2,\mathrm{R}}^{[1]} := \widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},2,\mathrm{R}}^{[1]} := \int_{\mathbb{R}^d} \frac{2\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right)\right\}^c \cdot \mathrm{R} \left\{ \frac{1}{\mathfrak{U}_1 - \mathfrak{U}_0} \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \varphi(u,s)\, du \right\} \mathbf{w}(s)\, ds. \tag{C.187}$$

Then, it holds for $T \to \infty$ and all $\mathrm{R} \in \{\Re, \Im\}$:

(i) $\sqrt{T} \left\| \widehat{\mathbb{D}}_{T,1,\mathrm{R}} - \mathbb{D}_{1,\mathrm{R}} - \widehat{\mathbb{D}}_{T,1,\mathrm{R}}^{[1]} \right\|_1 = o(1).$

(ii) $\sqrt{T} \left\| \widehat{\mathbb{D}}_{T,2,\mathrm{R}} - \mathbb{D}_{2,\mathrm{R}} - \widehat{\mathbb{D}}_{T,2,\mathrm{R}}^{[1]} \right\|_1 = o(1).$

Proof. (i) In the following, Lemma C.1 (i) with $\mathrm{R} = \Re$ will be proved. Lemma B.2 (iii) in combination with (3.15), Proposition 2.12 together with (3.11), (C.370), the fact that Lemma B.1 with $\kappa = 1$ provides $\left|\mathbb{E}\left[\Re\left\{\widehat{\varphi}\left(u,s\right)\right\}\right]\right| \leqslant C$ a. s. $\forall\, u \in [0,1]$, $s \in \mathbb{R}^d$ and $\delta \in (0,1]$ imply for all $s \in \mathbb{R}^d$ (see Definition 3.8 (i)):

$$\begin{aligned}
&\sqrt{T}\, \mathbb{E}\left[\left| \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\widehat{\varphi}\left(u_k,s\right)\right\}^2 - \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re\left\{\varphi(u,s)\right\}^2\, du \right.\right. \\
&\left.\left. - \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left(\Re\left\{\widehat{\varphi}\left(u_k,s\right)\right\}^2 - \left(\mathbb{E}\left[\Re\left\{\widehat{\varphi}\left(u_k,s\right)\right\}\right]\right)^2 \right) \right| \right] \\
&\leqslant \sqrt{T} \left| - \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re\left\{\varphi(u,s)\right\}^2\, du + \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\varphi\left(u_k,s\right)\right\}^2 \right| \\
&+ \sqrt{T} \left| - \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\varphi\left(u_k,s\right)\right\}^2 + \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left(\mathbb{E}\left[\Re\left\{\widehat{\varphi}\left(u_k,s\right)\right\}\right]\right)^2 \right| \\
&\leqslant C \left(\sqrt{T} b^{1+\delta} + \frac{1}{\sqrt{Tb}} \right) \left(|s|_1^2 + 1 \right).
\end{aligned} \tag{C.188}$$

Moreover, one observes for each real-valued random variable X with finite first moment:

$$X^2 - \left(\mathbb{E}\left[X\right]\right)^2 - 2X^c\, \mathbb{E}\left[X\right] = \left(X - \mathbb{E}\left[X\right]\right)^2, \tag{C.189}$$

which yields for all $s \in \mathbb{R}^d$ due to Proposition 2.14:

$$\begin{aligned}
&\sqrt{T}\, \mathbb{E}\left[\left| \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left(\Re\left\{\widehat{\varphi}\left(u_k,s\right)\right\}^2 - \left(\mathbb{E}\left[\Re\left\{\widehat{\varphi}\left(u_k,s\right)\right\}\right]\right)^2 \right) \right.\right. \\
&\left.\left. - \frac{2\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\widehat{\varphi}\left(u_k,s\right)\right\}^c\, \mathbb{E}\left[\Re\left\{\widehat{\varphi}\left(u_k,s\right)\right\}\right] \right| \right] \\
&\leqslant \frac{C}{\sqrt{Tb}} \left(|s|_1 + 1 \right).
\end{aligned} \tag{C.190}$$

Since $\left|\Re\left\{\widehat{\varphi}\left(u,s\right)\right\}^c\right| \leqslant C$ a. s. $\forall\, u \in [0,1]$, $s \in \mathbb{R}^d$ (which follows from Lemma B.1 with $\kappa_1 = 1$), Proposition 2.12 together with (3.11) provides for all $s \in \mathbb{R}^d$:

$$\begin{aligned}
&\sqrt{T}\, \mathbb{E}\left[\left| \frac{2\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\widehat{\varphi}\left(u_k,s\right)\right\}^c \left(\mathbb{E}\left[\Re\left\{\widehat{\varphi}\left(u_k,s\right)\right\}\right] - \Re\left\{\varphi\left(u_k,s\right)\right\} \right) \right| \right] \\
&\leqslant C \left(\sqrt{T} b^{1+\delta} + \frac{1}{\sqrt{Tb}} \right) \left(|s|_1^{1+\delta} + 1 \right).
\end{aligned} \tag{C.191}$$

Lemma C.1 (i) with $\mathrm{R} = \Re$ is an implication of (C.188), (C.190), (C.191) as well as the Assumptions 3.1 $[\mathbf{WEI}.\mathbf{1}]$ and 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii) (recall (C.187)). Similar arguments prove Lemma C.1 (i) with $\mathrm{R} = \Im$.

(ii) In the following, Lemma C.1 (ii) with $\mathrm{R} = \Re$ will be shown. One obtains for all $v, w \in (0,1)$, $s \in \mathbb{R}^d$ from (3.12), (3.14) with $q = (1+\delta)/\delta$, Assumption 2.2 $[\mathbf{StAp}]$ (ii), $\delta \in (0,1]$ and $|v-w| \leqslant |v-w|^{\delta}$:

$$\begin{aligned}
&|\partial_v \Re\{\varphi(v,s)\} - \partial_w \Re\{\varphi(w,s)\}| \\
&\leqslant \Big|\mathbb{E}\Big[\Big(-\sin\Big(\Big\langle s, \widetilde{X}_0(v)\Big\rangle\Big) + \sin\Big(\Big\langle s, \widetilde{X}_0(w)\Big\rangle\Big)\Big)\Big\langle s, \partial_v \widetilde{X}_0(v)\Big\rangle\Big] \\
&+ \mathbb{E}\Big[\Big(-\Big\langle s, \partial_v \widetilde{X}_0(v)\Big\rangle + \Big\langle s, \partial_w \widetilde{X}_0(w)\Big\rangle\Big)\sin\Big(\Big\langle s, \widetilde{X}_0(w)\Big\rangle\Big)\Big]\Big| \\
&\leqslant C\Big\|-\sin\Big(\Big\langle s, \widetilde{X}_0(v)\Big\rangle\Big) + \sin\Big(\Big\langle s, \widetilde{X}_0(w)\Big\rangle\Big)\Big\|_{\frac{1+\delta}{\delta}}\Big\|\Big\langle s, \partial_v \widetilde{X}_0(v)\Big\rangle\Big\|_{1+\delta} \\
&+ C\Big\|\Big\langle s, -\partial_v \widetilde{X}_0(v) + \partial_w \widetilde{X}_0(w)\Big\rangle\Big\|_1 \\
&\leqslant C\left(|s|_1^{1+\delta} + |s|_1\right)|v-w|^{\delta}.
\end{aligned} \tag{C.192}$$

It follows for all $s \in \mathbb{R}^d$ from (C.370), Lemma B.2 (iii) together with (C.192), $|\mathbb{E}[\widehat{\varphi}(u,s)]| \leqslant C$ a. s. $\forall\, u \in [0,1]$, $s \in \mathbb{R}^d$ (the latter holds due to Lemma B.1 with $\kappa_1 = 1$) and Proposition 2.12 in combination with (3.11) (see Definition 3.8 (i)):

$$\begin{aligned}
&\sqrt{T}\,\mathbb{E}\left[\left|\frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\widehat{\varphi}(u_k,s)\right\}^2 - \frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\varphi(u,s)\,du\right\}^2\right.\right. \\
&\left.\left.-\left(\frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\widehat{\varphi}(u_k,s)\right\}^2 - \frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{E}[\widehat{\varphi}(u_k,s)]\right\}^2\right)\right|\right] \\
&\leqslant \frac{\sqrt{T}}{\mathfrak{U}_1-\mathfrak{U}_0}\left|-\Re\left\{\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\varphi(u,s)\,du\right\}^2 + \Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\varphi(u_k,s)\right\}^2\right| \\
&+ \frac{\sqrt{T}}{\mathfrak{U}_1-\mathfrak{U}_0}\left|-\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\varphi(u_k,s)\right\}^2 + \Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{E}[\widehat{\varphi}(u_k,s)]\right\}^2\right| \\
&\leqslant C\left(\sqrt{T}b^{1+\delta} + \frac{1}{\sqrt{T}b}\right)\left(|s|_1^{1+\delta} + 1\right).
\end{aligned} \tag{C.193}$$

Moreover, (C.189) and Proposition 2.14 show for all $s \in \mathbb{R}^d$:

$$\begin{aligned}
&\sqrt{T}\,\mathbb{E}\left[\left|\frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\widehat{\varphi}(u_k,s)\right\}^2 - \frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{E}[\widehat{\varphi}(u_k,s)]\right\}^2\right.\right. \\
&\left.\left.-\frac{2}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\widehat{\varphi}(u_k,s) - \mathbb{E}[\widehat{\varphi}(u_k,s)]\right)\right\}\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{E}[\widehat{\varphi}(u_k,s)]\right\}\right|\right] \\
&\leqslant \frac{\sqrt{T}}{\mathfrak{U}_1-\mathfrak{U}_0}\left\|\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\widehat{\varphi}(u_k,s) - \mathbb{E}[\widehat{\varphi}(u_k,s)]\right)\right\}\right\|_2^2 \\
&\leqslant \frac{\sqrt{T}}{\mathfrak{U}_1-\mathfrak{U}_0}\left(\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left\|\widehat{\varphi}(u_k,s) - \mathbb{E}[\widehat{\varphi}(u_k,s)]\right\|_2\right)^2 \\
&\leqslant \frac{C}{\sqrt{T}b}\left(|s|_1 + 1\right).
\end{aligned} \tag{C.194}$$

One obtains for all $s \in \mathbb{R}^d$ from $|\widehat{\varphi}(u,s) - \mathbb{E}[\widehat{\varphi}(u,s)]| \leqslant C$ a. s. $\forall\, u \in [0,1]$ (which holds due to

Lemma B.1 with $\kappa_1 = 1$) and arguments which are similar to those that show (C.193):

$$\begin{aligned}
&\sqrt{T}\,\mathbb{E}\left[\left|\frac{2}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\widehat{\varphi}\left(u_k,s\right)-\mathbb{E}\left[\widehat{\varphi}\left(u_k,s\right)\right]\right)\right\}\right.\right.\\
&\quad\left.\left.\cdot\left(\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{E}\left[\widehat{\varphi}\left(u_k,s\right)\right]\right\}-\Re\left\{\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\varphi(u,s)\,du\right\}\right)\right|\right]\\
&\leqslant C\left(\sqrt{T}b^{1+\delta}+\frac{1}{\sqrt{Tb}}\right)\left(|s|_1^{1+\delta}+1\right).
\end{aligned} \tag{C.195}$$

Lemma C.1 (ii) with $\mathrm{R} = \Re$ is an implication of (C.193), (C.194), (C.195) and the Assumptions 3.1 $[\mathbf{WEI.1}]$ as well as 2.8 $[\mathbf{K\&b.1}]$ (ii) (note (C.187)). Similar arguments prove Lemma C.1 (ii) with $\mathrm{R} = \Im$. □

Lemma C.2. *Suppose that the Assumptions 2.4* $[\mathbf{DM.1}]$*, 3.1* $[\mathbf{WEI.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *hold. Define for all* $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$*,* $s \in \mathbb{R}^d$*,* $\mathrm{R} \in \{\Re, \Im\}$ *(recall (C.17)):*

$$\begin{aligned}
\varphi^{\circ}\left(u_k,s\right) &:= \varphi^{\circ}_{T,\mathfrak{U}_{0,1}}\left(u_k,s\right) := \frac{1}{\lfloor Tb\rfloor}\sum_{t=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}K\left(\frac{t-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)e^{i\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t}\left(\tilde{u}_{k,t}\right)\right\rangle},\\
\widehat{\mathbb{D}}^{[2]}_{T,1,\mathrm{R}} &:= \widehat{\mathbb{D}}^{[2]}_{T,\mathfrak{U}_{0,1},1,\mathrm{R}} := \int_{\mathbb{R}^d}\frac{2\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathrm{R}\left\{\varphi^{\circ}\left(u_k,s\right)\right\}^c\mathrm{R}\left\{\varphi\left(u_k,s\right)\right\}\mathbf{w}(s)\,ds \quad \text{and}\\
\widehat{\mathbb{D}}^{[2]}_{T,2,\mathrm{R}} &:= \widehat{\mathbb{D}}^{[2]}_{T,\mathfrak{U}_{0,1},2,\mathrm{R}} := \int_{\mathbb{R}^d}\frac{2\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathrm{R}\left\{\varphi^{\circ}\left(u_k,s\right)\right\}^c\mathrm{R}\left\{\frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\varphi(u,s)\,du\right\}\mathbf{w}(s)\,ds.
\end{aligned} \tag{C.196}$$

Then, one obtains for $T \to \infty$ *and all* $\mathrm{R} \in \{\Re, \Im\}$ *(see (C.187)):*

(i) $\sqrt{T}\left\|\widehat{\mathbb{D}}^{[1]}_{T,1,\mathrm{R}}-\widehat{\mathbb{D}}^{[2]}_{T,1,\mathrm{R}}\right\|_1 = o(1).$

(ii) $\sqrt{T}\left\|\widehat{\mathbb{D}}^{[1]}_{T,2,\mathrm{R}}-\widehat{\mathbb{D}}^{[2]}_{T,2,\mathrm{R}}\right\|_1 = o(1).$

Remark C.3. *Since (C.18) ensures that* $\tilde{u}_{k,t} \in [\mathfrak{U}_0, \mathfrak{U}_1] \subseteq [0,1]\ \forall\, k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$*,* $t \in \{1, \ldots, 2 \cdot \lfloor T_{\mathfrak{U}}b\rfloor\}$*,* $\varphi^{\circ}\left(u_k,s\right)$ *is well-defined for all* $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$*,* $s \in \mathbb{R}^d$*.*

Proof of Lemma C.2. Throughout this proof, it is assumed that T is large enough to ensure $2\lfloor T_{\mathfrak{U}}b\rfloor \geqslant 1$, which holds for sufficiently large T due to Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii) (recall (C.17)). Before verifying Lemma C.2 (i) and (ii), it will be shown for all $s \in \mathbb{R}^d$:

$$\sqrt{T}\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor}\left\|\widehat{\varphi}\left(u_k,s\right)-\varphi^{\circ}\left(u_k,s\right)\right\|_1 \leqslant \frac{C}{\sqrt{Tb}}+\frac{C}{\sqrt{T}}|s|_1. \tag{C.197}$$

It holds for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$ (see Definition 3.8 (i) as well as (C.17)):

$$\begin{aligned}
\lfloor u_kT-T_{\mathfrak{U}}b\rfloor+1 &\geqslant \left\lfloor \mathfrak{U}_0T+\frac{T_{\mathfrak{U}}}{2\lfloor 1/(2b)\rfloor}-\frac{T_{\mathfrak{U}}}{2\left(1/(2b)\right)}\right\rfloor+1 \geqslant 1 \quad \text{and}\\
\lceil u_kT+T_{\mathfrak{U}}b\rceil &\leqslant \left\lceil \mathfrak{U}_0T+T_{\mathfrak{U}}-\frac{T_{\mathfrak{U}}}{2\lfloor 1/(2b)\rfloor}+\frac{T_{\mathfrak{U}}}{2\left(1/(2b)\right)}\right\rceil \leqslant T.
\end{aligned} \tag{C.198}$$

Moreover, Assumption 2.8 $[\mathbf{K\&b.1}]$ (i) implies for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$ that $K\left(\left(t-u_kT\right)/Tb\right) = 0$ for $t \leqslant \lfloor u_kT-T_{\mathfrak{U}}b\rfloor$ or $t \geqslant \lceil u_kT+T_{\mathfrak{U}}b\rceil$. Thus, (C.198) shows for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^d$

(recall Definition 2.11):

$$\widehat{\varphi}\left(u_k, s\right) = \frac{1}{Tb} \sum_{t=\lfloor u_k T - T_{\mathfrak{U}} b \rfloor + 1}^{\lceil u_k T + T_{\mathfrak{U}} b \rceil} K\left(\frac{t - u_k T}{Tb}\right) e^{i\left\langle s, X_{t,T}\right\rangle}. \tag{C.199}$$

Further, one defines for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $s \in \mathbb{R}^d$:

$$\breve{\varphi}\left(u_k, s\right) := \breve{\varphi}_{T,\mathfrak{U}_{0,1}}\left(u_k, s\right) := \frac{1}{\lfloor Tb \rfloor} \sum_{t=1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} K\left(\frac{t - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) e^{i\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t, T}\right\rangle}. \tag{C.200}$$

According to (C.52), $\breve{\varphi}\left(u_k, s\right)$ just takes $X_{r,T}$ with $r \in \{1, \ldots, T\}$ into account. Therefore, $\breve{\varphi}\left(u_k, s\right)$ is well-defined for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $s \in \mathbb{R}^d$.
One obtains for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $s \in \mathbb{R}^d$ from shifting the index of a sum (see (C.200)):

$$\breve{\varphi}\left(u_k, s\right) = \frac{1}{\lfloor Tb \rfloor} \sum_{t=\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + 1}^{\lfloor u_k T \rfloor + \lfloor T_{\mathfrak{U}} b \rfloor} K\left(\frac{t - \lfloor u_k T \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) e^{i\left\langle s, X_{t,T}\right\rangle}. \tag{C.201}$$

Overall, (C.199), (C.201), $|1/(Tb) - 1/\lfloor Tb \rfloor| \leqslant C/(Tb)^2$, Assumption 2.8 $[\mathbf{K\&b.1}]$ (i), $|1/(T_{\mathfrak{U}} b) - 1/\lfloor T_{\mathfrak{U}} b \rfloor| \leqslant C/(Tb)^2$ (recall (C.17)) and $|t - u_k T| \leqslant CTb \; \forall\, t \in \{\lfloor u_k T - T_{\mathfrak{U}} b \rfloor + 1, \ldots, \lceil u_k T + T_{\mathfrak{U}} b \rceil\}$ provide for all $s \in \mathbb{R}^d$:

$$\begin{aligned}
&\sqrt{T} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \left\| \widehat{\varphi}\left(u_k, s\right) - \breve{\varphi}\left(u_k, s\right) \right\|_1 \\
&= \sqrt{T} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \left\| \left(\frac{1}{Tb} - \frac{1}{\lfloor Tb \rfloor}\right) \sum_{t=\lfloor u_k T - T_{\mathfrak{U}} b \rfloor + 1}^{\lceil u_k T + T_{\mathfrak{U}} b \rceil} K\left(\frac{t - u_k T}{Tb}\right) e^{i\left\langle s, X_{t,T}\right\rangle} + \frac{1}{\lfloor Tb \rfloor} \sum_{t=\lfloor u_k T - T_{\mathfrak{U}} b \rfloor + 1}^{\lceil u_k T + T_{\mathfrak{U}} b \rceil} \right. \\
&\left(K\left(\frac{t - u_k T}{T_{\mathfrak{U}} b}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) - K\left(\frac{t - u_k T}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) + K\left(\frac{t - u_k T}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right)\right. \\
&\left.-K\left(\frac{t - \lfloor u_k T \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right)\right) e^{i\left\langle s, X_{t,T}\right\rangle} + \frac{1}{\lfloor Tb \rfloor} \sum_{t=\lfloor u_k T - T_{\mathfrak{U}} b \rfloor + 1}^{\lceil u_k T + T_{\mathfrak{U}} b \rceil} K\left(\frac{t - \lfloor u_k T \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) e^{i\left\langle s, X_{t,T}\right\rangle} \\
&\left. -\frac{1}{\lfloor Tb \rfloor} \sum_{t=\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + 1}^{\lfloor u_k T \rfloor + \lfloor T_{\mathfrak{U}} b \rfloor} K\left(\frac{t - \lfloor u_k T \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) e^{i\left\langle s, X_{t,T}\right\rangle} \right\|_1 \\
&\leqslant \frac{C}{\sqrt{Tb}}.
\end{aligned} \tag{C.202}$$

Moreover, (C.65) with $q = 1 + \delta$ shows for all $s \in \mathbb{R}^d$ (see (C.200) and (C.196)):

$$\sqrt{T} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \left\| \breve{\varphi}\left(u_k, s\right) - \varphi^\circ\left(u_k, s\right) \right\|_1 \leqslant \frac{C}{\sqrt{T}} |s|_1. \tag{C.203}$$

In conclusion, (C.197) follows from (C.202) and (C.203). Lemma C.2 (i) as well as (ii) are implications of (C.197) and the Assumptions 3.1 $[\mathbf{WEI.1}]$ as well as 2.8 $[\mathbf{K\&b.1}]$ (ii) (note (C.187) and (C.196)). □

Lemma C.4. *Let the Assumptions 2.4* $[\mathbf{DM.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *(ii) be fulfilled. Then, it holds for all* $s \in \mathbb{R}^d$, $q \geqslant 1 + \delta$ *and expressions* $o\left(1/(Tb^2)\right)$ *which do not depend on* $s \in \mathbb{R}^d$ *(recall Definition A.1 (i) as well as (ii) and that* δ *originates from Assumption 2.2* $[\mathbf{StAp}]$*):*

(i)

$$\begin{aligned}
\sup_{t=1,\ldots,T} \left\| \left(e^{i\left\langle s, X_{t,T}\right\rangle}\right)_{\mathscr{m}} - e^{i\left\langle s, X_{t,T}\right\rangle} \right\|_q &\leqslant 2 \sup_{t=1,\ldots,T} \left\| e^{i\left\langle s, X_{t,T,\{\mathscr{m}\}}\right\rangle} - e^{i\left\langle s, X_{t,T}\right\rangle} \right\|_q \quad \forall\, T \in \mathbb{N} \\
&= o\left(\frac{1}{Tb^2}\right)^{\frac{1+\delta}{q}} |s|_1^{\frac{1+\delta}{q}} \;\; \textit{for } T \to \infty.
\end{aligned}$$

(ii)

$$\sup_{u\in[0,1]}\sup_{t\in\mathbb{Z}}\left\|\left(e^{i\left\langle s,\tilde{X}_t(u)\right\rangle}\right)_{\mathscr{m}}-e^{i\left\langle s,\tilde{X}_t(u)\right\rangle}\right\|_q \leqslant 2\sup_{u\in[0,1]}\sup_{t\in\mathbb{Z}}\left\|e^{i\left\langle s,\tilde{X}_{t,\{\mathscr{m}\}}(u)\right\rangle}-e^{i\left\langle s,\tilde{X}_t(u)\right\rangle}\right\|_q \quad \forall\, T\in\mathbb{N}$$

$$= o\left(\frac{1}{Tb^2}\right)^{\frac{1+\delta}{q}}|s|_1^{\frac{1+\delta}{q}} \text{ for } T\to\infty.$$

Proof. (i) First, one observes that $e^{i\left\langle s,X_{t,T,\{\mathscr{m}\}}\right\rangle}$ (with $s\in\mathbb{R}^d$, $t\in\{1,\ldots,T\}$) is measurable with respect to the sigma algebra generated by $\mathcal{F}_{t,t-\mathscr{m}}$ (see Definition A.1 (i)). Thus, it holds for all $s\in\mathbb{R}^d$, $q\geqslant 1+\delta$:

$$\begin{aligned}
&\sup_{t=1,\ldots,T}\left\|\left(e^{i\left\langle s,X_{t,T}\right\rangle}\right)_{\mathscr{m}}-e^{i\left\langle s,X_{t,T}\right\rangle}\right\|_q\\
&\leqslant \sup_{t=1,\ldots,T}\left\|\mathbb{E}\left[e^{i\left\langle s,X_{t,T}\right\rangle}-e^{i\left\langle s,X_{t,T,\{\mathscr{m}\}}\right\rangle}\middle|\mathcal{F}_{t,t-\mathscr{m}}\right]\right\|_q+\sup_{t=1,\ldots,T}\left\|e^{i\left\langle s,X_{t,T,\{\mathscr{m}\}}\right\rangle}-e^{i\left\langle s,X_{t,T}\right\rangle}\right\|_q\\
&\leqslant 2\sup_{t=1,\ldots,T}\left\|e^{i\left\langle s,X_{t,T,\{\mathscr{m}\}}\right\rangle}-e^{i\left\langle s,X_{t,T}\right\rangle}\right\|_q. \qquad\qquad (C.204)
\end{aligned}$$

Moreover, (3.14), Lemma B.4 (i), shifting the index of a sum and $l^2/\mathscr{m}^2\geqslant 1\;\forall\, l\geqslant \mathscr{m}+1$ imply for all $s\in\mathbb{R}^d$, $q\geqslant 1+\delta$ (recall Definition A.1 (i) as well as (ii)):

$$\begin{aligned}
&\sup_{t=1,\ldots,T}\left\|\Re\left\{e^{i\left\langle s,X_{t,T,\{\mathscr{m}\}}\right\rangle}-e^{i\left\langle s,X_{t,T}\right\rangle}\right\}\right\|_q\\
&\leqslant C\sup_{t=1,\ldots,T}\left(\mathbb{E}\left[\left|\left\langle s,X_{t,T,\{\mathscr{m}\}}-X_{t,T}\right\rangle\right|^{1+\delta}\right]\right)^{\frac{1+\delta}{q(1+\delta)}}\\
&\leqslant C|s|_1^{\frac{1+\delta}{q}}\left(\sum_{l=\mathscr{m}}^{\infty}\sup_{t=1,\ldots,T}\left\|\mathbb{E}\left[X_{t,T}\middle|\mathcal{F}_{t,t-l}\right]-\mathbb{E}\left[X_{t,T}\middle|\mathcal{F}_{t,t-l-1}\right]\right\|_{1+\delta}\right)^{\frac{1+\delta}{q}}\\
&\leqslant C|s|_1^{\frac{1+\delta}{q}}\left(\sum_{l=\mathscr{m}+1}^{\infty}\Delta_l\frac{l^2}{\mathscr{m}^2}\right)^{\frac{1+\delta}{q}}\\
&\leqslant C|s|_1^{\frac{1+\delta}{q}}\left(\sum_{l=\widetilde{\mathscr{m}}}^{\infty}\Delta_l l^2\left[\sqrt{T}b\left(\sum_{l=\widetilde{\mathscr{m}}}^{\infty}\Delta_l l^2\right)^{1/4}\right]^{-2}\right)^{\frac{1+\delta}{q}}\\
&\leqslant C|s|_1^{\frac{1+\delta}{q}}\left(\left(\frac{1}{Tb^2}\right)\left(\sum_{l=\widetilde{\mathscr{m}}}^{\infty}\Delta_l l^2\right)^{1-1/2}\right)^{\frac{1+\delta}{q}}. \qquad\qquad (C.205)
\end{aligned}$$

Further, $\widetilde{\mathscr{m}}\to\infty$ for $T\to\infty$ holds due to Definition A.1 (ii) and Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii). Thus, Assumption 2.4 $[\mathbf{DM.1}]$ provides $\sum_{l=\widetilde{\mathscr{m}}}^{\infty}\Delta_l l^2\to 0$ for $T\to\infty$. Hence, Lemma C.4 (i) follows from (C.204), (C.205) and similar arguments.

(ii) Lemma C.4 (ii) can be proved similarly to Lemma C.4 (i).

□

Lemma C.5. *Suppose that the Assumptions 2.4* $[\mathbf{DM.1}]$*, 3.1* $[\mathbf{WEI.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *hold. Then, one obtains for* $T\to\infty$ *and all* $\mathrm{R}\in\{\Re,\Im\}$ *(see (C.196) as well as (C.17)):*

(i) $$\left\|\sqrt{T}\,\widehat{\mathbb{D}}^{[2]}_{T,1,\mathrm{R}}-\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{D}^{\circ}_{T,k,1,\mathrm{R}}\right\|_1=o(1).$$

(ii) $$\left\|\sqrt{T}\,\widehat{\mathbb{D}}^{[2]}_{T,2,\mathrm{R}}-\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{D}^{\circ}_{T,k,2,\mathrm{R}}\right\|_1=o(1).$$

Proof. (i) In the following, Lemma C.5 (i) with $\mathrm{R}=\Re$ will be shown. It is supposed throughout this proof that T is large enough to ensure $2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}\geqslant 1+\mathscr{m}$ (recall Definition A.1 (ii) and (C.17)), which holds for sufficiently large T due to Remark A.2 (i). One defines for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $s \in \mathbb{R}^d$ (see Definition 3.8 (i)):

$$\overset{\times}{\varphi}(u_k, s) := \overset{\times}{\varphi}_{T,\mathfrak{U}_{0,1}}(u_k, s) := \frac{1}{\lfloor Tb \rfloor} \sum_{t=1+m}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - m} K\left(\frac{t - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right) e^{i\left\langle s, \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t}(\tilde{u}_{k,t})\right\rangle}, \tag{C.206}$$

whereby $\tilde{u}_{k,t} \in [\mathfrak{U}_0, \mathfrak{U}_1] \subseteq [0,1]$ (which holds according to (C.18)) ensures that $\overset{\times}{\varphi}(u_k, s)$ is well-defined for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $s \in \mathbb{R}^d$.
It holds for all $s \in \mathbb{R}^d$ due to Remark A.2 (i) (recall (C.196) and (C.206)):

$$\left\| \frac{2\sqrt{T}(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\varphi^\circ(u_k, s) - \overset{\times}{\varphi}(u_k, s)\right\}^c \cdot \Re\left\{\varphi(u_k, s)\right\} \right\|_1 \leqslant C \frac{m}{\sqrt{Tb}} = o(1), \tag{C.207}$$

whereby the expression $o(1)$ does not depend on $s \in \mathbb{R}^d$. Further, one observes for all $s \in \mathbb{R}^d$ (see (C.206) and (C.17)):

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{2\sqrt{T}(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\overset{\times}{\varphi}(u_k, s) - \varphi^\circ_{m}(u_k, s)\right\}^c \cdot \Re\left\{\varphi(u_k, s)\right\}\right)^2\right] \\
&\leqslant \frac{C}{T} \sum_{k_1,k_2=1}^{\lfloor 1/(2b) \rfloor} \sum_{t_1,t_2=1+m}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - m} \left| \mathrm{Cov}\left(\Re\left\{e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\tilde{u}_{k_1,t_1})\right\rangle} - \left(e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\tilde{u}_{k_1,t_1})\right\rangle}\right)_{m}\right\},\right.\right. \\
&\left.\left.\Re\left\{e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\tilde{u}_{k_2,t_2})\right\rangle} - \left(e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\tilde{u}_{k_2,t_2})\right\rangle}\right)_{m}\right\}\right)\right|.
\end{aligned} \tag{C.208}$$

It follows for all $s \in \mathbb{R}^d$ from (C.22), Lemma B.4 (iv) with $q = 1 + \delta$, Lemma C.4 (ii) with $q = (1+\delta)/\delta$ and shifting the index of a sum (see Definition A.1 (i) as well as (C.17)):

$$\begin{aligned}
&\sum_{\substack{k_1,k_2=1 \\ k_1 \geqslant k_2+1}}^{\lfloor 1/(2b) \rfloor} \sum_{t_1,t_2=1+m}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - m} \left| \mathrm{Cov}\left(\Re\left\{e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\tilde{u}_{k_1,t_1})\right\rangle} - \left(e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\tilde{u}_{k_1,t_1})\right\rangle}\right)_{m}\right\},\right.\right. \\
&\left.\left.\Re\left\{e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\tilde{u}_{k_2,t_2})\right\rangle} - \left(e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\tilde{u}_{k_2,t_2})\right\rangle}\right)_{m}\right\}\right)\right| \\
&\leqslant \sum_{\substack{k_1,k_2=1 \\ k_1 \geqslant k_2+1}}^{\lfloor 1/(2b) \rfloor} \sum_{t_1,t_2=1+m}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - m} \left| \mathbb{E}\left[\left(\mathbb{E}\left[\Re\left\{e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\tilde{u}_{k_1,t_1})\right\rangle} - \left(e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\tilde{u}_{k_1,t_1})\right\rangle}\right)_{m}\right\}\right|\right.\right.\right. \\
&\left.\mathcal{F}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}\right] - \mathbb{E}\left[\Re\left\{e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\tilde{u}_{k_1,t_1})\right\rangle} - \left(e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\tilde{u}_{k_1,t_1})\right\rangle}\right)_{m}\right\}\right| \\
&\left.\left.\mathcal{F}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, \lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2 + 1}\right]\right) \\
&\left.\left.\cdot \Re\left\{e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\tilde{u}_{k_2,t_2})\right\rangle} - \left(e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\tilde{u}_{k_2,t_2})\right\rangle}\right)_{m}\right\}\right]\right| \\
&\leqslant \sum_{\substack{k_1,k_2=1 \\ k_1 \geqslant k_2+1}}^{\lfloor 1/(2b) \rfloor} \sum_{t_1,t_2=1+m}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - m} \sum_{l = \lfloor u_{k_1} T \rfloor - \lfloor u_{k_2} T \rfloor + t_1 - t_2 - 1}^{\infty} \\
&\left(\left\| \mathbb{E}\left[\Re\left\{e^{i\left\langle s, \tilde{X}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\tilde{u}_{k_1,t_1})\right\rangle}\right\} \middle| \mathcal{F}_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, \lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1 - l}\right]\right.\right.
\end{aligned}$$

$$-\mathbb{E}\left[\Re\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k_1,t_1}\right)\right\rangle}\right\}\middle|\mathcal{F}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1-l-1}\right]\Bigg\|_{1+\delta}$$

$$+\Bigg\|\mathbb{E}\Bigg[-\mathbb{E}\left[\Re\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k_1,t_1}\right)\right\rangle}\right\}\middle|\mathcal{F}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1-l}\right]$$

$$+\mathbb{E}\left[\Re\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k_1,t_1}\right)\right\rangle}\right\}\middle|\mathcal{F}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1-l-1}\right]\Bigg|$$

$$\mathcal{F}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1-\mathscr{m}}\Bigg]\Bigg\|_{1+\delta}\Bigg)\sup_{u\in[0,1]}\sup_{t\in\mathbb{Z}}\left\|e^{i\left\langle s,\tilde{X}_t(u)\right\rangle}-\left(e^{i\left\langle s,\tilde{X}_t(u)\right\rangle}\right)_{\mathscr{m}}\right\|_{\frac{1+\delta}{\delta}}$$

$$\leqslant C\sum_{\substack{k_1,k_2=1\\k_1\geqslant k_2+1}}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\sum_{l=\lfloor u_{k_1}T\rfloor-\lfloor u_{k_2}T\rfloor+t_1-t_2}^{\infty}\Delta_l|s|_1\left(\frac{1}{Tb^2}\right)^{\delta}|s|_1^{\delta}. \tag{C.209}$$

One obtains for all $k_1,k_2\in\{1,\dots,\lfloor 1/(2b)\rfloor\}$, $t_1,t_2\in\{1+\mathscr{m},\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}\}$ with $k_1\geqslant k_2+1$ from $\mathscr{m}\in\mathbb{N}$ and $1/\lfloor 1/(2b)\rfloor\geqslant 2b$ (recall Definition 3.8 (i), (C.17) as well as Definition A.1 (ii)):

$$\begin{aligned}\lfloor u_{k_1}T\rfloor-\lfloor u_{k_2}T\rfloor+t_1-t_2&\geqslant u_{k_1}T-1-u_{k_2}T+t_1-2T_{\mathfrak{U}}b+1+\mathscr{m}\\&\geqslant\lceil u_{k_1}T-u_{k_2}T-2T_{\mathfrak{U}}b\rceil+t_1\\&=\left\lceil(k_1-k_2)\frac{T_{\mathfrak{U}}}{\lfloor 1/(2b)\rfloor}-2T_{\mathfrak{U}}b\right\rceil+t_1\\&\geqslant\lceil(k_1-k_2-1)\,2T_{\mathfrak{U}}b\rceil+t_1.\end{aligned} \tag{C.210}$$

It follows for all $s\in\mathbb{R}^d$ from (C.209), (C.210), $l^2/\left((k_1-k_2-1)\,2T_{\mathfrak{U}}b+t_1\right)^2\geqslant 1\;\forall\, l\geqslant\lceil(k_1-k_2-1)\cdot 2T_{\mathfrak{U}}b\rceil+t_1$, shifting the indices of sums and the Assumptions 2.4 $[\mathbf{DM.1}]$ as well as 2.8 $[\mathbf{K\&b.1}]$ (ii) (see (C.17)):

$$\sum_{\substack{k_1,k_2=1\\k_1\geqslant k_2+1}}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\left|\operatorname{Cov}\left(\Re\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k_1,t_1}\right)\right\rangle}-\left(e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k_1,t_1}\right)\right\rangle}\right)_{\mathscr{m}}\right\},\right.\right.$$

$$\left.\left.\Re\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\tilde{u}_{k_2,t_2}\right)\right\rangle}-\left(e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\tilde{u}_{k_2,t_2}\right)\right\rangle}\right)_{\mathscr{m}}\right\}\right)\right|$$

$$\leqslant C\left(\frac{1}{Tb^2}\right)^{\delta}|s|_1^{1+\delta}\sum_{\substack{k_1,k_2=1\\k_1\geqslant k_2+1}}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\sum_{l=\lceil(k_1-k_2-1)2T_{\mathfrak{U}}b\rceil+t_1}^{\infty}\Delta_l\cdot\frac{l^2}{\left((k_1-k_2-1)\,2T_{\mathfrak{U}}b+t_1\right)^2}$$

$$\leqslant C\left(\frac{1}{Tb^2}\right)^{\delta}|s|_1^{1+\delta}\left(\sum_{\substack{k_1,k_2=1\\k_1=k_2+1}}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\frac{1}{t_1^2}\sum_{l=1}^{\infty}\Delta_l l^2+\sum_{\substack{k_1,k_2=1\\k_1\geqslant k_2+2}}^{\lfloor 1/(2b)\rfloor}\frac{1}{(k_1-k_2-1)^2\,(2T_{\mathfrak{U}}b)^2}\sum_{t_1,t_2=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\sum_{l=1}^{\infty}\Delta_l l^2\right)$$

$$\leqslant C\left(\frac{1}{Tb^2}\right)^{\delta}|s|_1^{1+\delta}\left(\lfloor 1/(2b)\rfloor\sum_{t_1=1}^{\infty}\frac{1}{t_1^2}2\lfloor T_{\mathfrak{U}}b\rfloor\sum_{l=1}^{\infty}\Delta_l l^2+\sum_{k_2=1}^{\lfloor 1/(2b)\rfloor}\sum_{k_1=1}^{\infty}\frac{1}{k_1^2}\frac{\left(2\lfloor T_{\mathfrak{U}}b\rfloor\right)^2}{(2T_{\mathfrak{U}}b)^2}\sum_{l=1}^{\infty}\Delta_l l^2\right)$$

$$=o(T)|s|_1^{1+\delta}, \tag{C.211}$$

whereby the expression $o(T)$ does not depend on $s\in\mathbb{R}^d$. Moreover, one obtains for all $s\in\mathbb{R}^d$ similarly to (C.209) by using (B.45) and Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii) (recall (C.17)):

$$\sum_{\substack{k_1,k_2=1\\k_1=k_2}}^{\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,t_2=1+\mathscr{m}\\t_1>t_2}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\left|\operatorname{Cov}\left(\Re\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k_1,t_1}\right)\right\rangle}-\left(e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k_1,t_1}\right)\right\rangle}\right)_{\mathscr{m}}\right\},\right.\right.$$

$$\Re\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\tilde{u}_{k_2,t_2}\right)\right\rangle}-\left(e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\tilde{u}_{k_2,t_2}\right)\right\rangle}\right)_{\mathscr{m}}\right\}\Bigg)\Bigg|$$

$$\leqslant C\sum_{\substack{k_1,k_2=1\\k_1=k_2}}^{\lfloor 1/(2b)\rfloor}\sum_{t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}-1}\sum_{t_1=t_2+1}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\sum_{l=\lfloor u_{k_1}T\rfloor-\lfloor u_{k_2}T\rfloor+t_1-t_2}^{\infty}\Delta_l|s|_1\left(\frac{1}{Tb^2}\right)^{\delta}|s|_1^{\delta}$$

$$=o(T)|s|_1^{1+\delta},\tag{C.212}$$

whereby the expression $o(T)$ does not depend on $s\in\mathbb{R}^d$. Further, observe for all real-valued random variables X and Y which live on the same probability space and fulfil $\max(|X|,|Y|)\leqslant C$ a. s. that $\mathrm{Var}(X-Y)\leqslant\mathbb{E}[|X-Y|(|X|+|Y|)]+\mathbb{E}[|X-Y|]\,\mathbb{E}[|X|+|Y|]\leqslant C\|X-Y\|_{1+\delta}$. This provides for all $s\in\mathbb{R}^d$ by using Lemma C.4 (ii) with $q=1+\delta$ and Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii) (see (C.17)):

$$\sum_{\substack{k_1,k_2=1\\k_1=k_2}}^{\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,t_2=1+\mathscr{m}\\t_1=t_2}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\Bigg|\mathrm{Cov}\Bigg(\Re\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k_1,t_1}\right)\right\rangle}-\left(e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k_1,t_1}\right)\right\rangle}\right)_{\mathscr{m}}\right\},$$

$$\Re\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\tilde{u}_{k_2,t_2}\right)\right\rangle}-\left(e^{i\left\langle s,\tilde{X}_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\tilde{u}_{k_2,t_2}\right)\right\rangle}\right)_{\mathscr{m}}\right\}\Bigg)\Bigg|$$

$$\leqslant C\lfloor 1/(2b)\rfloor\lfloor T_{\mathfrak{U}}b\rfloor\sup_{u\in[0,1]}\sup_{t\in\mathbb{Z}}\left\|e^{i\langle s,\tilde{X}_t(u)\rangle}-\left(e^{i\langle s,\tilde{X}_t(u)\rangle}\right)_{\mathscr{m}}\right\|_{1+\delta}$$

$$=o(T)|s|_1,\tag{C.213}$$

whereby the expression $o(T)$ does not depend on $s\in\mathbb{R}^d$. Overall, it follows for all $s\in\mathbb{R}^d$ from (C.208), (C.211), (C.212) as well as similar arguments and (C.213):

$$\left\|\frac{2\sqrt{T}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\Re\left\{\overset{\times}{\varphi}\left(u_k,s\right)-\varphi^{\circ}_{\mathscr{m}}\left(u_k,s\right)\right\}^c\cdot\Re\left\{\varphi\left(u_k,s\right)\right\}\right\|_2=o(1)\left(|s|_1^{\frac{1+\delta}{2}}+|s|_1^{\frac{1}{2}}\right),\tag{C.214}$$

whereby the expression $o(1)$ does not depend on $s\in\mathbb{R}^d$. Lemma C.5 (i) with $\mathrm{R}=\Re$ is an implication of (C.207), (C.214) and Assumption 3.1 $[\mathbf{WEI}.\mathbf{1}]$ (note (C.196), (C.17) as well as (3.16)). Similar arguments show Lemma C.5 (i) with $\mathrm{R}=\Im$.

(ii) Lemma C.5 (ii) can be proved similarly to Lemma C.5 (i).

□

Lemma C.6. *Let the Assumptions 2.4* $[\mathbf{DM}.\mathbf{1}]$*, 3.1* $[\mathbf{WEI}.\mathbf{1}]$ *and 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *be fulfilled. Further, suppose that* $(\mathscr{G}_T)_{T\in\mathbb{N}}$ *is a sequence of deterministic functions with:*

$$\mathscr{G}_T\colon\mathbb{Z}\to\mathbb{R},\quad\mathscr{G}_T(h)=\mathscr{G}_T(-h)\quad\forall\,T\in\mathbb{N},h\in\mathbb{Z}\quad\text{as well as}\quad\sup_{T\in\mathbb{N}}\sup_{h\in\mathbb{Z}}|\mathscr{G}_T(h)|\leqslant C.\tag{C.215}$$

Moreover, define for all $\mathrm{R}_1,\mathrm{R}_2\in\{\Re,\Im\}$, $\gamma,\tilde{\gamma}\in\mathbb{R}^{1\times 2}$, $s_1,s_2\in\mathbb{R}^d$ *(recall (3.16), Definition 3.8 (i) as well as (C.17)):*

$$\mathrm{Cov}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}_{\mathrm{R}_1,\mathrm{R}_2,T}\left(s_1,s_2\right):=\frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\tau_{\mathfrak{U}_{0,1},\mathrm{R}_1}\left(\gamma,u_k,s_1\right)\tau_{\mathfrak{U}_{0,1},\mathrm{R}_2}\left(\tilde{\gamma},u_k,s_2\right)\sum_{t_1,t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\mathscr{G}_T\left(t_2-t_1\right)$$

$$\cdot K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)$$

$$\cdot\,\mathrm{Cov}\left(\mathrm{R}_1\left\{e^{i\left\langle s_1,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k,t_1}\right)\right\rangle}\right\}_{\mathscr{m}},\mathrm{R}_2\left\{e^{i\left\langle s_2,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\tilde{u}_{k,t_2}\right)\right\rangle}\right\}_{\mathscr{m}}\right)\mathbb{1}_{\{|t_2-t_1|\leqslant\mathscr{m}\}},$$

$$\mathrm{Cov}^{[\mathscr{G}_T]}_{\mathrm{R}_1,\mathrm{R}_2,T}(\gamma,\widetilde{\gamma}) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\mathrm{R}_1,\mathrm{R}_2,T}(s_1,s_2)\ \mathbf{w}(s_1)\ ds_1\,\mathbf{w}(s_2)\ ds_2,$$

$$\mathrm{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\mathrm{R}_1,\mathrm{R}_2}(s_1,s_2) := 8\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \tau_{\mathfrak{U}_{0,1},\mathrm{R}_1}(\gamma,u,s_1)\,\tau_{\mathfrak{U}_{0,1},\mathrm{R}_2}(\widetilde{\gamma},u,s_2)$$
$$\cdot \sum_{t=-\infty}^{\infty} \mathscr{G}_T(t)\,\mathrm{Cov}\left(\mathrm{R}_1\left\{e^{i\left\langle s_1,\tilde{X}_0(u)\right\rangle}\right\},\mathrm{R}_2\left\{e^{i\left\langle s_2,\tilde{X}_t(u)\right\rangle}\right\}\right)\,du \quad \textit{and}$$

$$\mathrm{Cov}^{[\mathscr{G}_T]}_{\mathrm{R}_1,\mathrm{R}_2}(\gamma,\widetilde{\gamma}) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\mathrm{R}_1,\mathrm{R}_2}(s_1,s_2)\ \mathbf{w}(s_1)\ ds_1\,\mathbf{w}(s_2)\ ds_2. \tag{C.216}$$

Then, it holds for all $\mathrm{R}_1,\mathrm{R}_2 \in \{\Re,\Im\}$*, all arbitrary but fixed* $\gamma,\widetilde{\gamma}\in\mathbb{R}^{1\times 2}$ *and for* $T\to\infty$*:*

$$\left|\mathrm{Cov}^{[\mathscr{G}_T]}_{\mathrm{R}_1,\mathrm{R}_2,T}(\gamma,\widetilde{\gamma}) - \mathrm{Cov}^{[\mathscr{G}_T]}_{\mathrm{R}_1,\mathrm{R}_2}(\gamma,\widetilde{\gamma})\right| = o(1).$$

Proof. Throughout this proof, it is assumed that $\gamma,\widetilde{\gamma}\in\mathbb{R}^{1\times 2}$ are arbitrary but fixed and T is large enough to ensure that $2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{m} \geqslant 1 + \mathscr{m}$ (see (C.17) as well as Definition A.1 (ii)), whereby the latter holds for sufficiently large T due to Remark A.2 (i). In the following, Lemma C.6 with $\mathrm{R}_1 = \Re$ and $\mathrm{R}_2 = \Im$ will be proved. Therefor, one defines for all $s_1,s_2\in\mathbb{R}^d$ (note (C.215), (3.16), (C.17) as well as Definition 3.8 (i) and that the following expression results from $\mathrm{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\Re,\Im,T}(s_1,s_2)$ (defined in (C.216)) by replacing t_2 contained in $K\left((t_2-\lfloor T_{\mathfrak{U}}b\rfloor)/\lfloor T_{\mathfrak{U}}b\rfloor(\mathfrak{U}_1-\mathfrak{U}_0)\right)$ by t_1 and each $\widetilde{u}_{k,t}$ (with $t\in\{t_1,t_2\}$) by u_k):

$$\mathrm{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\Re,\Im,T,1}(s_1,s_2) := \frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\tau_{\mathfrak{U}_{0,1},\Re}(\gamma,u_k,s_1)\,\tau_{\mathfrak{U}_{0,1},\Im}(\widetilde{\gamma},u_k,s_2)\sum_{t_1,t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\mathscr{G}_T(t_2-t_1)$$
$$\cdot K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2\mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}(u_k)\right\rangle\right)_{\mathscr{m}},\right.$$
$$\left.\sin\left(\left\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}(u_k)\right\rangle\right)_{\mathscr{m}}\right)\cdot\mathbb{1}_{\{|t_2-t_1|\leqslant\mathscr{m}\}}. \tag{C.217}$$

It holds for all real-valued random variables X_1,X_2,Y_1,Y_2 that live on the same probability space and fulfil $|Z|\leqslant C$ a. s. $\forall\,Z\in\{X_1,X_2,Y_1,Y_2\}$:

$$\begin{aligned}\left|\mathrm{Cov}(X_1,X_2)-\mathrm{Cov}(Y_1,Y_2)\right| &= \left|\mathrm{Cov}(X_1-Y_1,X_2)+\mathrm{Cov}(Y_1,X_2-Y_2)\right|\\ &\leqslant C\left\|X_1-Y_1\right\|_{1+\delta}+C\left\|X_2-Y_2\right\|_{1+\delta}.\end{aligned} \tag{C.218}$$

In conclusion, (C.215), (C.218), Remark 2.3, Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) and Remark A.2 (i) imply for all $s_1,s_2\in\mathbb{R}^d$ (recall (C.216), (C.217), (3.16) as well as (C.17)):

$$\left|\mathrm{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\Re,\Im,T}(s_1,s_2)-\mathrm{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\Re,\Im,T,1}(s_1,s_2)\right|$$
$$\leqslant \frac{C}{T}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left|\tau_{\mathfrak{U}_{0,1},\Re}(\gamma,u_k,s_1)\,\tau_{\mathfrak{U}_{0,1},\Im}(\widetilde{\gamma},u_k,s_2)\right|\sum_{\substack{t_1,t_2=1+\mathscr{m}\\|t_2-t_1|\leqslant\mathscr{m}}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\left|\mathscr{G}_T(t_2-t_1)\right|$$
$$\cdot K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)$$
$$\cdot\left|\mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}(\widetilde{u}_{k,t_1})\right\rangle\right)_{\mathscr{m}},\sin\left(\left\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}(\widetilde{u}_{k,t_2})\right\rangle\right)_{\mathscr{m}}\right)\right.$$
$$\left.-\mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}(u_k)\right\rangle\right)_{\mathscr{m}},\sin\left(\left\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}(u_k)\right\rangle\right)_{\mathscr{m}}\right)\right|$$
$$+\frac{C}{T}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left|\tau_{\mathfrak{U}_{0,1},\Re}(\gamma,u_k,s_1)\,\tau_{\mathfrak{U}_{0,1},\Im}(\widetilde{\gamma},u_k,s_2)\right|\sum_{\substack{t_1,t_2=1+\mathscr{m}\\|t_2-t_1|\leqslant\mathscr{m}}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\left|\mathscr{G}_T(t_1-t_2)\right|$$

$$
\begin{aligned}
&\cdot K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\left|K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)-K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\right|\\
&\cdot\left|\operatorname{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(u_k\right)\right\rangle\right)_{\mathscr{m}},\sin\left(\left\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(u_k\right)\right\rangle\right)_{\mathscr{m}}\right)\right|\\
&\leqslant\frac{C}{T}\lfloor 1/(2b)\rfloor\lfloor T_{\mathfrak{U}}b\rfloor\,\mathscr{m}\sup_{t_1,t_2=1,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor}\left(|s_1|_1\left|\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{T}\right|+|s_2|_1\left|\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{T}\right|\right)\\
&+\frac{C}{T}\lfloor 1/(2b)\rfloor\lfloor T_{\mathfrak{U}}b\rfloor\,\mathscr{m}\sup_{t_1,t_2=1,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor:|t_2-t_1|\leqslant\mathscr{m}}\left|\frac{t_2-t_1}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right|\\
&\leqslant C\left(\sqrt{T}b^2+b\right)\left(|s_1|_1+|s_2|_1+1\right),
\end{aligned}
\tag{C.219}
$$

such that the Assumptions 3.1 $[\mathbf{WEI.1}]$ and 2.8 $[\mathbf{K\&b.1}]$ (ii) together with $\delta\leqslant 1$ provide (see (C.216)):

$$
\left|\operatorname{Cov}^{[\mathscr{G}_T]}_{\Re,\Im,T}\left(\gamma,\widetilde{\gamma}\right)-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\operatorname{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\Re,\Im,T,1}\left(s_1,s_2\right)\,\mathbf{w}\left(s_1\right)\,ds_1\,\mathbf{w}\left(s_2\right)\,ds_2\right|=o(1).
\tag{C.220}
$$

Further, one defines for all $s_1,s_2\in\mathbb{R}^d$ (note that the conditional expectations of the $\cos$- and $\sin$-terms included in (C.217) (see Definition A.1 (i)) are omitted in the next expression):

$$
\begin{aligned}
\operatorname{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\Re,\Im,T,2}\left(s_1,s_2\right):=&\frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma,u_k,s_1\right)\tau_{\mathfrak{U}_{0,1},\Im}\left(\widetilde{\gamma},u_k,s_2\right)\sum_{t_1,t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\mathscr{G}_T\left(t_2-t_1\right)\\
&\cdot K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2\operatorname{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(u_k\right)\right\rangle\right),\right.\\
&\left.\sin\left(\left\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(u_k\right)\right\rangle\right)\right)\mathbb{1}_{\{|t_2-t_1|\leqslant\mathscr{m}\}}.
\end{aligned}
\tag{C.221}
$$

It follows for all $s_1,s_2\in\mathbb{R}^d$ from (C.215), (C.218), Lemma C.4 (ii) with $q=1+\delta$ and Remark A.2 (i) (recall (C.217), (C.221), (3.16) as well as (C.17)):

$$
\begin{aligned}
&\left|\operatorname{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\Re,\Im,T,1}\left(s_1,s_2\right)-\operatorname{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\Re,\Im,T,2}\left(s_1,s_2\right)\right|\\
&\leqslant\frac{C}{T}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left|\tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma,u_k,s_1\right)\tau_{\mathfrak{U}_{0,1},\Im}\left(\widetilde{\gamma},u_k,s_2\right)\right|\sum_{\substack{t_1,t_2=1+\mathscr{m}\\|t_1-t_2|\leqslant\mathscr{m}}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}}\left|\mathscr{G}_T\left(t_2-t_1\right)\right|K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2\\
&\cdot\left|\operatorname{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(u_k\right)\right\rangle\right)_{\mathscr{m}},\sin\left(\left\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(u_k\right)\right\rangle\right)_{\mathscr{m}}\right)\right.\\
&\left.-\operatorname{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(u_k\right)\right\rangle\right),\sin\left(\left\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(u_k\right)\right\rangle\right)\right)\right|\\
&\leqslant\frac{C}{T}\lfloor 1/(2b)\rfloor\lfloor T_{\mathfrak{U}}b\rfloor\,\mathscr{m}\frac{1}{Tb^2}\left(|s_1|_1+|s_2|_1\right)\\
&\leqslant\frac{C}{\sqrt{T}b}\left(|s_1|_1+|s_2|_1\right).
\end{aligned}
\tag{C.222}
$$

Thus, the Assumptions 3.1 $[\mathbf{WEI.1}]$ and 2.8 $[\mathbf{K\&b.1}]$ (ii) imply:

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\operatorname{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\Re,\Im,T,1}\left(s_1,s_2\right)\,\mathbf{w}\left(s_1\right)\,ds_1\,\mathbf{w}\left(s_2\right)\,ds_2-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\operatorname{Cov}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}_{\Re,\Im,T,2}\left(s_1,s_2\right)\,\mathbf{w}\left(s_1\right)\,ds_1\,\mathbf{w}\left(s_2\right)\,ds_2\right|\\
&=o(1).
\end{aligned}
\tag{C.223}
$$

The function $\mathbb{F}_{s_1,s_2}\colon\mathbb{R}^d\to\mathbb{R}^2, x\mapsto\left(\cos\left(\langle s_1,x\rangle\right),\sin\left(\langle s_2,x\rangle\right)\right)'$ is continuous for all $s_1,s_2\in\mathbb{R}^d$ and, therefore, measurable. Hence, Assumption 2.2 [$\mathbf{StAp}$] (iii) and Theorem 3.35 in [78, White (2001), p. 44] provide that $\left(\mathbb{F}_{s_1,s_2}\left(\widetilde{X}_t(u)\right)\right)_{t\in\mathbb{Z}}$ is stationary for all $s_1, s_2 \in \mathbb{R}^d$, $u \in [0,1]$, such that:

$$\begin{aligned}
&\mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1}\left(u_k\right)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2}\left(u_k\right)\right\rangle\right)\right) \\
&= \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0\left(u_k\right)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_2-t_1}\left(u_k\right)\right\rangle\right)\right) \\
&\qquad \forall\, s_1, s_2 \in \mathbb{R}^d,\ k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\},\ t_1, t_2 \in \mathbb{Z}, \qquad \text{(C.224)}
\end{aligned}$$

which implies (see (C.221)):

$$\begin{aligned}
\mathrm{Cov}_{\Re,\Im,T,2}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}\left(s_1, s_2\right) &= \frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)^2}{\lfloor 1/(2b)\rfloor^2 \lfloor Tb\rfloor^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma, u_k, s_1\right) \tau_{\mathfrak{U}_{0,1},\Im}\left(\widetilde{\gamma}, u_k, s_2\right) \sum_{t_1,t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}} \mathscr{G}_T\left(t_2-t_1\right) \\
&\cdot K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2 \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0\left(u_k\right)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_2-t_1}\left(u_k\right)\right\rangle\right)\right) \mathbb{1}_{\{|t_2-t_1|\leqslant \mathscr{m}\}}.
\end{aligned} \qquad \text{(C.225)}$$

Moreover, one defines for all $s_1, s_2 \in \mathbb{R}^d$ (note that the following expression results from the right side of (C.225) by replacing the contained Riemann sum with the indices $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, which is based on the evolution points u_k, by an integral with respect to $u \in [\mathfrak{U}_0, \mathfrak{U}_1]$):

$$\begin{aligned}
\mathrm{Cov}_{\Re,\Im,T,3}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}\left(s_1, s_2\right) &:= \frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma, u, s_1\right) \tau_{\mathfrak{U}_{0,1},\Im}\left(\widetilde{\gamma}, u, s_2\right) \sum_{t_1,t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}} \mathscr{G}_T\left(t_2-t_1\right) \\
&\cdot K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2 \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0\left(u\right)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_2-t_1}\left(u\right)\right\rangle\right)\right) \mathbb{1}_{\{|t_2-t_1|\leqslant \mathscr{m}\}}\, du.
\end{aligned} \qquad \text{(C.226)}$$

It holds for all $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$:

$$x_1x_2x_3 - y_1y_2y_3 = \left(x_1-y_1\right)x_2x_3 + \left(x_2-y_2\right)y_1x_3 + y_1y_2\left(x_3-y_3\right). \qquad \text{(C.227)}$$

One obtains for all $v, w \in [0,1]$, $s_1, s_2 \in \mathbb{R}^d$ from (C.215), (C.227), Remark 2.3 and (C.218) (recall (3.16) as well as (C.17)):

$$\begin{aligned}
&\Bigg| \frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma, v, s_1\right) \tau_{\mathfrak{U}_{0,1},\Im}\left(\widetilde{\gamma}, v, s_2\right) \sum_{t_1,t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}} \mathbb{1}_{\{|t_2-t_1|\leqslant \mathscr{m}\}} \mathscr{G}_T\left(t_2-t_1\right) \\
&\cdot K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2 \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0\left(v\right)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_2-t_1}\left(v\right)\right\rangle\right)\right) \\
&- \frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma, w, s_1\right) \tau_{\mathfrak{U}_{0,1},\Im}\left(\widetilde{\gamma}, w, s_2\right) \sum_{t_1,t_2=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}} \mathbb{1}_{\{|t_2-t_1|\leqslant \mathscr{m}\}} \mathscr{G}_T\left(t_2-t_1\right) \\
&\cdot K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2 \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0\left(w\right)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_2-t_1}\left(w\right)\right\rangle\right)\right) \Bigg| \\
&\leqslant \frac{C}{Tb} \sum_{\substack{t_1,t_2=1+\mathscr{m} \\ |t_2-t_1|\leqslant \mathscr{m}}}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}} \sup_{t\in\mathbb{Z}} \Bigg| \tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma, v, s_1\right) \tau_{\mathfrak{U}_{0,1},\Im}\left(\widetilde{\gamma}, v, s_2\right) \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0\left(v\right)\right\rangle\right),\right. \\
&\left.\sin\left(\left\langle s_2, \widetilde{X}_t\left(v\right)\right\rangle\right)\right) - \tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma, w, s_1\right) \tau_{\mathfrak{U}_{0,1},\Im}\left(\widetilde{\gamma}, w, s_2\right) \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0\left(w\right)\right\rangle\right),\right. \\
&\left.\sin\left(\left\langle s_2, \widetilde{X}_t\left(w\right)\right\rangle\right)\right) \Bigg| \\
&\leqslant C\mathscr{m}\left(\left|s_1\right|_1 + \left|s_2\right|_1\right)\left|v-w\right|.
\end{aligned} \qquad \text{(C.228)}$$

Overall, (C.225), Lemma B.2 (ii) together with (C.228), Assumption 3.1 [**WEI.1**], Remark A.2 (i), Assumption 2.8 [**K**&**b.1**] (ii) and $\delta \leqslant 1$ imply (see (C.226) as well as Definition 3.8 (i)):

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{Cov}_{\Re,\Im,T,2}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}(s_1,s_2)\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{Cov}_{\Re,\Im,T,3}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}(s_1,s_2)\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 \right| \leqslant \frac{C\mathscr{m}}{\lfloor 1/(2b) \rfloor} = o(1). \tag{C.229}$$

Shifting the index of a sum provides for all $s_1, s_2 \in \mathbb{R}^d$ (recall (C.226)):

$$\mathrm{Cov}_{\Re,\Im,T,3}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}(s_1,s_2) = \frac{4T(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b) \rfloor \lfloor Tb \rfloor^2} \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \tau_{\mathfrak{U}_{0,1},\Re}(\gamma,u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Im}(\tilde{\gamma},u,s_2) \sum_{t_1=1+\mathscr{m}}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\mathscr{m}} K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor} \cdot (\mathfrak{U}_1-\mathfrak{U}_0)\right)^2 \sum_{t_2=1+\mathscr{m}-t_1}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}-t_1} \mathbb{1}_{\{|t_2|\leqslant \mathscr{m}\}} \mathscr{G}_T(t_2)\, \mathrm{Cov}\left(\cos\left(\left\langle s_1, \tilde{X}_0(u)\right\rangle\right), \sin\left(\left\langle s_2, \tilde{X}_{t_2}(u)\right\rangle\right)\right) du. \tag{C.230}$$

For the remaining steps of this proof, it is assumed that T is large enough to ensure (see (C.17) and Definition A.1 (ii)):

$$2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - 2\mathscr{m} \geqslant 1 + 2\mathscr{m}, \tag{C.231}$$

which holds for sufficiently large T due to Remark A.2 (i). One defines for all $s_1, s_2 \in \mathbb{R}^d$ (in contrast to the right side of (C.230), the following expression contains a sum with the indices $t_1 \in \{1+2\mathscr{m}, \ldots, 2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - 2\mathscr{m}\}$ instead of $t_1 \in \{1+\mathscr{m}, \ldots, 2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{m}\}$):

$$\mathrm{Cov}_{\Re,\Im,T,4}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}(s_1,s_2) := \frac{4T(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b) \rfloor \lfloor Tb \rfloor^2} \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \tau_{\mathfrak{U}_{0,1},\Re}(\gamma,u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Im}(\tilde{\gamma},u,s_2) \sum_{t_1=1+2\mathscr{m}}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-2\mathscr{m}} K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor} \cdot (\mathfrak{U}_1-\mathfrak{U}_0)\right)^2 \sum_{t_2=1+\mathscr{m}-t_1}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}-t_1} \mathbb{1}_{\{|t_2|\leqslant \mathscr{m}\}} \mathscr{G}_T(t_2)\, \mathrm{Cov}\left(\cos\left(\left\langle s_1, \tilde{X}_0(u)\right\rangle\right), \sin\left(\left\langle s_2, \tilde{X}_{t_2}(u)\right\rangle\right)\right) du, \tag{C.232}$$

whereby (C.230), (C.215), Assumptions 3.1 [**WEI.1**], Remark A.2 (i) and Assumption 2.8 [**K**&**b.1**] (ii) provide (recall (3.16)):

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{Cov}_{\Re,\Im,T,3}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}(s_1,s_2)\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{Cov}_{\Re,\Im,T,4}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}(s_1,s_2)\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 \right| \leqslant \frac{CT\mathscr{m}^2}{\lfloor 1/(2b) \rfloor \lfloor Tb \rfloor^2} = o(1). \tag{C.233}$$

Moreover, it holds $\{-\mathscr{m}, \ldots, \mathscr{m}\} \subseteq \{1+\mathscr{m}-t_1, \ldots, 2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{m} - t_1\}$ $\forall\, t_1 \in \{1+2\mathscr{m}, \ldots, 2 \cdot \lfloor T_{\mathfrak{U}}b\rfloor - 1 - 2\mathscr{m}\}$ (see (C.231)), which implies for all $s_1, s_2 \in \mathbb{R}^d$ (recall (C.232)):

$$\mathrm{Cov}_{\Re,\Im,T,4}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}(s_1,s_2) = \frac{4T(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b) \rfloor \lfloor Tb \rfloor^2} \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \tau_{\mathfrak{U}_{0,1},\Re}(\gamma,u,s_1)\, \tau_{\mathfrak{U}_{0,1},\Im}(\tilde{\gamma},u,s_2) \sum_{t_1=1+2\mathscr{m}}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-2\mathscr{m}} K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor} \cdot (\mathfrak{U}_1-\mathfrak{U}_0)\right)^2 \sum_{t_2=-\mathscr{m}}^{\mathscr{m}} \mathscr{G}_T(t_2)\, \mathrm{Cov}\left(\cos\left(\left\langle s_1, \tilde{X}_0(u)\right\rangle\right), \sin\left(\left\langle s_2, \tilde{X}_{t_2}(u)\right\rangle\right)\right) du. \tag{C.234}$$

One defines for all $s_1, s_2 \in \mathbb{R}^d$ (note that the following expression results from the right side of (C.234) by replacing $1/\lfloor Tb\rfloor \sum_{t_1=1+2\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\mathscr{m}} K\left((t_1-\lfloor T_{\mathfrak{U}}b\rfloor)/\lfloor T_{\mathfrak{U}}b\rfloor(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2$ by $\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz$):

$$\operatorname{Cov}_{\mathfrak{R},\mathfrak{I},T,5}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}(s_1,s_2) := \frac{4T(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor}\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\tau_{\mathfrak{U}_{0,1},\mathfrak{R}}(\gamma,u,s_1)\,\tau_{\mathfrak{U}_{0,1},\mathfrak{I}}(\widetilde{\gamma},u,s_2)\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0}K(z)^2\,dz$$
$$\cdot\sum_{t_2=-\mathscr{m}}^{\mathscr{m}}\mathscr{G}_T(t_2)\operatorname{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_0(u)\right\rangle\right),\sin\left(\left\langle s_2,\widetilde{X}_{t_2}(u)\right\rangle\right)\right)du. \quad \text{(C.235)}$$

It follows from $|1/\lfloor Tb\rfloor-1/(Tb)|\leqslant C/(Tb)^2$, $|1/(T_{\mathfrak{U}}b)-1/\lfloor T_{\mathfrak{U}}b\rfloor|\leqslant C/(Tb)^2$ (see (C.17)), shifting the index of a sum, Assumption 2.8 $[\mathbf{K\&b.1}]$ (i) (which ensures $K(z)=K(-z)\ \forall z\in\mathbb{R}$ as well as $\left|K(z_1)^2-K(z_2)^2\right|\leqslant|K(z_1)+K(z_2)|\,|K(z_1)-K(z_2)|\leqslant C|z_1-z_2|\ \forall z_1,z_2\in\mathbb{R}$) and Lemma B.2 (i):

$$\left|\frac{1}{\lfloor Tb\rfloor}\sum_{t_1=1+2\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\mathscr{m}}K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2-(\mathfrak{U}_1-\mathfrak{U}_0)\int_{-1}^{1}K(x(\mathfrak{U}_1-\mathfrak{U}_0))^2\,dx\right|$$
$$\leqslant\left|\left(\frac{1}{\lfloor Tb\rfloor}-\frac{1}{Tb}+\frac{\mathfrak{U}_1-\mathfrak{U}_0}{T_{\mathfrak{U}}b}-\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor T_{\mathfrak{U}}b\rfloor}\right)\sum_{t_1=1+2\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\mathscr{m}}K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2\right|$$
$$+\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor T_{\mathfrak{U}}b\rfloor}\left|\sum_{t_1=1+2\mathscr{m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\mathscr{m}}K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2-\sum_{t_1=0}^{2\lfloor T_{\mathfrak{U}}b\rfloor}K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2\right|$$
$$+(\mathfrak{U}_1-\mathfrak{U}_0)\left|\frac{1}{\lfloor T_{\mathfrak{U}}b\rfloor}\sum_{t_1=0}^{2\lfloor T_{\mathfrak{U}}b\rfloor}K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2-\frac{1}{\lfloor T_{\mathfrak{U}}b\rfloor}\sum_{t_1=-\lfloor T_{\mathfrak{U}}b\rfloor}^{\lfloor T_{\mathfrak{U}}b\rfloor}K\left(\frac{t_1}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2\right|$$
$$+(\mathfrak{U}_1-\mathfrak{U}_0)\left|\frac{1}{\lfloor T_{\mathfrak{U}}b\rfloor}\sum_{t_1=-\lfloor T_{\mathfrak{U}}b\rfloor}^{-1}K\left(\frac{t_1}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2-\int_{-1}^{0}K(x(\mathfrak{U}_1-\mathfrak{U}_0))^2\,dx\right|$$
$$+\frac{(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor T_{\mathfrak{U}}b\rfloor}K(0)+(\mathfrak{U}_1-\mathfrak{U}_0)\left|\frac{1}{\lfloor T_{\mathfrak{U}}b\rfloor}\sum_{t_1=1}^{\lfloor T_{\mathfrak{U}}b\rfloor}K\left(\frac{t_1}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2-\int_{0}^{1}K(x(\mathfrak{U}_1-\mathfrak{U}_0))^2\,dx\right|$$
$$\leqslant\frac{C}{Tb}+\frac{C\mathscr{m}}{Tb}+2(\mathfrak{U}_1-\mathfrak{U}_0)\left|\frac{1}{\lfloor T_{\mathfrak{U}}b\rfloor}\sum_{t_1=1}^{\lfloor T_{\mathfrak{U}}b\rfloor}K\left(\frac{t_1}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2-\int_{0}^{1}K(x(\mathfrak{U}_1-\mathfrak{U}_0))^2\,dx\right|$$
$$\leqslant C\frac{1+\mathscr{m}}{Tb}. \quad \text{(C.236)}$$

The substitution $z:=x(\mathfrak{U}_1-\mathfrak{U}_0)$ provides:

$$(\mathfrak{U}_1-\mathfrak{U}_0)\int_{-1}^{1}K(x(\mathfrak{U}_1-\mathfrak{U}_0))^2\,dx=\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0}K(z)^2\,dz. \quad \text{(C.237)}$$

One obtains from (C.234), (C.236), (C.237), (C.215), Lemma 3.12, Assumption 3.1 $[\mathbf{WEI.1}]$ and Remark A.2 (i) (recall (C.235) as well as (3.16)):

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\operatorname{Cov}_{\mathfrak{R},\mathfrak{I},T,4}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}(s_1,s_2)\,\mathbf{w}(s_1)\,ds_1\,\mathbf{w}(s_2)\,ds_2-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\operatorname{Cov}_{\mathfrak{R},\mathfrak{I},T,5}^{[\mathscr{G}_T,\gamma,\widetilde{\gamma}]}(s_1,s_2)\,\mathbf{w}(s_1)\,ds_1\,\mathbf{w}(s_2)\,ds_2\right|$$
$$\leqslant\frac{CT\cdot(1+\mathscr{m})}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor Tb}=o(1). \quad \text{(C.238)}$$

Moreover, it follows for all $s_1,s_2\in\mathbb{R}^d$ from (C.215), Lemma B.4 (vi), $l/\mathscr{m}\geqslant 1\ \forall l\geqslant\mathscr{m}+1$, arguments which are similar to those that prove (B.45) as well as Assumption 2.4 $[\mathbf{DM}.1]$:

$$\begin{aligned}\sum_{t_2=\mathscr{m}+1}^{\infty}\left|\mathscr{G}_T\left(t_2\right)\right| \sup_{u\in[0,1]}\left|\operatorname{Cov}\left(\cos\left(\left\langle s_1,\tilde{X}_0\left(u\right)\right\rangle\right),\sin\left(\left\langle s_2,\tilde{X}_{t_2}\left(u\right)\right\rangle\right)\right)\right| &\leqslant C\sum_{t_2=\mathscr{m}+1}^{\infty}\sum_{l=t_2}^{\infty}\Delta_l\frac{l}{\mathscr{m}}\left|s_2\right|_1\\ &\leqslant \frac{C}{\mathscr{m}}\sum_{l=1}^{\infty}\Delta_l l^2\left|s_2\right|_1\\ &\leqslant \frac{C}{\mathscr{m}}\left|s_2\right|_1 \qquad \text{(C.239)}\end{aligned}$$

and analog arguments show for all $s_1, s_2 \in \mathbb{R}^d$:

$$\sum_{t_2=-\infty}^{-\mathscr{m}-1}\left|\mathscr{G}_T\left(t_2\right)\right| \sup_{u\in[0,1]}\left|\operatorname{Cov}\left(\cos\left(\left\langle s_1,\tilde{X}_0\left(u\right)\right\rangle\right),\sin\left(\left\langle s_2,\tilde{X}_{t_2}\left(u\right)\right\rangle\right)\right)\right| \leqslant \frac{C}{\mathscr{m}}\left|s_1\right|_1. \tag{C.240}$$

One obtains for all $s_1, s_2 \in \mathbb{R}^d$ due to (C.215), Lemma 3.12, (C.239) and (C.240) (note (C.235), (C.216) as well as (3.16)):

$$\begin{aligned}&\left|\operatorname{Cov}_{\Re,\Im,T,5}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}\left(s_1,s_2\right)-\operatorname{Cov}_{\Re,\Im}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}\left(s_1,s_2\right)\right|\\ &\leqslant\left|\frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{1/(2b)\cdot Tb}\left(\frac{1/(2b)\cdot Tb}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor}-1\right)\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma,u,s_1\right)\tau_{\mathfrak{U}_{0,1},\Im}\left(\tilde{\gamma},u,s_2\right)\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0}K(z)^2\,dz\right.\\ &\left.\cdot\sum_{t_2=-\mathscr{m}}^{\mathscr{m}}\mathscr{G}_T\left(t_2\right)\operatorname{Cov}\left(\cos\left(\left\langle s_1,\tilde{X}_0\left(u\right)\right\rangle\right),\sin\left(\left\langle s_2,\tilde{X}_{t_2}\left(u\right)\right\rangle\right)\right)du\right|\\ &+\left|\frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{1/(2b)\cdot Tb}\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0}K(z)^2\,dz\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma,u,s_1\right)\tau_{\mathfrak{U}_{0,1},\Im}\left(\tilde{\gamma},u,s_2\right)\right.\\ &\cdot\left(\sum_{t_2=-\mathscr{m}}^{\mathscr{m}}\mathscr{G}_T\left(t_2\right)\operatorname{Cov}\left(\cos\left(\left\langle s_1,\tilde{X}_0\left(u\right)\right\rangle\right),\sin\left(\left\langle s_2,\tilde{X}_{t_2}\left(u\right)\right\rangle\right)\right)\right.\\ &\left.\left.-\sum_{t_2=-\infty}^{\infty}\mathscr{G}_T\left(t_2\right)\operatorname{Cov}\left(\cos\left(\left\langle s_1,\tilde{X}_0\left(u\right)\right\rangle\right),\sin\left(\left\langle s_2,\tilde{X}_{t_2}\left(u\right)\right\rangle\right)\right)\right)du\right|\\ &\leqslant C\left|\frac{\left(1/(2b)-\lfloor 1/(2b)\rfloor\right)\cdot Tb+\lfloor 1/(2b)\rfloor\left(Tb-\lfloor Tb\rfloor\right)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor}\right|\left(1+\left|s_1\right|_1+\left|s_2\right|_1\right)+\frac{C}{\mathscr{m}}\left(\left|s_1\right|_1+\left|s_2\right|_1\right)\\ &\leqslant C\left(\frac{1}{\lfloor 1/(2b)\rfloor}+\frac{1}{\lfloor Tb\rfloor}+\frac{1}{\mathscr{m}}\right)\left(1+\left|s_1\right|_1+\left|s_2\right|_1\right).\end{aligned}$$

Thus, Assumption 3.1 $[\mathbf{WEI}.1]$, Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.1]$ (ii) as well as the fact that $\mathscr{m}\to\infty$ for $T\to\infty$ (which follows from Definition A.1 (ii) and $\widetilde{\mathscr{m}}\to\infty$ for $T\to\infty$, as stated in Remark A.2 (i)) provide (see (C.216)):

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\operatorname{Cov}_{\Re,\Im,T,5}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}\left(s_1,s_2\right)\,\mathbf{w}\left(s_1\right)\,ds_1\,\mathbf{w}\left(s_2\right)\,ds_2-\operatorname{Cov}_{\Re,\Im}^{[\mathscr{G}_T]}\left(\gamma,\tilde{\gamma}\right)\right|=o(1). \tag{C.241}$$

Lemma C.6 with $\mathrm{R}_1=\Re$ and $\mathrm{R}_2=\Im$ is an implication of (C.220), (C.223), (C.229), (C.233), (C.238) and (C.241) (note that $\gamma,\tilde{\gamma}\in\mathbb{R}^{1\times 2}$ were chosen arbitrary but fixed at the beginning of this proof). Similar arguments show Lemma C.6 with the other choices of R_1 and R_2. □

Corollary C.7. *Suppose that the Assumptions 2.4* $[\mathbf{DM}.1]$, *3.1* $[\mathbf{WEI}.1]$ *and 2.8* $[\mathbf{K}\&\mathbf{b}.1]$ *are fulfilled. Then, it follows for* $T\to\infty$ *and all* $j_1,j_2\in\{1,2\}$ *(recall (3.18) as well as (C.17), in particular,* $\gamma_1:=(1,0)$ *and* $\gamma_2:=(0,1)$*):*

$$\operatorname{Cov}\left(\mathbb{D}_{T,j_1}^{\circ},\mathbb{D}_{T,j_2}^{\circ}\right)=\sigma_{\mathfrak{U}_{0,1}}\left(\gamma_{j_1},\gamma_{j_2}\right)+o(1).$$

Proof. The indicator function $\mathbb{1}_{\{|t_2-t_1|\leqslant \mathscr{m}\}}$ contained in $\mathrm{Cov}^{[\mathscr{G}_T,\gamma,\tilde{\gamma}]}_{\mathrm{R}_1,\mathrm{R}_2,T}(s_1,s_2)$ (see (C.216)) can be omitted because the opposite condition generates addends which equal zero according to Definition A.1 (i). Thus, one obtains for all $j_1, j_2 \in \{1,2\}$ and for $\mathscr{G}_T \equiv 1$ from (C.23) (recall (C.17) as well as (C.216)):

$$\begin{aligned}\mathrm{Cov}\left(\mathbb{D}^{\circ}_{T,j_1}, \mathbb{D}^{\circ}_{T,j_2}\right) &= \mathrm{Cov}^{[\mathscr{G}_T]}_{\Re,\Re,T}(\gamma_{j_1},\gamma_{j_2}) + \mathrm{Cov}^{[\mathscr{G}_T]}_{\Im,\Im,T}(\gamma_{j_1},\gamma_{j_2}) + \mathrm{Cov}^{[\mathscr{G}_T]}_{\Re,\Im,T}(\gamma_{j_1},\gamma_{j_2}) \\ &\quad + \mathrm{Cov}^{[\mathscr{G}_T]}_{\Im,\Re,T}(\gamma_{j_1},\gamma_{j_2}),\end{aligned}$$

such that Corollary C.7 is an implication of Lemma C.6 with $\mathscr{G}_T \equiv 1$ (see (C.216), (3.18) and (3.17)). $\square$

Lemma C.8. *Let the Assumptions 2.4* $[\mathbf{DM.1}]$*, 2.8* $[\mathbf{K\&b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled.*

(i) It holds for all $T \in \mathbb{N}$*:*

$$\sup_{u,w\in[0,1]} \frac{1}{T^2}\sum_{t_1,t_2=1}^{T} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-w\right)\left|\mathbb{E}\left[W^*_{t_1}W^*_{t_2}\right]\right| \leqslant \frac{C\beta}{Tb}.$$

(ii) The following statement is valid for all $T \in \mathbb{N}$ *(recall Definition A.1 (ii) as well as (iii) and Assumption 3.15* $[\mathbf{W}^*]$ *(i)):*

$$\mathscr{m} \leqslant \mathscr{m}_\beta \leqslant C\beta\mathscr{m}.$$

(iii) One obtains for all $T \in \mathbb{N}$ *(see Definition A.1 (i)):*

$$\sup_{t\in\mathbb{Z}} \left\| W^*_{t,\{\mathscr{m}_\beta\}} - W^*_t \right\|_2 \leqslant \frac{C}{Tb^2}.$$

Proof. (i) At first, note that (C.112), Assumption 2.8 $[\mathbf{K\&b.1}]$ (i) and $|1/(Tb) - 1/\lfloor Tb\rfloor| \leqslant C/(Tb)^2$ provide (recall (C.17)):

$$\sup_{u,w\in[0,1]} \sup_{t_1,t_2=-\lfloor T_{\mathfrak{U}}b\rfloor-1,\ldots,\lfloor T_{\mathfrak{U}}b\rfloor+2} \left| K\left(\frac{\frac{t_1+\lfloor uT\rfloor}{T}-u}{b}\right) K\left(\frac{\frac{t_2+\lfloor wT\rfloor}{T}-w}{b}\right) - K\left(\frac{t_1}{\lfloor Tb\rfloor}\right) K\left(\frac{t_2}{\lfloor Tb\rfloor}\right)\right| \leqslant \frac{C}{Tb}. \tag{C.242}$$

Assumption 2.8 $[\mathbf{K\&b.1}]$ (i), which implies in the case $K_b(t/T-u) > 0$ that $uT - T_{\mathfrak{U}}b \leqslant t \leqslant uT + T_{\mathfrak{U}}b$ (see Definition 2.11 as well as (C.17)), shifting indices of sums, (C.242) and Assumption 3.15 $[\mathbf{W}^*]$ (iii) yield:

$$\begin{aligned}&\sup_{u,w\in[0,1]} \frac{1}{T^2}\sum_{t_1,t_2=1}^{T} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-w\right)\left|\mathbb{E}\left[W^*_{t_1}W^*_{t_2}\right]\right| \\ &\leqslant \sup_{u,w\in[0,1]} \frac{1}{T^2} \sum_{t_1=\lfloor uT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor-1}^{\lfloor uT\rfloor+\lfloor T_{\mathfrak{U}}b\rfloor+2} \sum_{t_2=\lfloor wT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor-1}^{\lfloor wT\rfloor+\lfloor T_{\mathfrak{U}}b\rfloor+2} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-w\right)\left|\mathbb{E}\left[W^*_{t_1}W^*_{t_2}\right]\right| \\ &= \sup_{u,w\in[0,1]} \frac{1}{(Tb)^2} \sum_{t_1,t_2=-\lfloor T_{\mathfrak{U}}b\rfloor-1}^{\lfloor T_{\mathfrak{U}}b\rfloor+2} K\left(\frac{\frac{t_1+\lfloor uT\rfloor}{T}-u}{b}\right) K\left(\frac{\frac{t_2+\lfloor wT\rfloor}{T}-w}{b}\right)\left|\mathbb{E}\left[W^*_{t_1+\lfloor uT\rfloor}W^*_{t_2+\lfloor wT\rfloor}\right]\right| \\ &\leqslant \frac{C}{Tb} + \sup_{u,w\in[0,1]} \frac{1}{(Tb)^2} \sum_{t_1=-\lfloor T_{\mathfrak{U}}b\rfloor}^{\lfloor T_{\mathfrak{U}}b\rfloor} \sum_{t_2=-\lfloor T_{\mathfrak{U}}b\rfloor-t_1}^{\lfloor T_{\mathfrak{U}}b\rfloor-t_1} K\left(\frac{t_1}{\lfloor Tb\rfloor}\right) K\left(\frac{t_2+t_1}{\lfloor Tb\rfloor}\right)\left|K^*\left(\frac{\lfloor uT\rfloor - t_2 - \lfloor wT\rfloor}{\beta}\right)\right| \\ &\leqslant \frac{C}{Tb} + \sup_{u,w\in[0,1]} \frac{C}{Tb} \sum_{t_2=-2\lfloor T_{\mathfrak{U}}b\rfloor}^{2\lfloor T_{\mathfrak{U}}b\rfloor} \left|K^*\left(\frac{\lfloor uT\rfloor - t_2 - \lfloor wT\rfloor}{\beta}\right)\right| \\ &\leqslant \frac{C}{Tb} + \sup_{u,w\in[0,1]} \frac{C}{Tb} \sum_{t_2=-2\lfloor T_{\mathfrak{U}}b\rfloor-|\lfloor uT\rfloor-\lfloor wT\rfloor|}^{2\lfloor T_{\mathfrak{U}}b\rfloor+|\lfloor uT\rfloor-\lfloor wT\rfloor|} \left|K^*\left(-\frac{t_2}{\beta}\right)\right|.\end{aligned} \tag{C.243}$$

This, $|\lfloor wT \rfloor - \lfloor uT \rfloor| \leqslant T \;\forall\, u, w \in [0,1]$ and Assumption 3.15 $[\mathbf{W}^*]$ (iii) as well as (i) show Lemma C.8 (i).

(ii) Definition A.1 (ii) provides $\ln(Tb^2)/\mathscr{m} \to 0$ for $T \to \infty$ and Assumption 3.15 $[\mathbf{W}^*]$ (iv) ensures $\rho_* \in (0,1)$, such that Remark A.2 (iv) yields $\mathscr{m}_* \leqslant C\mathscr{m}$ (recall Definition A.1 (iii)). Thus, one obtains from $\ln(\rho_*) < 0$ and Assumption 3.15 $[\mathbf{W}^*]$ (i) (see Definition A.1 (ii) as well as (iii)):

$$\mathscr{m} \leqslant \mathscr{m}_* \beta \leqslant \mathscr{m}_\beta \leqslant \beta C \mathscr{m} - \beta \frac{\ln\left(\beta_{\text{sup}}^{\text{inv}}\right) + \ln\left(CTb^2\right)}{\ln(\rho_*)} + 1 \leqslant C\beta\mathscr{m} + C\beta\widetilde{\mathscr{m}} \leqslant C\beta\mathscr{m},$$

which shows Lemma C.8 (ii).

(iii) Since $\ln(\rho_*) < 0$, Definition A.1 (iii) yields $\rho_*^{\mathscr{m}_\beta/\beta} = e^{\ln(\rho_*)\mathscr{m}_\beta/\beta} \leqslant e^{\ln(\rho_*)\mathscr{m}_* - \ln(\beta_{\text{sup}}^{\text{inv}}\beta)} \leqslant C/(Tb^2\beta)$. It follows from arguments which are similar to those that provide (B.53), Assumption 3.15 $[\mathbf{W}^*]$ (iv), shifting the index of a sum, $\rho_*^{\mathscr{m}_\beta/\beta} \leqslant C/(Tb^2\beta)$ (as shown immediately before) and the fact that Assumption 3.15 $[\mathbf{W}^*]$ (i) implies $\lim_{T\to\infty} 1/\beta \sum_{l=1}^{\infty} {\rho_*}^{l/\beta} = -\ln(\rho_*)^{-1}$ (note Definition A.1 (i)):

$$\sup_{t\in\mathbb{Z}} \left\| W^*_{t,\{\mathscr{m}_\beta\}} - W^*_t \right\|_2 \leqslant \sup_{t\in\mathbb{Z}} \sum_{l=\mathscr{m}_\beta}^{\infty} \left\| \mathbb{E}\left[W^*_t \middle| \mathcal{F}^*_{t,t-l}\right] - \mathbb{E}\left[W^*_t \middle| \mathcal{F}^*_{t,t-l-1}\right]\right\|_2 \leqslant C\rho_*^{\frac{\mathscr{m}_\beta}{\beta}} \sum_{l=1}^{\infty} {\rho_*}^{\frac{l}{\beta}} \leqslant \frac{C}{Tb^2},$$

which proves Lemma C.8 (iii).

□

Lemma C.9. *Let the Assumptions 2.4* $[\mathbf{DM.1}]$*, 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K\&b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled. Then, it holds for* $T \to \infty$ *(recall (3.38), Definition 3.17 (ii), (3.22) and (3.26)):*

$$\sqrt{T}\left|\widehat{\mathbb{D}}^*_T\left(\left(\widehat{\gamma}^{\text{norm}}_{T,1}, \widehat{\gamma}^{\text{norm}}_{T,2}\right)\right) - \widehat{\mathbb{D}}^*_T\left(\left(\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1}, \gamma^{\text{norm}}_{\mathfrak{U}_{0,1},2}\right)\right)\right| = o^*_{\mathbb{P}}(1).$$

Proof. At first, it will be shown (see (3.38)):

$$\sqrt{T}\left|\widehat{\mathbb{D}}^*_{T,\Re}\left(\left(\widehat{\gamma}^{\text{norm}}_{T,1}, \widehat{\gamma}^{\text{norm}}_{T,2}\right)\right) - \widehat{\mathbb{D}}^*_{T,\Re}\left(\left(\gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1}, \gamma^{\text{norm}}_{\mathfrak{U}_{0,1},2}\right)\right)\right| = o^*_{\mathbb{P}}(1). \tag{C.244}$$

Therefore, one defines (recall (3.38)):

$$\mathbb{A}_{T,1} := \left(\widehat{\gamma}^{\text{norm}}_{T,1} - \gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1}\right)\sqrt{T}\int_{\mathbb{R}^d} \widehat{\mathbb{D}}^*_{T,1,\Re}(s)\,\mathbf{w}(s)\,ds \quad \text{and}$$
$$\mathbb{A}_{T,2} := \left(\widehat{\gamma}^{\text{norm}}_{T,2} - \gamma^{\text{norm}}_{\mathfrak{U}_{0,1},2}\right)\sqrt{T}\int_{\mathbb{R}^d} \widehat{\mathbb{D}}^*_{T,2,\Re}(s)\,\mathbf{w}(s)\,ds, \tag{C.245}$$

whereby $|\Re\{\widehat{\varphi}(u,s)\}| \leqslant C \;\forall\, u \in [0,1]$, $s \in \mathbb{R}^d$, which holds due to Lemma B.1 with $\kappa_1 = 1$ (see Definition 2.11), Lemma C.8 (i), Assumption 3.15 $[\mathbf{W}^*]$ (i) and Assumption 3.1 $[\mathbf{WEI.1}]$ provide that the following inequalities hold almost surely for an absolute deterministic constant $\breve{A} \in (0,\infty)$ (see (3.38)):

$$\begin{aligned}
\mathbb{E}^*\left[\left|\mathbb{A}_{T,1}\right|^2\right] &\leqslant \left(\widehat{\gamma}^{\text{norm}}_{T,1} - \gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1}\right)^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{4T}{\lfloor 1/(2b)\rfloor^2} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \left|\Re\left\{\widehat{\varphi}\left(u_{k_1}, s_1\right)\right\}\right| \left|\Re\left\{\widehat{\varphi}\left(u_{k_2}, s_2\right)\right\}\right| \\
&\cdot \frac{1}{T^2}\sum_{t_1,t_2=1}^{T} K_b\left(\frac{t_1}{T} - u_{k_1}\right) K_b\left(\frac{t_2}{T} - u_{k_2}\right)\left|\Re\left\{e^{i\left\langle s_1, X_{t_1,T}\right\rangle} - \widehat{\varphi}\left(u_{k_1}, s_1\right)\right\}\right| \\
&\cdot \left|\Re\left\{e^{i\left\langle s_2, X_{t_2,T}\right\rangle} - \widehat{\varphi}\left(u_{k_2}, s_2\right)\right\}\right| \left|\mathbb{E}\left[W^*_{t_1} W^*_{t_2}\right]\right| \cdot \mathbf{w}\left(s_1\right)\, ds_1\, \mathbf{w}\left(s_2\right)\, ds_2 \\
&\leqslant \frac{\breve{A}T\,Tb^2}{Tb}\left(\widehat{\gamma}^{\text{norm}}_{T,1} - \gamma^{\text{norm}}_{\mathfrak{U}_{0,1},1}\right)^2.
\end{aligned} \tag{C.246}$$

Lemma 3.4 implies that the function $h_3\left((y_1,y_2)'\right) := y_2/y_1^2 \,\forall\, y_1 > 0, y_2 \in \mathbb{R}$ is partially differentiable at $(y_1,y_2)' = (\mathbb{D}_1,\mathbb{D}_2)'$ (recall Definition 3.3 (ii)), Lemma 3.9 yields for sufficiently large T that $\widehat{\gamma}_{T,1}^{\text{norm}} = h_3\big((\widehat{\mathbb{D}}_{T,1},\widehat{\mathbb{D}}_{T,2})'\big)$ (see Definition 3.17 (ii)) and $\gamma_{\mathfrak{U}_{0,1},1}^{\text{norm}} = h_3\left((\mathbb{D}_1,\mathbb{D}_2)'\right)$ holds according to (3.22). Thus, Theorem 3.13 (i) together with the delta method, the continuous mapping theorem and Slutzky's lemma provide for $T \to \infty$:

$$Tb\left(\widehat{\gamma}_{T,1}^{\text{norm}} - \gamma_{\mathfrak{U}_{0,1},1}^{\text{norm}}\right)^2 \xrightarrow{\text{d}} 0. \tag{C.247}$$

It follows for all $\epsilon_1,\epsilon_2 > 0$ from the conditional version of Markov's inequality, (C.246), (C.247) and the fact that convergence in distribution to a deterministic random variable implies convergence in probability to this random variables (recall that $\breve{A} \in (0,\infty)$ is deterministic):

$$\mathbb{P}\left(\mathbb{P}^*\left(|\mathbb{A}_{T,1}| > \epsilon_1\right) > \epsilon_2\right) \leqslant \mathbb{P}\left(\frac{1}{\epsilon_1^2}\mathbb{E}^*\left[|\mathbb{A}_{T,1}|^2\right] \geqslant \epsilon_2\right) \leqslant \mathbb{P}\left(Tb\left(\widehat{\gamma}_{T,1}^{\text{norm}} - \gamma_{\mathfrak{U}_{0,1},1}^{\text{norm}}\right)^2 \geqslant \frac{\epsilon_2\epsilon_1^2}{\breve{A}}\right) \xrightarrow{T\to\infty} 0. \tag{C.248}$$

Similar arguments prove for all $\epsilon_1,\epsilon_2 > 0$ (see (C.245), Definition 3.17 (ii), (3.22), (3.38) and (3.26)):

$$|\mathbb{A}_{T,2}| = o_{\mathbb{P}}^*(1). \tag{C.249}$$

In conclusion, (C.244) is an implication of (C.248), (C.249) and (3.28) (recall (C.245) as well as (3.38)). Lemma C.9 follows from (C.244), similar arguments and (3.28) (see (3.38)). □

Lemma C.10. *Suppose that the Assumptions 2.4* [**DM.1**], *3.1* [**WEI.1**], *2.8* [**K**&**b.1**] *and 3.15* [$\mathbf{W}^*$] *are fulfilled. Moreover, define for all* $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^d$, $\mathrm{R} \in \{\Re,\Im\}$ *(recall Assumption 3.15* [$\mathbf{W}^*$] *and* $X^c := X - \mathbb{E}[X]$ *for each random variable* X *with finite first moment):*

$$\begin{aligned}
\breve{\varphi}^*\left(u_k,s\right) &:= \breve{\varphi}^*_{T,\mathfrak{U}_{0,1}}\left(u_k,s\right) \\
&:= \frac{1}{\lfloor Tb\rfloor}\sum_{t=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor} K\left(\frac{t-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\left(e^{i\left\langle s,X_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t,T}\right\rangle}\right)^c W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t}, \\
\widehat{\mathbb{D}}^{*[1]}_{T,1,\mathrm{R}}(s) &:= \widehat{\mathbb{D}}^{*[1]}_{T,\mathfrak{U}_{0,1},1,\mathrm{R}}(s) := \frac{2\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathrm{R}\left\{\varphi\left(u_k,s\right)\right\}\mathrm{R}\left\{\breve{\varphi}^*\left(u_k,s\right)\right\} \quad \textit{as well as} \\
\widehat{\mathbb{D}}^{*[1]}_{T,2,\mathrm{R}}(s) &:= \widehat{\mathbb{D}}^{*[1]}_{T,\mathfrak{U}_{0,1},2,\mathrm{R}}(s) := \frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\mathrm{R}\left\{\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\varphi\left(u,s\right)\,du\right\}\frac{2\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathrm{R}\left\{\breve{\varphi}^*\left(u_k,s\right)\right\}.
\end{aligned} \tag{C.250}$$

Then, it holds for all $\mathrm{R} \in \{\Re,\Im\}$ *and* $T \to \infty$ *(see (3.38)):*

(i) $\sqrt{T}\int_{\mathbb{R}^d}\left\|\widehat{\mathbb{D}}^*_{T,1,\mathrm{R}}(s) - \widehat{\mathbb{D}}^{*[1]}_{T,1,\mathrm{R}}(s)\right\|_1 \mathbf{w}(s)\,ds = o(1).$

(ii) $\sqrt{T}\int_{\mathbb{R}^d}\left\|\widehat{\mathbb{D}}^*_{T,2,\mathrm{R}}(s) - \widehat{\mathbb{D}}^{*[1]}_{T,2,\mathrm{R}}(s)\right\|_1 \mathbf{w}(s)\,ds = o(1).$

Remark C.11. *Since (C.52) ensures that* $\breve{\varphi}^*\left(u_k,s\right)$ *just takes* $X_{t,T}$ *with* $t \in \{1,\ldots,T\}$ *into account,* $\breve{\varphi}^*\left(u_k,s\right)$ *is well-defined for all* $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^d$.

Proof of Lemma C.10. Throughout this proof, it is assumed that T is large enough to ensure $\lfloor T_{\mathfrak{U}}b\rfloor \geqslant 1$ (recall (C.17)), which holds for sufficiently large T due to Assumption 2.8 [**K**&**b.1**] (ii).

(i) In the following, Lemma C.10 (i) with $\mathrm{R} = \Re$ will be verified. Therefor, one defines at first for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^d$:

$$\widehat{\varphi}^*\left(u_k,s\right) := \widehat{\varphi}^*_{T,\mathfrak{U}_{0,1}}\left(u_k,s\right) := \frac{1}{T}\sum_{t=1}^{T}K_b\left(\frac{t}{T}-u_k\right)\left(e^{i\left\langle s,X_{t,T}\right\rangle}\right)^c W_t^* \quad \text{and}$$

$$\widehat{\mathbb{D}}^{*[0]}_{T,1,\Re}(s) := \widehat{\mathbb{D}}^{*[0]}_{T,\mathfrak{U}_0,1,1,\Re}(s) := \frac{2\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\varphi\left(u_k, s\right)\right\} \Re\left\{\widehat{\varphi}^*\left(u_k, s\right)\right\}. \tag{C.251}$$

Next, it will be shown that:

$$\sqrt{T} \int_{\mathbb{R}^d} \left\| \widehat{\mathbb{D}}^*_{T,1,\Re}(s) - \widehat{\mathbb{D}}^{*[0]}_{T,1,\Re}(s) \right\|_1 \mathbf{w}(s)\, ds = o(1). \tag{C.252}$$

One observes for all $s \in \mathbb{R}^d$ (see (3.38) and (C.251)):

$$\begin{aligned}
&\sqrt{T} \left\| \widehat{\mathbb{D}}^*_{T,1,\Re}(s) - \widehat{\mathbb{D}}^{*[0]}_{T,1,\Re}(s) \right\|_1 \\
&\leqslant \sqrt{T} \left\| \frac{2\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\widehat{\varphi}\left(u_k, s\right) - \varphi\left(u_k, s\right)\right\} \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right) \Re\left\{e^{i\left\langle s, X_{t,T}\right\rangle} - \widehat{\varphi}\left(u_k, s\right)\right\} W_t^* \right\|_1 \\
&+ \sqrt{T} \left\| \frac{2\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\varphi\left(u_k, s\right)\right\} \cdot \left(\frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right) \Re\left\{e^{i\left\langle s, X_{t,T}\right\rangle} - \widehat{\varphi}\left(u_k, s\right)\right\} W_t^* \right.\right. \\
&\left.\left. - \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right) \Re\left\{e^{i\left\langle s, X_{t,T}\right\rangle} - \mathbb{E}\left[e^{i\left\langle s, X_{t,T}\right\rangle}\right]\right\} W_t^* \right) \right\|_1 \\
&=: \mathrm{R}^*_{1,1,T}(s) + \mathrm{R}^*_{1,2,T}(s).
\end{aligned} \tag{C.253}$$

Let $(Z_{t,T})_{t \in \mathbb{Z}, T \in \mathbb{N}}$ be a sequence of random functions with $Z_{t,T} \colon [0,1] \times \mathbb{R}^d \to \mathbb{R}$, each $Z_{t,T}$ should be measurable with respect to the sigma algebra generated by $(\varepsilon_k)_{k \in \mathbb{Z}}$ (recall Definition 2.1) and suppose $\sup_{u \in [0,1]} \sup_{t=1,\ldots,T} \mathbb{E}\left[Z_{t,T}(u,s)^2\right] < \infty \; \forall\, T \in \mathbb{N},\, s \in \mathbb{R}^d$. According to Assumption 3.15 $[\mathbf{W}^*]$ (ii), $(Z_{t,T})_{t \in \mathbb{Z}, T \in \mathbb{N}}$ is independent of $(W_t^*)_{t \in \mathbb{Z}}$, such that Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) (which ensures $K(z) = 0 \; \forall\, z \in \mathbb{R} : |z| > 1$) and Lemma C.8 (i) imply for all $s \in \mathbb{R}^d$ (see the Definitions 2.11 as well as 3.8 (i)):

$$\begin{aligned}
&\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left| \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right) Z_{t,T}(u_k, s)\, W_t^* \right|^2 \right] \\
&\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \frac{1}{T^2} \sum_{t_1,t_2=1}^{T} K_b\left(\frac{t_1}{T} - u_k\right) K_b\left(\frac{t_2}{T} - u_k\right) \left|\mathbb{E}\left[W_{t_1}^* W_{t_2}^*\right]\right| \\
&\cdot \mathbb{1}_{\left\{\left|\frac{t_1}{T} - u_k\right| \leqslant b\right\}} \mathbb{1}_{\left\{\left|\frac{t_2}{T} - u_k\right| \leqslant b\right\}} \left|\mathbb{E}\left[Z_{t_1,T}(u_k,s) Z_{t_2,T}(u_k,s)\right]\right| \\
&\leqslant \frac{C\beta}{Tb} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t=1,\ldots,T : \left|\frac{t}{T} - u_k\right| \leqslant b} \mathbb{E}\left[Z_{t,T}(u_k,s)^2\right].
\end{aligned} \tag{C.254}$$

One obtains for all $s \in \mathbb{R}^d$ from (3.11), the Propositions 2.12 and 2.14, $\left|\Re\left\{\widehat{\varphi}(u,s)\right\}\right| \leqslant C \; \forall\, u \in [0,1]$, $s \in \mathbb{R}^d$ (which holds due to Lemma B.1 with $\kappa_1 = 1$ (note Definition 2.11)) as well as (C.254) (recall (C.253)):

$$\begin{aligned}
\mathrm{R}^*_{1,1,T}(s) &\leqslant C\sqrt{T} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \left\| \Re\left\{\widehat{\varphi}\left(u_k, s\right) - \varphi\left(u_k, s\right)\right\} \right\|_2 \\
&\cdot \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \left\| \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right) \Re\left\{e^{i\left\langle s, X_{t,T}\right\rangle} - \widehat{\varphi}\left(u_k, s\right)\right\} W_t^* \right\|_2 \\
&\leqslant C\sqrt{T} \left(b^{1+\delta} + \frac{1}{\sqrt{Tb}}\right) \left(|s|_1^{1+\delta} + 1\right) \sqrt{\frac{\beta}{Tb}}.
\end{aligned} \tag{C.255}$$

Further, it follows for all $s \in \mathbb{R}^d$ from (C.254), the Propositions 2.14 as well as 2.12 (in combination with (3.11)), Remark 2.3 and Assumption 2.2 $[\mathbf{StAp}]$ (i) (see (C.253) as well as Definition 2.6):

$$\mathrm{R}_{1,2,T}^{*}(s) \leqslant C\sqrt{T} \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \Bigg\| \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u_k\right) \Re\Bigg\{ -\widehat{\varphi}(u_k,s) + \mathbb{E}\left[\widehat{\varphi}(u_k,s)\right] - \mathbb{E}\left[\widehat{\varphi}(u_k,s)\right]$$
$$+\varphi(u_k,s) - \varphi(u_k,s) + \mathbb{E}\left[e^{i\langle s,\tilde{X}_0(\frac{t}{T})\rangle}\right] - \mathbb{E}\left[e^{i\langle s,\tilde{X}_t(\frac{t}{T})\rangle}\right] + \mathbb{E}\left[e^{i\langle s,X_{t,T}\rangle}\right]\Bigg\} W_t^{*}\Bigg\|_2$$
$$\leqslant C\sqrt{T}\sqrt{\frac{\beta}{Tb}}\left(\frac{1}{\sqrt{Tb}}\sqrt{|s|_1+1} + \left(b^{1+\delta}+\frac{1}{Tb}\right)\left(|s|_1^{1+\delta}+1\right) + b|s|_1 + \frac{1}{T}|s|_1\right). \tag{C.256}$$

In conclusion, (C.252) is an implication of (C.253), (C.255), (C.256) and the Assumptions 3.1 $[\mathbf{WEI.1}]$, 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) as well as 3.15 $[\mathbf{W}^{*}]$ (i).
Moreover, one observes that $\widehat{\varphi}^{*}$ and $\breve{\varphi}^{*}$ are defined very similarly to $\widehat{\varphi}$ and $\breve{\varphi}$, respectively (recall (C.251), (C.250), Definition 2.11 as well as (C.200)). Hence, it follows for all $s \in \mathbb{R}^d$ analogously to (C.199), (C.201) and (C.202) by using Assumption 3.15 $[\mathbf{W}^{*}]$ (iii):

$$\sqrt{T} \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \left\|\widehat{\varphi}^{*}(u_k,s) - \breve{\varphi}^{*}(u_k,s)\right\|_1 \leqslant \frac{C}{\sqrt{Tb}}. \tag{C.257}$$

This provides due to the Assumptions 3.1 $[\mathbf{WEI.1}]$ and 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (see (C.251) as well as (C.250)):

$$\sqrt{T}\int_{\mathbb{R}^d} \left\|\widehat{\mathbb{D}}_{T,1,\Re}^{*[0]}(s) - \widehat{\mathbb{D}}_{T,1,\Re}^{*[1]}(s)\right\|_1 \mathbf{w}(s)\,ds = o(1). \tag{C.258}$$

Lemma C.10 (i) with $\mathrm{R} = \Re$ is an implication of (C.252) and (C.258). Lemma C.10 (i) with $\mathrm{R} = \Im$ can be proved similarly.

(ii) In the following, Lemma C.10 (ii) with $\mathrm{R} = \Re$ will be verified. Therefor, one defines at first for all $s \in \mathbb{R}^d$ (recall (C.251)):

$$\widehat{\mathbb{D}}_{T,2,\Re}^{*[0]}(s) := \widehat{\mathbb{D}}_{T,\mathfrak{U}_0,1,2,\Re}^{*[0]}(s) := \frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\varphi(u,s)\,du\right\}\frac{2(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\Re\left\{\widehat{\varphi}^{*}(u_k,s)\right\}. \tag{C.259}$$

Next, it will be shown that:

$$\sqrt{T}\int_{\mathbb{R}^d} \left\|\widehat{\mathbb{D}}_{T,2,\Re}^{*}(s) - \widehat{\mathbb{D}}_{T,2,\Re}^{*[0]}(s)\right\|_1 \mathbf{w}(s)\,ds = o(1). \tag{C.260}$$

One observes for all $s \in \mathbb{R}^d$ (see (3.38), (C.259) and (C.251)):

$$\sqrt{T}\left\|\widehat{\mathbb{D}}_{T,2,\Re}^{*}(s) - \widehat{\mathbb{D}}_{T,2,\Re}^{*[0]}(s)\right\|_1$$
$$\leqslant \sqrt{T}\Bigg\|\frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k_1=1}^{\lfloor 1/(2b)\rfloor}\widehat{\varphi}(u_{k_1},s) - \int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\varphi(u,s)\,du\right\}$$
$$\cdot\frac{2(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor}\sum_{k_2=1}^{\lfloor 1/(2b)\rfloor}\frac{1}{T}\sum_{t=1}^{T}K_b\left(\frac{t}{T}-u_{k_2}\right)\Re\left\{e^{i\langle s,X_{t,T}\rangle} - \widehat{\varphi}(u_{k_2},s)\right\}W_t^{*}\Bigg\|_1$$
$$+\sqrt{T}\Bigg\|\frac{1}{\mathfrak{U}_1-\mathfrak{U}_0}\Re\left\{\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\varphi(u,s)\,du\right\}\frac{2(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor}\sum_{k_2=1}^{\lfloor 1/(2b)\rfloor}\frac{1}{T}\sum_{t=1}^{T}K_b\left(\frac{t}{T}-u_{k_2}\right)$$

$$\cdot \left(\Re \left\{ e^{i\langle s, X_{t,T} \rangle} - \widehat{\varphi}\left(u_{k_2}, s\right) \right\} - \Re \left\{ e^{i\langle s, X_{t,T} \rangle} - \mathbb{E}\left[e^{i\langle s, X_{t,T} \rangle} \right] \right\} \right) W_t^* \Bigg\|_1$$

$$=: \mathrm{R}^*_{2,1,T}(s) + \mathrm{R}^*_{2,2,T}(s). \tag{C.261}$$

It follows for all $s \in \mathbb{R}^d$ from (3.11), the Propositions 2.12 as well as 2.14 and Lemma B.2 (iii) together with (C.192):

$$\left\| \Re \left\{ \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k_1=1}^{\lfloor 1/(2b) \rfloor} \widehat{\varphi}\left(u_{k_1}, s\right) - \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \varphi(u, s)\, du \right\} \right\|_2$$

$$\leqslant \left\| \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k_1=1}^{\lfloor 1/(2b) \rfloor} \Re \left\{ \widehat{\varphi}\left(u_{k_1}, s\right) - \varphi\left(u_{k_1}, s\right) \right\} \right\|_2 + \left\| \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k_1=1}^{\lfloor 1/(2b) \rfloor} \Re \left\{ \varphi\left(u_{k_1}, s\right) \right\} - \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \Re \left\{ \varphi(u, s) \right\} du \right\|_2$$

$$\leqslant C \left(b^{1+\delta} + \frac{1}{\sqrt{Tb}} \right) \left(|s|_1^{1+\delta} + 1 \right) + C b^{1+\delta} \left(|s|_1^{1+\delta} + |s|_1 \right), \tag{C.262}$$

such that (C.254) and $|\Re\{\widehat{\varphi}(u, s)\}| \leqslant C \; \forall\, u \in [0, 1]$, $s \in \mathbb{R}^d$ (which holds due to Lemma B.1 with $\kappa_1 = 1$ (recall Definition 2.11)) yield for all $s \in \mathbb{R}^d$ (see (C.261)):

$$\mathrm{R}^*_{2,1,T}(s) \leqslant C\sqrt{T} \left(b^{1+\delta} + \frac{1}{\sqrt{Tb}} \right) \left(|s|_1^{1+\delta} + 1 \right) \sqrt{\frac{\beta}{Tb}}. \tag{C.263}$$

Moreover, one obtains for all $s \in \mathbb{R}^d$ similarly to (C.256) (recall (C.261) and (C.253)):

$$\mathrm{R}^*_{2,2,T}(s) \leqslant C\sqrt{T} \sqrt{\frac{\beta}{Tb}} \left(\frac{1}{\sqrt{Tb}} \sqrt{|s|_1 + 1} + \left(b^{1+\delta} + \frac{1}{Tb} \right) \left(|s|_1^{1+\delta} + 1 \right) + b|s|_1 + \frac{1}{T}|s|_1 \right). \tag{C.264}$$

In conclusion, (C.260) is an implication of (C.261), (C.263), (C.264) and the Assumptions 3.1 $[\mathbf{WEI.1}]$, 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) as well as 3.15 $[\mathbf{W}^*]$ (i). Further, one obtains from (C.257), Assumption 3.1 $[\mathbf{WEI.1}]$ and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (see (C.259) as well as (C.250)):

$$\sqrt{T} \int_{\mathbb{R}^d} \left\| \widehat{\mathbb{D}}^{*[0]}_{T,2,\Re}(s) - \widehat{\mathbb{D}}^{*[1]}_{T,2,\Re}(s) \right\|_1 \mathbf{w}(s)\, ds = o(1). \tag{C.265}$$

Lemma C.10 (ii) with $\mathrm{R} = \Re$ follows from (C.260) and (C.265). Lemma C.10 (ii) with $\mathrm{R} = \Im$ can be proved similarly. □

Lemma C.12. *Let the Assumptions 2.4* $[\mathbf{DM.1}]$*, 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled. Then, it holds for $T \to \infty$ and all $\mathrm{R} \in \{\Re, \Im\}$ (recall (C.250), (C.51) and that γ_1 as well as γ_2 originate from (C.17)):*

(i)

$$\left\| \sqrt{T} \int_{\mathbb{R}^d} \widehat{\mathbb{D}}^{*[1]}_{T,1,\mathrm{R}}(s)\, \mathbf{w}(s)\, ds - \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{D}^{\circ *}_{T,k,\gamma_1,\mathrm{R}} \right\|_1 = o(1).$$

(ii)

$$\left\| \sqrt{T} \int_{\mathbb{R}^d} \widehat{\mathbb{D}}^{*[1]}_{T,2,\mathrm{R}}(s)\, \mathbf{w}(s)\, ds - \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{D}^{\circ *}_{T,k,\gamma_2,\mathrm{R}} \right\|_1 = o(1).$$

Proof. (i) In the following, Lemma C.12 with $\mathrm{R} = \Re$ will be proved. It is supposed throughout this proof that T is large enough to ensure that (C.48) holds. One defines for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $s \in \mathbb{R}^d$ (note

$X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\breve{\varphi}^*_{m_\beta}(u_k, s) := \breve{\varphi}^*_{T,\mathfrak{U}_{0,1},m_\beta}(u_k, s)$$
$$:= \frac{1}{\lfloor Tb \rfloor} \sum_{t=1+m_\beta}^{2\lfloor T_\mathfrak{U} b\rfloor - 1 - m_\beta} K\left(\frac{t - \lfloor T_\mathfrak{U} b \rfloor}{\lfloor T_\mathfrak{U} b \rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right)\left(e^{i\left\langle s, X_{\lfloor u_k T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t, T}\right\rangle}\right)^c W^*_{\lfloor u_k T\rfloor - \lfloor T_\mathfrak{U} b \rfloor + t}$$

and $$\widehat{\mathbb{D}}^{*[1]}_{T,1,\Re,m_\beta}(s) := \widehat{\mathbb{D}}^{*[1]}_{T,\mathfrak{U}_{0,1},1,\Re,m_\beta}(s) := \frac{2(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \Re\{\varphi(u_k, s)\}\, \Re\left\{\breve{\varphi}^*_{m_\beta}(u_k, s)\right\}. \tag{C.266}$$

Since (C.52) ensures that $\breve{\varphi}^*_{m_\beta}(u_k, s)$ just takes $X_{t,T}$ with $t \in \{1, \ldots, T\}$ into account, $\breve{\varphi}^*_{m_\beta}(u_k, s)$ is well-defined for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $s \in \mathbb{R}^d$.
Next, it will be proved that:

$$\sqrt{T} \int_{\mathbb{R}^d} \left\| \widehat{\mathbb{D}}^{*[1]}_{T,1,\Re}(s) - \widehat{\mathbb{D}}^{*[1]}_{T,1,\Re,m_\beta}(s) \right\|_2 \mathbf{w}(s)\, ds = o(1). \tag{C.267}$$

Assumption 3.15 $[\mathbf{W}^*]$ (ii) provides for all $s \in \mathbb{R}^d$ (see (C.250) and (C.266)):

$$\begin{aligned}
&T\,\mathbb{E}\left[\left(\widehat{\mathbb{D}}^{*[1]}_{T,1,\Re}(s) - \widehat{\mathbb{D}}^{*[1]}_{T,1,\Re,m_\beta}(s)\right)^2\right] \\
&\leqslant 2T\,\mathbb{E}\Bigg[\Bigg(\frac{2(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \Re\{\varphi(u_k, s)\} \\
&\cdot \frac{1}{\lfloor Tb \rfloor} \sum_{t=1}^{m_\beta} K\left(\frac{t - \lfloor T_\mathfrak{U} b\rfloor}{\lfloor T_\mathfrak{U} b \rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right) \cos\left(\left\langle s, X_{\lfloor u_k T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t, T}\right\rangle\right)^c W^*_{\lfloor u_k T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t}\Bigg)^2\Bigg] \\
&+ 2T\,\mathbb{E}\Bigg[\Bigg(\frac{2(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \Re\{\varphi(u_k, s)\} \\
&\cdot \frac{1}{\lfloor Tb \rfloor} \sum_{t=2\lfloor T_\mathfrak{U} b\rfloor - m_\beta}^{2\lfloor T_\mathfrak{U} b\rfloor} K\left(\frac{t - \lfloor T_\mathfrak{U} b\rfloor}{\lfloor T_\mathfrak{U} b \rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right) \cos\left(\left\langle s, X_{\lfloor u_k T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t, T}\right\rangle\right)^c W^*_{\lfloor u_k T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t}\Bigg)^2\Bigg] \\
&\leqslant \frac{C}{T} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=1}^{m_\beta} \left|\mathbb{E}\left[W^*_{\lfloor u_{k_1} T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t_1} W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t_2}\right]\right| \\
&\cdot \left|\operatorname{Cov}\left(\cos\left(\left\langle s, X_{\lfloor u_{k_1} T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t_1, T}\right\rangle\right), \cos\left(\left\langle s, X_{\lfloor u_{k_2} T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t_2, T}\right\rangle\right)\right)\right| \\
&+ \frac{C}{T} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=2\lfloor T_\mathfrak{U} b\rfloor - m_\beta}^{2\lfloor T_\mathfrak{U} b\rfloor} \left|\mathbb{E}\left[W^*_{\lfloor u_{k_1} T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t_1} W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t_2}\right]\right| \\
&\cdot \left|\operatorname{Cov}\left(\cos\left(\left\langle s, X_{\lfloor u_{k_1} T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t_1, T}\right\rangle\right), \cos\left(\left\langle s, X_{\lfloor u_{k_2} T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t_2, T}\right\rangle\right)\right)\right|.
\end{aligned} \tag{C.268}$$

One obtains for all $k_1, k_2 \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $t_1, t_2 \in \{1, \ldots, 2\lfloor T_\mathfrak{U} b\rfloor\}$ with $k_1 \geqslant k_2 + 1$ analogously to (C.22):

$$\lfloor u_{k_1} T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t_1 \geqslant \lfloor u_{k_2} T\rfloor - \lfloor T_\mathfrak{U} b\rfloor + t_2 + 1 \tag{C.269}$$

and from (C.269) as well as arguments which are similar to those that show (C.210):

$$\lfloor u_{k_1} T\rfloor - \lfloor u_{k_2} T\rfloor + t_1 - t_2 \geqslant \max\{1, \lceil (k_1 - k_2 - 1)\, 2T_\mathfrak{U} b\rceil + t_1 - 2\}. \tag{C.270}$$

Further, define:

$$\mathscr{T}_{T,1} := \{1, \ldots, m_\beta\} \quad \text{and} \quad \mathscr{T}_{T,2} := \{2\lfloor T_\mathfrak{U} b\rfloor - m_\beta, \ldots, 2\lfloor T_\mathfrak{U} b\rfloor\}. \tag{C.271}$$

Lemma B.4 (v) together with (C.269), (C.270), shifting the indices of sums, $k_1 T_{\mathfrak{U}} b + t_1 - 2 \geqslant 0$ $\forall\, k_1 \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $t_1 \in \mathscr{T}_{T,1} \cup \mathscr{T}_{T,2}$ (the latter holds due to $T_{\mathfrak{U}} b \geqslant 1$, which follows from (C.48)) as well as Assumption 2.4 [**DM**.1] imply for all $\mathscr{T}_T \in \{\mathscr{T}_{T,1}, \mathscr{T}_{T,2}\}$, $s \in \mathbb{R}^d$ (recall (C.271) and (C.17)):

$$
\begin{aligned}
&\sum_{\substack{k_1,k_2=1\\ k_1 \geqslant k_2+1}}^{\lfloor 1/(2b) \rfloor} \sum_{t_1,t_2 \in \mathscr{T}_T} \left| \mathrm{Cov}\left(\cos\left(\left\langle s, X_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, T} \right\rangle \right), \cos\left(\left\langle s, X_{\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, T} \right\rangle \right) \right) \right| \\
&\leqslant C \sum_{\substack{k_1,k_2=1\\ k_1 \geqslant k_2+1}}^{\lfloor 1/(2b) \rfloor} \sum_{t_1,t_2 \in \mathscr{T}_T} \sum_{l = \lfloor u_{k_1} T \rfloor + t_1 - \lfloor u_{k_2} T \rfloor - t_2}^{\infty} \Delta_l |s|_1 \\
&\leqslant C \sum_{\substack{k_1,k_2=1\\ k_1 = k_2+1}}^{\lfloor 1/(2b) \rfloor} \sum_{t_1 \in \{1,2\}, t_2 \in \mathscr{T}_T} \sum_{l=1}^{\infty} \Delta_l |s|_1 + C \sum_{\substack{k_1,k_2=1\\ k_1 = k_2+1}}^{\lfloor 1/(2b) \rfloor} \sum_{\substack{t_1,t_2 \in \mathscr{T}_T\\ t_1 \geqslant 3}} \sum_{l=t_1-2}^{\infty} \Delta_l \frac{l^2}{(t_1-2)^2} |s|_1 \\
&+ C \sum_{\substack{k_1,k_2=1\\ k_1 \geqslant k_2+2}}^{\lfloor 1/(2b) \rfloor} \sum_{t_1,t_2 \in \mathscr{T}_T} \sum_{l = \lceil (k_1-k_2-1) 2 T_{\mathfrak{U}} b \rceil + t_1 - 2}^{\infty} \Delta_l \frac{l^2}{((k_1-k_2-1) \cdot 2 T_{\mathfrak{U}} b + t_1 - 2)^2} |s|_1 \\
&\leqslant C \lfloor 1/(2b) \rfloor \cdot \# \mathscr{T}_T \sum_{l=1}^{\infty} \Delta_l |s|_1 + C \lfloor 1/(2b) \rfloor \cdot \# \mathscr{T}_T \sum_{t_1=1}^{\infty} \frac{1}{t_1^2} \sum_{l=1}^{\infty} \Delta_l l^2 |s|_1 \\
&+ C \sum_{k_2=1}^{\lfloor 1/(2b) \rfloor} \sum_{k_1=1}^{\infty} \sum_{t_1,t_2 \in \mathscr{T}_T} \frac{1}{(k_1 T_{\mathfrak{U}} b + k_1 T_{\mathfrak{U}} b + t_1 - 2)^2} \sum_{l=1}^{\infty} \Delta_l l^2 |s|_1 \\
&\leqslant C \lfloor 1/(2b) \rfloor \# \mathscr{T}_T |s|_1 + C \lfloor 1/(2b) \rfloor \# \mathscr{T}_T^2 \sum_{k_1=1}^{\infty} \frac{1}{(k_1 T_{\mathfrak{U}} b)^2} |s|_1 \\
&\leqslant C \lfloor 1/(2b) \rfloor \left(m_\beta + 1 + \frac{(m_\beta+1)^2}{(T_{\mathfrak{U}} b)^2} \right) |s|_1. \qquad \text{(C.272)}
\end{aligned}
$$

Moreover, it follows for all $\mathscr{T}_T \in \{\mathscr{T}_{T,1}, \mathscr{T}_{T,2}\}$, $s \in \mathbb{R}^d$ from Lemma B.4 (v) and (B.45) (see (C.271)):

$$
\begin{aligned}
&\sum_{\substack{k_1,k_2=1\\ k_1 = k_2}}^{\lfloor 1/(2b) \rfloor} \sum_{\substack{t_1,t_2 \in \mathscr{T}_T\\ t_1 \geqslant t_2}} \left| \mathrm{Cov}\left(\cos\left(\left\langle s, X_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, T} \right\rangle \right), \cos\left(\left\langle s, X_{\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, T} \right\rangle \right) \right) \right| \\
&\leqslant C \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left(\sum_{\substack{t_1,t_2 \in \mathscr{T}_T\\ t_1 = t_2}} 1 + \sum_{t_2 \in \mathscr{T}_T} \sum_{t_1 = t_2+1}^{\infty} \sum_{l = t_1 - t_2}^{\infty} \Delta_l \, |s|_1 \right) \\
&\leqslant C \lfloor 1/(2b) \rfloor (m_\beta + 1) \left(1 + |s|_1\right). \qquad \text{(C.273)}
\end{aligned}
$$

In conclusion, (C.268), Assumption 3.15 [$\mathbf{W}^*$] (iii), (C.272), (C.273) as well as similar arguments, Lemma C.8 (ii) together with Assumption 3.15 [$\mathbf{W}^*$] (i) (the latter ensures $\beta = o(1/b)$), Remark A.2 (i) and Assumption 2.8 [**K**&**b**.1] (ii) imply for all $s \in \mathbb{R}^d$ (note (C.17)):

$$
\begin{aligned}
T\, \mathbb{E}\left[\left(\widehat{\mathbb{D}}^{*[1]}_{T,1,\Re}(s) - \widehat{\mathbb{D}}^{*[1]}_{T,1,\Re,m_\beta}(s) \right)^2 \right] &\leqslant \frac{C}{T} \lfloor 1/(2b) \rfloor \left(m_\beta + 1 + (m_\beta+1)^2/(T_{\mathfrak{U}} b)^2 \right) (|s|_1 + 1) \\
&= o(1) \left(|s|_1 + 1\right),
\end{aligned}
$$

whereby the expression $o(1)$ does not depend on $s \in \mathbb{R}^d$. Hence, Assumption 3.1 [**WEI**.1] yields (C.267).
Further, it holds (recall $\gamma_1 := (1, 0)$ - according to (C.17), (C.51) and (3.16)):

$$
\sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{D}^{\circ *}_{T,k,\gamma_1,\Re} = \int_{\mathbb{R}^d} \frac{2\sqrt{T}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{ \varphi\left(u_k, s\right) \right\} \Re\left\{ \varphi^{\circ *}_{m_\beta}\left(u_k, s\right) \right\} \mathbf{w}(s)\, ds. \qquad \text{(C.274)}
$$

One obtains for all $s \in \mathbb{R}^d$ by using Assumption 3.15 $[\mathbf{W}^*]$ (ii), arguments which are similar to those that show (C.272) as well as (C.273), Lemma C.8 (iii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii) (see (C.266), (C.51) as well as (C.17)):

$$\mathbb{E}\left[\left(\sqrt{T}\,\widehat{\mathbb{D}}^{*[1]}_{T,1,\Re,m_\beta}(s) - \frac{2\sqrt{T}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\Re\{\varphi(u_k,s)\}\,\Re\left\{\varphi^{\circ *}_{m_\beta}(u_k,s)\right\}\right)^2\right]$$
$$\leqslant \frac{C}{T}\sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=1+m_\beta}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - m_\beta}\Big|\mathrm{Cov}\left(\cos\left(\left\langle s, X_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1, T}\right\rangle\right),\right.$$
$$\left.\cos\left(\left\langle s, X_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2, T}\right\rangle\right)\right)\Big| \cdot \Bigg(\sup_{k_1,k_2=1,\ldots,\lfloor 1/(2b)\rfloor}\ \sup_{t_1,t_2=1,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor}\Big|\mathbb{E}\Big[\Big(W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1}$$
$$-W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{m_\beta\}}\Big)\cdot\Big(W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2} - W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{m_\beta\}}\Big)\Big]\Big|\Bigg)$$
$$\leqslant \frac{C}{T}\lfloor 1/(2b)\rfloor\left(T_{\mathfrak{U}} b + (T_{\mathfrak{U}} b)^2/(T_{\mathfrak{U}} b)^2\right)(|s|_1 + 1)\,\frac{C}{Tb^2}$$
$$= o(1)\,(|s|_1 + 1)\,,$$

whereby the expression $o(1)$ does not depend on $s \in \mathbb{R}^d$. Thus, Assumption 3.1 $[\mathbf{WEI}.\mathbf{1}]$ provides:

$$\int_{\mathbb{R}^d}\left\|\sqrt{T}\,\widehat{\mathbb{D}}^{*[1]}_{T,1,\Re,m_\beta}(s) - \frac{2\sqrt{T}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\Re\{\varphi(u_k,s)\}\,\Re\left\{\varphi^{\circ *}_{m_\beta}(u_k,s)\right\}\right\|_2 \mathbf{w}(s)\,ds = o(1). \tag{C.275}$$

Lemma C.12 (i) with $\mathrm{R} = \Re$ follows from (C.267), (C.275) and (C.274). Lemma C.12 (i) with $\mathrm{R} = \Im$ can be proved analogously.

(ii) Lemma C.12 (ii) can be shown similarly to Lemma C.12 (i). □

Lemma C.13. *Let the Assumptions 2.4* $[\mathbf{DM}.\mathbf{1}]$*, 3.1* $[\mathbf{WEI}.\mathbf{1}]$*, 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled. Then, it holds for* $T \to \infty$ *and all arbitrary but fixed* $\gamma := \left(\gamma^{[1]}, \gamma^{[2]}\right) \in \mathbb{R}^{1\times 2}$ *(recall (C.51) as well as (3.18)):*

$$\mathbb{E}\left[\left|\mathrm{Var}^*\left(\mathbb{D}^{\circ *}_{T,\gamma}\right) - \sigma_{\mathfrak{U}_{0,1}}(\gamma,\gamma)\right|\right] = o(1).$$

Proof. Throughout this proof, it is assumed that $\gamma \in \mathbb{R}^{1\times 2}$ is arbitrary but fixed and T is large enough to ensure that (C.48) holds. Further, one defines for all $\mathrm{R} \in \{\Re, \Im\}$ (see (C.51)):

$$\mathbb{D}^{\circ *}_{T,\gamma,\mathrm{R}} := \mathbb{D}^{\circ *}_{T,\mathfrak{U}_{0,1},\gamma,\mathrm{R}} := \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{D}^{\circ *}_{T,k,\gamma,\mathrm{R}} \tag{C.276}$$

and observes that the following equations hold almost surely (recall (C.51)):

$$\begin{aligned}\mathrm{Var}^*\left(\mathbb{D}^{\circ *}_{T,\gamma}\right) &= \mathrm{Var}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Re} + \mathbb{D}^{\circ *}_{T,\gamma,\Im}\right)\\ &= \mathrm{Var}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Re}\right) + \mathrm{Var}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Im}\right) + \mathrm{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Re}, \mathbb{D}^{\circ *}_{T,\gamma,\Im}\right) + \mathrm{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Im}, \mathbb{D}^{\circ *}_{T,\gamma,\Re}\right).\end{aligned} \tag{C.277}$$

Next, the asymptotic behaviour of $\mathrm{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Re}, \mathbb{D}^{\circ *}_{T,\gamma,\Im}\right)$ is investigated. Therefor, at first, it will be shown that:

$$\mathrm{Var}\left(\mathrm{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Re}, \mathbb{D}^{\circ *}_{T,\gamma,\Im}\right)\right) = o(1). \tag{C.278}$$

One obtains for all $k_1, k_2 \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $t_1, t_2 \in \{1 + m_\beta, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - m_\beta\}$ with $k_1 \neq k_2$ from (C.57) by recalling Definition A.1 (i) that $\mathbb{E}\Big[W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{m_\beta\}}W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{m_\beta\}}\Big] =$

$\mathbb{E}\Big[W^*_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\mathscr{m}_\beta\}}W^*_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\mathscr{m}_\beta\}}\Big]\mathbb{1}_{\{k_1=k_2\}}$. Thus, Assumption 3.15 $[\mathbf{W}^*]$ (ii) provides (see (C.276) and (C.51)):

$$\begin{aligned}
&\operatorname{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Re},\mathbb{D}^{\circ *}_{T,\gamma,\Im}\right)\\
&=\frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma,u_k,s_1\right)\tau_{\mathfrak{U}_{0,1},\Im}\left(\gamma,u_k,s_2\right)\sum_{t_1,t_2=1+\mathscr{m}_\beta}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}_\beta}K\bigg(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\cdot(\mathfrak{U}_1\\
&-\mathfrak{U}_0)\bigg)K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\cos\left(\left\langle s_1,X_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,T}\right\rangle\right)^c\sin\left(\left\langle s_2,X_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,T}\right\rangle\right)^c\\
&\cdot\mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\mathscr{m}_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\mathscr{m}_\beta\}}\right]\cdot\mathbf{w}\left(s_1\right)\,ds_1\,\mathbf{w}\left(s_2\right)\,ds_2. \qquad \text{(C.279)}
\end{aligned}$$

Hence, it follows (recall (3.16)):

$$\begin{aligned}
&\operatorname{Var}\left(\operatorname{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Re},\mathbb{D}^{\circ *}_{T,\gamma,\Im}\right)\right)\\
&\leqslant\frac{C}{T^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2,t_3,t_4=1+\mathscr{m}_\beta}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{m}_\beta}\bigg|\operatorname{Cov}\bigg(\cos\left(\left\langle s_1,X_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,T}\right\rangle\right)^c\\
&\cdot\sin\left(\left\langle s_2,X_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,T}\right\rangle\right)^c,\cos\left(\left\langle s_3,X_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_3,T}\right\rangle\right)^c\\
&\cdot\sin\left(\left\langle s_4,X_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_4,T}\right\rangle\right)^c\bigg)\bigg|\left|\mathbb{E}\left[W^*_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\mathscr{m}_\beta\}}W^*_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\mathscr{m}_\beta\}}\right]\right|\\
&\cdot\left|\mathbb{E}\left[W^*_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_3,\{\mathscr{m}_\beta\}}W^*_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_4,\{\mathscr{m}_\beta\}}\right]\right|\mathbf{w}\left(s_1\right)\,ds_1\,\mathbf{w}\left(s_2\right)\,ds_2\,\mathbf{w}\left(s_3\right)\,ds_3\,\mathbf{w}\left(s_4\right)\,ds_4. \qquad \text{(C.280)}
\end{aligned}$$

Further, one obtains for all $r_1,\ldots,r_4\in\{1,\ldots,T\}$, $s_1,\ldots,s_4\in\mathbb{R}^d$ with $r_1\geqslant r_2>r_3\geqslant r_4$ from arguments which are similar to those that show (B.53) by using (C.112), Lemma B.4 (iii) with $q=1+\delta$ and by shifting the indices of sums (see Definition A.1 (i)):

$$\begin{aligned}
&\bigg|\operatorname{Cov}\bigg(\cos\left(\left\langle s_1,X_{r_1,T}\right\rangle\right)^c\sin\left(\left\langle s_2,X_{r_2,T}\right\rangle\right)^c,\cos\left(\left\langle s_3,X_{r_3,T}\right\rangle\right)^c\sin\left(\left\langle s_4,X_{r_4,T}\right\rangle\right)^c\bigg)\bigg|\\
&=\bigg|\mathbb{E}\bigg[\bigg(\mathbb{E}\bigg[\cos\left(\left\langle s_1,X_{r_1,T}\right\rangle\right)^c\sin\left(\left\langle s_2,X_{r_2,T}\right\rangle\right)^c\bigg|\mathcal{F}_{r_1}\bigg]\\
&-\mathbb{E}\bigg[\cos\left(\left\langle s_1,X_{r_1,T}\right\rangle\right)^c\sin\left(\left\langle s_2,X_{r_2,T}\right\rangle\right)^c\bigg|\mathcal{F}_{r_1,r_3+1}\bigg]\bigg)\cos\left(\left\langle s_3,X_{r_3,T}\right\rangle\right)^c\sin\left(\left\langle s_4,X_{r_4,T}\right\rangle\right)^c\bigg]\bigg|\\
&\leqslant\sum_{l=r_1-r_3-1}^{\infty}\bigg\|\mathbb{E}\bigg[\cos\left(\left\langle s_1,X_{r_1,T}\right\rangle\right)^c\sin\left(\left\langle s_2,X_{r_2,T}\right\rangle\right)^c\bigg|\mathcal{F}_{r_1,r_1-l}\bigg]-\mathbb{E}\bigg[\cos\left(\left\langle s_1,X_{r_1,T}\right\rangle\right)^c\sin\left(\left\langle s_2,X_{r_2,T}\right\rangle\right)^c\\
&\bigg|\mathcal{F}_{r_1,r_1-l-1}\bigg]\bigg\|_1\\
&\leqslant\sum_{l=r_1-r_3-1}^{\infty}\bigg\|\cos\left(\left\langle s_1,X^{\times(r_1-l-1)}_{r_1,T}\right\rangle\right)^c\sin\left(\left\langle s_2,X^{\times(r_2-(l+r_2-r_1)-1)}_{r_2,T}\right\rangle\right)^c-\cos\left(\left\langle s_1,X_{r_1,T}\right\rangle\right)^c\\
&\cdot\sin\left(\left\langle s_2,X_{r_2,T}\right\rangle\right)^c\bigg\|_1\\
&\leqslant C\sum_{l=r_1-r_3-1}^{\infty}\bigg(\bigg\|\cos\left(\left\langle s_1,X^{\times(r_1-l-1)}_{r_1,T}\right\rangle\right)-\cos\left(\left\langle s_1,X_{r_1,T}\right\rangle\right)\bigg\|_{1+\delta}\\
&+\bigg\|\sin\left(\left\langle s_2,X^{\times(r_2-(l+r_2-r_1)-1)}_{r_2,T}\right\rangle\right)-\sin\left(\left\langle s_2,X_{r_2,T}\right\rangle\right)\bigg\|_{1+\delta}\bigg)\\
&\leqslant C\sum_{l=r_1-r_3-1}^{\infty}\left(\Delta_{l+1}\left|s_1\right|_1+\Delta_{l+r_2-r_1+1}\left|s_2\right|_1\right)\\
&=C\left(\sum_{l=r_1-r_3}^{\infty}\Delta_l\left|s_1\right|_1+\sum_{l=r_2-r_3}^{\infty}\Delta_l\left|s_2\right|_1\right)\\
&\leqslant C\sum_{l=r_2-r_3}^{\infty}\Delta_l\left(\left|s_1\right|_1+\left|s_2\right|_1\right). \qquad \text{(C.281)}
\end{aligned}$$

It follows for all $k_1, k_2 \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $t_1, t_2, t_3, t_4 \in \{1 + m_\beta, \ldots, 2\lfloor T_\mathfrak{U} b \rfloor - 1 - m_\beta\}$ with $k_1 \geqslant k_2 + 1$, $t_1 \geqslant t_2$ and $t_3 \geqslant t_4$ from (C.52) as well as (C.57) that $r_1 := \lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_1$, $r_2 := \lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_2$, $r_3 := \lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_3$, $r_4 := \lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_4$ fulfil $r_1, \ldots, r_4 \in \{1, \ldots, T\}$ and $r_1 \geqslant r_2 > r_3 \geqslant r_4$. Hence, (C.281), (C.210) (whereby $\{1 + m_\beta, \ldots, 2\lfloor T_\mathfrak{U} b \rfloor - 1 - m_\beta\} \subseteq \{1 + m, \ldots, 2\lfloor T_\mathfrak{U} b \rfloor - 1 - m\}$ holds according to Lemma C.8 (ii) and (C.48)) as well as Assumption 2.4 $[\mathbf{DM}.\mathbf{1}]$ imply:

$$\begin{aligned}
&\Big|\mathrm{Cov}\Big(\cos\Big(\Big\langle s_1, X_{\lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_1, T}\Big\rangle\Big)^c \sin\Big(\Big\langle s_2, X_{\lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_2, T}\Big\rangle\Big)^c, \\
&\cos\Big(\Big\langle s_3, X_{\lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_3, T}\Big\rangle\Big)^c \sin\Big(\Big\langle s_4, X_{\lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_4, T}\Big\rangle\Big)^c\Big)\Big| \\
&\leqslant C \sum_{l = \lfloor u_{k_1} T \rfloor + t_2 - \lfloor u_{k_2} T \rfloor - t_3}^{\infty} \Delta_l \left(|s_1|_1 + |s_2|_1\right) \\
&\leqslant C \sum_{l = \lceil (k_1 - k_2 - 1) 2 T_\mathfrak{U} b \rceil + t_2}^{\infty} \Delta_l \left(|s_1|_1 + |s_2|_1\right) \\
&\leqslant C \left(\mathbb{1}_{\{k_1 = k_2 + 1\}} \sum_{l = t_2}^{\infty} \Delta_l \frac{l^2}{t_2^2} + \mathbb{1}_{\{k_1 \geqslant k_2 + 2\}} \sum_{l = \lceil (k_1 - k_2 - 1) 2 T_\mathfrak{U} b \rceil}^{\infty} \Delta_l \frac{l^2}{(k_1 - k_2 - 1)^2 (2 T_\mathfrak{U} b)^2}\right) \left(|s_1|_1 + |s_2|_1\right) \\
&\leqslant \left(\frac{C}{t_2^2} \mathbb{1}_{\{k_1 = k_2 + 1\}} + \frac{C}{(k_1 - k_2 - 1)^2 (2 T_\mathfrak{U} b)^2} \mathbb{1}_{\{k_1 \geqslant k_2 + 2\}}\right) \left(|s_1|_1 + |s_2|_1\right). \qquad \text{(C.282)}
\end{aligned}$$

One obtains from (C.282), Assumption 3.1 $[\mathbf{WEI}.\mathbf{1}]$, (C.72) and shifting the index of a sum:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{\substack{k_1, k_2 = 1 \\ k_1 \geqslant k_2 + 1}}^{\lfloor 1/(2b) \rfloor} \sum_{\substack{t_1, t_2, t_3, t_4 = 1 + m_\beta \\ (t_1 \geqslant t_2) \wedge (t_3 \geqslant t_4)}}^{2\lfloor T_\mathfrak{U} b \rfloor - 1 - m_\beta} \Big|\mathrm{Cov}\Big(\cos\Big(\Big\langle s_1, X_{\lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_1, T}\Big\rangle\Big)^c \\
&\cdot \sin\Big(\Big\langle s_2, X_{\lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_2, T}\Big\rangle\Big)^c, \cos\Big(\Big\langle s_3, X_{\lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_3, T}\Big\rangle\Big)^c \sin\Big(\Big\langle s_4, X_{\lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_4, T}\Big\rangle\Big)^c\Big)\Big| \\
&\cdot \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_1, \{m_\beta\}} W^*_{\lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_2, \{m_\beta\}}\Big]\Big| \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_3, \{m_\beta\}} \\
&\cdot W^*_{\lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_4, \{m_\beta\}}\Big]\Big| \,\mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2\, \mathbf{w}(s_3)\, ds_3\, \mathbf{w}(s_4)\, ds_4 \\
&\leqslant C \sum_{\substack{k_1, k_2 = 1 \\ k_1 = k_2 + 1}}^{\lfloor 1/(2b) \rfloor} \sum_{t_2 = 1}^{\infty} \frac{1}{t_2^2} \cdot \Bigg(\sup_{k_1 = 1, \ldots, \lfloor 1/(2b) \rfloor} \sup_{t_2 = 1, \ldots, 2\lfloor T_\mathfrak{U} b \rfloor} \sum_{t_1 = 1}^{2\lfloor T_\mathfrak{U} b \rfloor} \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_1, \{m_\beta\}} \\
&\cdot W^*_{\lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_2, \{m_\beta\}}\Big]\Big|\Bigg) \\
&\cdot \sum_{t_3 = 1}^{2\lfloor T_\mathfrak{U} b \rfloor} \Bigg(\sup_{k_2 = 1, \ldots, \lfloor 1/(2b) \rfloor} \sup_{t_3 = 1, \ldots, 2\lfloor T_\mathfrak{U} b \rfloor} \sum_{t_4 = 1}^{2\lfloor T_\mathfrak{U} b \rfloor} \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_3, \{m_\beta\}} W^*_{\lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_4, \{m_\beta\}}\Big]\Big|\Bigg) \\
&+ C \sum_{k_2 = 1}^{\lfloor 1/(2b) \rfloor} \sum_{k_1 = k_2 + 2}^{\infty} \frac{1}{(k_1 - k_2 - 1)^2} \frac{1}{(2 T_\mathfrak{U} b)^2} \\
&\cdot \sum_{t_1 = 1}^{2\lfloor T_\mathfrak{U} b \rfloor} \Bigg(\sup_{k_1 = 1, \ldots, \lfloor 1/(2b) \rfloor} \sup_{t_1 = 1, \ldots, 2\lfloor T_\mathfrak{U} b \rfloor} \sum_{t_2 = 1}^{2\lfloor T_\mathfrak{U} b \rfloor} \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_1, \{m_\beta\}} W^*_{\lfloor u_{k_1} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_2, \{m_\beta\}}\Big]\Big|\Bigg) \\
&\cdot \sum_{t_3 = 1}^{2\lfloor T_\mathfrak{U} b \rfloor} \Bigg(\sup_{k_2 = 1, \ldots, \lfloor 1/(2b) \rfloor} \sup_{t_3 = 1, \ldots, 2\lfloor T_\mathfrak{U} b \rfloor} \sum_{t_4 = 1}^{2\lfloor T_\mathfrak{U} b \rfloor} \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_3, \{m_\beta\}} W^*_{\lfloor u_{k_2} T \rfloor - \lfloor T_\mathfrak{U} b \rfloor + t_4, \{m_\beta\}}\Big]\Big|\Bigg) \\
&\leqslant C \lfloor 1/(2b) \rfloor \beta \lfloor T_\mathfrak{U} b \rfloor \beta + C \sum_{k_2 = 1}^{\lfloor 1/(2b) \rfloor} \sum_{k_1 = 1}^{\infty} \frac{1}{k_1^2} \frac{(2\lfloor T_\mathfrak{U} b \rfloor)^2 \beta^2}{(2 T_\mathfrak{U} b)^2} \\
&\leqslant C \lfloor 1/(2b) \rfloor (\lfloor T_\mathfrak{U} b \rfloor + 1) \beta^2. \qquad \text{(C.283)}
\end{aligned}$$

It follows for all $r_1, r_2, r_3, r_4 \in \{1, \ldots, T\}$ with $r_1 \geqslant r_2$ and $r_1 \geqslant r_3 \geqslant r_4$ by using Lemma B.4 (iii) with $q = 1 + \delta$ (recall Definition A.1 (i)):

$$
\begin{aligned}
&\left|\mathbb{E}\Big[\cos\left(\langle s_1, X_{r_1,T}\rangle\right)^c \sin\left(\langle s_2, X_{r_2,T}\rangle\right)^c \cos\left(\langle s_3, X_{r_3,T}\rangle\right)^c \sin\left(\langle s_4, X_{r_4,T}\rangle\right)^c \Big]\right| \\
&= \mathbb{1}_{\{r_1 > \max\{r_2,r_3\}\}} \left|\mathbb{E}\Big[\Big(\mathbb{E}\left[\cos\left(\langle s_1, X_{r_1,T}\rangle\right) \,\middle|\, \mathcal{F}_{r_1}\right] - \mathbb{E}\left[\cos\left(\langle s_1, X_{r_1,T}\rangle\right) \,\middle|\, \mathcal{F}_{r_1, \max\{r_2,r_3\}+1}\right]\Big) \right. \\
&\left. \cdot \sin\left(\langle s_2, X_{r_2,T}\rangle\right)^c \cos\left(\langle s_3, X_{r_3,T}\rangle\right)^c \sin\left(\langle s_4, X_{r_4,T}\rangle\right)^c \Big]\right| + C\mathbb{1}_{\{r_1 = \max\{r_2,r_3\}\}} \\
&\leqslant C\mathbb{1}_{\{r_1 > \max\{r_2,r_3\}\}} \sum_{l = r_1 - \max\{r_2,r_3\}-1}^{\infty} \Delta_{l+1} \left|s_1\right|_1 + C\mathbb{1}_{\{r_1 = \max\{r_2,r_3\}\}}.
\end{aligned} \tag{C.284}
$$

Since $\left|\mathrm{Cov}\left(X, Y\right)\right| \leqslant \left|\mathbb{E}\left[XY\right]\right| + \left|\mathbb{E}\left[X\right]\right| \left|\mathbb{E}\left[Y\right]\right|$ for each random variable X and Y which live on the same probability space and own finite second moments, (C.284) (in combination with shifting the index of a sum) as well as Lemma B.4 (v) provide for all $r_1, r_2, r_3, r_4 \in \{1, \ldots, T\}$ with $r_1 \geqslant r_2$ and $r_1 \geqslant r_3 \geqslant r_4$:

$$
\begin{aligned}
&\left|\mathrm{Cov}\Big(\cos\left(\langle s_1, X_{r_1,T}\rangle\right)^c \sin\left(\langle s_2, X_{r_2,T}\rangle\right)^c, \cos\left(\langle s_3, X_{r_3,T}\rangle\right)^c \sin\left(\langle s_4, X_{r_4,T}\rangle\right)^c \Big)\right| \\
&\leqslant C\left(\mathbb{1}_{\{r_1 > r_2\}} \sum_{l = r_1 - r_2}^{\infty} \Delta_l \left|s_1\right|_1 + \mathbb{1}_{\{r_1 > r_3\}} \sum_{l = r_1 - r_3}^{\infty} \Delta_l \left|s_1\right|_1 + \mathbb{1}_{\{r_1 = \max\{r_2,r_3\}\}} \right) \\
&+ C\left(\mathbb{1}_{\{r_1 > r_2\}} \sum_{l = r_1 - r_2}^{\infty} \Delta_l \left|s_1\right|_1 + \mathbb{1}_{\{r_1 = r_2\}} \right) \left(\mathbb{1}_{\{r_3 > r_4\}} \sum_{l = r_3 - r_4}^{\infty} \Delta_l \left|s_3\right|_1 + \mathbb{1}_{\{r_3 = r_4\}} \right).
\end{aligned} \tag{C.285}
$$

One obtains from (C.285), Assumption 3.1 $[\mathbf{WEI.1}]$, Assumption 3.15 $[\mathbf{W}^*]$ (iii), especially $\sup_{j_1, j_2 \in \mathbb{Z}} \left|\mathbb{E}\left[W^*_{j_1,\{\mathscr{m}_\beta\}} W^*_{j_2,\{\mathscr{m}_\beta\}}\right]\right| \leqslant \|W^*_0\|_2^2 \leqslant C$ (note Definition A.1 (i)), (C.72) and (B.45):

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{\substack{k_1,k_2=1 \\ k_1 = k_2}}^{\lfloor 1/(2b)\rfloor} \sum_{\substack{t_1,t_2,t_3,t_4 = 1 + \mathscr{m}_\beta \\ t_1 \geqslant t_2, t_1 \geqslant t_3 \geqslant t_4}}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{m}_\beta} \left|\mathrm{Cov}\left(\cos\left(\left\langle s_1, X_{\lfloor u_{k_1} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1, T}\right\rangle\right)^c \right.\right. \\
&\left.\left. \cdot \sin\left(\left\langle s_2, X_{\lfloor u_{k_1} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2, T}\right\rangle\right)^c, \cos\left(\left\langle s_3, X_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_3, T}\right\rangle\right)^c \sin\left(\left\langle s_4, X_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_4, T}\right\rangle\right)^c \right)\right| \\
&\cdot \left|\mathbb{E}\left[W^*_{\lfloor u_{k_1} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1, \{\mathscr{m}_\beta\}} W^*_{\lfloor u_{k_1} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2, \{\mathscr{m}_\beta\}} \right]\right| \\
&\cdot \left|\mathbb{E}\left[W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_3, \{\mathscr{m}_\beta\}} W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_4, \{\mathscr{m}_\beta\}} \right]\right| \mathbf{w}\left(s_1\right) ds_1 \, \mathbf{w}\left(s_2\right) ds_2 \, \mathbf{w}\left(s_3\right) ds_3 \, \mathbf{w}\left(s_4\right) ds_4 \\
&\leqslant C \sum_{\substack{k_1,k_2=1 \\ k_1 = k_2}}^{\lfloor 1/(2b)\rfloor} \left\{ \sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{t_1 = t_2+1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{l = t_1 - t_2}^{\infty} \Delta_l \cdot \sum_{t_3,t_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left|\mathbb{E}\left[W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_3, \{\mathscr{m}_\beta\}} W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_4, \{\mathscr{m}_\beta\}} \right]\right| \right. \\
&+ C \sum_{t_3=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{t_1 = t_3+1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{l = t_1 - t_3}^{\infty} \Delta_l \cdot \sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left(\sup_{t_3 = 1, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{t_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left|\mathbb{E}\left[W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_3, \{\mathscr{m}_\beta\}} \right.\right.\right. \\
&\left.\left.\left. \cdot W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_4, \{\mathscr{m}_\beta\}} \right]\right| \right) + \sum_{t_2,t_3=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{t_1=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \mathbb{1}_{\{t_1 = \max\{t_2,t_3\}\}} \\
&\cdot \left(\sup_{t_3 = 1, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{t_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left|\mathbb{E}\left[W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_3, \{\mathscr{m}_\beta\}} W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_4, \{\mathscr{m}_\beta\}} \right]\right| \right) \\
&\left. + \left(\sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{t_1 = t_2+1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{l = t_1 - t_2}^{\infty} \Delta_l + \sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{t_1=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \mathbb{1}_{\{t_1 = t_2\}} \right) \left(\sum_{t_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{t_3 = t_4+1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{l = t_3 - t_4}^{\infty} \Delta_l + \sum_{t_4=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{t_3=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \mathbb{1}_{\{t_3 = t_4\}} \right) \right\} \\
&\leqslant C \lfloor 1/(2b)\rfloor \left\{ \lfloor T_{\mathfrak{U}} b\rfloor^2 \beta + \lfloor T_{\mathfrak{U}} b\rfloor^2 \beta + \lfloor T_{\mathfrak{U}} b\rfloor^2 \beta + \lfloor T_{\mathfrak{U}} b\rfloor^2 \right\} \\
&\leqslant C \lfloor 1/(2b)\rfloor \lfloor T_{\mathfrak{U}} b\rfloor^2 \beta.
\end{aligned} \tag{C.286}
$$

In conclusion, (C.280), (C.283), (C.286) and similar arguments, Assumption 3.15 $[\mathbf{W}^*]$ (i) as well as

Assumption 2.8 [**K**&**b**.**1**] (ii) imply (see (C.17)):

$$\operatorname{Var}\left(\operatorname{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Re}, \mathbb{D}^{\circ *}_{T,\gamma,\Im}\right)\right) \leqslant C \frac{T^2 \left(\lfloor 1/(2b) \rfloor \left(\lfloor T_{\mathfrak{U}} b \rfloor + 1\right) \beta^2 + \lfloor 1/(2b) \rfloor \lfloor T_{\mathfrak{U}} b \rfloor^2 \beta\right)}{\lfloor 1/(2b) \rfloor^4 \lfloor Tb \rfloor^4} = o(1). \tag{C.287}$$

Next, the asymptotic behaviour of $\mathbb{E}\big[\operatorname{Cov}^*\big(\mathbb{D}^{\circ *}_{T,\gamma,\Re}, \mathbb{D}^{\circ *}_{T,\gamma,\Im}\big)\big]$ is investigated. Therefor, one defines (recall (3.16) as well as (C.17) and note that the following expression results by replacing the cos- and sin-terms on the right side of (C.279) by a certain expectation):

$$\begin{aligned}
\mathscr{C}^{\circ *}_{T,\mathfrak{U}_{0,1},1} := {} & \frac{4T\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)^2}{\lfloor 1/(2b) \rfloor^2 \lfloor Tb \rfloor^2} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma, u_k, s_1\right) \tau_{\mathfrak{U}_{0,1},\Im}\left(\gamma, u_k, s_2\right) \sum_{t_1,t_2=1+\mathscr{m}_\beta}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \mathscr{m}_\beta} K\Bigg(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} \\
& \cdot \left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\Bigg) K\left(\frac{t_2 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) \mathbb{E}\Big[\cos\left(\left\langle s_1, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}\left(\widetilde{u}_{k,t_1}\right)\right\rangle\right)^c_{\mathscr{m}} \\
& \cdot \sin\left(\left\langle s_2, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}\left(\widetilde{u}_{k,t_2}\right)\right\rangle\right)^c_{\mathscr{m}}\Big] \\
& \cdot \mathbb{E}\left[W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, \{\mathscr{m}_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, \{\mathscr{m}_\beta\}}\right] \cdot \mathbf{w}\left(s_1\right)\, ds_1\, \mathbf{w}\left(s_2\right)\, ds_2.
\end{aligned} \tag{C.288}$$

It holds for all sets $\mathscr{T}, \mathscr{T}_1, \mathscr{T}_2 \subseteq \mathbb{N}_0$ and all deterministic functions $f\colon \mathscr{T} \to [0,\infty)$, $g, h\colon \mathscr{T} \times \mathscr{T}_1 \times \mathscr{T}_2 \to [0,\infty)$ for which $\sum_{k\in\mathscr{T}} f(k) \sum_{t_1\in\mathscr{T}_1} \sup_{t_2\in\mathscr{T}_2} g\left(k,t_1,t_2\right) < \infty$ and $\sup_{k\in\mathscr{T}} \sup_{t_1\in\mathscr{T}_1} \sum_{t_2\in\mathscr{T}_2} h(k, t_1,t_2) < \infty$:

$$\begin{aligned}
& \sum_{k\in\mathscr{T}} f(k) \sum_{t_1\in\mathscr{T}_1} \sum_{t_2\in\mathscr{T}_2} \left(g\left(k,t_1,t_2\right) \cdot h\left(k,t_1,t_2\right)\right) \\
& \leqslant \left(\sum_{k\in\mathscr{T}} f(k) \sum_{t_1\in\mathscr{T}} \sup_{t_2\in\mathscr{T}} g\left(k,t_1,t_2\right)\right) \cdot \left(\sup_{k\in\mathscr{T}} \sup_{t_1\in\mathscr{T}} \sum_{t_2\in\mathscr{T}} h\left(k,t_1,t_2\right)\right).
\end{aligned} \tag{C.289}$$

In conclusion, (C.279), (C.289), (C.66), Assumption 3.1 [**WEI**.**1**], (C.72) and Assumption 3.15 [$\mathbf{W}^*$] (i) provide (recall (C.288), (3.16) as well as (C.17)):

$$\begin{aligned}
& \left|\mathbb{E}\left[\operatorname{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\Re}, \mathbb{D}^{\circ *}_{T,\gamma,\Im}\right)\right] - \mathscr{C}^{\circ *}_{T,\mathfrak{U}_{0,1},1}\right| \\
& \leqslant \frac{C}{T} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left|\tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma, u_k, s_1\right) \tau_{\mathfrak{U}_{0,1},\Im}\left(\gamma, u_k, s_2\right)\right| \sum_{t_1=1+\mathscr{m}_\beta}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \mathscr{m}_\beta} \sup_{t_2=1+\mathscr{m}_\beta,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \mathscr{m}_\beta} \\
& \Bigg\{K\Bigg(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\Bigg) K\left(\frac{t_2 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) \Bigg|\mathbb{E}\Big[\Big(\cos\left(\left\langle s_1, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, T}\right\rangle\right)^c \\
& - \cos\left(\left\langle s_1, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}\left(\widetilde{u}_{k,t_1}\right)\right\rangle\right)^c_{\mathscr{m}}\Big) \sin\left(\left\langle s_2, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, T}\right\rangle\right)^c\Big] \\
& + \mathbb{E}\Big[\cos\left(\left\langle s_1, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}\left(\widetilde{u}_{k,t_1}\right)\right\rangle\right)^c_{\mathscr{m}} \Big(\sin\left(\left\langle s_2, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, T}\right\rangle\right)^c \\
& - \sin\left(\left\langle s_2, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}\left(\widetilde{u}_{k,t_2}\right)\right\rangle\right)^c_{\mathscr{m}}\Big)\Big]\Bigg|\Bigg\} \cdot \mathbf{w}\left(s_1\right)\, ds_1\, \mathbf{w}\left(s_2\right)\, ds_2 \\
& \cdot \left(\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t_1=1,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor} \sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \left|\mathbb{E}\left[W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, \{\mathscr{m}_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, \{\mathscr{m}_\beta\}}\right]\right|\right) \\
& \leqslant \frac{C \lfloor 1/(2b) \rfloor \lfloor T_{\mathfrak{U}} b \rfloor \beta}{T T b^2} = o(1).
\end{aligned} \tag{C.290}$$

Further, one defines (see (3.16) as well as Assumption 3.15 [$\mathbf{W}^*$] (iii) and note that the following expression results from (C.288) by replacing $\mathbb{E}\left[W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, \{\mathscr{m}_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, \{\mathscr{m}_\beta\}}\right]$ by

$K^*\left((t_1-t_2)/\beta\right)\big)$:

$$\mathscr{C}^{\circ *}_{T,\mathfrak{U}_{0,1},2} := \frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)^2}{\lfloor 1/(2b)\rfloor^2 \lfloor Tb\rfloor^2} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma,u_k,s_1\right)\tau_{\mathfrak{U}_{0,1},\Im}\left(\gamma,u_k,s_2\right) \sum_{t_1,t_2=1+m_\beta}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-m_\beta} K\bigg(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}$$
$$\cdot\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\bigg) K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\mathbb{E}\Big[\cos\Big(\Big\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\widetilde{u}_{k,t_1}\right)\Big\rangle\Big)^c_{m}$$
$$\cdot\sin\Big(\Big\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\widetilde{u}_{k,t_2}\right)\Big\rangle\Big)^c_{m}\Big]\cdot K^*\left(\frac{t_1-t_2}{\beta}\right)\cdot \mathbf{w}\left(s_1\right)\,ds_1\,\mathbf{w}\left(s_2\right)\,ds_2. \quad \text{(C.291)}$$

Assumption 3.15 $[\mathbf{W}^*]$ (iii) provides $K^*\left((t_1-t_2)/\beta\right) = \mathbb{E}\big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\big]\ \forall k\in\{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $t_1,t_2\in\mathbb{Z}$. Hence, it follows from Lemma C.8 (iii), Assumption 3.15 $[\mathbf{W}^*]$ (iii) (the latter ensures $\|W^*_{t,\{m_\beta\}}\|_2 \leqslant \|W^*_t\|_2 \leqslant C\ \forall t\in\mathbb{Z}$ (see Definition A.1 (i))), Assumption 3.1 $[\mathbf{WEI.1}]$, Remark A.2 (i) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (recall (C.288), (C.291), (3.16), Definition A.1 (i) as well as (C.17)):

$$\left|\mathscr{C}^{\circ *}_{T,\mathfrak{U}_{0,1},1}-\mathscr{C}^{\circ *}_{T,\mathfrak{U}_{0,1},2}\right|$$
$$= \frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)^2}{\lfloor 1/(2b)\rfloor^2 \lfloor Tb\rfloor^2} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \left|\tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma,u_k,s_1\right)\tau_{\mathfrak{U}_{0,1},\Im}\left(\gamma,u_k,s_2\right)\right| \sum_{\substack{t_1,t_2=1+m_\beta\\|t_1-t_2|\leqslant m}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-m_\beta} K\bigg(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}$$
$$\cdot\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\bigg) K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\bigg|\mathbb{E}\Big[\cos\Big(\Big\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\widetilde{u}_{k,t_1}\right)\Big\rangle\Big)^c_{m}$$
$$\cdot\sin\Big(\Big\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\widetilde{u}_{k,t_2}\right)\Big\rangle\Big)^c_{m}\Big]\bigg|$$
$$\cdot\bigg|\mathbb{E}\Big[\Big(W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{m_\beta\}}-W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\Big)W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{m_\beta\}}\Big]$$
$$+\mathbb{E}\Big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\Big(W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{m_\beta\}}-W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\Big)\Big]\bigg|\cdot \mathbf{w}\left(s_1\right)\,ds_1\,\mathbf{w}\left(s_2\right)\,ds_2$$
$$\leqslant \frac{C}{T}\lfloor 1/(2b)\rfloor\lfloor T_{\mathfrak{U}}b\rfloor\, m\frac{1}{Tb^2}$$
$$= o(1). \quad \text{(C.292)}$$

Further, one defines $\mathscr{T}_{T,m,m_\beta} := \{1+m,\ldots,m_\beta\}\cup\{2\lfloor T_{\mathfrak{U}}b\rfloor-m_\beta,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-m\}$. The Assumptions 3.15 $[\mathbf{W}^*]$ (iii) and 3.1 $[\mathbf{WEI.1}]$, $\#\mathscr{T}_{T,m,m_\beta}\leqslant C+C\beta m$ (which holds due to Lemma C.8 (ii)), Assumption 3.15 $[\mathbf{W}^*]$ (i) (the latter ensures $\beta = o(1/b)$) as well as Remark A.2 (i) imply (see (3.16)):

$$\frac{4T\left(\mathfrak{U}_1-\mathfrak{U}_0\right)^2}{\lfloor 1/(2b)\rfloor^2 \lfloor Tb\rfloor^2} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \left|\tau_{\mathfrak{U}_{0,1},\Re}\left(\gamma,u_k,s_1\right)\tau_{\mathfrak{U}_{0,1},\Im}\left(\gamma,u_k,s_2\right)\right| \sum_{\substack{t_1,t_2\in\mathscr{T}_{T,m,m_\beta}\\|t_1-t_2|\leqslant m}} K\bigg(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}$$
$$\cdot\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\bigg) K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\bigg|\mathbb{E}\Big[\cos\Big(\Big\langle s_1,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\widetilde{u}_{k,t_1}\right)\Big\rangle\Big)^c_{m}$$
$$\cdot\sin\Big(\Big\langle s_2,\widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\widetilde{u}_{k,t_2}\right)\Big\rangle\Big)^c_{m}\Big]\bigg|\cdot\left|K^*\left(\frac{t_1-t_2}{\beta}\right)\right|\cdot \mathbf{w}\left(s_1\right)\,ds_1\,\mathbf{w}\left(s_2\right)\,ds_2$$
$$\leqslant \frac{CT\lfloor 1/(2b)\rfloor\#\mathscr{T}_{T,m,m_\beta}\, m}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^2}$$
$$= o(1). \quad \text{(C.293)}$$

The condition $|t_1-t_2|\leqslant m$ contained in the double sum over t_1,t_2 on the left side of (C.293) can be omitted because the opposite condition generates addends which equal zero according to Definition A.1 (i). Thus, it follows for $\mathscr{G}_T(x) := K^*(x/\beta)\ \forall x\in\mathbb{Z}$ from (C.293) (recall (C.291) as well as (C.216) and observe that, on one hand, Lemma C.8 (ii) ensures $m\leqslant m_\beta$ and, on the other hand, if $m = m_\beta$,

$\mathscr{C}^{\circ *}_{T,\mathfrak{U}_{0,1},2} = \mathrm{Cov}^{[\mathscr{G}_T]}_{\mathfrak{R},\mathfrak{I},T}(\gamma,\gamma)\big)$:

$$\left|\mathscr{C}^{\circ *}_{T,\mathfrak{U}_{0,1},2} - \mathrm{Cov}^{[\mathscr{G}_T]}_{\mathfrak{R},\mathfrak{I},T}(\gamma,\gamma)\right| = o(1). \tag{C.294}$$

The fact that $\mathrm{Cov}^{[\mathscr{G}_T]}_{\mathfrak{R},\mathfrak{I}}(\gamma,\gamma)$ is deterministic (see (C.216)), (C.287), (C.290), (C.292), (C.294) and Lemma C.6 with $\mathrm{R}_1 = \mathfrak{R}$ as well as $\mathrm{R}_2 = \mathfrak{I}$ $\big($whereby Assumption 3.15 $[\mathbf{W}^*]$ (iii) yields that $\mathscr{G}_T\colon \mathbb{Z} \to \mathbb{R},\ x \mapsto K^*(x/\beta)$ fulfils (C.215)$\big)$ imply:

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathrm{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\mathfrak{R}}, \mathbb{D}^{\circ *}_{T,\gamma,\mathfrak{I}}\right) - \mathrm{Cov}^{[\mathscr{G}_T]}_{\mathfrak{R},\mathfrak{I}}(\gamma,\gamma)\right)^2\right] \\
&= \mathrm{Var}\left(\mathrm{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\mathfrak{R}}, \mathbb{D}^{\circ *}_{T,\gamma,\mathfrak{I}}\right) - \mathrm{Cov}^{[\mathscr{G}_T]}_{\mathfrak{R},\mathfrak{I}}(\gamma,\gamma)\right) + \left(\mathbb{E}\left[\mathrm{Cov}^*\left(\mathbb{D}^{\circ *}_{T,\gamma,\mathfrak{R}}, \mathbb{D}^{\circ *}_{T,\gamma,\mathfrak{I}}\right)\right] - \mathrm{Cov}^{[\mathscr{G}_T]}_{\mathfrak{R},\mathfrak{I}}(\gamma,\gamma)\right)^2 \\
&= o(1).
\end{aligned} \tag{C.295}$$

Assumption 3.15 $[\mathbf{W}^*]$ (iii), Lemma 3.12 and Assumption 3.1 $[\mathbf{WEI.1}]$ show (note (3.16)):

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \sum_{t=-\infty}^{\infty} \Big| \tau_{\mathfrak{U}_{0,1},\mathfrak{R}}(\gamma,u,s_1)\, \tau_{\mathfrak{U}_{0,1},\mathfrak{I}}(\gamma,u,s_2)\, K^*\left(\frac{t}{\beta}\right) \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u)\right\rangle\right),\right. \\
&\left.\sin\left(\left\langle s_2, \widetilde{X}_t(u)\right\rangle\right)\right)\Big|\, du\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 \leqslant C.
\end{aligned} \tag{C.296}$$

Moreover, Assumption 3.15 $[\mathbf{W}^*]$ (iii), Lemma B.4 (vi) and Assumption 3.1 $[\mathbf{WEI.1}]$ provide for all $t \in \mathbb{Z}$ (recall (3.16)):

$$\begin{aligned}
&\Bigg| K^*\left(\frac{t}{\beta}\right) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \tau_{\mathfrak{U}_{0,1},\mathfrak{R}}(\gamma,u,s_1)\, \tau_{\mathfrak{U}_{0,1},\mathfrak{I}}(\gamma,u,s_2)\, \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u)\right\rangle\right),\right. \\
&\left.\sin\left(\left\langle s_2, \widetilde{X}_t(u)\right\rangle\right)\right)\, du\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 \Bigg| \\
&\leqslant C \sum_{l=|t|}^{\infty} \Delta_l \mathbb{1}_{\{t\neq 0\}} + C\mathbb{1}_{\{t=0\}}.
\end{aligned} \tag{C.297}$$

The Fubini–Tonelli theorem in combination with (C.296), Lebesgue's dominated convergence theorem together with (C.297), (B.45) and (C.15) $\big($the latter two show $\sum_{t=-\infty}^{\infty}\left(\sum_{l=|t|}^{\infty} \Delta_l \mathbb{1}_{\{t\neq 0\}} + \mathbb{1}_{\{t=0\}}\right) \leqslant C\big)$ as well as Assumption 3.15 $[\mathbf{W}^*]$ (i) and (iii) $\big($which yield $\lim_{T\to\infty} K^*(x/\beta) = 1\ \forall\, x \in \mathbb{Z}\big)$ imply $\big($see (C.216) and recall $\mathscr{G}_T(x) := K^*(x/\beta)\ \forall\, x \in \mathbb{Z}$ as well as (3.17)$\big)$:

$$\begin{aligned}
\lim_{T\to\infty} \mathrm{Cov}^{[\mathscr{G}_T]}_{\mathfrak{R},\mathfrak{I}}(\gamma,\gamma) &= 8(\mathfrak{U}_1 - \mathfrak{U}_0) \int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\, dz \sum_{t=-\infty}^{\infty} \lim_{T\to\infty} K^*\left(\frac{t}{\beta}\right) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \tau_{\mathfrak{U}_{0,1},\mathfrak{R}}(\gamma,u,s_1) \\
&\quad \cdot \tau_{\mathfrak{U}_{0,1},\mathfrak{I}}(\gamma,u,s_2)\, \mathrm{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_t(u)\right\rangle\right)\right) du\, \mathbf{w}(s_1)\, ds_1 \\
&\quad \cdot \mathbf{w}(s_2)\, ds_2 \\
&= 8(\mathfrak{U}_1 - \mathfrak{U}_0) \int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\, dz \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \tau_{\mathfrak{U}_{0,1},\mathfrak{R}}(\gamma,u,s_1)\, \tau_{\mathfrak{U}_{0,1},\mathfrak{I}}(\gamma,u,s_2) \\
&\quad \cdot \sigma_{\infty,\mathfrak{R},\mathfrak{I}}(u,s_1,s_2)\, du\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2.
\end{aligned} \tag{C.298}$$

Since $\gamma \in \mathbb{R}^{1\times 2}$ was chosen arbitrary but fixed at the beginning of this proof, Lemma C.13 is an implication of (C.277), (C.295) and (C.298) as well as similar arguments (note (3.18)). □

Lemma C.14. *Let the Assumptions 2.4* $[\mathbf{DM.1}]$*, 3.1* $[\mathbf{WEI.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *be fulfilled. Moreover, define for all* $\mathrm{R} \in \{\mathfrak{R}, \mathfrak{I}\}$ *(recall that* $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ *according to Definition 3.3 (i) and the Definitions*

2.11, 2.6 as well as 3.8 (i)):

$$\widehat{\mathbb{T}}_{T,\mathrm{R}}^{[1]} := \widehat{\mathbb{T}}_{T,\mathfrak{U}_{0,1},\mathrm{R}}^{[1]} := \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathrm{R}\left\{\widehat{\varphi}\left(u_k, s\right) - \varphi\left(u_k, s\right)\right\}^2 \mathbf{w}(s)\, ds. \tag{C.299}$$

Then, it holds under the null hypothesis $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\mathrm{distr}}$ (see (3.49)) for $T \to \infty$:

$$T\sqrt{b}\,\mathbb{E}\left[\left|\widehat{\mathbb{D}}_T - \left(\widehat{\mathbb{T}}_{T,\Re}^{[1]} + \widehat{\mathbb{T}}_{T,\Im}^{[1]}\right)\right|\right] = o(1).$$

Proof. Throughout this proof, it is supposed that T is large enough to ensure (note (C.17)):

$$\lfloor T_{\mathfrak{U}} b \rfloor \geqslant 1 \text{ and } \lfloor 1/(2b) \rfloor \geqslant 6, \tag{C.300}$$

which is valid for sufficiently large T due to Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii).
At first, define for all $\mathrm{R} \in \{\Re, \Im\}$:

$$\widehat{\mathbb{D}}_{T,\mathrm{R}} := \widehat{\mathbb{D}}_{T,\mathfrak{U}_{0,1},\mathrm{R}} := \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{2\lfloor 1/(2b) \rfloor^2} \sum_{k_1,k_2=1}^{\lfloor 1/(2b) \rfloor} \mathrm{R}\left\{\widehat{\varphi}\left(u_{k_1}, s\right) - \widehat{\varphi}\left(u_{k_2}, s\right)\right\}^2 \mathbf{w}(s)\, ds. \tag{C.301}$$

Since $u_k \in \mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]\ \forall\, k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$ (recall Definition 3.8 (i)), one obtains almost surely under $\mathcal{H}_{0,\mathfrak{U}_{0,1}}^{\mathrm{distr}}$ (see (3.49), Definition 2.6 and (C.299)):

$$\begin{aligned}
\widehat{\mathbb{D}}_{T,\Re} &= \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{2\lfloor 1/(2b) \rfloor^2} \sum_{k_1,k_2=1}^{\lfloor 1/(2b) \rfloor} \left(\Re\left\{\widehat{\varphi}\left(u_{k_1}, s\right) - \varphi\left(u_{k_1}, s\right)\right\} - \Re\left\{\widehat{\varphi}\left(u_{k_2}, s\right) - \varphi\left(u_{k_2}, s\right)\right\}\right)^2 \mathbf{w}(s)\, ds \\
&= \widehat{\mathbb{T}}_{T,\Re}^{[1]} - (\mathfrak{U}_1 - \mathfrak{U}_0) \int_{\mathbb{R}^d} \left(\frac{1}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\widehat{\varphi}\left(u_k, s\right) - \varphi\left(u_k, s\right)\right\}\right)^2 \mathbf{w}(s)\, ds.
\end{aligned} \tag{C.302}$$

Moreover, it holds for all $s \in \mathbb{R}^d$ due to (C.25) with $M = 2$:

$$\begin{aligned}
&T\sqrt{b}\,\mathbb{E}\left[\left(\frac{1}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\widehat{\varphi}\left(u_k, s\right) - \varphi\left(u_k, s\right)\right\}\right)^2\right] \\
&\leqslant 2T\sqrt{b}\,\mathbb{E}\left[\left(\frac{1}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\widehat{\varphi}\left(u_k, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_k, s\right)\right]\right\}\right)^2\right] \\
&+ 2T\sqrt{b}\left(\frac{1}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Re\left\{\mathbb{E}\left[\widehat{\varphi}\left(u_k, s\right)\right] - \varphi\left(u_k, s\right)\right\}\right)^2 \\
&=: 2\,\mathrm{I}_{T,1}(s) + 2\,\mathrm{I}_{T,2}(s).
\end{aligned} \tag{C.303}$$

Further, one observes for all $s \in \mathbb{R}^d$:

$$\begin{aligned}
\mathrm{I}_{T,1}(s) \leqslant\ & \frac{T\sqrt{b}}{\lfloor 1/(2b) \rfloor^2} \sum_{k_1 \in \{1, \lfloor 1/(2b) \rfloor\}} \sum_{k_2=1}^{\lfloor 1/(2b) \rfloor} \left|\mathbb{E}\left[\Re\{\widehat{\varphi}\left(u_{k_1}, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_{k_1}, s\right)\right]\}\,\Re\{\widehat{\varphi}\left(u_{k_2}, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_{k_2}, s\right)\right]\}\right]\right| \\
&+ \frac{T\sqrt{b}}{\lfloor 1/(2b) \rfloor^2} \sum_{k_1=2}^{\lfloor 1/(2b) \rfloor - 1} \sum_{k_2 \in \{1, \lfloor 1/(2b) \rfloor\}} \left|\mathbb{E}\left[\Re\left\{\widehat{\varphi}\left(u_{k_1}, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_{k_1}, s\right)\right]\right\}\Re\left\{\widehat{\varphi}\left(u_{k_2}, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_{k_2}, s\right)\right]\right\}\right]\right| \\
&+ \frac{T\sqrt{b}}{\lfloor 1/(2b) \rfloor^2} \sum_{\substack{k_1,k_2=2 \\ |k_1-k_2| \leqslant 2}}^{\lfloor 1/(2b) \rfloor - 1} \left|\mathbb{E}\left[\Re\left\{\widehat{\varphi}\left(u_{k_1}, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_{k_1}, s\right)\right]\right\}\Re\left\{\widehat{\varphi}\left(u_{k_2}, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_{k_2}, s\right)\right]\right\}\right]\right|
\end{aligned}$$

$$+ \frac{T\sqrt{b}}{\lfloor 1/(2b) \rfloor^2} \sum_{\substack{k_1,k_2=2 \\ |k_1-k_2|\geqslant 3}}^{\lfloor 1/(2b)\rfloor -1} \left| \mathbb{E}\left[\Re\left\{ \widehat{\varphi}\left(u_{k_1}, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_{k_1}, s\right)\right]\right\} \Re\left\{ \widehat{\varphi}\left(u_{k_2}, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_{k_2}, s\right)\right]\right\}\right]\right|$$

$$=: \mathrm{I}_{T,1.1}(s) + \mathrm{I}_{T,1.2}(s) + \mathrm{I}_{T,1.3}(s) + \mathrm{I}_{T,1.4}(s), \tag{C.304}$$

whereby the Cauchy-Schwarz inequality and Proposition 2.14 show for all $s \in \mathbb{R}^d$:

$$\mathrm{I}_{T,1.1}(s) + \mathrm{I}_{T,1.2}(s) + \mathrm{I}_{T,1.3}(s) \leqslant C \frac{T\sqrt{b}}{\lfloor 1/(2b) \rfloor} \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \operatorname{Var}\left(\Re\left\{\widehat{\varphi}\left(u_k, s\right)\right\}\right) \leqslant C\sqrt{b}\left(|s|_1 + 1\right). \tag{C.305}$$

It follows for all $k \in \{2, \ldots, \lfloor 1/(2b) \rfloor - 1\}$ from $\lfloor x+y \rfloor \geqslant \lfloor x \rfloor + \lfloor y \rfloor$, $\lfloor xy \rfloor \geqslant \lfloor x \rfloor \lfloor y \rfloor \ \forall\, x, y \geqslant 0$ and (C.300) (recall Definition 3.8 (i) as well as (C.17)):

$$\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor - 1 \geqslant \left\lfloor \mathfrak{U}_0 T + \frac{3T_{\mathfrak{U}}}{2\left(1/(2b)\right)} \right\rfloor - \lfloor T_{\mathfrak{U}} b \rfloor - 1 \geqslant \lfloor \mathfrak{U}_0 T \rfloor + \lfloor 3 \rfloor \lfloor T_{\mathfrak{U}} b \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor - 1 \geqslant 1 \quad \text{and}$$
$$\lfloor u_k T \rfloor + \lfloor T_{\mathfrak{U}} b \rfloor + 2 \leqslant \mathfrak{U}_0 T + \frac{\left(\lfloor 1/(2b) \rfloor - 3/2\right) T_{\mathfrak{U}}}{\lfloor 1/(2b) \rfloor} + T_{\mathfrak{U}} b + 2 \leqslant T\mathfrak{U}_1 - \frac{3T_{\mathfrak{U}}}{2\left(1/(2b)\right)} + T_{\mathfrak{U}} b + 2 \leqslant T. \tag{C.306}$$

Further, one obtains for all $k_1, k_2 \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$ with $k_1 \geqslant k_2 + 3$ as well as all $t_1, t_2 \in \{0, \ldots, 2\lfloor T_{\mathfrak{U}} b \rfloor + 3\}$ from $\lfloor x+y \rfloor \geqslant \lfloor x \rfloor + \lfloor y \rfloor$, $\lfloor xy \rfloor \geqslant \lfloor x \rfloor \lfloor y \rfloor \ \forall\, x, y \geqslant 0$ and (C.300) (see Definition 3.8 (i) as well as (C.17)):

$$\lfloor u_{k_1} T \rfloor + t_1 \geqslant \left\lfloor \mathfrak{U}_0 T + \frac{k_1 - 3 - \frac{1}{2}}{\lfloor 1/(2b) \rfloor} T_{\mathfrak{U}} \right\rfloor + \left\lfloor \frac{3T_{\mathfrak{U}}}{1/(2b)} \right\rfloor \geqslant \lfloor u_{k_2} T \rfloor + \lfloor 6 \rfloor \lfloor T_{\mathfrak{U}} b \rfloor \geqslant \lfloor u_{k_2} T \rfloor + t_2 + \lfloor T_{\mathfrak{U}} b \rfloor. \tag{C.307}$$

Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) (which implies $K(z) = 0$ for all $z \in \mathbb{R}$ with $|z| \geqslant \mathfrak{U}_1 - \mathfrak{U}_0$) provides for all $s \in \mathbb{R}^d$ the following inequality, whereby (C.306) ensures that the right side of this inequality just takes $X_{t,T}$ into account which fulfil $t \in \{1, \ldots, T\}$, such that the right side of this inequality is well-defined (recall (C.304), Definition 2.11 and (C.17)):

$$\mathrm{I}_{T,1.4}(s) \leqslant \frac{C\sqrt{b}}{T} \sum_{\substack{k_1,k_2=2 \\ |k_1-k_2|\geqslant 3}}^{\lfloor 1/(2b)\rfloor -1} \sum_{t_1=\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor - 1}^{\lfloor u_{k_1} T \rfloor + \lfloor T_{\mathfrak{U}} b \rfloor + 2} \sum_{t_2=\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor - 1}^{\lfloor u_{k_2} T \rfloor + \lfloor T_{\mathfrak{U}} b \rfloor + 2} K\left(\frac{\frac{t_1}{T} - u_{k_1}}{b}\right) K\left(\frac{\frac{t_2}{T} - u_{k_2}}{b}\right)$$
$$\cdot \left|\operatorname{Cov}\left(\cos\left(\langle s, X_{t_1,T}\rangle\right), \cos\left(\langle s, X_{t_2,T}\rangle\right)\right)\right|. \tag{C.308}$$

It follows for all $s \in \mathbb{R}^d$ from shifting the indices of sums, Lemma B.4 (v) together with (C.307) and (C.300), $l^2 \geqslant \lfloor T_{\mathfrak{U}} b \rfloor^2$ for all $l \geqslant \lfloor T_{\mathfrak{U}} b \rfloor$ as well as Assumption 2.4 $[\mathbf{DM.1}]$ (note (C.17) and that, according to (C.300), the sums with respect to k_1 and k_2 contained in the following inequalities do not sum over the empty set):

$$\frac{C\sqrt{b}}{T} \sum_{\substack{k_1,k_2=2 \\ k_1\geqslant k_2+3}}^{\lfloor 1/(2b)\rfloor -1} \sum_{t_1=\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor - 1}^{\lfloor u_{k_1} T \rfloor + \lfloor T_{\mathfrak{U}} b \rfloor + 2} \sum_{t_2=\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor - 1}^{\lfloor u_{k_2} T \rfloor + \lfloor T_{\mathfrak{U}} b \rfloor + 2} K\left(\frac{\frac{t_1}{T} - u_{k_1}}{b}\right) K\left(\frac{\frac{t_2}{T} - u_{k_2}}{b}\right)$$
$$\cdot \left|\operatorname{Cov}\left(\cos\left(\langle s, X_{t_1,T}\rangle\right), \cos\left(\langle s, X_{t_2,T}\rangle\right)\right)\right|$$
$$\leqslant \frac{C\sqrt{b}}{T} \sum_{\substack{k_1,k_2=2 \\ k_1\geqslant k_2+3}}^{\lfloor 1/(2b)\rfloor -1} \sum_{t_1=0}^{2\lfloor T_{\mathfrak{U}} b \rfloor + 3} \sum_{t_2=0}^{2\lfloor T_{\mathfrak{U}} b \rfloor + 3} \left|\operatorname{Cov}\left(\cos\left(\left\langle s, X_{\lfloor u_{k_1} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor - 1 + t_1, T}\right\rangle\right),\right.\right.$$
$$\left.\left. \cdot \ \cos\left(\left\langle s, X_{\lfloor u_{k_2} T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor - 1 + t_2, T}\right\rangle\right)\right)\right|$$
$$\leqslant \frac{C\sqrt{b}}{T} \sum_{\substack{k_1,k_2=2 \\ k_1\geqslant k_2+3}}^{\lfloor 1/(2b)\rfloor -1} \sum_{t_1,t_2=0}^{2\lfloor T_{\mathfrak{U}} b \rfloor + 3} \sum_{l=\lfloor u_{k_1} T \rfloor + t_1 - \lfloor u_{k_2} T \rfloor - t_2}^{\infty} \Delta_l |s|_1$$

$$\leqslant \frac{C\sqrt{b}}{T} \sum_{\substack{k_1,k_2=2 \\ k_1 \geqslant k_2+3}}^{\lfloor 1/(2b) \rfloor -1} \sum_{t_1,t_2=0}^{2\lfloor T_{\mathfrak{U}} b \rfloor +3} \sum_{l=\lfloor T_{\mathfrak{U}} b \rfloor}^{\infty} \Delta_l |s|_1$$

$$\leqslant CT\sqrt{b} \sum_{l=\lfloor T_{\mathfrak{U}} b \rfloor}^{\infty} \Delta_l \frac{l^2}{\lfloor T_{\mathfrak{U}} b \rfloor^2} |s|_1$$

$$\leqslant \frac{C}{Tb^{\frac{3}{2}}} |s|_1. \tag{C.309}$$

One obtains for all $s \in \mathbb{R}^d$ from (C.308), (C.309) and similar arguments:

$$\mathrm{I}_{T,1.4}(s) \leqslant \frac{C}{Tb^{\frac{3}{2}}} |s|_1. \tag{C.310}$$

Proposition 2.12 together with (3.11) and (C.25) with $M = 2$ show for all $s \in \mathbb{R}^d$ (recall (C.303)):

$$\mathrm{I}_{T,2}(s) \leqslant T\sqrt{b} \left(\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \left| \Re \left\{ \mathbb{E}\left[\widehat{\varphi}\left(u_k, s\right) \right] - \varphi\left(u_k, s\right) \right\} \right| \right)^2 \leqslant C\sqrt{b} \left(Tb^{2+2\delta} + \frac{1}{Tb^2} \right) \left(|s|_1^{2+2\delta} + 1 \right). \tag{C.311}$$

In conclusion, (C.302), (C.303), (C.304), (C.305), (C.310), (C.311) and the Assumptions 3.1 [**WEI**.**1**] as well as 2.8 [**K**&**b**.**1**] (ii) imply:

$$T\sqrt{b}\, \mathbb{E}\left[\left| \widehat{\mathbb{D}}_{T,\Re} - \widehat{\mathbb{T}}_{T,\Re}^{[1]} \right| \right] = o(1). \tag{C.312}$$

Since Proposition 3.11 (ii) provides $\widehat{\mathbb{D}}_T = \widehat{\mathbb{D}}_{T,\Re} + \widehat{\mathbb{D}}_{T,\Im}$ (see (C.301)), Lemma C.14 follows from (C.312) and similar arguments. □

Lemma C.15. *Suppose that the Assumptions 2.4* [**DM**.**1**]*, 3.1* [**WEI**.**1**] *and 2.8* [**K**&**b**.**1**] *hold. Moreover, define for all* $\mathrm{R} \in \{\Re, \Im\}$ *(recall the Definitions 3.3 (i) and 2.11):*

$$\widehat{\mathbb{T}}_{T,\mathrm{R}}^{[2]} := \widehat{\mathbb{T}}_{T,\mathfrak{U}_{0,1},\mathrm{R}}^{[2]} := \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathrm{R}\left\{ \widehat{\varphi}\left(u_k, s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_k, s\right) \right] \right\}^2 \mathbf{w}(s)\, ds. \tag{C.313}$$

Then, it holds for $T \to \infty$ *(note (C.299)):*

$$T\sqrt{b}\, \mathbb{E}\left[\left| \widehat{\mathbb{T}}_{T,\Re}^{[1]} + \widehat{\mathbb{T}}_{T,\Im}^{[1]} - \left(\widehat{\mathbb{T}}_{T,\Re}^{[2]} + \widehat{\mathbb{T}}_{T,\Im}^{[2]} \right) \right| \right] = o(1).$$

Proof. At first, one obtains for all $u \in [0,1]$, $s \in \mathbb{R}^d$ as well as for arbitrary, not necessarily deterministic functions $f, g \colon [0,1] \times \mathbb{R}^d \to \mathbb{C}$ that live on the same probability space by using $\left|x^2\right| = |x|^2\ \forall\, x \in \mathbb{C}$:

$$\begin{aligned} \left| |f(u,s)|^2 - |g(u,s)|^2 \right| &\leqslant \left| (f(u,s) - g(u,s))^2 + 2\left(f(u,s) - g(u,s) \right) g(u,s) \right| \\ &\leqslant |f(u,s) - g(u,s)|^2 + 2\left| f(u,s) - g(u,s) \right| |g(u,s)|. \end{aligned} \tag{C.314}$$

If each of the expressions contained in (C.315) given below is well-defined, one will obtain from (C.314) as well as the Cauchy-Schwarz inequalities for expectations, sums and integrals (see Definition 3.8 (i)):

$$\int_{\mathbb{R}^d} \frac{1}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left| \left| f\left(u_k, s\right) \right|^2 - \left| g\left(u_k, s\right) \right|^2 \right| \right] \mathbf{w}(s)\, ds$$

$$\leqslant \int_{\mathbb{R}^d} \frac{1}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left| f\left(u_k, s\right) - g\left(u_k, s\right) \right|^2 \right] \mathbf{w}(s)\, ds$$

$$+ 2\sqrt{\int_{\mathbb{R}^d} \frac{1}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left| f\left(u_k, s\right) - g\left(u_k, s\right) \right|^2 \right] \mathbf{w}(s)\, ds}\sqrt{\int_{\mathbb{R}^d} \frac{1}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left| g\left(u_k, s\right) \right|^2 \right] \mathbf{w}(s)\, ds}. \tag{C.315}$$

Further, it follows for all $s \in \mathbb{R}^d$ similarly to the second inequality of (C.311) and by using Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii):

$$T\sqrt{b} \sup_{k=1,\ldots,[1/(2b)]} \left|-\varphi(u_k,s) + \mathbb{E}\left[\widehat{\varphi}(u_k,s)\right]\right|^2 = o\left(\sqrt{b}\right)\left(|s|_1^{2+2\delta}+1\right), \tag{C.316}$$

whereby the expression $o\left(\sqrt{b}\right)$ does not depend on $s \in \mathbb{R}^d$. Moreover, Proposition 2.14 provides for all $s \in \mathbb{R}^d$ (recall Definition 3.8 (i)):

$$T\sqrt{b} \sup_{k=1,\ldots,[1/(2b)]} \mathbb{E}\left[\left|\widehat{\varphi}(u_k,s) - \mathbb{E}\left[\widehat{\varphi}(u_k,s)\right]\right|^2\right] \leqslant \frac{C}{\sqrt{b}}\left(|s|_1+1\right). \tag{C.317}$$

In conclusion, Lemma C.15 follows from (C.315) with $f(u,s) := \widehat{\varphi}(u,s) - \varphi(u,s)$ and $g(u,s) := \widehat{\varphi}(u,s) - \mathbb{E}\left[\widehat{\varphi}(u,s)\right]\ \forall\, u \in [0,1]$, $s \in \mathbb{R}^d$, (C.316), (C.317) as well as Assumption 3.1 $[\mathbf{WEI}.\mathbf{1}]$ (see (C.299) and (C.313)). □

Lemma C.16. *Let the Assumptions 2.4* $[\mathbf{DM}.\mathbf{2}]$ *and 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *(ii) be fulfilled. Then, it holds for all* $s \in \mathbb{R}^d$, $q \geqslant 1+\delta$ *and expressions* $o(1/(Tb))$ *which do not depend on* $s \in \mathbb{R}^d$ *(recall that* $\delta \in (0,1]$ *originates from Assumption 2.2* $[\mathbf{StAp}]$ *and see also Definition A.1 (i) as well as (iv)):*

(i)

$$\begin{aligned}\sup_{t=1,\ldots,T}\left\|\left(e^{i\langle s,X_{t,T}\rangle}\right)_{\varkappa} - e^{i\langle s,X_{t,T}\rangle}\right\|_q &\leqslant 2\sup_{t=1,\ldots,T}\left\|e^{i\langle s,X_{t,T,\{\varkappa\}}\rangle} - e^{i\langle s,X_{t,T}\rangle}\right\|_q \quad \forall\, T \in \mathbb{N}\\ &= o\left(\frac{1}{Tb}\right)^{\frac{1+\delta}{\delta q}} |s|_1^{\frac{1+\delta}{q}} \quad \text{for } T \to \infty.\end{aligned} \tag{C.318}$$

(ii)

$$\begin{aligned}\sup_{u\in[0,1]}\sup_{t\in\mathbb{Z}}\left\|\left(e^{i\langle s,\tilde{X}_t(u)\rangle}\right)_{\varkappa} - e^{i\langle s,\tilde{X}_t(u)\rangle}\right\|_q &\leqslant 2\sup_{u\in[0,1]}\sup_{t\in\mathbb{Z}}\left\|e^{i\langle s,\tilde{X}_{t,\{\varkappa\}}(u)\rangle} - e^{i\langle s,\tilde{X}_t(u)\rangle}\right\|_q \quad \forall\, T \in \mathbb{N}\\ &= o\left(\frac{1}{Tb}\right)^{\frac{1+\delta}{\delta q}} |s|_1^{\frac{1+\delta}{q}} \quad \text{for } T \to \infty.\end{aligned} \tag{C.319}$$

Proof. (i) The first inequality of Lemma C.16 (i) can be proved analogously to (C.204). Further, one obtains for all $s \in \mathbb{R}^d$ similarly to (C.205) (recall Definition A.1 (i) as well as (iv)):

$$\begin{aligned}\sup_{t=1,\ldots,T}\left\|\Re\left\{e^{i\langle s,X_{t,T,\{\varkappa\}}\rangle} - e^{i\langle s,X_{t,T}\rangle}\right\}\right\|_q &\leqslant C|s|_1^{\frac{1+\delta}{q}}\left(\sum_{l=\widetilde{\varkappa}+1}^{\infty}\Delta_l\frac{l^{2/\delta}}{\varkappa^{2/\delta}}\right)^{\frac{1+\delta}{q}}\\ &\leqslant C|s|_1^{\frac{1+\delta}{q}}\left(\frac{1}{(Tb)^{1/\delta}}\left(\sum_{l=\widetilde{\varkappa}}^{\infty}\Delta_l l^{2/\delta}\right)^{1-1/2}\right)^{\frac{1+\delta}{q}}.\end{aligned} \tag{C.320}$$

The Assumptions 2.4 $[\mathbf{DM}.\mathbf{2}]$ and 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii) provide $\sum_{l=\widetilde{\varkappa}}^{\infty}\Delta_l l^{2/\delta} \to 0$ for $T \to \infty$ (see Definition A.1 (iv)). Thus, (C.320) and similar arguments finish the proof of Lemma C.16 (i).

(ii) Lemma C.16 (ii) can be shown analogously to Lemma C.16 (i).

□

Lemma C.17. *Suppose that the Assumptions 2.4* $[\mathbf{DM}.\mathbf{2}]$*, 3.1* $[\mathbf{WEI}.\mathbf{1}]$ *and 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *hold. Moreover, define for all* $u \in [0,1]$, $s \in \mathbb{R}^d$ *(recall Definition A.1 (iv), that* $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ *according to Definition 3.3 (i) and the Definitions 2.11 as well as A.1 (i)):*

$$\widehat{\varphi}_{\varkappa}(u,s) := \widehat{\varphi}_{T,\mathfrak{U}_{0,1},\varkappa}(u,s) := \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right)\left(e^{i\langle s,X_{t,T}\rangle}\right)_{\varkappa} \tag{C.321}$$

and for all $\mathrm{R} \in \{\Re, \Im\}$ *(see Definition 3.8 (i)):*

$$\widehat{\mathbb{T}}^{[2]}_{T,\mathrm{R},\varkappa} := \widehat{\mathbb{T}}^{[2]}_{T,\mathfrak{U}_{0,1},\mathrm{R},\varkappa} := \int\limits_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathrm{R}\left\{\widehat{\varphi}_{\varkappa}(u_k, s) - \mathbb{E}\left[\widehat{\varphi}_{\varkappa}(u_k, s)\right]\right\}^2 \mathbf{w}(s)\, ds. \tag{C.322}$$

Then, it holds for $T \to \infty$ *(recall (C.313)):*

$$T\sqrt{b}\,\mathbb{E}\left[\left|\widehat{\mathbb{T}}^{[2]}_{T,\Re} + \widehat{\mathbb{T}}^{[2]}_{T,\Im} - \left(\widehat{\mathbb{T}}^{[2]}_{T,\Re,\varkappa} + \widehat{\mathbb{T}}^{[2]}_{T,\Im,\varkappa}\right)\right|\right] = o(1).$$

Proof. At first, one observes for all $s \in \mathbb{R}^d$ (see (C.321), Definition 3.8 (i) as well as Definition 2.11 and recall that $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\begin{aligned}
&T\sqrt{b} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left(\Re\left\{\widehat{\varphi}_{\varkappa}(u_k, s) - \widehat{\varphi}(u_k, s)\right\}^c\right)^2\right] \\
&\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \frac{CT\sqrt{b}}{(Tb)^2} \sum_{t_1,t_2=1}^{T} K\left(\frac{\frac{t_1}{T} - u_k}{b}\right) K\left(\frac{\frac{t_2}{T} - u_k}{b}\right) \\
&\cdot \left|\mathrm{Cov}\left(\left(\cos\left(\langle s, X_{t_1,T}\rangle\right)\right)_{\varkappa} - \cos\left(\langle s, X_{t_1,T}\rangle\right), \left(\cos\left(\langle s, X_{t_2,T}\rangle\right)\right)_{\varkappa} - \cos\left(\langle s, X_{t_2,T}\rangle\right)\right)\right|.
\end{aligned} \tag{C.323}$$

It follows for all $s \in \mathbb{R}^d$ from Lemma C.16 (i) with $q = (1+\delta)/\delta$, Lemma B.4 (iii) with $q = 1+\delta$, Lemma B.1 with $\kappa_1 = 1$, (B.45) together with shifting the index of a sum and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (see the Definitions 3.8 (i), 2.1 as well as A.1 (i)):

$$\begin{aligned}
&\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \frac{CT\sqrt{b}}{(Tb)^2} \sum_{\substack{t_1,t_2=1 \\ t_1 \geqslant t_2+1}}^{T} K\left(\frac{\frac{t_1}{T} - u_k}{b}\right) K\left(\frac{\frac{t_2}{T} - u_k}{b}\right) \\
&\cdot \left|\mathrm{Cov}\left(\left(\cos\left(\langle s, X_{t_1,T}\rangle\right)\right)_{\varkappa} - \cos\left(\langle s, X_{t_1,T}\rangle\right), \left(\cos\left(\langle s, X_{t_2,T}\rangle\right)\right)_{\varkappa} - \cos\left(\langle s, X_{t_2,T}\rangle\right)\right)\right| \\
&\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \frac{CT\sqrt{b}}{(Tb)^2} \sum_{\substack{t_1,t_2=1 \\ t_1 \geqslant t_2+1}}^{T} K\left(\frac{\frac{t_2}{T} - u_k}{b}\right) \Bigg|\mathbb{E}\Bigg[\Bigg(\mathbb{E}\Big[\mathbb{E}\left[\cos\left(\langle s, X_{t_1,T}\rangle\right) \middle| \mathcal{F}_{t_1}\right] - \mathbb{E}\Big[\cos\left(\langle s, X_{t_1,T}\rangle\right) \Big| \\
&\mathcal{F}_{t_1,t_2+1}\Big] \Big| \mathcal{F}_{t_1,t_1-\varkappa}\Big] - \left(\mathbb{E}\left[\cos\left(\langle s, X_{t_1,T}\rangle\right) \middle| \mathcal{F}_{t_1}\right] - \mathbb{E}\left[\cos\left(\langle s, X_{t_1,T}\rangle\right) \middle| \mathcal{F}_{t_1,t_2+1}\right]\right)\Bigg) \\
&\cdot \left(\left(\cos\left(\langle s, X_{t_2,T}\rangle\right)\right)_{\varkappa} - \cos\left(\langle s, X_{t_2,T}\rangle\right)\right)\Bigg]\Bigg| \\
&\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \frac{CT\sqrt{b}}{(Tb)^2} \sum_{t_2=1}^{T-1} K\left(\frac{\frac{t_2}{T} - u_k}{b}\right) \left\|\left(\cos\left(\langle s, X_{t_2,T}\rangle\right)\right)_{\varkappa} - \cos\left(\langle s, X_{t_2,T}\rangle\right)\right\|_{\frac{1+\delta}{\delta}} \\
&\cdot 2 \sum_{t_1=t_2+1}^{T} \left\|\sum_{l=t_1-t_2-1}^{\infty} \left(\mathbb{E}\left[\cos\left(\langle s, X_{t_1,T}\rangle\right) \middle| \mathcal{F}_{t_1,t_1-l}\right] - \mathbb{E}\left[\cos\left(\langle s, X_{t_1,T}\rangle\right) \middle| \mathcal{F}_{t_1,t_1-l-1}\right]\right)\right\|_{1+\delta} \\
&\leqslant \frac{CT\sqrt{b}}{Tb} \left(\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \frac{1}{Tb} \sum_{t_2=1}^{T-1} K\left(\frac{\frac{t_2}{T} - u_k}{b}\right) \frac{1}{Tb} |s|_1^{\delta}\right) \sup_{t_2=1,\ldots,T-1} \sum_{t_1=t_2+1}^{T} \sum_{l=t_1-t_2-1}^{\infty} \Delta_{l+1} |s|_1 \\
&= o\left(\sqrt{b}\right) |s|_1^{1+\delta},
\end{aligned} \tag{C.324}$$

whereby the expression $o(\sqrt{b})$ does not depend on $s \in \mathbb{R}^d$. Moreover, Lemma B.1 with $\kappa_1 = 2$, Lemma C.16 (i) with $q = 2$, the inequality $(1+\delta)/\delta \geqslant 2$ (which holds because $\delta \in (0,1]$ according to Assumption 2.2 $[\mathbf{StAp}]$) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) imply for all $s \in \mathbb{R}^d$:

$$\begin{aligned}
&\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \frac{CT\sqrt{b}}{(Tb)^2} \sum_{\substack{t_1,t_2=1 \\ t_1 = t_2}}^{T} K\left(\frac{\frac{t_1}{T} - u_k}{b}\right) K\left(\frac{\frac{t_2}{T} - u_k}{b}\right) \\
&\cdot \left|\mathrm{Cov}\left(\left(\cos\left(\langle s, X_{t_1,T}\rangle\right)\right)_{\varkappa} - \cos\left(\langle s, X_{t_1,T}\rangle\right), \left(\cos\left(\langle s, X_{t_2,T}\rangle\right)\right)_{\varkappa} - \cos\left(\langle s, X_{t_2,T}\rangle\right)\right)\right| \\
&\leqslant C\,\frac{T\sqrt{b}}{Tb} \sup_{t=1,\ldots,T} \left\|\left(\cos\left(\langle s, X_{t,T}\rangle\right)\right)_{\varkappa} - \cos\left(\langle s, X_{t,T}\rangle\right)\right\|_2^2 \\
&= o\left(\sqrt{b}\right) |s|_1^{1+\delta},
\end{aligned} \tag{C.325}$$

whereby the expression $o(\sqrt{b})$ does not depend on $s \in \mathbb{R}^d$. It follows for all $s \in \mathbb{R}^d$ from (C.323), (C.324), (C.325) and similar arguments:

$$T\sqrt{b} \sup_{k=1,\dots,\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left|\widehat{\varphi}_{\mathscr{n}}(u_k,s) - \mathbb{E}\left[\widehat{\varphi}_{\mathscr{n}}(u_k,s)\right] - \left(\widehat{\varphi}\left(u_k,s\right) - \mathbb{E}\left[\widehat{\varphi}\left(u_k,s\right)\right]\right)\right|^2\right] = o\left(\sqrt{b}\right)|s|_1^{1+\delta}, \tag{C.326}$$

whereby the expression $o(\sqrt{b})$ does not depend on $s \in \mathbb{R}^d$. In conclusion, Lemma C.17 is an implication of (C.315) with $f(u,s) := \widehat{\varphi}_{\mathscr{n}}(u,s) - \mathbb{E}[\widehat{\varphi}_{\mathscr{n}}(u,s)]$ and $g(u,s) := \widehat{\varphi}(u,s) - \mathbb{E}[\widehat{\varphi}(u,s)]$ $\forall\, u \in [0,1]$, $s \in \mathbb{R}^d$, (C.326), (C.317) as well as Assumption 3.1 $[\mathbf{WEI}.\mathbf{1}]$ (note (C.313) and (C.322)). □

Lemma C.18. *Let the Assumptions 2.4* $[\mathbf{DM}.\mathbf{2}]$*, 3.1* $[\mathbf{WEI}.\mathbf{1}]$ *and 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *be fulfilled. Moreover, define for all* $k \in \{1,\dots,\lfloor 1/(2b)\rfloor\}$*,* $s \in \mathbb{R}^d$ *(recall Definition A.1 (iv), that* $\mathfrak{U}_{0,1} := [\mathfrak{U}_0,\mathfrak{U}_1]$ *according to Definition 3.3 (i), Definition 3.8 (i), (C.17) and Definition A.1 (i)):*

$$\varphi^{\circ}_{\mathscr{n}}(u_k,s) := \varphi^{\circ}_{T,\mathfrak{U}_{0,1},\mathscr{n}}(u_k,s) := \frac{1}{\lfloor Tb\rfloor}\sum_{t=1+\mathscr{n}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{n}} K\left(\frac{t-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\left(e^{i\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t}\left(\tilde{u}_{k,t}\right)\right\rangle}\right)_{\mathscr{n}} \tag{C.327}$$

as well as for all $\mathrm{R} \in \{\Re,\Im\}$*:*

$$\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R},\mathscr{n}} := \widehat{\mathbb{T}}^{[3]}_{T,\mathfrak{U}_{0,1},\mathrm{R},\mathscr{n}} := \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathrm{R}\left\{\varphi^{\circ}_{\mathscr{n}}(u_k,s) - \mathbb{E}\left[\varphi^{\circ}_{\mathscr{n}}(u_k,s)\right]\right\}^2 \mathbf{w}(s)\,ds. \tag{C.328}$$

Then, it holds for $T \to \infty$ *(see (C.322)):*

$$T\sqrt{b}\,\mathbb{E}\left[\left|\widehat{\mathbb{T}}^{[2]}_{T,\Re,\mathscr{n}} + \widehat{\mathbb{T}}^{[2]}_{T,\Im,\mathscr{n}} - \left(\widehat{\mathbb{T}}^{[3]}_{T,\Re,\mathscr{n}} + \widehat{\mathbb{T}}^{[3]}_{T,\Im,\mathscr{n}}\right)\right|\right] = o(1).$$

Remark C.19. *Since (C.18) ensures* $\tilde{u}_{k,t} \in [\mathfrak{U}_0,\mathfrak{U}_1] \subseteq [0,1]$ $\forall\, k \in \{1,\dots,\lfloor 1/(2b)\rfloor\}$*,* $t \in \{1,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor\}$*,* $\varphi^{\circ}_{\mathscr{n}}(u_k,s)$ *is well-defined for all* $k \in \{1,\dots,\lfloor 1/(2b)\rfloor\}$*,* $s \in \mathbb{R}^d$*.*

Proof of Lemma C.18. At first, one defines for all $k \in \{1,\dots,\lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^d$ (recall (C.17), Definition 3.8 (i) as well as Definition A.1 (i)):

$$\breve{\varphi}_{\mathscr{n}}(u_k,s) := \frac{1}{\lfloor Tb\rfloor}\sum_{t=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor} K\left(\frac{t-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\left(e^{i\left\langle s,X_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t,T}\right\rangle}\right)_{\mathscr{n}} \quad \text{and}$$
$$\varphi^{\circ}_{\mathscr{n},+}(u_k,s) := \frac{1}{\lfloor Tb\rfloor}\sum_{t=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor} K\left(\frac{t-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\left(e^{i\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t}\left(\tilde{u}_{k,t}\right)\right\rangle}\right)_{\mathscr{n}}. \tag{C.329}$$

Since $\widehat{\varphi}_{\mathscr{n}}$ (see (C.321)) as well as $\breve{\varphi}_{\mathscr{n}}$ are defined very similarly to $\widehat{\varphi}$ and $\breve{\varphi}$, respectively (recall Definition 2.11 as well as (C.200)), it follows for all $s \in \mathbb{R}^d$ analogously to (C.199), (C.201) and (C.202) by using (C.65) with $q = 2$ as well as Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii), whereby the latter shows $Tb/T^{(1+\delta)/2} = \sqrt{Tb^{2+2\delta}}/(Tb^2)^{\delta/2} = o(1)$, which implies $1/T^{(1+\delta)/2} = o(1/(Tb))$ (recall (C.329)):

$$\begin{aligned}
&\sup_{k=1,\dots,\lfloor 1/(2b)\rfloor}\left\|\widehat{\varphi}_{\mathscr{n}}(u_k,s) - \varphi^{\circ}_{\mathscr{n},+}(u_k,s)\right\|_2^2\\
&\leqslant \sup_{k=1,\dots,\lfloor 1/(2b)\rfloor}\left(\left\|\widehat{\varphi}_{\mathscr{n}}(u_k,s) - \breve{\varphi}_{\mathscr{n}}(u_k,s)\right\|_2 + \left\|\breve{\varphi}_{\mathscr{n}}(u_k,s) - \varphi^{\circ}_{\mathscr{n},+}(u_k,s)\right\|_2\right)^2\\
&\leqslant \frac{C}{(Tb)^2}\left(1+|s|_1^{\frac{1+\delta}{2}}\right)^2.
\end{aligned} \tag{C.330}$$

Moreover, one obtains for all $s \in \mathbb{R}^d$ and for $\mathscr{T}_{T,\mathscr{n}} := \{1,\dots,\mathscr{n}\} \cup \{2\lfloor T_{\mathfrak{U}}b\rfloor - \mathscr{n},\dots,2\lfloor T_{\mathfrak{U}}b\rfloor\}$ from (C.25) with $M = 2$, Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (i) (which ensures $K(-(\mathfrak{U}_1-\mathfrak{U}_0)) = K((\mathfrak{U}_1-\mathfrak{U}_0)) = 0$ and that K is Lipschitz continuous on $\mathbb{R}$), Lemma B.4 (viii), (B.45) and Remark A.2 (ii) (note (C.329), (C.327) as well as (C.17)):

$$
\begin{aligned}
&\sup_{k=1,\dots,\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\Re\left\{\varphi^{\circ}_{\mathscr{n},+}(u_k,s) - \mathbb{E}\left[\varphi^{\circ}_{\mathscr{n},+}(u_k,s)\right] - \left(\varphi^{\circ}_{\mathscr{n}}(u_k,s) - \mathbb{E}\left[\varphi^{\circ}_{\mathscr{n}}(u_k,s)\right]\right)\right\}^2\right] \\
&\leqslant \frac{1}{\lfloor Tb\rfloor^2} \sup_{k=1,\dots,\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=1}^{\mathscr{n}} \left(\left|K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right) - K\left(\frac{-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)\right|\right. \\
&\cdot \left.\left|K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right) - K\left(\frac{-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)\right|\right) \\
&\cdot \left|\mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}(\widetilde{u}_{k,t_1})\right\rangle\right)_{\mathscr{n}}, \cos\left(\left\langle s, \widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}(\widetilde{u}_{k,t_2})\right\rangle\right)_{\mathscr{n}}\right)\right| \\
&+ \frac{1}{\lfloor Tb\rfloor^2} \sup_{k=1,\dots,\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=2\lfloor T_{\mathfrak{U}}b\rfloor-\mathscr{n}}^{2\lfloor T_{\mathfrak{U}}b\rfloor} \left(\left|K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right) - K\left(\frac{\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)\right|\right. \\
&\cdot \left.\left|K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right) - K\left(\frac{\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)\right|\right) \\
&\cdot \left|\mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}(\widetilde{u}_{k,t_1})\right\rangle\right)_{\mathscr{n}}, \cos\left(\left\langle s, \widetilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}(\widetilde{u}_{k,t_2})\right\rangle\right)_{\mathscr{n}}\right)\right| \\
&\leqslant \frac{C}{T^2b^2}\frac{\mathscr{n}^2}{\lfloor T_{\mathfrak{U}}b\rfloor^2}\left(\sum_{t_2\in\mathscr{T}_{T,\mathscr{n}}}\sum_{t_1=t_2+1}^{\infty}\sum_{l=t_1-t_2}^{\infty}\Delta_l|s|_1 + \sum_{t_1\in\mathscr{T}_{T,\mathscr{n}}}\sum_{t_2=t_1+1}^{\infty}\sum_{l=t_2-t_1}^{\infty}\Delta_l|s|_1 + \sum_{t_1,t_2\in\mathscr{T}_{T,\mathscr{n}}}\mathbb{1}_{\{t_1=t_2\}}\right) \\
&\leqslant \frac{C}{T^{5/2}b^{5/2}}\left(|s|_1+1\right). \qquad\qquad (C.331)
\end{aligned}
$$

In conclusion, (C.25) with $M = 2$, (C.330), (C.331) and similar arguments as well as Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) provide for all $s \in \mathbb{R}^d$:

$$
\begin{aligned}
&T\sqrt{b} \sup_{k=1,\dots,\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left|\widehat{\varphi}_{\mathscr{n}}(u_k,s) - \mathbb{E}\left[\widehat{\varphi}_{\mathscr{n}}(u_k,s)\right] - \left(\varphi^{\circ}_{\mathscr{n}}(u_k,s) - \mathbb{E}\left[\varphi^{\circ}_{\mathscr{n}}(u_k,s)\right]\right)\right|^2\right] \\
&\leqslant 2 \sup_{k=1,\dots,\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left|\widehat{\varphi}_{\mathscr{n}}(u_k,s) - \mathbb{E}\left[\widehat{\varphi}_{\mathscr{n}}(u_k,s)\right] - \left(\varphi^{\circ}_{\mathscr{n},+}(u_k,s) - \mathbb{E}\left[\varphi^{\circ}_{\mathscr{n},+}(u_k,s)\right]\right)\right|^2\right] \\
&+ 2 \sup_{k=1,\dots,\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left|\varphi^{\circ}_{\mathscr{n},+}(u_k,s) - \mathbb{E}\left[\varphi^{\circ}_{\mathscr{n},+}(u_k,s)\right] - \left(\varphi^{\circ}_{\mathscr{n}}(u_k,s) - \mathbb{E}\left[\varphi^{\circ}_{\mathscr{n}}(u_k,s)\right]\right)\right|^2\right] \\
&= o\left(\sqrt{b}\right)\left(1+|s|_1^{1+\delta}\right), \qquad\qquad (C.332)
\end{aligned}
$$

whereby the expression $o(\sqrt{b})$ does not depend on $s \in \mathbb{R}^d$. Moreover, it follows for all $s \in \mathbb{R}^d$ from (C.25) with $M = 2$, (C.326) and (C.317):

$$
T\sqrt{b} \sup_{k=1,\dots,\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\left|\widehat{\varphi}_{\mathscr{n}}(u_k,s) - \mathbb{E}\left[\widehat{\varphi}_{\mathscr{n}}(u_k,s)\right]\right|^2\right] \leqslant \frac{C}{\sqrt{b}}\left(|s|_1^{1+\delta}+1\right). \qquad (C.333)
$$

Overall, (C.315) with $f(u,s) := \varphi^{\circ}_{\mathscr{n}}(u_k,s) - \mathbb{E}\left[\varphi^{\circ}_{\mathscr{n}}(u_k,s)\right]$ and $g(u,s) := \widehat{\varphi}_{\mathscr{n}}(u,s) - \mathbb{E}\left[\widehat{\varphi}_{\mathscr{n}}(u,s)\right]$ $\forall\, u \in [0,1]$, $s \in \mathbb{R}^d$, (C.332), (C.333) as well as Assumption 3.1 $[\mathbf{WEI.1}]$ show Lemma C.18 (see (C.328) and (C.322)). ☐

Lemma C.20. *Suppose that the Assumptions 2.4* $[\mathbf{DM.2}]$, *3.1* $[\mathbf{WEI.1}]$ *as well as 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *hold and assume that* $(\mathscr{G}_T)_{T\in\mathbb{N}}$ *is a sequence of deterministic functions which fulfils (C.215). Moreover, define for all* $\mathrm{R} \in \{\Re, \Im\}$, $s \in \mathbb{R}^d$ *(recall (C.17) as well as Definition A.1 (iv) and (i)):*

$$
\begin{aligned}
\widetilde{\mathbf{Bias}}^{[\mathscr{G}_T]}_{T,\mathrm{R}}(s) := \frac{T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2} &\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=1+\mathscr{n}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\mathscr{n}} \mathscr{G}_T(t_2-t_1)\,K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right) \\
&\cdot K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)\mathrm{Cov}\left(\mathrm{R}\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}(\tilde{u}_{k,t_1})\right\rangle}\right\}_{\mathscr{n}},\right. \\
&\left.\mathrm{R}\left\{e^{i\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}(\tilde{u}_{k,t_2})\right\rangle}\right\}_{\mathscr{n}}\right)\mathbb{1}_{\{|t_2-t_1|\leqslant\mathscr{n}\}} \qquad (C.334)
\end{aligned}
$$

as well as:

$$\mathbf{Bias}_{T,\mathrm{R}}^{[\mathscr{G}_T]}(s) := \frac{1}{\sqrt{b}} \int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\, dz \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \sum_{t=-\infty}^{\infty} \mathscr{G}_T(t) \mathrm{Cov}\Big(\mathrm{R}\left\{e^{i\langle s, \tilde{X}_0(u)\rangle}\right\}, \mathrm{R}\left\{e^{i\langle s, \tilde{X}_t(u)\rangle}\right\}\Big)\, du. \quad \text{(C.335)}$$

Then, one obtains for $T \to \infty$ *and all* $\mathrm{R} \in \{\Re, \Im\}$ *:*

$$\int_{\mathbb{R}^d} \left| \widetilde{\mathbf{Bias}}_{T,\mathrm{R}}^{[\mathscr{G}_T]}(s) - \mathbf{Bias}_{T,\mathrm{R}}^{[\mathscr{G}_T]}(s) \right| \mathbf{w}(s)\, ds = o(1).$$

Remark C.21. *Since (C.18) ensures that* $\widetilde{u}_{k,t} \in [\mathfrak{U}_0, \mathfrak{U}_1] \subseteq [0,1]\ \forall\, k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$*,* $t \in \{1, \ldots, 2 \cdot \lfloor T_{\mathfrak{U}} b \rfloor\}$*,* $\widetilde{\mathbf{Bias}}_{T,\mathrm{R}}^{[\mathscr{G}_T]}(s)$ *is well-defined for all* $\mathrm{R} \in \{\Re, \Im\}$*,* $s \in \mathbb{R}^d$*. Further,* $\mathbf{Bias}_{T,\mathrm{R}}^{[\mathscr{G}_T]}(s)$ *is well-defined due to (C.215), Lemma 3.12 and Assumption 3.1* $[\mathbf{WEI.1}]$.

Proof. Throughout this proof, T is supposed to be large enough to ensure (see (C.17) and Definition A.1 (iv)):

$$2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - 2\mathscr{n} \geqslant 1 + 2\mathscr{n}, \quad \text{(C.336)}$$

which holds for sufficiently large T due to Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii). In the following, Lemma C.20 with $\mathrm{R} = \Re$ will be shown. Therefor, one defines for all $s \in \mathbb{R}^d$ the expression $\widetilde{\mathbf{Bias}}_{T,\Re,1}^{[\mathscr{G}_T]}(s)$ given below, which results from $\widetilde{\mathbf{Bias}}_{T,\Re}^{[\mathscr{G}_T]}(s)$ (recall (C.334)) by replacing $\widetilde{u}_{k,t_2}$ by $\widetilde{u}_{k,t_1}$:

$$\begin{aligned}\widetilde{\mathbf{Bias}}_{T,\Re,1}^{[\mathscr{G}_T]}(s) &:= \frac{T\sqrt{b}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b) \rfloor \lfloor Tb \rfloor^2} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \sum_{t_1,t_2=1+\mathscr{n}}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \mathscr{n}} \mathscr{G}_T(t_2 - t_1)\, K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right) \\ &\cdot K\left(\frac{t_2 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right) \mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\widetilde{u}_{k,t_1})\right\rangle\right)_{\mathscr{n}},\right. \\ &\left.\cos\left(\left\langle s, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\widetilde{u}_{k,t_1})\right\rangle\right)_{\mathscr{n}}\right) \mathbb{1}_{\{|t_2 - t_1| \leqslant \mathscr{n}\}}. \end{aligned} \quad \text{(C.337)}$$

It holds for all real-valued random variables X as well as Y that live on the same probability space and own finite $(1+\delta)/\delta$ moments (whereby $\delta \in (0,1]$ originates from Assumption 2.2 $[\mathbf{StAp}]$):

$$|\mathrm{Cov}(X, Y)| \leqslant 2\, \|X\|_{1+\delta}\, \|Y\|_{\frac{1+\delta}{\delta}}\,. \quad \text{(C.338)}$$

One obtains for all $s \in \mathbb{R}^d$ from (C.215), (C.338) and the Remarks 2.3 as well as A.2 (ii) (see (C.334), (C.337), Definition A.1 (i) and (C.17)):

$$\begin{aligned}&\left|\widetilde{\mathbf{Bias}}_{T,\Re}^{[\mathscr{G}_T]}(s) - \widetilde{\mathbf{Bias}}_{T,\Re,1}^{[\mathscr{G}_T]}(s)\right| \\ &\leqslant \frac{C}{T\sqrt{b}} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \sum_{\substack{t_1,t_2=1 \\ |t_2-t_1| \leqslant \mathscr{n}}}^{2\lfloor T_{\mathfrak{U}} b \rfloor} |\mathscr{G}_T(t_2 - t_1)|\, K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right) K\left(\frac{t_2 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}(\mathfrak{U}_1\right. \\ &\left.-\mathfrak{U}_0)\right) \cdot \left|\mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1}(\widetilde{u}_{k,t_1})\right\rangle\right)_{\mathscr{n}},\right.\right. \\ &\left.\left.\cos\left(\left\langle s, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\widetilde{u}_{k,t_2})\right\rangle\right)_{\mathscr{n}} - \cos\left(\left\langle s, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\widetilde{u}_{k,t_1})\right\rangle\right)_{\mathscr{n}}\right)\right| \\ &\leqslant C \frac{\mathscr{n}}{\sqrt{b}} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t_1,t_2 \in \{1,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor\} : |t_2 - t_1| \leqslant \mathscr{n}} \left\| \cos\left(\left\langle s, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\widetilde{u}_{k,t_2})\right\rangle\right)\right. \\ &\left.- \cos\left(\left\langle s, \widetilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2}(\widetilde{u}_{k,t_1})\right\rangle\right)\right\|_{1+\delta} \\ &\leqslant C\sqrt{b}|s|_1. \end{aligned} \quad \text{(C.339)}$$

Moreover, define for all $s \in \mathbb{R}^d$ (the following expression results from (C.337) by replacing t_2 contained in $K\left(\left(t_2-\lfloor T_{\mathfrak{U}}b\rfloor\right)/\lfloor T_{\mathfrak{U}}b\rfloor\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)$ by t_1 and by manipulating the terms in the underlying covariance$\big)$:

$$\widetilde{\mathbf{Bias}}_{T,\mathfrak{R},2}^{[\mathscr{G}_T]}(s):=\frac{T\sqrt{b}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=1+\varkappa}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa}\mathscr{G}_T\left(t_2-t_1\right)K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2$$
$$\cdot\operatorname{Cov}\left(\cos\left(\left\langle s,\tilde{X}_{t_1}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right),\cos\left(\left\langle s,\tilde{X}_{t_2}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right)\right)\mathbb{1}_{\{|t_2-t_1|\leqslant\varkappa\}}. \tag{C.340}$$

It follows for all $s\in\mathbb{R}^d$ from (C.215), (C.338), Lemma C.16 (ii) with $q=1+\delta$, $1/\delta\geqslant 1$ (which holds due to $\delta\in(0,1]$) and Remark A.2 (ii) (recall (C.17)):

$$\frac{T\sqrt{b}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,t_2=1+\varkappa\\|t_2-t_1|\leqslant\varkappa}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa}\left|\mathscr{G}_T\left(t_2-t_1\right)\right|K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)$$
$$\cdot\left|\operatorname{Cov}\left(\cos\left(\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right)_{\varkappa},\cos\left(\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right)_{\varkappa}\right)\right.$$
$$\left.-\operatorname{Cov}\left(\cos\left(\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right),\cos\left(\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right)\right)\right|$$
$$\leqslant C\frac{T\sqrt{b}\varkappa}{\lfloor Tb\rfloor}\frac{1}{Tb}|s|_1$$
$$\leqslant\frac{C}{\sqrt{Tb}}|s|_1. \tag{C.341}$$

One obtains similarly to (C.224):

$$\operatorname{Cov}\left(\cos\left(\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right),\cos\left(\left\langle s,\tilde{X}_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right)\right)$$
$$=\operatorname{Cov}\left(\cos\left(\left\langle s,\tilde{X}_{t_1}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right),\cos\left(\left\langle s,\tilde{X}_{t_2}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right)\right)$$
$$\forall\, s\in\mathbb{R}^d,\ k\in\{1,\ldots,\lfloor 1/(2b)\rfloor\},\ t_1,t_2\in\{1,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor\}. \tag{C.342}$$

Further, (C.215), Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i), Lemma B.4 (vi), (B.45) and Remark A.2 (ii) yield for all $s\in\mathbb{R}^d$ (see (C.17)):

$$\frac{T\sqrt{b}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,t_2=1+\varkappa\\t_1\geqslant t_2}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa}\left|\mathscr{G}_T\left(t_2-t_1\right)\right|K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\left|K\left(\frac{t_2-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\right.$$
$$\left.-K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)\right|\mathbb{1}_{\{|t_2-t_1|\leqslant\varkappa\}}\left|\operatorname{Cov}\left(\cos\left(\left\langle s,\tilde{X}_{t_1}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right),\cos\left(\left\langle s,\tilde{X}_{t_2}\left(\tilde{u}_{k,t_1}\right)\right\rangle\right)\right)\right|$$
$$\leqslant\sum_{t_1,t_2=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}0\mathbb{1}_{\{t_1=t_2\}}+C\frac{T\sqrt{b}}{\lfloor Tb\rfloor^2}\left(\sup_{t_1,t_2\in\{1,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor\}:|t_2-t_1|\leqslant\varkappa}\left|\frac{t_2-t_1}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right|\right)\sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1}\sum_{t_1=t_2+1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\sum_{l=t_1-t_2}^{\infty}\Delta_l|s|_1$$
$$\leqslant\frac{C}{\sqrt{Tb}}|s|_1. \tag{C.343}$$

It follows for all $s\in\mathbb{R}^d$ from (C.341), (C.342), (C.343) and similar arguments (recall (C.337) as well as (C.340)):

$$\left|\widetilde{\mathbf{Bias}}_{T,\mathfrak{R},1}^{[\mathscr{G}_T]}(s)-\widetilde{\mathbf{Bias}}_{T,\mathfrak{R},2}^{[\mathscr{G}_T]}(s)\right|\leqslant\frac{C}{\sqrt{Tb}}|s|_1. \tag{C.344}$$

Moreover, one defines for all $s\in\mathbb{R}^d$ $\big($in contrast to (C.340), the next expression contains a sum with the indices $t_1\in\{1+2\varkappa,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\varkappa\}$ instead of $t_1\in\{1+\varkappa,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa\}\big)$:

$$\widetilde{\mathbf{Bias}}_{T,\mathfrak{R},3}^{[\mathscr{G}_T]}(s):=\frac{T\sqrt{b}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1=1+2\varkappa}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\varkappa}K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2$$

$$\cdot \sum_{t_2=1+\varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa} \mathscr{G}_T\left(t_2-t_1\right) \operatorname{Cov}\left(\cos \left(\left\langle s, \widetilde{X}_{t_1}\left(\tilde{u}_{k, t_1}\right)\right\rangle\right), \cos \left(\left\langle s, \widetilde{X}_{t_2}\left(\tilde{u}_{k, t_1}\right)\right\rangle\right)\right) \mathbb{1}_{\left\{\left|t_2-t_1\right| \leqslant \varkappa\right\}}$$

$$\text{and} \quad \widetilde{\mathscr{T}}_{T, \varkappa}:=\{1+\varkappa, \ldots, 2 \varkappa\} \cup\left\{2\left\lfloor T_{\mathfrak{U}} b\right\rfloor-2 \varkappa, \ldots, 2\left\lfloor T_{\mathfrak{U}} b\right\rfloor-1-\varkappa\right\} . \tag{C.345}$$

It follows for all $s \in \mathbb{R}^d$ from (C.336), shifting the index of a sum, (C.215), arguments which are similar to those that show (C.224), Lemma 3.12 and Remark A.2 (ii) (see (C.340) as well as (C.345)):

$$\begin{aligned}
&\left|\widetilde{\mathbf{Bias}}_{T, \Re, 2}^{\left[\mathscr{G}_T\right]}(s)-\widetilde{\mathbf{Bias}}_{T, \Re, 3}^{\left[\mathscr{G}_T\right]}(s)\right| \\
&\leqslant \frac{C}{T \sqrt{b}} \sum_{k=1}^{\lfloor 1 /(2 b)\rfloor} \sum_{t_1 \in \widetilde{\mathscr{T}}_{T, \varkappa}} K\left(\frac{t_1-\left\lfloor T_{\mathfrak{U}} b\right\rfloor}{\left\lfloor T_{\mathfrak{U}} b\right\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2 \\
&\cdot \sum_{t_2=1+\varkappa-t_1}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa-t_1}\left|\mathscr{G}_T\left(t_2\right)\right|\left|\operatorname{Cov}\left(\cos \left(\left\langle s, \widetilde{X}_{t_1}\left(\tilde{u}_{k, t_1}\right)\right\rangle\right), \cos \left(\left\langle s, \widetilde{X}_{t_2+t_1}\left(\tilde{u}_{k, t_1}\right)\right\rangle\right)\right)\right| \mathbb{1}_{\left\{\left|t_2\right| \leqslant \varkappa\right\}} \\
&\leqslant \frac{C \varkappa}{T b^{3 / 2}} \sum_{t_2=-\infty}^{\infty} \sup _{u \in[0,1]}\left|\operatorname{Cov}\left(\cos \left(\left\langle s, \widetilde{X}_0(u)\right\rangle\right), \cos \left(\left\langle s, \widetilde{X}_{t_2}(u)\right\rangle\right)\right)\right| \\
&\leqslant \frac{C}{\sqrt{T b}}\left(1+|s|_1\right) .
\end{aligned} \tag{C.346}$$

One obtains from (C.336) that $\{-\varkappa, \ldots, \varkappa\} \subseteq \left\{1+\varkappa-t_1, \ldots, 2\left\lfloor T_{\mathfrak{U}} b\right\rfloor-1-\varkappa-t_1\right\} \forall t_1 \in \{1+2 \varkappa, \ldots, 2\left\lfloor T_{\mathfrak{U}} b\right\rfloor-1-2 \varkappa\}$. Hence, shifting the index of a sum and arguments which are similar to those that imply (C.224) provide for all $s \in \mathbb{R}^d$ (recall (C.345)):

$$\begin{aligned}
\widetilde{\mathbf{Bias}}_{T, \Re, 3}^{\left[\mathscr{G}_T\right]}(s) &= \frac{T \sqrt{b}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1 /(2 b)\rfloor\lfloor T b\rfloor^2} \sum_{k=1}^{\lfloor 1 /(2 b)\rfloor} \sum_{t_1=1+2 \varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-2 \varkappa} K\left(\frac{t_1-\left\lfloor T_{\mathfrak{U}} b\right\rfloor}{\left\lfloor T_{\mathfrak{U}} b\right\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2 \sum_{\substack{t_2=1+\varkappa-t_1 \\ |t_2| \leqslant \varkappa}}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa-t_1} \mathscr{G}_T\left(t_2\right) \\
&\quad \cdot \operatorname{Cov}\left(\cos \left(\left\langle s, \widetilde{X}_{t_1}\left(\tilde{u}_{k, t_1}\right)\right\rangle\right), \cos \left(\left\langle s, \widetilde{X}_{t_2+t_1}\left(\tilde{u}_{k, t_1}\right)\right\rangle\right)\right) \\
&= \frac{T \sqrt{b}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1 /(2 b)\rfloor\lfloor T b\rfloor} \sum_{k=1}^{\lfloor 1 /(2 b)\rfloor} \frac{1}{\lfloor T b\rfloor} \sum_{t_1=1+2 \varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-2 \varkappa} K\left(\frac{t_1-\left\lfloor T_{\mathfrak{U}} b\right\rfloor}{\left\lfloor T_{\mathfrak{U}} b\right\rfloor}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2 \\
&\quad \cdot \sum_{t_2=-\varkappa}^{\varkappa} \mathscr{G}_T\left(t_2\right) \operatorname{Cov}\left(\cos \left(\left\langle s, \widetilde{X}_0\left(\tilde{u}_{k, t_1}\right)\right\rangle\right), \cos \left(\left\langle s, \widetilde{X}_{t_2}\left(\tilde{u}_{k, t_1}\right)\right\rangle\right)\right) .
\end{aligned} \tag{C.347}$$

Further, define for all $s \in \mathbb{R}^d$ (note that the following expression results from the right side of (C.347) by replacing $1 / \lfloor T b\rfloor \sum_{t_1=1+2 \varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-2 \varkappa} K\left(\left(t_1-\left\lfloor T_{\mathfrak{U}} b\right\rfloor\right) /\left\lfloor T_{\mathfrak{U}} b\right\rfloor\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\right)^2$ by $\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\, d z$ and each $\tilde{u}_{k, t}$ (with $t \in \{t_1, t_2\}$) by u_k):

$$\begin{aligned}
\widetilde{\mathbf{Bias}}_{T, \Re, 4}^{\left[\mathscr{G}_T\right]}(s) &:= \frac{T \sqrt{b}\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1 /(2 b)\rfloor\lfloor T b\rfloor} \sum_{k=1}^{\lfloor 1 /(2 b)\rfloor} \int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\, d z \sum_{t_2=-\varkappa}^{\varkappa} \mathscr{G}_T\left(t_2\right) \operatorname{Cov}\left(\cos \left(\left\langle s, \widetilde{X}_0\left(u_k\right)\right\rangle\right),\right. \\
&\quad \left.\cos \left(\left\langle s, \widetilde{X}_{t_2}\left(u_k\right)\right\rangle\right)\right) .
\end{aligned} \tag{C.348}$$

It follows from (C.215) and the mean value theorem together with (C.140) (see (C.17) as well as Definition 3.8 (i)):

$$\begin{aligned}
&\sup _{k=1, \ldots,\lfloor 1 /(2 b)\rfloor} \sup _{t_1=1+2 \varkappa, \ldots, 2\left\lfloor T_{\mathfrak{U}} b\right\rfloor-1-2 \varkappa} \sum_{t_2=-\varkappa}^{\varkappa}\left|\mathscr{G}_T\left(t_2\right)\right| \left\lvert\, \operatorname{Cov}\left(\cos \left(\left\langle s, \widetilde{X}_0\left(\tilde{u}_{k, t_1}\right)\right\rangle\right), \cos \left(\left\langle s, \widetilde{X}_{t_2}\left(\tilde{u}_{k, t_1}\right)\right\rangle\right)\right)\right. \\
&\left.-\operatorname{Cov}\left(\cos \left(\left\langle s, \widetilde{X}_0\left(u_k\right)\right\rangle\right), \cos \left(\left\langle s, \widetilde{X}_{t_2}\left(u_k\right)\right\rangle\right)\right)\right\rvert\, \\
&\leqslant C\left(|s|_1^{1+\delta}+|s|_1\right) \sup _{k=1, \ldots,\lfloor 1 /(2 b)\rfloor} \sup _{t_1=1+2 \varkappa, \ldots, 2\left\lfloor T_{\mathfrak{U}} b\right\rfloor-1-2 \varkappa}\left|\tilde{u}_{k, t_1}-u_k\right| \\
&\leqslant C\left(|s|_1^{1+\delta}+|s|_1\right) b .
\end{aligned} \tag{C.349}$$

One obtains similarly to (C.236) by using (C.237) and Remark A.2 (ii):

$$\left|\frac{1}{\lfloor Tb\rfloor}\sum_{t_1=1+2\mathscr{n}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\mathscr{n}} K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2-\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0}K(z)^2\,dz\right|\leqslant\frac{C}{\sqrt{Tb}}. \tag{C.350}$$

In conclusion, (C.347), (C.349), (C.350), (C.215) and Lemma 3.12 show for all $s\in\mathbb{R}^d$ (recall (C.348)):

$$\left|\widetilde{\mathbf{Bias}}_{T,\mathfrak{R},3}^{[\mathscr{G}_T]}(s)-\widetilde{\mathbf{Bias}}_{T,\mathfrak{R},4}^{[\mathscr{G}_T]}(s)\right|\leqslant C\sqrt{b}\left(|s|_1^{1+\delta}+|s|_1\right)+\frac{C}{\sqrt{b}\sqrt{Tb}}\left(1+|s|_1\right). \tag{C.351}$$

One defines for all $s\in\mathbb{R}^d$ (note that the following expression results from (C.348) by replacing the contained Riemann sum with the indices $k\in\{1,\ldots,\lfloor 1/(2b)\rfloor\}$, which is based on the evolution points u_k, by an integral with respect to $u\in[\mathfrak{U}_0,\mathfrak{U}_1]$):

$$\widetilde{\mathbf{Bias}}_{T,\mathfrak{R},5}^{[\mathscr{G}_T]}(s):=\frac{T\sqrt{b}}{\lfloor Tb\rfloor}\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0}K(z)^2\,dz\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\sum_{t_2=-\mathscr{n}}^{\mathscr{n}}\mathscr{G}_T(t_2)\,\mathrm{Cov}\left(\cos\left(\left\langle s,\tilde{X}_0(u)\right\rangle\right),\right.$$
$$\left.\cos\left(\left\langle s,\tilde{X}_{t_2}(u)\right\rangle\right)\right)du. \tag{C.352}$$

It follows for all $s\in\mathbb{R}^d$, $v,w\in[0,1]$ similarly to (C.349):

$$\begin{aligned}&\sum_{t_2=-\mathscr{n}}^{\mathscr{n}}|\mathscr{G}_T(t_2)|\left|\mathrm{Cov}\left(\cos\left(\left\langle s,\tilde{X}_0(v)\right\rangle\right),\cos\left(\left\langle s,\tilde{X}_{t_2}(v)\right\rangle\right)\right)\right.\\&\left.-\mathrm{Cov}\left(\cos\left(\left\langle s,\tilde{X}_0(w)\right\rangle\right),\cos\left(\left\langle s,\tilde{X}_{t_2}(w)\right\rangle\right)\right)\right|\\&\leqslant C\left(|s|_1^{1+\delta}+|s|_1\right)|v-w|,\end{aligned}$$

such that Lemma B.2 (ii) yields for all $s\in\mathbb{R}^d$ (recall (C.348), (C.352) as well as Definition 3.8 (i)):

$$\left|\widetilde{\mathbf{Bias}}_{T,\mathfrak{R},4}^{[\mathscr{G}_T]}(s)-\widetilde{\mathbf{Bias}}_{T,\mathfrak{R},5}^{[\mathscr{G}_T]}(s)\right|\leqslant C\sqrt{b}\left(|s|_1^{1+\delta}+|s|_1\right). \tag{C.353}$$

Definition A.1 (iv) implies $\mathscr{n}\sqrt{b}\geqslant\tilde{\mathscr{n}}\sqrt{b}\geqslant\mathscr{N}\in\mathbb{N}$. Moreover, Assumption 2.4 $[\mathbf{DM.2}]$, $\delta\in(0,1]$ and $\mathscr{n}\to\infty$ for $T\to\infty$ (which holds due to $\mathscr{n}\geqslant\tilde{\mathscr{n}}$ as well as $\tilde{\mathscr{n}}\to\infty$ for $T\to\infty$) yield $\sum_{l=\mathscr{n}+1}^{\infty}\Delta_l l^2\to 0$ for $T\to\infty$. One obtains from shifting the index of a sum, $l/(\mathscr{n}-t_2)\geqslant 1\;\forall\, l\geqslant\mathscr{n}-t_2$ (with $t_2\in\mathbb{Z}$), Tonelli's theorem for infinite series together with $\Delta_l\geqslant 0\;\forall\, l\in\mathbb{N}$ (the latter follows from the fact that (2.2) and (2.3) should hold according to Assumption 2.4 $[\mathbf{DM.2}]$) as well as the above-shown statements $\mathscr{n}\sqrt{b}\geqslant\mathscr{N}\in\mathbb{N}$ and $\sum_{l=\mathscr{n}+1}^{\infty}\Delta_l l^2\to 0$ for $T\to\infty$:

$$\begin{aligned}&\sum_{t_2=-\infty}^{-\mathscr{n}-1}\sum_{l=-t_2}^{\infty}\Delta_l\leqslant\sum_{t_2=-\infty}^{-1}\sum_{l=\mathscr{n}-t_2}^{\infty}\Delta_l\frac{l}{(\mathscr{n}-t_2)}\leqslant\frac{1}{\mathscr{n}}\sum_{t_2=-\infty}^{-1}\sum_{l=\mathscr{n}-t_2}^{\infty}\Delta_l l=\frac{1}{\mathscr{n}}\sum_{t_2=-\infty}^{-1}\sum_{l=\mathscr{n}+1}^{\infty}\mathbb{1}_{\{l\geqslant\mathscr{n}-t_2\}}\Delta_l l\\&\leqslant\frac{1}{\mathscr{n}}\sum_{l=\mathscr{n}+1}^{\infty}\sum_{t_2=-\infty}^{-1}\mathbb{1}_{\{-l\leqslant t_2\}}\Delta_l l=\frac{\sqrt{b}}{\mathscr{n}\sqrt{b}}\sum_{l=\mathscr{n}+1}^{\infty}l\Delta_l l=o\left(\sqrt{b}\right).\end{aligned} \tag{C.354}$$

It holds for all $s\in\mathbb{R}^d$ due to (C.215), Lemma B.4 (vi) and (C.354):

$$\frac{T\sqrt{b}}{\lfloor Tb\rfloor}\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\sum_{t_2=-\infty}^{-\mathscr{n}-1}|\mathscr{G}_T(t_2)|\left|\mathrm{Cov}\left(\cos\left(\left\langle s,\tilde{X}_0(u)\right\rangle\right),\cos\left(\left\langle s,\tilde{X}_{t_2}(u)\right\rangle\right)\right)\right|du=o(1)|s|_1, \tag{C.355}$$

whereby the expression $o(1)$ does not depend on $s\in\mathbb{R}^d$. Overall, (C.355) and similar arguments, $\left|T\sqrt{b}/\lfloor Tb\rfloor-1/\sqrt{b}\right|=o(1)$ (the latter follows from Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii)), (C.215) as well as Lemma 3.12 yield for all $s \in \mathbb{R}^d$ (see (C.352) and (C.335)):

$$\left| \widetilde{\mathbf{Bias}}^{[\mathscr{G}_T]}_{T,\Re,5}(s) - \widetilde{\mathbf{Bias}}^{[\mathscr{G}_T]}_{T,\Re}(s) \right| = o(1)\,(1 + |s|_1)\,, \tag{C.356}$$

whereby the expression $o(1)$ does not depend on $s \in \mathbb{R}^d$.
Lemma C.20 with $\mathrm{R} = \Re$ is an implication of (C.339), (C.344), (C.346), (C.351), (C.353), (C.356), Assumption 3.1 $[\mathbf{WEI.1}]$ and Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii). Similar arguments prove Lemma C.20 with $\mathrm{R} = \Im$. ☐

Corollary C.22. *Suppose that the Assumptions 2.4* $[\mathbf{DM.2}]$*, 3.1* $[\mathbf{WEI.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *hold. Then, one obtains for* $T \to \infty$ *and all* $\mathrm{R} \in \{\Re, \Im\}$ *(recall (C.328) as well as (3.51)):*

$$\left| T\sqrt{b}\,\mathbb{E}\left[\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R},\varkappa}\right] - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\mathrm{R}} \right| = o(1).$$

Proof. At first, note that the indicator function $\mathbb{1}_{\{|t_2 - t_1| \leqslant \varkappa\}}$ in the double sum with respect to t_1 and t_2 contained in $\widetilde{\mathbf{Bias}}^{[\mathscr{G}_T]}_{T,\mathfrak{U}_{0,1},\mathrm{R},1}(s)$ (see (C.334)) can be omitted because the opposite condition $|t_2 - t_1| > \varkappa$ belongs to addends of this double sum that equal zero due to Definition A.1 (i). Hence, it holds for $\mathscr{G}_T(x) := 1\ \forall\, x \in \mathbb{Z}$ (recall (C.328), (C.327) as well as (C.334)):

$$T\sqrt{b}\,\mathbb{E}\left[\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R},\varkappa}\right] = \int_{\mathbb{R}^d} \widetilde{\mathbf{Bias}}^{[\mathscr{G}_T]}_{T,\mathrm{R}}(s)\ \mathbf{w}(s)\,ds \quad \forall\,\mathrm{R} \in \{\Re, \Im\}$$

and $\mathscr{G}_T(x) := 1\ \forall\, x \in \mathbb{Z}$ obviously fulfils (C.215), such that Corollary C.22 follows from Lemma C.20 (note (C.335), (3.51) as well as (3.17)). ☐

Lemma C.23. *Let the Assumptions 2.4* $[\mathbf{DM.2}]$*, 3.1* $[\mathbf{WEI.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *be fulfilled. Then, it holds for* $T \to \infty$ *and all* $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$ *(see (C.328) as well as (3.52)):*

$$\mathrm{Cov}\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R}_1,\varkappa}, T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R}_2,\varkappa}\right) = \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1},\mathrm{R}_1,\mathrm{R}_2} + o(1).$$

Proof. Throughout this proof, T is supposed to be large enough to ensure that (C.336) (recall (C.17) as well as Definition A.1 (iv)) is fulfilled, which holds for sufficiently large T due to Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii). In the following, Lemma C.23 with $\mathrm{R}_1 = \Re$ and $\mathrm{R}_2 = \Im$ will be shown. Therefor, one defines for all $s_1, s_2 \in \mathbb{R}^d$ (note (C.17), (C.80) and that $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\begin{aligned}\widetilde{\mathrm{Cov}}_{T,1}(s_1, s_2) := \frac{T^2 b\,(\mathfrak{U}_1 - \mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2\,\lfloor Tb\rfloor^4} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,\ldots,t_4 = 1+\varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa} \Big(\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1, s_1)\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2, s_1)\, \widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3, s_2) \\ \cdot \widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4, s_2)\Big] - \mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1, s_1)\, \widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2, s_1)\Big]\, \mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3, s_2)\, \widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4, s_2)\Big]\Big).\end{aligned} \tag{C.357}$$

Let $\mathscr{K}_1, \mathscr{K}_2 \subseteq \{1, \ldots, \lfloor 1/(2b)\rfloor\}$ and $\mathfrak{T}_1, \mathfrak{T}_2 \subseteq \{1, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor\}$ be arbitrary sets and $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$. Then, it holds (see (C.80) as well as Definition A.1 (i) and recall that $\perp\!\!\!\perp$ is the abbreviated form of stochastic independence):

$$\begin{aligned}&\left(\widetilde{\mathbb{X}}_{T,k_1,\mathrm{R}_1}(t_1, s_1)\right)_{k_1 \in \mathscr{K}_1, t_1 \in \mathfrak{T}_1, s_1 \in \mathbb{R}^d} \perp\!\!\!\perp \left(\widetilde{\mathbb{X}}_{T,k_2,\mathrm{R}_2}(t_2, s_2)\right)_{k_2 \in \mathscr{K}_2, t_2 \in \mathfrak{T}_2, s_2 \in \mathbb{R}^d} \\ &\text{if} \quad \left|\lfloor u_{k_1} T\rfloor + t_1 - (\lfloor u_{k_2} T\rfloor + t_2)\right| > \varkappa \quad \forall\, k_1 \in \mathscr{K}_1,\ t_1 \in \mathfrak{T}_1,\ k_2 \in \mathscr{K}_2,\ t_2 \in \mathfrak{T}_2.\end{aligned} \tag{C.358}$$

One obtains from (C.358) and (C.85) (see (C.328), (C.327), (C.80), Definition A.1 (i) as well as (C.357)):

$$\begin{aligned}&\mathrm{Cov}\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\Re,\varkappa}, T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\Im,\varkappa}\right) \\ &= \frac{T^2 b\,(\mathfrak{U}_1 - \mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2\,\lfloor Tb\rfloor^4} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,\ldots,t_4=1+\varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa} \Big(\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k_1,\Re}(t_1, s_1)\, \widetilde{\mathbb{K}}^c_{T,k_1,\Re}(t_2, s_1)\, \widetilde{\mathbb{K}}^c_{T,k_2,\Im}(t_3, s_2)\end{aligned}$$

$$\cdot\widetilde{\mathbb{K}}^c_{T,k_2,\Im}(t_4,s_2)\Big] - \mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k_1,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k_1,\Re}(t_2,s_1)\Big]\,\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k_2,\Im}(t_3,s_2)\,\widetilde{\mathbb{K}}^c_{T,k_2,\Im}(t_4,s_2)\Big]\Big)$$
$$\cdot\,\mathbb{1}_{\{\exists\, r_1\in\{t_1,t_2\},\, r_2\in\{t_3,t_4\}:\,\left|\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+r_1-\left(\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+r_2\right)\right|\leqslant \varkappa\}}\,\mathbf{w}(s_1)\,ds_1\,\mathbf{w}(s_2)\,ds_2$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\widetilde{\mathrm{Cov}}_{T,1}(s_1,s_2)\,\mathbf{w}(s_1)\,ds_1\,\mathbf{w}(s_2)\,ds_2. \tag{C.359}$$

It follows for all $s_1,s_2\in\mathbb{R}^d$ from (C.358) (recall (C.357) and (C.80)):

$$\widetilde{\mathrm{Cov}}_{T,1}(s_1,s_2)$$
$$= \frac{T^2 b\,(\mathfrak{U}_1-\mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^4}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,\ldots,t_4=1+\varkappa\\ \forall\, l_1\in\{t_1,t_3\},l_2\in\{t_2,t_4\}:|l_1-l_2|>\varkappa}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa}\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)\Big]$$
$$\cdot\,\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_2)\Big]$$
$$+ \frac{T^2 b\,(\mathfrak{U}_1-\mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^4}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,\ldots,t_4=1+\varkappa\\ \exists l_1\in\{t_1,t_3\},l_2\in\{t_2,t_4\}:|l_1-l_2|\leqslant\varkappa\\ \forall\, o_1\in\{t_1,t_4\},o_2\in\{t_2,t_3\}:|o_1-o_2|>\varkappa}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa}\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_2)\Big]$$
$$\cdot\,\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)\Big]$$
$$+ \frac{T^2 b\,(\mathfrak{U}_1-\mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^4}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,\ldots,t_4=1+\varkappa\\ \exists l_1\in\{t_1,t_3\},l_2\in\{t_2,t_4\}:|l_1-l_2|\leqslant\varkappa\\ \exists o_1\in\{t_1,t_4\},o_2\in\{t_2,t_3\}:|o_1-o_2|\leqslant\varkappa\\ \forall\, r_1\in\{t_1,t_2\},r_2\in\{t_3,t_4\}:|r_1-r_2|>\varkappa}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa}\Big(\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_1)\Big]\,\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)$$
$$\cdot\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_2)\Big] - \mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_1)\Big]\,\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_2)\Big]\Big)$$
$$+ \frac{T^2 b\,(\mathfrak{U}_1-\mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^4}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,\ldots,t_4=1+\varkappa\\ \exists l_1\in\{t_1,t_3\},l_2\in\{t_2,t_4\}:|l_1-l_2|\leqslant\varkappa\\ \exists o_1\in\{t_1,t_4\},o_2\in\{t_2,t_3\}:|o_1-o_2|\leqslant\varkappa\\ \exists r_1\in\{t_1,t_2\},r_2\in\{t_3,t_4\}:|r_1-r_2|\leqslant\varkappa}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa}\Big(\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)$$
$$\cdot\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_2)\Big] - \mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_1)\Big]\,\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_2)\Big]\Big)$$
$$\cdot\Big(\mathbb{1}_{\{\forall\, p_1\in\{1,\ldots,4\}\,\exists\, p_2\in\{1,\ldots,4\}\setminus\{p_1\}:\,|t_{p_1}-t_{p_2}|\leqslant\varkappa\}} + \mathbb{1}_{\{\exists\, p_1\in\{1,\ldots,4\}:\,|t_{p_1}-t_{p_2}|>\varkappa\,\forall\, p_2\in\{1,\ldots,4\}\setminus\{p_1\}\}}\Big)$$
$$=: \widetilde{\mathrm{Cov}}_{T,1,1}(s_1,s_2)+\widetilde{\mathrm{Cov}}_{T,1,2}(s_1,s_2)+\widetilde{\mathrm{Cov}}_{T,1,3}(s_1,s_2)+\widetilde{\mathrm{Cov}}_{T,1,4}(s_1,s_2). \tag{C.360}$$

One observes that the condition $\exists l_1\in\{t_1,t_3\},l_2\in\{t_2,t_4\}:|l_1-l_2|\leqslant\varkappa$ contained in $\widetilde{\mathrm{Cov}}_{T,1,2}(s_1,s_2)$ can be omitted because the opposite generates summands which equal zero due to (C.358) (see (C.80)). Thus, it holds for all $s_1,s_2\in\mathbb{R}^d$:

$$\widetilde{\mathrm{Cov}}_{T,1,2}(s_1,s_2) = \widetilde{\mathrm{Cov}}_{T,1,1}(s_1,s_2). \tag{C.361}$$

Obviously, one obtains for all $s_1,s_2\in\mathbb{R}^d$:

$$\widetilde{\mathrm{Cov}}_{T,1,3}(s_1,s_2) = 0. \tag{C.362}$$

The condition $\exists\, p_1\in\{1,\ldots,4\}:|t_{p_1}-t_{p_2}|>\varkappa\,\forall\, p_2\in\{1,\ldots,4\}\setminus\{p_1\}$ belongs to summands of $\widetilde{\mathrm{Cov}}_{T,1,4}(s_1,s_2)$ which are zero due to (C.358) (recall (C.80)). All other conditions on t_1,t_2,t_3,t_4 together which are contained in $\widetilde{\mathrm{Cov}}_{T,1,4}(s_1,s_2)$ imply $|t_{q_1}-t_{q_2}|\leqslant 3\varkappa\,\forall\, q_1,q_2\in\{1,\ldots,4\}$. Thus, it holds for all $s_1,s_2\in\mathbb{R}^d$:

$$\left|\widetilde{\mathrm{Cov}}_{T,1,4}(s_1,s_2)\right|$$
$$\leqslant \frac{C}{T^2 b}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1\ldots,t_4=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\Big(\Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_2)\Big]\Big|$$

$$+\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_1)\right]\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_2)\right]\right|\Bigg)\mathbb{1}_{\left\{\forall\, q_1,q_2\in\{1,\dots,4\}:\left|t_{q_1}-t_{q_2}\right|\leqslant 3\varkappa\right\}}. \tag{C.363}$$

Moreover, Lemma B.4 (viii) and (B.45) provide for all $s_1, s_2 \in \mathbb{R}^d$ (see (C.80)):

$$\begin{aligned}
&\sup_{k=1,\dots,\lfloor 1/(2b)\rfloor}\sum_{t_1,\dots,t_4=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_1)\right]\right|\cdot\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_2)\right]\right|\\
&\cdot\mathbb{1}_{\left\{\left|t_{q_1}-t_{q_2}\right|\leqslant 3\varkappa\,\forall\, q_1,q_2\in\{1,\dots,4\}\right\}}\\
&\leqslant C\left(\sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\sum_{t_1=t_2+1}^{\infty}\sum_{l=t_1-t_2}^{\infty}\Delta_l\left|s_1\right|_1+\sum_{t_1=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\sum_{t_2=t_1+1}^{\infty}\sum_{l=t_2-t_1}^{\infty}\Delta_l\left|s_1\right|_1+\sum_{t_1,t_2=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\mathbb{1}_{\{t_1=t_2\}}\right)\\
&\cdot\left(\sup_{t_1=1,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor}\sum_{t_3,t_4=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\mathbb{1}_{\{|t_3-t_1|\leqslant 3\varkappa\}}\mathbb{1}_{\{|t_4-t_1|\leqslant 3\varkappa\}}\right)\\
&\leqslant C\left\lfloor T_{\mathfrak{U}}b\right\rfloor\varkappa^2\left(\left|s_1\right|_1+1\right).
\end{aligned}\tag{C.364}$$

In conclusion, (C.363), arguments which are similar to those that show (C.94) and (C.364) imply for all $s_1, s_2 \in \mathbb{R}^d$ (recall (C.17)):

$$\left|\widetilde{\operatorname{Cov}}_{T,1,4}(s_1,s_2)\right|\leqslant\frac{C\varkappa^2}{Tb}\left(1+\left|s_1\right|_1+\left|s_2\right|_1\right). \tag{C.365}$$

It follows from (C.360), (C.361), (C.362), (C.365), Assumption 3.1 $[\mathbf{WEI.1}]$ and Remark A.2 (ii):

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\widetilde{\operatorname{Cov}}_{T,1}(s_1,s_2)\;\mathbf{w}(s_1)\;ds_1\,\mathbf{w}(s_2)\;ds_2-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}2\widetilde{\operatorname{Cov}}_{T,1,1}(s_1,s_2)\;\mathbf{w}(s_1)\;ds_1\,\mathbf{w}(s_2)\;ds_2\right|=o(1). \tag{C.366}$$

Further, define for all $s_1, s_2 \in \mathbb{R}^d$:

$$\widetilde{\operatorname{Cov}}_{T,2}(s_1,s_2):=\frac{2T^2b\left(\mathfrak{U}_1-\mathfrak{U}_0\right)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^4}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\sum_{t_1,t_3=1+\varkappa}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa}\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)\right]\right)^2. \tag{C.367}$$

If the conditions $\exists\, l_1\in\{t_1,t_3\}, l_2\in\{t_2,t_4\}: |l_1-l_2|\leqslant\varkappa$ and $|t_1-t_3|\leqslant\varkappa$ as well as $|t_2-t_4|\leqslant\varkappa$ hold for some $t_1,\dots,t_4\in\{1+\varkappa,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa\}$, one will obtain $|t_{q_1}-t_{q_2}|\leqslant 3\varkappa\;\forall\, q_1,q_2\in\{1,\dots,4\}$. Hence, it follows for all $s_1,s_2\in\mathbb{R}^d$ from $\mathbb{E}\big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(r_2,s_2)\big]=\mathbb{E}\big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1,s_1)\widetilde{\mathbb{K}}^c_{T,k,\Im}(r_2,s_2)\big]\mathbb{1}_{\{|r_1-r_2|\leqslant\varkappa\}}\;\forall\, k\in\{1,\dots,\lfloor 1/(2b)\rfloor\}$, $r_1,r_2\in\{1+\varkappa,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa\}$ (which holds due to (C.358) (see (C.80))) and by using arguments which are similar to those that provide (C.364) (recall (C.367), (C.360) as well as (C.17)):

$$\begin{aligned}
&\left|\widetilde{\operatorname{Cov}}_{T,2}(s_1,s_2)-2\widetilde{\operatorname{Cov}}_{T,1,1}(s_1,s_2)\right|\\
&\leqslant\frac{C}{T^2b}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{\substack{t_1,\dots,t_4=1+\varkappa\\ \exists\, l_1\in\{t_1,t_3\},l_2\in\{t_2,t_4\}:|l_1-l_2|\leqslant\varkappa}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa}\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3,s_2)\right]\right|\mathbb{1}_{\{|t_1-t_3|\leqslant\varkappa\}}\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_2,s_1)\right.\right.\\
&\left.\left.\cdot\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_2)\right]\right|\mathbb{1}_{\{|t_2-t_4|\leqslant\varkappa\}}\\
&\leqslant\frac{C\varkappa^2}{Tb}\left(\left|s_1\right|_1+\left|s_2\right|_1+1\right).
\end{aligned}$$

Thus, Assumption 3.1 [**WEI.1**] and Remark A.2 (ii) provide:

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 2\widetilde{\mathrm{Cov}}_{T,1,1}(s_1, s_2)\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{\mathrm{Cov}}_{T,2}(s_1, s_2)\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 \right| = o(1). \tag{C.368}$$

Define for all $s_1, s_2 \in \mathbb{R}^d$ (see (C.80) and note that the following expression results from (C.367) by manipulating the expression contained in the parenthesis):

$$\begin{aligned} \widetilde{\mathrm{Cov}}_{T,3}(s_1, s_2) &:= \frac{2T^2 b\, (\mathfrak{U}_1 - \mathfrak{U}_0)^2}{\lfloor 1/(2b) \rfloor^2 \lfloor Tb \rfloor^4} \\ &\quad \cdot \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left(\sum_{t_1, t_3 = 1 + \varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa} K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0) \right)^2 \mathbb{E}\left[\widetilde{\mathbb{X}}^c_{T,k,\Re}(t_1, s_1)\, \widetilde{\mathbb{X}}^c_{T,k,\Im}(t_3, s_2) \right] \right)^2. \end{aligned} \tag{C.369}$$

Obviously, it holds:

$$\left| x^2 - y^2 \right| \leqslant |x - y| \left(|x| + |y| \right) \ \forall\, x, y \in \mathbb{R} \tag{C.370}$$

and one obtains for all $s_1, s_2 \in \mathbb{R}^d$ from Lemma B.4 (viii) as well as (B.45) (recall (C.80)):

$$\begin{aligned} &\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sum_{t_1, t_3 = 1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \left| \mathbb{E}\left[\widetilde{\mathbb{X}}^c_{T,k,\Re}(t_1, s_1)\, \widetilde{\mathbb{X}}^c_{T,k,\Im}(t_3, s_2) \right] \right| \\ &\leqslant C \left(\sum_{t_3=1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \sum_{t_1 = t_3 + 1}^{\infty} \sum_{l = t_1 - t_3}^{\infty} \Delta_l\, |s_1|_1 + \sum_{t_1=1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \sum_{t_3 = t_1 + 1}^{\infty} \sum_{l = t_3 - t_1}^{\infty} \Delta_l\, |s_2|_1 + \sum_{t_1, t_3 = 1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \mathbb{1}_{\{t_1 = t_3\}} \right) \\ &\leqslant C \lfloor T_{\mathfrak{U}} b \rfloor \left(|s_1|_1 + |s_2|_1 + 1 \right). \end{aligned} \tag{C.371}$$

In conclusion, (C.358), (C.370), (C.371) and Assumption 2.8 [**K**&**b.1**] (i) provide for all $s_1, s_2 \in \mathbb{R}^d$ (see (C.80) as well as (C.17)):

$$\begin{aligned} &\sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left| \left(\sum_{t_1, t_3 = 1 + \varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa} \mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1, s_1)\, \widetilde{\mathbb{K}}^c_{T,k,\Im}(t_3, s_2) \right] \right)^2 \right. \\ &\left. - \left(\sum_{t_1, t_3 = 1 + \varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa} K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0) \right)^2 \mathbb{E}\left[\widetilde{\mathbb{X}}^c_{T,k,\Re}(t_1, s_1)\, \widetilde{\mathbb{X}}^c_{T,k,\Im}(t_3, s_2) \right] \right)^2 \right| \\ &\leqslant \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \sum_{t_1, t_3 = 1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0) \right) \left| K\left(\frac{t_3 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0) \right) - K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0) \right) \right| \\ &\cdot \left| \mathbb{E}\left[\widetilde{\mathbb{X}}^c_{T,k,\Re}(t_1, s_1)\, \widetilde{\mathbb{X}}^c_{T,k,\Im}(t_3, s_2) \right] \right| \mathbb{1}_{\{|t_1 - t_3| \leqslant \varkappa\}} C \lfloor T_{\mathfrak{U}} b \rfloor \left(|s_1|_1 + |s_2|_1 + 1 \right) \\ &\leqslant C \left(\sup_{t_1, t_3 \in \{1, \ldots, 2\lfloor T_{\mathfrak{U}} b \rfloor\} : |t_1 - t_3| \leqslant \varkappa} \left| \frac{t_3 - t_1}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0) \right| \right) \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \sum_{t_1, t_3 = 1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \left| \mathbb{E}\left[\widetilde{\mathbb{X}}^c_{T,k,\Re}(t_1, s_1)\, \widetilde{\mathbb{X}}^c_{T,k,\Im}(t_3, s_2) \right] \right| \\ &\cdot \lfloor T_{\mathfrak{U}} b \rfloor \left(|s_1|_1 + |s_2|_1 + 1 \right) \\ &\leqslant C T \varkappa \left(|s_1|_1 + |s_2|_1 + 1 \right)^2. \end{aligned} \tag{C.372}$$

It follows from (C.372), Assumption 3.1 [**WEI.1**], Remark A.2 (ii) and Assumption 2.8 [**K**&**b.1**] (ii) (recall (C.367) as well as (C.369)):

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{\mathrm{Cov}}_{T,2}(s_1, s_2)\ \mathbf{w}(s_1)\ ds_1\, \mathbf{w}(s_2)\ ds_2 - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{\mathrm{Cov}}_{T,3}(s_1, s_2)\ \mathbf{w}(s_1)\ ds_1\, \mathbf{w}(s_2)\ ds_2 \right| = o(1). \tag{C.373}$$

Further, one defines for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $\mathrm{R} \in \{\Re, \Im\}$, $r_1, r_2 \in \{1, \ldots, 2\lfloor T_{\mathfrak{U}} b \rfloor\}$, $s \in \mathbb{R}^d$ (see (C.17)):

$$\widetilde{\mathbb{Y}}_{T,k,\mathrm{R}}(r_1, r_2, s) := \mathrm{R}\left\{ e^{i\left\langle s, \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + r_1}(\tilde{u}_{k,r_2}) \right\rangle} \right\} \tag{C.374}$$

and for all $s_1, s_2 \in \mathbb{R}^d$ (note that the following expression results from (C.369) by manipulating the expression in the expectation and observe also that, in contrast to (C.369), the following expression contains the indicator $\mathbb{1}_{\{|t_1 - t_3| \leqslant \varkappa\}}$):

$$\begin{aligned} \widetilde{\mathrm{Cov}}_{T,4}(s_1, s_2) := \frac{2T^2 b (\mathfrak{U}_1 - \mathfrak{U}_0)^2}{\lfloor 1/(2b) \rfloor^2 \lfloor Tb \rfloor^4} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \Bigg(\sum_{t_1, t_3 = 1+\varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa} K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0) \right)^2 \\ \cdot \mathbb{E}\left[\widetilde{\mathbb{Y}}^c_{T,k,\Re}(t_1, t_3, s_1)\, \widetilde{\mathbb{Y}}^c_{T,k,\Im}(t_3, t_3, s_2) \right] \cdot \mathbb{1}_{\{|t_1 - t_3| \leqslant \varkappa\}} \Bigg)^2, \end{aligned} \tag{C.375}$$

whereby $\tilde{u}_{k,r_2} \in [\mathfrak{U}_0, \mathfrak{U}_1] \subseteq [0,1]$ (which is provided by (C.18)) ensures that $\widetilde{\mathbb{Y}}_{T,k,\mathrm{R}}(r_1, r_2, s)$ is well-defined for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $\mathrm{R} \in \{\Re, \Im\}$, $r_1, r_2 \in \{1, \ldots, 2\lfloor T_{\mathfrak{U}} b \rfloor\}$, $s \in \mathbb{R}^d$.
It follows for all $s_1, s_2 \in \mathbb{R}^d$ similarly to (C.371) by using Lemma B.4 (vi) instead of Lemma B.4 (viii) (recall (C.80) and (C.374)):

$$\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sum_{t_1, t_3 = 1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \left| \mathbb{E}\left[\widetilde{\mathbb{Y}}^c_{T,k,\Re}(t_1, t_3, s_1)\, \widetilde{\mathbb{Y}}^c_{T,k,\Im}(t_3, t_3, s_2) \right] \right| \leqslant C \lfloor T_{\mathfrak{U}} b \rfloor (|s_1|_1 + |s_2|_1 + 1). \tag{C.376}$$

Moreover, Lemma C.16 (ii) with $q = 1 + \delta$ together with Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) and $1/\delta \geqslant 1$ (which holds due to $\delta \in (0, 1]$) provide for all $s_2 \in \mathbb{R}^d$ (see (C.80), (C.374) as well as (C.17)):

$$\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t_3 = 1,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor} \left\| \widetilde{\mathbb{X}}^c_{T,k,\Im}(t_3, s_2) - \widetilde{\mathbb{Y}}^c_{T,k,\Im}(t_3, t_3, s_2) \right\|_{1+\delta} \leqslant \frac{C}{Tb} |s_2|_1. \tag{C.377}$$

One obtains for all $s_1, s_2 \in \mathbb{R}^d$ similarly to (C.377) and by using Remark 2.3 (recall (C.80), (C.374) as well as (C.17)):

$$\begin{aligned} &\sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t_1, t_3 \in \{1,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor\} : |t_1 - t_3| \leqslant \varkappa} \left\| \widetilde{\mathbb{X}}^c_{T,k,\Re}(t_1, s_1) - \widetilde{\mathbb{Y}}^c_{T,k,\Re}(t_1, t_3, s_1) \right\|_{1+\delta} \\ &\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t_1 = 1,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor} \left\| \left(\widetilde{\mathbb{X}}_{T,k,\Re}(t_1, s_1) - \widetilde{\mathbb{Y}}_{T,k,\Re}(t_1, t_1, s_1) \right)^c \right\|_{1+\delta} \\ &+ \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sup_{t_1, t_3 \in \{1,\ldots,2\lfloor T_{\mathfrak{U}} b \rfloor\} : |t_1 - t_3| \leqslant \varkappa} \left\| \left(\widetilde{\mathbb{Y}}_{T,k,\Re}(t_1, t_1, s_1) - \widetilde{\mathbb{Y}}_{T,k,\Re}(t_1, t_3, s_1) \right)^c \right\|_{1+\delta} \\ &\leqslant \frac{C}{Tb} |s_1|_1 + \frac{C\varkappa}{T} |s_1|_1. \end{aligned} \tag{C.378}$$

Overall, it follows for all $s_1, s_2 \in \mathbb{R}^d$ from (C.370), (C.371), (C.376), (C.378) and (C.377) (see (C.80) as well as (C.374)):

$$\sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left| \left(\sum_{t_1, t_3 = 1+\varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa} K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} (\mathfrak{U}_1 - \mathfrak{U}_0) \right)^2 \mathbb{E}\left[\widetilde{\mathbb{X}}^c_{T,k,\Re}(t_1, s_1)\, \widetilde{\mathbb{X}}^c_{T,k,\Im}(t_3, s_2) \right] \mathbb{1}_{\{|t_1 - t_3| \leqslant \varkappa\}} \right)^2 \right.$$

$$- \left(\sum_{t_1, t_3 = 1 + \varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa} K \left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} \left(\mathfrak{U}_1 - \mathfrak{U}_0 \right) \right)^2 \mathbb{E} \left[\widetilde{\mathbb{Y}}^c_{T,k,\Re} \left(t_1, t_3, s_1 \right) \widetilde{\mathbb{Y}}^c_{T,k,\Im} \left(t_3, t_3, s_2 \right) \right] \mathbb{1}_{\{|t_1 - t_3| \leqslant \varkappa\}} \right)^2 \Bigg|$$

$$\leqslant \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \sum_{\substack{t_1, t_3 = 1 \\ |t_1 - t_3| \leqslant \varkappa}}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \left(\left| \mathbb{E} \left[\left(\widetilde{\mathbb{X}}^c_{T,k,\Re} \left(t_1, s_1 \right) - \widetilde{\mathbb{Y}}^c_{T,k,\Re} \left(t_1, t_3, s_1 \right) \right) \widetilde{\mathbb{X}}^c_{T,k,\Im} \left(t_3, s_2 \right) \right] \right| \right.$$

$$\left. + \left| \mathbb{E} \left[\widetilde{\mathbb{Y}}^c_{T,k,\Re} \left(t_1, t_3, s_1 \right) \left(\widetilde{\mathbb{X}}^c_{T,k,\Im} \left(t_3, s_2 \right) - \widetilde{\mathbb{Y}}^c_{T,k,\Im} \left(t_3, t_3, s_2 \right) \right) \right] \right| \right) C \lfloor T_{\mathfrak{U}} b \rfloor \left(|s_1|_1 + |s_2|_1 + 1 \right)$$

$$\leqslant C \lfloor 1/(2b) \rfloor \lfloor T_{\mathfrak{U}} b \rfloor^2 \varkappa \left(\frac{C}{Tb} + \frac{C \varkappa}{T} \right) \left(|s_1|_1 + |s_2|_1 \right) \left(|s_1|_1 + |s_2|_1 + 1 \right). \tag{C.379}$$

Since the expressions $\mathbb{1}_{\{|t_1 - t_3| \leqslant \varkappa\}}$ contained on the left side of (C.379) can be omitted due to (C.358), one obtains from Assumption 3.1 $[\mathbf{WEI.1}]$, Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (recall (C.369), (C.375) as well as (C.17)):

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{\mathrm{Cov}}_{T,3} \left(s_1, s_2 \right) \mathbf{w} \left(s_1 \right) ds_1 \mathbf{w} \left(s_2 \right) ds_2 - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{\mathrm{Cov}}_{T,4} \left(s_1, s_2 \right) \mathbf{w} \left(s_1 \right) ds_1 \mathbf{w} \left(s_2 \right) ds_2 \right| = o(1). \tag{C.380}$$

Define for all $s_1, s_2 \in \mathbb{R}^d$ (note that in contrast to (C.375), the next expression contains a sum with the indices $t_1 \in \{1 + 2\varkappa, \dots, 2 \lfloor T_{\mathfrak{U}} b \rfloor - 1 - 2\varkappa\}$ instead of $t_1 \in \{1 + \varkappa, \dots, 2 \lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa\}$ and observe that, as mentioned at the beginning of this proof, throughout this proof, T is supposed be large enough to ensure that (C.336) holds):

$$\widetilde{\mathrm{Cov}}_{T,5} \left(s_1, s_2 \right) := \frac{2T^2 b \left(\mathfrak{U}_1 - \mathfrak{U}_0 \right)^2}{\lfloor 1/(2b) \rfloor^2 \lfloor Tb \rfloor^4} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left(\sum_{t_1 = 1 + 2\varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - 2\varkappa} \sum_{t_3 = 1 + \varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa} K \left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} \left(\mathfrak{U}_1 - \mathfrak{U}_0 \right) \right)^2 \right.$$

$$\left. \cdot \mathbb{E} \left[\widetilde{\mathbb{Y}}^c_{T,k,\Re} \left(t_1, t_3, s_1 \right) \widetilde{\mathbb{Y}}^c_{T,k,\Im} \left(t_3, t_3, s_2 \right) \right] \cdot \mathbb{1}_{\{|t_1 - t_3| \leqslant \varkappa\}} \right)^2. \tag{C.381}$$

It follows for all $s_1, s_2 \in \mathbb{R}^d$ from (C.370) and (C.376) (whereby $\widetilde{\mathscr{T}}_{T,\varkappa}$ is defined as in (C.345)):

$$\sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left| \left(\sum_{t_1, t_3 = 1 + \varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa} K \left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} \left(\mathfrak{U}_1 - \mathfrak{U}_0 \right) \right)^2 \mathbb{E} \left[\widetilde{\mathbb{Y}}^c_{T,k,\Re} \left(t_1, t_3, s_1 \right) \widetilde{\mathbb{Y}}^c_{T,k,\Im} \left(t_3, t_3, s_2 \right) \right] \mathbb{1}_{\{|t_1 - t_3| \leqslant \varkappa\}} \right)^2 \right.$$

$$- \left(\sum_{t_1 = 1 + 2\varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - 2\varkappa} \sum_{t_3 = 1 + \varkappa}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa} K \left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor} \left(\mathfrak{U}_1 - \mathfrak{U}_0 \right) \right)^2 \mathbb{E} \left[\widetilde{\mathbb{Y}}^c_{T,k,\Re} \left(t_1, t_3, s_1 \right) \widetilde{\mathbb{Y}}^c_{T,k,\Im} \left(t_3, t_3, s_2 \right) \right] \right.$$

$$\left. \left. \cdot \mathbb{1}_{\{|t_1 - t_3| \leqslant \varkappa\}} \right)^2 \right|$$

$$\leqslant \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \sum_{t_1 \in \widetilde{\mathscr{T}}_{T,\varkappa}} \sum_{\substack{t_3 = 1 + \varkappa \\ |t_3 - t_1| \leqslant \varkappa}}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa} \left| \mathbb{E} \left[\widetilde{\mathbb{Y}}^c_{T,k,\Re} \left(t_1, t_3, s_1 \right) \widetilde{\mathbb{Y}}^c_{T,k,\Im} \left(t_3, t_3, s_2 \right) \right] \right| \cdot C \lfloor T_{\mathfrak{U}} b \rfloor \left(|s_1|_1 + |s_2|_1 + 1 \right)$$

$$\leqslant C \lfloor 1/(2b) \rfloor \varkappa^2 \lfloor T_{\mathfrak{U}} b \rfloor \left(|s_1|_1 + |s_2|_1 + 1 \right).$$

This provides by using Assumption 3.1 $[\mathbf{WEI.1}]$ and Remark A.2 (ii) (see (C.375), (C.381) as well as (C.17)):

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{\mathrm{Cov}}_{T,4} \left(s_1, s_2 \right) \mathbf{w} \left(s_1 \right) ds_1 \mathbf{w} \left(s_2 \right) ds_2 - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{\mathrm{Cov}}_{T,5} \left(s_1, s_2 \right) \mathbf{w} \left(s_1 \right) ds_1 \mathbf{w} \left(s_2 \right) ds_2 \right| = o(1). \tag{C.382}$$

One defines for all $s_1, s_2 \in \mathbb{R}^d$ (recall Definition 3.8 (i) and note that the following expression results from (C.381) by pulling one of the factors $1/\lfloor Tb\rfloor^2$ inside one of the parentheses (without squaring it in the parenthesis) and by manipulating the expression contained in this parenthesis):

$$\widetilde{\mathrm{Cov}}_{T,6}(s_1,s_2) := \frac{2T^2 b\,(\mathfrak{U}_1-\mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\frac{1}{\lfloor Tb\rfloor}\sum_{t_1=1+2\varkappa}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\varkappa} K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2\right.$$
$$\left.\cdot\sum_{t_3=-\varkappa}^{\varkappa}\mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_0(u_k)\right\rangle\right),\sin\left(\left\langle s_2,\widetilde{X}_{t_3}(u_k)\right\rangle\right)\right)\right)^2. \tag{C.383}$$

It follows similarly to (C.224) (see (C.374)):

$$\mathbb{E}\left[\widetilde{\mathbb{Y}}^c_{T,k,\Re}(t_1,t_3,s_1)\,\widetilde{\mathbb{Y}}^c_{T,k,\Im}(t_3,t_3,s_2)\right] = \mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_{t_1}(\tilde{u}_{k,t_3})\right\rangle\right),\sin\left(\left\langle s_2,\widetilde{X}_{t_3}(\tilde{u}_{k,t_3})\right\rangle\right)\right)$$
$$\forall\, k\in\{1,\dots,\lfloor 1/(2b)\rfloor\},\ t_1,t_3\in\{1,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor\},\ s_1,s_2\in\mathbb{R}^d. \tag{C.384}$$

As mentioned at the beginning of this proof, T is supposed be large enough to ensure that (C.336) holds. Thus, one obtains $\{-\varkappa,\dots,\varkappa\}\subseteq\{1+\varkappa-t_1,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa-t_1\}\ \forall\, t_1\in\{1+2\varkappa,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\varkappa\}$. Hence, shifting the index of a sum, (C.384) and similar arguments provide for all $s_1,s_2\in\mathbb{R}^d$ (recall (C.381)):

$$\widetilde{\mathrm{Cov}}_{T,5}(s_1,s_2) = \frac{2T^2 b\,(\mathfrak{U}_1-\mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\frac{1}{\lfloor Tb\rfloor}\sum_{t_1=1+2\varkappa}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\varkappa} K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2\right.$$
$$\left.\cdot\sum_{\substack{t_3=1+\varkappa-t_1\\|t_3|\leqslant\varkappa}}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa-t_1}\mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_{t_1}(\tilde{u}_{k,t_3+t_1})\right\rangle\right),\sin\left(\left\langle s_2,\widetilde{X}_{t_3+t_1}(\tilde{u}_{k,t_3+t_1})\right\rangle\right)\right)\right)^2$$
$$= \frac{2T^2 b\,(\mathfrak{U}_1-\mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\frac{1}{\lfloor Tb\rfloor}\sum_{t_1=1+2\varkappa}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\varkappa} K\left(\frac{t_1-\lfloor T_{\mathfrak{U}}b\rfloor}{\lfloor T_{\mathfrak{U}}b\rfloor}(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2\right.$$
$$\left.\cdot\sum_{t_3=-\varkappa}^{\varkappa}\mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_0(\tilde{u}_{k,t_3+t_1})\right\rangle\right),\sin\left(\left\langle s_2,\widetilde{X}_{t_3}(\tilde{u}_{k,t_3+t_1})\right\rangle\right)\right)\right)^2. \tag{C.385}$$

Further, it follows for all $s_1,s_2\in\mathbb{R}^d$ similarly to (C.140):

$$\sum_{t_3=-\infty}^{\infty}\sup_{u\in(0,1)}\left|\partial_u\mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_0(u)\right\rangle\right),\sin\left(\left\langle s_2,\widetilde{X}_{t_3}(u)\right\rangle\right)\right)\right|\leqslant C\left(|s_1|_1^{1+\delta}+|s_2|_1^{1+\delta}+|s_1|_1+|s_2|_1\right). \tag{C.386}$$

The mean value theorem together with (C.386) implies for all $s_1,s_2\in\mathbb{R}^d$ (see (C.17) and Definition 3.8 (i)):

$$\sup_{k=1,\dots,\lfloor 1/(2b)\rfloor}\ \sup_{t_1=1+2\varkappa,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\varkappa}\sum_{t_3=-\varkappa}^{\varkappa}\left|\mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_0(\tilde{u}_{k,t_3+t_1})\right\rangle\right),\right.\right.$$
$$\left.\sin\left(\left\langle s_2,\widetilde{X}_{t_3}(\tilde{u}_{k,t_3+t_1})\right\rangle\right)\right)-\mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_0(u_k)\right\rangle\right),\sin\left(\left\langle s_2,\widetilde{X}_{t_3}(u_k)\right\rangle\right)\right)\Big|$$
$$\leqslant C\frac{\lfloor T_{\mathfrak{U}}b\rfloor+\varkappa}{T}\left(|s_1|_1^{1+\delta}+|s_2|_1^{1+\delta}+|s_1|_1+|s_2|_1\right). \tag{C.387}$$

Lemma 3.12 shows for all $s_1,s_2\in\mathbb{R}^d$ (recall (C.17)):

$$\sup_{k=1,\dots,\lfloor 1/(2b)\rfloor}\ \sup_{t_1=1+2\varkappa,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-2\varkappa}\sum_{t_3=-\varkappa}^{\varkappa}\left|\mathrm{Cov}\left(\cos\left(\left\langle s_1,\widetilde{X}_0(\tilde{u}_{k,t_3+t_1})\right\rangle\right),\sin\left(\left\langle s_2,\widetilde{X}_{t_3}(\tilde{u}_{k,t_3+t_1})\right\rangle\right)\right)\right|$$
$$\leqslant C\left(1+|s_1|_1+|s_2|_1\right) \tag{C.388}$$

and (see Definition 3.8 (i)):

$$\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{t_3=-\varkappa}^{\varkappa} \left|\operatorname{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u_k)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_3}(u_k)\right\rangle\right)\right)\right| \leqslant C\left(1+|s_1|_1+|s_2|_1\right). \tag{C.389}$$

Overall, (C.385), (C.370), (C.387), (C.388), (C.389), Assumption 3.1 $[\mathbf{WEI.1}]$, Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) imply (note (C.383) as well as (C.17)):

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widetilde{\operatorname{Cov}}_{T,5}(s_1,s_2)\ \mathbf{w}(s_1)\ ds_1\,\mathbf{w}(s_2)\ ds_2 - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widetilde{\operatorname{Cov}}_{T,6}(s_1,s_2)\ \mathbf{w}(s_1)\ ds_1\,\mathbf{w}(s_2)\ ds_2\right| = o(1). \tag{C.390}$$

Further, one defines for all $s_1, s_2 \in \mathbb{R}^d$ (recall Definition 3.8 (i) and note that the following expression results from (C.383) by replacing $1/\lfloor Tb\rfloor \sum_{t_1=1+2\varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-2\varkappa} K\left((t_1-\lfloor T_{\mathfrak{U}} b\rfloor)/\lfloor T_{\mathfrak{U}} b\rfloor(\mathfrak{U}_1-\mathfrak{U}_0)\right)^2$ by $\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\ dz$):

$$\begin{aligned}\widetilde{\operatorname{Cov}}_{T,7}(s_1,s_2) &:= \frac{2T^2 b\left(\mathfrak{U}_1-\mathfrak{U}_0\right)^2}{\lfloor 1/(2b)\rfloor^2\lfloor Tb\rfloor^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz\right.\\ &\quad\left.\cdot\sum_{t_3=-\varkappa}^{\varkappa}\operatorname{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u_k)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_3}(u_k)\right\rangle\right)\right)\right)^2.\end{aligned} \tag{C.391}$$

It follows from (C.370), (C.350), (C.389), Assumption 3.1 $[\mathbf{WEI.1}]$, Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (see (C.383) as well as (C.391)):

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widetilde{\operatorname{Cov}}_{T,6}(s_1,s_2)\ \mathbf{w}(s_1)\ ds_1\,\mathbf{w}(s_2)\ ds_2 - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widetilde{\operatorname{Cov}}_{T,7}(s_1,s_2)\ \mathbf{w}(s_1)\ ds_1\,\mathbf{w}(s_2)\ ds_2\right| = o(1). \tag{C.392}$$

Furthermore, one defines for all $s_1, s_2 \in \mathbb{R}^d$ (note that the following expression results from (C.391) by replacing the contained Riemann sum with the indices $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, which is based on the evolution points u_k, by an integral with respect to $u \in [\mathfrak{U}_0, \mathfrak{U}_1]$ and by manipulating the first factor):

$$\begin{aligned}\widetilde{\operatorname{Cov}}_{T,8}(s_1,s_2) &:= 4\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\left(\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz\right)^2\\ &\quad\cdot\int_{\mathfrak{U}_0}^{\mathfrak{U}_1}\left(\sum_{t_3=-\varkappa}^{\varkappa}\operatorname{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_3}(u)\right\rangle\right)\right)\right)^2 du.\end{aligned} \tag{C.393}$$

It follows for all $s_1, s_2 \in \mathbb{R}^d$ from $\left|2T^2b/\left(\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2\right)-4\right| = o(1)$ (which holds due to Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii)) and (C.389) (recall (C.391)):

$$\begin{aligned}&\left|\widetilde{\operatorname{Cov}}_{T,7}(s_1,s_2) - 4\left(\mathfrak{U}_1-\mathfrak{U}_0\right)\left(\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz\right)^2\right.\\ &\left.\cdot\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\sum_{t_3=-\varkappa}^{\varkappa}\operatorname{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u_k)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_3}(u_k)\right\rangle\right)\right)\right)^2\right|\\ &= o(1)\left(1+|s_1|_1+|s_2|_1\right)^2,\end{aligned} \tag{C.394}$$

whereby the expression $o(1)$ does not depend on $s_1, s_2 \in \mathbb{R}^d$. Moreover, one obtains for all $s_1, s_2 \in \mathbb{R}^d$, $v, w \in [0,1]$ due to (C.370), the mean value theorem together with (C.386) and Lemma 3.12:

$$\left| \left(\sum_{t_3=-\varkappa}^{\varkappa} \operatorname{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(v)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_3}(v)\right\rangle\right)\right)\right)^2 - \left(\sum_{t_3=-\varkappa}^{\varkappa} \operatorname{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(w)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_3}(w)\right\rangle\right)\right)\right)^2 \right| \leqslant C\left(|s_1|_1^{1+\delta} + |s_2|_1^{1+\delta} + |s_1|_1 + |s_2|_1\right)\left(1 + |s_1|_1 + |s_2|_1\right)|v-w|,$$

such that Lemma B.2 (ii) yields for all $s_1, s_2 \in \mathbb{R}^d$ (see Definition 3.8 (i)):

$$\left| \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left(\sum_{t_3=-\varkappa}^{\varkappa} \operatorname{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u_k)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_3}(u_k)\right\rangle\right)\right)\right)^2 - \int_{\mathfrak{U}_0}^{\mathfrak{U}_1} \left(\sum_{t_3=-\varkappa}^{\varkappa} \operatorname{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_3}(u)\right\rangle\right)\right)\right)^2 du \right| \leqslant Cb\left(|s_1|_1^{1+\delta} + |s_2|_1^{1+\delta} + |s_1|_1 + |s_2|_1\right)\left(1 + |s_1|_1 + |s_2|_1\right). \tag{C.395}$$

Overall, (C.394), (C.395), Assumption 3.1 $[\mathbf{WEI.1}]$ and Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii) show (recall (C.391) as well as (C.393)):

$$\left| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widetilde{\operatorname{Cov}}_{T,7}(s_1, s_2)\ \mathbf{w}(s_1)\ ds_1\, \mathbf{w}(s_2)\ ds_2 - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widetilde{\operatorname{Cov}}_{T,8}(s_1, s_2)\ \mathbf{w}(s_1)\ ds_1\, \mathbf{w}(s_2)\ ds_2 \right| = o(1). \tag{C.396}$$

Further, one obtains for all $s_1, s_2 \in \mathbb{R}^d$ from (C.370), arguments which are similar to those that show (C.239) as well as (C.240) and Lemma 3.12 (see (3.17)):

$$\sup_{u\in[0,1]} \left| \sigma_{\infty,\Re,\Im}(u, s_1, s_2)^2 - \left(\sum_{t_3=-\varkappa}^{\varkappa} \operatorname{Cov}\left(\cos\left(\left\langle s_1, \widetilde{X}_0(u)\right\rangle\right), \sin\left(\left\langle s_2, \widetilde{X}_{t_3}(u)\right\rangle\right)\right)\right)^2 \right| \leqslant \frac{C}{\varkappa}\left(|s_1|_1 + |s_2|_1\right)\left(1 + |s_1|_1 + |s_2|_1\right). \tag{C.397}$$

It holds $\varkappa \to \infty$ due to $\widetilde{\varkappa} \to \infty$ for $T \to \infty$ (recall Definition A.1 (iv)). This and Assumption 3.1 $[\mathbf{WEI.1}]$ provide (note (C.393) as well as (3.52)):

$$\left| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widetilde{\operatorname{Cov}}_{T,8}(s_1, s_2)\ \mathbf{w}(s_1)\ ds_1\, \mathbf{w}(s_2)\ ds_2 - \sigma^{\text{distr}}_{\mathfrak{U}_{0,1},\Re,\Im} \right| = o(1). \tag{C.398}$$

Lemma C.23 with $\mathrm{R}_1 = \Re$ and $\mathrm{R}_2 = \Im$ is an implication of (C.359), (C.366), (C.368), (C.373), (C.380), (C.382), (C.390), (C.392), (C.396) and (C.398). The other versions of Lemma C.23 (with other choices of R_1 and R_2) follow similarly. □

Lemma C.24. *Suppose that the Assumptions 2.4* $[\mathbf{DM.2}]$, *3.1* $[\mathbf{WEI.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *hold. Then, one obtains for* $T \to \infty$ *and all* $\mathrm{R} \in \{\Re, \Im\}$ *(see (C.328), (3.51) as well as (C.80)):*

$$\mathbb{E}\left[\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R},\varkappa} - \mathbf{Bias}^{\text{distr}}_{T,\mathfrak{U}_{0,1},\mathrm{R}} - \widetilde{\mathbb{S}}_{T,\mathrm{R}}\right)^2\right] = o(1) \tag{C.399}$$

and:

$$\left|\operatorname{Var}\left(T\sqrt{b}\left(\widehat{\mathbb{T}}^{[3]}_{T,\Re,\varkappa} + \widehat{\mathbb{T}}^{[3]}_{T,\Im,\varkappa}\right)\right) - \operatorname{Var}\left(\widetilde{\mathbb{S}}_T\right)\right| = o(1). \tag{C.400}$$

Proof. Throughout this proof, T should be large enough to ensure (recall (C.17) as well as Definition A.1 (iv)):

$$2\lfloor T_{\mathfrak{U}} b\rfloor - 9\varkappa \geqslant 2\varkappa + 2, \tag{C.401}$$

which holds for sufficiently large T due to Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii).

In the following, (C.399) with $\mathrm{R} = \Re$ will be shown.
It holds $\mathbb{E}\left[\widetilde{\mathbb{S}}_{T,\Re}\right] = 0$ due to (C.84) (see (C.80)), such that Corollary C.22 with $\mathrm{R} = \Re$ provides:

$$\left|\mathbb{E}\left[T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\Re,\varkappa} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\Re} - \widetilde{\mathbb{S}}_{T,\Re}\right]\right| = o(1). \tag{C.402}$$

One obtains from $\widetilde{\mathbb{I}}_{T,k,\Re}(t,j) = \widetilde{\mathbb{I}}_{T,k,\Re}(j,t)$ (recall (C.80)) for all $k \in \{1,\dots,\lfloor 1/(2b)\rfloor\}$, $t,j \in \{1,\dots,2\cdot\lfloor T_{\mathfrak{U}} b\rfloor\}$ (see (C.328), (C.327) as well as (C.80)):

$$\begin{aligned}
T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\Re,\varkappa} &= \frac{T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t,j=1+\varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa}\widetilde{\mathbb{I}}_{T,k,\Re}(t,j)\\
&= \frac{T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t=1+\varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa}\widetilde{\mathbb{I}}_{T,k,\Re}(t,t)\\
&\quad+\frac{2T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t=2+\varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa}\sum_{j=1+\varkappa}^{t-1}\widetilde{\mathbb{I}}_{T,k,\Re}(t,j),
\end{aligned} \tag{C.403}$$

such that (note (C.80)):

$$\begin{aligned}
T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\Re,\varkappa} - \widetilde{\mathbb{S}}_{T,\Re} &= \frac{T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t=1+\varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa}\widetilde{\mathbb{I}}_{T,k,\Re}(t,t)\\
&\quad+\frac{2T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t=2+\varkappa}^{9\varkappa+1}\sum_{j=1+\varkappa}^{t-1}\widetilde{\mathbb{I}}_{T,k,\Re}(t,j)\\
&\quad+\frac{2T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t=9\varkappa+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa}\sum_{j=t-7\varkappa}^{t-1}\widetilde{\mathbb{I}}_{T,k,\Re}(t,j)\\
&\quad+\frac{2T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor\lfloor Tb\rfloor^2}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t=9\varkappa+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa}\sum_{j=1+\varkappa}^{2\varkappa}\widetilde{\mathbb{I}}_{T,k,\Re}(t,j)\\
&=: \mathbf{R}_{T,1}+\mathbf{R}_{T,2}+\mathbf{R}_{T,3}+\mathbf{R}_{T,4}.
\end{aligned} \tag{C.404}$$

All random variables X and Y, which live on the same probability space and own finite second moments, fulfil:

$$|\mathrm{Cov}(X,Y)| \leqslant \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} \leqslant \mathrm{Var}(X)+\mathrm{Var}(Y). \tag{C.405}$$

Moreover, (C.358), (C.85), Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) yield (recall (C.404), (C.80) as well as Definition A.1 (i)):

$$\begin{aligned}
&\mathrm{Var}(\mathbf{R}_{T,1})\\
&\leqslant \frac{C}{T^2 b}\sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=1+\varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa}\left|\mathrm{Cov}\left(\widetilde{\mathbb{I}}_{T,k_1,\Re}(t_1,t_1),\widetilde{\mathbb{I}}_{T,k_2,\Re}(t_2,t_2)\right)\right|\mathbb{1}_{\{|\lfloor u_{k_1}T\rfloor+t_1-(\lfloor u_{k_2}T\rfloor+t_2)|\leqslant\varkappa\}}\\
&\leqslant \frac{C}{T^2 b}\sum_{\substack{k_1,k_2=1\\k_1=k_2}}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=1+\varkappa}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa}\mathbb{1}_{\{|t_1-t_2|\leqslant\varkappa\}}\\
&= o(1).
\end{aligned} \tag{C.406}$$

Further, (C.401) provides $2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{n} \geqslant 9\mathscr{n}+1$, such that $\{1+\mathscr{n},\ldots,9\mathscr{n}+1\} \subseteq \{1+\mathscr{n},\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{n}\}$. Thus, (C.358), (C.85), Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) imply (see (C.404), (C.80) as well as Definition A.1 (i)):

$$\begin{aligned}
\operatorname{Var}\left(\mathbf{R}_{T,2}\right) &\leqslant \frac{C}{T^2 b} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=2+\mathscr{n}}^{9\mathscr{n}+1} \sum_{j_1,j_2=1+\mathscr{n}}^{9\mathscr{n}} \left|\operatorname{Cov}\left(\widetilde{\mathbb{I}}_{T,k_1,\Re}\left(t_1,j_1\right), \widetilde{\mathbb{I}}_{T,k_2,\Re}\left(t_2,j_2\right)\right)\right| \\
&\quad \cdot \mathbb{1}_{\left\{\exists o_1 \in \{t_1,j_1\}, o_2 \in \{t_2,j_2\} : \left|\left\lfloor u_{k_1}T\right\rfloor + o_1 - \left(\left\lfloor u_{k_2}T\right\rfloor + o_2\right)\right| \leqslant \mathscr{n}\right\}} \\
&\leqslant \frac{C}{T^2 b} \sum_{\substack{k_1,k_2=1 \\ k_1=k_2}}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=2+\mathscr{n}}^{9\mathscr{n}+1} \sum_{j_1,j_2=1+\mathscr{n}}^{9\mathscr{n}} C \\
&= o(1). \qquad\qquad \text{(C.407)}
\end{aligned}$$

If $t_1,t_2 \in \{9\mathscr{n}+2,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{n}\}$, $j_1 \in \{t_1 - 7\mathscr{n},\ldots,t_1-1\}$, $j_2 \in \{t_2 - 7\mathscr{n},\ldots,t_2-1\}$ and $\exists\, o_1 \in \{t_1,j_1\}, o_2 \in \{t_2,j_2\} : |o_1 - o_2| \leqslant \mathscr{n}$, it will hold for all $p_1,p_2 \in \{t_1,j_1,t_2,j_2\}$ that $|p_1 - p_2| \leqslant 15\mathscr{n}$. Hence, one obtains from (C.358), (C.85), (C.85), arguments which are similar to those that show (C.94) and (C.364), Assumption 3.1 $[\mathbf{WEI.1}]$, Remark A.2 (ii) as well as Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (recall (C.404), (C.80), Definition A.1 (i) and (C.17)):

$$\begin{aligned}
\operatorname{Var}\left(\mathbf{R}_{T,3}\right) &\leqslant \frac{C}{T^2 b} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=9\mathscr{n}+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{n}} \sum_{j_1=t_1-7\mathscr{n}}^{t_1-1} \sum_{j_2=t_2-7\mathscr{n}}^{t_2-1} \left|\operatorname{Cov}\left(\widetilde{\mathbb{I}}_{T,k_1,\Re}\left(t_1,j_1\right), \widetilde{\mathbb{I}}_{T,k_2,\Re}\left(t_2,j_2\right)\right)\right| \\
&\quad \cdot \mathbb{1}_{\left\{\exists\, o_1 \in \{t_1,j_1\}, o_2 \in \{t_2,j_2\} : \left|\left\lfloor u_{k_1}T\right\rfloor + o_1 - \left(\left\lfloor u_{k_2}T\right\rfloor + o_2\right)\right| \leqslant \mathscr{n}\right\}} \\
&\leqslant \frac{C}{T^2 b} \sum_{\substack{k_1,k_2=1 \\ k_1=k_2}}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2,j_1,j_2=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor} \left(\left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\Re}\left(t_1,j_1\right) \widetilde{\mathbb{I}}_{T,k_2,\Re}\left(t_2,j_2\right)\right]\right|\right. \\
&\quad \left. + \left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\Re}\left(t_1,j_1\right)\right] \mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_2,\Re}\left(t_2,j_2\right)\right]\right|\right) \cdot \mathbb{1}_{\left\{\forall\, p_1,p_2 \in \{t_1,j_1,t_2,j_2\} : |p_1 - p_2| \leqslant 15\mathscr{n}\right\}} \\
&= o(1). \qquad\qquad \text{(C.408)}
\end{aligned}$$

It follows from (C.358), (C.85), arguments which are similar to those that show (C.371), Assumption 3.1 $[\mathbf{WEI.1}]$, Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (see (C.404), (C.80), Definition A.1 (i) as well as (C.17)):

$$\begin{aligned}
\mathbb{E}\left[\mathbf{R}_{T,4}^2\right] &\leqslant \frac{C}{T^2 b} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=9\mathscr{n}+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{n}} \sum_{j_1,j_2=1+\mathscr{n}}^{2\mathscr{n}} \left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\Re}\left(t_1,j_1\right) \widetilde{\mathbb{I}}_{T,k_2,\Re}\left(t_2,j_2\right)\right]\right| \\
&= \frac{C}{T^2 b} \sum_{\substack{k_1,k_2=1 \\ k_1\neq k_2}}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=9\mathscr{n}+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{n}} \sum_{j_1,j_2=1+\mathscr{n}}^{2\mathscr{n}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_1,\Re}\left(t_1,s_1\right)\right] \mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_1,\Re}\left(j_1,s_1\right)\right]\right| \\
&\quad \cdot \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_2,\Re}\left(t_2,s_2\right)\right] \mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_2,\Re}\left(j_2,s_2\right)\right]\right| \mathbf{w}\left(s_1\right)\, ds_1\, \mathbf{w}\left(s_2\right)\, ds_2 \\
&\quad + \frac{C}{T^2 b} \sum_{\substack{k_1,k_2=1 \\ k_1=k_2}}^{\lfloor 1/(2b)\rfloor} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{t_1,t_2=9\mathscr{n}+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor - 1 - \mathscr{n}} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_1,\Re}\left(t_1,s_1\right) \widetilde{\mathbb{K}}^c_{T,k_2,\Re}\left(t_2,s_2\right)\right]\right| \\
&\quad \cdot \sum_{j_1,j_2=1+\mathscr{n}}^{2\mathscr{n}} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_1,\Re}\left(j_1,s_1\right) \widetilde{\mathbb{K}}^c_{T,k_2,\Re}\left(j_2,s_2\right)\right]\right| \mathbf{w}\left(s_1\right)\, ds_1\, \mathbf{w}\left(s_2\right)\, ds_2 \\
&= o(1). \qquad\qquad \text{(C.409)}
\end{aligned}$$

Since $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\Re}$ is deterministic (note (3.51) as well as (3.17)), one obtains from (C.404), (C.405), (C.406), (C.407), (C.408) and (C.409):

$$\begin{aligned}
\operatorname{Var}\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\Re,\mathscr{n}} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\Re} - \widetilde{\mathbb{S}}_{T,\Re}\right) &\leqslant C\left(\operatorname{Var}\left(\mathbf{R}_{T,1}\right) + \operatorname{Var}\left(\mathbf{R}_{T,2}\right) + \operatorname{Var}\left(\mathbf{R}_{T,3}\right) + \operatorname{Var}\left(\mathbf{R}_{T,4}\right)\right) \\
&= o(1). \qquad\qquad \text{(C.410)}
\end{aligned}$$

In conclusion, (C.399) with $\mathrm{R} = \Re$ is an implication of (C.402) and (C.410). Moreover, (C.399) with $\mathrm{R} = \Im$ can be proved similarly.

Next, (C.400) will be shown.
One obtains for all $\mathrm{R} \in \{\Re, \Im\}$ due to $\mathbb{E}[X^2] = (\mathbb{E}[X])^2 + \mathrm{Var}(X)$ (which holds for each random variable X with finite second moments), Corollary C.22, the fact that $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\mathrm{R}}$ is deterministic, Lemma C.23 with $\mathrm{R}_1 = \mathrm{R}_2 = \mathrm{R}$, Lemma 3.12 and Assumption 3.1 $[\mathbf{WEI.1}]$ (the latter two show $\sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1},\mathrm{R},\mathrm{R}} \leqslant C$ (recall (3.52) as well as (3.17))):

$$\mathbb{E}\left[\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R},\varkappa} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\mathrm{R}}\right)^2\right] = o(1) + \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1},\mathrm{R},\mathrm{R}} \leqslant C. \tag{C.411}$$

In addition, (C.25) with $M = 2$, (C.399) and (C.411) imply for all $\mathrm{R} \in \{\Re, \Im\}$:

$$\mathbb{E}\left[\widetilde{\mathbb{S}}^2_{T,\mathrm{R}}\right] \leqslant 2\,\mathbb{E}\left[\left(\widetilde{\mathbb{S}}_{T,\mathrm{R}} - \left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R},\varkappa} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\mathrm{R}}\right)\right)^2\right] + 2\,\mathbb{E}\left[\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R},\varkappa} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\mathrm{R}}\right)^2\right]$$
$$\leqslant C. \tag{C.412}$$

Moreover, it holds for all random variables X_1, X_2, Y_1, Y_2, which live on the same probability space and own finite second moments, that $|\mathrm{Cov}\,(X_1, X_2) - \mathrm{Cov}\,(Y_1, Y_2)| = |\mathrm{Cov}\,(X_1 - Y_1, X_2) + \mathrm{Cov}(Y_1, X_2 - Y_2)| \leqslant C\,\|X_1 - Y_1\|_2\,\|X_2\|_2 + C\,\|Y_1\|_2\,\|X_2 - Y_2\|_2$. Hence, (C.399), (C.411) and (C.412) show for all $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$:

$$\left|\mathrm{Cov}\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R}_1,\varkappa} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\mathrm{R}_1}, T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\mathrm{R}_2,\varkappa} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\mathrm{R}_2}\right) - \mathrm{Cov}\left(\widetilde{\mathbb{S}}_{T,\mathrm{R}_1}, \widetilde{\mathbb{S}}_{T,\mathrm{R}_2}\right)\right| = o(1). \tag{C.413}$$

This provides (see (C.80), in particular, $\widetilde{\mathbb{S}}_T = \widetilde{\mathbb{S}}_{T,\Re} + \widetilde{\mathbb{S}}_{T,\Im}$):

$$\left|\mathrm{Var}\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\Re,\varkappa} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\Re} + T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\Im,\varkappa} - \mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\Im}\right) - \mathrm{Var}\left(\widetilde{\mathbb{S}}_T\right)\right| = o(1),$$

which implies (C.400) because $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\Re}$ and $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\Im}$ are deterministic. □

Lemma C.25. *Suppose that the Assumptions 2.4* $[\mathbf{DM.2}]$ *and 3.15* $[\mathbf{W}^*]$ *(whereby the latter includes Assumption 2.8* $[\mathbf{K}\&\mathbf{b.1}]$ *(ii)) are fulfilled.*

(i) One obtains for all $T \in \mathbb{N}$ (recall Definition A.1 (iv) as well as (v) and Assumption 3.15 $[\mathbf{W}^*]$ *(i)):*

$$1 + \varkappa \leqslant \varkappa_\beta \leqslant C\,(\beta \ln(e + Tb) + \varkappa) \leqslant o\left(\sqrt{Tb}\right)\ln(e + Tb)\,.$$

(ii) It holds for all $T \in \mathbb{N}$ (see Definition A.1 (i)):

$$\sup_{t\in\mathbb{Z}} \left\|W^*_{t,\{\varkappa_\beta\}} - W^*_t\right\|_2 \leqslant \frac{C}{Tb}.$$

Proof. (i) Assumption 3.15 $[\mathbf{W}^*]$ (i) (in particular, $\beta > 0$), $e+Tb > e$ (that follows from Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii)), $\ln(\rho_*) < 0$ (which holds according to Assumption 3.15 $[\mathbf{W}^*]$ (iv)), $\beta^{\mathrm{inv}}_{\sup}\beta \geqslant e$ (recall Definition A.1 (iii)) and Definition A.1 (v) show the first inequality of Lemma C.25 (i). Moreover, the Assumptions 3.15 $[\mathbf{W}^*]$ (i) and 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) provide $\beta^{\mathrm{inv}}_{\sup}\beta/(e + Tb) = o(1)$, such that $\ln(\rho_*) < 0$, $\ln(e + Tb) > 0$ and $1 \leqslant C\beta \ln(e + Tb)$, whereby the latter holds due to Assumption 3.15 $[\mathbf{W}^*]$ (i), yield (see Definition A.1 (v) as well as (iv)):

$$\varkappa_\beta \leqslant \frac{\beta}{\ln(\rho_*)}\,(-C\ln(e + Tb)) + 1 + \varkappa \leqslant C\,(\beta \ln(e + Tb) + \varkappa)\,,$$

which implies the second inequality of Lemma C.25 (i). Furthermore, Assumption 3.15 $[\mathbf{W}^*]$ (i), that ensures $\beta = o\big(\sqrt{Tb^2}\sqrt{1/b}\big)$, Remark A.2 (ii) and $\ln(e + Tb) > 1$ show the last inequality of Lemma C.25 (i).

(ii) It follows from Definition A.1 (v) that $\rho_*^{\mathscr{n}_\beta/\beta} = e^{\ln(\rho_*)\mathscr{n}_\beta/\beta} \leqslant e^{-\ln(e+Tb)-\ln(\beta_{\sup}^{\mathrm{inv}}\beta)} = C/(Tb\beta)$. Thus, Lemma C.25 (ii) can be proved similarly to Lemma C.8 (iii).

□

Lemma C.26. *Let the Assumptions 2.4* [**DM.2**]*, 3.1* [**WEI.1**]*, 2.8* [**K**&**b.1**] *and 3.15* [$\mathbf{W}^*$] *be fulfilled. Moreover, define for all* $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$*,* $s \in \mathbb{R}^d$ *(recall that* $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ *according to Definition 3.3 (i) and see Definition A.1 (v) as well as (i), (C.17) and Definition 3.8 (i)):*

$$\begin{aligned}\breve{\varphi}^*_{\mathscr{n}_\beta}(u_k, s) &:= \breve{\varphi}^*_{T,\mathfrak{U}_{0,1},\mathscr{n}_\beta}(u_k, s)\\ &:= \frac{1}{\lfloor Tb \rfloor} \sum_{t=1+\mathscr{n}_\beta}^{2\lfloor T_{\mathfrak{U}}b \rfloor - 1 - \mathscr{n}_\beta} K\left(\frac{t - \lfloor T_{\mathfrak{U}}b \rfloor}{\lfloor T_{\mathfrak{U}}b \rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right)\left(e^{i\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}}b \rfloor + t, T}\right\rangle}\right)^c W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}}b \rfloor + t, \{\mathscr{n}_\beta\}}\end{aligned} \tag{C.414}$$

and for all $\mathrm{R} \in \{\Re, \Im\}$*:*

$$\widehat{\mathbb{T}}^{[1]*}_{T,\mathrm{R},\mathscr{n}_\beta} := \widehat{\mathbb{T}}^{[1]*}_{T,\mathfrak{U}_{0,1},\mathrm{R},\mathscr{n}_\beta} := \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathrm{R}\left\{\breve{\varphi}^*_{\mathscr{n}_\beta}(u_k, s)\right\}^2 \mathbf{w}(s)\, ds. \tag{C.415}$$

Then, it holds for $T \to \infty$ *(recall (3.56)):*

$$T\sqrt{b}\,\mathbb{E}\left[\left|\widehat{\mathbb{D}}^*_{T,\mathrm{Test}} - \left(\widehat{\mathbb{T}}^{[1]*}_{T,\Re,\mathscr{n}_\beta} + \widehat{\mathbb{T}}^{[1]*}_{T,\Im,\mathscr{n}_\beta}\right)\right|\right] = o(1).$$

Remark C.27. *Since (C.52) ensures that* $\breve{\varphi}^*_{\mathscr{n}_\beta}(u_k, s)$ *just takes* $X_{t,T}$ *with* $t \in \{1, \ldots, T\}$ *into account,* $\breve{\varphi}^*_{\mathscr{n}_\beta}(u_k, s)$ *is well-defined for all* $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$*,* $s \in \mathbb{R}^d$*.*

Proof of Lemma C.26. Throughout this proof, T should be large enough to ensure (see (C.17) and Definition A.1 (v)):

$$2\lfloor T_{\mathfrak{U}}b \rfloor - 1 - \mathscr{n}_\beta \geqslant 1 + \mathscr{n}_\beta, \tag{C.416}$$

which holds for sufficiently large T due to Lemma C.25 (i) and Assumption 2.8 [**K**&**b.1**] (ii). At first, it follows for all $s \in \mathbb{R}^d$ similarly to (C.256) (recall (C.253)) by using (C.25) with $M = 4$ and the Assumptions 3.15 [$\mathbf{W}^*$] (i) as well as 2.8 [**K**&**b.1**] (ii) (note (C.251) and that $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\begin{aligned}&T\sqrt{b} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right)\Re\left\{e^{i\langle s, X_{t,T}\rangle} - \widehat{\varphi}(u_k, s)\right\}W^*_t - \Re\{\widehat{\varphi}^*(u_k, s)\}\right|^2\right]\\ &\leqslant CT\sqrt{b}\left(\sqrt{\frac{\beta}{Tb}}\left(\frac{1}{\sqrt{Tb}}\sqrt{|s|_1 + 1} + \left(b^{1+\delta} + \frac{1}{Tb}\right)\left(|s|_1^{1+\delta} + 1\right) + b|s|_1 + \frac{1}{T}|s|_1\right)\right)^2\\ &\leqslant C\left(\frac{o(Tb^2)}{Tb^{\frac{3}{2}}} + o(1/b)b^{3/2}\right)\left(|s|_1^{2+2\delta} + 1\right)\\ &= o\left(\sqrt{b}\right)\left(|s|_1^{2+2\delta} + 1\right),\end{aligned} \tag{C.417}$$

whereby the expressions $o(Tb^2)$, $o(1/b)$ as well as $o(\sqrt{b})$ do not depend on $s \in \mathbb{R}^d$. Moreover, one defines for all $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $s \in \mathbb{R}^d$ (recall that $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ according to Definition 3.3 (i) and Definition 3.8 (i) as well as (C.17)):

$$\begin{aligned}\breve{\varphi}^*(u_k, s) &:= \breve{\varphi}^*_{T,\mathfrak{U}_{0,1}}(u_k, s)\\ &:= \frac{1}{\lfloor Tb \rfloor}\sum_{t=1}^{2\lfloor T_{\mathfrak{U}}b \rfloor} K\left(\frac{t - \lfloor T_{\mathfrak{U}}b \rfloor}{\lfloor T_{\mathfrak{U}}b \rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right)\left(e^{i\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}}b \rfloor + t, T}\right\rangle}\right)^c W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}}b \rfloor + t}.\end{aligned} \tag{C.418}$$

Since $\widehat{\varphi}^*$ (see (C.251)) as well as $\breve{\varphi}^*$ are defined similarly to $\widehat{\varphi}$ and $\breve{\varphi}$ (recall Definition 2.11 as well as

(C.200)), respectively, it follows for all $s \in \mathbb{R}^d$ analogously to (C.199), (C.201) and (C.202) by using the Assumptions 3.15 $[\mathbf{W}^*]$ (iii) as well as 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii):

$$T\sqrt{b} \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\Re\left\{\widehat{\varphi}^*\left(u_k, s\right) - \breve{\varphi}^*\left(u_k, s\right)\right\}^2\right] \leqslant T\sqrt{b}\,\frac{C}{T^2 b^2} = o\left(\sqrt{b}\right), \tag{C.419}$$

whereby the expression $o\big(\sqrt{b}\big)$ does not depend on $s \in \mathbb{R}^d$. Furthermore, one defines for all $k \in \{1, \ldots, \lfloor T_{\mathfrak{U}} b \rfloor\}$, $s \in \mathbb{R}^d$ (see Definition A.1 (v) as well as (i), Definition 3.8 (i) and (C.17)):

$$\begin{aligned} \breve{\varphi}^*_{\varkappa_\beta,+}\left(u_k, s\right) &:= \breve{\varphi}^*_{T,\mathfrak{U}_{0,1},\varkappa_\beta,+}\left(u_k, s\right) \\ &:= \frac{1}{\lfloor Tb \rfloor} \sum_{t=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} K\left(\frac{t - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right)\left(e^{i\left\langle s, X_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t, T}\right\rangle}\right)^c W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t,\{\varkappa_\beta\}}. \end{aligned} \tag{C.420}$$

Assumption 3.15 $[\mathbf{W}^*]$ (ii), Lemma C.25 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii) provide for all $s \in \mathbb{R}^d$ (recall (C.418), (C.420) as well as (C.17)):

$$\begin{aligned} &T\sqrt{b} \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\Re\left\{\breve{\varphi}^*\left(u_k, s\right) - \breve{\varphi}^*_{\varkappa_\beta,+}\left(u_k, s\right)\right\}^2\right] \\ &\leqslant T\sqrt{b} \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \left(\frac{1}{\lfloor Tb \rfloor}\sum_{t=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} K\left(\frac{t - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right)\left\|\cos\left(\left\langle s, X_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t, T}\right\rangle\right)^c\right\|_2 \right. \\ &\left. \cdot \left\|W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t} - W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t,\{\varkappa_\beta\}}\right\|_2\right)^2 \\ &= o\left(\sqrt{b}\right), \end{aligned} \tag{C.421}$$

whereby the expression $o\big(\sqrt{b}\big)$ does not depend on $s \in \mathbb{R}^d$.
Further, one observes for all $x_1, x_2 \geqslant 0$:

$$\left(x_1 + x_2\right)^3 \leqslant C\left(x_1^3 + x_2^3\right). \tag{C.422}$$

It follows for all $s \in \mathbb{R}^d$ and for $\mathscr{T}_{T,\varkappa_\beta} := \{1, \ldots, \varkappa_\beta\} \cup \{2\lfloor T_{\mathfrak{U}} b\rfloor - \varkappa_\beta, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor\}$ from (C.25) with $M = 2$, Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (i) $\big($which ensures $K\left(-\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) = K\left(\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) = 0$ and that K is Lipschitz continuous on $\mathbb{R}\big)$, Lemma B.4 (v), (B.45), the second inequality of Lemma C.25 (i) together with (C.422), Assumption 3.15 $[\mathbf{W}^*]$ (i), Remark A.2 (ii) as well as Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii) (see (C.420), (C.414) and (C.17)):

$$\begin{aligned} &T\sqrt{b} \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\Re\left\{\breve{\varphi}^*_{\varkappa_\beta,+}\left(u_k, s\right) - \breve{\varphi}^*_{\varkappa_\beta}\left(u_k, s\right)\right\}^2\right] \\ &\leqslant 2\frac{T\sqrt{b}}{\lfloor Tb\rfloor^2} \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=1}^{\varkappa_\beta} \left(\left|K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) - K\left(\frac{-\lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right)\right|\right. \\ &\left.\cdot \left|K\left(\frac{t_2 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) - K\left(\frac{-\lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right)\right|\right) \\ &\cdot \left|\operatorname{Cov}\left(\cos\left(\left\langle s, X_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1, T}\right\rangle\right), \cos\left(\left\langle s, X_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2, T}\right\rangle\right)\right)\right| \\ &\cdot \left|\mathbb{E}\left[W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\varkappa_\beta\}}\right]\right| \\ &+ 2\frac{T\sqrt{b}}{\lfloor Tb\rfloor^2} \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=2\lfloor T_{\mathfrak{U}} b\rfloor - \varkappa_\beta}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left(\left|K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) - K\left(\frac{\lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right)\right|\right. \\ &\left.\cdot \left|K\left(\frac{t_2 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) - K\left(\frac{\lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right)\right|\right) \\ &\cdot \left|\operatorname{Cov}\left(\cos\left(\left\langle s, X_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1, T}\right\rangle\right), \cos\left(\left\langle s, X_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2, T}\right\rangle\right)\right)\right| \end{aligned}$$

$$\cdot \left| \mathbb{E}\left[W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, \{\varkappa_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, \{\varkappa_\beta\}} \right] \right|$$
$$\leqslant \frac{CT\sqrt{b}}{T^2 b^2} \frac{\varkappa_\beta^2}{\lfloor T_{\mathfrak{U}} b \rfloor^2} \left(\sum_{t_2 \in \mathscr{T}_{T,\varkappa_\beta}} \sum_{t_1 = t_2 + 1}^{\infty} \sum_{l = t_1 - t_2}^{\infty} \Delta_l |s|_1 + \sum_{t_1 \in \mathscr{T}_{T,\varkappa_\beta}} \sum_{t_2 = t_1 + 1}^{\infty} \sum_{l = t_2 - t_1}^{\infty} \Delta_l |s|_1 + \sum_{t_1, t_2 \in \mathscr{T}_{T,\varkappa_\beta}} \mathbb{1}_{\{t_1 = t_2\}} \right)$$
$$\leqslant \frac{CT\sqrt{b}}{T^2 b^2} \frac{o\left(Tb^2 Tb^2 \frac{1}{b}\right) \ln\left(e + Tb\right)^3 + \varkappa^3}{\lfloor T_{\mathfrak{U}} b \rfloor^2} \left(|s|_1 + 1\right)$$
$$= o\left(\sqrt{b}\right) \left(|s|_1 + 1\right), \tag{C.423}$$

whereby the expression $o\big(\sqrt{b}\big)$ does not depend on $s \in \mathbb{R}^d$. In conclusion, (C.417), (C.419), (C.421) and (C.423) as well as similar arguments imply:

$$T\sqrt{b} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left| \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right) \left(e^{i\langle s, X_{t,T}\rangle} - \widehat{\varphi}\left(u_k, s\right)\right) W_t^* - \breve{\varphi}^*_{\varkappa_\beta}\left(u_k, s\right) \right|^2 \right]$$
$$= o\left(\sqrt{b}\right) \left(|s|_1^{2+2\delta} + 1\right), \tag{C.424}$$

whereby the expression $o\big(\sqrt{b}\big)$ does not depend on $s \in \mathbb{R}^d$. Further, one obtains for all $s \in \mathbb{R}^d$ from Assumption 3.15 $[\mathbf{W}^*]$ (ii) and (iii) (the latter shows $\sup_{t\in\mathbb{Z}} \big\|W^*_{t,\{\varkappa_\beta\}}\big\|_2 \leqslant \|W_0^*\|_2 \leqslant C$ (recall Definition A.1 (i))), Lemma B.4 (v) as well as (B.45) (see (C.414) and (C.17)):

$$T\sqrt{b} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \mathbb{E}\left[\left| \breve{\varphi}^*_{\varkappa_\beta}\left(u_k, s\right) \right|^2 \right]$$
$$\leqslant C \frac{T\sqrt{b}}{\lfloor Tb \rfloor^2} \sup_{k=1,\ldots,\lfloor 1/(2b) \rfloor} \sum_{t_1, t_2 = 1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \Big(\left| \mathbb{E}\left[\cos\left(\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, T}\right\rangle\right)^c \cos\left(\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, T}\right\rangle\right)^c \right] \right|$$
$$+ \left| \mathbb{E}\left[\sin\left(\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, T}\right\rangle\right)^c \sin\left(\left\langle s, X_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, T}\right\rangle\right)^c \right] \right| \Big)$$
$$\cdot \left| \mathbb{E}\left[W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_1, \{\varkappa_\beta\}} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t_2, \{\varkappa_\beta\}} \right] \right|$$
$$\leqslant C \frac{T\sqrt{b}}{\lfloor Tb \rfloor^2} \left(\sum_{t_2 = 1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \sum_{t_1 = t_2 + 1}^{\infty} \sum_{l = t_1 - t_2}^{\infty} \Delta_l |s|_1 + \sum_{t_1 = 1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \sum_{t_2 = t_1 + 1}^{\infty} \sum_{l = t_2 - t_1}^{\infty} \Delta_l |s|_1 + \sum_{t_1, t_2 = 1}^{2\lfloor T_{\mathfrak{U}} b \rfloor} \mathbb{1}_{\{t_1 = t_2\}} \right)$$
$$\leqslant \frac{C}{\sqrt{b}} \left(|s|_1 + 1\right). \tag{C.425}$$

Overall, (C.315) with $f\left(u_k, s\right) := 1/T \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u_k\right)\left(e^{i\langle s, X_{t,T}\rangle} - \widehat{\varphi}\left(u_k, s\right)\right) W_t^*$ and $g\left(u_k, s\right) := \breve{\varphi}^*_{\varkappa_\beta}\left(u_k, s\right) \forall\, k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $s \in \mathbb{R}^d$, (C.424), (C.425) as well as Assumption 3.1 $[\mathbf{WEI.1}]$ show Lemma C.26 (note (3.56) and (C.415)). □

Lemma C.28. *Suppose that the Assumptions 2.4* $[\mathbf{DM.2}]$*, 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K\&b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *hold. Moreover, define for all* $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$*,* $s \in \mathbb{R}^d$ *(recall that* $\mathfrak{U}_{0,1} := [\mathfrak{U}_0, \mathfrak{U}_1]$ *according to Definition 3.3 (i) and the Definitions A.1 (i), (iv), (v), 3.8 (i) as well as (C.17)):*

$$\varphi^{\circ *}_{\varkappa,\varkappa_\beta}\left(u_k, s\right) := \varphi^{\circ *}_{T,\mathfrak{U}_{0,1},\varkappa,\varkappa_\beta}\left(u_k, s\right)$$
$$:= \frac{1}{\lfloor Tb \rfloor} \sum_{t = 1 + \varkappa_\beta}^{2\lfloor T_{\mathfrak{U}} b \rfloor - 1 - \varkappa_\beta} K\left(\frac{t - \lfloor T_{\mathfrak{U}} b \rfloor}{\lfloor T_{\mathfrak{U}} b \rfloor}\left(\mathfrak{U}_1 - \mathfrak{U}_0\right)\right) \left(e^{i\left\langle s, \tilde{X}_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t}\left(\tilde{u}_{k,t}\right)\right\rangle}\right)^c_{\varkappa} W^*_{\lfloor u_k T \rfloor - \lfloor T_{\mathfrak{U}} b \rfloor + t, \{\varkappa_\beta\}} \tag{C.426}$$

and for all $\mathrm{R} \in \{\Re, \Im\}$*:*

$$\widehat{\mathbb{T}}^{[2]*}_{T,\mathrm{R},\varkappa,\varkappa_\beta} := \widehat{\mathbb{T}}^{[2]*}_{T,\mathfrak{U}_{0,1},\mathrm{R},\varkappa,\varkappa_\beta} := \int_{\mathbb{R}^d} \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \mathrm{R}\left\{\varphi^{\circ *}_{\varkappa,\varkappa_\beta}\left(u_k, s\right)\right\}^2 \mathbf{w}(s)\, ds. \tag{C.427}$$

Then, it holds for $T \to \infty$ (see (C.415)):

$$\mathbb{E}\left[\left|T\sqrt{b}\left(\widehat{\mathbb{T}}^{[1]*}_{T,\Re,\varkappa_\beta} + \widehat{\mathbb{T}}^{[1]*}_{T,\Im,\varkappa_\beta} - \left(\widehat{\mathbb{T}}^{[2]*}_{T,\Re,\varkappa,\varkappa_\beta} + \widehat{\mathbb{T}}^{[2]*}_{T,\Im,\varkappa,\varkappa_\beta}\right)\right)\right|^2\right] = o(1).$$

Remark C.29. *Since $\widetilde{u}_{k,t} \in [\mathfrak{U}_0, \mathfrak{U}_1] \subseteq [0,1]\ \forall\, k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $t \in \{1, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor\}$ (which holds according to (C.18)), $\varphi^{\circ *}_{\varkappa,\varkappa_\beta}(u_k, s)$ is well-defined for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $s \in \mathbb{R}^d$.*

Proof of Lemma C.28. Throughout this proof, T should be large enough to ensure that (C.416) is fulfilled, which holds for sufficiently large T due to Lemma C.25 (i) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii). At first, one observes (note (C.415), (C.414), (C.101), (C.427), (C.426), (C.108) as well as (C.80)):

$$T\sqrt{b}\,\widehat{\mathbb{T}}^{[1]*}_{T,\Re,\varkappa_\beta} = \frac{T\sqrt{b}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t,j=1+\varkappa_\beta}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta} \mathbb{I}^*_{T,k,\Re}(t,j) \quad \text{and}$$

$$T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\Re,\varkappa,\varkappa_\beta} = \frac{T\sqrt{b}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t,j=1+\varkappa_\beta}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta} \widetilde{\mathbb{I}}^*_{T,k,\Re}(t,j)\,. \tag{C.428}$$

Moreover, (C.105), (C.114) and (C.116) imply (see Definition A.1 (i)):

$$\begin{aligned}
&\sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,j_1,t_2,j_2=1+\varkappa_\beta}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta} \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\varkappa_\beta\}} \\
&\cdot W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_2,\{\varkappa_\beta\}}\Big]\Big| \\
&\leqslant \sum_{k_1=1}^{\lfloor 1/(2b)\rfloor} \left(\sum_{t_1,j_1=1+\varkappa_\beta}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta} \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_1,\{\varkappa_\beta\}}\Big]\Big| \right. \\
&\left. \cdot \sum_{\substack{k_2=1\\k_2\neq k_1}}^{\lfloor 1/(2b)\rfloor} \sum_{t_2,j_2=1+\varkappa_\beta}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta} \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_2,\{\varkappa_\beta\}}\Big]\Big| \right) \\
&+ \sum_{\substack{k_1,k_2=1\\k_1=k_2}}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,j_1,t_2,j_2=1+\varkappa_\beta}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta} \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\varkappa_\beta\}} \\
&\cdot W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_2,\{\varkappa_\beta\}}\Big]\Big| \\
&\leqslant C \lfloor 1/(2b)\rfloor^2 \lfloor T_{\mathfrak{U}} b\rfloor^2 \beta^2 + C \lfloor 1/(2b)\rfloor \left(\lfloor T_{\mathfrak{U}} b\rfloor\, \varkappa_\beta^3 + \lfloor T_{\mathfrak{U}} b\rfloor^2 \beta^2\right).
\end{aligned} \tag{C.429}$$

Overall, (C.428), Assumption 3.15 $[\mathbf{W}^*]$ (ii), (C.429), (C.113) with $q = 2$, Assumption 3.15 $[\mathbf{W}^*]$ (i) $\big($the latter ensures $\beta^2 = o\left(Tb^2 1/b\right)\big)$, Lemma C.25 (i) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) show (recall (C.101), (C.108) as well as (C.17)):

$$\begin{aligned}
&\mathbb{E}\left[\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[1]*}_{T,\Re,\varkappa_\beta} - T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\Re,\varkappa,\varkappa_\beta}\right)^2\right] \\
&\leqslant \frac{C}{T^2 b} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,j_1,t_2,j_2=1+\varkappa_\beta}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta} \Big|\mathbb{E}\Big[W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\varkappa_\beta\}} \\
&\cdot W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_2,\{\varkappa_\beta\}}\Big]\Big| \\
&\cdot \Big|\mathbb{E}\Big[\left(\mathbb{I}_{T,k_1,\Re}(t_1,j_1) - \widetilde{\mathbb{I}}_{T,k_1,\Re}(t_1,j_1)\right)\left(\mathbb{I}_{T,k_2,\Re}(t_2,j_2) - \widetilde{\mathbb{I}}_{T,k_2,\Re}(t_2,j_2)\right)\Big]\Big| \\
&\leqslant \frac{C}{T^2 b}\left(\lfloor 1/(2b)\rfloor^2 \lfloor T_{\mathfrak{U}} b\rfloor^2 \beta^2 + \lfloor 1/(2b)\rfloor \left(\lfloor T_{\mathfrak{U}} b\rfloor\, \varkappa_\beta^3 + \lfloor T_{\mathfrak{U}} b\rfloor^2 \beta^2\right)\right) \cdot o\left(\frac{1}{\sqrt{T}}\right)^2 \\
&\leqslant C\left(\frac{\beta^2}{b} + \frac{\varkappa_\beta^3}{Tb} + \beta^2\right) o\left(\frac{1}{T}\right) \\
&= o(1).
\end{aligned} \tag{C.430}$$

Lemma C.28 follows from (C.25) with $M = 2$, (C.430) and arguments which are similar to those that prove (C.430) (see (C.415) as well as (C.427)). $\square$

Lemma C.30. *Let the Assumptions 2.4* $[\mathbf{DM.2}]$, *3.1* $[\mathbf{WEI.1}]$, *2.8* $[\mathbf{K\&b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled. Then, one obtains for* $T \to \infty$ *and all* $\mathrm{R} \in \{\Re, \Im\}$ *(recall (C.427) as well as (3.57)):*

$$\left|\mathbb{E}\left[T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\mathrm{R},\mathscr{n},\mathscr{n}_\beta}\right] - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1},\mathrm{R}}\right| = o(1).$$

Proof. Throughout this proof, T should be large enough to ensure (see Definition A.1 (v) and (iv) as well as (C.17)):

$$\mathscr{n}_\beta \geqslant 1 + \mathscr{n} \quad \text{and} \quad 2\lfloor T_{\mathfrak{U}} b\rfloor - \mathscr{n} - \mathscr{n}_\beta \geqslant \mathscr{n} + \mathscr{n}_\beta, \tag{C.431}$$

which holds for sufficiently large T due to Lemma C.25 (i) as well as Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii). In the following, Lemma C.30 with $\mathrm{R} = \Re$ will be proved. Therefor, one defines for all $s \in \mathbb{R}^d$ (recall (C.17), the Definitions A.1 (v), (i) as well as (iv), 3.15 $[\mathbf{W}^*]$ (iii) and 3.8 (i)):

$$\begin{aligned}\widetilde{\mathbf{Bias}}^*_{T,\Re}(s) := {}& \frac{T\sqrt{b}\,(\mathfrak{U}_1 - \mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=1+\mathscr{n}_\beta}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{n}_\beta} K^*\left(\frac{t_1 - t_2}{\beta}\right) K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right) \\ & \cdot K\left(\frac{t_2 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right) \mathrm{Cov}\left(\cos\left(\left\langle s, \tilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1}(\tilde{u}_{k,t_1})\right\rangle\right)_{\mathscr{n}},\right. \\ & \left.\cos\left(\left\langle s, \tilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2}(\tilde{u}_{k,t_2})\right\rangle\right)_{\mathscr{n}}\right).\end{aligned} \tag{C.432}$$

It follows from (C.358) (see (C.80)), Assumption 3.15 $[\mathbf{W}^*]$ (ii) as well as (iii), (C.112), Lemma C.25 (ii), Assumption 3.1 $[\mathbf{WEI.1}]$, Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii) (note (C.427), (C.426) as well as (C.432)):

$$\begin{aligned}& \left|\mathbb{E}\left[T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\Re,\mathscr{n},\mathscr{n}_\beta}\right] - \int_{\mathbb{R}^d} \widetilde{\mathbf{Bias}}^*_{T,\Re}(s)\,\mathbf{w}(s)\,ds\right| \\ & \leqslant \frac{C}{T\sqrt{b}} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{\substack{t_1,t_2=1+\mathscr{n}_\beta \\ |t_1 - t_2| \leqslant \mathscr{n}}}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{n}_\beta} K\left(\frac{t_1 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right) K\left(\frac{t_2 - \lfloor T_{\mathfrak{U}} b\rfloor}{\lfloor T_{\mathfrak{U}} b\rfloor}(\mathfrak{U}_1 - \mathfrak{U}_0)\right) \\ & \cdot \int_{\mathbb{R}^d} \left|\mathrm{Cov}\left(\cos\left(\left\langle s, \tilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1}(\tilde{u}_{k,t_1})\right\rangle\right)_{\mathscr{n}}, \cos\left(\left\langle s, \tilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2}(\tilde{u}_{k,t_2})\right\rangle\right)_{\mathscr{n}}\right)\right| \mathbf{w}(s)\,ds \\ & \cdot \sup_{t_1,t_2 \in \mathbb{Z}} \left|\mathbb{E}\left[W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\mathscr{n}_\beta\}} W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\mathscr{n}_\beta\}}\right] - \mathbb{E}\left[W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1} W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2}\right]\right| \\ & \leqslant \frac{C}{T\sqrt{b}} \lfloor 1/(2b)\rfloor \lfloor T_{\mathfrak{U}} b\rfloor\, \mathscr{n} \frac{C}{Tb} \\ & = o(1).\end{aligned} \tag{C.433}$$

One defines the sets $\breve{\mathscr{T}}_{T,\mathscr{n},\mathscr{n}_\beta} := \{(t_1,t_2)' \in \mathbb{N}^2 : t_1, t_2 \in \{1 + \mathscr{n}, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{n}\} \wedge |t_1 - t_2| \leqslant \mathscr{n}\} \setminus \{(t_1,t_2)' \in \mathbb{N}^2 : t_1, t_2 \in \{1 + \mathscr{n}_\beta, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{n}_\beta\} \wedge |t_1 - t_2| \leqslant \mathscr{n}\}$ as well as $\breve{\mathscr{T}}^+_{T,\mathscr{n},\mathscr{n}_\beta} := \{1 + \mathscr{n}, \ldots, \mathscr{n}_\beta + \mathscr{n}\} \cup \{2\lfloor T_{\mathfrak{U}} b\rfloor - \mathscr{n}_\beta - \mathscr{n}, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \mathscr{n}\}$. It holds $\left|\mathrm{Cov}\left(\cos\left(\langle s, \tilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1}(\tilde{u}_{k,t_1})\rangle\right)_{\mathscr{n}}, \cos\left(\langle s, \tilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2}(\tilde{u}_{k,t_2})\rangle\right)_{\mathscr{n}}\right)\right| = 0$ for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $t_1, t_2 \in \{1, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor\}$ with $|t_1 - t_2| > \mathscr{n}$ (see Definition A.1 (i)). Thus, one obtains for $\mathscr{G}_T(x) := K^*(x/\beta)\ \forall\, x \in \mathbb{Z}$ from Assumption 3.15 $[\mathbf{W}^*]$ (iii), $\breve{\mathscr{T}}_{T,\mathscr{n},\mathscr{n}_\beta} \subseteq \breve{\mathscr{T}}^+_{T,\mathscr{n},\mathscr{n}_\beta} \times \breve{\mathscr{T}}^+_{T,\mathscr{n},\mathscr{n}_\beta}$ (which follows from (C.431)), Lemma B.4 (viii), Assumption 3.1 $[\mathbf{WEI.1}]$, (B.45), the second inequality of Lemma C.25 (i), Assumption 3.15 $[\mathbf{W}^*]$ (i), Remark A.2 (ii), (C.119) with $p = 2$ (the latter yields

$\sqrt{b}\ln(e+Tb) = o(1)\big)$ and Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii) (recall (C.334) as well as (C.432)):

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| \widetilde{\mathbf{Bias}}^{[\mathscr{G}_T]}_{T,\Re}(s) - \widetilde{\mathbf{Bias}}^{*}_{T,\Re}(s) \right| \mathbf{w}(s)\, ds \\
&\leqslant \frac{C}{T\sqrt{b}} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \sum_{(t_1,t_2)\in \widetilde{\mathscr{T}}_{T,\mathscr{n},\mathscr{n}_\beta}} \left| K^*\left(\frac{t_1-t_2}{\beta}\right)\right| \int_{\mathbb{R}^d} \Big|\mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1}\left(\widetilde{u}_{k,t_1}\right)\right\rangle\right)_{\mathscr{n}},\right. \\
&\left.\cos\left(\left\langle s, \widetilde{X}_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2}\left(\widetilde{u}_{k,t_2}\right)\right\rangle\right)_{\mathscr{n}}\right)\Big| \,\mathbf{w}(s)\, ds \\
&\leqslant \frac{C}{T\sqrt{b}} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \left(\sum_{t_2\in \widetilde{\mathscr{T}}^{+}_{T,\mathscr{n},\mathscr{n}_\beta}} \sum_{t_1=t_2+1}^{\infty} \sum_{l=t_1-t_2}^{\infty} \Delta_l + \sum_{t_1\in \widetilde{\mathscr{T}}^{+}_{T,\mathscr{n},\mathscr{n}_\beta}} \sum_{t_2=t_1+1}^{\infty} \sum_{l=t_2-t_1}^{\infty} \Delta_l + \sum_{t_1,t_2\in \widetilde{\mathscr{T}}^{+}_{T,\mathscr{n},\mathscr{n}_\beta}} \mathbb{1}_{\{t_1=t_2\}} \right) \\
&\leqslant \frac{C}{T\sqrt{b}} \lfloor 1/(2b) \rfloor \left(o\left(Tb^2\right)\ln(e+Tb) + o\left(\sqrt{Tb}\right)\right) \\
&= o(1). \qquad \text{(C.434)}
\end{aligned}$$

Lemma C.30 with $\mathrm{R} = \Re$ follows from (C.433), (C.434) and Lemma C.20 with $\mathscr{G}_T(x) := K^*(x/\beta)$ $\forall\, x \in \mathbb{Z}$ (see (C.335) as well as (3.57)), whereby $\mathbb{Z} \ni x \mapsto K^*(x/\beta)$ fulfils (C.215) due to Assumption 3.15 $[\mathbf{W}^*]$ (iii). Lemma C.30 with $\mathrm{R} = \Im$ can be proved similarly. □

Lemma C.31. *Suppose that the Assumptions 2.4* $[\mathbf{DM}.\mathbf{2}]$*, 3.1* $[\mathbf{WEI}.\mathbf{1}]$*, 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *and 3.15* $[\mathbf{W}^*]$ *hold. Then, one obtains for* $T \to \infty$ *(recall (C.415), (3.57) as well as (C.101)):*

$$\mathbb{E}\left[\left(T\sqrt{b}\left(\widehat{\mathbb{T}}^{[1]*}_{T,\Re,\mathscr{n}_\beta} + \widehat{\mathbb{T}}^{[1]*}_{T,\Im,\mathscr{n}_\beta}\right) - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}} - \mathbb{S}^*_T\right)^2\right] = o(1).$$

Proof. Throughout this proof, T should be large enough to ensure (see (C.17) as well as Definition A.1 (v)):

$$2\lfloor T_{\mathfrak{U}} b\rfloor - 9\mathscr{n}_\beta \geqslant 2\mathscr{n}_\beta + 2, \qquad \text{(C.435)}$$

which holds for sufficiently large T due to Lemma C.25 (i) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii).
At first, one observes that (C.25) with $M = 5$ provides (recall (C.415), (3.57), (C.101) and (C.108)):

$$\begin{aligned}
&\mathbb{E}\left[\left(T\sqrt{b}\left(\widehat{\mathbb{T}}^{[1]*}_{T,\Re,\mathscr{n}_\beta} + \widehat{\mathbb{T}}^{[1]*}_{T,\Im,\mathscr{n}_\beta}\right) - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1}} - \mathbb{S}^*_T\right)^2\right] \\
&\leqslant 5\,\mathbb{E}\left[\left|T\sqrt{b}\left(\widehat{\mathbb{T}}^{[1]*}_{T,\Re,\mathscr{n}_\beta} + \widehat{\mathbb{T}}^{[1]*}_{T,\Im,\mathscr{n}_\beta} - \left(\widehat{\mathbb{T}}^{[2]*}_{T,\Re,\mathscr{n},\mathscr{n}_\beta} + \widehat{\mathbb{T}}^{[2]*}_{T,\Im,\mathscr{n},\mathscr{n}_\beta}\right)\right)\right|^2\right] \\
&+ 5\,\mathbb{E}\left[\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\Re,\mathscr{n},\mathscr{n}_\beta} - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1},\Re} - \widetilde{\mathbb{S}}^*_{T,\Re}\right)^2\right] + 5\,\mathbb{E}\left[\left(\widetilde{\mathbb{S}}^*_{T,\Re} - \mathbb{S}^*_{T,\Re}\right)^2\right] \\
&+ 5\,\mathbb{E}\left[\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\Im,\mathscr{n},\mathscr{n}_\beta} - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1},\Im} - \widetilde{\mathbb{S}}^*_{T,\Im}\right)^2\right] + 5\,\mathbb{E}\left[\left(\widetilde{\mathbb{S}}^*_{T,\Im} - \mathbb{S}^*_{T,\Im}\right)^2\right]. \qquad \text{(C.436)}
\end{aligned}$$

In the following, it is shown (see (C.427), (3.57) and (C.108)):

$$\mathbb{E}\left[\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\Re,\mathscr{n},\mathscr{n}_\beta} - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1},\Re} - \widetilde{\mathbb{S}}^*_{T,\Re}\right)^2\right] = o(1). \qquad \text{(C.437)}$$

One obtains $\mathbb{E}\big[\widetilde{\mathbb{S}}^*_{T,\Re}\big] = 0$ due to Assumption 3.15 $[\mathbf{W}^*]$ (ii) as well as (iii) (recall (C.108), (C.80) and Definition A.1 (i)), such that Lemma C.30 with $\mathrm{R} = \Re$ provides:

$$\left|\mathbb{E}\left[T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\Re,\mathscr{n},\mathscr{n}_\beta} - \mathbf{Bias}^{\mathrm{distr}*}_{T,\mathfrak{U}_{0,1},\Re} - \widetilde{\mathbb{S}}^*_{T,\Re}\right]\right| = o(1). \qquad \text{(C.438)}$$

The first equation of (C.403) as well as (C.428) yield that $T\sqrt{b}\,\widehat{\mathbb{T}}^{[3]}_{T,\Re,\mathscr{n}}$ and $T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\Re,\mathscr{n},\mathscr{n}_\beta}$ own similar representations. Moreover, $\widetilde{\mathbb{S}}_{T,\Re}$ (see (C.80)) and $\widetilde{\mathbb{S}}^*_{T,\Re}$ (recall (C.108)) are defined very similarly. Thus, one obtains from $\widetilde{\mathbb{I}}^*_{T,k,\mathfrak{R}}(t,j) = \widetilde{\mathbb{I}}^*_{T,k,\mathfrak{R}}(j,t)$ for all $k \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $t,j \in \{1,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor\}$ (see (C.108) as well as (C.80)) analogously to the second equation of (C.403) and to (C.404):

$$\begin{aligned}
T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\mathfrak{R},\varkappa,\varkappa_\beta} - \widetilde{\mathbb{S}}^*_{T,\mathfrak{R}} &= \frac{T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t=1+\varkappa_\beta}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta} \widetilde{\mathbb{I}}^*_{T,k,\mathfrak{R}}(t,t) \\
&+ \frac{2T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t=2+\varkappa_\beta}^{9\varkappa_\beta+1} \sum_{j=1+\varkappa_\beta}^{t-1} \widetilde{\mathbb{I}}^*_{T,k,\mathfrak{R}}(t,j) \\
&+ \frac{2T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta} \sum_{j=t-7\varkappa_\beta}^{t-1} \widetilde{\mathbb{I}}^*_{T,k,\mathfrak{R}}(t,j) \\
&+ \frac{2T\sqrt{b}\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor \lfloor Tb\rfloor^2} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta} \sum_{j=1+\varkappa_\beta}^{2\varkappa_\beta} \widetilde{\mathbb{I}}^*_{T,k,\mathfrak{R}}(t,j) \\
&=: \mathbf{R}^*_{T,1} + \mathbf{R}^*_{T,2} + \mathbf{R}^*_{T,3} + \mathbf{R}^*_{T,4}. \qquad \text{(C.439)}
\end{aligned}$$

It follows for all $k_1, k_2 \in \{1,\ldots,\lfloor 1/(2b)\rfloor\}$, $t_1, j_1 \in \{1,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor\}$ from Assumption 3.15 $[\mathbf{W}^*]$ (ii), (C.112), Assumption 3.15 $[\mathbf{W}^*]$ (iii) $\big($the latter yields $\sup_{t\in\mathbb{Z}} \|W^*_{t,\{\varkappa_\beta\}}\|_4 \leqslant C$ due to Definition A.1 (v)$\big)$, (C.358) and Lemma C.25 (i), which provides $\varkappa \leqslant \varkappa_\beta$ (recall (C.108), (C.80) as well as Definition A.1 (i) and (v)):

$$\begin{aligned}
&\left|\mathrm{Cov}\left(\widetilde{\mathbb{I}}^*_{T,k_1,\mathfrak{R}}(t_1,j_1), \widetilde{\mathbb{I}}^*_{T,k_2,\mathfrak{R}}(t_2,j_2)\right)\right| \\
&= \Big|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\mathfrak{R}}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k_2,\mathfrak{R}}(t_2,j_2)\right] \mathbb{E}\Big[W^*_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\varkappa_\beta\}} \\
&\cdot W^*_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\varkappa_\beta\}}\Big] - \mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\mathfrak{R}}(t_1,j_1)\right] \mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_2,\mathfrak{R}}(t_2,j_2)\right] \\
&\cdot \mathbb{E}\Big[W^*_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\varkappa_\beta\}}\Big]\mathbb{E}\Big[W^*_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\varkappa_\beta\}}\Big]\Big| \\
&\leqslant C \left|\mathrm{Cov}\left(\widetilde{\mathbb{I}}_{T,k_1,\mathfrak{R}}(t_1,j_1), \widetilde{\mathbb{I}}_{T,k_2,\mathfrak{R}}(t_2,j_2)\right)\right| \mathbb{1}_{\{\exists o_1\in\{t_1,j_1\},o_2\in\{t_2,j_2\}:\left|\lfloor u_{k_1}T\rfloor+o_1-(\lfloor u_{k_2}T\rfloor+o_2)\right|\leqslant\varkappa\}} \\
&+ \Big|\mathrm{Cov}\Big(W^*_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_1}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\varkappa_\beta\}}, W^*_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_2}T\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\varkappa_\beta\}}\Big)\Big| \\
&\cdot \mathbb{1}_{\{\exists o_1\in\{t_1,j_1\},o_2\in\{t_2,j_2\}:\left|\lfloor u_{k_1}T\rfloor+o_1-(\lfloor u_{k_2}T\rfloor+o_2)\right|\leqslant\varkappa_\beta\}} \left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\mathfrak{R}}(t_1,j_1)\right] \mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_2,\mathfrak{R}}(t_2,j_2)\right]\right| \\
&\leqslant C \left(\left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\mathfrak{R}}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k_2,\mathfrak{R}}(t_2,j_2)\right]\right| + \left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\mathfrak{R}}(t_1,j_1)\right] \mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_2,\mathfrak{R}}(t_2,j_2)\right]\right|\right) \\
&\cdot \mathbb{1}_{\{\exists o_1\in\{t_1,j_1\},o_2\in\{t_2,j_2\}:\left|\lfloor u_{k_1}T\rfloor+o_1-(\lfloor u_{k_2}T\rfloor+o_2)\right|\leqslant\varkappa_\beta\}}. \qquad \text{(C.440)}
\end{aligned}$$

One obtains from (C.440), (C.105), Lemma C.25 (i) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (see (C.439) as well as (C.17)):

$$\mathrm{Var}\left(\mathbf{R}^*_{T,1}\right) \leqslant \frac{C}{T^2 b} \sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=1+\varkappa_\beta}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta} \mathbb{1}_{\{k_1=k_2\}} \mathbb{1}_{\{|t_1-t_2|\leqslant\varkappa_\beta\}} = o(1). \qquad \text{(C.441)}$$

Further, it follows for all $s_{q_1}, s_{q_2} \in \mathbb{R}^d$ similarly to (C.371) (recall (C.80)):

$$\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{\substack{r_1,r_2=1\\ r_1\geqslant r_2}}^{9\varkappa_\beta+1} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\mathfrak{R}}(r_1,s_{q_1})\,\widetilde{\mathbb{K}}^c_{T,k,\mathfrak{R}}(r_2,s_{q_2})\right]\right| \leqslant C\varkappa_\beta\left(|s_{q_1}|_1+1\right). \qquad \text{(C.442)}$$

Moreover, one obtains for all $s_{q_1}, s_{q_2}, s_{q_3}, s_{q_4} \in \mathbb{R}^d$ analogously to (C.93) by using (C.442) as well as similar arguments (see (C.80) and note that $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\begin{aligned}
&\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{\substack{r_1,\ldots,r_4=1\\ r_1\geqslant\cdots\geqslant r_4}}^{9\varkappa_\beta+1} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1,s_{q_1})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2,s_{q_2})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3,s_{q_3})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4,s_{q_4})\right]\right| \\
&\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{\substack{r_1,\ldots,r_4=1\\ r_1\geqslant\cdots\geqslant r_4}}^{9\varkappa_\beta+1} \Bigg(\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1,s_{q_1})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2,s_{q_2})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3,s_{q_3})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4,s_{q_4})\right]\right| \\
&\cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_1,\ldots,r_4\}}|o_1-o_2|\leqslant 3\varkappa\}} + \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1,s_{q_1})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2,s_{q_2})\right]\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3,s_{q_3})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4,s_{q_4})\right]\right| \\
&\cdot \mathbb{1}_{\{\exists o_1,o_2\in\{r_1,\ldots,r_4\}:|o_1-o_2|>3\varkappa\}}\Bigg) \\
&\leqslant C\varkappa_\beta\varkappa^2\left(|s_{q_1}|_1+1\right) + C\varkappa_\beta^2\left(|s_{q_1}|_1+1\right)\left(|s_{q_3}|_1+1\right). \qquad \text{(C.443)}
\end{aligned}$$

It holds $\{1+\varkappa_\beta,\ldots,9\varkappa_\beta+1\}\subseteq\{1+\varkappa_\beta,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta\}$ due to (C.435). Thus, (C.440), (C.105), (C.443), (C.442), Assumption 3.1 $[\mathbf{WEI.1}]$, Lemma C.25 (i), Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii) imply (recall (C.439) as well as (C.80)):

$$\begin{aligned}
\operatorname{Var}\left(\mathbf{R}^*_{T,2}\right) &\leqslant \frac{C}{T^2 b} \sum_{\substack{k_1,k_2=1\\ k_1=k_2}}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2,j_1,j_2=1+\varkappa_\beta}^{9\varkappa_\beta+1} \Bigg(\left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\Re}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k_2,\Re}(t_2,j_2)\right]\right| \\
&\quad + \left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\Re}(t_1,j_1)\right]\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_2,\Re}(t_2,j_2)\right]\right|\Bigg) \\
&\leqslant \frac{C}{T^2b^2}\left(\varkappa_\beta\varkappa^2+\varkappa_\beta^2\right) \\
&= o(1). \qquad \text{(C.444)}
\end{aligned}$$

Further, one obtains for all $s_{q_1},s_{q_2},s_{q_3},s_{q_4}\in\mathbb{R}^d$ from Lemma B.4 (viii) and (B.45) (see (C.80)):

$$\begin{aligned}
&\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{\substack{r_1,\ldots,r_4=1\\ r_1\geqslant\cdots\geqslant r_4}}^{2\lfloor T_{\mathfrak{U}}b\rfloor} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1,s_{q_1})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2,s_{q_2})\right]\right|\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3,s_{q_3})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4,s_{q_4})\right]\right| \\
&\cdot \mathbb{1}_{\{\forall p_1,p_2\in\{r_1,\ldots,r_4\}:|p_1-p_2|\leqslant 15\varkappa_\beta\}} \\
&\leqslant C\sum_{r_2=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\left(\sum_{r_1=r_2+1}^{\infty}\sum_{l=r_1-r_2}^{\infty}\Delta_l\,|s_{q_1}|_1+\sum_{r_1=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\mathbb{1}_{\{r_1=r_2\}}\right)\sum_{\substack{r_4=1\\ |r_4-r_2|\leqslant 15\varkappa_\beta}}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\left(\sum_{r_3=r_4+1}^{\infty}\sum_{l=r_3-r_4}^{\infty}\Delta_l\,|s_{q_3}|_1+\sum_{r_3=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\mathbb{1}_{\{r_3=r_4\}}\right) \\
&\leqslant C\lfloor T_{\mathfrak{U}}b\rfloor\left(|s_{q_1}|+1\right)\varkappa_\beta\left(|s_{q_3}|+1\right). \qquad \text{(C.445)}
\end{aligned}$$

It follows for all $s_{q_1},s_{q_2},s_{q_3},s_{q_4}\in\mathbb{R}^d$ similarly to (C.93) by using (C.445) (recall (C.80) and Definition A.1 (i)):

$$\begin{aligned}
&\sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{\substack{r_1,\ldots,r_4=1\\ r_1\geqslant\cdots\geqslant r_4}}^{2\lfloor T_{\mathfrak{U}}b\rfloor} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1,s_{q_1})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2,s_{q_2})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3,s_{q_3})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4,s_{q_4})\right]\right| \\
&\cdot \mathbb{1}_{\{\forall p_1,p_2\in\{r_1,\ldots,r_4\}:|p_1-p_2|\leqslant 15\varkappa_\beta\}} \\
&\leqslant \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{\substack{r_1,\ldots,r_4=1\\ r_1\geqslant\cdots\geqslant r_4}}^{2\lfloor T_{\mathfrak{U}}b\rfloor} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1,s_{q_1})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2,s_{q_2})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3,s_{q_3})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4,s_{q_4})\right]\right| \\
&\cdot \mathbb{1}_{\{\forall p_1,p_2\in\{r_1,\ldots,r_4\}:|p_1-p_2|\leqslant 15\varkappa_\beta\}}\mathbb{1}_{\{\forall o_1,o_2\in\{r_1,\ldots,r_4\}:|o_1-o_2|\leqslant 3\varkappa\}} \\
&+ \sup_{k=1,\ldots,\lfloor 1/(2b)\rfloor} \sum_{\substack{r_1,\ldots,r_4=1\\ r_1\geqslant\cdots\geqslant r_4}}^{2\lfloor T_{\mathfrak{U}}b\rfloor} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_1,s_{q_1})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_2,s_{q_2})\right]\right|\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_3,s_{q_3})\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(r_4,s_{q_4})\right]\right|
\end{aligned}$$

$$\cdot \mathbb{1}_{\{\forall\, p_1,p_2\in\{r_1,\ldots,r_4\}:|p_1-p_2|\leqslant 15\varkappa_\beta\}}\mathbb{1}_{\{\exists\, o_1,o_2\in\{r_1,\ldots,r_4\}:|o_1-o_2|>3\varkappa\}}$$
$$\leqslant C\left\lfloor T_{\mathfrak{U}}b\right\rfloor \varkappa^2\left(|s_{q_1}|+1\right)+C\left\lfloor T_{\mathfrak{U}}b\right\rfloor \varkappa_\beta\left(|s_{q_1}|+1\right)\left(|s_{q_3}|+1\right). \tag{C.446}$$

If $t_1,t_2\in\{9\varkappa_\beta+2,\ldots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta\}$, $j_1\in\{t_1-7\varkappa_\beta,\ldots,t_1-1\}$, $j_2\in\{t_2-7\varkappa_\beta,\ldots,t_2-1\}$ and $\exists\, o_1\in\{t_1,j_1\}, o_2\in\{t_2,j_2\}:|o_1-o_2|\leqslant\varkappa_\beta$, one will obtain for all $p_1,p_2\in\{t_1,j_1,t_2,j_2\}$ that $|p_1-p_2|\leqslant 15\varkappa_\beta$. Hence, (C.440), (C.105), (C.446), (C.445), Assumption 3.1 $[\mathbf{WEI}.\mathbf{1}]$, Lemma C.25 (i), Remark A.2 (ii) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii) provide (see (C.439), (C.80) as well as (C.17)):

$$\begin{aligned}
\operatorname{Var}\left(\mathbf{R}^*_{T,3}\right) &\leqslant \frac{C}{T^2b}\sum_{\substack{k_1,k_2=1\\k_1=k_2}}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\sum_{j_1=t_1-7\varkappa_\beta}^{t_1-1}\sum_{j_2=t_2-7\varkappa_\beta}^{t_2-1}\Big(\left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\Re}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k_2,\Re}(t_2,j_2)\right]\right|\\
&\quad+\left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_1,\Re}(t_1,j_1)\right]\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k_2,\Re}(t_2,j_2)\right]\right|\Big)\mathbb{1}_{\{\forall\, p_1,p_2\in\{t_1,j_1,t_2,j_2\}:|p_1-p_2|\leqslant 15\varkappa_\beta\}}\\
&\leqslant \frac{C}{T^2b^2}\left(\lfloor T_{\mathfrak{U}}b\rfloor\varkappa^2+\lfloor T_{\mathfrak{U}}b\rfloor\varkappa_\beta\right)\\
&= o(1).
\end{aligned}\tag{C.447}$$

It follows from $\varkappa_\beta\geqslant 1$ (note Definition A.1 (v)) and (C.435) that $1+\varkappa_\beta\leqslant 2\varkappa_\beta\leqslant 9\varkappa_\beta+2\leqslant 2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta$. Thus, (C.358) together with $\varkappa_\beta\geqslant 1$, (C.105), Assumption 3.15 $[\mathbf{W}^*]$ (ii) and (iii) (the latter ensures $\sup_{t\in\mathbb{Z}}\left\|W^*_{t,\{\varkappa_\beta\}}\right\|_2\leqslant C$), arguments which are similar to those that show (C.442), Assumption 3.1 $[\mathbf{WEI}.\mathbf{1}]$, Lemma C.25 (i) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (ii) yield (recall (C.439), (C.108), (C.80), Definition A.1 (i) as well as (v) and (C.17)):

$$\begin{aligned}
\mathbb{E}\left[\left(\mathbf{R}^*_{T,4}\right)^2\right] &\leqslant \frac{C}{T^2b}\sum_{k_1,k_2=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\sum_{j_1,j_2=1+\varkappa_\beta}^{2\varkappa_\beta}\left|\mathbb{E}\left[\widetilde{\mathbb{I}}^*_{T,k_1,\Re}(t_1,j_1)\,\widetilde{\mathbb{I}}^*_{T,k_2,\Re}(t_2,j_2)\right]\right|\\
&\leqslant \frac{C}{T^2b}\sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{\lfloor 1/(2b)\rfloor}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{t_1,t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\sum_{j_1,j_2=1+\varkappa_\beta}^{2\varkappa_\beta}\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_1,\Re}(t_1,s_1)\right]\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_1,\Re}(j_1,s_1)\right]\right|\\
&\quad\cdot\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_2,\Re}(t_2,s_2)\right]\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_2,\Re}(j_2,s_2)\right]\right|\left|\mathbb{E}\left[W^*_{\lceil u_{k_1}T\rceil-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}}\right]\right.\\
&\quad\left.\cdot\mathbb{E}\left[W^*_{\lceil u_{k_1}T\rceil-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\varkappa_\beta\}}\right]\right|\left|\mathbb{E}\left[W^*_{\lceil u_{k_2}T\rceil-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}}\right]\mathbb{E}\left[W^*_{\lceil u_{k_2}T\rceil-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\varkappa_\beta\}}\right]\right|\\
&\quad\cdot\mathbf{w}(s_1)\,ds_1\mathbf{w}(s_2)\,ds_2\\
&\quad+\frac{C}{T^2b}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{\substack{k_1,k_2=1\\k_1=k_2}}^{\lfloor 1/(2b)\rfloor}\sum_{t_1,t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_1,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k_2,\Re}(t_2,s_2)\right]\right|\\
&\quad\cdot\sum_{j_1,j_2=1+\varkappa_\beta}^{2\varkappa_\beta}\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k_1,\Re}(j_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k_2,\Re}(j_2,s_2)\right]\right|\left|\mathbb{E}\left[W^*_{\lceil u_{k_1}T\rceil-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}}\right.\right.\\
&\quad\left.\left.\cdot W^*_{\lceil u_{k_2}T\rceil-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}}\right]\right|\left|\mathbb{E}\left[W^*_{\lceil u_{k_1}T\rceil-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\varkappa_\beta\}}W^*_{\lceil u_{k_2}T\rceil-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\varkappa_\beta\}}\right]\right|\mathbf{w}(s_1)\,ds_1\\
&\quad\cdot\mathbf{w}(s_2)\,ds_2\\
&\leqslant \frac{C}{T^2b^2}\lfloor T_{\mathfrak{U}}b\rfloor\varkappa_\beta\\
&= o(1).
\end{aligned}\tag{C.448}$$

Since $\mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1},\Re}$ is deterministic (note (3.57)), (C.439), (C.405), (C.441), (C.444), (C.447) and (C.448) imply:

$$\operatorname{Var}\left(T\sqrt{b}\,\widehat{\mathbb{T}}^{[2]*}_{T,\Re,\varkappa,\varkappa_\beta}-\mathbf{Bias}^{\text{distr}*}_{T,\mathfrak{U}_{0,1},\Re}-\widetilde{\mathbb{S}}^*_{T,\Re}\right)=o(1). \tag{C.449}$$

In conclusion, (C.438) and (C.449) provide (C.437). Moreover, one obtains similarly to (C.430) (see

(C.108) and (C.101)):

$$\mathbb{E}\left[\left(\widetilde{\mathbb{S}}^*_{T,\Re} - \mathbb{S}^*_{T,\Re}\right)^2\right] = o(1). \tag{C.450}$$

Overall, (C.436), Lemma C.28, (C.437) and (C.450) as well as analog arguments prove Lemma C.31. □

Lemma C.32. *Let the Assumptions 2.4* $[\mathbf{DM.2}]$*, 3.1* $[\mathbf{WEI.1}]$*, 2.8* $[\mathbf{K\&b.1}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled. Then, one obtains for* $T \to \infty$ *(recall (C.101) as well as (3.52)):*

$$\mathbb{E}\left[\left|\mathrm{Var}^*\left(\mathbb{S}^*_T\right) - \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1}}\right|\right] = o(1).$$

Proof. Throughout this proof, T should be large enough to ensure (see (C.17) as well as Definition A.1 (v)):

$$2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta \geqslant 9\varkappa_\beta + 2,$$

which holds for sufficiently large T due to Lemma C.25 (i) and Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii).
In the following, it will be proved (note (C.101) and (3.52)):

$$\mathbb{E}\left[\left|\mathrm{Cov}^*\left(\mathbb{S}^*_{T,\Re}, \mathbb{S}^*_{T,\Im}\right) - \sigma^{\mathrm{distr}}_{\mathfrak{U}_{0,1},\Re,\Im}\right|\right] = o(1). \tag{C.451}$$

One obtains for all $k_1, k_2 \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $t_1, j_1, t_2, j_2 \in \{1 + \varkappa_\beta, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta\}$ with $k_1 \geqslant k_2 + 1$ from (C.105) (recall Definition A.1 (i)):

$$\begin{aligned}&\mathrm{Cov}\left(W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_1,\{\varkappa_\beta\}}, W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_2,\{\varkappa_\beta\}}\right)\\ &= 0\end{aligned} \tag{C.452}$$

and it follows for all $k \in \{1, \ldots, \lfloor 1/(2b)\rfloor\}$, $t_1, j_1, t_2, j_2 \in \{1 + \varkappa_\beta, \ldots, 2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta\}$ with $|t_h - j_h| \geqslant 7\varkappa_\beta + 1 \; \forall\, h \in \{1, 2\}$ from Assumption 3.15 $[\mathbf{W}^*]$ (iii), which ensures $\mathbb{E}\big[W^*_{t,\{\varkappa_\beta\}}\big] = \mathbb{E}[W^*_t] = 0$ $\forall\, t \in \mathbb{Z}$ (see Definition A.1 (i)):

$$\begin{aligned}&\mathrm{Cov}\left(W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_1,\{\varkappa_\beta\}}, W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_2} T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_2,\{\varkappa_\beta\}}\right)\\ &= \mathbb{E}\left[W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\varkappa_\beta\}}\right] \mathbb{E}\left[W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_1,\{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_2,\{\varkappa_\beta\}}\right].\end{aligned} \tag{C.453}$$

Assumption 3.15 $[\mathbf{W}^*]$ (ii), (C.452) and (C.453) imply (recall (C.101)):

$$\begin{aligned}\mathrm{Cov}^*\left(\mathbb{S}^*_{T,\Re}, \mathbb{S}^*_{T,\Im}\right) &= \frac{4T^2 b\,(\mathfrak{U}_1 - \mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2 \lfloor Tb\rfloor^4} \sum_{\substack{k_1,k_2=1\\ k_1=k_2}}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta} \sum_{j_1=1+2\varkappa_\beta}^{t_1 - 7\varkappa_\beta - 1} \sum_{j_2=1+2\varkappa_\beta}^{t_2 - 7\varkappa_\beta - 1} \mathbb{I}_{T,k_1,\Re}(t_1, j_1)\\ &\cdot \mathbb{I}_{T,k_2,\Im}(t_2, j_2)\,\mathrm{Cov}\Big(W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_{k_1}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_1,\{\varkappa_\beta\}}, W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\varkappa_\beta\}}\\ &\cdot W^*_{\lfloor u_{k_2}T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_2,\{\varkappa_\beta\}}\Big)\\ &= \frac{4T^2 b\,(\mathfrak{U}_1 - \mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2 \lfloor Tb\rfloor^4} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor - 1 - \varkappa_\beta} \sum_{j_1=1+2\varkappa_\beta}^{t_1 - 7\varkappa_\beta - 1} \sum_{j_2=1+2\varkappa_\beta}^{t_2 - 7\varkappa_\beta - 1} \mathbb{I}_{T,k,\Re}(t_1, j_1)\,\mathbb{I}_{T,k,\Im}(t_2, j_2)\\ &\cdot \mathbb{E}\left[W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + t_2,\{\varkappa_\beta\}}\right] \mathbb{E}\left[W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_1,\{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor - \lfloor T_{\mathfrak{U}} b\rfloor + j_2,\{\varkappa_\beta\}}\right].\end{aligned} \tag{C.454}$$

Moreover, one defines the following expression, which results from the right side of (C.454) by replacing

$\mathbb{I}_{T,k,\Re}(t_1,j_1)\,\mathbb{I}_{T,k,\Im}(t_2,j_2)$ by $\widetilde{\mathbb{I}}_{T,k,\Re}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k,\Im}(t_2,j_2)$ (see (C.80)):

$$\widetilde{\mathrm{Cov}}^*_{T,1} := \frac{4T^2 b\,(\mathfrak{U}_1-\mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2 \lfloor Tb\rfloor^4} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa_\beta} \sum_{j_1=1+2\varkappa_\beta}^{t_1-7\varkappa_\beta-1} \sum_{j_2=1+2\varkappa_\beta}^{t_2-7\varkappa_\beta-1} \widetilde{\mathbb{I}}_{T,k,\Re}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k,\Im}(t_2,j_2)$$
$$\cdot\,\mathbb{E}\left[W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_2,\{\varkappa_\beta\}}\right] \mathbb{E}\left[W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+j_1,\{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+j_2,\{\varkappa_\beta\}}\right]. \tag{C.455}$$

It follows from (C.454), (C.112), the first inequality of (C.113) with $q = 1+\delta$, $1/\delta \geqslant 1$ (which holds because Assumption 2.2 $[\mathbf{StAp}]$ supposes $\delta \in (0,1]$), (C.114), Assumption 3.15 $[\mathbf{W}^*]$ (i) (the latter ensures $\beta = o\big(\sqrt{Tb^2 1/b}\big)$) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (recall (C.455), (C.17), (C.101) as well as (C.80)):

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathrm{Cov}^*\left(\mathbb{S}^*_{T,\Re},\mathbb{S}^*_{T,\Im}\right) - \widetilde{\mathrm{Cov}}^*_{T,1}\right|\right]\\
&\leqslant \frac{C}{T^2 b} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \sum_{j_1,j_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left(\frac{1}{(Tb)^{1/\delta}} + \frac{1}{T}\right) \left|\mathbb{E}\left[W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+t_2,\{\varkappa_\beta\}}\right]\right|\\
&\cdot \left|\mathbb{E}\left[W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+j_1,\{\varkappa_\beta\}} W^*_{\lfloor u_k T\rfloor-\lfloor T_{\mathfrak{U}} b\rfloor+j_2,\{\varkappa_\beta\}}\right]\right|\\
&= o(1).
\end{aligned} \tag{C.456}$$

One obtains from Assumption 3.15 $[\mathbf{W}^*]$ (iii), that yields $\sup_{t\in\mathbb{Z}} \left\|W^*_{t,\{\varkappa_\beta\}}\right\|_2 \leqslant C$, (C.85), $\varkappa_\beta \geqslant \varkappa$ (which holds due to Lemma C.25 (i)) and similarly to arguments that show the second inequality and last equality of (C.96), i. e., arguments which are analog to (C.91) to (C.95) (see (C.455), (C.80), Definition A.1 (i), (C.17) as well as (C.90)):

$$\begin{aligned}
&\mathrm{Var}\left(\widetilde{\mathrm{Cov}}^*_{T,1}\right)\\
&\leqslant \frac{C}{T^4 b^2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{\substack{k_1,k_2=1\\k_1=k_2}}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2,t_3,t_4=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa_\beta} \sum_{j_1=1+2\varkappa_\beta}^{t_1-7\varkappa_\beta-1} \sum_{j_2=1+2\varkappa_\beta}^{t_2-7\varkappa_\beta-1} \sum_{j_3=1+2\varkappa_\beta}^{t_3-7\varkappa_\beta-1} \sum_{j_4=1+2\varkappa_\beta}^{t_4-7\varkappa_\beta-1}\\
&\Big|\mathrm{Cov}\Big(\widetilde{\mathbb{K}}^c_{T,k_1,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k_1,\Re}(j_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k_1,\Im}(t_2,s_2)\,\widetilde{\mathbb{K}}^c_{T,k_1,\Im}(j_2,s_2), \widetilde{\mathbb{K}}^c_{T,k_2,\Re}(t_3,s_3)\\
&\cdot \widetilde{\mathbb{K}}^c_{T,k_2,\Re}(j_3,s_3)\,\widetilde{\mathbb{K}}^c_{T,k_2,\Im}(t_4,s_4)\,\widetilde{\mathbb{K}}^c_{T,k_2,\Im}(j_4,s_4)\Big)\Big|\ \mathbf{w}(s_1)\ ds_1\,\mathbf{w}(s_2)\ ds_2\,\mathbf{w}(s_3)\ ds_3\,\mathbf{w}(s_4)\ ds_4\\
&\leqslant \frac{C}{T^4 b^2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2,t_3,t_4=9\varkappa+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa} \sum_{j_1=1+2\varkappa}^{t_1-7\varkappa-1} \sum_{j_2=1+2\varkappa}^{t_2-7\varkappa-1} \sum_{j_3=1+2\varkappa}^{t_3-7\varkappa-1} \sum_{j_4=1+2\varkappa}^{t_4-7\varkappa-1}\\
&\Bigg(\Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_2,s_2)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(j_2,s_2)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_3,s_3)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_3,s_3)\\
&\cdot \widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_4)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(j_4,s_4)\Big]\Big| + \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_2,s_2)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(j_2,s_2)\Big]\Big|\\
&\cdot \Big|\mathbb{E}\Big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_3,s_3)\,\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_3,s_3)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_4,s_4)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(j_4,s_4)\Big]\Big|\Bigg)\ \mathbf{w}(s_1)\ ds_1\,\mathbf{w}(s_2)\ ds_2\\
&\cdot \mathbf{w}(s_3)\ ds_3\,\mathbf{w}(s_4)\ ds_4\\
&= o(1).
\end{aligned} \tag{C.457}$$

Further, one defines (recall (C.17), Definition A.1 (i) as well as (C.80) and note that the following expression results from (C.455) by omitting the terms contained in the second line of (C.455)):

$$\widetilde{\mathrm{Cov}}^*_{T,2} := \frac{4T^2 b\,(\mathfrak{U}_1-\mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2 \lfloor Tb\rfloor^4} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa_\beta} \sum_{j_1=1+2\varkappa_\beta}^{t_1-7\varkappa_\beta-1} \sum_{j_2=1+2\varkappa_\beta}^{t_2-7\varkappa_\beta-1} \widetilde{\mathbb{I}}_{T,k,\Re}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k,\Im}(t_2,j_2). \tag{C.458}$$

It holds $\sup_{t\in\mathbb{Z}} |K^*(t/\beta) - 1| \leqslant C$ due to Assumption 3.15 $[\mathbf{W}^*]$ (iii), such that iterative applying of the Fatou–Lebesgue theorem together with (B.45), Assumption 2.4 $[\mathbf{DM.2}]$ and Assumption 3.15 $[\mathbf{W}^*]$ (i) as well as (iii) (the latter two ensure $K^*(t/\beta) \to 1$ for $T \to \infty$ and all $t \in \mathbb{Z}$) provide:

$$\limsup_{T\to\infty} \sum_{t=1}^{\infty}\sum_{l=t}^{\infty} \Delta_l \left| K^*\left(\frac{t}{\beta}\right) - 1\right| \leqslant \sum_{t=1}^{\infty}\sum_{l=t}^{\infty} \Delta_l \limsup_{T\to\infty} \left| K^*\left(\frac{t}{\beta}\right) - 1\right| = 0. \tag{C.459}$$

Lemma B.4 (viii), Assumption 3.1 $[\mathbf{WEI.1}]$, (C.112), Lemma C.25 (ii), Assumption 3.15 $[\mathbf{W}^*]$ (iii) $\big($which yields $\sup_{t\in\mathbb{Z}} \|W^*_{t,\{\varkappa_\beta\}}\|_2 \leqslant C$, $\sup_{t\in\mathbb{Z}} |K^*(t/\beta) - 1| \leqslant C$, $K^*(0) = 1$ as well as $K^*(x/\beta) = K^*(-x/\beta)\ \forall\, x \in \mathbb{Z}\big)$, shifting the indices of sums, (B.45), Assumption 2.8 $[\mathbf{K\&b.1}]$ (ii) and (C.459) show (see (C.17) as well as (C.80)):

$$\begin{aligned}
&\frac{4T^2 b\,(\mathfrak{U}_1 - \mathfrak{U}_0)^2}{\lfloor 1/(2b)\rfloor^2 \lfloor Tb\rfloor^4} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{\substack{t_1,t_2=9\varkappa_\beta+2\\ t_1\geqslant t_2}}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa_\beta} \sum_{j_1=1+2\varkappa_\beta}^{t_1-7\varkappa_\beta-1} \sum_{\substack{j_2=1+2\varkappa_\beta\\ j_2\geqslant j_1}}^{t_2-7\varkappa_\beta-1} \\
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_2,s_2)\right]\right| \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(j_2,s_2)\right]\right| \mathbf{w}(s_1)\,ds_1\,\mathbf{w}(s_2)\,ds_2 \\
&\cdot \Bigg(\bigg|\mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}} W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}}\right] - \mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1} W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2}\right] \\
&+ K^*\left(\frac{t_1-t_2}{\beta}\right) - 1\bigg| \left|\mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\varkappa_\beta\}} W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\varkappa_\beta\}}\right]\right| \\
&+ \bigg|\mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\varkappa_\beta\}} W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\varkappa_\beta\}}\right] - \mathbb{E}\left[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_1} W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_2}\right] \\
&+ K^*\left(\frac{j_1-j_2}{\beta}\right) - 1\bigg|\Bigg) \\
&\leqslant \frac{C}{T^2 b} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Bigg(\sum_{t_1=t_2+1}^{\infty} \sum_{l=t_1-t_2}^{\infty} \Delta_l \left(\frac{1}{Tb} + \left|K^*\left(\frac{t_1-t_2}{\beta}\right) - 1\right|\right) + \sum_{t_1=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \mathbb{1}_{\{t_1=t_2\}} \bigg(\frac{1}{Tb} \\
&+ \left|K^*\left(\frac{t_1-t_2}{\beta}\right) - 1\right|\bigg)\Bigg) \cdot \sum_{j_1=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left(\sum_{j_2=j_1+1}^{\infty} \sum_{l=j_2-j_1}^{\infty} \Delta_l + \sum_{j_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \mathbb{1}_{\{j_2=j_1\}} \right) \\
&+ \frac{C}{T^2 b} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left(\sum_{t_1=t_2+1}^{\infty} \sum_{l=t_1-t_2}^{\infty} \Delta_l + \sum_{t_1=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \mathbb{1}_{\{t_1=t_2\}} \right) \cdot \sum_{j_1=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \Bigg(\sum_{j_2=j_1+1}^{\infty} \sum_{l=j_2-j_1}^{\infty} \Delta_l \\
&\cdot \left(\frac{1}{Tb} + \left|K^*\left(\frac{j_1-j_2}{\beta}\right) - 1\right|\right) + \sum_{j_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \mathbb{1}_{\{j_2=j_1\}} \left(\frac{1}{Tb} + \left|K^*\left(\frac{j_1-j_2}{\beta}\right) - 1\right|\right)\Bigg) \\
&\leqslant \frac{C}{T^2 b^2} \sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left(\sum_{t_1=1}^{\infty} \sum_{l=t_1}^{\infty} \Delta_l \frac{1}{Tb} + \sum_{t_1=1}^{\infty} \sum_{l=t_1}^{\infty} \Delta_l \left|K^*\left(\frac{t_1}{\beta}\right) - 1\right| + \frac{1}{Tb} + 0\right) \sum_{j_1=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} C \\
&+ \frac{C}{T^2 b^2} \sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} C \sum_{j_1=1}^{2\lfloor T_{\mathfrak{U}} b\rfloor} \left(\sum_{j_2=1}^{\infty} \sum_{l=j_2}^{\infty} \Delta_l \frac{1}{Tb} + \sum_{j_2=1}^{\infty} \sum_{l=j_2}^{\infty} \Delta_l \left|K^*\left(\frac{-j_2}{\beta}\right) - 1\right| + \frac{1}{Tb} + 0\right) \\
&= o(1).
\end{aligned} \tag{C.460}$$

One obtains from (C.112), (C.460) together with $\mathbb{E}\big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+l_1} W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+l_2}\big] = K^*\big((l_1 - l_2)/\beta\big)\ \forall\, l_1, l_2 \in \mathbb{Z}$ (whereby the latter holds due to Assumption 3.15 $[\mathbf{W}^*]$ (iii)) and similar arguments (recall (C.455), (C.458), (C.80) as well as Definition A.1 (i)):

$$\begin{aligned}
&\left|\mathbb{E}\left[\widetilde{\mathrm{Cov}}^*_{T,1}\right] - \mathbb{E}\left[\widetilde{\mathrm{Cov}}^*_{T,2}\right]\right| \\
&\leqslant \frac{C}{T^2 b} \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \sum_{t_1,t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa_\beta} \sum_{j_1=1+2\varkappa_\beta}^{t_1-7\varkappa_\beta-1} \sum_{j_2=1+2\varkappa_\beta}^{t_2-7\varkappa_\beta-1}
\end{aligned}$$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_2,s_2)\right]\right| \left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(j_2,s_2)\right]\right| \mathbf{w}(s_1)\,ds_1\,\mathbf{w}(s_2)\,ds_2$$
$$\cdot \left|\mathbb{E}\Big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_1,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+t_2,\{\varkappa_\beta\}}\Big]\,\mathbb{E}\Big[W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_1,\{\varkappa_\beta\}}W^*_{\lfloor u_kT\rfloor-\lfloor T_{\mathfrak{U}}b\rfloor+j_2,\{\varkappa_\beta\}}\Big]-1\cdot 1\right|$$
$$= o(1). \tag{C.461}$$

Further, $\varkappa_\beta \geqslant \varkappa + 1$, which holds due to Lemma C.25 (i), implies (see (C.80) and (C.458)):

$$\left|\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathbb{E}\left[\widetilde{\mathbb{S}}_{T,k,\Re}\widetilde{\mathbb{S}}_{T,k,\Im}\right]-\mathbb{E}\left[\widetilde{\mathrm{Cov}}^*_{T,2}\right]\right|$$
$$\leqslant \frac{C}{T^2b}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1\in\{9\varkappa+2,\dots,9\varkappa_\beta+1\}\cup\{2\lfloor T_{\mathfrak{U}}b\rfloor-\varkappa_\beta,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa\}}\ \sum_{t_2=9\varkappa+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa}\ \sum_{j_1=1+2\varkappa}^{t_1-7\varkappa-1}\ \sum_{j_2=1+2\varkappa}^{t_2-7\varkappa-1}$$
$$\left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k,\Re}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k,\Im}(t_2,j_2)\right]\right|$$
$$+\frac{C}{T^2b}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\ \sum_{t_2\in\{9\varkappa+2,\dots,9\varkappa_\beta+1\}\cup\{2\lfloor T_{\mathfrak{U}}b\rfloor-\varkappa_\beta,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa\}}\ \sum_{j_1=1+2\varkappa}^{t_1-7\varkappa-1}\ \sum_{j_2=1+2\varkappa}^{t_2-7\varkappa-1}$$
$$\left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k,\Re}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k,\Im}(t_2,j_2)\right]\right|$$
$$+\frac{C}{T^2b}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\ \sum_{t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\ \sum_{j_1\in\{1+2\varkappa,\dots,2\varkappa_\beta\}\cup\{t_1-7\varkappa_\beta,\dots,t_1-7\varkappa-1\}}\ \sum_{j_2=1+2\varkappa}^{t_2-7\varkappa-1}$$
$$\left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k,\Re}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k,\Im}(t_2,j_2)\right]\right|$$
$$+\frac{C}{T^2b}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\ \sum_{t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\ \sum_{j_1=1+2\varkappa_\beta}^{t_1-7\varkappa_\beta-1}\ \sum_{j_2\in\{1+2\varkappa,\dots,2\varkappa_\beta\}\cup\{t_2-7\varkappa_\beta,\dots,t_1-7\varkappa-1\}}$$
$$\left|\mathbb{E}\left[\widetilde{\mathbb{I}}_{T,k,\Re}(t_1,j_1)\,\widetilde{\mathbb{I}}_{T,k,\Im}(t_2,j_2)\right]\right|$$
$$=: \widetilde{\mathbf{R}}^*_{T,1}+\widetilde{\mathbf{R}}^*_{T,2}+\widetilde{\mathbf{R}}^*_{T,3}+\widetilde{\mathbf{R}}^*_{T,4}. \tag{C.462}$$

In the following, it will be just shown that $\widetilde{\mathbf{R}}^*_{T,4}$ vanishes asymptotically because similar arguments imply that $\widetilde{\mathbf{R}}^*_{T,1}$, $\widetilde{\mathbf{R}}^*_{T,2}$ and $\widetilde{\mathbf{R}}^*_{T,3}$ converge to zero for $T\to\infty$.

Let $\widehat{\mathscr{T}}_{T,\varkappa,\varkappa_\beta}(t_2) := \{1+2\varkappa,\dots,2\varkappa_\beta\}\cup\{t_2-7\varkappa_\beta,\dots,t_2-7\varkappa-1\}$ and $\widehat{\mathscr{T}}^+_{T,\varkappa,\varkappa_\beta}(t_2) := \{x\in\mathbb{N} : \exists y\in\widehat{\mathscr{T}}_{T,\varkappa,\varkappa_\beta}(t_2) : |y-x|\leqslant\varkappa\}\ \forall\, t_2\in\{9\varkappa_\beta+2,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta\}$. It holds for all $k\in\{1,\dots,\lfloor 1/(2b)\rfloor\}$, $l_1,l_2\in\{1,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor\}$, $s_1,s_2\in\mathbb{R}^d$ with $|l_1-l_2|>\varkappa$ that $\mathbb{E}\big[\widetilde{\mathbb{K}}^c_{T,k,\Re}(l_1,s_1)\cdot\widetilde{\mathbb{K}}^c_{T,k,\Im}(l_2,s_2)\big]=0$ (recall (C.80) as well as Definition A.1 (i)) and, obviously, $\widehat{\mathscr{T}}_{T,\varkappa,\varkappa_\beta}(t_2)\subseteq\widehat{\mathscr{T}}^+_{T,\varkappa,\varkappa_\beta}(t_2)\ \forall\, t_2\in\{9\varkappa_\beta+2,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta\}$. Thus, one obtains from Lemma B.4 (viii), Assumption 3.1 [**WEI.1**], (B.45), Lemma C.25 (i) and Assumption 2.8 [**K**&**b.1**] (ii) (see (C.462) as well as (C.80)):

$$\widetilde{\mathbf{R}}^*_{T,4}\leqslant\frac{C}{T^2b}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{t_1,t_2=9\varkappa_\beta+2}^{2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta}\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(t_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(t_2,s_2)\right]\right|$$
$$\cdot\sup_{\tilde{t}_2\in\{9\varkappa_\beta+2,\dots,2\lfloor T_{\mathfrak{U}}b\rfloor-1-\varkappa_\beta\}}\ \sum_{j_1\in\widehat{\mathscr{T}}^+_{T,\varkappa,\varkappa_\beta}(\tilde{t}_2)}\ \sum_{j_2\in\widehat{\mathscr{T}}_{T,\varkappa,\varkappa_\beta}(\tilde{t}_2)}\left|\mathbb{E}\left[\widetilde{\mathbb{K}}^c_{T,k,\Re}(j_1,s_1)\,\widetilde{\mathbb{K}}^c_{T,k,\Im}(j_2,s_2)\right]\right|$$
$$\cdot\,\mathbf{w}(s_1)\,ds_1\,\mathbf{w}(s_2)\,ds_2$$
$$\leqslant\frac{C}{T^2b}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\sum_{t_2=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\ \sum_{t_1=t_2+1}^{\infty}\ \sum_{l=t_1-t_2}^{\infty}\Delta_l+\sum_{t_1=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\ \sum_{t_2=t_1+1}^{\infty}\ \sum_{l=t_2-t_1}^{\infty}\Delta_l+\sum_{t_1,t_2=1}^{2\lfloor T_{\mathfrak{U}}b\rfloor}\mathbb{1}_{\{t_1=t_2\}}\right)$$

$$\cdot \sup_{\tilde{t}_2 \in \{9\varkappa_\beta+2,\ldots,2\lfloor T_{\mathfrak{U}} b\rfloor-1-\varkappa_\beta\}} \left(\sum_{j_1 \in \widehat{\mathscr{T}}^{+}_{T,\varkappa,\varkappa_\beta}(\tilde{t}_2)} \sum_{j_2=j_1+1}^{\infty} \sum_{l=j_2-j_1}^{\infty} \Delta_l + \sum_{j_2 \in \widehat{\mathscr{T}}_{T,\varkappa,\varkappa_\beta}(\tilde{t}_2)} \sum_{j_1=j_2+1}^{\infty} \sum_{l=j_1-j_2}^{\infty} \Delta_l \right.$$

$$\left. + \sum_{j_1,j_2 \in \widehat{\mathscr{T}}_{T,\varkappa,\varkappa_\beta}(\tilde{t}_2)} \mathbb{1}_{\{j_2=j_1\}} \right)$$

$$\leqslant \frac{C\varkappa_\beta}{Tb} = o(1). \tag{C.463}$$

It follows from (C.86), (C.84), (C.462), (C.463) and similar arguments (recall (C.458) as well as (C.80)):

$$\left|\mathbb{E}\left[\widetilde{\mathrm{Cov}}^*_{T,2}\right] - \mathrm{Cov}\left(\widetilde{\mathbb{S}}_{T,\Re}, \widetilde{\mathbb{S}}_{T,\Im}\right)\right| = \left|\mathbb{E}\left[\widetilde{\mathrm{Cov}}^*_{T,2}\right] - \sum_{k=1}^{\lfloor 1/(2b)\rfloor} \mathbb{E}\left[\widetilde{\mathbb{S}}_{T,k,\Re}\widetilde{\mathbb{S}}_{T,k,\Im}\right]\right| = o(1). \tag{C.464}$$

In conclusion, (C.456), (C.457), (C.461), (C.464), (C.413) with $\mathrm{R}_1 = \Re$ and $\mathrm{R}_2 = \Im$, the fact that $\mathbf{Bias}^{\mathrm{distr}}_{T,\mathfrak{U}_{0,1},\mathrm{R}}$ is deterministic for all $\mathrm{R} \in \{\Re, \Im\}$ (note (3.51)) and Lemma C.23 with $\mathrm{R}_1 = \Re$ as well as $\mathrm{R}_2 = \Im$ provide (C.451). Lemma C.32 holds due to $\mathrm{Var}^*\left(\mathbb{S}^*_T\right) = \mathrm{Var}^*\left(\mathbb{S}^*_{T,\Re} + \mathbb{S}^*_{T,\Im}\right) = \mathrm{Var}^*\left(\mathbb{S}^*_{T,\Re}\right) + \mathrm{Var}^*\left(\mathbb{S}^*_{T,\Im}\right) + \mathrm{Cov}^*\left(\mathbb{S}^*_{T,\Re}, \mathbb{S}^*_{T,\Im}\right) + \mathrm{Cov}^*\left(\mathbb{S}^*_{T,\Im}, \mathbb{S}^*_{T,\Re}\right)$ (see (C.101)), (C.451) and similar arguments (recall (3.52)). ☐

Lemma C.33. *Let the Assumptions 2.4* $[\mathbf{DM.1}]$ *and 2.8* $[\mathbf{K\&b.1}]$ *be fulfilled. Then, one obtains for* $T \in \mathbb{N}\backslash\{1\}$ *and all* $h \in \{-\lfloor T/2\rfloor+1,\ldots,\lfloor T/2\rfloor-1\}$, $\mathrm{R} \in \{\Re,\Im\}$, $s \in \mathbb{R}^d$ *(see (2.9) as well as (3.66) and observe that* $\delta \in (0,1]$ *originates from Assumption 2.2* $[\mathbf{StAp}]$*):*

$$\sup_{u \in \mathfrak{U}_{0,1,b}} \left\|\widehat{\sigma}_{h,T,\mathrm{R}}(u,s) - \mathrm{Cov}\left(\mathrm{R}\left\{e^{i\langle s,\tilde{X}_0(u)\rangle}\right\}, \mathrm{R}\left\{e^{i\langle s,\tilde{X}_h(u)\rangle}\right\}\right)\right\|_{1+\delta}$$
$$\leqslant \frac{C}{Tb}(|h|+1) + C\left(b + \frac{1}{\sqrt{Tb}} + \frac{|h|}{T}\right)\left(|s|_1^{1+\delta}+1\right).$$

Proof. In the following, Lemma C.33 with $\mathrm{R} = \Re$ will be proved. Therefor, one defines for all $h \in \{-\lfloor T/2\rfloor+1,\ldots,\lfloor T/2\rfloor-1\}$, $u \in [0,1]$, $s \in \mathbb{R}^d$ (recall (3.66) and Definition 2.11):

$$\widehat{\sigma}_{1,h,T,\Re}(u,s) := \frac{1}{T}\sum_{t=1+|h|}^{T-|h|} K_b\left(\frac{t}{T}-u\right) \cdot \cos\left(\langle s, X_{t,T}\rangle\right)^{\widehat{c}(u)} \cdot \cos\left(\langle s, X_{t+h,T}\rangle\right)^{\widehat{c}(u)}. \tag{C.465}$$

Since Lemma B.1 with $\kappa_1 = 1$ shows (see (3.66) and Definition 2.11):

$$\sup_{s\in\mathbb{R}^d}\sup_{u\in[0,1]}\sup_{t=1,\ldots,T}\left|\cos\left(\langle s,X_{t,T}\rangle\right)^{\widehat{c}(u)}\right| \leqslant C \quad \text{a. s.}, \tag{C.466}$$

it holds for all $h \in \{-\lfloor T/2\rfloor+1,\ldots,\lfloor T/2\rfloor-1\}$, $s \in \mathbb{R}^d$ (recall (3.66), (C.465) as well as Definition 2.11):

$$\sup_{u\in\mathfrak{U}_{0,1,b}}\left\|\widehat{\sigma}_{h,T,\Re}(u,s) - \widehat{\sigma}_{1,h,T,\Re}(u,s)\right\|_2 \leqslant \frac{C|h|}{Tb}. \tag{C.467}$$

Moreover, one defines for all $h \in \{-\lfloor T/2\rfloor+1,\ldots,\lfloor T/2\rfloor-1\}$, $u \in [0,1]$, $s \in \mathbb{R}^d$:

$$\widehat{\sigma}_{2,h,T,\Re}(u,s) := \frac{1}{T}\sum_{t=1+|h|}^{T-|h|} K_b\left(\frac{t}{T}-u\right)\cdot\left(\cos\left(\langle s,X_{t,T}\rangle\right) - \mathbb{E}\left[\cos\left(\left\langle s,\tilde{X}_t(u)\right\rangle\right)\right]\right)$$
$$\cdot\left(\cos\left(\langle s,X_{t+h,T}\rangle\right) - \mathbb{E}\left[\cos\left(\left\langle s,\tilde{X}_{t+h}(u)\right\rangle\right)\right]\right). \tag{C.468}$$

It follows for all $h \in \{-\lfloor T/2\rfloor+1,\ldots,\lfloor T/2\rfloor-1\}$, $s \in \mathbb{R}^d$ from (C.112), Lemma B.1 with $\kappa_1 = 1$, whereby the latter provides $\sup_{u\in[0,1]} 1/T\sum_{t=1+|h|}^{T-|h|} K_b\left(\frac{t}{T}-u\right) \leqslant C$, (C.466), the Propositions 2.12

and 2.14 together with the Definitions 2.11 as well as 2.6 (see (C.465), (C.468) and (3.66)):

$$\sup_{u\in\mathfrak{U}_{0,1,b}} \left\|\widehat{\sigma}_{1,h,T,\Re}(u,s) - \widehat{\sigma}_{2,h,T,\Re}(u,s)\right\|_2 \leqslant C\left(b^{1+\delta} + \frac{1}{\sqrt{Tb}}\right)\left(|s|_1^{1+\delta} + 1\right). \tag{C.469}$$

Further, one defines for all $h \in \{-\lfloor T/2\rfloor+1,\ldots,\lfloor T/2\rfloor-1\}$, $u \in [0,1]$, $s \in \mathbb{R}^d$ the following expression, which results from (C.468) by replacing $X_{t,T}$ by $\widetilde{X}_t(u)$ and $X_{t+h,T}$ by $\widetilde{X}_{t+h}(u)$:

$$\begin{aligned}\widehat{\sigma}_{3,h,T,\Re}(u,s) := \frac{1}{T}\sum_{t=1+|h|}^{T-|h|} K_b\left(\frac{t}{T}-u\right)\cdot\left(\cos\left(\left\langle s,\widetilde{X}_t(u)\right\rangle\right) - \mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_t(u)\right\rangle\right)\right]\right)\\ \cdot\left(\cos\left(\left\langle s,\widetilde{X}_{t+h}(u)\right\rangle\right) - \mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_{t+h}(u)\right\rangle\right)\right]\right).\end{aligned} \tag{C.470}$$

It holds for all $h \in \{-\lfloor T/2\rfloor+1,\ldots,\lfloor T/2\rfloor-1\}$, $s \in \mathbb{R}^d$ due to (C.112), Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) (the latter ensures $K(z) = 0$ for all $z \in \mathbb{R}$ with $|z| > 1$), Assumption 2.2 $[\mathbf{StAp}]$ (i), Remark 2.3 and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (ii) (recall (C.468), (C.470) as well as Definition 2.11):

$$\begin{aligned}&\sup_{u\in\mathfrak{U}_{0,1,b}} \left\|\widehat{\sigma}_{2,h,T,\Re}(u,s) - \widehat{\sigma}_{3,h,T,\Re}(u,s)\right\|_{1+\delta}\\ &\leqslant \sup_{u\in\mathfrak{U}_{0,1,b}} \frac{1}{Tb}\sum_{t=1+|h|}^{T-|h|} K\left(\frac{\frac{t}{T}-u}{b}\right)\mathbb{1}_{\{|\frac{t}{T}-u|\leqslant b\}}\cdot\Bigg\|\Bigg\{\cos\left(\langle s,X_{t,T}\rangle\right) - \cos\left(\left\langle s,\widetilde{X}_t\left(\frac{t}{T}\right)\right\rangle\right)\\ &+\cos\left(\left\langle s,\widetilde{X}_t\left(\frac{t}{T}\right)\right\rangle\right) - \cos\left(\left\langle s,\widetilde{X}_t(u)\right\rangle\right)\Bigg\}\Bigg\{\cos\left(\langle s,X_{t+h,T}\rangle\right) - \mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_{t+h}(u)\right\rangle\right)\right]\Bigg\}\\ &+\Bigg\{\cos\left(\langle s,X_{t+h,T}\rangle\right) - \cos\left(\left\langle s,\widetilde{X}_{t+h}\left(\frac{t+h}{T}\right)\right\rangle\right) + \cos\left(\left\langle s,\widetilde{X}_{t+h}\left(\frac{t+h}{T}\right)\right\rangle\right)\\ &-\cos\left(\left\langle s,\widetilde{X}_{t+h}\left(\frac{t}{T}\right)\right\rangle\right) + \cos\left(\left\langle s,\widetilde{X}_{t+h}\left(\frac{t}{T}\right)\right\rangle\right) - \cos\left(\left\langle s,\widetilde{X}_{t+h}(u)\right\rangle\right)\Bigg\}\\ &\cdot\Bigg\{\cos\left(\left\langle s,\widetilde{X}_t(u)\right\rangle\right) - \mathbb{E}\left[\cos\left(\left\langle s,\widetilde{X}_t(u)\right\rangle\right)\right]\Bigg\}\Bigg\|_{1+\delta}\\ &\leqslant \sup_{u\in\mathfrak{U}_{0,1,b}} \frac{1}{Tb}\sum_{t=\max\{1+|h|,\lfloor uT-Tb\rfloor\}}^{\min\{T-|h|,\lceil uT+Tb\rceil\}} \mathbb{1}_{\{|\frac{t}{T}-u|\leqslant b\}} C\left(\frac{1}{T} + \left|\frac{t}{T}-u\right| + \frac{1}{T} + \frac{|h|}{T} + \left|\frac{t}{T}-u\right|\right)|s|_1\\ &\leqslant C\left(b + \frac{|h|}{T}\right)|s|_1.\end{aligned} \tag{C.471}$$

According to (B.41), $1 = \int_0^1 1/bK\big((z-u)/b\big)dz\ \forall\, u \in \mathfrak{U}_{0,1,b}$ (see (2.9)), such that one obtains for all $h \in \{-\lfloor T/2\rfloor+1,\ldots,\lfloor T/2\rfloor-1\}$ from (B.40):

$$\begin{aligned}&\sup_{u\in\mathfrak{U}_{0,1,b}}\left|\frac{1}{T}\sum_{t=1+|h|}^{T-|h|} K_b\left(\frac{t}{T}-u\right) - 1\right|\\ &\leqslant \sup_{u\in\mathfrak{U}_{0,1,b}}\left(\frac{1}{Tb}\left|\sum_{t=1+|h|}^{T-|h|} K\left(\frac{\frac{t}{T}-u}{b}\right) - \sum_{t=1}^{T} K\left(\frac{\frac{t}{T}-u}{b}\right)\right| + \left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) - \int_0^1 \frac{1}{b}K\left(\frac{z-u}{b}\right)dz\right|\right)\\ &\leqslant \frac{C}{Tb}\left(|h|+1\right).\end{aligned} \tag{C.472}$$

Arguments which are similar to those that yield (C.224) provide $\mathrm{Cov}\big(\cos\big(\langle s,\widetilde{X}_t(u)\rangle\big), \cos\big(\langle s,\widetilde{X}_{t+h}(u)\rangle\big)\big) = \mathrm{Cov}\big(\cos\big(\langle s,\widetilde{X}_0(u)\rangle\big), \cos\big(\langle s,\widetilde{X}_h(u)\rangle\big)\big)\ \forall\, s \in \mathbb{R}^d, t \in \mathbb{Z}, h \in \{-\lfloor T/2\rfloor+1,\ldots,\lfloor T/2\rfloor-1\}$, $u \in [0,1]$. Thus, (C.472) implies for all $h \in \{-\lfloor T/2\rfloor+1,\ldots,\lfloor T/2\rfloor-1\}$, $s \in \mathbb{R}^d$ (recall (C.470)):

$$\sup_{u\in\mathfrak{U}_{0,1,b}}\left|\mathbb{E}\left[\widehat{\sigma}_{3,h,T,\Re}(u,s)\right] - \mathrm{Cov}\left(\cos\left(\left\langle s,\widetilde{X}_0(u)\right\rangle\right), \cos\left(\left\langle s,\widetilde{X}_h(u)\right\rangle\right)\right)\right| \leqslant \frac{C}{Tb}\left(|h|+1\right). \tag{C.473}$$

In the following, it will be shown for all $h \in \{-\lfloor T/2 \rfloor + 1, \ldots, \lfloor T/2 \rfloor - 1\}$, $s \in \mathbb{R}^d$ (see (C.470)):

$$\sup_{u \in [0,1]} \operatorname{Var}\left(\widehat{\sigma}_{3,h,T,\Re}(u,s)\right) \leqslant \frac{C}{Tb}\left(1 + |s|_1 + |s|_1^2\right). \tag{C.474}$$

One obtains for all $s \in \mathbb{R}^d$ as well as all $r_1, \ldots, r_4 \in \mathbb{N}$ with $r_1 \geqslant r_2 > r_3 \geqslant r_4$ by using arguments which are analog to those that show (B.53), (C.112), Lemma B.4 (iv) with $q = 1 + \delta$, by shifting the indices of sums and from $r_1 - r_3 \geqslant r_2 - r_3$ (recall the Definitions 2.1 as well as A.1 (i), that $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment and that $\widetilde{X}_t^{\times(t-l)}(u)$ is defined in Assumption 2.4 $[\mathbf{DM}]$ (ii)):

$$\begin{aligned}
&\sup_{u \in [0,1]} \left|\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_2}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{r_3}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_4}(u)\right\rangle\right)^c\right)\right| \\
&\leqslant \sup_{u \in [0,1]} \left|\mathbb{E}\left[\left(\mathbb{E}\left[\cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_2}(u)\right\rangle\right)^c \middle| \mathcal{F}_{r_1}\right] - \mathbb{E}\left[\cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right)^c\right.\right.\right. \\
&\left.\left.\left.\cdot \cos\left(\left\langle s, \widetilde{X}_{r_2}(u)\right\rangle\right)^c \middle| \mathcal{F}_{r_1, r_3+1}\right]\right) \cdot \cos\left(\left\langle s, \widetilde{X}_{r_3}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_4}(u)\right\rangle\right)^c\right]\right| \\
&\leqslant \sup_{u \in [0,1]} C \sum_{l=r_1-r_3-1}^{\infty} \left\|\mathbb{E}\left[\cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_2}(u)\right\rangle\right)^c \middle| \mathcal{F}_{r_1, r_1-l}\right]\right. \\
&\left.-\mathbb{E}\left[\cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_2}(u)\right\rangle\right)^c \middle| \mathcal{F}_{r_1, r_1-l-1}\right]\right\|_{1+\delta} \\
&\leqslant \sup_{u \in [0,1]} C \sum_{l=r_1-r_3-1}^{\infty} \left\|\cos\left(\left\langle s, \widetilde{X}_{r_1}^{\times(r_1-l-1)}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_2}^{\times(r_2-(r_2-r_1+l+1))}(u)\right\rangle\right)^c\right. \\
&\left.- \cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_2}(u)\right\rangle\right)^c\right\|_{1+\delta} \\
&\leqslant C \sum_{l=r_1-r_3-1}^{\infty} \left(\Delta_{l+1} |s|_1 + \Delta_{r_2-r_1+l+1} |s|_1\right) \\
&= C \sum_{l=r_1-r_3}^{\infty} \Delta_l |s|_1 + C \sum_{l=r_2-r_3}^{\infty} \Delta_l |s|_1 \\
&\leqslant C \sum_{l=r_2-r_3}^{\infty} \Delta_l |s|_1.
\end{aligned} \tag{C.475}$$

Further, if $r_1 > r_2 > r_3 > r_4$, it will follow from (C.475), Lemma B.4 (vi), $r_3 - r_4 \geqslant 1$, $r_1 - r_3 \geqslant r_1 - r_2$ as well as $r_2 - r_4 \geqslant 1$ (note that the first line of (C.476) given below results from the first line of (C.475) by interchanging r_2 and r_3):

$$\begin{aligned}
&\sup_{u \in [0,1]} \left|\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_3}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{r_2}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_4}(u)\right\rangle\right)^c\right)\right| \\
&= \sup_{u \in [0,1]} \left|\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_2}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{r_3}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{r_4}(u)\right\rangle\right)^c\right)\right. \\
&+\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right), \cos\left(\left\langle s, \widetilde{X}_{r_2}(u)\right\rangle\right)\right) \cdot \operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{r_3}(u)\right\rangle\right), \cos\left(\left\langle s, \widetilde{X}_{r_4}(u)\right\rangle\right)\right) \\
&\left.-\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{r_1}(u)\right\rangle\right), \cos\left(\left\langle s, \widetilde{X}_{r_3}(u)\right\rangle\right)\right) \cdot \operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{r_2}(u)\right\rangle\right), \cos\left(\left\langle s, \widetilde{X}_{r_4}(u)\right\rangle\right)\right)\right| \\
&\leqslant C \sum_{l=r_2-r_3}^{\infty} \Delta_l |s|_1 + C \sum_{l=r_1-r_2}^{\infty} \Delta_l |s|_1 \cdot \sum_{l=r_3-r_4}^{\infty} \Delta_l |s|_1 + C \sum_{l=r_1-r_3}^{\infty} \Delta_l |s|_1 \cdot \sum_{l=r_2-r_4}^{\infty} \Delta_l |s|_1 \\
&\leqslant C \sum_{l=r_2-r_3}^{\infty} \Delta_l |s|_1 + C \sum_{l=r_1-r_2}^{\infty} \Delta_l |s|_1^2.
\end{aligned} \tag{C.476}$$

Assumption 2.8 $[\mathbf{K\&b.1}]$ (i) (which ensures $K(z) = 0$ for all $z \in \mathbb{R}$ with $|z| > 1$) provides for all

$h \in \{-\lfloor T/2\rfloor + 1, \ldots, \lfloor T/2\rfloor - 1\}$, $s \in \mathbb{R}^d$:

$$\sup_{u\in[0,1]} \frac{1}{T^2} \sum_{t_1,t_2=1+|h|}^{T-|h|} \mathbb{1}_{\{t_1=t_2\}} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-u\right)$$
$$\cdot \left|\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{t_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_1+h}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{t_2}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_2+h}(u)\right\rangle\right)^c\right)\right|$$
$$\leqslant \sup_{u\in[0,1]} \frac{C}{T^2 b^2} \sum_{t=\max\{1+|h|,\lfloor uT-Tb\rfloor\}}^{\min\{T-|h|,\lceil uT+Tb\rceil\}} K\left(\frac{\frac{t}{T}-u}{b}\right)^2$$
$$\leqslant \frac{C}{Tb}. \tag{C.477}$$

One obtains for all $h \in \{-\lfloor T/2\rfloor + 1, \ldots, 0\}$, $s \in \mathbb{R}^d$ from (C.475) with $r_1 := t_1 \geqslant r_2 := t_1 + h > r_3 := t_2 \geqslant r_4 := t_2 + h$, shifting the index of a sum, (B.45) and Lemma B.1 with $\kappa_1 = 1$:

$$\sup_{u\in[0,1]} \frac{1}{T^2} \sum_{t_1,t_2=1+|h|}^{T-|h|} \mathbb{1}_{\{t_1>t_2\}} \mathbb{1}_{\{t_1+h>t_2\}} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-u\right)$$
$$\cdot \left|\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{t_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_1+h}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{t_2}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_2+h}(u)\right\rangle\right)^c\right)\right|$$
$$\leqslant \sup_{u\in[0,1]} \frac{C}{T^2 b^2} \sum_{t_1,t_2=1-h}^{T+h} \mathbb{1}_{\{t_1+h>t_2\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_2}{T}-u}{b}\right) \sum_{l=t_1+h-t_2}^{\infty} \Delta_l \left|s\right|_1$$
$$= \sup_{u\in[0,1]} \frac{C}{T^2 b^2} \sum_{t_2=1-h}^{T+h} K\left(\frac{\frac{t_2}{T}-u}{b}\right) \sum_{t_1=1}^{T+2h} \mathbb{1}_{\{t_1>t_2\}} K\left(\frac{\frac{t_1-h}{T}-u}{b}\right) \sum_{l=t_1-t_2}^{\infty} \Delta_l \left|s\right|_1$$
$$\leqslant \sup_{u\in[0,1]} \frac{C}{T^2 b^2} \sum_{t_2=1-h}^{T+h} K\left(\frac{\frac{t_2}{T}-u}{b}\right) \sum_{t_1=t_2+1}^{\infty} \sum_{l=t_1-t_2}^{\infty} \Delta_l \left|s\right|_1$$
$$\leqslant \frac{C}{Tb} \left|s\right|_1. \tag{C.478}$$

Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) (which ensures $K(z) = 0$ for all $z \in \mathbb{R}$ with $|z| > 1$) implies for all $h \in \{-\lfloor T/2\rfloor + 1, \ldots, 0\}$, $s \in \mathbb{R}^d$:

$$\sup_{u\in[0,1]} \frac{1}{T^2} \sum_{t_1,t_2=1+|h|}^{T-|h|} \mathbb{1}_{\{t_1>t_2\}} \mathbb{1}_{\{t_1+h=t_2\}} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-u\right)$$
$$\cdot \left|\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{t_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_1+h}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{t_2}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_2+h}(u)\right\rangle\right)^c\right)\right|$$
$$\leqslant \sup_{u\in[0,1]} \frac{C}{T^2 b^2} \sum_{t_1=\max\{1-h,\lfloor uT-Tb\rfloor\}}^{\min\{T+h,\lceil uT+Tb\rceil\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_1+h}{T}-u}{b}\right)$$
$$\leqslant \frac{C}{Tb}. \tag{C.479}$$

It follows for all $h \in \{-\lfloor T/2\rfloor + 1, \ldots, -1\}$, $s \in \mathbb{R}^d$ from (C.476) with $r_1 := t_1 > r_2 := t_2 > r_3 := t_1 + h > r_4 := t_2 + h$, shifting the index of a sum, Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) (which ensures $K(z) = 0$ for all $z \in \mathbb{R}$ with $|z| > 1$), (B.45) and Lemma B.1 with $\kappa_1 = 1$:

$$\sup_{u\in[0,1]} \frac{1}{T^2} \sum_{t_1,t_2=1+|h|}^{T-|h|} \mathbb{1}_{\{t_1>t_2\}} \mathbb{1}_{\{t_1+h<t_2\}} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-u\right)$$
$$\cdot \left|\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{t_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_1+h}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{t_2}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_2+h}(u)\right\rangle\right)^c\right)\right|$$
$$\leqslant \sup_{u\in[0,1]} \frac{C}{T^2 b^2} \sum_{t_1,t_2=1-h}^{T+h} \mathbb{1}_{\{t_1>t_2\}} \mathbb{1}_{\{t_2>t_1+h\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_2}{T}-u}{b}\right) \left(\sum_{l=t_2-t_1-h}^{\infty} \Delta_l \left|s\right|_1 + \sum_{l=t_1-t_2}^{\infty} \Delta_l \left|s\right|_1^2\right)$$

$$
\begin{aligned}
&\leqslant \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_1=1}^{T+2h} K\left(\frac{\frac{t_1-h}{T}-u}{b}\right) \sum_{t_2=1-h}^{T+h} \mathbb{1}_{\{t_1-h>t_1\}}\mathbb{1}_{\{t_2>t_1\}} K\left(\frac{\frac{t_2}{T}-u}{b}\right) \sum_{l=t_2-t_1}^{\infty} \Delta_l \left|s\right|_1 \\
&+ \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_2=1-h}^{T+h} K\left(\frac{\frac{t_2}{T}-u}{b}\right) \sum_{t_1=1-h}^{T+h} \mathbb{1}_{\{t_1>t_2\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) \sum_{l=t_1-t_2}^{\infty} \Delta_l \left|s\right|_1^2 \\
&\leqslant \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_1=\max\{1,\lfloor uT+h-Tb\rfloor\}}^{\min\{T+2h,\lceil uT+h+Tb\rceil\}} K\left(\frac{\frac{t_1}{T}-\left(u+\frac{h}{T}\right)}{b}\right) \sum_{t_2=t_1+1}^{\infty} \sum_{l=t_2-t_1}^{\infty} \Delta_l \left|s\right|_1 \\
&+ \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_2=1-h}^{T+h} K\left(\frac{\frac{t_2}{T}-u}{b}\right) \sum_{t_1=t_2+1}^{\infty} \sum_{l=t_1-t_2}^{\infty} \Delta_l \left|s\right|_1^2 \\
&\leqslant \frac{C}{Tb}\left(\left|s\right|_1 + \left|s\right|_1^2\right). \qquad \text{(C.480)}
\end{aligned}
$$

Since $t_1 > t_2$ is a contradiction to $t_1 + h < t_2$ for $h = 0$ and all $t_1, t_2 \in \{1 + |h|, \ldots, T - |h|\}$, one obtains for $h = 0$:

$$
\begin{aligned}
&\sup_{u\in[0,1]} \frac{1}{T^2} \sum_{t_1,t_2=1+|h|}^{T-|h|} \mathbb{1}_{\{t_1>t_2\}}\mathbb{1}_{\{t_1+h<t_2\}} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-u\right) \\
&\cdot \left|\mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{t_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_1+h}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{t_2}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_2+h}(u)\right\rangle\right)^c\right)\right| \\
&= 0. \qquad \text{(C.481)}
\end{aligned}
$$

It follows for all $h \in \{1, \ldots, \lfloor T/2\rfloor - 1\}$, $s \in \mathbb{R}^d$ from (C.475) with $r_1 := t_1 + h \geqslant r_2 := t_1 > r_3 := t_2 + h \geqslant r_4 := t_2$, by shifting the index of a sum and by using Assumption 2.8 $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ (i) (which ensures $K(z) = 0$ for all $z \in \mathbb{R}$ with $|z| > 1$) as well as (B.45):

$$
\begin{aligned}
&\sup_{u\in[0,1]} \frac{1}{T^2} \sum_{t_1,t_2=1+|h|}^{T-|h|} \mathbb{1}_{\{t_1>t_2\}}\mathbb{1}_{\{t_1>t_2+h\}} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-u\right) \\
&\cdot \left|\mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{t_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_1+h}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{t_2}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_2+h}(u)\right\rangle\right)^c\right)\right| \\
&\leqslant \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_1,t_2=1+h}^{T-h} \mathbb{1}_{\{t_1>t_2+h\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_2}{T}-u}{b}\right) \sum_{l=t_1-t_2-h}^{\infty} \Delta_l \left|s\right|_1 \\
&= \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_2=1+2h}^{T} K\left(\frac{\frac{t_2-h}{T}-u}{b}\right) \sum_{t_1=1+h}^{T-h} \mathbb{1}_{\{t_1>t_2\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) \sum_{l=t_1-t_2}^{\infty} \Delta_l \left|s\right|_1 \\
&\leqslant \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_2=\max\{1+2h,\lfloor uT+h-Tb\rfloor\}}^{\min\{T,\lceil uT+h+Tb\rceil\}} K\left(\frac{\frac{t_2}{T}-\left(u+\frac{h}{T}\right)}{b}\right) \sum_{t_1=t_2+1}^{\infty} \sum_{l=t_1-t_2}^{\infty} \Delta_l \left|s\right|_1 \\
&\leqslant \frac{C}{Tb}\left|s\right|_1. \qquad \text{(C.482)}
\end{aligned}
$$

Arguments which are similar to those that show (C.479), together with $t_1 = t_2 + h \iff t_1 - h = t_2$ imply for all $h \in \{1, \ldots, \lfloor T/2\rfloor - 1\}$, $s \in \mathbb{R}^d$:

$$
\begin{aligned}
&\sup_{u\in[0,1]} \frac{1}{T^2} \sum_{t_1,t_2=1+|h|}^{T-|h|} \mathbb{1}_{\{t_1>t_2\}}\mathbb{1}_{\{t_1=t_2+h\}} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-u\right) \\
&\cdot \left|\mathrm{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{t_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_1+h}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{t_2}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_2+h}(u)\right\rangle\right)^c\right)\right| \\
&\leqslant \frac{C}{Tb}. \qquad \text{(C.483)}
\end{aligned}
$$

One obtains for all $h \in \{1, \ldots, \lfloor T/2\rfloor - 1\}$, $s \in \mathbb{R}^d$ from (C.476) with $r_1 := t_1 + h > r_2 := t_2 + h >$

$r_3 := t_1 > r_4 := t_2$, shifting the index of a sum, (B.45) and Lemma B.1 with $\kappa_1 = 1$:

$$\sup_{u\in[0,1]} \frac{1}{T^2} \sum_{t_1,t_2=1+|h|}^{T-|h|} \mathbb{1}_{\{t_1>t_2\}} \mathbb{1}_{\{t_1<t_2+h\}} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-u\right)$$
$$\cdot \left|\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{t_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_1+h}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{t_2}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_2+h}(u)\right\rangle\right)^c\right)\right|$$
$$\leqslant \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_1,t_2=1+h}^{T-h} \mathbb{1}_{\{t_1>t_2\}} \mathbb{1}_{\{t_2+h>t_1\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_2}{T}-u}{b}\right) \left(\sum_{l=t_2+h-t_1}^{\infty} \Delta_l |s|_1 + \sum_{l=t_1-t_2}^{\infty} \Delta_l |s|_1^2\right)$$
$$\leqslant \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_1=1+h}^{T-h} K\left(\frac{\frac{t_1}{T}-u}{b}\right) \sum_{t_2=1+2h}^{T} \mathbb{1}_{\{t_1>t_2-h\}} \mathbb{1}_{\{t_2>t_1\}} K\left(\frac{\frac{t_2-h}{T}-u}{b}\right) \sum_{l=t_2-t_1}^{\infty} \Delta_l |s|_1$$
$$+ \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_2=1+h}^{T-h} K\left(\frac{\frac{t_2}{T}-u}{b}\right) \sum_{t_1=1+h}^{T-h} \mathbb{1}_{\{t_1>t_2\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) \sum_{l=t_1-t_2}^{\infty} \Delta_l |s|_1^2$$
$$\leqslant \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_1=1+h}^{T-h} K\left(\frac{\frac{t_1}{T}-u}{b}\right) \sum_{t_2=t_1+1}^{\infty} \sum_{l=t_2-t_1}^{\infty} \Delta_l |s|_1$$
$$+ \sup_{u\in[0,1]} \frac{C}{T^2b^2} \sum_{t_2=1+h}^{T-h} K\left(\frac{\frac{t_2}{T}-u}{b}\right) \sum_{t_1=t_2+1}^{\infty} \sum_{l=t_1-t_2}^{\infty} \Delta_l |s|_1^2$$
$$\leqslant \frac{C}{Tb}\left(|s|_1 + |s|_1^2\right). \tag{C.484}$$

In conclusion, (C.477) to (C.484) as well as similar arguments provide for all $h \in \{-\lfloor T/2\rfloor + 1, \ldots, \lfloor T/2\rfloor - 1\}$:

$$\sup_{u\in[0,1]} \frac{1}{T^2} \sum_{t_1,t_2=1+|h|}^{T-|h|} \mathbb{1}_{\{t_1\geqslant t_2\}} K_b\left(\frac{t_1}{T}-u\right) K_b\left(\frac{t_2}{T}-u\right)$$
$$\cdot \left|\operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_{t_1}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_1+h}(u)\right\rangle\right)^c, \cos\left(\left\langle s, \widetilde{X}_{t_2}(u)\right\rangle\right)^c \cos\left(\left\langle s, \widetilde{X}_{t_2+h}(u)\right\rangle\right)^c\right)\right|$$
$$\leqslant \frac{C}{Tb}\left(1 + |s|_1 + |s|_1^2\right). \tag{C.485}$$

It follows from (C.485) and similar arguments that (C.474) holds (see (C.470)).
Furthermore, (C.473), (C.474) and $\sqrt{x+y+z} \leqslant \sqrt{x} + \sqrt{y} + \sqrt{z}\ \forall\, x, y, z \geqslant 0$ show for all $h \in \{-\lfloor T/2\rfloor + 1, \ldots, \lfloor T/2\rfloor - 1\}$, $s \in \mathbb{R}^d$ (recall (2.9)):

$$\sup_{u\in\mathfrak{U}_{0,1,b}} \left\|\widehat{\sigma}_{3,h,T,\Re}(u,s) - \operatorname{Cov}\left(\cos\left(\left\langle s, \widetilde{X}_0(u)\right\rangle\right), \cos\left(\left\langle s, \widetilde{X}_h(u)\right\rangle\right)\right)\right\|_2$$
$$\leqslant \frac{C}{Tb}(|h|+1) + \frac{C}{\sqrt{Tb}}\left(1 + \sqrt{|s|_1} + |s|_1\right). \tag{C.486}$$

Lemma C.33 with $\mathrm{R} = \Re$ is an implication of (C.467), (C.469), (C.471) and (C.486) (note that $\delta \in (0,1]$ originates from Assumption 2.2 $[\mathbf{StAp}]$). Lemma C.33 with $\mathrm{R} = \Im$ can be proved similarly. □

D. Appendix to Chapter 4

D.1. Proofs of the statements given in Chapter 4

Proof of Lemma 4.8. At first, one observes that Assumption 2.4 $[\mathbf{DM.3}]$ (which is contained in Assumption 4.1 $[\mathbf{INDEP}]$) provides by shifting the index of a sum:

$$\sum_{l=|z|}^{\infty} \Delta_l \leqslant C\rho^{|z|} \quad \forall\, z \in \mathbb{Z}. \tag{D.1}$$

It follows for all $\mathrm{R}_1, \mathrm{R}_2 \in \{\Re, \Im\}$, $s_1, s_2 \in \mathbb{R}^d$, $t \in \mathbb{Z}$ from the bilinearity of the covariance, Lemma B.4 (vi) and (D.1) (see (4.11) as well as (4.6)):

$$\sup_{u\in[0,1]} \left|\mathrm{Cov}\left(\widetilde{\mathbf{G}}_{0,\mathrm{R}_1}(u,s_1), \widetilde{\mathbf{G}}_{t,\mathrm{R}_2}(u,s_2)\right)\right| \leqslant C|s_2|_1\rho^t \mathbb{1}_{\{t\in\mathbb{N}\}} + C\mathbb{1}_{\{t=0\}} + C|s_1|_1\rho^{-t}\mathbb{1}_{\{t\in\mathbb{Z}\setminus\mathbb{N}_0\}}, \tag{D.2}$$

which implies Lemma 4.8 due to $\rho \in (0,1)$ (the latter is supposed in Assumption 2.4 $[\mathbf{DM.3}]$). □

Proof of Theorem 4.9. (i) In order to prove Theorem 4.9 (i), define at first sequences $(\rho_T)_{T\in\mathbb{N}}$, $(L_T)_{T\in\mathbb{N}}$ and $(\mathfrak{K}_T)_{T\in\mathbb{N}}$ of deterministic real numbers which fulfil the following properties (recall that $x_T \ll y_T$ means $x_T/y_T \overset{T\to\infty}{\longrightarrow} 0$ for sequences $(x_T)_{T\in\mathbb{N}}$ as well as $(y_T)_{T\in\mathbb{N}}$ of real numbers and that $\mathscr{a}$ originates from Definition A.1 (vi)):

$$\begin{aligned}
&\rho_T, L_T \in \mathbb{N} \text{ with } \lceil 2Tb\rceil \leqslant \rho_T \leqslant L_T \leqslant T\ \forall\, T\in\mathbb{N} \text{ and } \rho_T \ll Tb\mathscr{a}^2 \ll Tb\mathscr{a}^4 \ll L_T \ll \frac{T\sqrt{b}}{\mathscr{a}^2} \text{ for } T\to\infty,\\
&\mathfrak{K}_T := \left\lfloor \frac{T-L_T}{L_T+\rho_T}\right\rfloor + 1 \quad \left(\text{note that } \mathfrak{K}_T = \mathcal{O}\left(\frac{T}{L_T}\right)\right) \quad \text{and define}\\
&l_{T,k} := (k-1)(L_T+\rho_T) + 7\mathscr{a} + 2 \ \text{ as well as } \ o_{T,k} := (k-1)(L_T+\rho_T) + L_T\ \ \forall\, k\in\{1,\ldots,\mathfrak{K}_T\}.
\end{aligned} \tag{D.3}$$

The following considerations show that sequences $(\rho_T)_{T\in\mathbb{N}}$ as well as $(L_T)_{T\in\mathbb{N}}$ (and, hence, also a sequence $(\mathfrak{K}_T)_{T\in\mathbb{N}}$) exist which fulfil these properties:
It holds $\lceil 2Tb\rceil \leqslant T$ due to $b \in (0, 1/2)$ (as demanded in Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii)), such that $\lceil 2Tb\rceil \leqslant \rho_T \leqslant L_T \leqslant T\ \forall\, T\in\mathbb{N}$ is feasible. Moreover, Definition A.1 (vi) (which provides $\mathscr{a}\to\infty$ for $T\to\infty$) and (A.1) (that implies $\mathscr{a} \ll b^{-1/12}$) show $\lceil 2Tb\rceil \ll Tb\mathscr{a}^2 \ll Tb\mathscr{a}^4 \ll (T\sqrt{b})/\mathscr{a}^2 \ll T$. Thus, sequences $(\rho_T)_{T\in\mathbb{N}}$ and $(L_T)_{T\in\mathbb{N}}$ for which (D.3) is valid exist.
For the remaining steps of the present proof, assume that T is large enough to ensure:

$$7\mathscr{a} + 2 \leqslant 2Tb,$$

which is possible due to (A.1) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii). This yields $l_{T,k} \leqslant o_{T,k}\ \forall\, k \in \{1,\ldots,\mathfrak{K}_T\}$ due to $\lceil 2Tb\rceil \leqslant L_T$.
Further, note that $\{\widetilde{X}_t(u)\}$ (with $u\in[0,1]$) is defined in Assumption 4.1 $[\mathbf{INDEP}]$ and introduce the following notations for all $r,t,j\in\mathbb{Z}$, $\mathrm{n}\in\mathbb{N}_0$, $u\in[0,1]$, $s\in\mathbb{R}^d$, $\mathrm{R}\in\{\Re,\Im\}$, $k\in\{1,\ldots,\mathfrak{K}_T\}$ (see (4.11), Definition A.1 (i) as well as (vi), (D.3) and recall that $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\begin{aligned}
&\widetilde{\mathbf{G}}_{r,\{\mathrm{n}\}}(u,s) := \mathbf{g}_{u,s}\left(\widetilde{X}_{r,\{\mathrm{n}\}}(u)\right), \quad \widetilde{\mathbf{G}}_{r,\{\mathrm{n}\},\mathrm{R}}(u,s) := \mathrm{R}\left\{\widetilde{\mathbf{G}}_{r,\{\mathrm{n}\}}(u,s)\right\},\\
&\widetilde{\mathfrak{H}}_{T,\mathrm{R}}(t,j) := \frac{2}{Tb^{\frac{3}{2}}}\int\limits_{\mathbb{R}^d}\int\limits_{b}^{1-b} K\left(\frac{\frac{t}{T}-u}{b}\right)\widetilde{\mathbf{G}}^c_{t,\{\mathscr{a}\},\mathrm{R}}(u,s)\cdot K\left(\frac{\frac{j}{T}-u}{b}\right)\widetilde{\mathbf{G}}^c_{j,\{\mathscr{a}\},\mathrm{R}}(u,s)\,du\,\mathbf{w}(s)\,ds,\\
&\widetilde{\mathfrak{H}}_T(t,j) := \widetilde{\mathfrak{H}}_{T,\Re}(t,j) + \widetilde{\mathfrak{H}}_{T,\Im}(t,j), \qquad \widetilde{\mathbb{H}}_{T,k,\mathrm{R}} := \sum_{t=l_{T,k}}^{o_{T,k}}\ \sum_{j=1}^{t-7\mathscr{a}-1}\widetilde{\mathfrak{H}}_{T,\mathrm{R}}(t,j),\\
&\widetilde{\mathbb{H}}_{T,k} := \widetilde{\mathbb{H}}_{T,k,\Re} + \widetilde{\mathbb{H}}_{T,k,\Im} \quad \text{and} \quad \widetilde{\mathbb{H}}_T := \sum_{k=1}^{\mathfrak{K}_T}\widetilde{\mathbb{H}}_{T,k}.
\end{aligned} \tag{D.4}$$

By assumption of Theorem 4.9 (i) (which is currently proved), $\mathcal{H}^{\mathrm{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ (note (4.1)) is valid, such that the Lemmata D.3, D.4 and D.5 as well as (D.118) provide:

$$\left\|T\sqrt{b}\,\widehat{\mathfrak{Q}}_T - \mathbf{Bias}^{\mathrm{indep}}_T - \widetilde{\mathbb{H}}_T\right\|_1 = o(1). \tag{D.5}$$

Hence, Theorem 4.9 (i) will be proved if the following statement holds for $T \to \infty$ (see (4.14)):

$$\widetilde{\mathbb{H}}_T \xrightarrow{\mathrm{d}} Z^{\mathrm{indep}}. \tag{D.6}$$

From now on, the validity of the assumptions demanded in Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274 et seq.] will be examined in order to use this theorem to verify (D.6), whereby the expression $\widetilde{\mathbb{H}}_{T,k}$ takes the role of $X_{n,k}$ which originates from Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274 et seq.].

Therefor, note at first that it follows for all $t, j, l_1, \ldots, l_4 \in \mathbb{Z}$, $\mathrm{R} \in \{\Re, \Im\}$ from Assumption 4.5 $[\mathbf{K\&b.2}]$ (i) (recall (D.4), (4.11) as well as Definition A.1 (i)):

$$\sup_{t,j\in\mathbb{Z}} \left|\widetilde{\mathfrak{H}}_{T,\mathrm{R}}(t,j)\right| \leqslant \sup_{t\in\mathbb{Z}} \frac{C}{Tb^{\frac{3}{2}}} \int_b^{1-b} \mathbb{1}_{\{u\in[t/T-b,t/T+b]\}}\, du \leqslant \frac{C}{T\sqrt{b}}, \quad \sup_{t,j\in\mathbb{Z}} \left|\widetilde{\mathfrak{H}}_T(t,j)\right| \leqslant \frac{C}{T\sqrt{b}}, \tag{D.7}$$

$$|t-j| > 2Tb \Longrightarrow \left(\widetilde{\mathfrak{H}}_{T,\mathrm{R}}(t,j) = 0 \ \wedge \ \widetilde{\mathfrak{H}}_T(t,j) = 0\right), \tag{D.8}$$

$$|t-j| > \alpha \Longrightarrow \left(\mathbb{E}\left[\widetilde{\mathfrak{H}}_{T,\mathrm{R}}(t,j)\right] = 0 \ \wedge \ \mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t,j)\right] = 0\right) \quad \text{and} \tag{D.9}$$

$$\text{if } \exists n_1 \in \{1,\ldots,4\} : |l_{n_1} - l_{n_2}| > \alpha\, \forall\, n_2 \in \{1,\ldots,4\} \setminus \{n_1\}, \text{then } \mathbb{E}\left[\widetilde{\mathfrak{H}}_T(l_1,l_2)\, \widetilde{\mathfrak{H}}_T(l_3,l_4)\right] = 0. \tag{D.10}$$

One obtains from (D.9) (see (D.4)):

$$\mathbb{E}\left[\widetilde{\mathbb{H}}_{T,k}\right] = 0 \quad \forall\, k \in \{1,\ldots,\mathfrak{K}_T\}, \tag{D.11}$$

which provides that the first assumption of Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274] is valid.
Further, (D.8) yields:

$$\widetilde{\mathbb{H}}_{T,k} = \sum_{t=l_{T,k}}^{o_{T,k}} \sum_{j=\max\{1,t-\lceil 2Tb\rceil\}}^{t-7\alpha-1} \widetilde{\mathfrak{H}}_T(t,j) \quad \forall\, k \in \{1,\ldots,\mathfrak{K}_T\} \tag{D.12}$$

and it holds (recall (D.3)):

$$l_{T,k+1} - o_{T,k} = \rho_T + 7\alpha + 2 \quad \forall\, k \in \{1,\ldots,\mathfrak{K}_T - 1\}. \tag{D.13}$$

According to (D.12), the random variable $\widetilde{\mathbb{H}}_{T,k}$ is measurable with respect to the sigma algebra generated by $\mathcal{F}_{o_{T,k},l_{T,k}-\lceil 2Tb\rceil-\alpha}$ (see Definition A.1 (i)), such that (D.13) and (D.3) $\big($which ensure $l_{T,k+1} - \lceil 2Tb\rceil - \alpha \geqslant o_{T,k} + 1\ \forall\, k \in \{1,\ldots,\mathfrak{K}_T - 1\}\big)$ show:

$$\left(\widetilde{\mathbb{H}}_{T,k}\right)_{k=1}^{\mathfrak{K}_T} \text{ is a sequence of independent random variables.} \tag{D.14}$$

Hence, (D.11) implies (note (D.4)):

$$\mathrm{Var}\left(\widetilde{\mathbb{H}}_T\right) = \sum_{k=1}^{\mathfrak{K}_T} \mathbb{E}\left[\widetilde{\mathbb{H}}_{T,k}^2\right]. \tag{D.15}$$

Thus, one obtains from (D.119) and Lemma 4.8 together with Assumption 4.3 $[\mathbf{WEI.2}]$ (whereby the latter two provide $\sigma^{\mathrm{indep}} < \infty$ (recall (4.13))) that $\big(\sum_{k=1}^{\mathfrak{K}_T} \mathbb{E}\big[\widetilde{\mathbb{H}}_{T,k}^2\big]\big)_{T\in\mathbb{N}}$ is a convergent sequence for $T \to \infty$, such that it is bounded from above by an absolute constant $\nu_0 < \infty$, which yields that the second assumption of Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274] holds.
In addition, (D.119) implies (6.25) in [52, Leucht and Neumann (2013), p. 274] with $\Sigma = \sigma^{\mathrm{indep}}$, whereby Σ originates from [52, Leucht and Neumann (2013), p. 274].
Next, the validity of (6.26) in [52, Leucht and Neumann (2013), p. 275] is proved. Therefor, it is shown

at first for all $\mathrm{R} \in \{\Re, \Im\}$ that:

$$\sum_{k=1}^{\mathfrak{K}_T} \mathbb{E}\left[\left(\widetilde{\mathbb{H}}_{T,k,\mathrm{R}}\right)^4\right] = o(1). \tag{D.16}$$

One obtains for all $\mathrm{R} \in \{\Re, \Im\}$ due to (D.8), Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) and arguments which are similar to those that show (C.91) (see (D.4) as well as Definition A.1 (i) and note that the eightfold sum over $r_1, \ldots, r_8 \in \mathbb{N} : \{r_1, \ldots, r_8\} = (t_1, \ldots, t_4, j_1, \ldots, j_4)$ should mean to sum over all $r_1, \ldots, r_8 \in \mathbb{N}$ which fulfil that a tuple with the elements $r_1, \ldots, r_8$ (in arbitrary order) exists that equals the tuple $(t_1, \ldots, t_4, j_1, \ldots, j_4)$):

$$\begin{aligned}
&\sum_{k=1}^{\mathfrak{K}_T} \mathbb{E}\left[\left(\widetilde{\mathbb{H}}_{T,k,\mathrm{R}}\right)^4\right] \\
&\leqslant \frac{C}{T^4 b^6} \sum_{k=1}^{\mathfrak{K}_T} \sum_{t_1,\ldots,t_4=l_{T,k}}^{o_{T,k}} \sum_{j_1=\max\{1,t_1-\lceil 2Tb\rceil\}}^{t_1-7\mathscr{a}-1} \sum_{j_2=\max\{1,t_2-\lceil 2Tb\rceil\}}^{t_2-7\mathscr{a}-1} \sum_{j_3=\max\{1,t_3-\lceil 2Tb\rceil\}}^{t_3-7\mathscr{a}-1} \sum_{j_4=\max\{1,t_4-\lceil 2Tb\rceil\}}^{t_4-7\mathscr{a}-1} \\
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \int_{\max\{b,t_4/T-b\}}^{\min\{1-b,t_4/T+b\}} \int_{\max\{b,t_3/T-b\}}^{\min\{1-b,t_3/T+b\}} \int_{\max\{b,t_2/T-b\}}^{\min\{1-b,t_2/T+b\}} \int_{\max\{b,t_1/T-b\}}^{\min\{1-b,t_1/T+b\}} K\left(\frac{\frac{t_1}{T}-u_1}{b}\right) K\left(\frac{\frac{t_2}{T}-u_2}{b}\right) \\
&\cdot K\left(\frac{\frac{t_3}{T}-u_3}{b}\right) K\left(\frac{\frac{t_4}{T}-u_4}{b}\right) K\left(\frac{\frac{j_1}{T}-u_1}{b}\right) K\left(\frac{\frac{j_2}{T}-u_2}{b}\right) K\left(\frac{\frac{j_3}{T}-u_3}{b}\right) K\left(\frac{\frac{j_4}{T}-u_4}{b}\right) \\
&\cdot \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{t_1,\{\mathscr{a}\},\mathrm{R}}(u_1,s_1)\, \widetilde{\mathbf{G}}^c_{t_2,\{\mathscr{a}\},\mathrm{R}}(u_2,s_2)\, \widetilde{\mathbf{G}}^c_{t_3,\{\mathscr{a}\},\mathrm{R}}(u_3,s_3)\, \widetilde{\mathbf{G}}^c_{t_4,\{\mathscr{a}\},\mathrm{R}}(u_4,s_4)\, \widetilde{\mathbf{G}}^c_{j_1,\{\mathscr{a}\},\mathrm{R}}(u_1,s_1)\right.\right. \\
&\left.\left.\cdot \widetilde{\mathbf{G}}^c_{j_2,\{\mathscr{a}\},\mathrm{R}}(u_2,s_2)\, \widetilde{\mathbf{G}}^c_{j_3,\{\mathscr{a}\},\mathrm{R}}(u_3,s_3)\, \widetilde{\mathbf{G}}^c_{j_4,\{\mathscr{a}\},\mathrm{R}}(u_4,s_4)\right]\right| du_1\, du_2\, du_3\, du_4\, \mathbf{w}(s_1)\, ds_1\, \mathbf{w}(s_2)\, ds_2 \\
&\cdot \mathbf{w}(s_3)\, ds_3\, \mathbf{w}(s_4)\, ds_4 \\
&\leqslant \frac{C}{T^4 b^2} \sum_{k=1}^{\mathfrak{K}_T} \sum_{t_1,\ldots,t_4=l_{T,k}}^{o_{T,k}} \sum_{j_1=\max\{1,t_1-\lceil 2Tb\rceil\}}^{t_1-7\mathscr{a}-1} \sum_{j_2=\max\{1,t_2-\lceil 2Tb\rceil\}}^{t_2-7\mathscr{a}-1} \sum_{j_3=\max\{1,t_3-\lceil 2Tb\rceil\}}^{t_3-7\mathscr{a}-1} \sum_{j_4=\max\{1,t_4-\lceil 2Tb\rceil\}}^{t_4-7\mathscr{a}-1} \\
&\sum_{\substack{r_1,\ldots,r_8\in\mathbb{N}: \\ \{r_1,\ldots,r_8\}=(t_1,\ldots,t_4,j_1,\ldots,j_4)}} \sup_{\widetilde{s}_1,\ldots,\widetilde{s}_8\in\mathbb{R}^d} \sup_{\widetilde{u}_1,\ldots,\widetilde{u}_8\in[0,1]} \left\{ \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_1,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_1,\widetilde{s}_1)\, \widetilde{\mathbf{G}}^c_{r_2,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_2,\widetilde{s}_2)\, \widetilde{\mathbf{G}}^c_{r_3,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_3,\widetilde{s}_3)\right.\right.\right. \\
&\left.\left.\cdot \widetilde{\mathbf{G}}^c_{r_4,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_4,\widetilde{s}_4)\right]\right| \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_1,\ldots,r_4\}}|o_1-o_2|\leqslant 3\mathscr{a}\}} \cdot \Big(\left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_5,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_5,\widetilde{s}_5)\, \widetilde{\mathbf{G}}^c_{r_6,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_6,\widetilde{s}_6)\right.\right. \\
&\left.\left.\cdot \widetilde{\mathbf{G}}^c_{r_7,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_7,\widetilde{s}_7)\, \widetilde{\mathbf{G}}^c_{r_8,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_8,\widetilde{s}_8)\right]\right| \cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_5,\ldots,r_8\}}|o_1-o_2|\leqslant 3\mathscr{a}\}} \\
&+ \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_5,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_5,\widetilde{s}_5)\, \widetilde{\mathbf{G}}^c_{r_6,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_6,\widetilde{s}_6)\right]\right| \cdot \mathbb{1}_{\{|r_5-r_6|\leqslant \mathscr{a}\}} \\
&\cdot \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_7,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_7,\widetilde{s}_7)\, \widetilde{\mathbf{G}}^c_{r_8,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_8,\widetilde{s}_8)\right]\right| \mathbb{1}_{\{|r_7-r_8|\leqslant \mathscr{a}\}} \Big) \\
&+ \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_1,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_1,\widetilde{s}_1)\, \widetilde{\mathbf{G}}^c_{r_2,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_2,\widetilde{s}_2)\, \widetilde{\mathbf{G}}^c_{r_3,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_3,\widetilde{s}_3)\right]\right| \cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_1,\ldots,r_3\}}|o_1-o_2|\leqslant 2\mathscr{a}\}} \\
&\cdot \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_4,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_4,\widetilde{s}_4)\, \widetilde{\mathbf{G}}^c_{r_5,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_5,\widetilde{s}_5)\, \widetilde{\mathbf{G}}^c_{r_6,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_6,\widetilde{s}_6)\right]\right| \cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_4,\ldots,r_6\}}|o_1-o_2|\leqslant 2\mathscr{a}\}} \\
&\left.\cdot \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_7,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_7,\widetilde{s}_7)\, \widetilde{\mathbf{G}}^c_{r_8,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_8,\widetilde{s}_8)\right]\right| \mathbb{1}_{\{|r_7-r_8|\leqslant \mathscr{a}\}} \right\} \\
&+ \frac{C}{T^4 b^2} \sum_{k=1}^{\mathfrak{K}_T} \sum_{t_1,\ldots,t_4=l_{T,k}}^{o_{T,k}} \sum_{j_1=\max\{1,t_1-\lceil 2Tb\rceil\}}^{t_1-7\mathscr{a}-1} \sum_{j_2=\max\{1,t_2-\lceil 2Tb\rceil\}}^{t_2-7\mathscr{a}-1} \sum_{j_3=\max\{1,t_3-\lceil 2Tb\rceil\}}^{t_3-7\mathscr{a}-1} \sum_{j_4=\max\{1,t_4-\lceil 2Tb\rceil\}}^{t_4-7\mathscr{a}-1} \\
&\sum_{\substack{r_1,\ldots,r_8\in\mathbb{N}: \\ \{r_1,\ldots,r_8\}=(t_1,\ldots,t_4,j_1,\ldots,j_4)}} \sup_{\widetilde{s}_1,\ldots,\widetilde{s}_8\in\mathbb{R}^d} \sup_{\widetilde{u}_1,\ldots,\widetilde{u}_8\in[0,1]} \left\{ \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_1,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_1,\widetilde{s}_1)\, \widetilde{\mathbf{G}}^c_{r_2,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_2,\widetilde{s}_2)\right]\right| \mathbb{1}_{\{|r_1-r_2|\leqslant \mathscr{a}\}} \right. \\
&\cdot \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_3,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_3,\widetilde{s}_3)\, \widetilde{\mathbf{G}}^c_{r_4,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_4,\widetilde{s}_4)\right]\right| \mathbb{1}_{\{|r_3-r_4|\leqslant \mathscr{a}\}} \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_5,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_5,\widetilde{s}_5)\, \widetilde{\mathbf{G}}^c_{r_6,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_6,\widetilde{s}_6)\right]\right|
\end{aligned}$$

$$\cdot \mathbb{1}_{\{|r_5-r_6|\leqslant \mathscr{a}\}} \left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_7,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_7,\widetilde{s}_7)\,\widetilde{\mathbf{G}}^c_{r_8,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_8,\widetilde{s}_8)\right]\right| \mathbb{1}_{\{|r_7-r_8|\leqslant \mathscr{a}\}}\Bigg\}$$
$$=: \mathbb{F}_{T,1,\mathrm{R}} + \mathbb{F}_{T,2,\mathrm{R}}. \tag{D.17}$$

It follows for all $\mathrm{R} \in \{\Re, \Im\}$ by using (D.3), (A.1) $\big($which ensures $\mathscr{a} \ll b^{-1/12}\big)$ and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii):

$$\mathbb{F}_{T,1,\mathrm{R}} \leqslant \frac{C}{T^4 b^2} \sum_{k=1}^{\mathfrak{K}_T} \sum_{t_1,\dots,t_4,j_1,\dots,j_4=\max\{1,l_{T,k}-\lceil 2Tb\rceil\}}^{o_{T,k}} \sum_{\substack{r_1,\dots,r_8\in\mathbb{N}:\\ \{r_1,\dots,r_8\}=(t_1,\dots,t_4,j_1,\dots,j_4)}} \Big\{\mathbb{1}_{\{\sup_{o_1,o_2\in\{r_1,\dots,r_4\}}|o_1-o_2|\leqslant 3\mathscr{a}\}}$$
$$\cdot \Big(\mathbb{1}_{\{\sup_{o_1,o_2\in\{r_5,\dots,r_8\}}|o_1-o_2|\leqslant 3\mathscr{a}\}} + \mathbb{1}_{\{|r_5-r_6|\leqslant \mathscr{a}\}}\mathbb{1}_{\{|r_7-r_8|\leqslant \mathscr{a}\}}\Big) + \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_1,\dots,r_3\}}|o_1-o_2|\leqslant 2\mathscr{a}\}}$$
$$\cdot \mathbb{1}_{\{\sup_{o_1,o_2\in\{r_4,\dots,r_6\}}|o_1-o_2|\leqslant 2\mathscr{a}\}}\mathbb{1}_{\{|r_7-r_8|\leqslant \mathscr{a}\}}\Big\}$$
$$\leqslant \frac{C}{T^4 b^2}\frac{T}{L_T}\left\{L_T \mathscr{a}^3\left(L_T\mathscr{a}^3 + L_T\mathscr{a}L_T\mathscr{a}\right) + L_T\mathscr{a}^2 L_T \mathscr{a}^2 L_T \mathscr{a}\right\}$$
$$= o(1). \tag{D.18}$$

In order to bound $\mathbb{F}_{T,2,\mathrm{R}}$, observe that the condition of the eightfold sum contained in $\mathbb{F}_{T,2,\mathrm{R}}$ implies that one of the indices $r_1,\dots,r_8$ has to equal j_1. In the following, just the case $r_1 = j_1$ is considered since the other cases can be handled similarly (note that the only difference between the left side of (D.19) given below and $\mathbb{F}_{T,2,\mathrm{R}}$ is that the eightfold sum with respect to $r_1,\dots,r_8$ contained on the left side of (D.19) is additionally restricted by the condition $r_1 = j_1$).
If $r_1 = j_1$, $j_1 \leqslant t_1 - 7\mathscr{a} - 1$ and $|r_1 - r_2| \leqslant \mathscr{a}$, one will obtain $r_2 \neq t_1$, such that the existence of (at least) one element in $\{t_2,t_3,t_4,j_2,j_3,j_4\}$ which differs at most $\mathscr{a}$ from t_1 and of another element that differs at most $\mathscr{a}$ from j_1 is necessary to ensure that the left side of (D.19) is non-zero. Therefore, $\#\{\max\{1,t_1-\lceil 2Tb\rceil\},\dots,t_1-7\mathscr{a}-1\} \leqslant CTb$ and (D.3) provide for all $\mathrm{R}\in\{\Re,\Im\}$:

$$\frac{C}{T^4b^2}\sum_{k=1}^{\mathfrak{K}_T}\sum_{t_1,\dots,t_4=l_{T,k}}^{o_{T,k}}\sum_{j_1=\max\{1,t_1-\lceil 2Tb\rceil\}}^{t_1-7\mathscr{a}-1}\sum_{j_2=\max\{1,t_2-\lceil 2Tb\rceil\}}^{t_2-7\mathscr{a}-1}\sum_{j_3=\max\{1,t_3-\lceil 2Tb\rceil\}}^{t_3-7\mathscr{a}-1}\sum_{j_4=\max\{1,t_4-\lceil 2Tb\rceil\}}^{t_4-7\mathscr{a}-1}$$
$$\sum_{\substack{r_1,\dots,r_8\in\mathbb{N}:\\ \{r_1,\dots,r_8\}=(t_1,\dots,t_4,j_1,\dots,j_4)\\ r_1=j_1}} \sup_{\widetilde{s}_1,\dots,\widetilde{s}_8\in\mathbb{R}^d}\ \sup_{\widetilde{u}_1,\dots,\widetilde{u}_8\in[0,1]} \Big\{\left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_1,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_1,\widetilde{s}_1)\,\widetilde{\mathbf{G}}^c_{r_2,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_2,\widetilde{s}_2)\right]\right|\mathbb{1}_{\{|r_1-r_2|\leqslant\mathscr{a}\}}$$
$$\cdot\left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_3,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_3,\widetilde{s}_3)\,\widetilde{\mathbf{G}}^c_{r_4,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_4,\widetilde{s}_4)\right]\right|\mathbb{1}_{\{|r_3-r_4|\leqslant\mathscr{a}\}}\left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_5,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_5,\widetilde{s}_5)\,\widetilde{\mathbf{G}}^c_{r_6,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_6,\widetilde{s}_6)\right]\right|$$
$$\cdot\mathbb{1}_{\{|r_5-r_6|\leqslant\mathscr{a}\}}\left|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{r_7,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_7,\widetilde{s}_7)\,\widetilde{\mathbf{G}}^c_{r_8,\{\mathscr{a}\},\mathrm{R}}(\widetilde{u}_8,\widetilde{s}_8)\right]\right|\mathbb{1}_{\{|r_7-r_8|\leqslant\mathscr{a}\}}\Big\}$$
$$\leqslant \frac{C}{T^4b^2}\sum_{k=1}^{\mathfrak{K}_T}\sum_{t_1=l_{T,k}}^{o_{T,k}}\sum_{j_1=\max\{1,t_1-\lceil 2Tb\rceil\}}^{t_1-7\mathscr{a}-1}\sum_{t_2,\dots,t_4,j_2,\dots,j_4=\max\{1,l_{T,k}-\lceil 2Tb\rceil\}}^{o_{T,k}}\sum_{\substack{r_2,\dots,r_8\in\mathbb{N}:\\ \{r_2,\dots,r_8\}=(t_1,\dots,t_4,j_2,\dots,j_4)}}$$
$$\mathbb{1}_{\{(|j_1-r_2|\leqslant\mathscr{a})\wedge(r_2\neq t_1)\}}\mathbb{1}_{\{|r_3-r_4|\leqslant\mathscr{a}\}}\mathbb{1}_{\{|r_5-r_6|\leqslant\mathscr{a}\}}\mathbb{1}_{\{|r_7-r_8|\leqslant\mathscr{a}\}}$$
$$\leqslant \frac{C}{T^4b^2}\frac{T}{L_T}L_T\mathscr{a}Tb\mathscr{a}L_T\mathscr{a}L_T\mathscr{a}$$
$$= o(1). \tag{D.19}$$

In conclusion, (D.17), (D.18), (D.19) and analog arguments imply (D.16). It follows similarly to (C.27) by using (D.16) that (6.26) in [52, Leucht and Neumann (2013), p. 275] is valid.
The conditions (6.27) and (6.28) in [52, Leucht and Neumann (2013), p. 275] are valid with $\theta_r = 0$ $\forall\, r\in\mathbb{N}$ due to (D.14), whereby θ_r is defined in [52, Leucht and Neumann (2013), p. 275].
Overall, Theorem 6.1 in [52, Leucht and Neumann (2013), p. 274 et seq.] shows (D.6) (see (4.14)) and, therefore, (D.5) provides Theorem 4.9 (i).

(ii) In order to prove Theorem 4.9 (ii), note at first that it holds (recall (4.6)):

$$\begin{aligned}\mathbb{P}\left(T\sqrt{b}\,\widehat{\mathfrak{Q}}_T-\mathbf{Bias}_T^{\text{indep}}>\tau_T\right)&\geqslant\mathbb{P}\left(\mathfrak{Q}-\left|\mathfrak{Q}-\widehat{\mathfrak{Q}}_T\right|-\frac{\left|\mathbf{Bias}_T^{\text{indep}}\right|}{T\sqrt{b}}\geqslant\frac{\tau_T+1}{T\sqrt{b}},\left|\mathfrak{Q}-\widehat{\mathfrak{Q}}_T\right|\leqslant C\sqrt{b}\right)\\&\geqslant\mathbb{P}\left(\mathfrak{Q}\geqslant C\sqrt{b}+\frac{\left|\mathbf{Bias}_T^{\text{indep}}\right|}{T\sqrt{b}}+\frac{\tau_T+1}{T\sqrt{b}},\left|\mathfrak{Q}-\widehat{\mathfrak{Q}}_T\right|\leqslant C\sqrt{b}\right).\end{aligned}\tag{D.20}$$

By assumption of Theorem 4.9 (ii), $\mathcal{H}_{1,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ with $\mathfrak{D}_1=\mathfrak{D}_2=\{0\}$ (see (4.1)) is valid, such that the Assumptions 2.2 $[\mathbf{StAp}]$ (which is contained in Assumption 4.1 $[\mathbf{INDEP}]$) and 4.3 $[\mathbf{WEI.2}]$ yield $\mathfrak{Q}>0$ (note (4.6) as well as Definition 2.6). In addition, it holds $\left|\mathbf{Bias}_T^{\text{indep}}\right|\leqslant C/\sqrt{b}$ (recall (4.12)), whereby the latter is an implication of Lemma 4.8 and Assumption 4.3 $[\mathbf{WEI.2}]$. These considerations, Assumption 4.5 $[\mathbf{K\&b.2}]$ (ii) and (3.54) provide:

$$\lim_{T\to\infty}\mathbb{P}\left(\mathfrak{Q}\geqslant C\sqrt{b}+\frac{\left|\mathbf{Bias}_T^{\text{indep}}\right|}{T\sqrt{b}}+\frac{\tau_T}{T\sqrt{b}}\right)=1.\tag{D.21}$$

Moreover, one obtains from the inequality $\left||x|^2-|y|^2\right|\leqslant(|x|+|y|)\,|x-y|\ \forall\,x,y\in\mathbb{C}$, Lemma B.1 with $\kappa_1=1$, whereby the latter shows $\left|\widehat{\mathbb{Q}}_T(u,s)\right|+|\mathbb{Q}(u,s)|\leqslant C\ \forall\,u\in[0,1],\ s\in\mathbb{R}^d$ (see (4.9), Definition 2.11, (4.8) as well as (4.6)) and by using the Propositions 2.12 (with $\mathfrak{U}_{0,1,b}=[b,1-b]$ under Assumption 4.5 $[\mathbf{K\&b.2}]$ (i)) as well as 2.14 together with Remark 4.2 (i) and Assumption 4.3 $[\mathbf{WEI.2}]$:

$$\begin{aligned}\mathbb{E}\left[\left|\widehat{\mathfrak{Q}}_T-\mathfrak{Q}\right|\right]&\leqslant\int_b^{1-b}\int_{\mathbb{R}^d}\mathbb{E}\left[\left|\left|\widehat{\mathbb{Q}}_T(u,s)\right|^2-|\mathbb{Q}(u,s)|^2\right|\right]\mathbf{w}(s)\,ds\,du+\int_{[0,b]\cup[1-b,1]}\int_{\mathbb{R}^d}|\mathbb{Q}(u,s)|^2\,\mathbf{w}(s)\,ds\,du\\&\leqslant C\int_b^{1-b}\int_{\mathbb{R}^d}\mathbb{E}\Big[\Big|\widehat{\varphi}\,(u,s)-\varphi(u,s)+\widehat{\varphi}^{[1]}\left(u,s^{[1]}\right)\cdot\left(-\widehat{\varphi}^{[2]}\left(u,s^{[2]}\right)+\varphi^{[2]}\left(u,s^{[2]}\right)\right)\\&\quad+\left(-\widehat{\varphi}^{[1]}\left(u,s^{[1]}\right)+\varphi^{[1]}\left(u,s^{[1]}\right)\right)\cdot\varphi^{[2]}\left(u,s^{[2]}\right)\Big|\Big]\,\mathbf{w}(s)\,ds\,du+Cb\\&\leqslant Cb+\frac{C}{\sqrt{Tb}}.\end{aligned}$$

Therefore, Markov's inequality and Assumption 4.5 $[\mathbf{K\&b.2}]$ (ii) imply:

$$\lim_{T\to\infty}\mathbb{P}\left(\left|\mathfrak{Q}-\widehat{\mathfrak{Q}}_T\right|\leqslant C\sqrt{b}\right)=1.\tag{D.22}$$

Overall, (D.20), (C.98), (D.21) and (D.22) prove Theorem 4.9 (ii). □

Proof of Theorem 4.11. Throughout the present proof, assume that T is large enough to ensure (note Definition A.1 (vii) as well as (vi)):

$$7\mathscr{a}_\beta+2\leqslant 2Tb,$$

which is possible due to Lemma D.11 (i), Assumption 3.15 $[\mathbf{W}^*]$ (iii), (A.1) and Assumption 4.5 $[\mathbf{K\&b.2}]$ (ii). This ensures $7\mathscr{a}_\beta+2\leqslant L_T$ due to (D.3).
In order to verify Theorem 4.11, it will be shown at first for $T\to\infty$ that:

$$T\sqrt{b}\,\widehat{\mathfrak{Q}}_T^*-\mathbf{Bias}_T^{\text{indep}*}\xrightarrow{\text{d}}Z^{\text{indep}}\quad\text{in probability,}\tag{D.23}$$

which means that the distance (quantified by the Prokhorov metric) between the conditional distribution of $T\sqrt{b}\,\widehat{\mathfrak{Q}}_T^*-\mathbf{Bias}_T^{\text{indep}*}$ (conditioned on $X_{1,T},\dots,X_{T,T}$) and the distribution of Z^{indep} converges to zero in probability.

To verify (D.23), define for all $r, t, j \in \{1, \ldots, T\}$, $\mathrm{R} \in \{\Re, \Im\}$, $\mathrm{n} \in \mathbb{N}_0$, $u \in [0,1]$, $s \in \mathbb{R}^d$, $k \in \{1, \ldots, \mathfrak{K}_T\}$ (see (4.11), Definition A.1 (i), (D.3) and recall that $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\mathbf{G}_{r,T}(u,s) := \mathbf{g}_{u,s}\left(X_{r,T}\right), \quad \mathbf{G}_{r,T,\mathrm{R}}(u,s) := \mathrm{R}\left\{\mathbf{G}_{r,T}(u,s)\right\}, \quad \mathbf{G}_{r,T,\{\mathrm{n}\mathscr{a}\}}(u,s) := \mathbf{g}_{u,s}\left(X_{r,T,\{\mathrm{n}\mathscr{a}\}}\right),$$
$$\mathbf{G}_{r,T,\{\mathrm{n}\mathscr{a}\},\mathrm{R}}(u,s) := \mathrm{R}\left\{\mathbf{G}_{r,T,\{\mathrm{n}\mathscr{a}\}}(u,s)\right\},$$
$$\mathfrak{H}_{T,\mathrm{R}}\left(t,j\right) := \frac{2}{Tb^{\frac{3}{2}}}\int_{\mathbb{R}^d}\int_b^{1-b} K\left(\frac{\frac{t}{T}-u}{b}\right)\mathbf{G}^c_{t,T,\mathrm{R}}(u,s)\cdot K\left(\frac{\frac{j}{T}-u}{b}\right)\mathbf{G}^c_{j,T,\mathrm{R}}(u,s)\,du\,\mathbf{w}(s)\,ds,$$
$$\mathfrak{H}_T\left(t,j\right) := \mathfrak{H}_{T,\Re}\left(t,j\right) + \mathfrak{H}_{T,\Im}\left(t,j\right),$$
$$\mathfrak{H}^{\{\mathrm{n}\}}_{T,\mathrm{R}}\left(t,j\right) := \frac{2}{Tb^{\frac{3}{2}}}\int_{\mathbb{R}^d}\int_b^{1-b} K\left(\frac{\frac{t}{T}-u}{b}\right)\mathbf{G}^c_{t,T,\{\mathrm{n}\mathscr{a}\},\mathrm{R}}(u,s)\cdot K\left(\frac{\frac{j}{T}-u}{b}\right)\mathbf{G}^c_{j,T,\{\mathrm{n}\mathscr{a}\},\mathrm{R}}(u,s)\,du\,\mathbf{w}(s)\,ds,$$
$$\mathfrak{H}^{\{\mathrm{n}\}}_T\left(t,j\right) := \mathfrak{H}^{\{\mathrm{n}\}}_{T,\Re}\left(t,j\right) + \mathfrak{H}^{\{\mathrm{n}\}}_{T,\Im}\left(t,j\right),$$
$$l^*_{T,k} := (k-1)\cdot(L_T+\rho_T) + 7\mathscr{a}_\beta + 2, \qquad \mathbb{H}^*_{T,k,\mathrm{R}} := \sum_{t=l^*_{T,k}}^{o_{T,k}}\ \sum_{j=1}^{t-7\mathscr{a}_\beta-1} \mathfrak{H}_{T,\mathrm{R}}\left(t,j\right) W^*_{t,\{\mathscr{a}_\beta\}} W^*_{j,\{\mathscr{a}_\beta\}}$$
$$\mathbb{H}^*_{T,k} := \mathbb{H}^*_{T,k,\Re} + \mathbb{H}^*_{T,k,\Im} \quad \text{as well as} \quad \mathbb{H}^*_T := \sum_{k=1}^{\mathfrak{K}_T} \mathbb{H}^*_{T,k}. \tag{D.24}$$

One obtains for all $\mathrm{R} \in \{\Re, \Im\}$ analogously to (D.7):

$$\sup_{t,j=1,\ldots,T}\left|\mathfrak{H}_{T,\mathrm{R}}\left(t,j\right)\right| \leqslant \frac{C}{T\sqrt{b}}, \quad \sup_{t,j=1,\ldots,T}\left|\mathfrak{H}_T\left(t,j\right)\right| \leqslant \frac{C}{T\sqrt{b}}, \quad \sup_{\mathrm{n}\in\mathbb{N}_0}\sup_{t,j=1,\ldots,T}\left|\mathfrak{H}^{\{\mathrm{n}\}}_{T,\mathrm{R}}\left(t,j\right)\right| \leqslant \frac{C}{T\sqrt{b}},$$
$$\sup_{\mathrm{n}\in\mathbb{N}_0}\sup_{t,j=1,\ldots,T}\left|\mathfrak{H}^{\{\mathrm{n}\}}_T\left(t,j\right)\right| \leqslant \frac{C}{T\sqrt{b}} \tag{D.25}$$

and for all $\mathrm{R} \in \{\Re, \Im\}$, $t, j \in \{1, \ldots, T\}$, $\mathrm{n} \in \mathbb{N}_0$ similarly to (D.8):

$$|t-j| > 2Tb \implies \left(\mathfrak{H}_{T,\mathrm{R}}\left(t,j\right) = 0 \ \wedge \ \mathfrak{H}_T\left(t,j\right) = 0 \ \wedge \ \mathfrak{H}^{\{\mathrm{n}\}}_{T,\mathrm{R}}\left(t,j\right) = 0 \ \wedge \ \mathfrak{H}^{\{\mathrm{n}\}}_T\left(t,j\right) = 0\right). \tag{D.26}$$

The Lemmata D.12, D.13 as well as D.16 and (3.27) imply (note (3.26)):

$$T\sqrt{b}\,\widehat{\mathfrak{Q}}^*_T - \mathbf{Bias}^{\mathrm{indep}*}_T - \mathbb{H}^*_T = o^*_{\mathbb{P}}(1). \tag{D.27}$$

Thus, in order to prove (D.23), it suffices to show for $T \to \infty$ that:

$$\mathbb{H}^*_T \xrightarrow{\mathrm{d}} Z^{\mathrm{indep}} \quad \text{in probability.} \tag{D.28}$$

From now on, the validity of the assumptions demanded in Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275 et seq.] will be examined in order to use this corollary to verify (D.28), whereby the expression $\mathbb{H}^*_{T,k}$ takes the role of $X^*_{n,k}$ which originates from Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275 et seq.].

The Assumptions 3.15 $[\mathbf{W}^*]$ (ii) and (iii) imply (see Definition A.1 (i)):

$$\mathbb{E}^*\left[\mathbb{H}^*_{T,k}\right] = 0 \quad \forall\, k \in \{1, \ldots, \mathfrak{K}_T\}, \tag{D.29}$$

such that the first assumption of Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275] is valid. Further, it holds $l^*_{T,k+1} - \lceil 2Tb \rceil - \mathscr{a}_\beta \geqslant o_{T,k} + 1\ \forall\, k \in \{1, \ldots, \mathfrak{K}_T - 1\}$ (note (D.24) and (D.3)). Thus, one obtains from (D.26) similarly to (D.12) by recalling Definition A.1 (i) as well as using Assumption 3.15 $[\mathbf{W}^*]$ (ii) that $\left(\mathbb{H}^*_{T,k}\right)_{k=1}^{\mathfrak{K}_T}$ is a sequence of conditioned on $(X_{t,T})_{t=1}^T$ independent random variables. Therefore, one obtains from (D.29) (recall (D.24)):

$$\mathrm{Var}^*\left(\mathbb{H}_T^*\right) = \sum_{k=1}^{\mathfrak{K}_T} \mathbb{E}^*\left[\left(\mathbb{H}_{T,k}^*\right)^2\right] \quad \text{a. s.} \tag{D.30}$$

Markov's inequality, (D.30) and Lemma D.17 show:

$$\mathbb{P}\left(\sum_{k=1}^{\mathfrak{K}_T} \mathbb{E}^*\left[\left(\mathbb{H}_{T,k}^*\right)^2\right] > \sigma^{\text{indep}} + 1\right) \leqslant \mathbb{E}\left[\left|\sum_{k=1}^{\mathfrak{K}_T} \mathbb{E}^*\left[\left(\mathbb{H}_{T,k}^*\right)^2\right] - \sigma^{\text{indep}}\right|\right] \overset{T\to\infty}{\longrightarrow} 0. \tag{D.31}$$

The second assumption of Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275] holds due to (D.31), Lemma 4.8 and Assumption 4.3 $[\mathbf{WEI.2}]$, whereby the latter two provide $\sigma^{\text{indep}} < \infty$ (see (4.13)). Lemma D.17 shows (by recalling (D.24)) the validity of (6.29) in [52, Leucht and Neumann (2013), p. 275] with $\Sigma = \sigma^{\text{indep}}$ (note that Σ originates from (6.29) in [52, Leucht and Neumann (2013), p. 275]). Next, it is proved that the condition (6.30) in [52, Leucht and Neumann (2013), p. 275] is fulfilled. Therefor, it is shown at first that:

$$\sum_{k=1}^{\mathfrak{K}_T} \mathbb{E}\left[\left(\mathbb{H}_{T,k,\Re}^*\right)^4\right] = o(1). \tag{D.32}$$

It follows from Assumption 3.15 $[\mathbf{W}^*]$ (ii), arguments which are similar to those that show (C.120), (D.26), (C.64), Lemma D.15 (ii) with $\mathrm{n} = 8$, (D.25), the first inequality of Lemma D.11 (i), Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) and arguments which are analog to those that yield (D.16), i. e., which are similar to those that show (D.17), (D.18) as well as (D.19) (recall (D.24), (D.3) and Definition A.1 (vi)):

$$\begin{aligned}
&\sum_{k=1}^{\mathfrak{K}_T} \mathbb{E}\left[\left(\mathbb{H}_{T,k,\Re}^*\right)^4\right] \\
&\leqslant C \sum_{k=1}^{\mathfrak{K}_T} \sum_{t_1,\ldots,t_4=l_{T,k}^*}^{o_{T,k}} \sum_{\substack{j_1=1\\|j_1-t_1|\leqslant 2Tb}}^{t_1-7a_\beta-1} \sum_{\substack{j_2=1\\|j_2-t_2|\leqslant 2Tb}}^{t_2-7a_\beta-1} \sum_{\substack{j_3=1\\|j_3-t_3|\leqslant 2Tb}}^{t_3-7a_\beta-1} \sum_{\substack{j_4=1\\|j_4-t_4|\leqslant 2Tb}}^{t_4-7a_\beta-1} \Big|\mathbb{E}\left[\mathfrak{H}_{T,\Re}(t_1,j_1)\,\mathfrak{H}_{T,\Re}(t_2,j_2)\right.\\
&\left.\cdot\mathfrak{H}_{T,\Re}(t_3,j_3)\,\mathfrak{H}_{T,\Re}(t_4,j_4)\right] - \mathbb{E}\left[\mathfrak{H}_{T,\Re}^{\{8\}}(t_1,j_1)\,\mathfrak{H}_{T,\Re}^{\{8\}}(t_2,j_2)\,\mathfrak{H}_{T,\Re}^{\{8\}}(t_3,j_3)\,\mathfrak{H}_{T,\Re}^{\{8\}}(t_4,j_4)\right]\Big| \\
&+ C \sum_{k=1}^{\mathfrak{K}_T} \sum_{t_1,\ldots,t_4=l_{T,k}^*}^{o_{T,k}} \sum_{\substack{j_1=1\\|j_1-t_1|\leqslant 2Tb}}^{t_1-7a_\beta-1} \sum_{\substack{j_2=1\\|j_2-t_2|\leqslant 2Tb}}^{t_2-7a_\beta-1} \sum_{\substack{j_3=1\\|j_3-t_3|\leqslant 2Tb}}^{t_3-7a_\beta-1} \sum_{\substack{j_4=1\\|j_4-t_4|\leqslant 2Tb}}^{t_4-7a_\beta-1} \Big|\mathbb{E}\left[\mathfrak{H}_{T,\Re}^{\{8\}}(t_1,j_1)\right.\\
&\left.\cdot\mathfrak{H}_{T,\Re}^{\{8\}}(t_2,j_2)\,\mathfrak{H}_{T,\Re}^{\{8\}}(t_3,j_3)\,\mathfrak{H}_{T,\Re}^{\{8\}}(t_4,j_4)\right]\Big| \\
&\leqslant C\frac{T}{L_T} L_T^4 (Tb)^4 \frac{C}{T^{1+4\cdot(1+\delta)}\sqrt{b}} \frac{1}{\left(T\sqrt{b}\right)^3} \\
&+ C \sum_{k=1}^{\mathfrak{K}_T} \sum_{t_1,\ldots,t_4=l_{T,k}^*}^{o_{T,k}} \sum_{j_1=\max\{1,t_1-\lceil 2Tb\rceil\}}^{t_1-7\cdot 8a-1} \sum_{j_2=\max\{1,t_2-\lceil 2Tb\rceil\}}^{t_2-7\cdot 8a-1} \sum_{j_3=\max\{1,t_3-\lceil 2Tb\rceil\}}^{t_3-7\cdot 8a-1} \sum_{j_4=\max\{1,t_4-\lceil 2Tb\rceil\}}^{t_4-7\cdot 8a-1} \\
&\left|\mathbb{E}\left[\mathfrak{H}_{T,\Re}^{\{8\}}(t_1,j_1)\,\mathfrak{H}_{T,\Re}^{\{8\}}(t_2,j_2)\,\mathfrak{H}_{T,\Re}^{\{8\}}(t_3,j_3)\,\mathfrak{H}_{T,\Re}^{\{8\}}(t_4,j_4)\right]\right| \\
&= o(1).
\end{aligned} \tag{D.33}$$

One obtains from (C.26), (D.33) and similar arguments (see (D.24)):

$$\sum_{k=1}^{\mathfrak{K}_T} \mathbb{E}\left[\left(\mathbb{H}_{T,k}^*\right)^4\right] = o(1).$$

This and (C.24) show for all $\epsilon > 0$:

$$\mathbb{E}\left[\left|\sum_{k=1}^{\mathfrak{K}_T}\mathbb{E}^*\left[\left(\mathbb{H}_{T,k}^*\right)^2\mathbb{1}_{\{|\mathbb{H}_{T,k}^*|>\epsilon\}}\right]-0\right|\right]=\sum_{k=1}^{\mathfrak{K}_T}\mathbb{E}\left[\left(\mathbb{H}_{T,k}^*\right)^2\mathbb{1}_{\{|\mathbb{H}_{T,k}^*|>\epsilon\}}\right]=o(1),$$

such that (6.30) in [52, Leucht and Neumann (2013), p. 275] holds.

The validity of (6.31) as well as (6.32)[8] in [52, Leucht and Neumann (2013), p. 275] with $\theta_r = 0$ $\forall\, r \in \mathbb{N}$ can be shown by using that $\left(\mathbb{H}_{T,k}^*\right)_{k=1}^{\mathfrak{K}_T}$ is a sequence of conditioned on $(X_{t,T})_{t=1}^T$ independent random variables (as explained above (D.30)), whereby θ_r is defined in [52, Leucht and Neumann (2013), p. 275].
Overall, Corollary 6.1 in [52, Leucht and Neumann (2013), p. 275 et seq.] proves (D.28) (recall (D.24) and (4.14)), such that (D.27) yields (D.23).

In the case $\sigma^{\text{indep}} > 0$, (4.22) follows from (D.23) and a chaining argument that is similar to (C.77).

Instead, if $\sigma^{\text{indep}} = 0$, the Lemmata D.12, D.13 and D.16 as well as Lemma D.17 together with (D.29) will imply:

$$\mathbb{E}\left[\left|T\sqrt{b}\,\hat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}*}\right|\right] \leqslant \mathbb{E}\left[\left|T\sqrt{b}\,\hat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}*} - \mathbb{H}_T^*\right|\right] + \sqrt{\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{H}_T^*\right)^2\right]\right]} = o(1),$$

which proves (4.23) due to (3.27) (note (3.26)). □

Proof of Theorem 4.13. At first, one obtains from Assumption 3.15 $[\mathbf{W}^*]$ (iii) (which provides $K^*(0) = 1$), (D.2), Assumption 4.3 $[\mathbf{WEI.2}]$ as well as (3.60) (recall (4.12), (4.21) and that $\rho \in (0,1)$ according to Assumption 2.4 $[\mathbf{DM.3}]$, which is contained in Assumption 4.1 $[\mathbf{INDEP}]$):

$$\begin{aligned}
&\mathbb{E}\left[\left|T\sqrt{b}\,\hat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}} - \left(T\sqrt{b}\,\hat{\mathfrak{Q}}_T^* - \mathbf{Bias}_T^{\text{indep}*}\right)\right|\right]\\
&\leqslant \frac{C}{\sqrt{b}}\int\limits_{\mathbb{R}^d}\int\limits_0^1\sum_{t\in\mathbb{Z}\setminus\{0\}}\left|K^*\left(\frac{t}{\beta}\right)-1\right|\left|\gamma_{\tilde{\mathbf{G}}}(u,s,t)\right|\,du\,\mathbf{w}(s)\,ds\\
&\leqslant \frac{C}{\sqrt{b}\beta}\sum_{t\in\mathbb{Z}\setminus\{0\}}\frac{\left|K^*\left(\frac{t}{\beta}\right)-1\right|}{|t|/\beta}|t|\rho^{|t|} = o(1).
\end{aligned}\tag{D.34}$$

It follows from (D.34), (3.27), (D.23) and a chaining argument which is similar to (C.77) that (4.24) holds. Moreover, (D.34), (3.27), (4.23) and (3.28) yield (4.25). □

Proof of Theorem 4.18. Throughout the present proof, assume that T is large enough to ensure $T \geqslant 1 + \mathfrak{D}_{\max}$ (note (4.28)).
(i) In order to verify Theorem 4.18 (i), define at first for all $u \in [0,1]$, $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1} \times \mathbb{R}^{d_2\cdot\#\mathfrak{D}_2}$, $k \in \{1,2\}$ (see (4.26), (4.2) as well as (4.4)):

$$\begin{aligned}
\hat{\varphi}^+_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) &:= \frac{1}{T}\sum_{t=1}^T K_b\left(\frac{t}{T}-u\right)e^{i\left\langle s^{[1]},X^{[1]+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\right\rangle}e^{i\left\langle s^{[2]},X^{[2]+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\right\rangle},\\
\hat{\varphi}^{[k]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[k]}\right) &:= \frac{1}{T}\sum_{t=1}^T K_b\left(\frac{t}{T}-u\right)e^{i\left\langle s^{[k]},X^{[k]+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\right\rangle} \quad\text{and}\\
\hat{\mathfrak{Q}}^+_{\mathfrak{D}_1,\mathfrak{D}_2,T} &:= \int\limits_{\mathbb{R}^{d_1\cdot\#\mathfrak{D}_1+d_2\cdot\#\mathfrak{D}_2}}\int\limits_b^{1-b}\left|\hat{\varphi}^+_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s)-\hat{\varphi}^{[1]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[1]}\right)\hat{\varphi}^{[2]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[2]}\right)\right|^2\,du\,\mathbf{w}(s)\,ds.
\end{aligned}\tag{D.35}$$

Note that $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$, which holds by assumption of Theorem 4.18 (i), is equivalent to $\widetilde{X}_{\mathfrak{D}_1,0}^{[1]}(u) \perp\!\!\!\perp \widetilde{X}_{\mathfrak{D}_2,0}^{[2]}(u)\ \forall\, u \in [0,1]$ (recall (4.1) as well as (4.2)). Hence, applying Theorem 4.9 (i) (note thereby (4.9) as well as (4.11)) to the locally stationary process $\{X_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}^{+}\}$ (defined in (4.26)), whereby this application is justified due to Remark 4.16, implies (see (4.30) and (4.31)):

$$T\sqrt{b}\,\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}^{+} - \mathbf{Bias}_{\mathfrak{D}_1,\mathfrak{D}_2,T}^{\text{indep}} \xrightarrow{\text{d}} Z_{\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}. \tag{D.36}$$

Let $f, g\colon [0,1] \times \mathbb{R}^r \to \mathbb{C}$ (with $r \in \mathbb{N}$) be arbitrary, not necessarily deterministic functions that live on the same probability space. If each expression contained in (D.37) given below is well-defined, it will follow similarly to (C.315):

$$\begin{aligned}
&\mathbb{E}\left[\int\limits_{\mathbb{R}^r}\int\limits_{b}^{1-b}\left||f(u,s)|^2 - |g(u,s)|^2\right| du\,\mathbf{w}(s)\,ds\right] \\
&\leqslant \int\limits_{\mathbb{R}^r}\sup_{u\in[b,1-b]}\mathbb{E}\left[|f(u,s)-g(u,s)|^2\right]\mathbf{w}(s)\,ds + 2\sqrt{\int\limits_{\mathbb{R}^r}\sup_{u\in[b,1-b]}\mathbb{E}\left[|f(u,s)-g(u,s)|^2\right]\mathbf{w}(s)\,ds} \\
&\cdot\sqrt{\int\limits_{\mathbb{R}^r}\sup_{u\in[b,1-b]}\mathbb{E}\left[|g(u,s)|^2\right]\mathbf{w}(s)\,ds}.
\end{aligned} \tag{D.37}$$

One obtains for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1} \times \mathbb{R}^{d_2\cdot\#\mathfrak{D}_2}$ from the fact that $X_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}^{[k]+} = X_{\mathfrak{D}_k,t,T}^{[k]}$ $\forall\, t \in \left\{1 + \max_{\mathfrak{d}\in\mathfrak{D}_1\cup\mathfrak{D}_2}\mathfrak{d}, \ldots, T\right\}$ (which is valid due to (4.26)) and from Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) (recall (D.35), (4.28) as well as Definition 2.11):

$$\begin{aligned}
&\sup_{u\in[b,1-b]}\left\|\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}^{+}(u,s) - \widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s)\right\|_2 \\
&\leqslant \sup_{u\in[b,1-b]}\left\|\frac{1}{Tb}\sum_{t=1}^{\mathfrak{D}_{\max}} K\left(\frac{\frac{t}{T}-u}{b}\right) e^{i\left\langle s^{[1]}, X_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}^{[1]+}\right\rangle} e^{i\left\langle s^{[2]}, X_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}^{[2]+}\right\rangle}\right\|_2 \\
&+ \sup_{u\in[b,1-b]}\frac{1}{Tb}\sum_{t=\max\{1+\mathfrak{D}_{\max},\lfloor uT-Tb\rfloor\}}^{\min\{T,\lceil\mathfrak{D}_{\text{mean}}+uT+Tb\rceil\}}\left|K\left(\frac{\frac{t}{T}-u}{b}\right) - K\left(\frac{\frac{t-\mathfrak{D}_{\text{mean}}}{T}-u}{b}\right)\right| \\
&\leqslant \frac{C}{Tb}.
\end{aligned} \tag{D.38}$$

Lemma B.1 with $\kappa_1 = 1$ as well as similar considerations imply:

$$\left|\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}^{+}(u,s)\right| \leqslant C,\quad \left|\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}^{[k]+}\left(u,s^{[k]}\right)\right| \leqslant C,\quad \left|\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s)\right| \leqslant C \text{ and } \left|\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}^{[k]}\left(u,s^{[k]}\right)\right| \leqslant C$$
$$\forall\, u \in [0,1],\ s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1} \times \mathbb{R}^{d_2\cdot\#\mathfrak{D}_2},\ k \in \{1,2\}. \tag{D.39}$$

It follows for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1} \times \mathbb{R}^{d_2\cdot\#\mathfrak{D}_2}$ from (C.112), (C.25) with $M = 3$, (D.38) as well as similar considerations, (D.39) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) (see (4.28)):

$$\begin{aligned}
&T\sqrt{b}\sup_{u\in[b,1-b]}\mathbb{E}\left[\left|\widehat{\mathbb{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s) - \left(\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}^{+}(u,s) - \widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}^{[1]+}\left(u,s^{[1]}\right)\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}^{[2]+}\left(u,s^{[2]}\right)\right)\right|^2\right] \\
&\leqslant \frac{CT\sqrt{b}}{(Tb)^2} = o\left(\sqrt{b}\right).
\end{aligned} \tag{D.40}$$

In addition, recalling that $\mathcal{H}_{0,\mathfrak{D}_1,\mathfrak{D}_2}^{\text{indep}}$ is equivalent to $\widetilde{X}_{\mathfrak{D}_1,0}^{[1]}(u) \perp\!\!\!\perp \widetilde{X}_{\mathfrak{D}_2,0}^{[2]}(u)\ \forall\, u \in [0,1]$ (see (4.1) and (4.2)), using (C.112), (C.25) with $M = 3$, applying the Propositions 2.12 as well as 2.14 to the locally stationary process $\{X_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}^{+}\}$ (note (4.26)), whereby this application is justified due to Remark 4.16, and using (D.39) as well as Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) yield for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1} \times$

$\mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}$ (recall (D.35)):

$$T\sqrt{b} \sup_{u \in [b,1-b]} \mathbb{E}\left[\left|\widehat{\varphi}^{+}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) - \widehat{\varphi}^{[1]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[1]}\right) \widehat{\varphi}^{[2]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[2]}\right)\right|^2\right]$$
$$\leqslant T\sqrt{b} \sup_{u \in [b,1-b]} \mathbb{E}\left[\left|\widehat{\varphi}^{+}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) - \mathbb{E}\left[e^{i\left\langle s^{[1]}, \tilde{X}^{[1]}_{\mathfrak{D}_1,0}(u)\right\rangle} e^{i\left\langle s^{[2]}, \tilde{X}^{[2]}_{\mathfrak{D}_2,0}(u)\right\rangle}\right] + \mathbb{E}\left[e^{i\left\langle s^{[1]}, \tilde{X}^{[1]}_{\mathfrak{D}_1,0}(u)\right\rangle}\right]\right.\right.$$
$$\left.\left.\cdot \mathbb{E}\left[e^{i\left\langle s^{[2]}, \tilde{X}^{[2]}_{\mathfrak{D}_2,0}(u)\right\rangle}\right] - \widehat{\varphi}^{[1]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[1]}\right) \widehat{\varphi}^{[2]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[2]}\right)\right|^2\right]$$
$$\leqslant \frac{C}{\sqrt{b}}\left(|s|_1^{2+2\delta} + 1\right). \tag{D.41}$$

In summary, (D.36), (D.37) with $f(u,s) := \widehat{\mathbb{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s)$ as well as $g(u,s) := \widehat{\varphi}^{+}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) -\widehat{\varphi}^{[1]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[1]}\right)\widehat{\varphi}^{[2]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[2]}\right) \forall u \in [0,1]$, $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}$, (D.40), (D.41) and Assumption 4.17 [**WEI.3**] imply Theorem 4.18 (i) (see (4.28)).

(ii) In order to prove Theorem 4.18 (ii), note at first that one obtains for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}$ due to (C.98):

$$\mathbb{P}\left(T\sqrt{b}\, \widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \mathbf{Bias}^{\text{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} > \tau_T\right)$$
$$\geqslant \mathbb{P}\Bigg(-T\sqrt{b}\left|-\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} + \widehat{\mathfrak{Q}}^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,T}\right| + T\sqrt{b}\, \widehat{\mathfrak{Q}}^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \mathbf{Bias}^{\text{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} > \tau_T,$$
$$\left|\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \widehat{\mathfrak{Q}}^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,T}\right| \leqslant T^{-1/4}\Bigg)$$
$$\geqslant \mathbb{P}\left(T\sqrt{b}\, \widehat{\mathfrak{Q}}^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \mathbf{Bias}^{\text{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} > \tau_T + T\sqrt{b}\, T^{-1/4}\right) + \mathbb{P}\left(\left|\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \widehat{\mathfrak{Q}}^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,T}\right| \leqslant T^{-1/4}\right) - 1. \tag{D.42}$$

It follows from (3.54) (that holds by supposition of Theorem 4.18 (ii), which is currently proved) and from Assumption 4.5 [**K&b.2**] (ii) that $\left(\tau_T + T\sqrt{b}\, T^{-1/4}\right)_{T \in \mathbb{N}}$ is a sequence of positive numbers which grows to ∞ for $T \to \infty$ slower than $T\sqrt{b}$. Thus, applying Theorem 4.9 (ii) to the locally stationary process $\left\{X^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\right\}$ defined in (4.26) (which is justified by Remark 4.16) shows that the first summand on the right side of (D.42) converges to 1 for $T \to \infty$.
Moreover, $\mathbb{E}\left[\left|\, |X_1|^2 - |X_2|^2 \,\right|\right] \leqslant \|X_1 - X_2\|_2 \left(\|X_1\|_2 + \|X_2\|_2\right)$ holds for all random variables X_1 and X_2 that own two finite moments. Hence, the first inequality of (D.40) (divided by $T\sqrt{b}$), (D.39) and Assumption 4.5 [**K&b.2**] (ii) show $\mathbb{E}\left[\left|\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \widehat{\mathfrak{Q}}^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,T}\right|\right] \leqslant C/(Tb) = o\left(T^{-1/4}\right)$ (recall (4.28) as well as (D.35)). Therefore, Markov's inequality provides that the second summand on the right side of (D.42) converges to 1 for $T \to \infty$.
Overall, these arguments prove Theorem 4.18 (ii).

(iii) In order to prove Theorem 4.18 (iii), define at first for all $u \in [0,1]$, $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}$ (note (4.26) and (D.35)):

$$\widehat{\mathbb{Q}}^{*+}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s) := \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right)\Bigg(e^{i\left\langle s, X^{+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\right\rangle} - \widehat{\varphi}^{+}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) + \widehat{\varphi}^{[1]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[1]}\right)$$
$$\cdot\left(\widehat{\varphi}^{[2]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[2]}\right) - e^{i\left\langle s^{[2]}, X^{[2]+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\right\rangle}\right) + \left(\widehat{\varphi}^{[1]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[1]}\right) - e^{i\left\langle s^{[1]}, X^{[1]+}_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\right\rangle}\right)\widehat{\varphi}^{[2]+}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[2]}\right)\Bigg) W^{*}_t$$

$$\text{as well as } \widehat{\mathfrak{Q}}^{*+}_{\mathfrak{D}_1,\mathfrak{D}_2,T} := \int_{\mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1 + d_2 \cdot \#\mathfrak{D}_2}} \int_{b}^{1-b} \left|\widehat{\mathbb{Q}}^{*+}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s)\right|^2 du\, \mathbf{w}(s)\, ds. \tag{D.43}$$

Moreover, (C.112), the same arguments which have been used to verify (D.38), (D.39) and Assumption

3.15 $[\mathbf{W}^*]$ (ii), (iii) as well as (i) $\big($the latter provides $\beta = o\big(Tb^2\big)\big)$ show for all $s := \big(s^{[1]\prime}, s^{[2]\prime}\big)' \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}$ (see (4.29)):

$$\sqrt{T\sqrt{b}} \sup_{u \in [b,1-b]} \left\| \widehat{\mathbb{Q}}^{*+}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s) - \widehat{\mathbb{Q}}^{*}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s) \right\|_2 \leqslant \sqrt{T\sqrt{b}} \frac{C}{Tb} = o\left(\sqrt{\frac{\sqrt{b}}{\beta}}\right). \tag{D.44}$$

Let $(Z_{t,T})_{t \in \mathbb{Z}, T \in \mathbb{N}}$ be a sequence of random functions with $Z_{t,T} \colon [0,1] \times \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1 + d_2 \cdot \#\mathfrak{D}_2} \to \mathbb{C}$, each $Z_{t,T}$ should be measurable with respect to the sigma algebra generated by $(\varepsilon_k)_{k \in \mathbb{Z}}$ (introduced in Definition 2.1 that underlies Assumption 4.1 $[\mathbf{INDEP}]$) and suppose $\sup_{u \in [b,1-b]} \sup_{t=1,\dots,T} \mathbb{E}\big[|Z_{t,T}(u,s)|^2\big] < \infty \ \forall\, s \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1 + d_2 \cdot \#\mathfrak{D}_2}$. Then, it follows similarly to (C.254) for all $s \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1 + d_2 \cdot \#\mathfrak{D}_2}$:

$$\sup_{u \in [b,1-b]} \mathbb{E}\left[\left| \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) Z_{t,T}(u,s)\, W_t^* \right|^2 \right] \leqslant \frac{C\beta}{Tb} \sup_{u \in [b,1-b]} \sup_{t=1,\dots,T : \left|\frac{t}{T} - u\right| \leqslant b} \mathbb{E}\left[|Z_{t,T}(u,s)|^2 \right]. \tag{D.45}$$

Hence, one obtains for all $s \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}$ from (D.39) (recall (D.43)):

$$T\sqrt{b} \sup_{u \in [b,1-b]} \mathbb{E}\left[\left| \widehat{\mathbb{Q}}^{*+}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s) \right|^2 \right] \leqslant \frac{C\beta}{\sqrt{b}}. \tag{D.46}$$

In conclusion, (D.37) with $f(u,s) := \widehat{\mathbb{Q}}^{*}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s)$ and $g(u,s) := \widehat{\mathbb{Q}}^{*+}_{\mathfrak{D}_1,\mathfrak{D}_2,T}(u,s)\ \forall\, u \in [0,1],\ s \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}$, (D.44), (D.46), Assumption 3.15 $[\mathbf{W}^*]$ (i) (which provides $\beta \to \infty$ for $T \to \infty$) as well as (3.27) imply (see (4.29), (D.43) and (3.26)):

$$T\sqrt{b}\left(\widehat{\mathfrak{Q}}^{*}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \widehat{\mathfrak{Q}}^{*+}_{\mathfrak{D}_1,\mathfrak{D}_2,T} \right) = o^*_{\mathbb{P}}(1). \tag{D.47}$$

In the case $\sigma^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2} > 0$, applying (4.24) to the locally stationary process $\big\{X^+_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\big\}$ defined in (4.26) (which is possible due to Remark 4.16) yields $T\sqrt{b}\, \widehat{\mathfrak{Q}}^{*+}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \mathbf{Bias}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} \xrightarrow{\mathrm{d}} Z^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2}$ in probability (in particular, note thereby the similarity between (4.20) as well as (D.43) and recall (4.31)). Thus, in this case, (D.47) and a chaining argument which is similar to (C.77) show (4.32). Instead, if $\sigma^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2} = 0$, applying (4.25) to the locally stationary process $\big\{X^+_{\mathfrak{D}_1,\mathfrak{D}_2,t,T}\big\}$ implies $T\sqrt{b}\, \widehat{\mathfrak{Q}}^{*+}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \mathbf{Bias}^{\mathrm{indep}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} = o^*_{\mathbb{P}}(1)$, which verifies (4.33) due to (D.47) and (3.28). □

D.2. Auxiliary results belonging to Chapter 4 and their proofs

Lemma D.1. *Let the Assumptions 2.4* $[\mathbf{DM.3}]$ *and 4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *be fulfilled. Moreover, define the random functions* $\widehat{\varphi}_{\{\mathscr{A}\}} := \widehat{\varphi}_{T,\{\mathscr{A}\}}$ *and* $\widetilde{\varphi}_{\{\mathscr{A}\}} := \widetilde{\varphi}_{T,\{\mathscr{A}\}}$ *as follows (note the Definitions A.1 (i) as well as (vi)):*

$$\widehat{\varphi}_{\{\mathscr{A}\}}(u,s) := \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) e^{i\left\langle s, X_{t,T,\{\mathscr{A}\}}\right\rangle} \quad \text{and} \quad \widetilde{\varphi}_{\{\mathscr{A}\}}(u,s) := \frac{1}{T} \sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) e^{i\left\langle s, \widetilde{X}_{t,\{\mathscr{A}\}}(u)\right\rangle} \quad \forall\, u \in [0,1],\ s \in \mathbb{R}^d. \tag{D.48}$$

Then, it holds for all $s \in \mathbb{R}^d$, $q \geqslant 1 + \delta$ *(recall the Definitions 2.11, 2.6 as well as (B.29) and that* δ *originates from Assumption 2.2* $[\mathbf{StAp}]$*, which is contained in Assumption 2.4* $[\mathbf{DM.3}]$*):*

(i)

$$\sup_{t=1,\dots,T} \left\| e^{i\left\langle s, X_{t,T,\{\mathscr{A}\}}\right\rangle} - e^{i\left\langle s, X_{t,T}\right\rangle} \right\|_q \leqslant \frac{C}{T^{(1+\delta)/q}} |s|_1^{\frac{1+\delta}{q}}.$$

(ii)

$$\sup_{u\in[0,1]} \left\| \widehat{\varphi}_{\{\mathscr{a}\}}(u,s) - \widehat{\varphi}(u,s) \right\|_q \leqslant \frac{C}{T^{(1+\delta)/q}} |s|_1^{\frac{1+\delta}{q}}.$$

(iii)

$$\sup_{u\in[b,1-b]} \left| \mathbb{E}\left[\widehat{\varphi}_{\{\mathscr{a}\}}(u,s)\right] - \varphi(u,s) \right| \leqslant C \left(b^{1+\delta} + \frac{1}{Tb} \right) \left(|s|_1^{1+\delta} + 1 \right).$$

(iv)

$$\sup_{u\in[0,1]} \mathbb{E}\left[\left| \widehat{\varphi}_{\{\mathscr{a}\}}(u,s) - \mathbb{E}\left[\widehat{\varphi}_{\{\mathscr{a}\}}(u,s)\right] \right|^2 \right] \leqslant \frac{C}{Tb} \left(|s|_1 + 1 \right).$$

(v)

$$\sup_{u\in[0,1]} \sup_{t\in\mathbb{Z}} \left\| e^{i\left\langle s, \tilde{X}_{t,\{\mathscr{a}\}}(u) \right\rangle} - e^{i\left\langle s, \tilde{X}_t(u) \right\rangle} \right\|_q \leqslant \frac{C}{T^{(1+\delta)/q}} |s|_1^{\frac{1+\delta}{q}}.$$

(vi)

$$\sup_{u\in[0,1]} \left\| \widetilde{\varphi}_{\{\mathscr{a}\}}(u,s) - \widetilde{\varphi}(u,s) \right\|_q \leqslant \frac{C}{T^{(1+\delta)/q}} |s|_1^{\frac{1+\delta}{q}}.$$

(vii)

$$\sup_{u\in[b,1-b]} \left| \mathbb{E}\left[\widetilde{\varphi}_{\{\mathscr{a}\}}(u,s)\right] - \varphi(u,s) \right| \leqslant \frac{C}{Tb} \left(|s|_1 + 1 \right).$$

(viii)

$$\sup_{u\in[0,1]} \mathbb{E}\left[\left| \widetilde{\varphi}_{\{\mathscr{a}\}}(u,s) - \mathbb{E}\left[\widetilde{\varphi}_{\{\mathscr{a}\}}(u,s)\right] \right|^2 \right] \leqslant \frac{C}{Tb} \left(|s|_1 + 1 \right).$$

Proof. (i) One obtains for all $s \in \mathbb{R}^d$ similarly to the first two inequalities of (C.205) by using Lemma B.4 (i), shifting the index of a sum and due to Assumption 2.4 $[\mathbf{DM.3}]$:

$$\sup_{t=1,\dots,T} \left\| \Re\left\{ e^{i\left\langle s, X_{t,T,\{\mathscr{a}\}} \right\rangle} - e^{i\left\langle s, X_{t,T} \right\rangle} \right\} \right\|_q \leqslant C |s|_1^{\frac{1+\delta}{q}} \left(\sum_{l=\mathscr{a}}^{\infty} \Delta_{l+1} \right)^{\frac{1+\delta}{q}} \leqslant C |s|_1^{\frac{1+\delta}{q}} \rho^{\frac{\mathscr{a}(1+\delta)}{q}} \left(\sum_{l=1}^{\infty} \rho^l \right)^{\frac{1+\delta}{q}}. \tag{D.49}$$

Lemma D.1 (i) follows from (D.49) and Definition A.1 (vi) (note that $\rho \in (0,1)$ according to Assumption 2.4 $[\mathbf{DM.3}]$).

(ii) The Lemmata D.1 (i) and B.1 with $\kappa_1 = 1$ yield Lemma D.1 (ii) (see (D.48) as well as Definition 2.11).

(iii) Lemma D.1 (iii) is an implication of Lemma D.1 (ii) with $q = 1 + \delta$, Proposition 2.12 (with $\mathfrak{U}_{0,1,b} = [b, 1-b]$ under Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i)) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii).

(iv) One obtains for all $s \in \mathbb{R}^d$ from (C.370) and Lemma B.1 with $\kappa_1 = 1$, whereby the latter provides $\left|\Re\{\widehat{\varphi}_{\{\mathscr{a}\}}(u,s)\}\right| + \left|\Re\{\widehat{\varphi}(u,s)\}\right| \leqslant C \;\forall\, u \in [0,1],\, s \in \mathbb{R}^d$:

$$\sup_{u\in[0,1]} \left| \mathrm{Var}\left(\Re\left\{ \widehat{\varphi}_{\{\mathscr{a}\}}(u,s) \right\} \right) - \mathrm{Var}\left(\Re\left\{ \widehat{\varphi}(u,s) \right\} \right) \right| \leqslant C \sup_{u\in[0,1]} \left\| \Re\left\{ \widehat{\varphi}_{\{\mathscr{a}\}}(u,s) - \widehat{\varphi}(u,s) \right\} \right\|_1. \tag{D.50}$$

Lemma D.1 (iv) follows from (D.50) and similar arguments, Lemma D.1 (ii) with $q = 1 + \delta$ as well as Proposition 2.14.

(v) Lemma D.1 (v) can be proved similarly to Lemma D.1 (i).

(vi) The Lemmata D.1 (v) and B.1 with $\kappa_1 = 1$ show Lemma D.1 (vi) (note (D.48) as well as (B.29)).

(vii) The Lemmata D.1 (vi) with $q = 1 + \delta$ and B.3 imply Lemma D.1 (vii) since $\mathfrak{U}_{0,1,b} = [b, 1-b]$ (recall (2.9)) under Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i).

(viii) It follows analogously to the proof of Proposition 2.14 by using Lemma B.4 (vi) and Assumption 2.4 $[\mathbf{DM.3}]$:

$$\sup_{u\in[0,1]} \mathbb{E}\left[\left|\widetilde{\varphi}(u,s) - \mathbb{E}\left[\widetilde{\varphi}(u,s)\right]\right|^2\right] \leqslant \frac{C}{Tb}\left(|s|_1 + 1\right) \quad \forall\, s \in \mathbb{R}^d. \tag{D.51}$$

One obtains similarly to the proof of Lemma D.1 (iv) by using Lemma D.1 (vi) with $q = 1 + \delta$ and (D.51) that Lemma D.1 (viii) is valid.

□

Lemma D.2. *Suppose that the Assumptions 4.1* $[\mathbf{INDEP}]$ *and 4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *hold. Moreover, define for all* $t \in \{1,\ldots,T\}$, $r \in \mathbb{Z}$, $u \in [0,1]$, $s^{[k]} \in \mathbb{R}^{d_k}$ *with* $k \in \{1,2\}$ *(see the Definitions A.1 (i) as well as (vi)):*

$$\begin{aligned}
&X^{[k]}_{t,T,\{\mathscr{a}\}} := \mathbb{E}\left[X^{[k]}_{t,T}\middle|\mathcal{F}_{t,t-\mathscr{a}}\right], \quad \widetilde{X}^{[k]}_{r,\{\mathscr{a}\}}(u) := \mathbb{E}\left[\widetilde{X}^{[k]}_{r}(u)\middle|\mathcal{F}_{r,r-\mathscr{a}}\right],\\
&\widehat{\varphi}^{[k]}_{\{\mathscr{a}\}}\left(u, s^{[k]}\right) := \widehat{\varphi}^{[k]}_{T,\{\mathscr{a}\}}\left(u, s^{[k]}\right) := \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) e^{i\left\langle s^{[k]}, X^{[k]}_{t,T,\{\mathscr{a}\}}\right\rangle} \quad \text{and}\\
&\widetilde{\varphi}^{[k]}_{\{\mathscr{a}\}}\left(u, s^{[k]}\right) := \widetilde{\varphi}^{[k]}_{T,\{\mathscr{a}\}}\left(u, s^{[k]}\right) := \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right) e^{i\left\langle s^{[k]}, \widetilde{X}^{[k]}_{t,\{\mathscr{a}\}}(u)\right\rangle}.
\end{aligned} \tag{D.52}$$

Then, one obtains for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$*:*

(i)

$$\sup_{u\in[0,1]}\left|\mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right)\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right)\right] - \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right)\right]\mathbb{E}\left[\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right)\right]\right| \leqslant \frac{C}{Tb}\left(|s|_1 + 1\right).$$

(ii)

$$\sup_{u\in[0,1]}\left|\mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right)\widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right)\right] - \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right)\right]\mathbb{E}\left[\widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right)\right]\right| \leqslant \frac{C}{Tb}\left(|s|_1 + 1\right).$$

(iii)

$$\sup_{u\in[0,1]}\left|\mathbb{E}\left[\widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right)\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right)\right] - \mathbb{E}\left[\widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right)\right]\mathbb{E}\left[\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right)\right]\right| \leqslant \frac{C}{Tb}\left(|s|_1 + 1\right).$$

Proof. (i) Applying the Lemmata D.1 (iv) and (viii) to the locally stationary processes $\left\{X^{[k]}_{t,T}\right\}$ with $k \in \{1,2\}$ (which is justified due to Remark 4.2 (i)) provides for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$:

$$\begin{aligned}
&\sup_{u\in[0,1]}\left|\mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right)\cdot\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right)\right] - \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right)\right]\cdot\mathbb{E}\left[\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right)\right]\right|\\
&= \sup_{u\in[0,1]}\left|\mathbb{E}\left[\left(\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right) - \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right)\right]\right)\cdot\left(\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right) - \mathbb{E}\left[\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right)\right]\right)\right]\right|\\
&\leqslant \frac{C}{\sqrt{Tb}}\sqrt{\left|s^{[1]}\right|_1 + 1}\cdot\frac{C}{\sqrt{Tb}}\sqrt{\left|s^{[2]}\right|_1 + 1},
\end{aligned}$$

that proves Lemma D.2 (i).
Lemma D.2 (ii) and (iii) can be verified similarly to Lemma D.2 (i). □

Lemma D.3. *Let the Assumptions 4.1* $[\mathbf{INDEP}]$ *(in particular, note* $d := d_1 + d_2$*), 4.3* $[\mathbf{WEI.2}]$ *and 4.5* $[\mathbf{K\&b.2}]$ *be fulfilled. Moreover, define for all* $s := \left(s^{[1]'}, s^{[2]'}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ *(recall (D.48) and (D.52)):*

$$\widehat{\mathbb{Q}}_{T,\{\mathscr{a}\}}(u,s) := \widehat{\varphi}_{\{\mathscr{a}\}}(u,s) - \widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u, s^{[1]}\right) \cdot \widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u, s^{[2]}\right). \tag{D.53}$$

Then, it holds for $T \to \infty$ *(see (4.9)):*

$$T\sqrt{b}\,\mathbb{E}\left[\left|\widehat{\mathfrak{Q}}_T - \int_{\mathbb{R}^d}\int_b^{1-b} \left|\widehat{\mathbb{Q}}_{T,\{\mathscr{a}\}}(u,s)\right|^2 du\,\mathbf{w}(s)\,ds\right|\right] = o(1).$$

Proof. The inequality $\left||x|^2 - |y|^2\right| \leqslant |x-y|\big(|x| + |y|\big)\ \forall\, x,y \in \mathbb{C}$, Lemma B.1 with $\kappa_1 = 1$ (whereby the latter yields $\left|\widehat{\mathbb{Q}}_T(u,s)\right| + \left|\widehat{\mathbb{Q}}_{T,\{\mathscr{a}\}}(u,s)\right| \leqslant C\ \forall\, u \in [0,1],\ s \in \mathbb{R}^d$ (recall (4.9))) and (C.112) imply:

$$\begin{aligned}
&T\sqrt{b}\,\mathbb{E}\left[\left|\widehat{\mathfrak{Q}}_T - \int_{\mathbb{R}^d}\int_b^{1-b} \left|\widehat{\mathbb{Q}}_{T,\{\mathscr{a}\}}(u,s)\right|^2 du\,\mathbf{w}(s)\,ds\right|\right] \\
&\leqslant CT\sqrt{b}\int_{\mathbb{R}^d} \sup_{u\in[0,1]} \mathbb{E}\left[\left|\widehat{\varphi}(u,s) - \widehat{\varphi}_{\{\mathscr{a}\}}(u,s)\right|\right] + \sup_{u\in[0,1]} \mathbb{E}\left[\left|-\widehat{\varphi}^{[1]}\left(u,s^{[1]}\right) + \widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right|\right] \\
&+ \sup_{u\in[0,1]} \mathbb{E}\left[\left|\widehat{\varphi}^{[2]}\left(u,s^{[2]}\right) - \widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right|\right] \mathbf{w}(s)\,ds.
\end{aligned}$$

Thus, Lemma D.1 (ii) with $q = 1 + \delta$ together with Remark 4.2 (i), Assumption 4.3 $[\mathbf{WEI.2}]$ and Assumption 4.5 $[\mathbf{K\&b.2}]$ (ii) show Lemma D.3. □

Lemma D.4. *Suppose that the Assumptions 4.1* $[\mathbf{INDEP}]$*, 4.3* $[\mathbf{WEI.2}]$ *as well as 4.5* $[\mathbf{K\&b.2}]$ *are fulfilled and assume that the null hypothesis* $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ *(defined in (4.1)) with* $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ *is valid. Moreover, define for all* $u \in [0,1]$*,* $s := \left(s^{[1]'}, s^{[2]'}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ *(see (D.48), (D.52) and Definition A.1 (i) as well as (vi)):*

$$\begin{aligned}
\widetilde{\mathbb{Q}}^{\circ}_{T,\{\mathscr{a}\}}(u,s) &:= \widetilde{\varphi}_{\{\mathscr{a}\}}(u,s) - \mathbb{E}\left[\widetilde{\varphi}_{\{\mathscr{a}\}}(u,s)\right] + \mathbb{E}\left[\widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right] \cdot \mathbb{E}\left[\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right] \\
&\quad - \widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right) \cdot \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right).
\end{aligned} \tag{D.54}$$

Then, it holds for $T \to \infty$*:*

$$T\sqrt{b}\,\mathbb{E}\left[\left|\int_{\mathbb{R}^d}\int_b^{1-b} \left|\widehat{\mathbb{Q}}_{T,\{\mathscr{a}\}}(u,s)\right|^2 du\,\mathbf{w}(s)\,ds - \int_{\mathbb{R}^d}\int_b^{1-b} \left|\widetilde{\mathbb{Q}}^{\circ}_{T,\{\mathscr{a}\}}(u,s)\right|^2 du\,\mathbf{w}(s)\,ds\right|\right] = o(1).$$

Proof. At first, note that one obtains for all $s := \left(s^{[1]'}, s^{[2]'}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ from (C.25) with $M = 4$ (recall (D.53) as well as (D.54) and that $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\begin{aligned}
&\sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\widehat{\mathbb{Q}}_{T,\{\mathscr{a}\}}(u,s) - \widetilde{\mathbb{Q}}^{\circ}_{T,\{\mathscr{a}\}}(u,s)\right|^2\right] \\
&\leqslant 4 \sup_{u\in[b,1-b]} \left\| \left(\widehat{\varphi}_{\{\mathscr{a}\}}(u,s) - \widetilde{\varphi}_{\{\mathscr{a}\}}(u,s)\right)^c \right\|_2^2 \\
&+ 4 \sup_{u\in[b,1-b]} \left\| \left(\widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right) \cdot \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right) - \widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right) \cdot \widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right)^c \right\|_2^2
\end{aligned}$$

$$+ 4 \sup_{u\in[b,1-b]} \left|\mathbb{E}\left[\widehat{\varphi}_{\{\mathscr{a}\}}(u,s) - \widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\cdot \widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right]\right|^2$$

$$+ 4 \sup_{u\in[b,1-b]} \left|\mathbb{E}\left[\widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\cdot \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right] - \mathbb{E}\left[\widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right]\cdot \mathbb{E}\left[\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right]\right|^2$$

$$=: 4\mathrm{E}^{\perp\!\!\!\perp}_{1,T}(s) + 4\mathrm{E}^{\perp\!\!\!\perp}_{2,T}(s) + 4\mathrm{E}^{\perp\!\!\!\perp}_{3,T}(s) + 4\mathrm{E}^{\perp\!\!\!\perp}_{4,T}(s). \tag{D.55}$$

Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i), the inequality $|\mathrm{Cov}(X,Y)| \leqslant C\,\|X\|_2\,\|Y\|_2$ (which holds for all real-valued random variables X and Y with finite second moments that live on the same probability space), (3.14) with $q = 2$, Assumption 2.2 $[\mathbf{StAp}]$ (i) as well as Remark 2.3 yield for all $s \in \mathbb{R}^{d_1+d_2}$ (note (D.48), Definition 2.11 and Definition A.1 (i)):

$$\sup_{u\in[b,1-b]} \left\|\Re\left\{\widehat{\varphi}_{\{\mathscr{a}\}}(u,s) - \widetilde{\varphi}_{\{\mathscr{a}\}}(u,s)\right\}^c\right\|_2^2$$

$$\leqslant \sup_{u\in[b,1-b]} \frac{1}{(Tb)^2} \sum_{\substack{t_1,t_2=\max\{1,\lfloor uT-Tb\rfloor\}\\ |t_1-t_2|\leqslant \mathscr{a}}}^{\min\{T,\lfloor uT+Tb\rfloor\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_2}{T}-u}{b}\right)\left|\mathrm{Cov}\Big(\cos\left(\left\langle s, X_{t_1,T,\{\mathscr{a}\}}\right\rangle\right) -\right.$$

$$\left.-\cos\left(\left\langle s, \widetilde{X}_{t_1,\{\mathscr{a}\}}(u)\right\rangle\right), \cos\left(\left\langle s, X_{t_2,T,\{\mathscr{a}\}}\right\rangle\right) - \cos\left(\left\langle s, \widetilde{X}_{t_2,\{\mathscr{a}\}}(u)\right\rangle\right)\Big)\right|$$

$$\leqslant \frac{C\mathscr{a}}{Tb} \sup_{u\in[0,1]} \sup_{t=1,\ldots,T:\left|\frac{t}{T}-u\right|\leqslant b} \mathbb{E}\left[\left|\cos\left(\left\langle s, X_{t,T,\{\mathscr{a}\}}\right\rangle\right) - \cos\left(\left\langle s, \widetilde{X}_{t,\{\mathscr{a}\}}(u)\right\rangle\right)\right|^2\right]$$

$$\leqslant \frac{C\mathscr{a}}{Tb} \sup_{u\in[0,1]} \sup_{t=1,\ldots,T:\left|\frac{t}{T}-u\right|\leqslant b} \mathbb{E}\left[\left|\mathbb{E}\left[\left\langle s, X_{t,T} - \widetilde{X}_t(u)\right\rangle \middle| \mathcal{F}_{t,t-\mathscr{a}}\right]\right|^{1+\delta}\right]$$

$$\leqslant C\frac{\mathscr{a}b^\delta}{T}\,|s|_1^{1+\delta}. \tag{D.56}$$

It follows for all $s \in \mathbb{R}^{d_1+d_2}$ from (D.56) and similar arguments (see (D.55)):

$$\mathrm{E}^{\perp\!\!\!\perp}_{1,T}(s) \leqslant C\frac{\mathscr{a}b^\delta}{T}\,|s|_1^{1+\delta}. \tag{D.57}$$

Further, one obtains for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ due to (C.25) with $M = 2$ (recall (D.55)):

$$\mathrm{E}^{\perp\!\!\!\perp}_{2,T}(s) \leqslant 2 \sup_{u\in[b,1-b]} \mathbb{E}\Bigg[\Bigg|\mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\cdot\left(\widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right) - \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right)\right]$$

$$- \widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\cdot\left(\widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right) - \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right)\Bigg|^2\Bigg]$$

$$+ 2 \sup_{u\in[b,1-b]} \mathbb{E}\Bigg[\Bigg|\mathbb{E}\Big[\left(\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right) - \widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right)\cdot \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\Big]$$

$$- \left(\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right) - \widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right)\cdot \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\Bigg|^2\Bigg]$$

$$=: 2\mathrm{E}^{\perp\!\!\!\perp}_{2.1,T}(s) + 2\mathrm{E}^{\perp\!\!\!\perp}_{2.2,T}(s). \tag{D.58}$$

Moreover, (C.25) with $M = 4$ provides for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$:

$$\mathrm{E}^{\perp\!\!\!\perp}_{2.1,T}(s)$$

$$\leqslant 4 \sup_{u\in[b,1-b]} \left|\mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\cdot \widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right] - \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right]\cdot \mathbb{E}\left[\widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right]\right|^2$$

$$+ 4 \sup_{u\in[b,1-b]} \left| - \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\cdot \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right] + \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right]\cdot \mathbb{E}\left[\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right]\right|^2$$

$$+ 4 \sup_{u\in[b,1-b]} \mathbb{E}\left[\left| - \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right]\cdot\left(\widehat{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right) - \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right)^c\right|^2\right]$$

$$+ 4 \sup_{u \in [b,1-b]} \mathbb{E}\left[\left| \left(\mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\varkappa\}}\left(u, s^{[1]}\right)\right] - \widehat{\varphi}^{[1]}_{\{\varkappa\}}\left(u, s^{[1]}\right) \right) \cdot \left(\widehat{\varphi}^{[2]}_{\{\varkappa\}}\left(u, s^{[2]}\right) - \widetilde{\varphi}^{[2]}_{\{\varkappa\}}\left(u, s^{[2]}\right) \right) \right|^2 \right]$$

$$=: 4\mathrm{E}^{\perp\!\!\!\perp}_{2.1.1,T}(s) + 4\mathrm{E}^{\perp\!\!\!\perp}_{2.1.2,T}(s) + 4\mathrm{E}^{\perp\!\!\!\perp}_{2.1.3,T}(s) + 4\mathrm{E}^{\perp\!\!\!\perp}_{2.1.4,T}(s). \tag{D.59}$$

Lemma D.2 (ii) and (i) as well as (C.25) with $M = 2$ imply for all $s \in \mathbb{R}^{d_1+d_2}$:

$$\mathrm{E}^{\perp\!\!\!\perp}_{2.1.1,T}(s) + \mathrm{E}^{\perp\!\!\!\perp}_{2.1.2,T}(s) \leqslant \frac{C}{(Tb)^2}\left(|s|_1^2 + 1\right). \tag{D.60}$$

Lemma B.1 with $\kappa_1 = 1$ yields (note (D.52) and Definition 2.11):

$$\left| \widehat{\varphi}^{[k]}_{\{\varkappa\}}\left(u, s^{[k]}\right)\right| \leqslant C \quad \forall\, u \in [0,1],\ s \in \mathbb{R}^{d_k},\ k \in \{1,2\}, \tag{D.61}$$

such that arguments which are similar to those that show (D.56) provide for all $s \in \mathbb{R}^{d_1+d_2}$ (see (D.59)):

$$\mathrm{E}^{\perp\!\!\!\perp}_{2.1.3,T}(s) \leqslant C\frac{\varkappa b^{\delta}}{T} |s|_1^{1+\delta}. \tag{D.62}$$

Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) and the same arguments which prove the last two inequalities of (D.56) yield for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (recall (D.59), (D.52) as well as Definition A.1 (i)):

$$\begin{aligned}
&\mathrm{E}^{\perp\!\!\!\perp}_{2.1.4,T}(s) \\
&\leqslant \sup_{u \in [b,1-b]} \frac{1}{(Tb)^4} \sum_{\substack{t_1,t_2,t_3,t_4 = \max\{1, \lfloor uT - Tb \rfloor\} \\ \exists r \in \{2,3,4\}: |t_1 - t_r| \leqslant \varkappa}}^{\min\{T, \lfloor uT+Tb \rfloor\}} K\left(\frac{\frac{t_1}{T} - u}{b}\right) K\left(\frac{\frac{t_2}{T} - u}{b}\right) K\left(\frac{\frac{t_3}{T} - u}{b}\right) K\left(\frac{\frac{t_4}{T} - u}{b}\right) \\
&\cdot \Bigg| \mathbb{E}\Bigg[\left(\mathbb{E}\left[e^{i\left\langle s^{[1]}, X^{[1]}_{t_1,T,\{\varkappa\}}\right\rangle}\right] - e^{i\left\langle s^{[1]}, X^{[1]}_{t_1,T,\{\varkappa\}}\right\rangle} \right) \cdot \left(e^{i\left\langle s^{[2]}, X^{[2]}_{t_2,T,\{\varkappa\}}\right\rangle} - e^{i\left\langle s^{[2]}, \tilde{X}^{[2]}_{t_2,\{\varkappa\}}(u)\right\rangle} \right) \\
&\cdot \overline{\left(\mathbb{E}\left[e^{i\left\langle s^{[1]}, X^{[1]}_{t_3,T,\{\varkappa\}}\right\rangle}\right] - e^{i\left\langle s^{[1]}, X^{[1]}_{t_3,T,\{\varkappa\}}\right\rangle} \right) \cdot \left(e^{i\left\langle s^{[2]}, X^{[2]}_{t_4,T,\{\varkappa\}}\right\rangle} - e^{i\left\langle s^{[2]}, \tilde{X}^{[2]}_{t_4,\{\varkappa\}}(u)\right\rangle} \right)} \Bigg] \Bigg| \\
&\leqslant C\frac{\varkappa}{Tb} \sup_{u \in [b,1-b]} \sup_{t_2 = 1,\ldots,T: \left|\frac{t_2}{T} - u\right| \leqslant b} \sup_{t_4 = 1,\ldots,T: \left|\frac{t_4}{T} - u\right| \leqslant b} \\
&\sqrt{\mathbb{E}\left[\left| e^{i\left\langle s^{[2]}, X^{[2]}_{t_2,T,\{\varkappa\}}\right\rangle} - e^{i\left\langle s^{[2]}, \tilde{X}^{[2]}_{t_2,\{\varkappa\}}(u)\right\rangle} \right|^2 \right]} \cdot \sqrt{\mathbb{E}\left[\left| e^{i\left\langle -s^{[2]}, X^{[2]}_{t_4,T,\{\varkappa\}}\right\rangle} - e^{i\left\langle -s^{[2]}, \tilde{X}^{[2]}_{t_4,\{\varkappa\}}(u)\right\rangle} \right|^2 \right]} \\
&\leqslant C\frac{\varkappa b^{\delta}}{T} |s|_1^{1+\delta}.
\end{aligned} \tag{D.63}$$

Replace on the right side of the inequality contained in (D.59) all $\widehat{\varphi}^{[1]}_{\{\varkappa\}}\left(u, s^{[1]}\right)$ by $\widetilde{\varphi}^{[2]}_{\{\varkappa\}}\left(u, s^{[2]}\right)$, all $\widehat{\varphi}^{[2]}_{\{\varkappa\}}\left(u, s^{[2]}\right)$ by $\widehat{\varphi}^{[1]}_{\{\varkappa\}}\left(u, s^{[1]}\right)$ as well as all $\widetilde{\varphi}^{[2]}_{\{\varkappa\}}\left(u, s^{[2]}\right)$ by $\widetilde{\varphi}^{[1]}_{\{\varkappa\}}\left(u, s^{[1]}\right)$ to obtain for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ similarly to (D.59), (D.60) (in particular, note also Lemma D.2 (iii)), (D.62) and (D.63) (see (D.58)):

$$\mathrm{E}^{\perp\!\!\!\perp}_{2.2,T}(s) \leqslant \frac{C}{(Tb)^2}\left(|s|_1^2 + 1\right) + C\frac{\varkappa b^{\delta}}{T} |s|_1^{1+\delta}. \tag{D.64}$$

In summary, (D.58), (D.59), (D.60), (D.62), (D.63) and (D.64) imply for all $s \in \mathbb{R}^{d_1+d_2}$:

$$\mathrm{E}^{\perp\!\!\!\perp}_{2,T}(s) \leqslant C\left(\frac{1}{(Tb)^2} + \frac{\varkappa b^{\delta}}{T}\right)\left(|s|_1^2 + 1\right). \tag{D.65}$$

The null hypothesis $\mathcal{H}^{\text{indep}}_{0,\mathfrak{D}_1,\mathfrak{D}_2}$ (defined in (4.1)) with $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ holds by assumption of the lemma which is currently proved. Therefore, $\varphi(u,s) = \varphi^{[1]}\left(u, s^{[1]}\right)\varphi^{[2]}\left(u, s^{[2]}\right)$ is valid for all $u \in [0,1]$, $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (recall Definition 2.6 and (4.6)). Thus, (C.25) with $M \in \{3,2\}$, Lemma

D.1 (iii) together with (C.112), Remark 4.2 (i) and (D.61) as well as Lemma D.2 (ii) yield for all $s := \left(s^{[1]'}, s^{[2]'}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (note (D.55)):

$$\begin{aligned}
\mathrm{E}_{3,T}^{\perp\!\!\perp}(s) &\leqslant 3 \sup_{u \in [b,1-b]} \left| \mathbb{E}\left[\widehat{\varphi}_{\{\varkappa\}}(u,s)\right] - \varphi(u,s) \right|^2 \\
&\quad + 3 \sup_{u \in [b,1-b]} \left| \varphi^{[1]}\left(u, s^{[1]}\right) \varphi^{[2]}\left(u, s^{[2]}\right) - \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\varkappa\}}\left(u, s^{[1]}\right)\right] \cdot \mathbb{E}\left[\widehat{\varphi}^{[2]}_{\{\varkappa\}}\left(u, s^{[2]}\right)\right] \right|^2 \\
&\quad + 3 \sup_{u \in [b,1-b]} \left| \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\varkappa\}}\left(u, s^{[1]}\right)\right] \cdot \mathbb{E}\left[\widehat{\varphi}^{[2]}_{\{\varkappa\}}\left(u, s^{[2]}\right)\right] - \mathbb{E}\left[\widehat{\varphi}^{[1]}_{\{\varkappa\}}\left(u, s^{[1]}\right) \cdot \widehat{\varphi}^{[2]}_{\{\varkappa\}}\left(u, s^{[2]}\right)\right] \right|^2 \\
&\leqslant C\left(b^{2+2\delta} + \frac{1}{(Tb)^2}\right)\left(|s|_1^{2+2\delta} + 1\right). \qquad \text{(D.66)}
\end{aligned}$$

Lemma D.2 (iii) implies for all $s \in \mathbb{R}^{d_1+d_2}$ (recall (D.55)):

$$\mathrm{E}_{4,T}^{\perp\!\!\perp}(s) \leqslant \frac{C}{(Tb)^2}\left(|s|_1^2 + 1\right). \qquad \text{(D.67)}$$

One obtains for all $s \in \mathbb{R}^{d_1+d_2}$ from (D.55), (D.57), (D.65), (D.66), (D.67), (A.1) (the latter ensures $\varkappa b^{\delta} \to 0$ for $T \to \infty$) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii):

$$T\sqrt{b} \sup_{u \in [b,1-b]} \mathbb{E}\left[\left|\widehat{\mathbb{Q}}_{T,\{\varkappa\}}(u,s) - \widetilde{\mathbb{Q}}^{\circ}_{T,\{\varkappa\}}(u,s)\right|^2\right] \leqslant o\left(\sqrt{b}\right)\left(|s|_1^{2+2\delta} + 1\right), \qquad \text{(D.68)}$$

whereby the expression $o\big(\sqrt{b}\big)$ does not depend on $s \in \mathbb{R}^{d_1+d_2}$. Further, (C.112), (C.25) with $M = 3$, Lemma B.1 with $\kappa_1 = 1$, which yields $\left|\widetilde{\varphi}^{[k]}_{\{\varkappa\}}\left(u, s^{[k]}\right)\right| \leqslant C$ a.s. $\forall\, u \in [0,1]$, $s \in \mathbb{R}^{d_k}$ (note (D.52)) and Lemma D.1 (viii) together with the first paragraph of Remark 4.2 (i) show for all $s \in \mathbb{R}^{d_1+d_2}$ (see (D.54)):

$$T\sqrt{b} \sup_{u \in [b,1-b]} \mathbb{E}\left[\left|\widetilde{\mathbb{Q}}^{\circ}_{T,\{\varkappa\}}(u,s)\right|^2\right] \leqslant \frac{C}{\sqrt{b}}\left(|s|_1 + 1\right). \qquad \text{(D.69)}$$

Lemma D.4 follows from (D.37) with $f(u,s) := \widehat{\mathbb{Q}}_{T,\{\varkappa\}}(u,s)$ as well as $g(u,s) := \widetilde{\mathbb{Q}}^{\circ}_{T,\{\varkappa\}}(u,s)$ $\forall\, u \in [0,1]$, $s \in \mathbb{R}^{d_1+d_2}$, (D.68), (D.69) and Assumption 4.3 $[\mathbf{WEI.2}]$. ☐

Lemma D.5. *Let the Assumptions 4.1* $[\mathbf{INDEP}]$*, 4.3* $[\mathbf{WEI.2}]$ *and 4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *be valid. Moreover, define for all* $u \in [0,1]$*,* $s := \left(s^{[1]'}, s^{[2]'}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ *(recall (D.4), Definition A.1 (i) as well as (vi) and that* $X^c := X - \mathbb{E}[X]$ *for each random variable* X *with finite first moment):*

$$\widetilde{\mathbb{Q}}_{T,\{\varkappa\}}(u,s) := \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T} - u\right)\widetilde{\mathbf{G}}^{c}_{t,\{\varkappa\}}(u,s) \quad \text{and} \quad \widetilde{\mathfrak{Q}}_{T,\{\varkappa\}} := \int_{\mathbb{R}^d}\int_b^{1-b}\left|\widetilde{\mathbb{Q}}_{T,\{\varkappa\}}(u,s)\right|^2 du\, \mathbf{w}(s)\, ds. \qquad \text{(D.70)}$$

Then, it holds for $T \to \infty$ *(see (D.54)):*

$$T\sqrt{b}\,\mathbb{E}\left[\left|\widetilde{\mathfrak{Q}}_{T,\{\varkappa\}} - \int_{\mathbb{R}^d}\int_b^{1-b}\left|\widetilde{\mathbb{Q}}^{\circ}_{T,\{\varkappa\}}(u,s)\right|^2 du\, \mathbf{w}(s)\, ds\right|\right] = o(1).$$

Proof. At first, observe for all $u \in [0,1]$, $s := \left(s^{[1]'}, s^{[2]'}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (recall (D.70), (D.4), (4.11), (D.48) as well as (D.52)):

$$\widetilde{\mathbb{Q}}_{T,\{\varkappa\}}(u,s) = \left(\widetilde{\varphi}_{\{\varkappa\}}(u,s) - \varphi^{[1]}\left(u, s^{[1]}\right)\cdot\widetilde{\varphi}^{[2]}_{\{\varkappa\}}\left(u, s^{[2]}\right) - \varphi^{[2]}\left(u, s^{[2]}\right)\cdot\widetilde{\varphi}^{[1]}_{\{\varkappa\}}\left(u, s^{[1]}\right)\right)^c,$$

such that (C.25) with $M = 3$ provides for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (note (D.54)):

$$\begin{aligned}
&\sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\widetilde{\mathbb{Q}}_{T,\{\mathscr{a}\}}(u,s) - \widetilde{\mathbb{Q}}^{\circ}_{T,\{\mathscr{a}\}}(u,s)\right|^2\right] \\
&\leqslant 3 \sup_{u\in[b,1-b]} \left|\varphi^{[1]}\left(u,s^{[1]}\right) - \mathbb{E}\left[\widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right]\right|^2 \left\|\mathbb{E}\left[\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right] - \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right\|_2^2 \\
&+ 3 \sup_{u\in[b,1-b]} \left\|\mathbb{E}\left[\widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right] - \widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right\|_2^2 \left|\varphi^{[2]}\left(u,s^{[2]}\right) - \mathbb{E}\left[\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right]\right|^2 \\
&+ 3 \sup_{u\in[b,1-b]} \left\|\left(\mathbb{E}\left[\widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right] - \widetilde{\varphi}^{[1]}_{\{\mathscr{a}\}}\left(u,s^{[1]}\right)\right)\left(\mathbb{E}\left[\widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right] - \widetilde{\varphi}^{[2]}_{\{\mathscr{a}\}}\left(u,s^{[2]}\right)\right)\right\|_2^2 \\
&=: 3\breve{\mathrm{E}}^{\perp\!\!\!\perp}_{1,T}(s) + 3\breve{\mathrm{E}}^{\perp\!\!\!\perp}_{2,T}(s) + 3\breve{\mathrm{E}}^{\perp\!\!\!\perp}_{3,T}(s). \qquad \text{(D.71)}
\end{aligned}$$

Lemma D.1 (vii) as well as (viii) together with Remark 4.2 (i) and (C.422) yield for all $s \in \mathbb{R}^{d_1+d_2}$:

$$\breve{\mathrm{E}}^{\perp\!\!\!\perp}_{1,T}(s) + \breve{\mathrm{E}}^{\perp\!\!\!\perp}_{2,T}(s) \leqslant \frac{C}{(Tb)^3} \cdot \left(|s|_1^3 + 1\right). \qquad \text{(D.72)}$$

One obtains for all $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ from Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) (see (D.52) and Definition 2.11):

$$\begin{aligned}
\breve{\mathrm{E}}^{\perp\!\!\!\perp}_{3,T}(s) &\leqslant \sup_{u\in[b,1-b]} \frac{1}{(Tb)^4} \sum_{\substack{t_1,t_2,t_3,t_4=\max\{1,\lfloor uT-Tb\rfloor\} \\ \forall\, r\in\{1,\ldots,4\}\exists j\in\{1,\ldots,4\}\setminus\{r\}: |t_r-t_j|\leqslant \mathscr{a}}}^{\min\{T,\lceil uT+Tb\rceil\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_2}{T}-u}{b}\right) K\left(\frac{\frac{t_3}{T}-u}{b}\right) K\left(\frac{\frac{t_4}{T}-u}{b}\right) \\
&\cdot \left|\mathbb{E}\left[\left(-e^{i\left\langle s^{[1]},\tilde{X}^{[1]}_{t_1,\{\mathscr{a}\}}(u)\right\rangle}\right)^c\left(-e^{i\left\langle s^{[2]},\tilde{X}^{[2]}_{t_2,\{\mathscr{a}\}}(u)\right\rangle}\right)^c \overline{\left(-e^{i\left\langle s^{[1]},\tilde{X}^{[1]}_{t_3,\{\mathscr{a}\}}(u)\right\rangle}\right)^c\left(-e^{i\left\langle s^{[2]},\tilde{X}^{[2]}_{t_4,\{\mathscr{a}\}}(u)\right\rangle}\right)^c}\right]\right| \\
&\leqslant \frac{C\mathscr{a}^2}{(Tb)^2}. \qquad \text{(D.73)}
\end{aligned}$$

In summary, (D.71), (D.72), (D.73), (A.1) $\left(\text{the latter ensures } \mathscr{a} \ll \left(Tb^2\right)^{1/3}\right)$ and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) imply for all $s \in \mathbb{R}^{d_1+d_2}$:

$$T\sqrt{b} \sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\widetilde{\mathbb{Q}}_{T,\{\mathscr{a}\}}(u,s) - \widetilde{\mathbb{Q}}^{\circ}_{T,\{\mathscr{a}\}}(u,s)\right|^2\right] \leqslant o\left(\sqrt{b}\right)\left(1 + |s|_1^3\right), \qquad \text{(D.74)}$$

whereby the expression $o\big(\sqrt{b}\big)$ does not depend on $s \in \mathbb{R}^{d_1+d_2}$.
Lemma D.5 follows from (D.37) with $f(u,s) := \widetilde{\mathbb{Q}}_{T,\{\mathscr{a}\}}(u,s)$ as well as $g(u,s) := \widetilde{\mathbb{Q}}^{\circ}_{T,\{\mathscr{a}\}}(u,s)\ \forall\, u \in [0,1]$, $s \in \mathbb{R}^d$, (D.74), (D.69) and Assumption 4.3 $[\mathbf{WEI.2}]$ (recall (D.70)). ☐

Lemma D.6. *Suppose that the Assumptions 4.1* $[\mathbf{INDEP}]$, *4.3* $[\mathbf{WEI.2}]$ *and 4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *are valid. Moreover, assume that* $(\mathscr{G}_T)_{T\in\mathbb{N}}$ *is a sequence of deterministic functions which fulfils (C.215). Define for all* $\mathrm{R} \in \{\Re, \Im\}$ *(note (D.4) and (4.11)):*

$$\begin{aligned}
\mathfrak{B}^{\text{indep}}_{T,\mathscr{G}_T,\mathrm{R}} &:= \frac{1}{2}\sum_{t,j=1}^{T} \mathscr{G}_T(t-j)\, \tilde{\mathfrak{H}}_{T,\mathrm{R}}(t,j) \quad \textit{as well as} \\
\mathbf{Bias}^{\text{indep}}_{T,\mathscr{G}_T,\mathrm{R}} &:= \frac{1}{\sqrt{b}}\int_{-1}^{1} K(z)^2\, dz \cdot \int_{\mathbb{R}^d}\int_0^1 \sum_{j=-\infty}^{\infty} \mathscr{G}_T(j)\mathrm{Cov}\left(\tilde{\mathbf{G}}_{0,\mathrm{R}}(u,s), \tilde{\mathbf{G}}_{j,\mathrm{R}}(u,s)\right) du\, \mathbf{w}(s)\, ds. \qquad \text{(D.75)}
\end{aligned}$$

Then, the following statement holds for $T \to \infty$ *and all* $\mathrm{R} \in \{\Re, \Im\}$*:*

$$\mathbb{E}\left[\mathfrak{B}^{\text{indep}}_{T,\mathscr{G}_T,\mathrm{R}}\right] = \mathbf{Bias}^{\text{indep}}_{T,\mathscr{G}_T,\mathrm{R}} + o(1).$$

Remark D.7. *The expression* $\mathbf{Bias}^{\text{indep}}_{T,\mathscr{G}_T,\mathrm{R}}$ *is well-defined for all* $\mathrm{R} \in \{\Re, \Im\}$ *due to (C.215), Lemma 4.8* *and Assumption 4.3* $[\mathbf{WEI.2}]$.

Proof of Lemma D.6. In the following, Lemma D.6 with $\mathrm{R} = \Re$ is verified. Throughout this proof, let T be large enough to ensure that:

$$\mathscr{a} \leqslant \lfloor Tb \rfloor, \tag{D.76}$$

which is possible due to (A.1) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii).
Further, define:

$$\mathfrak{B}^{\text{indep}[1]}_{T,\mathscr{G}_T,\Re} := \frac{1}{2}\sum_{t,j=-1}^{T+2} \mathscr{G}_T(t-j)\,\mathbb{E}\left[\tilde{\mathfrak{H}}_{T,\Re}(t,j)\right]. \tag{D.77}$$

One obtains from (D.9), (D.7) and (C.215) (recall (D.75)):

$$\begin{aligned}\left|\mathfrak{B}^{\text{indep}[1]}_{T,\mathscr{G}_T,\Re} - \mathbb{E}\left[\mathfrak{B}^{\text{indep}}_{T,\mathscr{G}_T,\Re}\right]\right| &\leqslant \frac{1}{2}\sum_{t\in\{-1,0,T+1,T+2\}}\sum_{\substack{j=-1\\|t-j|\leqslant\mathscr{a}}}^{T+2}\left|\mathscr{G}_T(t-j)\,\mathbb{E}\left[\tilde{\mathfrak{H}}_{T,\Re}(t,j)\right]\right|\\ &\quad+\frac{1}{2}\sum_{j\in\{-1,0,T+1,T+2\}}\sum_{\substack{t=1\\|t-j|\leqslant\mathscr{a}}}^{T}\left|\mathscr{G}_T(t-j)\,\mathbb{E}\left[\tilde{\mathfrak{H}}_{T,\Re}(t,j)\right]\right|\\ &\leqslant \frac{C\mathscr{a}}{T\sqrt{b}}.\end{aligned} \tag{D.78}$$

Moreover, define (see (D.4)):

$$\begin{aligned}\mathfrak{B}^{\text{indep}[2]}_{T,\mathscr{G}_T,\Re} := \frac{1}{Tb^{\frac{3}{2}}}\int_{\mathbb{R}^d}\int_b^{1-b}\sum_{\substack{t,j=\lfloor uT\rfloor-\lfloor Tb\rfloor-1\\|t-j|\leqslant\mathscr{a}}}^{\lfloor uT\rfloor+\lfloor Tb\rfloor+2}\mathscr{G}_T(t-j)\,K\left(\frac{t-\lfloor uT\rfloor}{\lfloor Tb\rfloor}\right)^2\mathbb{E}\left[\tilde{\mathbf{G}}^c_{t,\{\mathscr{a}\},\Re}(u,s)\tilde{\mathbf{G}}^c_{j,\{\mathscr{a}\},\Re}(u,s)\right]\\ du\,\mathbf{w}(s)\,ds.\end{aligned} \tag{D.79}$$

It follows from (C.112), Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) (which provides the implication $K((t/T-u)/b) > 0 \Longrightarrow t \in [\lfloor uT\rfloor - \lfloor Tb\rfloor - 1, \lfloor uT\rfloor + \lfloor Tb\rfloor + 2]$ and that K is Lipschitz continuous), $|1/Tb - 1/\lfloor Tb\rfloor| \leqslant C/(Tb)^2$ as well as (C.215) (recall (D.77), (D.4), (4.11) and Definition A.1 (i)):

$$\begin{aligned}&\left|\mathfrak{B}^{\text{indep}[1]}_{T,\mathscr{G}_T,\Re} - \mathfrak{B}^{\text{indep}[2]}_{T,\mathscr{G}_T,\Re}\right|\\ &\leqslant \frac{C}{Tb^{\frac{3}{2}}}\int_{\mathbb{R}^d}\int_b^{1-b}\sum_{\substack{t,j=\lfloor uT\rfloor-\lfloor Tb\rfloor-1\\|t-j|\leqslant\mathscr{a}}}^{\lfloor uT\rfloor+\lfloor Tb\rfloor+2}|\mathscr{G}_T(t-j)|\left(\left|K\left(\frac{t-uT}{Tb}\right)K\left(\frac{j-uT}{Tb}\right)-K\left(\frac{t-\lfloor uT\rfloor}{\lfloor Tb\rfloor}\right)K\left(\frac{j-\lfloor uT\rfloor}{\lfloor Tb\rfloor}\right)\right|\right.\\ &\left.+\left|K\left(\frac{t-\lfloor uT\rfloor}{\lfloor Tb\rfloor}\right)K\left(\frac{j-\lfloor uT\rfloor}{\lfloor Tb\rfloor}\right)-K\left(\frac{t-\lfloor uT\rfloor}{\lfloor Tb\rfloor}\right)K\left(\frac{t-\lfloor uT\rfloor}{\lfloor Tb\rfloor}\right)\right|\mathbb{1}_{\{|t-j|\leqslant\mathscr{a}\}}\right)du\,\mathbf{w}(s)\,ds\\ &\leqslant \frac{C}{Tb^{\frac{3}{2}}}Tb\mathscr{a}\left(\frac{1}{Tb}+\frac{\mathscr{a}}{Tb}\right).\end{aligned} \tag{D.80}$$

Further, define (see (4.11)):

$$\mathfrak{B}^{\text{indep}[3]}_{T,\mathscr{G}_T,\Re} := \frac{1}{\sqrt{b}}\frac{1}{\lfloor Tb\rfloor}\int_{\mathbb{R}^d}\int_b^{1-b}\sum_{t=-\lfloor Tb\rfloor}^{\lfloor Tb\rfloor}\sum_{\substack{j=-\lfloor Tb\rfloor-1-t\\|j|\leqslant\mathscr{a}}}^{\lfloor Tb\rfloor+2-t}\mathscr{G}_T(j)\,K\left(\frac{t}{\lfloor Tb\rfloor}\right)^2\mathbb{E}\left[\tilde{\mathbf{G}}^c_{0,\Re}(u,s)\tilde{\mathbf{G}}^c_{j,\Re}(u,s)\right]du\,\mathbf{w}(s)\,ds. \tag{D.81}$$

Shifting the indices of sums, Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) and (C.215) yield (recall (D.79)):

$$\mathfrak{B}^{\text{indep}[2]}_{T,\mathscr{G}_T,\mathfrak{R}} = \frac{1}{Tb^{\frac{3}{2}}} \int_{\mathbb{R}^d} \int_b^{1-b} \sum_{\substack{t,j=-\lfloor Tb \rfloor -1 \\ |t-j| \leqslant \mathscr{a}}}^{\lfloor Tb \rfloor +2} \mathscr{G}_T\,(t-j)\,K\left(\frac{t}{\lfloor Tb \rfloor}\right)^2 \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t+\lfloor uT \rfloor,\{\mathscr{a}\},\mathfrak{R}}(u,s)\tilde{\mathbf{G}}^c_{j+\lfloor uT \rfloor,\{\mathscr{a}\},\mathfrak{R}}(u,s)\right]$$
$$du\,\mathbf{w}(s)\,ds$$
$$= \frac{1}{Tb^{\frac{3}{2}}} \int_{\mathbb{R}^d} \int_b^{1-b} \sum_{t=-\lfloor Tb \rfloor}^{\lfloor Tb \rfloor} \sum_{\substack{j=-\lfloor Tb \rfloor -1-t \\ |j| \leqslant \mathscr{a}}}^{\lfloor Tb \rfloor +2-t} \mathscr{G}_T\,(j)\,K\left(\frac{t}{\lfloor Tb \rfloor}\right)^2 \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t+\lfloor uT \rfloor,\{\mathscr{a}\},\mathfrak{R}}(u,s)\tilde{\mathbf{G}}^c_{j+t+\lfloor uT \rfloor,\{\mathscr{a}\},\mathfrak{R}}(u,s)\right]$$
$$du\,\mathbf{w}(s)\,ds. \tag{D.82}$$

Assumption 2.2 $[\mathbf{StAp}]$ (iii) and Theorem 3.35 in [78, White (2001), p. 44] imply that $\left(\tilde{\mathbf{G}}_{t,\mathfrak{R}}(u,s)\right)_{t\in\mathbb{Z}}$ (note (4.11)) is stationary for all $u \in [0,1]$, $s \in \mathbb{R}^d$, such that:

$$\mathbb{E}\left[\tilde{\mathbf{G}}^c_{0,\mathfrak{R}}(u,s)\tilde{\mathbf{G}}^c_{j,\mathfrak{R}}(u,s)\right] = \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t+\lfloor uT \rfloor,\mathfrak{R}}(u,s)\tilde{\mathbf{G}}^c_{j+t+\lfloor uT \rfloor,\mathfrak{R}}(u,s)\right] \quad \forall\, u \in [0,1],\ s \in \mathbb{R}^d,\ t,j \in \mathbb{Z}. \tag{D.83}$$

Moreover, one obtains for all $s \in \mathbb{R}^d$ from (C.112) and Lemma D.1 (v) with $q = 1+\delta$ together with Remark 4.2 (i) (see (D.4) as well as (4.11)):

$$\sup_{u\in[0,1]} \sup_{r_1,r_2\in\mathbb{Z}} \left|\mathbb{E}\left[\tilde{\mathbf{G}}^c_{r_1,\{\mathscr{a}\},\mathfrak{R}}(u,s)\tilde{\mathbf{G}}^c_{r_2,\{\mathscr{a}\},\mathfrak{R}}(u,s)\right] - \mathbb{E}\left[\tilde{\mathbf{G}}^c_{r_1,\mathfrak{R}}(u,s)\tilde{\mathbf{G}}^c_{r_2,\mathfrak{R}}(u,s)\right]\right| \leqslant \frac{C}{T}|s|_1. \tag{D.84}$$

It follows from (D.82), (C.112), $|1/Tb - 1/\lfloor Tb \rfloor| \leqslant C/(Tb)^2$, (C.215), (D.83), (D.84) as well as Assumption 4.3 $[\mathbf{WEI.2}]$ (recall (D.81)):

$$\left|\mathfrak{B}^{\text{indep}[2]}_{T,\mathscr{G}_T,\mathfrak{R}} - \mathfrak{B}^{\text{indep}[3]}_{T,\mathscr{G}_T,\mathfrak{R}}\right| \leqslant \frac{C}{\sqrt{b}}\frac{1}{(Tb)^2}Tb\mathscr{a} + \frac{C}{Tb^{\frac{3}{2}}}Tb\mathscr{a}\frac{1}{T}. \tag{D.85}$$

Further, define:

$$\mathfrak{B}^{\text{indep}[4]}_{T,\mathscr{G}_T,\mathfrak{R}} := \frac{1}{\sqrt{b}} \int_{-1}^{1} K(z)^2\,dz \int_{\mathbb{R}^d} \int_b^{1-b} \sum_{j=-\infty}^{\infty} \mathscr{G}_T\,(j)\,\mathbb{E}\left[\tilde{\mathbf{G}}^c_{0,\mathfrak{R}}(u,s)\tilde{\mathbf{G}}^c_{j,\mathfrak{R}}(u,s)\right] du\,\mathbf{w}(s)\,ds. \tag{D.86}$$

One obtains by using (C.215) and $\{-\mathscr{a},\ldots,\mathscr{a}\} \subseteq \{-\lfloor Tb \rfloor - 1 - t,\ldots,\lfloor Tb \rfloor + 2 - t\}\ \forall\, t \in \{-\lfloor Tb \rfloor + \mathscr{a},\ldots,\lfloor Tb \rfloor - \mathscr{a}\}$, whereby (D.76) should be noted (see (D.81)):

$$\mathfrak{B}^{\text{indep}[3]}_{T,\mathscr{G}_T,\mathfrak{R}} = \frac{1}{\sqrt{b}}\frac{1}{\lfloor Tb \rfloor} \int_{\mathbb{R}^d} \int_b^{1-b} \sum_{t=-\lfloor Tb \rfloor+\mathscr{a}}^{\lfloor Tb \rfloor-\mathscr{a}} \sum_{\substack{j=-\lfloor Tb \rfloor -1-t \\ |j| \leqslant \mathscr{a}}}^{\lfloor Tb \rfloor +2-t} \mathscr{G}_T\,(j)K\left(\frac{t}{\lfloor Tb \rfloor}\right)^2 \mathbb{E}\left[\tilde{\mathbf{G}}^c_{0,\mathfrak{R}}(u,s)\tilde{\mathbf{G}}^c_{j,\mathfrak{R}}(u,s)\right] du\,\mathbf{w}(s)\,ds$$
$$+ \mathcal{O}\left(\frac{\mathscr{a}^2}{\sqrt{b}Tb}\right)$$
$$= \frac{1}{\sqrt{b}}\frac{1}{\lfloor Tb \rfloor} \int_{\mathbb{R}^d} \int_b^{1-b} \sum_{t=-\lfloor Tb \rfloor+\mathscr{a}}^{\lfloor Tb \rfloor-\mathscr{a}} \sum_{j=-\mathscr{a}}^{\mathscr{a}} \mathscr{G}_T\,(j)K\left(\frac{t}{\lfloor Tb \rfloor}\right)^2 \mathbb{E}\left[\tilde{\mathbf{G}}^c_{0,\mathfrak{R}}(u,s)\tilde{\mathbf{G}}^c_{j,\mathfrak{R}}(u,s)\right] du\,\mathbf{w}(s)\,ds$$
$$+ \mathcal{O}\left(\frac{\mathscr{a}^2}{\sqrt{b}Tb}\right). \tag{D.87}$$

In addition, Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) (which ensures $K(z) = K(-z)\ \forall\, z \in \mathbb{R}$ as well as $\big|K\,(z_1)^2 -$

$K(z_2)^2\,\big| \leqslant |K(z_1)+K(z_2)|\,|K(z_1)-K(z_2)| \leqslant C\,|z_1-z_2|\ \forall\, z_1,z_2\in\mathbb{R}\big)$ and Lemma B.2 (i) yield:

$$\begin{aligned}
&\left|\frac{1}{\lfloor Tb\rfloor}\sum_{t=-\lfloor Tb\rfloor+\mathscr{a}}^{\lfloor Tb\rfloor-\mathscr{a}} K\left(\frac{t}{\lfloor Tb\rfloor}\right)^2-\int_{-1}^{1} K(z)^2\,dz\right| \\
&\leqslant \frac{C\mathscr{a}}{Tb}+\frac{1}{\lfloor Tb\rfloor}K\left(\frac{0}{\lfloor Tb\rfloor}\right)^2+2\left|\frac{1}{\lfloor Tb\rfloor}\sum_{t=1}^{\lfloor Tb\rfloor} K\left(\frac{t}{\lfloor Tb\rfloor}\right)^2-\int_{0}^{1} K(z)^2\,dz\right| \\
&\leqslant \frac{C\,(\mathscr{a}+1)}{Tb}. \qquad\qquad \text{(D.88)}
\end{aligned}$$

Moreover, (C.215), (D.2) and Definition A.1 (vi) imply for all $s\in\mathbb{R}^d$:

$$\begin{aligned}
&\sup_{u\in[b,1-b]}\left|\sum_{j=-\infty}^{\infty}\mathscr{G}_T(j)\mathbb{E}\Big[\tilde{\mathbf{G}}^c_{0,\Re}(u,s)\,\tilde{\mathbf{G}}^c_{j,\Re}(u,s)\Big]-\sum_{j=-\mathscr{a}}^{\mathscr{a}}\mathscr{G}_T(j)\mathbb{E}\Big[\tilde{\mathbf{G}}^c_{0,\Re}(u,s)\,\tilde{\mathbf{G}}^c_{j,\Re}(u,s)\Big]\right| \\
&\leqslant C\rho^{\mathscr{a}}|s|_1 \leqslant \frac{C}{T}|s|_1. \qquad\qquad \text{(D.89)}
\end{aligned}$$

One obtains from (D.87), (C.112), (D.88), (C.215), Lemma 4.8, (D.89) and Assumption 4.3 $[\mathbf{WEI.2}]$ (recall (D.86)):

$$\left|\mathfrak{B}^{\mathrm{indep}[4]}_{T,\mathscr{G}_T,\Re}-\mathfrak{B}^{\mathrm{indep}[3]}_{T,\mathscr{G}_T,\Re}\right| \leqslant \mathcal{O}\left(\frac{\mathscr{a}^2}{\sqrt{b}Tb}\right)+\frac{C\,(\mathscr{a}+1)}{\sqrt{b}Tb}+\frac{C}{\sqrt{b}T}. \qquad \text{(D.90)}$$

Overall, (D.78), (D.80), (D.85), (D.90), (A.1) as well as Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) show Lemma D.6 with $\mathrm{R}=\Re$ because the difference between $\mathfrak{B}^{\mathrm{indep}[4]}_{T,\mathscr{G}_T,\Re}$ and $\mathbf{Bias}^{\mathrm{indep}}_{T,\mathscr{G}_T,\Re}$ (see (D.86) as well as (D.75)) is asymptotically negligible due to (C.215), Lemma 4.8, Assumption 4.3 $[\mathbf{WEI.2}]$ and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii).
Lemma D.6 with $\mathrm{R}=\Im$ can be proved similarly. ☐

Corollary D.8. *Let the Assumptions 4.1* $[\mathbf{INDEP}]$, *4.3* $[\mathbf{WEI.2}]$ *and 4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *be fulfilled. Then, it holds for* $T\to\infty$*:*

$$\mathbb{E}\left[T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}}\right]=\mathbf{Bias}^{\mathrm{indep}}_T+o(1).$$

Proof. One obtains from $|z|^2=\Re\{z\}^2+\Im\{z\}^2$ (recall (D.70) and (D.4)):

$$T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}}=\frac{1}{2}\sum_{t,j=1}^{T}\left(\tilde{\mathfrak{H}}_{T,\Re}(t,j)+\tilde{\mathfrak{H}}_{T,\Im}(t,j)\right). \qquad \text{(D.91)}$$

Thus, Lemma D.6 with $\mathscr{G}_T(x):=1\ \forall\, x\in\mathbb{Z}$ yields Corollary D.8 (see (D.75) and (4.12)). ☐

Lemma D.9. *Suppose that the Assumptions 4.1* $[\mathbf{INDEP}]$, *4.3* $[\mathbf{WEI.2}]$ *and 4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *are valid. Then, it holds for* $T\to\infty$ *(recall (D.70) as well as (4.13)):*

$$\mathrm{Var}\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}}\right)=\sigma^{\mathrm{indep}}+o(1).$$

Proof. Throughout this proof, it is assumed that T is large enough to ensure:

$$2b\leqslant 1-2b,\quad \lceil 3Tb\rceil\leqslant T,\quad 1-\lfloor 2Tb\rfloor+\lfloor Tb\rfloor+1\leqslant -\mathscr{a}\ \text{ and }\ T-\lfloor T-2Tb\rfloor-\lfloor Tb\rfloor-2\geqslant \mathscr{a}, \qquad \text{(D.92)}$$

which is possible due to Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) and (A.1).
Motivated by (D.91), the asymptotic behaviour of $\mathrm{Cov}\big(1/2\sum_{t,j=1}^{T}\tilde{\mathfrak{H}}_{T,\Re}(t,j)\,,1/2\sum_{t,j=1}^{T}\tilde{\mathfrak{H}}_{T,\Im}(t,j)\big)$ is investigated in the following.

Therefor, define for all $t \in \mathbb{Z}$, $\mathrm{R} \in \{\Re, \Im\}$, $u \in [0,1]$, $s \in \mathbb{R}^d$ (see (D.4)):

$$\widetilde{\mathbf{G}}^{\mathbf{K}}_{t,T,\{\alpha\},\mathrm{R}}(u,s) := K\left(\frac{\frac{t}{T}-u}{b}\right)\widetilde{\mathbf{G}}_{t,\{\alpha\},\mathrm{R}}(u,s). \tag{D.93}$$

One obtains similarly to (C.360) (note (C.357)), (C.361) as well as (C.362) and by using Assumption 4.5 $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ (i), whereby the latter provides the implication $K((t/T-u)/b) > 0 \implies u \in [t/T - b, t/T+b]$ (recall (D.4) as well as Definition A.1 (i) and that $X^c := X - \mathbb{E}[X]$ for each random variable X with finite first moment):

$$\begin{aligned}
&\operatorname{Cov}\left(\frac{1}{2}\sum_{t,j=1}^{T}\widetilde{\mathfrak{H}}_{T,\Re}(t,j), \frac{1}{2}\sum_{t,j=1}^{T}\widetilde{\mathfrak{H}}_{T,\Im}(t,j)\right)\\
&= \frac{1}{T^2b^3}\sum_{t_1,\dots,t_4=1}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_b^{1-b}\int_b^{1-b}\Big(\mathbb{E}\Big[\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_1,T,\{\alpha\},\Re}(u,s_1)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_2,T,\{\alpha\},\Re}(u,s_1)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_3,T,\{\alpha\},\Im}(w,s_2)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_4,T,\{\alpha\},\Im}(w,\\
&s_2)\Big] - \mathbb{E}\Big[\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_1,T,\{\alpha\},\Re}(u,s_1)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_2,T,\{\alpha\},\Re}(u,s_1)\Big]\cdot\mathbb{E}\Big[\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_3,T,\{\alpha\},\Im}(w,s_2)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_4,T,\{\alpha\},\Im}(w,s_2)\Big]\Big)\\
&dw\,du\,\mathbf{w}(s_2)\,ds_2\,\mathbf{w}(s_1)\,ds_1\\
&= \frac{2}{T^2b^3}\sum_{\substack{t_1,\dots,t_4=1\\ \forall\, l_1\in\{t_1,t_3\},l_2\in\{t_2,t_4\}:|l_1-l_2|>\alpha}}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_b^{1-b}\int_b^{1-b}\mathbb{E}\Big[\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_1,T,\{\alpha\},\Re}(u,s_1)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_3,T,\{\alpha\},\Im}(w,s_2)\Big]\\
&\cdot\mathbb{E}\Big[\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_2,T,\{\alpha\},\Re}(u,s_1)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_4,T,\{\alpha\},\Im}(w,s_2)\Big]\,dw\,du\,\mathbf{w}(s_2)\,ds_2\,\mathbf{w}(s_1)\,ds_1\\
&+ \frac{1}{T^2b^3}\sum_{\substack{t_1,\dots,t_4=1\\ \exists l_1\in\{t_1,t_3\},l_2\in\{t_2,t_4\}:|l_1-l_2|\leqslant\alpha\\ \exists o_1\in\{t_1,t_4\},o_2\in\{t_2,t_3\}:|o_1-o_2|\leqslant\alpha\\ \exists r_1\in\{t_1,t_2\},r_2\in\{t_3,t_4\}:|r_1-r_2|\leqslant\alpha}}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\max\{b,\frac{t_1}{T}-b\}}^{\min\{1-b,\frac{t_1}{T}+b\}}\int_{\max\{b,\frac{t_3}{T}-b\}}^{\min\{1-b,\frac{t_3}{T}+b\}}\Big(\mathbb{E}\Big[\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_1,T,\{\alpha\},\Re}(u,s_1)\\
&\cdot\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_2,T,\{\alpha\},\Re}(u,s_1)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_3,T,\{\alpha\},\Im}(w,s_2)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_4,T,\{\alpha\},\Im}(w,s_2)\Big] - \mathbb{E}\Big[\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_1,T,\{\alpha\},\Re}(u,s_1)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_2,T,\{\alpha\},\Re}(u,s_1)\Big]\\
&\cdot\mathbb{E}\Big[\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_3,T,\{\alpha\},\Im}(w,s_2)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_4,T,\{\alpha\},\Im}(w,s_2)\Big]\Big)\,dw\,du\,\mathbf{w}(s_2)\,ds_2\,\mathbf{w}(s_1)\,ds_1\\
&\cdot\Big(\mathbb{1}_{\{\forall\, p_1\in\{1,\dots,4\}\,\exists\, p_2\in\{1,\dots,4\}\setminus\{p_1\}\,:\,|t_{p_1}-t_{p_2}|\leqslant\alpha\}} + \mathbb{1}_{\{\exists\, p_1\in\{1,\dots,4\}\,:\,|t_{p_1}-t_{p_2}|>\alpha\,\forall\, p_2\in\{1,\dots,4\}\setminus\{p_1\}\}}\Big)\\
&=: \widetilde{\sigma}^{\perp\!\!\!\perp}_{T,1,1} + \widetilde{\sigma}^{\perp\!\!\!\perp}_{T,1,2}.
\end{aligned}\tag{D.94}$$

The condition $\exists\, p_1 \in \{1,\dots,4\}\ :\ |t_{p_1} - t_{p_2}| > \alpha\,\forall\, p_2 \in \{1,\dots,4\}\setminus\{p_1\}$ belongs to summands of $\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,1,2}$ which are zero. All other conditions on t_1, t_2, t_3, t_4 together which are contained in $\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,1,2}$ imply $|t_{q_1} - t_{q_2}| \leqslant 3\alpha\,\forall\, q_1, q_2 \in \{1,\dots,4\}$. Thus, it holds:

$$\left|\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,1,2}\right| \leqslant \frac{C}{T^2b^3}T\alpha^3b^2. \tag{D.95}$$

Further, define (note that the following expression results from $\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,1,1}$ by omitting the restriction $\forall\, l_1 \in \{t_1,t_3\}, l_2 \in \{t_2,t_4\} : |l_1 - l_2| > \alpha$):

$$\begin{aligned}
\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,2} := \frac{2}{T^2b^3}\sum_{t_1,\dots,t_4=1}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_b^{1-b}\int_b^{1-b}\mathbb{E}\Big[\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_1,T,\{\alpha\},\Re}(u,s_1)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_3,T,\{\alpha\},\Im}(w,s_2)\Big]&\\
\cdot\mathbb{E}\Big[\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_2,T,\{\alpha\},\Re}(u,s_1)\,\widetilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_4,T,\{\alpha\},\Im}(w,s_2)\Big]\,dw\,du\,\mathbf{w}(s_2)\,ds_2\,\mathbf{w}(s_1)\,ds_1.&
\end{aligned}\tag{D.96}$$

Assumption 4.5 $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ (i) provides (see (D.94) and observe that all conditions on t_1, t_2, t_3, t_4 together which are contained on the right side of the first inequality of (D.97) yield $|t_{q_1} - t_{q_2}| \leqslant 3\alpha\,\forall\, q_1, q_2 \in$

$\{1, \ldots, 4\}$):

$$
\begin{aligned}
&\left|\tilde{\sigma}_{T,2}^{\perp\!\!\!\perp} - \tilde{\sigma}_{T,1,1}^{\perp\!\!\!\perp}\right| \\
&\leqslant \Bigg| \frac{2}{T^2 b^3} \sum_{\substack{t_1,\ldots,t_4=1 \\ \exists l_1 \in \{t_1,t_3\}, l_2 \in \{t_2,t_4\}: |l_1 - l_2| \leqslant \mathscr{a}}}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\max\left\{b, \frac{t_1}{T} - b\right\}}^{\min\left\{1-b, \frac{t_1}{T} + b\right\}} \int_{\max\left\{b, \frac{t_3}{T} - b\right\}}^{\min\left\{1-b, \frac{t_3}{T} + b\right\}} \mathbb{E}\left[\tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_1,T,\{\mathscr{a}\},\Re}(u, s_1)\, \tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_3,T,\{\mathscr{a}\},\Im}(w, s_2)\right] \\
&\cdot \mathbb{E}\left[\tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_2,T,\{\mathscr{a}\},\Re}(u, s_1)\, \tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_4,T,\{\mathscr{a}\},\Im}(w, s_2)\right] \mathbb{1}_{\{|t_1 - t_3| \leqslant \mathscr{a} \wedge |t_2 - t_4| \leqslant \mathscr{a}\}}\, dw\, du\, \mathbf{w}(s_2)\ ds_2\, \mathbf{w}(s_1)\ ds_1 \Bigg| \\
&\leqslant \frac{C}{T^2 b^3} T \mathscr{a}^3 b^2.
\end{aligned} \tag{D.97}
$$

Moreover, define (recall (D.92) and note that the following expression differs from (D.96) just in the integration intervals belonging to the integrals with respect to u and w):

$$
\begin{aligned}
\tilde{\sigma}_{T,3}^{\perp\!\!\!\perp} := \frac{2}{T^2 b^3} \sum_{t_1,\ldots,t_4=1}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{2b}^{1-2b} \int_{0}^{1} \mathbb{E}\left[\tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_1,T,\{\mathscr{a}\},\Re}(u, s_1)\, \tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_3,T,\{\mathscr{a}\},\Im}(w, s_2)\right] \\
\cdot \mathbb{E}\left[\tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_2,T,\{\mathscr{a}\},\Re}(u, s_1)\, \tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_4,T,\{\mathscr{a}\},\Im}(w, s_2)\right] dw\, du\, \mathbf{w}(s_2)\ ds_2\, \mathbf{w}(s_1)\ ds_1.
\end{aligned} \tag{D.98}
$$

If $u \in [b, 2b]$ and $|t/T - u| \leqslant b$ are fulfilled for a $t \in \{1, \ldots, T\}$, it will hold $1 \leqslant t \leqslant 3Tb$ for this t. Thus, Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) yields (see (D.93)):

$$
\begin{aligned}
&\Bigg| \frac{2}{T^2 b^3} \sum_{t_1,\ldots,t_4=1}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{b}^{2b} \int_{b}^{1-b} \mathbb{E}\left[\tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_1,T,\{\mathscr{a}\},\Re}(u, s_1)\, \tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_3,T,\{\mathscr{a}\},\Im}(w, s_2)\right] \\
&\cdot \mathbb{E}\left[\tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_2,T,\{\mathscr{a}\},\Re}(u, s_1)\, \tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_4,T,\{\mathscr{a}\},\Im}(w, s_2)\right] dw\, du\, \mathbf{w}(s_2)\ ds_2\, \mathbf{w}(s_1)\ ds_1 \Bigg| \\
&\leqslant \frac{2}{T^2 b^3} \sum_{t_1,t_2=1}^{\lceil 3Tb \rceil} \sum_{\substack{t_3=1 \\ |t_3 - t_1| \leqslant \mathscr{a}}}^{T} \sum_{\substack{t_4=1 \\ |t_4 - t_2| \leqslant \mathscr{a}}}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{b}^{2b} \int_{\max\left\{b, \frac{t_3}{T} - b\right\}}^{\min\left\{1-b, \frac{t_3}{T} + b\right\}} \left|\mathbb{E}\left[\tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_1,T,\{\mathscr{a}\},\Re}(u, s_1)\, \tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_3,T,\{\mathscr{a}\},\Im}(w, s_2)\right]\right| \\
&\cdot \left|\mathbb{E}\left[\tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_2,T,\{\mathscr{a}\},\Re}(u, s_1)\, \tilde{\mathbf{G}}^{\mathbf{K}\,c}_{t_4,T,\{\mathscr{a}\},\Im}(w, s_2)\right]\right| dw\, du\, \mathbf{w}(s_2)\ ds_2\, \mathbf{w}(s_1)\ ds_1 \\
&\leqslant \frac{C}{T^2 b^3} (Tb)^2 \mathscr{a}^2 b^2.
\end{aligned} \tag{D.99}
$$

One obtains from (D.99) and similar arguments (note (D.96) as well as (D.98)):

$$
\left|\tilde{\sigma}_{T,2}^{\perp\!\!\!\perp} - \tilde{\sigma}_{T,3}^{\perp\!\!\!\perp}\right| \leqslant \frac{C}{T^2 b^3} (Tb)^2 \mathscr{a}^2 b^2. \tag{D.100}
$$

Further, define (recall (4.11)):

$$
\begin{aligned}
\tilde{\sigma}_{T,4}^{\perp\!\!\!\perp} := \frac{2}{T^2 b^3} \sum_{t_1,\ldots,t_4=1}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{2b}^{1-2b} \int_{0}^{1} K\left(\frac{\frac{t_1}{T} - u}{b}\right) K\left(\frac{\frac{t_2}{T} - u}{b}\right) K\left(\frac{\frac{t_1}{T} - w}{b}\right) K\left(\frac{\frac{t_2}{T} - w}{b}\right) \\
\cdot \mathbb{E}\left[\tilde{\mathbf{G}}^{c}_{t_1,\Re}(u, s_1)\, \tilde{\mathbf{G}}^{c}_{t_3,\Im}(w, s_2)\right] \mathbb{E}\left[\tilde{\mathbf{G}}^{c}_{t_2,\Re}(u, s_1)\, \tilde{\mathbf{G}}^{c}_{t_4,\Im}(w, s_2)\right] \mathbb{1}_{\{|t_1 - t_3| \leqslant \mathscr{a}\}} \mathbb{1}_{\{|t_2 - t_4| \leqslant \mathscr{a}\}} \\
dw\, du\, \mathbf{w}(s_2)\ ds_2\, \mathbf{w}(s_1)\ ds_1.
\end{aligned} \tag{D.101}
$$

It follows for all $s_1, s_2 \in \mathbb{R}^d$ similarly to (D.84) (see (D.4) as well as (4.11)):

$$
\begin{aligned}
&\sup_{u,w \in [0,1]} \sup_{r_1, r_2 \in \mathbb{Z}} \left|\mathbb{E}\left[\tilde{\mathbf{G}}^{c}_{r_1,\{\mathscr{a}\},\Re}(u, s_1)\, \tilde{\mathbf{G}}^{c}_{r_2,\{\mathscr{a}\},\Im}(w, s_2)\right] - \mathbb{E}\left[\tilde{\mathbf{G}}^{c}_{r_1,\Re}(u, s_1)\, \tilde{\mathbf{G}}^{c}_{r_2,\Im}(w, s_2)\right]\right| \\
&\leqslant \frac{C}{T} \left(|s_1|_1 + |s_2|_1\right).
\end{aligned} \tag{D.102}
$$

If $|t_3 - t_1| \leqslant \mathscr{a}$, the condition $w \notin \left[(t_1 - \mathscr{a})/T - b, (t_1 + \mathscr{a})/T + b\right]$ will provide $K\left(\left(t_1/T - w\right)/b\right) = 0$ and $K\left(\left(t_3/T - w\right)/b\right) = 0$ due to Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i). These considerations, some other obvious conclusions from Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i), (C.112), (D.102) and Assumption 4.3 $[\mathbf{WEI.2}]$ imply (recall (D.98), (D.93) as well as (D.101)):

$$
\begin{aligned}
&\left|\tilde{\sigma}^{\perp\!\!\!\perp}_{T,3} - \tilde{\sigma}^{\perp\!\!\!\perp}_{T,4}\right| \\
&\leqslant \frac{2}{T^2 b^3} \sum_{t_1,t_2=1}^{T} \sum_{\substack{t_3=1\\|t_3-t_1|\leqslant \mathscr{a}}}^{T} \sum_{\substack{t_4=1\\|t_4-t_2|\leqslant \mathscr{a}}}^{T} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \int_{\max\left\{2b,\frac{t_1}{T}-b\right\}}^{\min\left\{1-2b,\frac{t_1}{T}+b\right\}} \int_{\max\left\{0,\frac{t_3}{T}-b\right\}}^{\min\left\{1,\frac{t_3}{T}+b\right\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_2}{T}-u}{b}\right) \\
&\cdot K\left(\frac{\frac{t_3}{T}-w}{b}\right) K\left(\frac{\frac{t_4}{T}-w}{b}\right) \Bigg| \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t_1,\{\mathscr{a}\},\Re}(u,s_1)\, \tilde{\mathbf{G}}^c_{t_3,\{\mathscr{a}\},\Im}(w,s_2)\right] \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t_2,\{\mathscr{a}\},\Re}(u,s_1)\right. \\
&\left.\cdot \tilde{\mathbf{G}}^c_{t_4,\{\mathscr{a}\},\Im}(w,s_2)\right] - \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t_1,\Re}(u,s_1)\, \tilde{\mathbf{G}}^c_{t_3,\Im}(w,s_2)\right] \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t_2,\Re}(u,s_1)\, \tilde{\mathbf{G}}^c_{t_4,\Im}(w,s_2)\right] \Bigg| \\
&dw\, du\, \mathbf{w}(s_2)\ ds_2\, \mathbf{w}(s_1)\ ds_1 \\
&+ \frac{2}{T^2 b^3} \sum_{t_1,t_2=1}^{T} \sum_{\substack{t_3=1\\|t_3-t_1|\leqslant \mathscr{a}}}^{T} \sum_{\substack{t_4=1\\|t_4-t_2|\leqslant \mathscr{a}}}^{T} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \int_{\max\left\{2b,\frac{t_1}{T}-b\right\}}^{\min\left\{1-2b,\frac{t_1}{T}+b\right\}} \int_{\max\left\{0,\frac{t_1-\mathscr{a}}{T}-b\right\}}^{\min\left\{1,\frac{t_1+\mathscr{a}}{T}+b\right\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_2}{T}-u}{b}\right) \\
&\cdot \left| K\left(\frac{\frac{t_3}{T}-w}{b}\right) K\left(\frac{\frac{t_4}{T}-w}{b}\right) - K\left(\frac{\frac{t_1}{T}-w}{b}\right) K\left(\frac{\frac{t_2}{T}-w}{b}\right) \right| \cdot \mathbb{1}_{\{|t_3-t_1|\leqslant \mathscr{a}\}} \mathbb{1}_{\{|t_4-t_2|\leqslant \mathscr{a}\}} \\
&\cdot \left| \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t_1,\Re}(u,s_1)\, \tilde{\mathbf{G}}^c_{t_3,\Im}(w,s_2)\right] \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t_2,\Re}(u,s_1)\, \tilde{\mathbf{G}}^c_{t_4,\Im}(w,s_2)\right] \right| dw\, du\, \mathbf{w}(s_2)\ ds_2\, \mathbf{w}(s_1)\ ds_1 \\
&\leqslant \frac{C}{T^2 b^3} T^2 \mathscr{a}^2 b^2 \frac{1}{T} + \frac{C}{T^2 b^3} T^2 \mathscr{a}^2 b \left(\frac{\mathscr{a}}{T} + b\right) \frac{\mathscr{a}}{Tb}. \qquad \text{(D.103)}
\end{aligned}
$$

Further, define (note that the only difference between the following expression and (D.101) is that $\tilde{\mathbf{G}}^c_{t,\Im}(w,s_2)$ contained in (D.101) is replaced by $\tilde{\mathbf{G}}^c_{t,\Im}(u,s_2)$ for all $t \in \{t_3, t_4\}$):

$$
\begin{aligned}
\tilde{\sigma}^{\perp\!\!\!\perp}_{T,5} &:= \frac{2}{T^2 b^3} \sum_{t_1,\ldots,t_4=1}^{T} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \int_{2b}^{1-2b} \int_{0}^{1} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_2}{T}-u}{b}\right) K\left(\frac{\frac{t_1}{T}-w}{b}\right) K\left(\frac{\frac{t_2}{T}-w}{b}\right) \\
&\cdot \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t_1,\Re}(u,s_1)\, \tilde{\mathbf{G}}^c_{t_3,\Im}(u,s_2)\right] \mathbb{E}\left[\tilde{\mathbf{G}}^c_{t_2,\Re}(u,s_1)\, \tilde{\mathbf{G}}^c_{t_4,\Im}(u,s_2)\right] \mathbb{1}_{\{|t_1-t_3|\leqslant \mathscr{a}\}} \mathbb{1}_{\{|t_2-t_4|\leqslant \mathscr{a}\}} \\
&dw\, du\, \mathbf{w}(s_2)\ ds_2\, \mathbf{w}(s_1)\ ds_1. \qquad \text{(D.104)}
\end{aligned}
$$

It follows for all $u_1, u_2 \in [0,1]$, $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (recall that $d_1 + d_2 = d$ according to Assumption 4.1 $[\mathbf{INDEP}]$) by using (C.112) and Remark 2.3 together with Remark 4.2 (i) (see (4.11) as well as (4.6)):

$$
\begin{aligned}
&\sup_{t\in\mathbb{Z}} \left\| \tilde{\mathbf{G}}_t(u_1,s) - \tilde{\mathbf{G}}_t(u_2,s) \right\|_{1+\delta} \\
&\leqslant C \sup_{t\in\mathbb{Z}} \Bigg(|s|_1 \left\| \tilde{X}_t(u_1) - \tilde{X}_t(u_2) \right\|_{1+\delta} + \left|s^{[1]}\right|_1 \left\| -\tilde{X}^{[1]}_t(u_1) + \tilde{X}^{[1]}_t(u_2) \right\|_1 + \left|s^{[2]}\right|_1 \Big\| -\tilde{X}^{[2]}_t(u_1) \\
&+ \tilde{X}^{[2]}_t(u_2) \Big\|_{1+\delta} + \left|s^{[2]}\right|_1 \left\| -\tilde{X}^{[2]}_t(u_1) + \tilde{X}^{[2]}_t(u_2) \right\|_1 + \left|s^{[1]}\right|_1 \left\| -\tilde{X}^{[1]}_t(u_1) + \tilde{X}^{[1]}_t(u_2) \right\|_{1+\delta} \Bigg) \\
&\leqslant C |s|_1 \left|u_1 - u_2\right|. \qquad \text{(D.105)}
\end{aligned}
$$

Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) provides that $|t_1 - t_2| \leqslant 2Tb$ is necessary to ensure $K\left(\left(t_1/T - u\right)/b\right) \cdot K\left(\left(t_2/T - u\right)/b\right) > 0$ and that $|u - w| \leqslant 2b$ is necessary to ensure $K\left(\left(t_1/T - u\right)/b\right) \cdot K\left(\left(t_1/T - w\right)/b\right) > 0$. These considerations, some other obvious conclusions resulting from Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i), (C.112), (D.105) and Assumption 4.3 $[\mathbf{WEI.2}]$ yield (note (D.101), (D.104) as well as (4.11)):

$$
\begin{aligned}
&\left|\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,4}-\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,5}\right| \\
&\leqslant \frac{2}{T^2b^3}\sum_{t_1=1}^{T}\sum_{\substack{t_2=1\\|t_2-t_1|\leqslant 2Tb}}^{T}\sum_{\substack{t_3=1\\|t_3-t_1|\leqslant \mathscr{a}}}^{T}\sum_{\substack{t_4=1\\|t_4-t_2|\leqslant \mathscr{a}}}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\max\left\{2b,\frac{t_1}{T}-b\right\}}^{\min\left\{1-2b,\frac{t_1}{T}+b\right\}}\int_{\max\left\{0,\frac{t_1}{T}-b\right\}}^{\min\left\{1,\frac{t_1}{T}+b\right\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right)K\left(\frac{\frac{t_2}{T}-u}{b}\right) \\
&\cdot K\left(\frac{\frac{t_1}{T}-w}{b}\right)K\left(\frac{\frac{t_2}{T}-w}{b}\right)\Bigg|\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{t_1,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_3,\Im}(w,s_2)\right]\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{t_2,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_4,\Im}(w,s_2)\right] \\
&-\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{t_1,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_3,\Im}(u,s_2)\right]\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{t_2,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_4,\Im}(u,s_2)\right]\Bigg|\mathbb{1}_{\{|u-w|\leqslant 2b\}}\,dw\,du\,\mathbf{w}(s_2)\;ds_2 \\
&\cdot\mathbf{w}(s_1)\;ds_1 \\
&\leqslant \frac{C}{T^2b^3}T\,Tb\,\mathscr{a}^2\,b^2b. \qquad \text{(D.106)}
\end{aligned}
$$

Assumption 4.5 [**K&b.2**] (i) in combination with $[uT-Tb,uT+Tb]\cap\mathbb{N}\subseteq\{\lfloor uT\rfloor-\lfloor Tb\rfloor-1,\ldots,\lfloor uT\rfloor+\lfloor Tb\rfloor+2\}\subseteq\{1,\ldots,T\}\;\forall\,u\in[2b,1-2b]$ (whereby the latter holds because (D.92) and $\mathscr{a}\in\mathbb{N}$ (see Definition A.1 (vi)) ensure $-\lfloor 2Tb\rfloor+\lfloor Tb\rfloor+1\leqslant -1$ ($\Leftrightarrow \lfloor 2Tb\rfloor-\lfloor Tb\rfloor-1\geqslant 1$) as well as $T\geqslant\lfloor T-2Tb\rfloor+\lfloor Tb\rfloor+2$), shifting the indices of sums and the fact that $\left(\left(\widetilde{\mathbf{G}}_{t,\Re}(u,s_1),\widetilde{\mathbf{G}}_{t,\Im}(u,s_2)\right)'\right)_{t\in\mathbb{Z}}$ (recall (4.11)) is stationary for all $u\in[0,1]$, $s_1,s_2\in\mathbb{R}^d$ (which follows from Theorem 3.35 in [78, White (2001), p. 44]) imply (note (D.104)):

$$
\begin{aligned}
\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,5} &= \frac{2}{T^2b^3}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{2b}^{1-2b}\int_{0}^{1}\sum_{t_1,t_2=\lfloor uT\rfloor-\lfloor Tb\rfloor-1}^{\lfloor uT\rfloor+\lfloor Tb\rfloor+2}\sum_{\substack{t_3=1\\|t_3-t_1|\leqslant\mathscr{a}}}^{T}\sum_{\substack{t_4=1\\|t_4-t_2|\leqslant\mathscr{a}}}^{T}K\left(\frac{\frac{t_1}{T}-u}{b}\right)K\left(\frac{\frac{t_2}{T}-u}{b}\right)K\left(\frac{\frac{t_1}{T}-w}{b}\right) \\
&\cdot K\left(\frac{\frac{t_2}{T}-w}{b}\right)\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{t_1,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_3,\Im}(u,s_2)\right]\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{t_2,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_4,\Im}(u,s_2)\right] \\
&dw\,du\,\mathbf{w}(s_2)\;ds_2\,\mathbf{w}(s_1)\;ds_1 \\
&= \frac{2}{T^2b^3}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{2b}^{1-2b}\int_{0}^{1}\sum_{t_1,t_2=\lfloor uT\rfloor-\lfloor Tb\rfloor-1}^{\lfloor uT\rfloor+\lfloor Tb\rfloor+2}\sum_{\substack{t_3=1-t_1\\|t_3|\leqslant\mathscr{a}}}^{T-t_1}\sum_{\substack{t_4=1-t_2\\|t_4|\leqslant\mathscr{a}}}^{T-t_2}K\left(\frac{\frac{t_1}{T}-u}{b}\right)K\left(\frac{\frac{t_2}{T}-u}{b}\right)K\left(\frac{\frac{t_1}{T}-w}{b}\right) \\
&\cdot K\left(\frac{\frac{t_2}{T}-w}{b}\right)\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_3,\Im}(u,s_2)\right]\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_4,\Im}(u,s_2)\right] \\
&dw\,du\,\mathbf{w}(s_2)\;ds_2\,\mathbf{w}(s_1)\;ds_1. \qquad \text{(D.107)}
\end{aligned}
$$

Moreover, (D.92) shows for all $t\in\{\lfloor uT\rfloor-\lfloor Tb\rfloor-1,\ldots,\lfloor uT\rfloor+\lfloor Tb\rfloor+2\}$ with $u\in[2b,1-2b]$ that $\{-\mathscr{a},\ldots,\mathscr{a}\}\subseteq\{1-t,\ldots,T-t\}$. Thus, (D.107) and shifting the indices of sums provides:

$$
\begin{aligned}
\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,5} &= \frac{2}{T^2b^3}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{2b}^{1-2b}\int_{0}^{1}\sum_{t_1,t_2=\lfloor uT\rfloor-\lfloor Tb\rfloor-1}^{\lfloor uT\rfloor+\lfloor Tb\rfloor+2}\sum_{t_3=-\mathscr{a}}^{\mathscr{a}}\sum_{t_4=-\mathscr{a}}^{\mathscr{a}}K\left(\frac{\frac{t_1}{T}-u}{b}\right)K\left(\frac{\frac{t_2}{T}-u}{b}\right)K\left(\frac{\frac{t_1}{T}-w}{b}\right) \\
&\cdot K\left(\frac{\frac{t_2}{T}-w}{b}\right)\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_3,\Im}(u,s_2)\right]\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_4,\Im}(u,s_2)\right] \\
&dw\,du\,\mathbf{w}(s_2)\;ds_2\,\mathbf{w}(s_1)\;ds_1 \\
&= \frac{2}{b(Tb)^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{2b}^{1-2b}\int_{0}^{1}\sum_{t_1,t_2=-\lfloor Tb\rfloor-1}^{\lfloor Tb\rfloor+2}K\left(\frac{\frac{t_1+\lfloor uT\rfloor}{T}-u}{b}\right)K\left(\frac{\frac{t_2+\lfloor uT\rfloor}{T}-u}{b}\right)K\left(\frac{\frac{t_1+\lfloor uT\rfloor}{T}-w}{b}\right) \\
&\cdot K\left(\frac{\frac{t_2+\lfloor uT\rfloor}{T}-w}{b}\right)\sum_{t_3=-\mathscr{a}}^{\mathscr{a}}\sum_{t_4=-\mathscr{a}}^{\mathscr{a}}\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_3,\Im}(u,s_2)\right]\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_4,\Im}(u,s_2)\right] \\
&dw\,du\,\mathbf{w}(s_2)\;ds_2\,\mathbf{w}(s_1)\;ds_1. \qquad \text{(D.108)}
\end{aligned}
$$

Further, define:

$$\widetilde{\sigma}_{T,6}^{\perp\!\!\perp} := \frac{2}{b\lfloor Tb\rfloor^2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\int\limits_{2b}^{1-2b}\int\limits_{0}^{1}\sum_{t_1,t_2=-\lfloor Tb\rfloor-1}^{\lfloor Tb\rfloor+2} K\left(\frac{t_1}{\lfloor Tb\rfloor}\right)K\left(\frac{t_2}{\lfloor Tb\rfloor}\right)K\left(\frac{t_1}{\lfloor Tb\rfloor}+\frac{u-w}{b}\right)K\left(\frac{t_2}{\lfloor Tb\rfloor}+\frac{u-w}{b}\right)$$
$$\cdot\sum_{t_3=-\mathscr{a}}^{\mathscr{a}}\sum_{t_4=-\mathscr{a}}^{\mathscr{a}}\mathbb{E}\Big[\widetilde{\mathbf{G}}_{0,\Re}^{c}(u,s_1)\,\widetilde{\mathbf{G}}_{t_3,\Im}^{c}(u,s_2)\Big]\,\mathbb{E}\Big[\widetilde{\mathbf{G}}_{0,\Re}^{c}(u,s_1)\,\widetilde{\mathbf{G}}_{t_4,\Im}^{c}(u,s_2)\Big]\,dw\,du\,\mathbf{w}(s_2)\,ds_2\,\mathbf{w}(s_1)\,ds_1. \tag{D.109}$$

One obtains from (D.108), (C.112), $\left|1/(Tb)^2-1/(\lfloor Tb\rfloor)^2\right|\leqslant C/(Tb)^3$, arguments which are similar to those that yield (C.242) and Lemma 4.8 together with Assumption 4.3 $[\mathbf{WEI.2}]$:

$$\left|\widetilde{\sigma}_{T,5}^{\perp\!\!\perp}-\widetilde{\sigma}_{T,6}^{\perp\!\!\perp}\right|\leqslant\frac{C}{bTb}. \tag{D.110}$$

Moreover, define:

$$g_{T,u,w}(q) := K(q)K\left(q+\frac{u-w}{b}\right)\quad\forall\,u,w,q\in\mathbb{R}\quad\text{as well as}$$
$$\widetilde{\sigma}_{T,7}^{\perp\!\!\perp} := \frac{2}{b}\int\limits_{2b}^{1-2b}\int\limits_{0}^{1}\left(\left(\int\limits_{-1}^{1}g_{T,u,w}(q)\,dq\right)^2dw\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\left(\sum_{t=-\mathscr{a}}^{\mathscr{a}}\mathbb{E}\Big[\widetilde{\mathbf{G}}_{0,\Re}^{c}(u,s_1)\,\widetilde{\mathbf{G}}_{t,\Im}^{c}(u,s_2)\Big]\right)^2\mathbf{w}(s_2)\,ds_2\,\mathbf{w}(s_1)\,ds_1\,du \tag{D.111}$$

and observe that Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) yields (see (D.109)):

$$\widetilde{\sigma}_{T,6}^{\perp\!\!\perp} = \frac{2}{b}\int\limits_{2b}^{1-2b}\int\limits_{0}^{1}\left(\frac{1}{\lfloor Tb\rfloor}\sum_{t=-\lfloor Tb\rfloor}^{\lfloor Tb\rfloor}g_{T,u,w}\left(\frac{t}{\lfloor Tb\rfloor}\right)\right)^2dw\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\left(\sum_{t=-\mathscr{a}}^{\mathscr{a}}\mathbb{E}\Big[\widetilde{\mathbf{G}}_{0,\Re}^{c}(u,s_1)\,\widetilde{\mathbf{G}}_{t,\Im}^{c}(u,s_2)\Big]\right)^2$$
$$\cdot\,\mathbf{w}(s_2)\,ds_2\,\mathbf{w}(s_1)\,ds_1\,du. \tag{D.112}$$

It follows from (C.370), the substitution $z := -q$ for $q\in[-1,0]$ and Lemma B.2 (i) together with (C.112) as well as Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) $\big($whereby the latter two show $\sup_{u,w\in\mathbb{R}}\left|g_{T,u,w}(q_1)-g_{T,u,w}(q_2)\right|\leqslant C|q_1-q_2|\ \forall\,q_1,q_2\in\mathbb{R}\big)$:

$$\sup_{u,w\in\mathbb{R}}\left|\left(\frac{1}{\lfloor Tb\rfloor}\sum_{t=-\lfloor Tb\rfloor}^{\lfloor Tb\rfloor}g_{T,u,w}\left(\frac{t}{\lfloor Tb\rfloor}\right)\right)^2-\left(\int\limits_{-1}^{1}g_{T,u,w}(q)\,dq\right)^2\right|$$
$$\leqslant C\left(\sup_{u,w\in\mathbb{R}}\left|\frac{1}{\lfloor Tb\rfloor}\sum_{t=1}^{\lfloor Tb\rfloor}g_{T,u,w}\left(\frac{-t}{\lfloor Tb\rfloor}\right)-\int\limits_{0}^{1}g_{T,u,w}(-z)\,dz\right|+\sup_{u,w\in\mathbb{R}}\left|\frac{1}{\lfloor Tb\rfloor}g_{T,u,w}\left(\frac{0}{\lfloor Tb\rfloor}\right)\right|\right.$$
$$\left.+\sup_{u,w\in\mathbb{R}}\left|\frac{1}{\lfloor Tb\rfloor}\sum_{t=1}^{\lfloor Tb\rfloor}g_{T,u,w}\left(\frac{t}{\lfloor Tb\rfloor}\right)-\int\limits_{0}^{1}g_{T,u,w}(q)\,dq\right|\right)$$
$$\leqslant\frac{C}{Tb}. \tag{D.113}$$

Thus, (D.112) and Lemma 4.8 in combination with Assumption 4.3 $[\mathbf{WEI.2}]$ imply (recall (D.111)):

$$\left|\widetilde{\sigma}_{T,6}^{\perp\!\!\perp}-\widetilde{\sigma}_{T,7}^{\perp\!\!\perp}\right|\leqslant\frac{C}{b\,Tb}. \tag{D.114}$$

Further, transforming the integral with respect to $w\in[0,1]$ contained in $\widetilde{\sigma}_{T,7}^{\perp\!\!\perp}$ by using the substitution

$v = (u-w)/b$ provides (see (D.111)):

$$\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,7} = -2 \int_{2b}^{1-2b} \int_{u/b}^{(u-1)/b} \left(\int_{-1}^{1} K(q) K(q+v)\, dq \right)^2 dv \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\sum_{t=-\mathscr{a}}^{\mathscr{a}} \mathbb{E}\Big[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\, \widetilde{\mathbf{G}}^c_{t,\Im}(u,s_2) \Big] \right)^2$$
$$\cdot\, \mathbf{w}(s_2)\, ds_2 \mathbf{w}(s_1)\, ds_1\, du$$
$$= 2 \int_{2b}^{1-2b} \int_{(u-1)/b}^{u/b} \left(\int_{-1}^{1} K(q) K(q+v)\, dq \right)^2 dv \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\sum_{t=-\mathscr{a}}^{\mathscr{a}} \mathbb{E}\Big[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\, \widetilde{\mathbf{G}}^c_{t,\Im}(u,s_2) \Big] \right)^2$$
$$\cdot\, \mathbf{w}(s_2)\, ds_2 \mathbf{w}(s_1)\, ds_1\, du. \tag{D.115}$$

In addition, (C.370), Lemma 4.8 and arguments which are similar to those that imply (D.89) yield for all $s_1, s_2 \in \mathbb{R}^d$:

$$\sup_{u \in [0,1]} \left| \left(\sum_{t=-\infty}^{\infty} \mathbb{E}\Big[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\; \widetilde{\mathbf{G}}^c_{t,\Im}(u,s_2) \Big] \right)^2 - \left(\sum_{t=-\mathscr{a}}^{\mathscr{a}} \mathbb{E}\Big[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\; \widetilde{\mathbf{G}}^c_{t,\Im}(u,s_2) \Big] \right)^2 \right|$$
$$\leqslant \frac{C}{T} \left(|s_1|_1 + |s_2|_1 \right) \left(|s_1|_1 + |s_2|_1 + 1 \right). \tag{D.116}$$

Assumption 4.5 $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ (i) provides for all $q \in [-1,1]$ that $K(q+v) = 0$ if $|v| > 2$. Moreover, it holds for all $u \in [2b, 1-2b]$ that $(u-1)/b \leqslant -2$ and $u/b \geqslant 2$ (note (D.92)). Thus, the integral with respect to $v \in [(u-1)/b, u/b]$ contained on the right side of (D.115) can be replaced by an integral with respect to $v \in [-2,2]$. Hence, (D.116) and Lemma 4.8 together with Assumption 4.3 $[\mathbf{WEI}.\mathbf{2}]$ show (recall (4.13)):

$$\widetilde{\sigma}^{\perp\!\!\!\perp}_{T,7} = 2 \int_{0}^{1} \int_{-2}^{2} \left(\int_{-1}^{1} K(q) K(q+v)\, dq \right)^2 dv \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{Cov}^{\text{indep}}_{\Re,\Im}(u,s_1,s_2)^2\, \mathbf{w}(s_2)\, ds_2 \mathbf{w}(s_1)\, ds_1\, du + o(1). \tag{D.117}$$

The equation $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X,Y)$, which is valid for all random variables X and Y with finite second moments that live on the same probability space, (D.91), (D.94), (D.95), (D.97), (D.100), (D.103), (D.106), (D.110), (D.114), (A.1), Assumption 4.5 $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ (ii), (D.117) and similar arguments prove Lemma D.9 (see (4.13)). □

Lemma D.10. *Let the Assumptions 4.1* $[\mathbf{INDEP}]$*, 4.3* $[\mathbf{WEI}.\mathbf{2}]$ *and 4.5* $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ *be fulfilled. Then, it holds for* $T \to \infty$ *(recall (D.70), (4.12) as well as (D.4)):*

$$\mathbb{E}\left[\left(T\sqrt{b}\, \widetilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}} - \mathbf{Bias}^{\text{indep}}_T - \widetilde{\mathbb{H}}_T \right)^2 \right] = o(1) \tag{D.118}$$

and (note (4.13)):

$$\left| \text{Var}\left(\widetilde{\mathbb{H}}_T \right) - \sigma^{\text{indep}} \right| = o(1). \tag{D.119}$$

Proof. Throughout this proof, assume that T is large enough to ensure (see Definition A.1 (vi) as well as (D.3)):

$$7\mathscr{a} + 2 \leqslant 1 + 8\mathscr{a} \leqslant \rho_T \leqslant \rho_T + 1 \leqslant L_T \leqslant L_T + 1 \leqslant T, \tag{D.120}$$

which is possible due to (A.1) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ (ii).
At first, (D.118) is proved. Therefor, define (recall (D.4)):

$$\widetilde{\mathfrak{T}}_T := \frac{1}{2} \sum_{t=1}^{T} \widetilde{\mathfrak{H}}_T(t,t) + \sum_{t=L_T+1}^{T} \sum_{j=1}^{t-1} \widetilde{\mathfrak{H}}_T(t,j). \tag{D.121}$$

In the following, it is shown that:

$$\mathbb{E}\left[\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}}-\tilde{\mathfrak{T}}_T\right)^2\right]=o(1). \tag{D.122}$$

One obtains from (D.91), $\tilde{\mathfrak{H}}_T(t,j):=\tilde{\mathfrak{H}}_{T,\Re}(t,j)+\tilde{\mathfrak{H}}_{T,\Im}(t,j)$ and $\tilde{\mathfrak{H}}_T(t,j)=\tilde{\mathfrak{H}}_T(j,t)\ \forall\, t,j\in\mathbb{Z}$ (whereby the latter two hold due to (D.4)):

$$T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}}=\frac{1}{2}\sum_{t=1}^{T}\tilde{\mathfrak{H}}_T(t,t)+\sum_{t=2}^{T}\sum_{j=1}^{t-1}\tilde{\mathfrak{H}}_T(t,j). \tag{D.123}$$

Furthermore, (D.9), (D.7), (D.3) and $\mathscr{a}\to\infty$ for $T\to\infty$ (the latter is valid according to Definition A.1 (vi)) provide:

$$\left|\mathbb{E}\left[\sum_{t=2}^{L_T}\sum_{j=1}^{t-1}\tilde{\mathfrak{H}}_T(t,j)\right]\right|\leqslant\sum_{t=2}^{L_T}\sum_{\substack{j=1\\|j-t|\leqslant\mathscr{a}}}^{t-1}\left|\mathbb{E}\left[\tilde{\mathfrak{H}}_T(t,j)\right]\right|\leqslant C\frac{L_T\mathscr{a}}{T\sqrt{b}}=o(1). \tag{D.124}$$

In addition, (D.10), (D.9), (D.7), (D.3) and $\mathscr{a}\to\infty$ for $T\to\infty$ imply:

$$\begin{aligned}\operatorname{Var}\left(\sum_{t=2}^{L_T}\sum_{j=1}^{t-1}\tilde{\mathfrak{H}}_T(t,j)\right)&\leqslant\sum_{l_1,\dots,l_4=1}^{L_T}\left(\left|\mathbb{E}\left[\tilde{\mathfrak{H}}_T(l_1,l_2)\,\tilde{\mathfrak{H}}_T(l_3,l_4)\right]\right|+\left|\mathbb{E}\left[\tilde{\mathfrak{H}}_T(l_1,l_2)\right]\mathbb{E}\left[\tilde{\mathfrak{H}}_T(l_3,l_4)\right]\right|\right)\\&\quad\cdot\mathbb{1}_{\{\forall\,n_1\in\{1,\dots,4\}\exists n_2\in\{1,\dots,4\}\setminus\{n_1\}:\left|l_{n_1}-l_{n_2}\right|\leqslant\mathscr{a}\}}\\&\leqslant\frac{CL_T^2\mathscr{a}^2}{T^2b}=o(1).\end{aligned} \tag{D.125}$$

It follows from (D.123), (D.124) and (D.125) that (D.122) holds (see (D.121)). Hence, (C.25) with $M=2$ yields that (D.118) will be valid if:

$$\mathbb{E}\left[\left(\tilde{\mathfrak{T}}_T-\mathbf{Bias}_T^{\mathrm{indep}}-\tilde{\mathbb{H}}_T\right)^2\right]=o(1), \tag{D.126}$$

whereby (D.126) is shown next.
In order to verify (D.126), note at first that (D.120) provides (recall (D.121)):

$$\begin{aligned}\tilde{\mathfrak{T}}_T&=\frac{1}{2}\sum_{t=1}^{T}\tilde{\mathfrak{H}}_T(t,t)+\sum_{t=L_T+1}^{T}\sum_{j=t-7\mathscr{a}}^{t-1}\tilde{\mathfrak{H}}_T(t,j)+\sum_{t=L_T+1}^{T}\sum_{j=1}^{t-7\mathscr{a}-1}\tilde{\mathfrak{H}}_T(t,j)\\&=:\tilde{\mathfrak{T}}_{T,\mathrm{Bias},1}+\tilde{\mathfrak{T}}_{T,\mathrm{Bias},2}+\tilde{\mathfrak{T}}_{T,\mathrm{Var}}.\end{aligned} \tag{D.127}$$

One obtains from (D.9), (D.122) and Corollary D.8:

$$\begin{aligned}\left|\mathbb{E}\left[\tilde{\mathfrak{T}}_{T,\mathrm{Bias},1}+\tilde{\mathfrak{T}}_{T,\mathrm{Bias},2}\right]-\mathbf{Bias}_T^{\mathrm{indep}}\right|&\leqslant\left|\mathbb{E}\left[\tilde{\mathfrak{T}}_T-T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}}\right]\right|+\left|\mathbb{E}\left[T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}}\right]-\mathbf{Bias}_T^{\mathrm{indep}}\right|\\&=o(1).\end{aligned} \tag{D.128}$$

If $t_1,t_2\in\{L_T+1,\dots,T\}$, $j_k\in\{t_k-7\mathscr{a},\dots,t_k-1\}\ \forall\,k\in\{1,2\}$ and $|t_1-t_2|\leqslant\mathscr{a}\vee|t_1-j_2|\leqslant\mathscr{a}\vee|j_1-t_2|\leqslant\mathscr{a}\vee|j_1-j_2|\leqslant\mathscr{a}$, it will hold $|l_1-l_2|\leqslant 15\mathscr{a}\,\forall\,l_1,l_2\in\{t_1,t_2,j_1,j_2\}$. Hence, (C.405), which yields $\operatorname{Var}(\tilde{\mathfrak{T}}_{T,\mathrm{Bias},1}+\tilde{\mathfrak{T}}_{T,\mathrm{Bias},2})\leqslant 3\operatorname{Var}(\tilde{\mathfrak{T}}_{T,\mathrm{Bias},1})+3\operatorname{Var}(\tilde{\mathfrak{T}}_{T,\mathrm{Bias},2})$, (D.7), (A.1) as well as Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) show (see (D.127) and note that $\tilde{\mathfrak{H}}_T(t,j)$ contains products of m-dependent random variables (with $m=\mathscr{a}$) according to (D.4) as well as Definition A.1 (i)):

$$\begin{aligned}\operatorname{Var}\left(\tilde{\mathfrak{T}}_{T,\mathrm{Bias},1}+\tilde{\mathfrak{T}}_{T,\mathrm{Bias},2}\right)&\leqslant C\sum_{t_1,t_2=1}^{T}\left|\operatorname{Cov}\left(\tilde{\mathfrak{H}}_T(t_1,t_1),\tilde{\mathfrak{H}}_T(t_2,t_2)\right)\right|\mathbb{1}_{\{|t_1-t_2|\leqslant\mathscr{a}\}}\\&\quad+C\sum_{t_1,t_2=L_T+1}^{T}\sum_{j_1=t_1-7\mathscr{a}}^{t_1-1}\sum_{j_2=t_2-7\mathscr{a}}^{t_2-1}\left|\operatorname{Cov}\left(\tilde{\mathfrak{H}}_T(t_1,j_1),\tilde{\mathfrak{H}}_T(t_2,j_2)\right)\right|\end{aligned}$$

$$\cdot \mathbb{1}_{\{|t_1-t_2|\leqslant \alpha \vee |t_1-j_2|\leqslant \alpha \vee |j_1-t_2|\leqslant \alpha \vee |j_1-j_2|\leqslant \alpha\}}$$
$$\leqslant \frac{CT\alpha}{T^2 b} + \frac{CT\alpha^3}{T^2 b} = o(1). \tag{D.129}$$

It follows from (D.120) (recall (D.127), (D.4) and (D.3)):

$$\begin{aligned}
\widetilde{\mathfrak{T}}_{T,\mathrm{Var}} - \widetilde{\mathbb{H}}_T &= \sum_{k=2}^{\mathfrak{K}_T} \sum_{t=o_{T,k-1}+1}^{o_{T,k}} \sum_{j=1}^{t-7\alpha-1} \widetilde{\mathfrak{H}}_T(t,j) + \sum_{\substack{t\in\{L_T+1,\dots,T\}:\\ t\geqslant o_{T,\mathfrak{K}_T}+1}} \sum_{j=1}^{t-7\alpha-1} \widetilde{\mathfrak{H}}_T(t,j) \\
&- \sum_{t=l_{T,1}}^{o_{T,1}} \sum_{j=1}^{t-7\alpha-1} \widetilde{\mathfrak{H}}_T(t,j) - \sum_{k=2}^{\mathfrak{K}_T} \sum_{t=l_{T,k}}^{o_{T,k}} \sum_{j=1}^{t-7\alpha-1} \widetilde{\mathfrak{H}}_T(t,j) \\
&= \sum_{k=2}^{\mathfrak{K}_T} \sum_{t=o_{T,k-1}+1}^{l_{T,k}-1} \sum_{j=1}^{o_{T,k-1}-\rho_T} \widetilde{\mathfrak{H}}_T(t,j) + \sum_{k=2}^{\mathfrak{K}_T} \sum_{t=o_{T,k-1}+1}^{l_{T,k}-1} \sum_{j=o_{T,k-1}-\rho_T+1}^{t-7\alpha-1} \widetilde{\mathfrak{H}}_T(t,j) \\
&+ \sum_{\substack{t\in\{L_T+1,\dots,T\}:\\ t\geqslant o_{T,\mathfrak{K}_T}+1}} \sum_{j=1}^{t-7\alpha-1} \widetilde{\mathfrak{H}}_T(t,j) - \sum_{t=l_{T,1}}^{o_{T,1}} \sum_{j=1}^{t-7\alpha-1} \widetilde{\mathfrak{H}}_T(t,j) \\
&=: \mathcal{R}^{\perp\!\!\perp}_{T,1} + \mathcal{R}^{\perp\!\!\perp}_{T,2} + \mathcal{R}^{\perp\!\!\perp}_{T,3} - \mathcal{R}^{\perp\!\!\perp}_{T,4}.
\end{aligned} \tag{D.130}$$

One obtains due to $\lceil 2Tb \rceil \leqslant \rho_T$ (as stated in (D.3)) and (D.8):

$$\mathcal{R}^{\perp\!\!\perp}_{T,1} = 0. \tag{D.131}$$

Further, (D.3) and (D.120) imply for all $k, k' \in \{2, \dots, \mathfrak{K}_T\}$ with $k \geqslant k' + 1$ that $o_{T,k-1} + 1 > l_{T,k'} - 1 + \alpha$. Hence, (D.10), (D.8), (D.3), (D.120) and (D.7) yield:

$$\begin{aligned}
\mathbb{E}\left[\left(\mathcal{R}^{\perp\!\!\perp}_{T,2}\right)^2\right] &\leqslant \sum_{\substack{k_1,k_2=2\\ k_1\neq k_2}}^{\mathfrak{K}_T} \sum_{t_1=o_{T,k_1-1}+1}^{l_{T,k_1}-1} \sum_{t_2=o_{T,k_2-1}+1}^{l_{T,k_2}-1} \sum_{j_1=o_{T,k_1-1}-\rho_T+1}^{t_1-7\alpha-1} \sum_{j_2=o_{T,k_2-1}-\rho_T+1}^{t_2-7\alpha-1} \left|\mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t_1,j_1)\,\widetilde{\mathfrak{H}}_T(t_2,j_2)\right]\right| \\
&+ \sum_{\substack{k_1,k_2=2\\ k_1=k_2}}^{\mathfrak{K}_T} \sum_{t_1=o_{T,k_1-1}+1}^{l_{T,k_1}-1} \sum_{\substack{t_2=o_{T,k_2-1}+1\\ |t_2-t_1|\leqslant\alpha}}^{l_{T,k_2}-1} \sum_{\substack{j_1=o_{T,k_1-1}-\rho_T+1\\ |j_1-t_1|\leqslant 2Tb}}^{t_1-7\alpha-1} \sum_{\substack{j_2=o_{T,k_2-1}-\rho_T+1\\ |j_2-j_1|\leqslant\alpha}}^{t_2-7\alpha-1} \left|\mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t_1,j_1)\,\widetilde{\mathfrak{H}}_T(t_2,j_2)\right]\right| \\
&= 0 + C\frac{T}{L_T}\,\rho_T\,\alpha Tb\,\alpha\frac{1}{T^2 b} = o(1).
\end{aligned} \tag{D.132}$$

Moreover, (D.10), (D.8), (D.3) together with $\mathfrak{K}_T \geqslant (T - L_T)/(L_T + \rho_T)$ as well as (D.7) show:

$$\mathbb{E}\left[\left(\mathcal{R}^{\perp\!\!\perp}_{T,3}\right)^2\right] \leqslant \sum_{\substack{t_1,t_2\in\{L_T+1,\dots,T\}:\\ t_1,t_2\geqslant o_{T,\mathfrak{K}_T}+1\\ |t_1-t_2|\leqslant\alpha}} \sum_{\substack{j_1=1\\ |j_1-t_1|\leqslant 2Tb}}^{t_1-7\alpha-1} \sum_{\substack{j_2=1\\ |j_2-j_1|\leqslant\alpha}}^{t_2-7\alpha-1} \left|\mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t_1,j_1)\,\widetilde{\mathfrak{H}}_T(t_2,j_2)\right]\right| \leqslant C\frac{L_T\,\alpha Tb\alpha}{T^2 b} = o(1) \tag{D.133}$$

and one obtains similarly to (D.133):

$$\mathbb{E}\left[\left(\mathcal{R}^{\perp\!\!\perp}_{T,4}\right)^2\right] = o(1). \tag{D.134}$$

In conclusion, (D.126) follows from (D.127), (C.25) with $M = 2$, (D.128), (D.129), (D.130), (C.25) with $M = 4$, (D.131), (D.132), (D.133) and (D.134):

$$\begin{aligned}
\mathbb{E}\left[\left(\widetilde{\mathfrak{T}}_T - \mathbf{Bias}_T^{\mathrm{indep}} - \widetilde{\mathbb{H}}_T\right)^2\right] &\leqslant 2\mathbb{E}\left[\left(\widetilde{\mathfrak{T}}_{T,\mathrm{Bias},1} + \widetilde{\mathfrak{T}}_{T,\mathrm{Bias},2} - \mathbf{Bias}_T^{\mathrm{indep}}\right)^2\right] + 2\mathbb{E}\left[\left(\widetilde{\mathfrak{T}}_{T,\mathrm{Var}} - \widetilde{\mathbb{H}}_T\right)^2\right] \\
&= o(1).
\end{aligned} \tag{D.135}$$

Overall, (D.118) is an implication of (C.25) with $M = 2$, (D.122) and (D.126).

Next, (D.119) is proved. Therefor, observe at first that (C.25) with $M = 2$, Lemma D.9, $\sigma^{\text{indep}} < \infty$ (see (4.13)), whereby the latter holds due to Lemma 4.8 as well as Assumption 4.3 $[\mathbf{WEI.2}]$ and Corollary D.8 yield:

$$\begin{aligned}\mathbb{E}\left[\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}} - \mathbf{Bias}_T^{\text{indep}}\right)^2\right] &\leqslant 2\mathbb{E}\left[\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}} - \mathbb{E}\left[T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}}\right]\right)^2\right] \\ &\quad + 2\left(\mathbb{E}\left[T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}}\right] - \mathbf{Bias}_T^{\text{indep}}\right)^2 \\ &\leqslant C. \end{aligned} \tag{D.136}$$

Moreover, it follows from (C.25) with $M = 2$, (D.118) and (D.136):

$$\mathbb{E}\left[\widetilde{\mathbb{H}}_T^2\right] \leqslant 2\mathbb{E}\left[\left(\widetilde{\mathbb{H}}_T - T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}} + \mathbf{Bias}_T^{\text{indep}}\right)^2\right] + 2\mathbb{E}\left[\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}} - \mathbf{Bias}_T^{\text{indep}}\right)^2\right] \leqslant C. \tag{D.137}$$

One obtains for all real-valued random variables X and Y with finite second moments which live on the same probability space that $|\text{Var}(X) - \text{Var}(Y)| = |\text{Cov}(X - Y, X + Y)| \leqslant \|X - Y\|_2 \cdot (\|X\|_2 + \|Y\|_2)$. Therefore, (D.118), (D.137) and (D.136) prove (note that $\mathbf{Bias}_T^{\text{indep}}$ (see (4.12)) is deterministic):

$$\left|\text{Var}\left(\widetilde{\mathbb{H}}_T\right) - \text{Var}\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}}\right)\right| = \left|\text{Var}\left(\widetilde{\mathbb{H}}_T\right) - \text{Var}\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,\{\mathscr{a}\}} - \mathbf{Bias}_T^{\text{indep}}\right)\right| = o(1).$$

This and Lemma D.9 show (D.119). □

Lemma D.11. *Suppose that the Assumptions 2.4* $[\mathbf{DM.3}]$, *4.5* $[\mathbf{K\&b.2}]$ *(ii) and 3.15* $[\mathbf{W}^*]$ *hold.*

(i) It follows for all $T \in \mathbb{N}$ *(recall Definition A.1 (vi) and (vii) as well as Assumption 3.15* $[\mathbf{W}^*]$ *(i)):*

$$8\mathscr{a} \leqslant \mathscr{a}_\beta \leqslant C\beta\mathscr{a}.$$

(ii) One obtains for all $T \in \mathbb{N}$ *(see Definition A.1 (i)):*

$$\sup_{t\in\mathbb{Z}} \left\|W^*_{t,\{\mathscr{a}_\beta\}} - W^*_t\right\|_2 \leqslant \frac{C}{T}.$$

Proof. (i) Assumption 3.15 $[\mathbf{W}^*]$ (i) provides that $\beta_{\text{sup}}^{\text{inv}} < \infty$ (note Definition A.1 (iii)) and that $\ln(\beta) \leqslant C\ln(T)$. This implies (recall Definition A.1 (vi), (vii) as well as that $\ln(\rho_*) < 0$ according to Assumption 3.15 $[\mathbf{W}^*]$ (iv)):

$$\begin{aligned} 8\mathscr{a} &\leqslant 8C_2\ln(C_3T)\,\beta_{\text{sup}}^{\text{inv}}\beta \leqslant \mathscr{a}_*\beta \leqslant \mathscr{a}_\beta \\ &\leqslant \beta\mathscr{a}_* - \beta\frac{\ln\left(\beta_{\text{sup}}^{\text{inv}}\right) + \ln(\beta)}{\ln(\rho_*)} + 1 \leqslant C\beta\mathscr{a}, \end{aligned}$$

which proves Lemma D.11 (i).

(ii) One obtains from $\rho_* \in (0, 1)$, Definition A.1 (vii) and $\beta_{\text{sup}}^{\text{inv}} > 1$ that $\rho_*^{\mathscr{a}_\beta/\beta} \leqslant \rho_*^{\mathscr{a}_* - \ln\left(\beta_{\text{sup}}^{\text{inv}}\beta\right)/\ln(\rho_*)} = e^{\mathscr{a}_*\ln(\rho_*) - \ln(\beta_{\text{sup}}^{\text{inv}}\beta)} \leqslant e^{-\ln(C_3T) - \ln(\beta_{\text{sup}}^{\text{inv}}\beta)} \leqslant C/(T\beta)$. Hence, Lemma D.11 (ii) follows similarly to Lemma C.8 (iii).

□

Lemma D.12. *Let the Assumptions 4.1* $[\mathbf{INDEP}]$, *4.3* $[\mathbf{WEI.2}]$, *4.5* $[\mathbf{K\&b.2}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled. Moreover, define for all* $u \in [0, 1]$, $s \in \mathbb{R}^d$ *(note (D.24), (4.11) and that* $X^c := X - \mathbb{E}[X]$ *for each* *random variable X with finite first moment):*

$$\mathbb{Q}^*_{T,1}(u,s) := \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right)\mathbf{G}^c_{t,T}(u,s)W^*_t. \tag{D.138}$$

Then, it holds for $T \to \infty$ (see (4.20)):

$$T\sqrt{b}\,\mathbb{E}\left[\left|\widehat{\mathfrak{Q}}^*_T - \int_{\mathbb{R}^d}\int_b^{1-b}\left|\mathbb{Q}^*_{T,1}(u,s)\right|^2\,du\,\mathbf{w}(s)\,ds\right|\right] = o(1).$$

Proof. At first, define for all $u \in [0,1]$, $s := \left(s^{[1]'}, s^{[2]'}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (recall the Definitions 2.11, (4.6) as well as (4.8) and that $d = d_1 + d_2$ according to Assumption 4.1 $[\mathbf{INDEP}]$):

$$\begin{aligned}\mathbb{Q}^{**}_{T,1}(u,s) := \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right)\Bigg(&e^{i\left\langle s,X_{t,T}\right\rangle} - \widehat{\varphi}\left(u,s\right) + \varphi^{[1]}\left(u,s^{[1]}\right)\cdot\left(\widehat{\varphi}^{[2]}\left(u,s^{[2]}\right)\right.\\ &\left.- e^{i\left\langle s^{[2]},X^{[2]}_{t,T}\right\rangle}\right) + \varphi^{[2]}\left(u,s^{[2]}\right)\cdot\left(\widehat{\varphi}^{[1]}\left(u,s^{[1]}\right) - e^{i\left\langle s^{[1]},X^{[1]}_{t,T}\right\rangle}\right)\Bigg)W^*_t.\end{aligned} \tag{D.139}$$

In the following, it will be shown for all $s \in \mathbb{R}^d$ (see (4.20)):

$$T\sqrt{b}\sup_{u\in[b,1-b]}\mathbb{E}\left[\left|\widehat{\mathbb{Q}}^*_T(u,s) - \mathbb{Q}^{**}_{T,1}(u,s)\right|^2\right] \leqslant o\left(\sqrt{b}\right)\left(|s|_1^{2+2\delta}+1\right) \quad \text{and} \tag{D.140}$$

$$T\sqrt{b}\sup_{u\in[b,1-b]}\mathbb{E}\left[\left|\mathbb{Q}^{**}_{T,1}(u,s) - \mathbb{Q}^{*}_{T,1}(u,s)\right|^2\right] \leqslant o\left(\sqrt{b}\right)\left(|s|_1^{2+2\delta}+1\right), \tag{D.141}$$

whereby the contained expressions $o\left(\sqrt{b}\right)$ do not depend on $s \in \mathbb{R}^d$.
One obtains for all $s := \left(s^{[1]'}, s^{[2]'}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ from (D.45), Lemma B.1 with $\kappa_1 = 1$ as well as the Propositions 2.12 (with $\mathfrak{U}_{0,1,b} = [b, 1-b]$ under Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$) and 2.14 in combination with Remark 4.2 (i) (recall (4.20) as well as (D.139)):

$$\begin{aligned}&\sup_{u\in[b,1-b]}\mathbb{E}\left[\left|\widehat{\mathbb{Q}}^*_T(u,s) - \mathbb{Q}^{**}_{T,1}(u,s)\right|^2\right]\\ &\leqslant \frac{C\beta}{Tb}\sup_{u\in[b,1-b]}\left(\mathbb{E}\left[\left|\widehat{\varphi}^{[1]}\left(u,s^{[1]}\right) - \varphi^{[1]}\left(u,s^{[1]}\right)\right|^2\right] + \mathbb{E}\left[\left|\widehat{\varphi}^{[2]}\left(u,s^{[2]}\right) - \varphi^{[2]}\left(u,s^{[2]}\right)\right|^2\right]\right)\\ &\leqslant \frac{C\beta}{Tb}\left(b^{2+2\delta} + \frac{1}{Tb}\right)\left(|s|_1^{2+2\delta}+1\right),\end{aligned} \tag{D.142}$$

which shows (D.140) by using the Assumptions 3.15 $[\mathbf{W}^*]$ (i) (that ensures $\beta = o\left(Tb^2\right)$) and 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii).
It holds for all $s := \left(s^{[1]'}, s^{[2]'}\right)' \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ due to (D.45), (C.25) with $M = 6$, the Propositions 2.12 (with $\mathfrak{U}_{0,1,b} = [b, 1-b]$ under Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$) and 2.14 together with Remark 4.2 (i), Assumption 2.2 $[\mathbf{StAp}]$ (i), Remark 2.3 as well as Assumption 3.15 $[\mathbf{W}^*]$ (i) (see (D.139), (D.138), (D.24) and (4.11)):

$$\begin{aligned}&\sup_{u\in[b,1-b]}\mathbb{E}\left[\left|\mathbb{Q}^{**}_{T,1}(u,s) - \mathbb{Q}^{*}_{T,1}(u,s)\right|^2\right]\\ &= \sup_{u\in[b,1-b]}\mathbb{E}\Bigg[\Bigg|\frac{1}{T}\sum_{t=1}^{T}K_b\left(\frac{t}{T}-u\right)\Bigg(-\widehat{\varphi}\left(u,s\right) + \mathbb{E}\left[e^{i\left\langle s,X_{t,T}\right\rangle}\right] + \varphi^{[1]}\left(u,s^{[1]}\right)\\ &\cdot\left(\widehat{\varphi}^{[2]}\left(u,s^{[2]}\right) - \mathbb{E}\left[e^{i\left\langle s^{[2]},X^{[2]}_{t,T}\right\rangle}\right]\right) + \varphi^{[2]}\left(u,s^{[2]}\right)\cdot\left(\widehat{\varphi}^{[1]}\left(u,s^{[1]}\right) - \mathbb{E}\left[e^{i\left\langle s^{[1]},X^{[1]}_{t,T}\right\rangle}\right]\right)\Bigg)W^*_t\Bigg|^2\Bigg]\end{aligned}$$

$$\leqslant \frac{C\beta}{Tb} \sup_{u\in[b,1-b]} \sup_{t=1,\ldots,T:\left|\frac{t}{T}-u\right|\leqslant b} 6\Bigg(\mathbb{E}\left[\left|-\widehat{\varphi}\left(u,s\right)+\mathbb{E}\left[e^{i\left\langle s,\tilde{X}_t(u)\right\rangle}\right]\right|^2\right]+\left|\mathbb{E}\left[-e^{i\left\langle s,\tilde{X}_t(u)\right\rangle}+e^{i\left\langle s,X_{t,T}\right\rangle}\right]\right|^2$$
$$+\mathbb{E}\left[\left|\widehat{\varphi}^{[2]}\left(u,s^{[2]}\right)-\mathbb{E}\left[e^{i\left\langle s^{[2]},\tilde{X}^{[2]}_t(u)\right\rangle}\right]\right|^2\right]+\left|\mathbb{E}\left[e^{i\left\langle s^{[2]},\tilde{X}^{[2]}_t(u)\right\rangle}-e^{i\left\langle s^{[2]},X^{[2]}_{t,T}\right\rangle}\right]\right|^2$$
$$+\mathbb{E}\left[\left|\widehat{\varphi}^{[1]}\left(u,s^{[1]}\right)-\mathbb{E}\left[e^{i\left\langle s^{[1]},\tilde{X}^{[1]}_t(u)\right\rangle}\right]\right|^2\right]+\left|\mathbb{E}\left[e^{i\left\langle s^{[1]},\tilde{X}^{[1]}_t(u)\right\rangle}-e^{i\left\langle s^{[1]},X^{[1]}_{t,T}\right\rangle}\right]\right|^2\Bigg)$$
$$\leqslant \frac{C\beta}{Tb}\left(b^{2+2\delta}+\frac{1}{Tb}+b^2\right)\left(|s|_1^{2+2\delta}+1\right)$$
$$=\left(\frac{o\left(Tb^2\right)}{Tb}\left(b^{2+2\delta}+\frac{1}{Tb}\right)+\frac{o\left(1/b\right)}{Tb}b^2\right)\left(|s|_1^{2+2\delta}+1\right), \tag{D.143}$$

which proves (D.141) by using Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii).
Further, note that if X_1, X_2, Y_1, Y_2 are real-valued random variables with finite second moments which fulfil that X_1, X_2 are centered and that $(X_1, X_2)'$ is independent of $(Y_1, Y_2)'$, it will hold $|\mathbb{E}[X_1Y_1X_2Y_2]| \leqslant |\mathrm{Cov}(X_1, X_2)|\, \|Y_1\|_2 \|Y_2\|_2$. Thus, one obtains for all $s \in \mathbb{R}^d$ analogously to (B.46) by using Assumption 3.15 $[\mathbf{W}^*]$ (ii) and (iii):

$$\sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right)\Re\left\{e^{i\left\langle s,X_{t,T}\right\rangle}-\mathbb{E}\left[e^{i\left\langle s,X_{t,T}\right\rangle}\right]\right\}W_t^*\right|^2\right]\leqslant \frac{C}{Tb}\left(1+|s|_1\right). \tag{D.144}$$

It follows for all $s \in \mathbb{R}^d$ from (C.25) with $M = 3$, (D.144) and similar arguments (recall (D.138), (D.24) as well as (4.11)):

$$T\sqrt{b} \sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\mathbb{Q}^*_{T,1}(u,s)\right|^2\right] \leqslant \frac{C}{\sqrt{b}}\left(1+|s|_1\right). \tag{D.145}$$

Lemma D.12 results from (D.37) with $f(u,s) := \widehat{\mathbb{Q}}^*_T(u,s)$ and $g(u,s) := \mathbb{Q}^*_{T,1}(u,s)$ $\forall\, u \in [0,1]$, $s \in \mathbb{R}^d$ by using (C.25) with $M = 2$ together with (D.140) as well as (D.141), (D.145) and Assumption 4.3 $[\mathbf{WEI.2}]$. □

Lemma D.13. *Suppose that the Assumptions 4.1* $[\mathbf{INDEP}]$, *4.3* $[\mathbf{WEI.2}]$, *4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *and 3.15* $[\mathbf{W}^*]$ *hold. Moreover, define (see (D.4) and Definition A.1 (i), (vi) as well as (vii)):*

$$\tilde{\mathfrak{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta} := \int_{\mathbb{R}^d}\int_b^{1-b}\left|\tilde{\mathbb{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta}(u,s)\right|^2 du\,\mathbf{w}(s)\,ds \quad \textit{with}$$
$$\tilde{\mathbb{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta}(u,s) := \frac{1}{T}\sum_{t=1}^{T}K_b\left(\frac{t}{T}-u\right)\tilde{\mathbf{G}}^c_{t,\{\mathscr{a}\}}(u,s)W^*_{t,\{\mathscr{a}_\beta\}} \quad \forall\, u\in[0,1],\ s\in\mathbb{R}^d. \tag{D.146}$$

Then, one obtains for $T \to \infty$ *(recall (D.138)):*

$$T\sqrt{b}\,\mathbb{E}\left[\left|\int_{\mathbb{R}^d}\int_b^{1-b}\left|\mathbb{Q}^*_{T,1}(u,s)\right|^2 du\,\mathbf{w}(s)\,ds-\tilde{\mathfrak{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta}\right|\right]=o(1).$$

Proof. At first, observe that (C.25) with $M = 3$ provides for all $s \in \mathbb{R}^d$ (see Definition A.1 (i)):

$$\sup_{u\in[b,1-b]}\mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T}K_b\left(\frac{t}{T}-u\right)\left(\Re\left\{e^{i\left\langle s,X_{t,T}\right\rangle}\right\}^cW_t^*-\Re\left\{e^{i\left\langle s,\tilde{X}_{t,\{\mathscr{a}\}}(u)\right\rangle}\right\}^cW^*_{t,\{\mathscr{a}_\beta\}}\right)\right|^2\right]$$
$$\leqslant 3\sup_{u\in[b,1-b]}\mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T}K_b\left(\frac{t}{T}-u\right)\Re\left\{e^{i\left\langle s,X_{t,T}\right\rangle}-e^{i\left\langle s,X_{t,T,\{\mathscr{a}\}}\right\rangle}\right\}^cW_t^*\right|^2\right]$$

$$+ 3 \sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) \Re\left\{e^{i\langle s,X_{t,T,\{a\}}\rangle} - e^{i\langle s,\tilde{X}_{t,\{a\}}(u)\rangle}\right\}^c W_t^*\right|^2\right]$$

$$+ 3 \sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) \Re\left\{e^{i\langle s,\tilde{X}_{t,\{a\}}(u)\rangle}\right\}^c \left(W_t^* - W_{t,\{a_\beta\}}^*\right)\right|^2\right]. \tag{D.147}$$

It follows for all $s \in \mathbb{R}^d$ from (D.45) and Lemma D.1 (i) with $q = 2$:

$$\sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) \Re\left\{e^{i\langle s,X_{t,T}\rangle} - e^{i\langle s,X_{t,T,\{a\}}\rangle}\right\}^c W_t^*\right|^2\right] \leqslant \frac{C\beta}{T^{2+\delta}b}\cdot |s|_1^{1+\delta}. \tag{D.148}$$

One obtains similarly to (D.56) by using Assumption 3.15 $[\mathbf{W}^*]$ (ii) and (iii) (recall (D.48)):

$$\sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) \Re\left\{e^{i\langle s,X_{t,T,\{a\}}\rangle} - e^{i\langle s,\tilde{X}_{t,\{a\}}(u)\rangle}\right\}^c W_t^*\right|^2\right] \leqslant \frac{C a b^\delta}{T}|s|_1^{1+\delta}. \tag{D.149}$$

The Assumptions 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) as well as 3.15 $[\mathbf{W}^*]$ (ii) and Lemma D.11 (ii) imply:

$$\begin{aligned}
&\sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) \Re\left\{e^{i\langle s,\tilde{X}_{t,\{a\}}(u)\rangle}\right\}^c \left(W_t^* - W_{t,\{a_\beta\}}^*\right)\right|^2\right] \\
&\leqslant \frac{1}{(Tb)^2} \sup_{u\in[b,1-b]} \sum_{\substack{t_1,t_2=\lfloor uT-Tb\rfloor \\ |t_1-t_2|\leqslant a}}^{\lceil uT+Tb\rceil} K\left(\frac{\frac{t_1}{T}-u}{b}\right) K\left(\frac{\frac{t_2}{T}-u}{b}\right) \left|\mathrm{Cov}\left(\Re\left\{e^{i\langle s,\tilde{X}_{t_1,\{a\}}(u)\rangle}\right\},\right.\right. \\
&\left.\left.\Re\left\{e^{i\langle s,\tilde{X}_{t_2,\{a\}}(u)\rangle}\right\}\right)\right| \left|\mathbb{E}\left[\left(W_{t_1}^* - W_{t_1,\{a_\beta\}}^*\right)\left(W_{t_2}^* - W_{t_2,\{a_\beta\}}^*\right)\right]\right| \\
&\leqslant \frac{C a}{Tb}\frac{1}{T^2}.
\end{aligned} \tag{D.150}$$

Overall, (D.147), (D.148), (D.149), (D.150), Assumption 3.15 $[\mathbf{W}^*]$ (i) (which ensures $\beta = o(Tb^2)$), (A.1) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) show for all $s \in \mathbb{R}^d$:

$$\begin{aligned}
&T\sqrt{b} \sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) \left(\Re\left\{e^{i\langle s,X_{t,T}\rangle}\right\}^c W_t^* - \Re\left\{e^{i\langle s,\tilde{X}_{t,\{a\}}(u)\rangle}\right\}^c W_{t,\{a_\beta\}}^*\right)\right|^2\right] \\
&\leqslant o\left(\sqrt{b}\right)\left(|s|_1^{1+\delta}+1\right),
\end{aligned}$$

whereby the contained expression $o(\sqrt{b})$ does not depend on $s \in \mathbb{R}^d$. This and similar arguments together with (C.25) with $M = 3$ yield (recall (D.146), (D.4), (D.138), (D.24) as well as (4.11)):

$$T\sqrt{b} \sup_{u\in[b,1-b]} \mathbb{E}\left[\left|\mathbb{Q}_{T,1}^*(u,s) - \widetilde{\mathbb{Q}}_{T,a,a_\beta}^*(u,s)\right|^2\right] \leqslant o\left(\sqrt{b}\right)\left(|s|_1^{1+\delta}+1\right), \tag{D.151}$$

whereby the contained expression $o(\sqrt{b})$ does not depend on $s \in \mathbb{R}^d$.
Lemma D.13 is an implication of (D.37) with $f(u,s) := \widetilde{\mathbb{Q}}_{T,a,a_\beta}^*(u,s)$ as well as $g(u,s) := \mathbb{Q}_{T,1}^*(u,s)$ $\forall\, u \in [0,1]$, $s \in \mathbb{R}^d$, (D.151), (D.145) and Assumption 4.3 $[\mathbf{WEI.2}]$. □

Corollary D.14. *Let the Assumptions 4.1* $[\mathbf{INDEP}]$*, 4.3* $[\mathbf{WEI.2}]$*, 4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *and 3.15* $[\mathbf{W}^*]$ *be fulfilled. Then, it holds for $T \to \infty$ (see (D.146) and (4.21)):*

$$\mathbb{E}\left[T\sqrt{b}\,\tilde{\mathfrak{Q}}_{T,a,a_\beta}^*\right] = \mathbf{Bias}_T^{\mathrm{indep}*} + o(1).$$

Proof. One obtains from the equality $|z|^2 = \Re\{z\}^2 + \Im\{z\}^2\ \forall\, z \in \mathbb{C}$, Assumption 3.15 $[\mathbf{W}^*]$ (ii) and

(D.9) (recall (D.146), (D.4), Definition 2.11 as well as Assumption 3.15 $[\mathbf{W}^*]$ (iii)):

$$\begin{aligned}\mathbb{E}\Big[T\sqrt{b}\,\widetilde{\mathfrak{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta}\Big] &= \frac{1}{2}\sum_{\substack{t,j=1\\|t-j|\leqslant \mathscr{a}}}^{T}\Big(\mathbb{E}\Big[W^*_{t,\{\mathscr{a}_\beta\}}W^*_{j,\{\mathscr{a}_\beta\}}\Big]-\mathbb{E}\big[W^*_tW^*_j\big]\Big)\,\mathbb{E}\Big[\widetilde{\mathfrak{H}}_{T,\Re}(t,j)+\widetilde{\mathfrak{H}}_{T,\Im}(t,j)\Big]\\
&+\frac{1}{2}\sum_{t,j=1}^{T}K^*\left(\frac{t-j}{\beta}\right)\mathbb{E}\Big[\widetilde{\mathfrak{H}}_{T,\Re}(t,j)+\widetilde{\mathfrak{H}}_{T,\Im}(t,j)\Big]\\
&=: \mathfrak{B}^{\text{indep}*}_{T,1}+\mathfrak{B}^{\text{indep}*}_{T,2}.\end{aligned}$$

Lemma D.11 (ii) together with (C.112), Assumption 3.15 $[\mathbf{W}^*]$ (iii), (D.7), (A.1) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ (ii) imply $\mathfrak{B}^{\text{indep}*}_{T,1} = o(1)$. Moreover, Lemma D.6 with $\mathscr{G}_T(h) := K^*(h/\beta)\ \forall\, h\in\mathbb{Z}$ shows $\mathfrak{B}^{\text{indep}*}_{T,2} = \mathbf{Bias}^{\text{indep}*}_T + o(1)$ (see (4.21) and (4.12)), whereby this choice of $\mathscr{G}_T$ fulfils (C.215) due to Assumption 3.15 $[\mathbf{W}^*]$ (iii).
Overall, these considerations prove Corollary D.14. □

Lemma D.15. *Suppose that the Assumptions 4.1* $[\mathbf{INDEP}]$*, 4.3* $[\mathbf{WEI}.\mathbf{2}]$ *and 4.5* $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ *are valid. Then, it holds for all* $T\in\mathbb{N}$ *(recall (D.24) and (D.4)):*

(i)

$$\sup_{t,j=1,\ldots,T}\left\|\mathfrak{H}^{\{1\}}_T(t,j)-\widetilde{\mathfrak{H}}_T(t,j)\right\|_2 \leqslant C\frac{b^{\delta/2}}{T}.$$

(ii)

$$\sup_{t,j=1,\ldots,T}\left\|\mathfrak{H}^{\{\mathrm{n}\}}_T(t,j)-\mathfrak{H}_T(t,j)\right\|_2 \leqslant \frac{C}{T^{1+\mathrm{n}(1+\delta)/2}\sqrt{b}}\qquad \forall\,\mathrm{n}\in\mathbb{N}_0.$$

Proof. (i) At first, one obtains for all $u\in[0,1]$, $s:=\big(s^{[1]\prime},s^{[2]\prime}\big)'\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$, all random variables X and Y with realizations in $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ for which $\|X\|_{1+\delta}<\infty$ as well as $\|Y\|_{1+\delta}<\infty$ and all $q\geqslant 1+\delta$ by using (3.14) as well as $\big|z^{[k]}\big|_1\leqslant |z|_1\ \forall\, z:=\big(z^{[1]\prime},z^{[2]\prime}\big)'\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$, $k\in\{1,2\}$ (note (4.11)):

$$\|\mathbf{g}_{u,s}(X)-\mathbf{g}_{u,s}(Y)\|_q \leqslant C\,|s|_1^{\frac{1+\delta}{q}}\left(\mathbb{E}\left[|X-Y|_1^{1+\delta}\right]\right)^{\frac{1}{q}}. \tag{D.152}$$

Assumption 4.5 $[\mathbf{K}\&\mathbf{b}.\mathbf{2}]$ (i), (C.112), $|\mathbf{g}_{u,s}(x)|\leqslant C\ \forall\, u\in[0,1]$, $s,x\in\mathbb{R}^d$, the inequality $|\Re\{z\}|\leqslant|z|$ $\forall\, z\in\mathbb{C}$, (D.152) with $q=2$, Assumption 4.3 $[\mathbf{WEI}.\mathbf{2}]$, Assumption 2.2 $[\mathbf{StAp}]$ (i) and Remark 2.3 imply (see (D.24), (D.4) as well as Definition A.1 (i)):

$$\begin{aligned}&\sup_{t,j=1,\ldots,T}\left\|\mathfrak{H}^{\{1\}}_{T,\Re}(t,j)-\widetilde{\mathfrak{H}}_{T,\Re}(t,j)\right\|_2\\
&\leqslant \frac{C}{Tb^{\frac{3}{2}}}\sup_{t,j=1,\ldots,T}\int_{\mathbb{R}^d}\int_{\max\{b,t/T-b\}}^{\min\{1-b,t/T+b\}}K\left(\frac{\frac{t}{T}-u}{b}\right)K\left(\frac{\frac{j}{T}-u}{b}\right)\mathbb{1}_{\{|\frac{t}{T}-u|\leqslant b\}}\mathbb{1}_{\{|\frac{j}{T}-u|\leqslant b\}}\Big(\Big\|\mathbf{G}^c_{t,T,\{\mathscr{a}\},\Re}(u,s)\\
&-\widetilde{\mathbf{G}}^c_{t,\{\mathscr{a}\},\Re}(u,s)\Big\|_2+\Big\|\mathbf{G}^c_{j,T,\{\mathscr{a}\},\Re}(u,s)-\widetilde{\mathbf{G}}^c_{j,\{\mathscr{a}\},\Re}(u,s)\Big\|_2\Big)\,du\,\mathbf{w}(s)\,ds\\
&\leqslant \frac{C}{Tb^{\frac{3}{2}}}b\sup_{u\in[b,1-b]}\ \sup_{t\in\{1,\ldots,T\}:\left|\frac{t}{T}-u\right|\leqslant b}\left\|\mathbb{E}\left[X_{t,T}-\widetilde{X}_t\left(\frac{t}{T}\right)+\widetilde{X}_t\left(\frac{t}{T}\right)-\widetilde{X}_t(u)\Big|\mathcal{F}_{t,t-\mathscr{a}}\right]\right\|_{1+\delta}^{\frac{1+\delta}{2}}\\
&\leqslant \frac{C}{Tb^{\frac{3}{2}}}bb^{(1+\delta)/2}.\end{aligned}\tag{D.153}$$

Lemma D.15 (i) follows from (D.153) and similar arguments.

(ii) Lemma B.4 (i), Assumption 2.4 [**DM.3**] and shifting the index of a sum yield (recall Definition A.1 (i) as well as (vi)):

$$\sup_{t=1,\dots,T} \mathbb{E}\left[\left|X_{t,T,\{n\mathscr{a}\}} - X_{t,T}\right|_1^{1+\delta}\right] \leqslant \left(\sup_{t=1,\dots,T} \sum_{l=n\mathscr{a}}^{\infty} \left\|\mathbb{E}\left[X_{t,T}\middle|\mathcal{F}_{t,t-l}\right] - \mathbb{E}\left[X_{t,T}\middle|\mathcal{F}_{t,t-l-1}\right]\right\|_{1+\delta}\right)^{1+\delta}$$
$$\leqslant \left(\sum_{l=n\mathscr{a}}^{\infty} \Delta_{l+1}\right)^{1+\delta} \leqslant C\left((\rho^{\mathscr{a}})^{n}\sum_{l=1}^{\infty}\rho^{l}\right)^{1+\delta} \leqslant \frac{C}{T^{n(1+\delta)}}. \quad \text{(D.154)}$$

Lemma D.15 (ii) follows similarly to (D.153) by using (D.154) (see (D.24)). □

Lemma D.16. *Let the Assumptions 4.1* [**INDEP**], *4.3* [**WEI.2**], *4.5* [**K**&**b.2**] *and 3.15* [$\mathbf{W}^*$] *be fulfilled. Then, one obtains for $T \to \infty$ (note (D.146), (4.21) and (D.24)):*

$$\mathbb{E}\left[\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta} - \mathbf{Bias}_T^{\text{indep}*} - \mathbb{H}_T^*\right)^2\right] = o(1).$$

Proof. Throughout this proof, assume that T is large enough to ensure (recall Definition A.1 (vii) as well as (D.3)):

$$7\mathscr{a}_\beta + 2 \leqslant 1 + 8\mathscr{a}_\beta \leqslant \rho_T \leqslant \rho_T + 1 \leqslant L_T \leqslant L_T + 1 \leqslant T, \quad \text{(D.155)}$$

which is possible due to Lemma D.11 (i), Assumption 3.15 [$\mathbf{W}^*$] (i), (A.1) and Assumption 4.5 [**K**&**b.2**] (ii).
Define (see (D.4) as well as Definition A.1 (i)):

$$\tilde{\mathfrak{T}}_T^* := \frac{1}{2}\sum_{t=1}^{T} \tilde{\mathfrak{H}}_T^W(t,t) + \sum_{t=L_T+1}^{T}\sum_{j=1}^{t-1} \tilde{\mathfrak{H}}_T^W(t,j) \quad \text{with} \quad \tilde{\mathfrak{H}}_T^W(t,j) := \tilde{\mathfrak{H}}_T(t,j)\, W^*_{t,\{\mathscr{a}_\beta\}} W^*_{j,\{\mathscr{a}_\beta\}}$$
$$\forall\, t,j \in \mathbb{Z}. \quad \text{(D.156)}$$

In the following, it is shown:

$$\mathbb{E}\left[\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta} - \tilde{\mathfrak{T}}_T^*\right)^2\right] = o(1). \quad \text{(D.157)}$$

One obtains from the equalities $|z|^2 = \Re\{z\}^2 + \Im\{z\}^2\ \forall\, z \in \mathbb{C}$ and $\tilde{\mathfrak{H}}_T^W(t,j) = \tilde{\mathfrak{H}}_T^W(j,t)\ \forall\, t,j \in \mathbb{Z}$ (note (D.146) as well as (D.4)):

$$T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta} = \frac{1}{2}\sum_{t,j=1}^{T} \tilde{\mathfrak{H}}_T^W(t,j) = \frac{1}{2}\sum_{t=1}^{T}\tilde{\mathfrak{H}}_T^W(t,t) + \sum_{t=2}^{T}\sum_{j=1}^{t-1}\tilde{\mathfrak{H}}_T^W(t,j). \quad \text{(D.158)}$$

It follows similarly to (D.124) and to (D.125) by using Assumption 3.15 [$\mathbf{W}^*$] (ii) as well as (iii):

$$\left|\mathbb{E}\left[\sum_{t=2}^{L_T}\sum_{j=1}^{t-1}\tilde{\mathfrak{H}}_T^W(t,j)\right]\right| = o(1) \quad \text{and} \quad \operatorname{Var}\left(\sum_{t=2}^{L_T}\sum_{j=1}^{t-1}\tilde{\mathfrak{H}}_T^W(t,j)\right) = o(1). \quad \text{(D.159)}$$

In summary, (D.158) and (D.159) yield (D.157).
Further, define (recall (D.3) and (D.24)):

$$\tilde{\mathbb{H}}_T^* := \sum_{k=1}^{\mathfrak{K}_T}\sum_{t=l^*_{T,k}}^{o_{T,k}}\sum_{j=1}^{t-7\mathscr{a}_\beta-1}\tilde{\mathfrak{H}}_T^W(t,j). \quad \text{(D.160)}$$

In the following, it is proved:

$$\mathbb{E}\left[\left(\tilde{\mathfrak{T}}_T^* - \mathbf{Bias}_T^{\text{indep}*} - \tilde{\mathbb{H}}_T^*\right)^2\right] = o(1). \quad \text{(D.161)}$$

Therefor, note at first that (D.155) yields:

$$\begin{aligned}\widetilde{\mathfrak{T}}_T^* &= \frac{1}{2}\sum_{t=1}^{T}\widetilde{\mathfrak{H}}_T^W(t,t) + \sum_{t=L_T+1}^{T}\sum_{j=t-7\mathscr{a}_\beta}^{t-1}\widetilde{\mathfrak{H}}_T^W(t,j) + \sum_{t=L_T+1}^{T}\sum_{j=1}^{t-7\mathscr{a}_\beta-1}\widetilde{\mathfrak{H}}_T^W(t,j)\\ &=: \widetilde{\mathfrak{T}}_{T,\text{Bias},1}^* + \widetilde{\mathfrak{T}}_{T,\text{Bias},2}^* + \widetilde{\mathfrak{T}}_{T,\text{Var}}^*,\end{aligned} \tag{D.162}$$

whereby Assumption 3.15 $[\mathbf{W}^*]$ (ii) and (iii) provide (see Definition A.1 (i)):

$$\mathbb{E}\left[\widetilde{\mathfrak{T}}_T^*\right] = \mathbb{E}\left[\widetilde{\mathfrak{T}}_{T,\text{Bias},1}^* + \widetilde{\mathfrak{T}}_{T,\text{Bias},2}^*\right]. \tag{D.163}$$

It holds due to (D.163), (D.157) and Corollary D.14:

$$\begin{aligned}&\left|\mathbb{E}\left[\widetilde{\mathfrak{T}}_{T,\text{Bias},1}^* + \widetilde{\mathfrak{T}}_{T,\text{Bias},2}^*\right] - \mathbf{Bias}_T^{\text{indep}*}\right|\\ &\leqslant \left|\mathbb{E}\left[\widetilde{\mathfrak{T}}_T^*\right] - \mathbb{E}\left[T\sqrt{b}\,\widetilde{\mathfrak{Q}}_{T,\mathscr{a},\mathscr{a}_\beta}^*\right]\right| + \left|\mathbb{E}\left[T\sqrt{b}\,\widetilde{\mathfrak{Q}}_{T,\mathscr{a},\mathscr{a}_\beta}^*\right] - \mathbf{Bias}_T^{\text{indep}*}\right| = o(1).\end{aligned} \tag{D.164}$$

One obtains for all real-valued random variables Y_1, Y_2, Z_1, Z_2 which own finite second moments and fulfil $(Y_l)_{l=1}^2 \perp\!\!\!\perp (Z_l)_{l=1}^2$:

$$\operatorname{Cov}(Y_1Z_1, Y_2Z_2) = \operatorname{Cov}(Y_1, Y_2)\,\mathbb{E}[Z_1Z_2] + \mathbb{E}[Y_1]\,\mathbb{E}[Y_2]\operatorname{Cov}(Z_1, Z_2). \tag{D.165}$$

Further, if $t_1, t_2 \in \{L_T+1, \ldots, T\}$, $t_3 \in \{t_1 - 7\mathscr{a}_\beta, \ldots, t_1 - 1\}$, $t_4 \in \{t_2 - 7\mathscr{a}_\beta, \ldots, t_2 - 1\}$ as well as $|t_1 - t_2| \leqslant \mathscr{a} \vee |t_1 - t_4| \leqslant \mathscr{a} \vee |t_3 - t_2| \leqslant \mathscr{a} \vee |t_3 - t_4| \leqslant \mathscr{a}$, it will hold $|t_1 - t_2| \leqslant 14\mathscr{a}_\beta + \mathscr{a}$ and a valid implication also results by replacing all contained $\mathscr{a}$ by $\mathscr{a}_\beta$. Thus, (C.405), (D.165) together with Assumption 3.15 $[\mathbf{W}^*]$ (ii), (D.10), (D.9), (D.7), Assumption 3.15 $[\mathbf{W}^*]$ (iii), Lemma D.11 (i), Assumption 3.15 $[\mathbf{W}^*]$ (i) $\big($the latter ensures $\beta = o(1/b)\big)$, (A.1) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) yield (recall (D.162) as well as (D.156)):

$$\begin{aligned}&\operatorname{Var}\left(\widetilde{\mathfrak{T}}_{T,\text{Bias},1}^* + \widetilde{\mathfrak{T}}_{T,\text{Bias},2}^*\right) \leqslant 3\operatorname{Var}\left(\widetilde{\mathfrak{T}}_{T,\text{Bias},1}^*\right) + 3\operatorname{Var}\left(\widetilde{\mathfrak{T}}_{T,\text{Bias},2}^*\right)\\ &\leqslant C\sum_{t_1,t_2=1}^{T}\left|\operatorname{Cov}\left(\widetilde{\mathfrak{H}}_T(t_1,t_1), \widetilde{\mathfrak{H}}_T(t_2,t_2)\right)\right|\mathbb{1}_{\{|t_1-t_2|\leqslant\mathscr{a}\}}\left|\mathbb{E}\left[{W_{t_1,\{\mathscr{a}_\beta\}}^*}^2\,{W_{t_2,\{\mathscr{a}_\beta\}}^*}^2\right]\right|\\ &+ C\sum_{t_1,t_2=1}^{T}\left|\mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t_1,t_1)\right]\mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t_2,t_2)\right]\right|\left|\operatorname{Cov}\left({W_{t_1,\{\mathscr{a}_\beta\}}^*}^2, {W_{t_2,\{\mathscr{a}_\beta\}}^*}^2\right)\right|\mathbb{1}_{\{|t_1-t_2|\leqslant\mathscr{a}_\beta\}}\\ &+ C\sum_{\substack{t_1,t_2=L_T+1\\|t_1-t_2|\leqslant14\mathscr{a}_\beta+\mathscr{a}}}^{T}\;\sum_{t_3=t_1-7\mathscr{a}_\beta}^{t_1-1}\;\sum_{t_4=t_2-7\mathscr{a}_\beta}^{t_2-1}\left|\mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t_1,t_3)\,\widetilde{\mathfrak{H}}_T(t_2,t_4)\right] - \mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t_1,t_3)\right]\mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t_2,t_4)\right]\right|\\ &\cdot \mathbb{1}_{\{\forall\, n_1\in\{1,\ldots,4\}\exists n_2\in\{1,\ldots,4\}\setminus\{n_1\}:\,|t_{n_1}-t_{n_2}|\leqslant\mathscr{a}\}}\mathbb{1}_{\{|t_1-t_2|\leqslant\mathscr{a}\vee|t_1-t_4|\leqslant\mathscr{a}\vee|t_3-t_2|\leqslant\mathscr{a}\vee|t_3-t_4|\leqslant\mathscr{a}\}}\\ &\cdot\left|\mathbb{E}\left[W_{t_1,\{\mathscr{a}_\beta\}}^* W_{t_3,\{\mathscr{a}_\beta\}}^* W_{t_2,\{\mathscr{a}_\beta\}}^* W_{t_4,\{\mathscr{a}_\beta\}}^*\right]\right|\\ &+ C\sum_{\substack{t_1,t_2=L_T+1\\|t_1-t_2|\leqslant15\mathscr{a}_\beta}}^{T}\;\sum_{\substack{t_3=t_1-7\mathscr{a}_\beta\\|t_3-t_1|\leqslant\mathscr{a}}}^{t_1-1}\;\sum_{\substack{t_4=t_2-7\mathscr{a}_\beta\\|t_4-t_2|\leqslant\mathscr{a}}}^{t_2-1}\left|\mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t_1,t_3)\right]\mathbb{E}\left[\widetilde{\mathfrak{H}}_T(t_2,t_4)\right]\right|\\ &\cdot\left|\operatorname{Cov}\left(W_{t_1,\{\mathscr{a}_\beta\}}^* W_{t_3,\{\mathscr{a}_\beta\}}^*, W_{t_2,\{\mathscr{a}_\beta\}}^* W_{t_4,\{\mathscr{a}_\beta\}}^*\right)\right|\cdot\mathbb{1}_{\{|t_1-t_2|\leqslant\mathscr{a}_\beta\vee|t_1-t_4|\leqslant\mathscr{a}_\beta\vee|t_3-t_2|\leqslant\mathscr{a}_\beta\vee|t_3-t_4|\leqslant\mathscr{a}_\beta\}}\\ &\leqslant C\frac{T\mathscr{a}}{T^2b} + C\frac{T\mathscr{a}_\beta}{T^2b} + C\frac{T\mathscr{a}_\beta\mathscr{d}^2}{T^2b} + C\frac{T\mathscr{a}_\beta\mathscr{d}^2}{T^2b} = o(1).\end{aligned} \tag{D.166}$$

In summary, (D.164) and (D.166) provide $\big($note that $\mathbf{Bias}_T^{\text{indep}*}$ defined in (4.21) is deterministic$\big)$:

$$\mathbb{E}\left[\left(\widetilde{\mathfrak{T}}_{T,\text{Bias},1}^* + \widetilde{\mathfrak{T}}_{T,\text{Bias},2}^* - \mathbf{Bias}_T^{\text{indep}*}\right)^2\right] = o(1). \tag{D.167}$$

Further, observe that $\widetilde{\mathfrak{T}}_{T,\text{Var}}^*$ and $\widetilde{\mathbb{H}}_T^*$ (see (D.162) as well as (D.160)) are defined similarly to $\widetilde{\mathfrak{T}}_{T,\text{Var}}$ and

$\widetilde{\mathbb{H}}_T$, respectively (recall (D.127) as well as (D.4)). Hence, by using (D.155) and Assumption 3.15 $[\mathbf{W}^*]$ (ii) as well as (iii), it follows analogously to (D.130) to (D.134) (in particular, replace in (D.130) each $l_{t,k}$ (see (D.3)) with $k \in \{1, \ldots, \mathfrak{K}_T\}$ by $l^*_{t,k}$ (note (D.24)), each $7\mathscr{a}$ by $7\mathscr{a}_\beta$ and each $\widetilde{\mathfrak{H}}_T$ by $\widetilde{\mathfrak{H}}^W_T$):

$$\mathbb{E}\left[\left(\widetilde{\mathfrak{T}}^*_{T,\mathrm{Var}} - \widetilde{\mathbb{H}}^*_T\right)^2\right] = o(1). \tag{D.168}$$

Overall, (D.162), (C.25) with $M = 2$, (D.167) and (D.168) imply (D.161).
Next, it is shown:

$$\mathbb{E}\left[\left(\widetilde{\mathbb{H}}^*_T - \mathbb{H}^*_T\right)^2\right] = o(1). \tag{D.169}$$

Therefor, observe at first for all $l_1, \ldots, l_4 \in \{1, \ldots, T\}$, $\mathrm{n} \in \mathbb{N}$ (see (D.24) as well as (D.4)):

$$\begin{aligned}&\text{If } \exists n_1 \in \{1, \ldots, 4\} : |l_{n_1} - l_{n_2}| > \mathrm{n}\mathscr{a} \ \forall\, n_2 \in \{1, \ldots, 4\} \setminus \{n_1\}\,, \text{then}\\ &\mathbb{E}\left[\mathfrak{H}^{\{\mathrm{n}\}}_T(l_1, l_2)\, \widetilde{\mathfrak{H}}_T(l_3, l_4)\right] = 0 \quad \text{and} \quad \mathbb{E}\left[\mathfrak{H}^{\{\mathrm{n}\}}_T(l_1, l_2)\, \mathfrak{H}^{\{\mathrm{n}\}}_T(l_3, l_4)\right] = 0.\end{aligned} \tag{D.170}$$

One obtains from Lemma D.11 (i) that $l^*_{T,k} > o_{T,k'} + \mathscr{a}_\beta > o_{T,k'} + \mathscr{a}$ holds for all $k, k' \in \{1, \ldots, \mathfrak{K}_T\}$ with $k > k'$ (recall (D.24) and (D.3)). Thus, (C.25) with $M = 2$, Assumption 3.15 $[\mathbf{W}^*]$ (ii), (D.10), (D.170) with $\mathrm{n} = 1$, (D.8), (D.26) with $\mathrm{n} = 1$, Assumption 3.15 $[\mathbf{W}^*]$ (iii), Lemma D.15 (i) and (ii) with $\mathrm{n} = 1$, Lemma D.11 (i), Assumption 3.15 $[\mathbf{W}^*]$ (i) (the latter ensures $\beta = o(1/b)$), (A.1) as well as Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) yield (see (D.160), (D.156), (D.24), (D.3) and Definition A.1 (i)):

$$\begin{aligned}\mathbb{E}\left[\left(\widetilde{\mathbb{H}}^*_T - \mathbb{H}^*_T\right)^2\right] &\leqslant 2\,\mathbb{E}\left[\left(\widetilde{\mathbb{H}}^*_T - \sum_{k=1}^{\mathfrak{K}_T} \sum_{t=l^*_{T,k}}^{o_{T,k}} \sum_{j=1}^{t-7\mathscr{a}_\beta-1} \mathfrak{H}^{\{1\}}_T(t,j)\, W^*_{t,\{\mathscr{a}_\beta\}} W^*_{j,\{\mathscr{a}_\beta\}}\right)^2\right]\\ &\quad + 2\,\mathbb{E}\left[\left(\sum_{k=1}^{\mathfrak{K}_T} \sum_{t=l^*_{T,k}}^{o_{T,k}} \sum_{j=1}^{t-7\mathscr{a}_\beta-1} \mathfrak{H}^{\{1\}}_T(t,j)\, W^*_{t,\{\mathscr{a}_\beta\}} W^*_{j,\{\mathscr{a}_\beta\}} - \mathbb{H}^*_T\right)^2\right]\\ &\leqslant 2 \sum_{\substack{k_1,k_2=1\\k_1=k_2}}^{\mathfrak{K}_T} \sum_{t_1=l^*_{T,k_1}}^{o_{T,k_1}} \sum_{\substack{t_2=l^*_{T,k_2}\\|t_2-t_1|\leqslant\mathscr{a}}}^{o_{T,k_2}} \sum_{\substack{j_1=1\\|j_1-t_1|\leqslant 2Tb}}^{t_1-7\mathscr{a}_\beta-1} \sum_{\substack{j_2=1\\|j_2-j_1|\leqslant\mathscr{a}}}^{t_2-7\mathscr{a}_\beta-1} \Big|\mathbb{E}\Big[W^*_{t_1,\{\mathscr{a}_\beta\}} W^*_{j_1,\{\mathscr{a}_\beta\}} W^*_{t_2,\{\mathscr{a}_\beta\}}\\ &\quad \cdot W^*_{j_2,\{\mathscr{a}_\beta\}}\Big]\Big|\,\Big|\mathbb{E}\Big[\Big(\widetilde{\mathfrak{H}}_T(t_1,j_1) - \mathfrak{H}^{\{1\}}_T(t_1,j_1)\Big)\Big(\widetilde{\mathfrak{H}}_T(t_2,j_2) - \mathfrak{H}^{\{1\}}_T(t_2,j_2)\Big)\Big]\Big|\\ &\quad + 2 \sum_{\substack{k_1,k_2=1\\k_1=k_2}}^{\mathfrak{K}_T} \sum_{t_1=l^*_{T,k_1}}^{o_{T,k_1}} \sum_{\substack{t_2=l^*_{T,k_2}\\|t_2-t_1|\leqslant\mathscr{a}_\beta}}^{o_{T,k_2}} \sum_{\substack{j_1=1\\|j_1-t_1|\leqslant 2Tb}}^{t_1-7\mathscr{a}_\beta-1} \sum_{\substack{j_2=1\\|j_2-j_1|\leqslant\mathscr{a}_\beta}}^{t_2-7\mathscr{a}_\beta-1} \Big|\mathbb{E}\Big[W^*_{t_1,\{\mathscr{a}_\beta\}} W^*_{j_1,\{\mathscr{a}_\beta\}} W^*_{t_2,\{\mathscr{a}_\beta\}}\\ &\quad \cdot W^*_{j_2,\{\mathscr{a}_\beta\}}\Big]\Big|\,\Big|\mathbb{E}\Big[\Big(\mathfrak{H}^{\{1\}}_T(t_1,j_1) - \mathfrak{H}_T(t_1,j_1)\Big)\Big(\mathfrak{H}^{\{1\}}_T(t_2,j_2) - \mathfrak{H}_T(t_2,j_2)\Big)\Big]\Big|\\ &\quad \leqslant C\frac{T}{L_T} L_T \mathscr{a} T b \mathscr{a} \sup_{t,j=1,\ldots,T} \mathbb{E}\left[\left|\widetilde{\mathfrak{H}}_T(t,j) - \mathfrak{H}^{\{1\}}_T(t,j)\right|^2\right]\\ &\quad + C\frac{T}{L_T} L_T \mathscr{a}_\beta T b \mathscr{a}_\beta \sup_{t,j=1,\ldots,T} \mathbb{E}\left[\left|\mathfrak{H}^{\{1\}}_T(t,j) - \mathfrak{H}_T(t,j)\right|^2\right]\\ &\quad = o(1),\end{aligned} \tag{D.171}$$

which proves (D.169).
Lemma D.16 follows from (C.25) with $M = 3$, (D.157), (D.161) and (D.169). □

Lemma D.17. *Suppose that the Assumptions 4.1* $[\mathbf{INDEP}]$*, 4.3* $[\mathbf{WEI.2}]$*, 4.5* $[\mathbf{K}\&\mathbf{b.2}]$ *and 3.15* $[\mathbf{W}^*]$ *hold. Then, one obtains for $T \to \infty$ (recall (D.24) and (4.13)):*

$$\mathbb{E}\left[\left|\mathrm{Var}^*\left(\mathbb{H}^*_T\right) - \sigma^{\mathrm{indep}}\right|\right] = o(1).$$

Proof. Throughout this proof, assume that T is large enough to ensure $7\alpha_\beta + 2 \leqslant L_T$ (which is possible due to Lemma D.11 (i), Assumption 3.15 $[\mathbf{W}^*]$ (i), (A.1), Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) and (D.3)). This provides $l^*_{T,k} \leqslant o_{T,k}\ \forall\, k \in \{1, \ldots, \mathfrak{K}_T\}$ (see (D.24) and (D.3)).
Further, define:

$$\mathbb{V}_T^{\perp\!\!\!\perp *} := \sum_{k=1}^{\mathfrak{K}_T} \sum_{t_1,t_2=l^*_{T,k}}^{o_{T,k}} \sum_{j_1=1}^{t_1-7\alpha_\beta-1} \sum_{j_2=1}^{t_2-7\alpha_\beta-1} \mathfrak{H}_T^{\{8\}}(t_1,j_1)\, \mathfrak{H}_T^{\{8\}}(t_2,j_2)\, \mathbb{E}\Big[W^*_{t_1,\{\alpha_\beta\}} W^*_{j_1,\{\alpha_\beta\}} W^*_{t_2,\{\alpha_\beta\}} \cdot W^*_{j_2,\{\alpha_\beta\}}\Big]. \tag{D.172}$$

The following assertions are shown below:

$$\mathbb{E}\left[\left|\operatorname{Var}^*(\mathbb{H}_T^*) - \mathbb{V}_T^{\perp\!\!\!\perp *}\right|\right] = o(1), \quad \mathbb{E}\left[\left|\mathbb{V}_T^{\perp\!\!\!\perp *} - \mathbb{E}\left[\mathbb{V}_T^{\perp\!\!\!\perp *}\right]\right|\right] = o(1),$$
$$\mathbb{E}\left[\operatorname{Var}^*\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\alpha,\alpha_\beta}\right)\right] = \sigma^{\text{indep}} + o(1) \quad \text{and} \quad \mathbb{E}\left[\left|\operatorname{Var}^*(\mathbb{H}_T^*) - \operatorname{Var}^*\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\alpha,\alpha_\beta}\right)\right|\right] = o(1). \tag{D.173}$$

Overall, these statements imply:

$$\begin{aligned}
&\mathbb{E}\left[\left|\operatorname{Var}^*(\mathbb{H}_T^*) - \sigma^{\text{indep}}\right|\right] \\
&\leqslant \mathbb{E}\left[\left|\operatorname{Var}^*(\mathbb{H}_T^*) - \mathbb{V}_T^{\perp\!\!\!\perp *}\right|\right] + \mathbb{E}\left[\left|\mathbb{V}_T^{\perp\!\!\!\perp *} - \mathbb{E}\left[\mathbb{V}_T^{\perp\!\!\!\perp *}\right]\right|\right] + \left|\mathbb{E}\left[\mathbb{V}_T^{\perp\!\!\!\perp *}\right] - \mathbb{E}\left[\operatorname{Var}^*(\mathbb{H}_T^*)\right]\right| \\
&+ \left|\mathbb{E}\left[\operatorname{Var}^*(\mathbb{H}_T^*)\right] - \mathbb{E}\left[\operatorname{Var}^*\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\alpha,\alpha_\beta}\right)\right]\right| + \left|\mathbb{E}\left[\operatorname{Var}^*\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\alpha,\alpha_\beta}\right)\right] - \sigma^{\text{indep}}\right| \\
&= o(1),
\end{aligned} \tag{D.174}$$

which proves Lemma D.17.
It holds due to (D.30), Assumption 3.15 $[\mathbf{W}^*]$ (ii), (D.26) with $\mathrm{n} = 8$, (C.112), Lemma D.15 (ii) with $\mathrm{n} = 8$, (D.25) with $\mathrm{n} = 8$, Lemma D.11 (i), Assumption 3.15 $[\mathbf{W}^*]$ (i) $\big($which ensures $\beta = o(Tb^2)\big)$, (A.1) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) (note (D.24), (D.172) as well as (D.3)):

$$\begin{aligned}
&\mathbb{E}\left[\left|\operatorname{Var}^*(\mathbb{H}_T^*) - \mathbb{V}_T^{\perp\!\!\!\perp *}\right|\right] \\
&\leqslant \sum_{k=1}^{\mathfrak{K}_T} \sum_{\substack{t_1,t_2=l^*_{T,k} \\ |t_2-t_1|\leqslant \alpha_\beta}}^{o_{T,k}} \sum_{\substack{j_1=1 \\ |t_1-j_1|\leqslant 2Tb}}^{t_1-7\alpha_\beta-1} \sum_{\substack{j_2=1 \\ |j_2-j_1|\leqslant \alpha_\beta}}^{t_2-7\alpha_\beta-1} \mathbb{E}\left[\left|\mathfrak{H}_T(t_1,j_1)\,\mathfrak{H}_T(t_2,j_2) - \mathfrak{H}_T^{\{8\}}(t_1,j_1)\,\mathfrak{H}_T^{\{8\}}(t_2,j_2)\right|\right] \\
&\cdot \left|\mathbb{E}\left[W^*_{t_1,\{\alpha_\beta\}} W^*_{j_1,\{\alpha_\beta\}} W^*_{t_2,\{\alpha_\beta\}} W^*_{j_2,\{\alpha_\beta\}}\right]\right| \\
&\leqslant C \frac{T}{L_T} L_T \alpha_\beta T b \alpha_\beta \frac{1}{T^{5+4\delta}\sqrt{b}} \frac{1}{T\sqrt{b}} = o(1),
\end{aligned} \tag{D.175}$$

which yields the first assertion of (D.173).
Further, one obtains from (D.26) with $\mathrm{n} = 8$ and Assumption 3.15 $[\mathbf{W}^*]$ (iii) (recall (D.172) as well as (D.24)):

$$\begin{aligned}
\operatorname{Var}\left(\mathbb{V}_T^{\perp\!\!\!\perp *}\right) \leqslant C \sum_{k_1,k_2=1}^{\mathfrak{K}_T} \sum_{t_1,t_2=l^*_{T,k_1}}^{o_{T,k_1}} \sum_{t_3,t_4=l^*_{T,k_2}}^{o_{T,k_2}} \sum_{\substack{j_1=1 \\ |j_1-t_1|\leqslant 2Tb}}^{t_1-7\alpha_\beta-1} \sum_{\substack{j_2=1 \\ |j_2-t_2|\leqslant 2Tb}}^{t_2-7\alpha_\beta-1} \sum_{\substack{j_3=1 \\ |j_3-t_3|\leqslant 2Tb}}^{t_3-7\alpha_\beta-1} \sum_{\substack{j_4=1 \\ |j_4-t_4|\leqslant 2Tb}}^{t_4-7\alpha_\beta-1} \Big|\operatorname{Cov}\Big(\mathfrak{H}_T^{\{8\}}(t_1,j_1) \\
\cdot \mathfrak{H}_T^{\{8\}}(t_2,j_2), \mathfrak{H}_T^{\{8\}}(t_3,j_3)\,\mathfrak{H}_T^{\{8\}}(t_4,j_4)\Big)\Big| \mathbb{1}_{\{\exists l_1\in\{t_1,j_1,t_2,j_2\}\wedge l_2\in\{t_3,j_3,t_4,j_4\}:|l_1-l_2|\leqslant 8\alpha\}}.
\end{aligned} \tag{D.176}$$

Moreover, if $k > k'$ (with $k, k' \in \{1, \ldots, \mathfrak{K}_T\}$), it will follow from $\rho_T \geqslant \lceil 2Tb \rceil$ and Lemma D.11 (i) that $l^*_{T,k} - 2Tb - 8\alpha > o_{T,k'}$ (see (D.3) as well as (D.24)). Thus, $k_1 = k_2$ is necessary on the right side of (D.176) to ensure that the condition in the contained indicator is fulfilled. Therefore, one obtains:

$$\operatorname{Var}\left(\mathbb{V}_T^{\perp\!\!\!\perp *}\right) \leqslant C \sum_{k=1}^{\mathfrak{K}_T} \sum_{t_1,\ldots,t_4=l^*_{T,k}}^{o_{T,k}} \sum_{\substack{j_1=1 \\ |j_1-t_1|\leqslant 2Tb}}^{t_1-7\alpha_\beta-1} \sum_{\substack{j_2=1 \\ |j_2-t_2|\leqslant 2Tb}}^{t_2-7\alpha_\beta-1} \sum_{\substack{j_3=1 \\ |j_3-t_3|\leqslant 2Tb}}^{t_3-7\alpha_\beta-1} \sum_{\substack{j_4=1 \\ |j_4-t_4|\leqslant 2Tb}}^{t_4-7\alpha_\beta-1}$$

$$\left|\mathbb{E}\left[\mathfrak{H}_T^{\{8\}}(t_1,j_1)\,\mathfrak{H}_T^{\{8\}}(t_2,j_2)\,\mathfrak{H}_T^{\{8\}}(t_3,j_3)\,\mathfrak{H}_T^{\{8\}}(t_4,j_4)\right]\right|$$

$$+\,C\sum_{k=1}^{\mathfrak{K}_T}\sum_{t_1,\ldots,t_4=l^*_{T,k}}^{o_{T,k}}\sum_{\substack{j_1=1\\|j_1-t_1|\leqslant 2Tb}}^{t_1-7\alpha_\beta-1}\sum_{\substack{j_2=1\\|j_2-t_2|\leqslant 2Tb}}^{t_2-7\alpha_\beta-1}\sum_{\substack{j_3=1\\|j_3-t_3|\leqslant 2Tb}}^{t_3-7\alpha_\beta-1}\sum_{\substack{j_4=1\\|j_4-t_4|\leqslant 2Tb}}^{t_4-7\alpha_\beta-1}$$

$$\left|\mathbb{E}\left[\mathfrak{H}_T^{\{8\}}(t_1,j_1)\,\mathfrak{H}_T^{\{8\}}(t_2,j_2)\right]\mathbb{E}\left[\mathfrak{H}_T^{\{8\}}(t_3,j_3)\,\mathfrak{H}_T^{\{8\}}(t_4,j_4)\right]\right|$$

$$=:\mathbb{V}^{\perp\!\!\!\perp *}_{T,1}+\mathbb{V}^{\perp\!\!\!\perp *}_{T,2}. \tag{D.177}$$

It follows from $8\alpha\leqslant\alpha_\beta$ (which is valid according to Lemma D.11 (i)) analogously to the last equality of (D.33) (note that the second summand on the left side of this equality is very similar to $\mathbb{V}^{\perp\!\!\!\perp *}_{T,1}$):

$$\mathbb{V}^{\perp\!\!\!\perp *}_{T,1}=o(1). \tag{D.178}$$

Moreover, $8\alpha\leqslant\alpha_\beta$, (D.170) with $\mathrm{n}=8$, (D.26) with $\mathrm{n}=8$, (D.3), (D.25) with $\mathrm{n}=8$, (A.1) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) yield:

$$\mathbb{V}^{\perp\!\!\!\perp *}_{T,2}\leqslant C\sum_{k=1}^{\mathfrak{K}_T}\sum_{\substack{t_1,t_2=l^*_{T,k}\\|t_2-t_1|\leqslant 8\alpha}}^{o_{T,k}}\sum_{\substack{j_1=1\\|j_1-t_1|\leqslant 2Tb}}^{t_1-7\alpha_\beta-1}\sum_{\substack{j_2=1\\|j_2-j_1|\leqslant 8\alpha}}^{t_2-7\alpha_\beta-1}\sum_{\substack{t_3,t_4=l^*_{T,k}\\|t_4-t_3|\leqslant 8\alpha}}^{o_{T,k}}\sum_{\substack{j_3=1\\|j_3-t_3|\leqslant 2Tb}}^{t_3-7\alpha_\beta-1}\sum_{\substack{j_4=1\\|j_4-j_3|\leqslant 8\alpha}}^{t_4-7\alpha_\beta-1}$$

$$\left|\mathbb{E}\left[\mathfrak{H}_T^{\{8\}}(t_1,j_1)\,\mathfrak{H}_T^{\{8\}}(t_2,j_2)\right]\mathbb{E}\left[\mathfrak{H}_T^{\{8\}}(t_3,j_3)\,\mathfrak{H}_T^{\{8\}}(t_4,j_4)\right]\right|$$

$$\leqslant C\frac{T}{L_T}L_T\alpha Tb\alpha L_T\alpha Tb\alpha\frac{1}{T^4b^2}=o(1). \tag{D.179}$$

The second statement of (D.173) is an implication of (D.177), (D.178) and (D.179).
In the following, the third assertion of (D.173) is proved.
Therefor, observe at first that $\mathbf{Bias}_T^{\text{indep}*}$ is deterministic (recall (4.21)) and that (D.29) provides $\mathbb{E}^*\left[\mathbb{H}^*_T\right]=0$ (see (D.24)). Thus, it holds due to Jensen's inequality for conditional expectations and Lemma D.16:

$$\begin{aligned}\operatorname{Var}\left(\mathbb{E}^*\left[T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\alpha,\alpha_\beta}\right]\right)&=\operatorname{Var}\left(\mathbb{E}^*\left[T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\alpha,\alpha_\beta}-\mathbf{Bias}_T^{\text{indep}*}-\mathbb{H}^*_T\right]\right)\\&\leqslant\mathbb{E}\left[\left(\mathbb{E}^*\left[T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\alpha,\alpha_\beta}-\mathbf{Bias}_T^{\text{indep}*}-\mathbb{H}^*_T\right]\right)^2\right]\\&=o(1).\end{aligned}$$

Hence, the law of total variance and the first equality of (D.158) show (recall (D.156) and (D.4)):

$$\begin{aligned}\mathbb{E}\left[\operatorname{Var}^*\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\alpha,\alpha_\beta}\right)\right]&=\operatorname{Var}\left(T\sqrt{b}\,\tilde{\mathfrak{Q}}^*_{T,\alpha,\alpha_\beta}\right)+o(1)\\&=\operatorname{Var}\left(\frac{1}{2}\sum_{t,j=1}^{T}\left(\tilde{\mathfrak{H}}_{T,\Re}(t,j)+\tilde{\mathfrak{H}}_{T,\Im}(t,j)\right)W^*_{t,\{\alpha_\beta\}}W^*_{j,\{\alpha_\beta\}}\right)+o(1).\end{aligned} \tag{D.180}$$

In order to analyze the asymptotic behaviour of the right side of (D.180), one observes at first that (D.165) and Assumption 3.15 $[\mathbf{W}^*]$ (ii) provide:

$$\operatorname{Cov}\left(\frac{1}{2}\sum_{t,j=1}^{T}\tilde{\mathfrak{H}}_{T,\Re}(t,j)\,W^*_{t,\{\alpha_\beta\}}W^*_{j,\{\alpha_\beta\}},\frac{1}{2}\sum_{t,j=1}^{T}\tilde{\mathfrak{H}}_{T,\Im}(t,j)\,W^*_{t,\{\alpha_\beta\}}W^*_{j,\{\alpha_\beta\}}\right)$$

$$=\frac{1}{4}\sum_{t_1,t_2,t_3,t_4=1}^{T}\operatorname{Cov}\left(\tilde{\mathfrak{H}}_{T,\Re}(t_1,t_2),\tilde{\mathfrak{H}}_{T,\Im}(t_3,t_4)\right)\mathbb{E}\left[W^*_{t_1,\{\alpha_\beta\}}W^*_{t_2,\{\alpha_\beta\}}W^*_{t_3,\{\alpha_\beta\}}W^*_{t_4,\{\alpha_\beta\}}\right]$$

$$+\frac{1}{4}\sum_{t_1,t_2,t_3,t_4=1}^{T}\mathbb{E}\left[\tilde{\mathfrak{H}}_{T,\Re}(t_1,t_2)\right]\mathbb{E}\left[\tilde{\mathfrak{H}}_{T,\Im}(t_3,t_4)\right]\operatorname{Cov}\left(W^*_{t_1,\{\alpha_\beta\}}W^*_{t_2,\{\alpha_\beta\}},W^*_{t_3,\{\alpha_\beta\}}W^*_{t_4,\{\alpha_\beta\}}\right)$$

$$=:\mathrm{J}^{\perp\!\!\!\perp *}_{T,1}+\mathrm{J}^{\perp\!\!\!\perp *}_{T,2}. \tag{D.181}$$

Further, define (see (4.11)):

$$\mathrm{J}_{T,1.1}^{\perp\!\!\!\perp *} := \frac{2}{T^2b^3}\sum_{t_1,\dots,t_4=1}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{2b}^{1-2b}\int_{0}^{1} K\left(\frac{\frac{t_1}{T}-u}{b}\right)K\left(\frac{\frac{t_2}{T}-u}{b}\right)K\left(\frac{\frac{t_1}{T}-w}{b}\right)K\left(\frac{\frac{t_2}{T}-w}{b}\right)$$
$$\cdot\,\mathbb{E}\Big[\widetilde{\mathbf{G}}^c_{t_1,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_3,\Im}(u,s_2)\Big]\,\mathbb{E}\Big[\widetilde{\mathbf{G}}^c_{t_2,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_4,\Im}(u,s_2)\Big]\,\mathbb{1}_{\{|t_1-t_3|\leqslant \alpha\}}\mathbb{1}_{\{|t_2-t_4|\leqslant \alpha\}}$$
$$\cdot\,\mathbb{E}\big[W^*_{t_1}W^*_{t_3}\big]\,\mathbb{E}\big[W^*_{t_2}W^*_{t_4}\big]\;dw\,du\,\mathbf{w}\,(s_2)\;ds_2\,\mathbf{w}\,(s_1)\;ds_1, \tag{D.182}$$

$$\mathrm{J}_{T,1.2}^{\perp\!\!\!\perp *} := \frac{2}{T^2b^3}\sum_{t_1,\dots,t_4=1}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\max\{2b,t_1/T-b\}}^{\min\{1-2b,t_1/T+b\}}\int_{\max\{0,t_1/T-b\}}^{\min\{1,t_1/T+b\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right)K\left(\frac{\frac{t_2}{T}-u}{b}\right)K\left(\frac{\frac{t_1}{T}-w}{b}\right)$$
$$\cdot\,K\left(\frac{\frac{t_2}{T}-w}{b}\right)\mathbb{E}\Big[\widetilde{\mathbf{G}}^c_{t_1,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_3,\Im}(u,s_2)\Big]\,\mathbb{E}\Big[\widetilde{\mathbf{G}}^c_{t_2,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_4,\Im}(u,s_2)\Big]\,\mathbb{1}_{\{|t_1-t_3|\leqslant \alpha\}}\mathbb{1}_{\{|t_2-t_4|\leqslant \alpha\}}$$
$$\cdot\left(\mathbb{E}\Big[W^*_{t_1,\{\alpha_\beta\}}W^*_{t_3,\{\alpha_\beta\}}\Big]\,\mathbb{E}\Big[W^*_{t_2,\{\alpha_\beta\}}W^*_{t_4,\{\alpha_\beta\}}\Big]-\mathbb{E}\big[W^*_{t_1}W^*_{t_3}\big]\,\mathbb{E}\big[W^*_{t_2}W^*_{t_4}\big]\right)dw\,du\,\mathbf{w}\,(s_2)\;ds_2$$
$$\cdot\,\mathbf{w}\,(s_1)\;ds_1 \quad \text{and} \tag{D.183}$$

$$\mathrm{J}_{T,1.3}^{\perp\!\!\!\perp *} := \frac{2}{T^2b^3}\sum_{t_1,\dots,t_4=1}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\max\{2b,t_1/T-b\}}^{\min\{1-2b,t_1/T+b\}}\int_{\max\{0,t_1/T-b\}}^{\min\{1,t_1/T+b\}} K\left(\frac{\frac{t_1}{T}-u}{b}\right)K\left(\frac{\frac{t_2}{T}-u}{b}\right)K\left(\frac{\frac{t_1}{T}-w}{b}\right)$$
$$\cdot\,K\left(\frac{\frac{t_2}{T}-w}{b}\right)\mathbb{E}\Big[\widetilde{\mathbf{G}}^c_{t_1,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_3,\Im}(u,s_2)\Big]\,\mathbb{E}\Big[\widetilde{\mathbf{G}}^c_{t_2,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_4,\Im}(u,s_2)\Big]\,\mathbb{1}_{\{|t_1-t_3|\leqslant \alpha\}}\mathbb{1}_{\{|t_2-t_4|\leqslant \alpha\}}$$
$$\cdot\,\mathrm{Cov}\Big(W^*_{t_1,\{\alpha_\beta\}}W^*_{t_3,\{\alpha_\beta\}},W^*_{t_2,\{\alpha_\beta\}}W^*_{t_4,\{\alpha_\beta\}}\Big)\;dw\,du\,\mathbf{w}(s_2)\;ds_2\,\mathbf{w}(s_1)\;ds_1. \tag{D.184}$$

It follows analogously to (D.94), (D.95), (D.97), (D.100), (D.103) and (D.106) (recall also (D.104)) by using Assumption 3.15 $[\mathbf{W}^*]$ (iii), (A.1), Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (ii) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i), whereby the latter yields that the integrals with respect to $w \in [\max\{0, t_1/T-b\}, \min\{1, t_1/T+b\}]$ as well as $u \in [\max\{2b, t_1/T-b\}, \min\{1-2b, t_1/T+b\}]$ contained in $\mathrm{J}_{T,1.2}^{\perp\!\!\!\perp *}$ and $\mathrm{J}_{T,1.3}^{\perp\!\!\!\perp *}$ can be replaced by integrals with respect to $w \in [0,1]$ as well as $u \in [2b, 1-2b]$, respectively (see (D.181)):

$$\mathrm{J}_{T,1}^{\perp\!\!\!\perp *} = \frac{2}{T^2b^3}\sum_{t_1,\dots,t_4=1}^{T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{2b}^{1-2b}\int_{0}^{1} K\left(\frac{\frac{t_1}{T}-u}{b}\right)K\left(\frac{\frac{t_2}{T}-u}{b}\right)K\left(\frac{\frac{t_1}{T}-w}{b}\right)K\left(\frac{\frac{t_2}{T}-w}{b}\right)$$
$$\cdot\,\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{t_1,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_3,\Im}(u,s_2)\right]\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{t_2,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t_4,\Im}(u,s_2)\right]\mathbb{1}_{\{|t_1-t_3|\leqslant \alpha\}}\mathbb{1}_{\{|t_2-t_4|\leqslant \alpha\}}$$
$$\cdot\,\mathbb{E}\left[W^*_{t_1,\{\alpha_\beta\}}W^*_{t_2,\{\alpha_\beta\}}W^*_{t_3,\{\alpha_\beta\}}W^*_{t_4,\{\alpha_\beta\}}\right]\;dw\,du\,\mathbf{w}\,(s_2)\;ds_2\,\mathbf{w}\,(s_1)\;ds_1+o(1)$$
$$= \mathrm{J}_{T,1.1}^{\perp\!\!\!\perp *}+\mathrm{J}_{T,1.2}^{\perp\!\!\!\perp *}+\mathrm{J}_{T,1.3}^{\perp\!\!\!\perp *}+o(1). \tag{D.185}$$

Since $\mathrm{J}_{T,1.1}^{\perp\!\!\!\perp *}$ (recall (D.182)) and $\widetilde{\sigma}_{T,5}^{\perp\!\!\!\perp}$ (see (D.104)) are defined very similarly, one obtains analogously to (D.110), (D.114) as well as (D.117) due to Assumption 3.15 $[\mathbf{W}^*]$ (iii) (note (4.13)):

$$\mathrm{J}_{T,1.1}^{\perp\!\!\!\perp *} = 2\int_0^1\int_{-2}^{2}\left(\int_{-1}^{1}K(q)K(q+v)\,dq\right)^2 dv\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\sum_{t=-\infty}^{\infty}K^*\left(\frac{t}{\beta}\right)\mathbb{E}\left[\widetilde{\mathbf{G}}^c_{0,\Re}(u,s_1)\,\widetilde{\mathbf{G}}^c_{t,\Im}(u,s_2)\right]\right)^2$$
$$\cdot\,\mathbf{w}(s_2)\,ds_2\mathbf{w}(s_1)\,ds_1\,du+o(1),$$

such that Lebesgue's dominated convergence theorem (together with Lemma 4.8 and the Assumptions

4.3 [**WEI.2**] as well as 3.15 [$\mathbf{W}^*$] (i) and (iii)) shows (see (4.13)):

$$\lim_{T\to\infty} \mathrm{J}_{T,1.1}^{\perp\!\!\!\perp *} = 2\int_0^1 \int_{-2}^2 \left(\int_{-1}^1 K(q)K(q+v)\, dq \right)^2 dv \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{Cov}^{\text{indep}}_{\mathfrak{R},\mathfrak{I}}(u, s_1, s_2)^2\, \mathbf{w}(s_2)\, ds_2 \mathbf{w}(s_1)\, ds_1\, du. \tag{D.186}$$

Further, (C.112) in combination with Lemma D.11 (ii) and Assumption 3.15 [$\mathbf{W}^*$] (iii), (A.1) as well as Assumption 4.5 [**K**&**b.2**] (ii) provide (recall (D.183)):

$$\left|\mathrm{J}_{T,1.2}^{\perp\!\!\!\perp *}\right| \leqslant \frac{C}{T^2 b^3} T T \mathscr{a} \mathscr{a} b b \frac{C}{T} = o(1). \tag{D.187}$$

Moreover, if the conditions $r_1, \ldots, r_4 \in \mathbb{Z}$, $|r_1 - r_2| \leqslant \mathscr{a}$ and $|r_3 - r_4| \leqslant \mathscr{a}$ are fulfilled, it will hold (see Definition A.1 (i)):

$$\left|\mathrm{Cov}\left(W^*_{r_1,\{\mathscr{a}_\beta\}} W^*_{r_2,\{\mathscr{a}_\beta\}}, W^*_{r_3,\{\mathscr{a}_\beta\}} W^*_{r_4,\{\mathscr{a}_\beta\}}\right)\right| \mathbb{1}_{\{|r_1 - r_3| > \mathscr{a}_\beta + 2\mathscr{a}\}} = 0. \tag{D.188}$$

One obtains from (D.188) with $r_1 := t_1$, $r_2 := t_3$, $r_3 := t_2$ as well as $r_4 := t_4$, Lemma D.11 (i), Assumption 3.15 [$\mathbf{W}^*$] (i) (the latter ensures $\beta = o(1/b)$), (A.1) and Assumption 4.5 [**K**&**b.2**] (ii) (recall (D.184)):

$$\left|\mathrm{J}_{T,1.3}^{\perp\!\!\!\perp *}\right| \leqslant \frac{C}{T^2 b^3} T \left(\mathscr{a}_\beta + 2\mathscr{a}\right) \mathscr{a}\mathscr{a} b b = o(1). \tag{D.189}$$

Further, (D.9), (D.188) with $r_l := t_l\ \forall\, l \in \{1, \ldots, 4\}$, (D.7), Lemma D.11 (i), Assumption 3.15 [$\mathbf{W}^*$] (i) (which ensures $\beta = o(1/b)$) and (iii), (A.1) as well as Assumption 4.5 [**K**&**b.2**] (ii) imply (see (D.181)):

$$\left|\mathrm{J}_{T,2}^{\perp\!\!\!\perp *}\right| \leqslant C T \mathscr{a} (\mathscr{a}_\beta + 2\mathscr{a})\, \mathscr{a} \frac{1}{T^2 b} = o(1). \tag{D.190}$$

Overall, (D.180), (D.181), (D.185), (D.186), (D.187), (D.189) as well as (D.190) and similar arguments show the third assertion of (D.173) (recall (4.13)).
In order to prove the fourth statement of (D.173), one observes at first for all real-valued random variables X and Y which live on the probability space Ω (that originates from Definition 2.1) and own finite second moments that $|\mathrm{Cov}^*(X - Y, 2Y)| \leqslant \mathrm{Var}^*(X - Y) + \mathrm{Var}^*(2Y)$ is valid, which can be verified similarly to (C.405). This inequality provides $\mathrm{Var}^*(X - Y + 2Y) \leqslant 3\mathrm{Var}^*(X - Y) + 3\mathrm{Var}^*(2Y)$, such that:

$$\begin{aligned}
\mathbb{E}\left[|\mathrm{Var}^*(X) - \mathrm{Var}^*(Y)|\right] &= \mathbb{E}\left[|\mathrm{Cov}^*(X - Y, X + Y)|\right] \\
&\leqslant \mathbb{E}\left[\sqrt{\mathrm{Var}^*(X - Y)}\sqrt{\mathrm{Var}^*(X + Y)}\right] \\
&\leqslant \sqrt{\mathbb{E}\left[\mathrm{Var}^*(X - Y)\right]}\sqrt{\mathbb{E}\left[\mathrm{Var}^*(X - Y + 2Y)\right]} \\
&\leqslant \sqrt{\mathbb{E}\left[\mathbb{E}^*\left[(X - Y)^2\right]\right]}\sqrt{\mathbb{E}\left[3\mathrm{Var}^*(X - Y) + 3 \cdot 4\mathrm{Var}^*(Y)\right]} \\
&\leqslant \sqrt{\mathbb{E}\left[(X - Y)^2\right]}\sqrt{3\mathbb{E}\left[\mathbb{E}^*\left[(X - Y)^2\right]\right] + 3 \cdot 4\mathbb{E}\left[\mathrm{Var}^*(Y)\right]}.
\end{aligned} \tag{D.191}$$

Since $\mathbf{Bias}_T^{\text{indep}*}$ is deterministic (see (4.21)), $\mathrm{Var}^*\big(T\sqrt{b}\,\widetilde{\mathfrak{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta}\big) = \mathrm{Var}^*\big(T\sqrt{b}\,\widetilde{\mathfrak{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta} - \mathbf{Bias}_T^{\text{indep}*}\big)$ holds. Thus, the fourth assertion of (D.173) is an implication of (D.191) with $X = \mathbb{H}_T^*$ as well as $Y = T\sqrt{b}\,\widetilde{\mathfrak{Q}}^*_{T,\mathscr{a},\mathscr{a}_\beta} - \mathbf{Bias}_T^{\text{indep}*}$, Lemma D.16 and the third statement of (D.173) together with $\sigma^{\text{indep}} < \infty$ (whereby the latter follows from Lemma 4.8 as well as Assumption 4.3 [**WEI.2**] by recalling (4.13)).
Overall, all assertions of (D.173) are shown, such that (D.174) proves Lemma D.17. □

E. Supplementary material belonging to the numerical examples

In the first proposition of this appendix, new representations of the statistics and associated bootstrap counterparts that underlie the algorithms evolved in Chapter 3 are introduced. Next, approximations of the statistic and its bootstrap counterpart contained in Algorithm **TEST.INDEP.2**, which equals Algorithm **TEST.INDEP.1** in the case $\mathfrak{D}_1 = \mathfrak{D}_2 = \{0\}$ (as stated in Remark 4.20), are constructed based on Riemann sums. These representations and approximations, respectively, are used in the programming codes that belong to Section 3.4, Section 4.4 as well as Chapter 5.
To improve the readability, the dependence on some parameters (like T) is repressed in several notations introduced in this appendix.

Proposition E.1. *Let the suppositions of Definition 2.1 and the Assumptions 3.1* $[\mathbf{WEI}.\mathbf{1}]$*, 2.8* $[\mathbf{K}\&\mathbf{b}.\mathbf{1}]$ *as well as 3.15* $[\mathbf{W}^*]$ *be fulfilled, whereby the function* $\mathbf{w}$ *that is introduced in Assumption 3.1* $[\mathbf{WEI}.\mathbf{1}]$ *should be even. Define for all* $t, t_1, t_2 \in \{1, \ldots, T\}$*,* $k, k_1, k_2 \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$ *(note that* K_b *originates from Definition 2.11,* $T_{\mathfrak{U}}$ *from (C.17),* u_k *from Definition 3.8 (i) as well as* $\mathfrak{F}\mathbf{w}$ *from (5.2)):*

$$\begin{aligned}
&\mathrm{U_{diff}} := \mathfrak{U}_1 - \mathfrak{U}_0,\ \mathrm{Nr} := \lfloor 1/(2b) \rfloor,\ \mathrm{Ker}(t,k) := K_b\left(\frac{t}{T} - u_k\right),\ \mathrm{Fw}(t_1,t_2) := \mathfrak{F}\mathbf{w}(X_{t_2,T} - X_{t_1,T}),\\
&\mathrm{low}(k) := \max\{1, \lfloor u_k T - T_{\mathfrak{U}} b \rfloor\},\ \ \mathrm{up}(k) := \min\{T, \lceil u_k T + T_{\mathfrak{U}} b \rceil\},\\
&\mathrm{x}_1(t_1,k) := \sum_{t_2=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}(t_2,k)\,\mathrm{Fw}(t_1,t_2),\ \ \mathrm{x}_2(k_1,k_2) := \sum_{t_1=\mathrm{low}(k_1)}^{\mathrm{up}(k_1)} \mathrm{Ker}(t_1,k_1)\,\mathrm{x}_1(t_1,k_2),\\
&\mathrm{x}_3(k) := \mathrm{x}_2(k,k),\ \ \mathrm{x}_4(t_1) := \sum_{k_2=1}^{\mathrm{Nr}} \mathrm{x}_1(t_1,k_2),\ \ \mathrm{x}_5(k) := \sum_{t_1=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}(t_1,k)\,\mathrm{x}_4(t_1),\\
&\mathrm{x}_6(k_1) := \sum_{k_2=1}^{\mathrm{Nr}} \mathrm{x}_2(k_1,k_2),\ \ \mathrm{x}_7(t_1,k) := \mathrm{x}_1(t_1,k) - \frac{1}{T}\mathrm{x}_3(k),\ \ \mathrm{x}_8(t_1,k_1) := \mathrm{x}_4(t_1) - \frac{1}{T}\mathrm{x}_6(k_1),\\
&\mathrm{x}_{\mathrm{B}.1}(k) := \sum_{t_1=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}(t_1,k)\,\mathrm{x}_7(t_1,k)\,W_{t_1}^*,\ \ \mathrm{x}_{\mathrm{B}.2} := \frac{2\mathrm{U_{diff}}}{\mathrm{Nr}}\frac{1}{T^2}\sum_{k=1}^{\mathrm{Nr}} \mathrm{x}_{\mathrm{B}.1}(k),\\
&\mathrm{x}_{\mathrm{B}.3}(k_1) := \sum_{t_1=\mathrm{low}(k_1)}^{\mathrm{up}(k_1)} \mathrm{Ker}(t_1,k_1)\,\mathrm{x}_8(t_1,k_1)\,W_{t_1}^*,\ \ \mathrm{x}_{\mathrm{B}.4} := \frac{2\mathrm{U_{diff}}}{\mathrm{Nr}^2}\frac{1}{T^2}\sum_{k_1=1}^{\mathrm{Nr}} \mathrm{x}_{\mathrm{B}.3}(k_1),\\
&\mathrm{x}_{\mathrm{B}.5}(t_1,k) := \sum_{t_2=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}(t_2,k)\,\mathrm{Fw}(t_1,t_2)\,W_{t_2}^*,\ \ \mathrm{x}_{\mathrm{B}.6}(k) := \sum_{t_1=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}(t_1,k)\,\mathrm{x}_{\mathrm{B}.5}(t_1,k)\,W_{t_1}^*,\\
&\mathrm{x}_{\mathrm{B}.7}(k) := \sum_{j=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}(j,k)\,W_j^* \ \ \textit{and}\ \ \mathrm{x}_{\mathrm{B}.8}(k) := \sum_{t_1=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}(t_1,k)\,\mathrm{x}_1(t_1,k)\,W_{t_1}^*. \qquad \text{(E.1)}
\end{aligned}$$

Then, it holds (recall Definition 3.8 (i), (3.38), Definition 3.17 as well as (3.56)):

(i)

$$\widehat{\mathbb{D}}_{T,1} = \frac{\mathrm{U_{diff}}}{\mathrm{Nr}}\frac{1}{T^2}\sum_{k=1}^{\mathrm{Nr}} \mathrm{x}_3(k) \quad \textit{and} \quad \widehat{\mathbb{D}}_{T,2} = \frac{\mathrm{U_{diff}}}{\mathrm{Nr}^2}\frac{1}{T^2}\sum_{k_1=1}^{\mathrm{Nr}} \mathrm{x}_5(k_1).$$

(ii)

$$\widehat{\mathbb{D}}_T^*((1,-1)) = \mathrm{x}_{\mathrm{B}.2} - \mathrm{x}_{\mathrm{B}.4} \ \ \textit{and if}\ \ \widehat{\mathbb{D}}_{T,1} > 0, \quad \widehat{\mathbb{D}}_T^*\left(\left(\widehat{\gamma}_{T,1}^{\mathrm{norm}}, \widehat{\gamma}_{T,2}^{\mathrm{norm}}\right)\right) = \frac{\widehat{\mathbb{D}}_{T,2}}{\widehat{\mathbb{D}}_{T,1}^2}\mathrm{x}_{\mathrm{B}.2} - \frac{1}{\widehat{\mathbb{D}}_{T,1}}\mathrm{x}_{\mathrm{B}.4}.$$

(iii)

$$\widehat{\mathbb{D}}_{T,\mathrm{Test}}^* = \frac{\mathrm{U_{diff}}}{\mathrm{Nr}}\frac{1}{T^2}\sum_{k=1}^{\mathrm{Nr}}\left(\mathrm{x}_{\mathrm{B}.6}(k) + \left(\frac{1}{T}\mathrm{x}_{\mathrm{B}.7}(k)\right)^2 \mathrm{x}_3(k) - \frac{2}{T}\mathrm{x}_{\mathrm{B}.7}(k)\mathrm{x}_{\mathrm{B}.8}(k)\right).$$

(iv) *In addition, suppose that Assumption 3.30* $[\mathbf{NW}]$ *and* $T \geqslant 2\mathbf{B}_T + 1$ *are fulfilled. Moreover, define for all* $t \in \{1+\mathbf{B}_T, \ldots, T-\mathbf{B}_T\}$, $h \in \{1, \ldots, 2\mathbf{B}_T+1\}$ *(note that* K^* *originates from Assumption 3.15* $[\mathbf{W}^*]$*):*

$$\mathrm{Ker.B.NW}(h) := \left(K^*\left(\frac{h-\mathbf{B}_T-1}{\beta}\right)-1\right)\mathbb{K}_{\mathbf{NW}}\left(\frac{h-\mathbf{B}_T-1}{\mathbf{B}_T}\right),$$

$$\mathrm{low.NW}(k) := \max\left\{1+\mathbf{B}_T, \lfloor u_k T - T_{\mathfrak{U}} b\rfloor\right\}, \quad \mathrm{up.NW}(k) := \min\left\{T-\mathbf{B}_T, 1+\mathbf{B}_T+\lceil u_k T + T_{\mathfrak{U}} b\rceil\right\},$$

$$\mathrm{x_{NW.1}}(t) := \sum_{h=1}^{2\mathbf{B}_T+1}\mathrm{Ker.B.NW}(h)\mathrm{Fw}(t, t+h-\mathbf{B}_T-1),$$

$$\mathrm{x_{NW.2}} := \sum_{h=1}^{2\mathbf{B}_T+1}\mathrm{Ker.B.NW}(h), \quad \mathrm{x_{NW.3}}(t,k) := \sum_{h=1}^{2\mathbf{B}_T+1}\mathrm{Ker.B.NW}(h)\mathrm{x}_1\left(t+h-\mathbf{B}_T-1, k\right) \quad \text{and}$$

$$\mathrm{x_{NW.4}}(k) := \sum_{t=\mathrm{low.NW}(k)}^{\mathrm{up.NW}(k)}\mathrm{Ker}(t,k)\left(\mathrm{x_{NW.1}}(t) - \frac{1}{T}\mathrm{x_{NW.2}}\cdot \mathrm{x}_7(t,k) - \frac{1}{T}\mathrm{x_{NW.3}}(t,k)\right). \tag{E.2}$$

Then, one obtains (see (3.68)):

$$\widehat{\mathbf{Bias}}_T^{\text{error}} = \frac{1}{\sqrt{b}}\int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\,dz\cdot\frac{\mathrm{U_{diff}}}{\mathrm{Nr}}\frac{1}{T-2\mathbf{B}_T}\sum_{k=1}^{\mathrm{Nr}}\mathrm{x_{NW.4}}(k).$$

Remark E.2. *The suppositions of Proposition E.1 which concern the weight function* $\mathbf{w}$ *are fulfilled by the weight functions that are used for the numerical examples contained in the present work which belong to the algorithms evolved in Chapter 3.*

Proof of Proposition E.1. (i) Proposition E.1 (i) follows from (5.3) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i), whereby the latter yields $K(z) = 0$ for all $z \in \mathbb{R}$ with $|z| > \mathfrak{U}_1 - \mathfrak{U}_0$ (recall also (E.1)).

(ii) In order to prove Proposition E.1 (ii), observe at first that one obtains for all $\mathrm{R} \in \{\Re, \Im\}$, $s \in \mathbb{R}^d$ by interchanging k_1 and k_2 in the definition of $\widehat{\mathbb{D}}^*_{T,2,\mathrm{R}}(s)$ (see (3.38) as well as Definition 2.11):

$$\begin{aligned}\widehat{\mathbb{D}}^*_{T,2,\mathrm{R}}(s) &= \frac{2\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor^2}\sum_{k_2=1}^{\lfloor 1/(2b)\rfloor}\frac{1}{T}\sum_{t_2=1}^{T}K_b\!\left(\frac{t_2}{T}-u_{k_2}\right)\mathrm{R}\left\{e^{i\left\langle s, X_{t_2,T}\right\rangle}\right\}\sum_{k_1=1}^{\lfloor 1/(2b)\rfloor}\left(\frac{1}{T}\sum_{t_1=1}^{T}K_b\!\left(\frac{t_1}{T}-u_{k_1}\right)\right.\\ &\left.\cdot\,\mathrm{R}\left\{e^{i\left\langle s, X_{t_1,T}\right\rangle}\right\}W^*_{t_1} - \frac{1}{T}\sum_{t_1=1}^{T}K_b\!\left(\frac{t_1}{T}-u_{k_1}\right)\frac{1}{T}\sum_{t_3=1}^{T}K_b\!\left(\frac{t_3}{T}-u_{k_1}\right)\mathrm{R}\left\{e^{i\left\langle s, X_{t_3,T}\right\rangle}\right\}W^*_{t_1}\right)\\ &= \frac{2\left(\mathfrak{U}_1-\mathfrak{U}_0\right)}{\lfloor 1/(2b)\rfloor^2}\frac{1}{T^2}\sum_{k_1=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_1=1}^{T}K_b\!\left(\frac{t_1}{T}-u_{k_1}\right)\left(\sum_{k_2=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_2=1}^{T}K_b\!\left(\frac{t_2}{T}-u_{k_2}\right)\mathrm{R}\left\{e^{i\left\langle s, X_{t_1,T}\right\rangle}\right\}\mathrm{R}\left\{e^{i\left\langle s, X_{t_2,T}\right\rangle}\right\}\right.\\ &\left.-\frac{1}{T}\sum_{k_2=1}^{\lfloor 1/(2b)\rfloor}\sum_{t_2,t_3=1}^{T}K_b\left(\frac{t_2}{T}-u_{k_2}\right)K_b\left(\frac{t_3}{T}-u_{k_1}\right)\mathrm{R}\left\{e^{i\left\langle s, X_{t_3,T}\right\rangle}\right\}\mathrm{R}\left\{e^{i\left\langle s, X_{t_2,T}\right\rangle}\right\}\right)W^*_{t_1}.\end{aligned} \tag{E.3}$$

Further, since the Fourier transform of an even integrable real-valued function is an even real-valued function, the assumptions on $\mathbf{w}$ which are supposed in Proposition E.1 provide (note (E.1) as well as (5.2)):

$$\mathrm{Fw}(t_1,t_2) \in \mathbb{R} \quad\text{and}\quad \mathrm{Fw}(t_1,t_2) = \mathrm{Fw}(t_2,t_1)\ \forall\, t_1,t_2 \in \{1,\ldots,T\}. \tag{E.4}$$

It follows from (C.147), Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) and (E.4) (recall (5.2) as well as (E.1)):

$$\begin{aligned}&\int_{\mathbb{R}^d}\widehat{\mathbb{D}}^*_{T,2,\Re}(s) + \widehat{\mathbb{D}}^*_{T,2,\Im}(s)\,\mathbf{w}(s)\,ds\\ &= \frac{2\mathrm{U_{diff}}}{\mathrm{Nr}^2}\frac{1}{T^2}\sum_{k_1=1}^{\mathrm{Nr}}\sum_{t_1=\mathrm{low}(k_1)}^{\mathrm{up}(k_1)}\mathrm{Ker}\left(t_1,k_1\right)\left(\sum_{k_2=1}^{\mathrm{Nr}}\sum_{t_2=\mathrm{low}(k_2)}^{\mathrm{up}(k_2)}\mathrm{Ker}\left(t_2,k_2\right)\mathrm{Fw}\left(t_1,t_2\right)\right.\end{aligned}$$

$$-\frac{1}{T}\sum_{k_2=1}^{\mathrm{Nr}}\sum_{t_3=\mathrm{low}(k_1)}^{\mathrm{up}(k_1)}\mathrm{Ker}\,(t_3,k_1)\sum_{t_2=\mathrm{low}(k_2)}^{\mathrm{up}(k_2)}\mathrm{Ker}\,(t_2,k_2)\,\mathrm{Fw}\,(t_3,t_2)\Bigg)W^*_{t_1}$$

$$=\frac{2\mathrm{U_{diff}}}{\mathrm{Nr}^2}\frac{1}{T^2}\sum_{k_1=1}^{\mathrm{Nr}}\sum_{t_1=\mathrm{low}(k_1)}^{\mathrm{up}(k_1)}\mathrm{Ker}\,(t_1,k_1)\left(\mathrm{x}_4\,(t_1)-\frac{1}{T}\sum_{k_2=1}^{\mathrm{Nr}}\mathrm{x}_2\,(k_1,k_2)\right)W^*_{t_1}$$

$$=\frac{2\mathrm{U_{diff}}}{\mathrm{Nr}^2}\frac{1}{T^2}\sum_{k_1=1}^{\mathrm{Nr}}\sum_{t_1=\mathrm{low}(k_1)}^{\mathrm{up}(k_1)}\mathrm{Ker}\,(t_1,k_1)\,\mathrm{x}_8\,(t_1,k_1)\,W^*_{t_1}$$

$$=\mathrm{x_{B.4}}. \tag{E.5}$$

Further, observe for all $\mathrm{R}\in\{\Re,\Im\}$, $s\in\mathbb{R}^d$ (see (3.38) and Definition 2.11):

$$\widehat{\mathbb{D}}^*_{T,1,\mathrm{R}}(s)=\frac{2\,(\mathfrak{U}_1-\mathfrak{U}_0)}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\frac{1}{T}\sum_{t_2=1}^{T}K_b\Big(\frac{t_2}{T}-u_k\Big)\,\mathrm{R}\left\{e^{i\langle s,X_{t_2,T}\rangle}\right\}\left(\frac{1}{T}\sum_{t_1=1}^{T}K_b\Big(\frac{t_1}{T}-u_k\Big)\right.$$
$$\left.\cdot\,\mathrm{R}\left\{e^{i\langle s,X_{t_1,T}\rangle}\right\}W^*_{t_1}-\frac{1}{T}\sum_{t_1=1}^{T}K_b\Big(\frac{t_1}{T}-u_k\Big)\frac{1}{T}\sum_{t_3=1}^{T}K_b\Big(\frac{t_3}{T}-u_k\Big)\,\mathrm{R}\left\{e^{i\langle s,X_{t_3,T}\rangle}\right\}W^*_{t_1}\right). \tag{E.6}$$

Note that the right side of (E.6) is very similar to the right side of the first equality of (E.3). Hence, one obtains analogously to (E.3) and (E.5) (recall (E.1)):

$$\int_{\mathbb{R}^d}\widehat{\mathbb{D}}^*_{T,1,\Re}(s)+\widehat{\mathbb{D}}^*_{T,1,\Im}(s)\,\mathbf{w}(s)\,ds=\mathrm{x_{B.2}}. \tag{E.7}$$

Since (3.38) provides for all $\gamma:=\left(\gamma^{[1]},\gamma^{[2]}\right)$ with arbitrary real-valued random variables $\gamma^{[1]},\gamma^{[2]}$:

$$\widehat{\mathbb{D}}^*_T\,(\gamma)=\gamma^{[1]}\int_{\mathbb{R}^d}\widehat{\mathbb{D}}^*_{T,1,\Re}(s)+\widehat{\mathbb{D}}^*_{T,1,\Im}(s)\,\mathbf{w}(s)\,ds+\gamma^{[2]}\int_{\mathbb{R}^d}\widehat{\mathbb{D}}^*_{T,2,\Re}(s)+\widehat{\mathbb{D}}^*_{T,2,\Im}(s)\,\mathbf{w}(s)\,ds,$$

(E.7) and (E.5) imply Proposition E.1 (ii) (see Definition 3.17 (ii)).

(iii) In order to prove Proposition E.1 (iii), observe at first for all $\mathrm{R}\in\{\Re,\Im\}$, $s\in\mathbb{R}^d$ (recall Definition 2.11):

$$\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\mathrm{R}\left\{\frac{1}{T}\sum_{t=1}^{T}K_b\left(\frac{t}{T}-u_k\right)\left(e^{i\langle s,X_{t,T}\rangle}-\widehat{\varphi}\,(u_k,s)\right)W^*_t\right\}^2$$
$$=\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\left(\frac{1}{T}\sum_{t_1=1}^{T}K_b\left(\frac{t_1}{T}-u_k\right)\mathrm{R}\left\{e^{i\langle s,X_{t_1,T}\rangle}\right\}W^*_{t_1}\right.$$
$$\left.-\frac{1}{T}\sum_{t_2=1}^{T}K_b\left(\frac{t_2}{T}-u_k\right)W^*_{t_2}\frac{1}{T}\sum_{t_3=1}^{T}K_b\left(\frac{t_3}{T}-u_k\right)\mathrm{R}\left\{e^{i\langle s,X_{t_3,T}\rangle}\right\}\right)^2$$
$$=\frac{\mathfrak{U}_1-\mathfrak{U}_0}{\lfloor 1/(2b)\rfloor}\sum_{k=1}^{\lfloor 1/(2b)\rfloor}\Bigg(\frac{1}{T^2}\sum_{t_1,t_1'=1}^{T}K_b\Big(\frac{t_1}{T}-u_k\Big)\,K_b\Big(\frac{t_1'}{T}-u_k\Big)\,\mathrm{R}\left\{e^{i\langle s,X_{t_1,T}\rangle}\right\}\mathrm{R}\left\{e^{i\langle s,X_{t_1',T}\rangle}\right\}W^*_{t_1}W^*_{t_1'}$$
$$+\left(\frac{1}{T}\sum_{t_2=1}^{T}K_b\left(\frac{t_2}{T}-u_k\right)W^*_{t_2}\right)^2\frac{1}{T^2}\sum_{t_3,t_3'=1}^{T}K_b\Big(\frac{t_3}{T}-u_k\Big)\,K_b\Big(\frac{t_3'}{T}-u_k\Big)\,\mathrm{R}\left\{e^{i\langle s,X_{t_3,T}\rangle}\right\}\mathrm{R}\left\{e^{i\langle s,X_{t_3',T}\rangle}\right\}$$
$$-2\frac{1}{T}\sum_{t_2=1}^{T}K_b\Big(\frac{t_2}{T}-u_k\Big)W^*_{t_2}\frac{1}{T^2}\sum_{t_1,t_3=1}^{T}K_b\Big(\frac{t_1}{T}-u_k\Big)\,K_b\Big(\frac{t_3}{T}-u_k\Big)\,\mathrm{R}\left\{e^{i\langle s,X_{t_1,T}\rangle}\right\}\mathrm{R}\left\{e^{i\langle s,X_{t_3,T}\rangle}\right\}W^*_{t_1}\Bigg).$$
$$\tag{E.8}$$

One obtains from $|z|^2=\Re\{z\}^2+\Im\{z\}^2\;\forall\,z\in\mathbb{C}$, (E.8), (C.147), (E.4) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) (see (3.56), (5.2) as well as (E.1)):

$$\widehat{\mathbb{D}}^{*}_{T,\mathrm{Test}} = \frac{\mathrm{U_{diff}}}{\mathrm{Nr}} \frac{1}{T^2} \sum_{k=1}^{\mathrm{Nr}} \Bigg(\sum_{t_1=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}\,(t_1,k) \sum_{t_2=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}\,(t_2,k)\,\mathrm{Fw}\,(t_1,t_2)\, W^{*}_{t_2} W^{*}_{t_1}$$
$$+ \left(\frac{1}{T} \sum_{j=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}\,(j,k)\, W^{*}_{j} \right)^2 \sum_{t_1=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}\,(t_1,k) \sum_{t_2=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}\,(t_2,k)\,\mathrm{Fw}\,(t_1,t_2)$$
$$- \frac{2}{T} \sum_{j=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}\,(j,k) W^{*}_{j} \sum_{t_1=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}\,(t_1,k) \sum_{t_2=\mathrm{low}(k)}^{\mathrm{up}(k)} \mathrm{Ker}\,(t_2,k)\,\mathrm{Fw}\,(t_1,t_2)\, W^{*}_{t_1} \Bigg), \tag{E.9}$$

which implies Proposition E.1 (iii) by recalling (E.1).

(iv) In order to prove Proposition E.1 (iv), note at first that it holds for all $s \in \mathbb{R}^d$, $t \in \{1+\mathbf{B}_T, \ldots, T-\mathbf{B}_T\}$, $k \in \{1, \ldots, \lfloor 1/(2b) \rfloor\}$, $h \in \{-\mathbf{B}_T, \ldots, \mathbf{B}_T\}$ due to (E.4) and Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) (see (3.66) as well as (E.1)):

$$\Re\left\{ \int_{\mathbb{R}^d} \left(e^{i\langle s, X_{t,T}\rangle}\right)^{\widehat{c}(u_k)} \overline{\left(e^{i\langle s, X_{t+h,T}\rangle}\right)^{\widehat{c}(u_k)}}\, \mathbf{w}(s)\, ds \right\}$$
$$= \Re\left\{ \int_{\mathbb{R}^d} e^{i\langle s, X_{t,T} - X_{t+h,T}\rangle}\, \mathbf{w}(s)\, ds \right\} - \frac{1}{T} \sum_{t_2=1}^{T} K_b\left(\frac{t_2}{T} - u_k\right) \Re\left\{ \int_{\mathbb{R}^d} e^{i\langle s, X_{t,T} - X_{t_2,T}\rangle}\, \mathbf{w}(s)\, ds \right\}$$
$$- \frac{1}{T} \sum_{t_2=1}^{T} K_b\left(\frac{t_2}{T} - u_k\right) \Re\left\{ \int_{\mathbb{R}^d} e^{i\langle s, X_{t_2,T} - X_{t+h,T}\rangle}\, \mathbf{w}(s)\, ds \right\}$$
$$+ \frac{1}{T^2} \sum_{t_1=1}^{T} K_b\left(\frac{t_1}{T} - u_k\right) \sum_{t_2=1}^{T} K_b\left(\frac{t_2}{T} - u_k\right) \Re\left\{ \int_{\mathbb{R}^d} e^{i\langle s, X_{t_1,T} - X_{t_2,T}\rangle}\, \mathbf{w}(s)\, ds \right\}$$
$$= \mathrm{Fw}(t, t+h) - \frac{1}{T}\mathrm{x}_1(t,k) - \frac{1}{T}\mathrm{x}_1(t+h,k) + \frac{1}{T^2}\mathrm{x}_3(k). \tag{E.10}$$

It follows from (E.10), Assumption 2.8 $[\mathbf{K}\&\mathbf{b.1}]$ (i) and shifting the index of a sum (recall (3.68) as well as (C.17)):

$$\widehat{\mathbf{Bias}}^{\mathrm{error}}_{T} = \frac{1}{\sqrt{b}} \int_{\mathfrak{U}_0-\mathfrak{U}_1}^{\mathfrak{U}_1-\mathfrak{U}_0} K(z)^2\, dz \cdot \frac{\mathfrak{U}_1 - \mathfrak{U}_0}{\lfloor 1/(2b) \rfloor} \sum_{k=1}^{\lfloor 1/(2b) \rfloor} \frac{1}{T - 2\mathbf{B}_T} \sum_{t=\max\{1+\mathbf{B}_T, \lfloor u_k T - T_{\mathfrak{U}} b \rfloor\}}^{\min\{T-\mathbf{B}_T, 1+\mathbf{B}_T+\lceil u_k T + T_{\mathfrak{U}} b \rceil\}} K_b\left(\frac{t}{T} - u_k\right)$$
$$\cdot \sum_{h=1}^{2\mathbf{B}_T+1} \left(K^*\left(\frac{h - \mathbf{B}_T - 1}{\beta}\right) - 1 \right) \mathbb{K}_{\mathbf{NW}}\left(\frac{h - \mathbf{B}_T - 1}{\mathbf{B}_T}\right) \Bigg(\mathrm{Fw}(t, t+h-\mathbf{B}_T-1)$$
$$- \frac{1}{T}\left(\mathrm{x}_1(t,k) - \frac{1}{T}\mathrm{x}_3(k)\right) - \frac{1}{T}\mathrm{x}_1(t+h-\mathbf{B}_T-1, k) \Bigg),$$

which shows Proposition E.1 (iv) (see (E.2)). □

Lemma E.3. *Assume that $N \in \mathbb{N}$ and that $x, y \in \mathbb{R}$ are deterministic, whereby $x < y$ holds. In addition, $f\colon [x,y] \to \mathbb{R}$ should be a random function which fulfils for all deterministic $q_1, q_2 \in [x,y]$ and a fixed $L_f \in [0,\infty)$:*

$$\sup_{q \in [x,y]} \mathbb{E}\left[|f(q)|\right] < \infty \quad \textit{as well as} \quad \mathbb{E}\left[|f(q_1) - f(q_2)|\right] \leqslant L_f\, |q_1 - q_2|\,. \tag{E.11}$$

Moreover, let z_k with $k \in \{0, \ldots, N\}$ originate from (B.48). Then, it holds:

$$\mathbb{E}\left[\left| \int_x^y f(q)\, dq - \frac{y-x}{N} \sum_{k=1}^{N} f(z_k) \right|\right] \leqslant \frac{L_f |y-x|^2}{N}.$$

Proof. One obtains similarly to [77, Walter (2007), p. 233 et seq.] by using (E.11) (recall (B.48)):

$$\mathbb{E}\left[\left|\int_x^y f(q)\,dq - \frac{y-x}{N}\sum_{k=1}^{N} f(z_k)\right|\right] = \mathbb{E}\left[\left|\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k} f(q)\,dq - \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k} f(z_k)\,dq\right|\right]$$
$$\leqslant \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}\mathbb{E}\left[|f(q)-f(z_k)|\right]dq$$
$$\leqslant N\Delta z L_f \Delta z,$$

which proves Lemma E.3 due to $\Delta z := (y-x)/N$ (see (B.48)). □

Proposition E.4. *Suppose that $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbb{N}_0$ are arbitrary, non-empty, fixed as well as finite sets and that $T \geqslant 1 + \mathfrak{D}_{\max}$ (recall (4.28)). In addition, let the Assumptions 4.1 $[\mathbf{INDEP}]$, 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ and 3.15 $[\mathbf{W}^*]$ be fulfilled. Moreover, assume for $d_1, d_2 \in \mathbb{N}$ introduced in Assumption 4.1 $[\mathbf{INDEP}]$, $\delta \in (0,1]$ originating from Assumption 2.2 $[\mathbf{StAp}]$ as well as each $l \in \{1,2\}$ that $\mathbf{w}^{[l]}\colon \mathbb{R}^{d_l\cdot\#\mathfrak{D}_l} \to [0,\infty)$ is a Riemann integrable function which is Lebesgue almost everywhere positive and fulfils:*

$$\int_{\mathbb{R}^{d_l\cdot\#\mathfrak{D}_l}}\left(1+\left|s^{[l]}\right|_1^{2+2\delta}+\left|s^{[l]}\right|_1^{3}\right)\mathbf{w}^{[l]}\left(s^{[l]}\right)ds^{[l]}<\infty \text{ as well as } \mathbf{w}^{[l]}\left(s^{[l]}\right)=\mathbf{w}^{[l]}\left(-s^{[l]}\right)\ \forall\, s^{[l]}\in\mathbb{R}^{d_l\cdot\#\mathfrak{D}_l}. \tag{E.12}$$

Further, define for all $N \in \mathbb{N}$, $k \in \{1,\ldots,N\}$, $t, t_1, t_2 \in \{1+\mathfrak{D}_{\max},\ldots,T\}$, $l \in \{1,2\}$ (see Definition 2.11, (5.2), (4.4) as well as (4.28)):

$$\begin{aligned}
&\mathrm{v}_{k.N} := b + k\frac{1-2b}{N}, \quad \mathrm{KER}_N(t,k) := K_b\left(\frac{t-\mathfrak{D}_{\text{mean}}}{T} - \mathrm{v}_{k.N}\right),\\
&\mathrm{Fw}_l(t_1,t_2) := \mathfrak{F}\mathbf{w}^{[l]}\left(X^{[l]}_{\mathfrak{D}_l,t_2,T} - X^{[l]}_{\mathfrak{D}_l,t_1,T}\right),\\
&\mathrm{LOW}_N(k) := \max\left\{1+\mathfrak{D}_{\max}, \lfloor \mathrm{v}_{k.N}T + \mathfrak{D}_{\text{mean}} - Tb\rfloor\right\},\\
&\mathrm{UP}_N(k) := \min\left\{T, 1+\mathfrak{D}_{\max} + \lceil \mathrm{v}_{k.N}T + \mathfrak{D}_{\text{mean}} + Tb\rceil\right\},\\
&\mathrm{y}_N(t,k) := \sum_{j=\mathrm{LOW}_N(k)}^{\mathrm{UP}_N(k)} \mathrm{KER}_N(j,k)\,\mathrm{Fw}_1(t,j)\mathrm{Fw}_2(t,j), \quad \mathrm{y}_{N.l}(t,k) := \sum_{j=\mathrm{LOW}_N(k)}^{\mathrm{UP}_N(k)} \mathrm{KER}_N(j,k)\,\mathrm{Fw}_l(t,j),\\
&Y_{N.1}(k) := \sum_{t=\mathrm{LOW}_N(k)}^{\mathrm{UP}_N(k)} \mathrm{KER}_N(t,k)\,\mathrm{y}_N(t,k), \quad Y_{N.2}(k) := \sum_{t=\mathrm{LOW}_N(k)}^{\mathrm{UP}_N(k)} \mathrm{KER}_N(t,k)\,\mathrm{y}_{N.1}(t,k)\mathrm{y}_{N.2}(t,k)\\
&\textit{and}\quad Y_{N.3.l}(k) := \sum_{t=\mathrm{LOW}_N(k)}^{\mathrm{UP}_N(k)} \mathrm{KER}_N(t,k)\,\mathrm{y}_{N.l}(t,k).
\end{aligned} \tag{E.13}$$

(i) Let $(d_T)_{T\in\mathbb{N}}$ be an arbitrary sequence of natural numbers for which $T/\left(\sqrt{b}d_T\right) \overset{T\to\infty}{\longrightarrow} 0$ holds and define:

$$\widehat{Q}_{T,\text{apprx}} := \frac{1-2b}{d_T}\sum_{k=1}^{d_T}\left(\frac{1}{T^2}Y_{d_T.1}(k) - \frac{2}{T^3}Y_{d_T.2}(k) + \frac{1}{T^4}Y_{d_T.3.1}(k)Y_{d_T.3.2}(k)\right). \tag{E.14}$$

In addition, $\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ should originate from (4.28), whereby $\mathbf{w}\colon \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1}\times\mathbb{R}^{d_2\cdot\#\mathfrak{D}_2} \to [0,\infty)$, defined as $\mathbf{w}(s) := \mathbf{w}^{[1]}\left(s^{[1]}\right)\mathbf{w}^{[2]}\left(s^{[2]}\right)\ \forall\, s := \left(s^{[1]\prime}, s^{[2]\prime}\right)^\prime \in \mathbb{R}^{d_1\cdot\#\mathfrak{D}_1}\times\mathbb{R}^{d_2\cdot\#\mathfrak{D}_2}$, is the underlying weight function. Then, one obtains for $T\to\infty$:

$$T\sqrt{b}\,\mathbb{E}\left[\left|\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \widehat{Q}_{T,\text{apprx}}\right|\right] = o(1).$$

(ii) Assume that $(e_T)_{T\in\mathbb{N}}$ is an arbitrary sequence of natural numbers with $T\sqrt{b}/e_T \overset{T\to\infty}{\longrightarrow} E \in [0,\infty)$

and define for all $k \in \{1, \ldots, e_T\}$, $t, p \in \{1, \ldots, T\}$:

$$\begin{aligned}
\mathrm{y}_{\mathrm{B}.1}(k) &:= \frac{4}{T^4} Y_{e_T.3.1}(k) Y_{e_T.3.2}(k) - \frac{4}{T^3} Y_{e_T.2}(k) + \frac{1}{T^2} Y_{e_T.1}(k), \\
\mathrm{y}_{\mathrm{B}.2}(t,k) &:= \left(\frac{2}{T^2} \mathrm{y}_{e_T.1}(t,k) - \frac{2}{T^3} \mathrm{Y}_{e_T.3.1}(k) \right) \mathrm{y}_{e_T.2}(t,k) - \frac{2}{T^3} \mathrm{y}_{e_T.1}(t,k) \mathrm{Y}_{e_T.3.2}(k) \\
&\quad - \frac{1}{T} \mathrm{y}_{e_T}(t,k) + \frac{1}{T^2} \sum_{j=\mathrm{LOW}_{e_T}(k)}^{\mathrm{UP}_{e_T}(k)} \mathrm{KER}_{e_T}(j,k) \left(\mathrm{y}_{e_T.1}(j,k) \mathrm{Fw}_2(j,t) + \mathrm{Fw}_1(j,t) \mathrm{y}_{e_T.2}(j,k) \right), \\
\mathrm{y}_{\mathrm{B}.3}(t,p,k) &:= \mathrm{Fw}_1(p,t) \mathrm{Fw}_2(p,t) + \frac{1}{T} \Big(\Big(- \mathrm{y}_{e_T.1}(p,k) - \mathrm{y}_{e_T.1}(t,k) + \frac{1}{T} \mathrm{Y}_{e_T.3.1}(k) \Big) \mathrm{Fw}_2(p,t) \\
&\quad + \mathrm{Fw}_1(p,t) \Big(- \mathrm{y}_{e_T.2}(p,k) - \mathrm{y}_{e_T.2}(t,k) + \frac{1}{T} \mathrm{Y}_{e_T.3.2}(k) \Big) \Big) \\
&\quad + \frac{1}{T^2} \big(\mathrm{y}_{e_T.1}(t,k) \mathrm{y}_{e_T.2}(p,k) + \mathrm{y}_{e_T.1}(p,k) \mathrm{y}_{e_T.2}(t,k) \big)
\end{aligned} \tag{E.15}$$

as well as:

$$\mathrm{Y}_{\mathrm{B}.1}(k) := \sum_{p=\mathrm{LOW}_{e_T}(k)}^{\mathrm{UP}_{e_T}(k)} \mathrm{KER}_{e_T}(p,k) W_p^*, \quad \mathrm{Y}_{\mathrm{B}.2}(k) := \sum_{t=\mathrm{LOW}_{e_T}(k)}^{\mathrm{UP}_{e_T}(k)} \mathrm{KER}_{e_T}(t,k) W_t^* \mathrm{y}_{\mathrm{B}.2}(t,k),$$

$$\mathrm{Y}_{\mathrm{B}.3.1}(p,k) := \sum_{t=\mathrm{LOW}_{e_T}(k)}^{\mathrm{UP}_{e_T}(k)} \mathrm{KER}_{e_T}(t,k) W_t^* \mathrm{y}_{\mathrm{B}.3}(t,p,k),$$

$$\mathrm{Y}_{\mathrm{B}.3.2}(k) := \sum_{p=\mathrm{LOW}_{e_T}(k)}^{\mathrm{UP}_{e_T}(k)} \mathrm{KER}_{e_T}(p,k) W_p^* \mathrm{Y}_{\mathrm{B}.3.1}(p,k)$$

$$\text{and } \widehat{Q}^*_{T,\mathrm{apprx}} := \frac{1-2b}{e_T} \frac{1}{T^2} \sum_{k=1}^{e_T} \left(\mathrm{Y}_{\mathrm{B}.1}(k)^2 \, \mathrm{y}_{\mathrm{B}.1}(k) + 2 \mathrm{Y}_{\mathrm{B}.1}(k) \mathrm{Y}_{\mathrm{B}.2}(k) + \mathrm{Y}_{\mathrm{B}.3.2}(k) \right). \tag{E.16}$$

In addition, $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ should originate from (4.29), whereby $\mathbf{w}\colon \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2} \to [0,\infty)$, defined as $\mathbf{w}(s) := \mathbf{w}^{[1]}\big(s^{[1]}\big) \mathbf{w}^{[2]}\big(s^{[2]}\big) \; \forall \, s := \Big(s^{[1]\prime}, s^{[2]\prime} \Big)' \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}$, is the underlying weight function. Then, one obtains for $T \to \infty$:

$$T\sqrt{b}\, \mathbb{E}\left[\left| \widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \widehat{Q}^*_{T,\mathrm{apprx}} \right| \right] = o(1).$$

Remark E.5. *The suppositions of Proposition E.4 which concern the weight function $\mathbf{w}$ are fulfilled by the weight functions that are used for the numerical examples contained in the present work which belong to the algorithms evolved in Chapter 4. In addition, note also that these suppositions ensure that $\mathbf{w}$ fulfils Assumption 4.17* [**WEI.3**]*, which follows from some straightforward arguments and by using that $(x_1 + x_2)^y \leqslant \max\left((2x_1)^y, (2x_2)^y\right) \leqslant 2^y \left(x_1^y + x_2^y\right) \; \forall \, x_1, x_2 \geqslant 0, \; y > 0$ yields for all $s := \Big(s^{[1]\prime}, s^{[2]\prime} \Big)' \in \mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}$:*

$$\begin{aligned}
&\int_{\mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1 + d_2 \cdot \#\mathfrak{D}_2}} \Big(1 + |s|_1^{2+2\delta} + |s|_1^3 \Big) \mathbf{w}(s) \, ds \\
&\leqslant C \int_{\mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1}} \int_{\mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}} \left(1 + \left| s^{[1]} \right|_1^{2+2\delta} + \left| s^{[2]} \right|_1^{2+2\delta} + \left| s^{[1]} \right|_1^3 + \left| s^{[2]} \right|_1^3 \right) \mathbf{w}^{[1]}\Big(s^{[1]}\Big) \mathbf{w}^{[2]}\Big(s^{[2]}\Big) \, ds^{[2]} \, ds^{[1]} \\
&\leqslant C \int_{\mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1}} \left(1 + \left| s^{[1]} \right|_1^{2+2\delta} + \left| s^{[1]} \right|_1^3 \right) \mathbf{w}^{[1]}\Big(s^{[1]}\Big) \, ds^{[1]} \int_{\mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}} \mathbf{w}^{[2]}\Big(s^{[2]}\Big) \, ds^{[2]} \\
&+ C \int_{\mathbb{R}^{d_1 \cdot \#\mathfrak{D}_1}} \mathbf{w}^{[1]}\Big(s^{[1]}\Big) \, ds^{[1]} \int_{\mathbb{R}^{d_2 \cdot \#\mathfrak{D}_2}} \left(1 + \left| s^{[2]} \right|_1^{2+2\delta} + \left| s^{[2]} \right|_1^3 \right) \mathbf{w}^{[2]}\Big(s^{[2]}\Big) \, ds^{[2]}.
\end{aligned}$$

Proof of Proposition E.4. (i) Proposition E.4 (i) will be proved by using Lemma E.3. Therefore, some considerations that allow to apply Lemma E.3 are stated next.
Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) implies that $w \in [(t-\mathfrak{D}_{\text{mean}})/T - b, (t-\mathfrak{D}_{\text{mean}})/T + b]$ and $t \in [\lfloor wT + \mathfrak{D}_{\text{mean}} \rfloor - \lfloor Tb \rfloor - 1, \lfloor wT + \mathfrak{D}_{\text{mean}} \rfloor + \lfloor Tb \rfloor + 2]$ are necessary to ensure $K(((t-\mathfrak{D}_{\text{mean}})/T - w)/b) \neq 0$. Moreover, $|u-v| \leqslant 2b$ holds in the case $u, v \in [(t-\mathfrak{D}_{\text{mean}})/T - b, (t-\mathfrak{D}_{\text{mean}})/T + b]$. Thus, in this case, Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) yields that $t \in [\lfloor uT + \mathfrak{D}_{\text{mean}} \rfloor - \lfloor 3Tb \rfloor - 1, \lfloor uT + \mathfrak{D}_{\text{mean}} \rfloor + \lfloor 3Tb \rfloor + 2]$ is necessary to ensure $|K(((t-\mathfrak{D}_{\text{mean}})/T - u)/b) - K(((t-\mathfrak{D}_{\text{mean}})/T - v)/b)| \neq 0$. These considerations and the fact that K is Lipschitz continuous (according to Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i)) provide for all $u, v \in [b, 1-b]$ (see (4.28) and Definition 2.11):

$$\begin{aligned}
&\sup_{s \in \mathbb{R}^{d_1 \# \mathfrak{D}_1 + d_2 \# \mathfrak{D}_2}} \left|\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) - \widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(v,s)\right| \\
&\leqslant \frac{1}{Tb} \sum_{t=\lfloor uT+\mathfrak{D}_{\text{mean}}\rfloor - \lfloor 3Tb \rfloor - 1}^{\lfloor uT+\mathfrak{D}_{\text{mean}}\rfloor + \lfloor 3Tb \rfloor + 2} \left| K\left(\frac{\frac{t-\mathfrak{D}_{\text{mean}}}{T} - u}{b}\right) - K\left(\frac{\frac{t-\mathfrak{D}_{\text{mean}}}{T} - v}{b}\right)\right| \\
&\cdot \mathbb{1}_{\{u,v \in [(t-\mathfrak{D}_{\text{mean}})/T - b, (t-\mathfrak{D}_{\text{mean}})/T + b]\}} \\
&+ \frac{1}{Tb} \sum_{t=\lfloor uT+\mathfrak{D}_{\text{mean}}\rfloor - \lfloor Tb \rfloor - 1}^{\lfloor uT+\mathfrak{D}_{\text{mean}}\rfloor + \lfloor Tb \rfloor + 2} \left| K\left(\frac{\frac{t-\mathfrak{D}_{\text{mean}}}{T} - u}{b}\right) - K\left(\frac{\frac{t-\mathfrak{D}_{\text{mean}}}{T} - v}{b}\right)\right| \\
&\cdot \mathbb{1}_{\{u \in [(t-\mathfrak{D}_{\text{mean}})/T - b, (t-\mathfrak{D}_{\text{mean}})/T + b], v \notin [(t-\mathfrak{D}_{\text{mean}})/T - b, (t-\mathfrak{D}_{\text{mean}})/T + b]\}} \\
&+ \frac{1}{Tb} \sum_{t=\lfloor vT+\mathfrak{D}_{\text{mean}}\rfloor - \lfloor Tb \rfloor - 1}^{\lfloor vT+\mathfrak{D}_{\text{mean}}\rfloor + \lfloor Tb \rfloor + 2} \left| K\left(\frac{\frac{t-\mathfrak{D}_{\text{mean}}}{T} - u}{b}\right) - K\left(\frac{\frac{t-\mathfrak{D}_{\text{mean}}}{T} - v}{b}\right)\right| \\
&\cdot \mathbb{1}_{\{u \notin [(t-\mathfrak{D}_{\text{mean}})/T - b, (t-\mathfrak{D}_{\text{mean}})/T + b], v \in [(t-\mathfrak{D}_{\text{mean}})/T - b, (t-\mathfrak{D}_{\text{mean}})/T + b]\}} \\
&\leqslant \frac{C}{b} |u-v| .
\end{aligned} \tag{E.17}$$

The inequality $\left||z_1|^2 - |z_2|^2\right| \leqslant |z_1 - z_2| (|z_1| + |z_2|) \ \forall\, z_1, z_2 \in \mathbb{C}$, (C.112), (D.39), (E.17) and similar arguments show for all $u, v \in [b, 1-b]$ (note (4.28)):

$$\begin{aligned}
&\sup_{s \in \mathbb{R}^{d_1 \# \mathfrak{D}_1 + d_2 \# \mathfrak{D}_2}} \mathbb{E}\Bigg[\Bigg|\left|\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) - \widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[1]}\right) \widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u, s^{[2]}\right)\right|^2 \\
&- \left|\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(v,s) - \widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(v, s^{[1]}\right) \widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(v, s^{[2]}\right)\right|^2 \Bigg|\Bigg] \\
&\leqslant \frac{C}{b} |u-v| .
\end{aligned} \tag{E.18}$$

Lemma E.3 in combination with (E.18) and $T/\left(\sqrt{b} d_T\right) \overset{T\to\infty}{\longrightarrow} 0$ yield (recall (4.28) as well as (E.13)):

$$\begin{aligned}
&T\sqrt{b}\,\mathbb{E}\Bigg[\Bigg|\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T} - \frac{1-2b}{d_T} \sum_{k=1}^{d_T} \int_{\mathbb{R}^{d_1 \# \mathfrak{D}_1 + d_2 \# \mathfrak{D}_2}} \left|\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(\mathrm{v}_{k.d_T}, s) - \widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\mathrm{v}_{k.d_T}, s^{[1]}\right) \widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\mathrm{v}_{k.d_T}, s^{[2]}\right)\right|^2 \\
&\cdot \mathbf{w}(s)\, ds \Bigg|\Bigg] \\
&\leqslant T\sqrt{b} \frac{C}{b\, d_T} = o(1) .
\end{aligned} \tag{E.19}$$

Next, it is shown that the Riemann approximation of $\widehat{\mathfrak{Q}}_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ contained in (E.19) equals $\widehat{Q}_{T,\text{apprx}}$ (defined in (E.14)), which finishes the proof of Proposition E.4 (i).
Therefor, note at first that one obtains similarly to (E.4) by using (E.12) (see (E.13) as well as (5.2)):

$$\mathrm{Fw}_l(t_1,t_2) \in \mathbb{R} \quad \text{and} \quad \mathrm{Fw}_l(t_1,t_2) = \mathrm{Fw}_l(t_2,t_1) \ \forall\, l \in \{1,2\},\, t_1,t_2 \in \{1,\ldots,T\}. \tag{E.20}$$

It follows for all $k \in \{1,\ldots,d_T\}$, $s := \left(s^{[1]\prime}, s^{[2]\prime}\right)' \in \mathbb{R}^{d_1 \cdot \# \mathfrak{D}_1} \times \mathbb{R}^{d_2 \cdot \# \mathfrak{D}_2}$ from $|z|^2 = z\overline{z}$

$\forall\, z \in \mathbb{C}$, (E.20) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) $\big($recall (4.28), (E.13), (5.2) as well as $\mathbf{w}(s) := \mathbf{w}^{[1]}\big(s^{[1]}\big)\mathbf{w}^{[2]}\big(s^{[2]}\big)\big)$:

$$\int_{\mathbb{R}^{d_1\cdot\#\mathfrak{D}_1+d_2\cdot\#\mathfrak{D}_2}} \left|\widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\mathrm{v}_{k.d_T}, s\right) - \widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\mathrm{v}_{k.d_T}, s^{[1]}\right)\widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\mathrm{v}_{k.d_T}, s^{[2]}\right)\right|^2 \mathbf{w}(s)\, ds$$
$$= \frac{1}{T^2}\sum_{t=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(t,k) \sum_{j=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(j,k)\Re\left\{\mathrm{Fw}_1(t,j)\mathrm{Fw}_2(t,j)\right\}$$
$$- \frac{1}{T^3}\sum_{t=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(t,k)\left(\sum_{j_1=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(j_1,k)\,\mathrm{Fw}_1(t,j_1)\sum_{j_2=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(j_2,k)\,\mathrm{Fw}_2(t,j_2)\right)$$
$$- \frac{1}{T^3}\sum_{t=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(t,k)\left(\sum_{j_1=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(j_1,k)\,\mathrm{Fw}_1(j_1,t)\sum_{j_2=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(j_2,k)\,\mathrm{Fw}_2(j_2,t)\right)$$
$$+ \frac{1}{T^4}\left(\sum_{t_1=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(t_1,k)\sum_{j_1=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(j_1,k)\,\mathrm{Fw}_1(t_1,j_1)\right)$$
$$\cdot\left(\sum_{t_2=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(t_2,k)\sum_{j_2=1+\mathfrak{D}_{\max}}^{T}\mathrm{KER}_{d_T}(j_2,k)\,\mathrm{Fw}_2(t_2,j_2)\right)$$
$$= \frac{1}{T^2}Y_{d_T.1}(k) - \frac{2}{T^3}Y_{d_T.2}(k) + \frac{1}{T^4}Y_{d_T.3.1}(k)Y_{d_T.3.2}(k). \tag{E.21}$$

Overall, (E.19) and (E.21) prove Proposition E.4 (i).

(ii) To verify Proposition E.4 (ii), define at first for all $t \in \{1+\mathfrak{D}_{\max},\dots,T\}$, $u \in [0,1]$, $s := \big(s^{[1]\prime}, s^{[2]\prime}\big)' \in \mathbb{R}^{d_1\#\mathfrak{D}_1}\times\mathbb{R}^{d_2\#\mathfrak{D}_2}$ (see (4.28) and (4.4)):

$$\Phi_{t,T}(u,s) := 2\widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\Big(u,s^{[1]}\Big)\widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\Big(u,s^{[2]}\Big) - \widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(u,s) + e^{i\left\langle s^{[1]},X^{[1]}_{\mathfrak{D}_1,t,T}\right\rangle}e^{i\left\langle s^{[2]},X^{[2]}_{\mathfrak{D}_2,t,T}\right\rangle}$$
$$- \widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[1]}\right)e^{i\left\langle s^{[2]},X^{[2]}_{\mathfrak{D}_2,t,T}\right\rangle} - e^{i\left\langle s^{[1]},X^{[1]}_{\mathfrak{D}_1,t,T}\right\rangle}\widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(u,s^{[2]}\right), \tag{E.22}$$

whereby (D.39) shows:

$$|\Phi_{t,T}(u,s)| \leqslant C\ \forall\, t\in\{1+\mathfrak{D}_{\max},\dots,T\},\ u\in[0,1],\ s := \big(s^{[1]\prime}, s^{[2]\prime}\big)' \in \mathbb{R}^{d_1\#\mathfrak{D}_1}\times\mathbb{R}^{d_2\#\mathfrak{D}_2}. \tag{E.23}$$

It holds (recall (4.29)):

$$\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T} = \int_b^{1-b}\int_{\mathbb{R}^{d_1\#\mathfrak{D}_1+d_2\#\mathfrak{D}_2}}\left|\frac{1}{T}\sum_{t=1+\mathfrak{D}_{\max}}^{T}K_b\left(\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-u\right)W_t^*\Phi_{t,T}(u,s)\right|^2\mathbf{w}(s)\,ds\,du. \tag{E.24}$$

In the following, Proposition E.4 (ii) will be proved by using (E.24) and Lemma E.3. Therefore, some considerations that allow to apply Lemma E.3 are stated next.
Arguments which are similar to those that provide (E.17), (C.25) with $M = 3$, $|z|^2 = z\overline{z}\ \forall\, z \in \mathbb{C}$, Assumption 3.15 $[\mathbf{W}^*]$ (ii), (E.23), shifting the indices of sums and Assumption 3.15 $[\mathbf{W}^*]$ (iii) in combination with $2\lfloor 3Tb\rfloor + 3 \leqslant 2\lfloor Tb\rfloor + T$ for sufficiently large T yield for all $u, v \in [b, 1-b]$ and $s \in \mathbb{R}^{d_1\#\mathfrak{D}_1}\times\mathbb{R}^{d_2\#\mathfrak{D}_2}$:

$$\mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1+\mathfrak{D}_{\max}}^{T}\left(K_b\left(\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-u\right)-K_b\left(\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-v\right)\right)W_t^*\Phi_{t,T}(u,s)\right|^2\right]$$
$$\leqslant 3\mathbb{E}\left[\left|\frac{1}{Tb}\sum_{t=\max\{1+\mathfrak{D}_{\max},\lfloor uT+\mathfrak{D}_{\mathrm{mean}}\rfloor-\lfloor 3Tb\rfloor-1\}}^{\min\{T,\lfloor uT+\mathfrak{D}_{\mathrm{mean}}\rfloor+\lfloor 3Tb\rfloor+2\}}\left(K\left(\frac{\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-u}{b}\right)-K\left(\frac{\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-v}{b}\right)\right)\right.\right.$$

$$\cdot \mathbb{1}_{\{u,v\in[(t-\mathfrak{D}_{\mathrm{mean}})/T-b,(t-\mathfrak{D}_{\mathrm{mean}})/T+b]\}} W_t^* \Phi_{t,T}(u,s)\Bigg|^2\Bigg]$$

$$+ 3\mathbb{E}\Bigg[\Bigg|\frac{1}{Tb}\sum_{t=\max\{1+\mathfrak{D}_{\max},\lfloor uT+\mathfrak{D}_{\mathrm{mean}}\rfloor-\lfloor Tb\rfloor-1\}}^{\min\{T,\lfloor uT+\mathfrak{D}_{\mathrm{mean}}\rfloor+\lfloor Tb\rfloor+2\}}\left(K\left(\frac{\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-u}{b}\right)-K\left(\frac{\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-v}{b}\right)\right)$$

$$\cdot \mathbb{1}_{\{u\in[(t-\mathfrak{D}_{\mathrm{mean}})/T-b,(t-\mathfrak{D}_{\mathrm{mean}})/T+b],v\notin[(t-\mathfrak{D}_{\mathrm{mean}})/T-b,(t-\mathfrak{D}_{\mathrm{mean}})/T+b]\}} W_t^* \Phi_{t,T}(u,s)\Bigg|^2\Bigg]$$

$$+ 3\mathbb{E}\Bigg[\Bigg|\frac{1}{Tb}\sum_{t=\max\{1+\mathfrak{D}_{\max},\lfloor vT+\mathfrak{D}_{\mathrm{mean}}\rfloor-\lfloor Tb\rfloor-1\}}^{\min\{T,\lfloor vT+\mathfrak{D}_{\mathrm{mean}}\rfloor+\lfloor Tb\rfloor+2\}}\left(K\left(\frac{\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-u}{b}\right)-K\left(\frac{\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-v}{b}\right)\right)$$

$$\cdot \mathbb{1}_{\{u\notin[(t-\mathfrak{D}_{\mathrm{mean}})/T-b,(t-\mathfrak{D}_{\mathrm{mean}})/T+b],v\in[(t-\mathfrak{D}_{\mathrm{mean}})/T-b,(t-\mathfrak{D}_{\mathrm{mean}})/T+b]\}} W_t^* \Phi_{t,T}(u,s)\Bigg|^2\Bigg]$$

$$\leqslant |u-v|^2\frac{C}{b^2}\frac{1}{(Tb)^2}\left(\sum_{t_1,t_2=\lfloor uT+\mathfrak{D}_{\mathrm{mean}}\rfloor-\lfloor 3Tb\rfloor-1}^{\lfloor uT+\mathfrak{D}_{\mathrm{mean}}\rfloor+\lfloor 3Tb\rfloor+2}\left|K^*\left(\frac{t_1-t_2}{\beta}\right)\right|\right.$$

$$\left.+\sum_{t_1,t_2=\lfloor uT+\mathfrak{D}_{\mathrm{mean}}\rfloor-\lfloor Tb\rfloor-1}^{\lfloor uT+\mathfrak{D}_{\mathrm{mean}}\rfloor+\lfloor Tb\rfloor+2}\left|K^*\left(\frac{t_1-t_2}{\beta}\right)\right|+\sum_{t_1,t_2=\lfloor vT+\mathfrak{D}_{\mathrm{mean}}\rfloor-\lfloor Tb\rfloor-1}^{\lfloor vT+\mathfrak{D}_{\mathrm{mean}}\rfloor+\lfloor Tb\rfloor+2}\left|K^*\left(\frac{t_1-t_2}{\beta}\right)\right|\right)$$

$$\leqslant |u-v|^2\frac{C}{T^2b^4}\sum_{t_2=-\lfloor 3Tb\rfloor-1}^{\lfloor 3Tb\rfloor+2}\sum_{t_1=-\lfloor 3Tb\rfloor-1-t_2}^{\lfloor 3Tb\rfloor+2-t_2}\left|K^*\left(\frac{t_1}{\beta}\right)\right|$$

$$\leqslant |u-v|^2\frac{C\beta}{Tb^3}. \tag{E.25}$$

In addition, one obtains for all $u,v\in[b,1-b]$, $s\in\mathbb{R}^{d_1\#\mathfrak{D}_1}\times\mathbb{R}^{d_2\#\mathfrak{D}_2}$ analogously to (C.254) by using (C.112), (E.17), arguments which are similar to those that show (E.17) and (D.39) (see (E.22) as well as (4.28)):

$$\mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1+\mathfrak{D}_{\max}}^{T}K_b\left(\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-v\right)W_t^*\left(\Phi_{t,T}(u,s)-\Phi_{t,T}(v,s)\right)\right|^2\right]$$

$$\leqslant \frac{C\beta}{Tb}\sup_{t=1+\mathfrak{D}_{\max},\ldots,T}\mathbb{E}\left[|\Phi_{t,T}(u,s)-\Phi_{t,T}(v,s)|^2\right]$$

$$\leqslant \frac{C\beta}{Tb^3}|u-v|^2. \tag{E.26}$$

Moreover, it follows for all $s\in\mathbb{R}^{d_1\#\mathfrak{D}_1}\times\mathbb{R}^{d_2\#\mathfrak{D}_2}$ due to (E.23) analogously to (E.26):

$$\sup_{u\in[b,1-b]}\mathbb{E}\left[\left|\frac{1}{T}\sum_{t=1+\mathfrak{D}_{\max}}^{T}K_b\left(\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-u\right)W_t^*\Phi_{t,T}(u,s)\right|^2\right]\leqslant\frac{C\beta}{Tb}. \tag{E.27}$$

Overall, (E.24), Lemma E.3, $\mathbb{E}\left[\left||X|^2-|Y|^2\right|\right]\leqslant\|X-Y\|_2\left(\|X\|_2+\|Y\|_2\right)$ (which is valid for all complex-valued random variables X as well as Y that live on the same probability space and own finite second moments), (E.25), (E.26), (E.27), Assumption 3.15 $[\mathbf{W}^*]$ (i) (which ensures $\beta=o(Tb^2)$) and $T\sqrt{b}/e_T\overset{T\to\infty}{\longrightarrow}E\in[0,\infty)$ yield (recall (E.13)):

$$T\sqrt{b}\,\mathbb{E}\left[\left|\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}-\frac{1-2b}{e_T}\sum_{k=1}^{e_T}\int_{\mathbb{R}^{d_1\#\mathfrak{D}_1+d_2\#\mathfrak{D}_2}}\left|\frac{1}{T}\sum_{t=1+\mathfrak{D}_{\max}}^{T}K_b\left(\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T}-\mathrm{v}_{k.e_T}\right)W_t^*\Phi_{t,T}\left(\mathrm{v}_{k.e_T},s\right)\right|^2\mathbf{w}(s)\,ds\right|\right]$$

$$\leqslant T\sqrt{b}\,\frac{C}{e_T}\sqrt{\frac{\beta}{Tb^3}}\sqrt{\frac{\beta}{Tb}}=o(1). \tag{E.28}$$

Next, it is shown that the Riemann approximation of $\widehat{\mathfrak{Q}}^*_{\mathfrak{D}_1,\mathfrak{D}_2,T}$ contained in (E.28) equals $\widehat{Q}^*_{T,\text{apprx}}$ (defined in (E.16)), which finishes the proof of Proposition E.4 (ii).
Therefor, note at first that one obtains for all $k \in \{1, \ldots, e_T\}$ by using $z\overline{z} = |z|^2 \;\forall\, z \in \mathbb{C}$ similarly to (E.21) (see (E.13), (E.15) as well as (E.16)):

$$\int\limits_{\mathbb{R}^{d_1\#\mathfrak{D}_1+d_2\#\mathfrak{D}_2}} \left(\frac{1}{T}\sum_{p=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(p,k)\, W^*_p \left(2\widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\text{v}_{k.e_T}, s^{[1]}\right) \widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\text{v}_{k.e_T}, s^{[2]}\right) - \widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(\text{v}_{k.e_T}, s)\right)\right)$$
$$\cdot \left(\frac{1}{T}\sum_{q=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(q,k)\, W^*_q \overline{\left(2\widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\text{v}_{k.e_T}, s^{[1]}\right) \widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\text{v}_{k.e_T}, s^{[2]}\right) - \widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(\text{v}_{k.e_T}, s)\right)}\right) \mathbf{w}(s)\, ds$$
$$= \frac{1}{T^2}\text{Y}_{\text{B}.1}(k)^2\, \text{y}_{\text{B}.1}(k). \tag{E.29}$$

Moreover, (E.20) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) provide for all $k \in \{1, \ldots, e_T\}$ (recall (4.28), (E.13), (E.15) as well as (E.16)):

$$\int\limits_{\mathbb{R}^{d_1\#\mathfrak{D}_1+d_2\#\mathfrak{D}_2}} \left(\frac{1}{T}\sum_{p=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(p,k) W^*_p \left(2\widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\text{v}_{k.e_T}, s^{[1]}\right) \widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\text{v}_{k.e_T}, s^{[2]}\right)\right.\right.$$
$$\left.\left. - \widehat{\varphi}_{\mathfrak{D}_1,\mathfrak{D}_2}(\text{v}_{k.e_T}, s)\right)\right) \cdot \left(\frac{1}{T}\sum_{t=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(t,k) W^*_t \overline{\left(e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,t,T}\right\rangle} e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,t,T}\right\rangle}\right.}\right.$$
$$\left.\overline{\left. - \widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\text{v}_{k.e_T}, s^{[1]}\right) e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,t,T}\right\rangle} - e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,t,T}\right\rangle} \cdot \widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\text{v}_{k.e_T}, s^{[2]}\right)\right)}\right) \mathbf{w}(s)\, ds$$
$$= \frac{1}{T}\sum_{p=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(p,k) W^*_p \frac{1}{T}\sum_{t=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(t,k) W^*_t$$
$$\cdot \left(\frac{2}{T^2}\sum_{j_1=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(j_1,k)\, \text{Fw}_1(j_1,t) \sum_{j_2=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(j_2,k)\, \text{Fw}_2(j_2,t)\right.$$
$$- \frac{2}{T^3}\sum_{t_1,j_1=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(t_1,k)\, \text{KER}_{e_T}(j_1,k)\, \text{Fw}_1(j_1,t_1) \sum_{j_2=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(j_2,k)\, \text{Fw}_2(j_2,t)$$
$$- \frac{2}{T^3}\sum_{j_1=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(j_1,k)\, \text{Fw}_1(j_1,t) \sum_{t_2,j_2=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(t_2,k)\, \text{KER}_{e_T}(j_2,k)\, \text{Fw}_2(t_2,j_2)$$
$$- \frac{1}{T}\sum_{j=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(j,k)\, \text{Fw}_1(j,t)\, \text{Fw}_2(j,t)$$
$$+ \frac{1}{T^2}\sum_{j=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(j,k) \left(\sum_{t_1=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(t_1,k)\, \text{Fw}_1(j,t_1)\right) \text{Fw}_2(j,t)$$
$$\left. + \frac{1}{T^2}\sum_{j=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(j,k)\, \text{Fw}_1(j,t) \sum_{t_2=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(t_2,k)\, \text{Fw}_2(j,t_2)\right)$$
$$= \frac{1}{T^2}\text{Y}_{\text{B}.1}(k)\text{Y}_{\text{B}.2}(k). \tag{E.30}$$

In addition, it holds for all $k \in \{1, \ldots, e_T\}$ due to (E.20) and Assumption 4.5 $[\mathbf{K}\&\mathbf{b.2}]$ (i) (see (4.28), (E.13), (E.15) as well as (E.16)):

$$\int\limits_{\mathbb{R}^{d_1\#\mathfrak{D}_1+d_2\#\mathfrak{D}_2}} \left(\frac{1}{T}\sum_{p=1+\mathfrak{D}_{\max}}^{T} \text{KER}_{e_T}(p,k) W^*_p \left(e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,p,T}\right\rangle} e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,p,T}\right\rangle}\right.\right.$$
$$\left.\left. - \widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\text{v}_{k.e_T}, s^{[1]}\right) e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,p,T}\right\rangle} - e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,p,T}\right\rangle} \cdot \widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2}\left(\text{v}_{k.e_T}, s^{[2]}\right)\right)\right)$$

$$
\cdot \left(\frac{1}{T} \sum_{t=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}(t,k) W_t^* \left(\overline{e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,t,T}\right\rangle} e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,t,T}\right\rangle}} \right.\right.
$$

$$
\left.\left. \overline{- \widehat{\varphi}^{[1]}_{\mathfrak{D}_1,\mathfrak{D}_2} \left(\mathrm{v}_{k.e_T}, s^{[1]}\right) e^{i\left\langle s^{[2]}, X^{[2]}_{\mathfrak{D}_2,t,T}\right\rangle} - e^{i\left\langle s^{[1]}, X^{[1]}_{\mathfrak{D}_1,t,T}\right\rangle} \cdot \widehat{\varphi}^{[2]}_{\mathfrak{D}_1,\mathfrak{D}_2} \left(\mathrm{v}_{k.e_T}, s^{[2]}\right)} \right) \right) \mathbf{w}(s)\, ds \Bigg\}
$$

$$
\begin{aligned}
&= \frac{1}{T} \sum_{p=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}(p,k) W_p^* \frac{1}{T} \sum_{t=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}(t,k) W_t^* \cdot \Bigg(\mathrm{Fw}_1(p,t) \mathrm{Fw}_2(p,t) \\
&- \frac{1}{T} \sum_{j_1=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}\left(j_1,k\right) \mathrm{Fw}_1\left(p,j_1\right) \mathrm{Fw}_2(p,t) \\
&- \frac{1}{T} \mathrm{Fw}_1(p,t) \sum_{j_2=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}\left(j_2,k\right) \mathrm{Fw}_2\left(p,j_2\right) \\
&- \frac{1}{T} \sum_{t_1=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}\left(t_1,k\right) \mathrm{Fw}_1\left(t_1,t\right) \mathrm{Fw}_2(p,t) \\
&+ \frac{1}{T^2} \sum_{t_1,j_1=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}\left(t_1,k\right) \mathrm{KER}_{e_T}\left(j_1,k\right) \mathrm{Fw}_1\left(t_1,j_1\right) \mathrm{Fw}_2(p,t) \\
&+ \frac{1}{T^2} \sum_{t_1=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}\left(t_1,k\right) \mathrm{Fw}_1\left(t_1,t\right) \sum_{j_2=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}\left(j_2,k\right) \mathrm{Fw}_2\left(p,j_2\right) \\
&- \frac{1}{T} \mathrm{Fw}_1(p,t) \sum_{t_2=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}\left(t_2,k\right) \mathrm{Fw}_2\left(t_2,t\right) \\
&+ \frac{1}{T^2} \sum_{j_1=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}\left(j_1,k\right) \mathrm{Fw}_1\left(p,j_1\right) \sum_{t_2=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}\left(t_2,k\right) \mathrm{Fw}_2(t_2,t) \\
&+ \frac{1}{T^2} \mathrm{Fw}_1\left(p,t\right) \sum_{t_2,j_2=1+\mathfrak{D}_{\max}}^{T} \mathrm{KER}_{e_T}\left(t_2,k\right) \mathrm{KER}_{e_T}\left(j_2,k\right) \mathrm{Fw}_2\left(t_2,j_2\right) \Bigg) \\
&= \frac{1}{T^2} \mathrm{Y}_{\mathrm{B.3.2}}(k). \qquad \text{(E.31)}
\end{aligned}
$$

One obtains for all $k \in \{1,\ldots,e_T\}$ from $|z|^2 = z\overline{z}\ \forall\, z \in \mathbb{C}$, $1/T^2 \mathrm{Y}_{\mathrm{B.1}}(k)\mathrm{Y}_{\mathrm{B.2}}(k) = \overline{1/T^2 \mathrm{Y}_{\mathrm{B.1}}(k)} \cdot \overline{\mathrm{Y}_{\mathrm{B.2}}(k)}$ (the latter is valid due to (E.20)), (E.29), (E.30) and (E.31) (recall (E.22)):

$$
\begin{aligned}
&\int_{\mathbb{R}^{d_1\#\mathfrak{D}_1 + d_2\#\mathfrak{D}_2}} \left| \frac{1}{T} \sum_{t=1+\mathfrak{D}_{\max}}^{T} K_b\!\left(\frac{t-\mathfrak{D}_{\mathrm{mean}}}{T} - \mathrm{v}_{k.e_T}\right) W_t^* \Phi_{t,T}\left(\mathrm{v}_{k.e_T}, s\right) \right|^2 \mathbf{w}(s)\, ds \\
&= \frac{1}{T^2} \mathrm{Y}_{\mathrm{B.1}}(k)^2\, \mathrm{y}_{\mathrm{B.1}}(k) + \frac{2}{T^2} \mathrm{Y}_{\mathrm{B.1}}(k) \mathrm{Y}_{\mathrm{B.2}}(k) + \frac{1}{T^2} \mathrm{Y}_{\mathrm{B.3.2}}(k). \qquad \text{(E.32)}
\end{aligned}
$$

Proposition E.4 (ii) is an implication of (E.28) and (E.32). □

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