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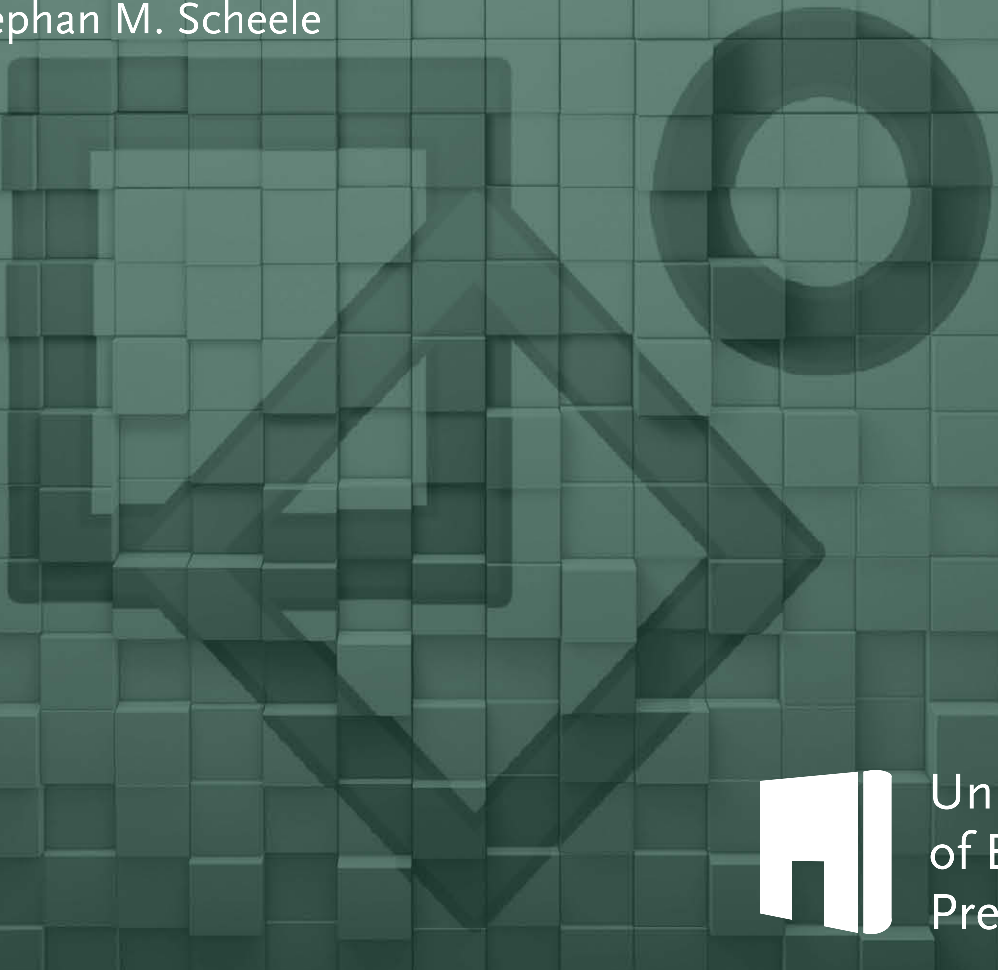
Model and Proof Theory of Constructive $\mathcal{ALC}$

Constructive Description Logics

Stephan M. Scheele



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Abstract

Description logics (DLs) represent a widely studied logical formalism with a significant impact in the field of knowledge representation that was put to practice in numerous cases of semantic-driven applications. They constitute a family of knowledge representation languages to specify and reason about the knowledge of a particular application domain in a structured and formally well-understood way [16; 24]. The orientation of the main branch of research in DLs is towards languages equipped with *classical descriptive semantics*, which is characterised by extensional truth and model structure, and by a platonic notion of truth. However, DLs are insufficiently expressive to deal with evolving information, as from data streams or ongoing processes. Such knowledge is only partially determined and incomplete, and represents abstractions of real objects, whose properties are evolving and defined merely up to construction. Only recently, non-classical interpretations of DLs have been investigated from a constructive perspective. Constructive (or intuitionistic) logic [265] is arguably one of the most intriguing modal theories, which has first been conceived by Glivenko, Heyting and Kolmogorov as a formalisation of the notion of constructive proof, based on the demand for explicit evidence for the justification of mathematical statements. Constructive logic is not only of a great value from a philosophical perspective, but also it has found widespread applications in the constructive foundations of mathematics as well as in computer science [264]. An important example is the fundamental correspondence, called the *Curry-Howard isomorphism* [149; 248], which closely relates intuitionistic logic and type theory, and allows to interpret proofs as computations in the simply-typed λ-calculus. Semantically, intuitionistic logic can be viewed as a logic of states of knowledge or context [170; 265], where information monotonically increases over time. This semantic characterisation, which is according to the process of constructing mathematical objects, is suitable to express and reason about partial and incomplete information.

We aim at combining constructive logic with description logic, two logical systems having an independent philosophical motivation, in order to find DLs that can be accepted by both, a logician of the DL domain addressing semantic applications, as well as by a constructivist mathematician as a suitable theory to represent and reason about knowledge according to constructively accepted principles. Our key conceptual contribution is the investigation of the model theory and proof theory of a *constructive* variant of the basic description logic $\mathcal{ALC}$ (Attributive Language with Complement), which will be denoted by *constructive* $\mathcal{ALC}$ ($c\mathcal{ALC}$). The semantic dimension of constructive DLs is investigated, by replacing the traditional binary truth interpretation of $\mathcal{ALC}$ with a constructive notion of truth. On the one hand, we argue that such a refined interpretation is crucial to represent applications with partial information adequately, and to achieve both consistency under abstraction as well as robustness under refinement, and on the other hand is compatible with the Curry-Howard isomorphism in order to form the cornerstone for a DL-based typing system. In particular, we provide the following key contributions to justify our claims: (i) We introduce the constructive description logic $c\mathcal{ALC}$, that uses the same language as $\mathcal{ALC}$, but with semantically independent logical operators, and is characterised in terms of an intuitionistic birelational semantics that is non-normal w.r.t. existential restriction (possibility modality). We present results regarding its expressiveness, study its constructive properties like the disjunction property, and provide a finite semantical characterisation for $c\mathcal{ALC}$, by proving the finite model property. Several examples illustrate situations, where the constructive semantics are essential to represent partial and incomplete information. We suggest applications in the domain of auditing, to use $c\mathcal{ALC}$ as a typing system for data streams, and also by showing a direct interpretation of proofs as computations following the Curry-Howard isomorphism. Furthermore, we discuss the open world assumption from a constructive point of view. (ii) The proof theory of $c\mathcal{ALC}$ is investigated by giving a sound and complete Hilbert-style axiomatisation and a Gentzen-style sequent calculus showing finite model property and decidability. We focus on reasoning w.r.t. TBoxes and show that both calculi admit the standard DL-style reasoning services w.r.t. TBoxes. We establish a modal deduction theorem that generalises the Hilbert calculus to allow for derivations w.r.t. global and local premises. Based on the Gentzen sequent calculus, we discuss the strengthening of $c\mathcal{ALC}$ towards normal intuitionistic modal logics and $\mathcal{ALC}$. (iii) We address the relation between $c\mathcal{ALC}$ and classical DLs by demonstrating an embedding of $c\mathcal{ALC}$ into a classical modal logic, which allows us to transfer the finite model property, decidability and explore the complexity of the satisfiability and subsumption problem. Moreover, we explore the sub-Boolean $\{\sqcup, \exists\}$-fragment $\mathcal{UL}$ of $\mathcal{ALC}$ w.r.t. general TBoxes, which turns out to be tractable under the constructive semantics, while it is intractable under the classical descriptive semantics. (iv) We introduce a tableau calculus for $c\mathcal{ALC}$ and prove its soundness and completeness w.r.t. the birelational semantics of $c\mathcal{ALC}$. By giving a termination proof for this calculus, we obtain an effectively decidable tableau algorithm that allows goal-directed proof search as well as countermodel construction w.r.t. TBoxes, and can be seen as a first step towards an implementation. Finally, we elaborate the difficulties of interpreting ABox assertions and nominals under the birelational semantics of $c\mathcal{ALC}$ and sketch an approach towards constructive ABox reasoning. In summary, we define a constructive version of $\mathcal{ALC}$, which on the one hand supports a constructive notion of truth and on the other hand is compatible with the Curry-Howard isomorphism.

Zusammenfassung

Beschreibungslogiken (BLen) stellen einen vieluntersuchten logischen Formalismus dar, der den Bereich der Wissensrepräsentation signifikant geprägt hat und vielfach in semantik-gesteuerten Anwendungen zum Einsatz kommt. Sie begründen eine Familie von logikbasierten Sprachen, die es erlauben, das Wissen eines bestimmten Anwendungsbereiches in einer strukturierten und formal wohlfundierten Weise zu repräsentieren, und Schlussfolgerungen daraus abzuleiten [16; 24]. Die Hauptforschungsrichtung von BLen konzentriert sich auf Sprachen, die auf *klassischer deskriptiver Semantik* basieren und durch eine extensionale Wahrheits- und Modellstruktur, und einen idealisierten Wahrheitsbegriff nach Platons Ideenlehre gekennzeichnet sind. Allerdings sind Beschreibungslogiken unzureichend, um in Entwicklung befindliches Wissen zu repräsentieren, wie es beispielsweise durch Datenströme oder fortlaufende Prozesse generiert wird. Derartiges Wissen ist lediglich partiell festgelegt und unvollständig. Es stellt Abstraktionen realer Gegenstände dar, deren Eigenschaften einer ständigen Änderung unterliegen und nur durch den Zustand einer Konstruktion bestimmt sind. Erst neuerdings beschäftigt sich die Forschung gezielt mit nicht-klassischen Interpretationen für Beschreibungslogiken aus einer konstruktiven Sicht. Konstruktive (oder intuitionistische) Logik [265] stellt wohl eine der faszinierendsten modalen Theorien dar, die auf die Forschungsarbeiten von Glivenko, Heyting und Kolmogorov zurückgeht, mit dem Ziel der Formalisierung eines konstruktiven Beweisbegriffs, der auf der Basis expliziter Belege den Nachweis mathematischer Aussagen erbringt. Konstruktive Logik ist nicht nur ein philosophisches Gedankenexperiment, sondern hat auch seine Berechtigung durch vielseitige Anwendungen in den Grundlagen der Mathematik sowie der Informatik [264] gefunden. Eine bedeutende Errungenschaft stellt der *Curry-Howard Isomorphismus* [149; 248] dar, ein fundamentales Korrespondenzergebnis der Beweistheorie, das die intuitionistische Logik mit der Typentheorie verbindet, und es so ermöglicht, Beweise als Programme des einfach typisierten λ-Kalküls zu interpretieren. Aus semantischer Sicht lässt sich intuitionistische Logik als eine Theorie charakterisieren, in der Wissen zustands- oder kontextbasiert interpretiert wird, und der Wahrheitsgehalt einer Aussage kontinuierlich zunimmt [170; 265]. Diese semantische Charakterisierung entspricht dem Grundgedanken konstruktiver Logiken, Aussagen durch die explizite Konstruktion mathematischer Objekte zu belegen, und eignet sich dazu, partielles und unvollständiges Wissen zu repräsentieren, sowie daraus Schlussfolgerungen abzuleiten. Das Ziel der Arbeit ist es, konstruktive Logik und Beschreibungslogik – zwei logische Formalismen mit einer differierenden philosophischen Motivation – zu einer *konstruktiven Beschreibungslogik* zu kombinieren, die einerseits den Erfordernissen semantischer Anwendungen im Bereich der BLen entspricht, und es andererseits erlaubt, in Übereinstimmung mit akzeptierten

Prinzipien konstruktiver Mathematik, Wissen zu repräsentieren und daraus Schlussfolgerungen zu ziehen. Der wesentliche konzeptionelle Beitrag dieser Arbeit ist die Untersuchung der Modell- und Beweistheorie einer *konstruktiven* Variante der Basis-BL $\mathcal{ALC}$ (Attributive Language with Complement), die im Folgenden als *constructive* $\mathcal{ALC}$ ($c\mathcal{ALC}$) bezeichnet wird. Die Semantik dieser konstruktiven Beschreibungslogik resultiert daraus, die traditionelle zweiwertige Interpretation logischer Aussagen des Systems $\mathcal{ALC}$ durch einen konstruktiven Wahrheitsbegriff zu ersetzen. Wir argumentieren, dass eine derart verfeinerte Interpretation die Voraussetzung dafür ist, einerseits, Anwendungen mit partiellem Wissen angemessen zu repräsentieren, und sowohl die Konsistenz logischer Aussagen unter Abstraktion als auch ihre Robustheit unter Verfeinerung zu gewährleisten, andererseits, den Grundstein für ein Beschreibungslogik-basiertes Typsystem gemäß dem Curry-Howard Isomorphismus zu legen. Die wesentlichen Ergebnisse zur Bestätigung unserer Thesen sind wie folgt: (i) Wir führen die konstruktive BL $c\mathcal{ALC}$ ein, die dieselbe Sprache wie $\mathcal{ALC}$ verwendet, aber über semantisch unabhängige logische Operatoren verfügt, und deren Semantik in der Form einer birelationalen Kripke Semantik mit einer unorthodoxen Interpretation existentieller Restriktionen (Modalität $\Diamond$ in der Modallogik) definiert ist. Die Ergebnisse dieser Arbeit untersuchen die Ausdrucksmächtigkeit von $c\mathcal{ALC}$, deren konstruktive Eigenschaften wie die Disjunktions-Eigenschaft, und geben eine endliche semantische Charakterisierung an, durch Nachweis der endlichen Modelleigenschaft. Mehrere Beispiele veranschaulichen Situationen, in denen die konstruktive Semantik unverzichtbar ist, um partielles und unvollständiges Wissen zu repräsentieren. Für den Bereich der Wirtschaftsprüfung schlagen wir Anwendungsfälle vor, einerseits $c\mathcal{ALC}$ für die Typisierung von Datenströmen zu verwenden, und andererseits entsprechend dem Curry-Howard Isomorphismus Beweise als Programme zu interpretieren. Darüber hinaus diskutieren wir die Open World Assumption aus konstruktiver Sicht. (ii) Die Ergebnisse der Untersuchung der Beweistheorie von $c\mathcal{ALC}$ umfassen eine vollständige und korrekte Hilbert Axiomatisierung und einen Gentzen Sequenzenkalkül, sowie Beweise zur endlichen Modelleigenschaft und Entscheidbarkeit. Dabei wird der Fokus auf TBox-Theorien gelegt und gezeigt, dass beide Kalküle die standard Inferenzdienste bzgl. TBoxen umsetzen. Für den Hilbert Kalkül wird ein modales Deduktionstheorem nachgewiesen, das die Ableitung unter globalen und lokalen Prämissen erlaubt. Basierend auf dem Gentzen Sequenzenkalkül werden logische Systeme diskutiert, die zwischen $c\mathcal{ALC}$, intuitionistischer Modallogik und der klassischen Beschreibungslogik $\mathcal{ALC}$ liegen. (iii) Die Beziehung zwischen $c\mathcal{ALC}$ und klassischen BLen wird anhand einer Einbettung von $c\mathcal{ALC}$ in eine klassische Modallogik betrachtet. Dies erlaubt den Transfer der Ergebnisse zur endlichen Modelleigenschaft, Entscheidbarkeit, sowie Resultate zur Komplexität der Inferenzprobleme zu Erfüllbarkeit und Subsumption. Des Weiteren wird das $\{\sqcup, \exists\}$-Teilfragment $\mathcal{UL}$ von

$\mathcal{ALC}$ bzgl. unbeschränkter TBoxen beleuchtet, dessen Inferenzproblem der Subsumption sich unter konstruktiver Semantik als effizient lösbar herausstellt, ganz im Gegensatz zur deskriptiven klassischen Semantik. (iv) Schließlich wird ein Tableaukalkül für $c\mathcal{ALC}$ vorgestellt und dessen Vollständigkeit und Korrektheit in Relation zur birelationalen Semantik von $c\mathcal{ALC}$ nachgewiesen. Aus dem Nachweis der Terminierung des Kalküls ergibt sich ein effektiv entscheidbarer Tableau-Algorithmus, der neben der zielorientierten Beweissuche die Konstruktion von Gegenmodellen erlaubt, und damit eine Grundlage für zukünftige Implementierungen bildet. Abschließend werden die wesentlichen Probleme der Interpretation von ABox-Aussagen und Nominalen unter der birelationalen Semantik von $c\mathcal{ALC}$ diskutiert, und eine mögliche Lösung für konstruktive Inferenzdienste bzgl. ABoxen skizziert. Zusammenfassend stellen wir eine konstruktive Variante der Beschreibungslogik $\mathcal{ALC}$ vor, die einerseits einen konstruktiven Wahrheitsbegriff unterstützt, und andererseits mit dem Curry-Howard Isomorphismus vereinbar ist.

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CHAPTER 1

Introduction

Description logics (DLs) form a family of knowledge representation formalisms to explicitly represent and reason about knowledge of a particular application domain in a structured and formally well-understood way [16; 24]. Historically, DLs emerged from early formalisms in the field of knowledge representation (KR), namely semantic networks [233] and frame systems [203]. These systems, mostly motivated by cognitive science, specified the knowledge of an application domain in terms of network-based structures, by specifying classes of objects and their interrelationship, but were criticised by reason of their lack of a precise semantic characterisation. DLs were introduced to overcome the deficiencies and ambiguities of previous semi-formal approaches, by building on a logical foundation with a formally well-understood semantics that allows to justify the correctness of inference mechanisms (see [16, Chap. 1] for a detailed historical retrospect on DLs). DLs are used to capture the meaning of natural language statements of a specific domain in knowledge bases, specified by formal ontologies, which represent knowledge in terms of a TBox and an ABox. The former introduces the terminological vocabulary (taxonomy) of a domain in terms of *concept descriptions*, which define classes of individuals and *roles* express relations between them. The ABox expresses the extensional knowledge that specifies the current state of affairs regarding concrete named *individuals* in terms of *assertional facts*. The key DL $\mathcal{ALC}$, standing for *Attribute Language with Complement*, allows to express concept descriptions build from atomic concept names and roles by means of the logical constructors from the propositional language, such as conjunction ($\sqcap$), disjunction ($\sqcup$) and negation ($\neg$), and indexed modal operators represented by existential ($\exists$) and universal restriction ($\forall$) of roles. For instance, the concept of a mother who is having at least one son who is a PhD-student can be expressed by

$$\mathsf{Female} \sqcap \forall \mathsf{hasChild}.\mathsf{Human} \sqcap \exists \mathsf{hasChild}.(\mathsf{Male} \sqcap \mathsf{PhDstudent}),$$

where $\mathsf{Female}, \mathsf{Human}, \mathsf{Male}$ and $\mathsf{PhDstudent}$ are concept names, and $\mathsf{hasChild}$ stands for a role. Moreover, the TBox is a set of axioms that allows to assign names to concept descriptions, *e.g.*, we can introduce the name $\mathsf{HappyMother}$ for the above concept, and we can express subsumption relationships such as

$$\exists \mathsf{supervises}.\mathsf{PhDstudent} \sqsupset \mathsf{Professor},$$

stating that only professors can supervise PhD-students. The assertional part of a knowledge base specifies a concrete situation by asserting facts about named individuals, *e.g.*,

$$\begin{gathered} \mathsf{BETTY} : \mathsf{HappyMother}, \ \ (\mathsf{BETTY}, \mathsf{BOB}) : \mathsf{hasChild}, \\ \mathsf{BOB} : \mathsf{PhDstudent}, \ \ (\mathsf{BILL}, \mathsf{BOB}) : \mathsf{supervises}, \end{gathered}$$

which express that the individual BETTY is an instance of the concept $\mathsf{HappyMother}$, that BOB is one of her children, that BOB is a $\mathsf{PhDstudent}$, and that BILL supervises BOB. The semantics of DLs is defined in a model theoretic way using a Tarski-style set theoretic *interpretation* $\mathcal{I}$, consisting of a *domain* $\Delta^{\mathcal{I}}$ and an *interpretation function* $\cdot^{\mathcal{I}}$, which is based on classical first-order logic, *i.e.*, concepts are interpreted as unary predicates over $\Delta^{\mathcal{I}}$, while roles are interpreted as binary predicates over the domain. The semantics of ABox assertions is by means of interpreting individual names, *i.e.*, by mapping each name to an element of the domain.

According to Schild [245], $\mathcal{ALC}$ is a notational variant of the basic multimodal logic K_m, *i.e.*, existential restriction $\exists R$ corresponds to the *possibility* modality $\Diamond_R$, and universal restriction $\forall R$ to the *necessity* modality $\Box_R$, where the modal operators are labelled by means of role names R. Schild's results lets us view $\mathcal{ALC}$ from an axiomatic perspective [245, p. 4; 234, p. 14], that is, $\mathcal{ALC}$ is sound and complete w.r.t. the Hilbert axiomatisation of classical propositional logic [103, p. 6], extended by the axioms $\forall R.(C \sqcap D) \equiv (\forall R.C \sqcap \forall R.D)$ and $\forall R.\top$, and the rule of necessitation, stating *if* C *then* $\forall R.C$. Existential restriction is usually defined in terms of negation and $\forall R$, as $\exists R.C =_{\mathit{df}} \neg\forall R.\neg C$, and from this duality one easily observes that $\exists R.(C \sqcup D) \equiv (\exists R.C \sqcup \exists R.D)$ and $\neg\exists R.\bot$ are theorems of $\mathcal{ALC}$. In the terminology of modal logics [33, pp. 191 f.], a logical system including the above formulæ is denoted to be *normal.* DLs provide their users with reasoning services to infer new implicit consequences from the existing knowledge, and to detect inconsistencies. For instance, the above situation lets us infer from the facts $\mathsf{BOB} : \mathsf{PhDstudent}$ and $(\mathsf{BILL}, \mathsf{BOB}) : \mathsf{supervises}$ that BILL is a Professor. The standard reasoning services [16, pp. 9 ff.] of DLs usually include the task of deciding (i) whether a concept C is *satisfiable* w.r.t. a TBox, *i.e.*, whether we can instantiate concept C in a given TBox; (ii) if a concept C is *subsumed* by a concept D, *i.e.*, whether all instances of C are necessarily instances of D; (iii) whether a given ABox is *consistent* (possibly w.r.t. a given TBox); and (iv) *instance checking*, *i.e.*, whether a given individual name is an instance of a specified concept. While early DL reasoners were based on structural algorithms and translation-based approaches (borrowing methods from modal or first-order logic), modern reasoning systems are usually based on highly optimised tableau algorithms (see [25] and [16, Chap. 8] for a survey), and recently on hypertableau-based calculi [207].

Nowadays, DLs are used in semantic databases, in life sciences, in applications of the Semantic Web and as formal grounding for the W3C-endorsed Web Ontology Language (OWL) [250]. The achievement of DLs in the many domains of semantic information processing is based on their flexibility to strike a carefully crafted trade-off between expressiveness and implementation efficiency. In particular, DLs encapsulate the semantic complexity in a compact syntactical notation that supports humans as well as automated inference services to handle complex logical specifications in a more efficient way, compared to plain vanilla first-order logic, where the complete quantification structure is made explicit. From a technical perspective, there is a strong connection between DLs and multimodal generalisations of modal logic [4; 103], and under this view they essentially correspond to guarded fragments of first-order logic. This class of fragments has turned out to be a breeding-ground of very well-behaved classes of logical formalisms, aiming at offering sufficient expressivity while maintaining decidability of the inference tasks.

1.1 Motivation: When Constructiveness matters

The main branch of research in DLs focusses on languages equipped with classical descriptive semantics. While the existing DLs are covering a broad selection of languages with different levels of expressivity, they are somehow limited in the sense that knowledge represented by concepts is assumed to be static [16, Chap. 1].

However, it is sometimes the case that knowledge is dynamic and incomplete. Individuals specific for an application domain may not be fixed and tangible but abstractions of real individuals whose properties are subject to refinement. Concrete knowledge about individuals may evolve and therefore correspond to a process of construction which is enriching the knowledge over time or up to available resources. Especially natural language statements are difficult to specify and interpret in a static way, since they are usually context dependent or subject to negotiation. Classical DLs have limited support for incomplete knowledge about individuals when assuming the *open world assumption*, *i.e.*, the interpretation of a concept is assumed to be static and at the outset either includes a given individual or not. Though, either option may be inconsistent, if the individual or the concept is only partially defined until a later state of knowledge revealing further evidence becomes available.

In the following, we illustrate the significance of constructive semantics, on the one hand to allow for the handling of incomplete knowledge in data- and process-driven applications, and on the other hand to provide a computational interpretation for DLs. For instance, DLs can be used to represent knowledge about a regional fauna, *e.g.*, the concept of the *European tree frog* may be defined as a frog belonging to the family of tree frogs. They are commonly found across Europe and their dorsal skin is usually coloured.

$$\mathsf{EuropeanTreeFrog} = \mathsf{Frog} \sqcap \exists \mathsf{indigenous.EuropenCountry} \sqcap \exists \mathsf{hasColour.Colour}.$$

We can give an interpretation of the above concept expressed by the following assertional facts about an individual called IGGY:

$$\begin{aligned} &\mathsf{IGGY} : \mathsf{EuropeanTreeFrog},\ \ \mathsf{GREEN} : \mathsf{Colour},\ \ \mathsf{GERMANY} : \mathsf{EuropenCountry},\\ &\qquad\qquad (\mathsf{IGGY}, \mathsf{GERMANY}) : \mathsf{indigenous},\ \ (\mathsf{IGGY}, \mathsf{GREEN}) : \mathsf{hasColour}. \end{aligned}$$

The interpretation of the concept $\mathsf{EuropeanTreeFrog}$ takes into account one state of knowledge only, *i.e.*, the facts about IGGY are assumed to be static, in particular the assertions specify that IGGY is always to be found in $\mathsf{GERMANY}$ and has always the colour GREEN. Classical description logic interprets statements based on the notion of absolute truth and allows not to abstract from an individual such that its semantic interpretation incorporates not only the actual state, but also the future states IGGY may evolve to. For instance, on the one hand, the habitat of IGGY may shift due to climate change in the future, on the other hand, the dorsal skin of the European tree frog is known to depend on contextual properties and is subject to change on a short-term basis. The variation ranges from light grey to kelly-green to tan, *i.e.*, it may change depending on the temperature or humidity of the environment, the structure of the underground or the animal's mood. These dynamic and context dependent properties cannot be expressed by the classical semantics of DLs.

An application area where this aspect is particularly prominent is business auditing, which has been motivated in [195] and [155]. It refers to the process of digital auditing of business data in order to verify the validity of the accounting, the absence of fraud or the conformance to regulations and financial process standards like SOX[1] and IFRS[2]. Like natural language statements, audit concepts rarely have a fixed interpretation but are subject to refinement or context. For instance, audit data are usually created by ongoing business processes within information systems and an audit can only cover

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[1] Sarbanes-Oxley Act, US law of 2002 on business reporting in reaction to Enron and WorldCom scandals.

[2] International Financial Reporting Standard.

a finite snapshot thereof. An audit statement like '*each delivery order must have an associated invoice*' has to consider the case when '*for some delivery order the invoice is still pending*' and will only become available after refinement of the audit data. Also, audit statements may contain partially determined concepts or roles, *e.g.*, consider the concept of a *solvent company*. Its interpretation may be subject to negotiations as it may depend on the rating of a credit check agency, and will be further refined as the audit case progresses. Entities may be abstractions of real individuals, *e.g.*, the notion of the '*chief financial officer of company X*' is an abstraction of a physical person who may be replaced while the audit proceeds. Abstraction plays an important role in the auditing domain due to the typical confrontation with vast amounts of business data, which demands to abstract from irrelevant details in order to reduce the complexity of audit problems. Another aspect is driven from research in the audit community to interpret auditing by means of game-theoretic techniques [36; 85; 274]. This can also be played from the perspective of constructive DLs, by expressing interactive games in the spirit of Lorenzen's Dialogical Logic [178; 239] between an auditee (proponent) and an auditor (opponent) as a game-theoretic decision procedure. In order to ensure the validity of audit statements, the notion of constructive proof relying on concrete evidence becomes an important aspect. For instance, in order to prove the existence of fraud, an auditor is required to provide objective evidence in the form of concrete facts or witnessing data. To summarise, auditing requires a constructive approach based on positive evidence that is robust under abstraction and refinement.

Another example where a constructive interpretation becomes prominent is the process of requirements engineering [237; 240; 279] in the software engineering lifecycle [218], which refers to the process of specifying, maintaining and documenting the requirements of a piece of software. Usually, a software development project begins with a requirements analysis, consisting of requirements elicitation, analysis and specification, with the goal to define at an abstract level the necessary software components, the goals and invariants that must be met to satisfy the customers' demands. In the subsequent software engineering process, these abstract software components are refined and evolve, *e.g.*, by explicitly specifying a software architecture, developing interfaces between components to allow for communication, or by implementing individual parts, which are later linked to form larger compound components. For example, let us consider the software component that controls the door system of a train. An important security requirement is that during a journey the doors of the train cannot be opened accidentally. Later on, this software component may be reused and combined with the software components of a control system as used in a slow urban train as well as its counterpart that is used in a high-speed train. In either case, it is important that the security requirement is still satisfied by these implementations. At every stage of a software development project it may be important that the initially defined requirements are still satisfied (requirements validation) by the intermediate development results in the sense that a requirement holds robustly w.r.t. abstraction and refinement of software components. For instance, if a requirement is specified in terms of a DL formula, then its interpretation has to take into account not only the finished (final) software component, but also all intermediate and future stages in which the software component may evolve. Classical semantics of DLs only allow reasoning about one stage of a component in the software development process, and assume that each such component corresponds to a final state. However, a software component may be subject to a steady update cycle, *e.g.*, by adding new features and/or fixing defects. Moreover, a component can be developed in different branches to account for mutually exclusive demands and may possibly never reach a final state. In order to ensure the validity of requirements, the semantics and the notion of proof need to support the interpretation of components relative to states of knowledge, where once established facts accumulate over time in the stages of a development process.

The problem of dynamic knowledge has recently been addressed for classical DLs by allowing to update the actual state of affairs at the instance level (ABox) [75; 76; 129; 177] or by extending the calculus with actions designed to dynamically change parts of a knowledge base [57; 202; 236]. For a comprehensive survey of the update problem specific to DLs see [177; 202]. However, while these approaches allow to represent the dynamic behaviour and the evolution of knowledge by applying modifications to the knowledge base, they do still rely on classical semantics and are therefore not compatible with the notion of positive evidence and realisability [262; 265]. One fundamental constructive interpretation (although informal) is the *Brouwer–Heyting–Kolmogorov* (BHK) interpretation [252, Chap. 2],[265] which is based on the notion of *proof*. For instance, in constructive logics the proof of a disjunction $C \sqcup D$ is by providing a positive evidence in the form of a proof for either C or D. In classical DLs the law of the Excluded Middle $C \sqcup \neg C$ is assumed to be true independently of the choice of any of the disjuncts. However, for an arbitrary concept C (which may refer to an open problem in mathematics) we do not know in general whether C or its negation has a proof. Constructive logic meets our expectations in that it only accepts constructive reasoning based on the notion of proof.

Finally, another important domain, which is prominent in the context of computer science, concerns the computational interpretation of DLs. Such constructive interpretations allow to exploit the computational properties of constructive DLs by relating their proof systems with type theories following the Curry-Howard isomorphism [113; 139; 248; 265], also known as the *proofs-as-computations* or *formulas-as-types interpretation*. Under this view, DL concept descriptions can specify the type of programs, *e.g.*, the constructive interpretation of an implication $C \supset D$ corresponds to a function of type $C \rightarrow D$, conjunction $C \sqcap D$ to the product type $C \times D$ and disjunction $C \sqcup D$ to the type of a disjoint sum $C + D$. In a similar vein, the proof of an implication $C \supset D$ can be seen as a construction, which gives a function from C to D, and according to the BHK interpretation [252, Chap. 2], this can also be viewed as a procedure that transforms proofs of C to proofs of D. The general benefits of the Curry-Howard isomorphism in the context of DLs have been argued in [41; 78; 196] and its development is in line with computational interpretations of intuitionistic modal logics (see Sec. 2.2.2). Based on this connection, constructive DLs can establish the base of a DL-based programming language system that serves as a typing system for an extension of the simply typed λ-calculus [74; 194; 208], or adapts the proofs-as-programs correspondence towards program synthesis [39; 41; 201].

To summarise, classic DLs are not sufficient to represent or reason about partial or incomplete knowledge and to achieve consistency under abstraction as well as robustness under refinement. From a proof-theoretical point of view, constructive logic is compatible with the idea of positive evidence and it does not infer the presence of objects from the absence of others. Instead, it insists on the existence of computational witnesses. Model-theoretically, constructive semantics support the notion of stages of knowledge with the property that once established evidence remains persistent under refinement and may only potentially increase. These features of constructive logic can be very important in the application domains of DLs (knowledge representation, life sciences, health care, software engineering, auditing, programming) such that it seems worth to explore constructive variants of DLs.

1.2 Aims and Contributions

This thesis is devoted to the investigation of the model theory and proof theory of a *constructive* variant of the basic description logic $\mathcal{ALC}$, which will be called $c\mathcal{ALC}$. Our approach is to replace the traditional binary truth interpretation of the basic DL $\mathcal{ALC}$ by a constructive notion of truth. We require this interpretation to be compatible with the idea of concepts comprising abstract entities with hidden fine structure, to represent partial and incomplete information, and on the other hand to comply with the Curry-Howard isomorphism in order to lay the grounds for a DL-based typing system.

Our investigation considers the strong relationship between the description logic $\mathcal{ALC}$ and the basic modal logic K_m, but lifted to a constructive perspective. Hereby, we follow de Paiva's [78] proposal, by basing $c\mathcal{ALC}$ on a constructive analogue of the modal logic K_m. From the viewpoint of a constructivist, the notion of construction and positive evidence are essential to argue the proof or to give a witness for existential statements. For instance, whenever $C \sqcup D$ is a theorem then constructive logic demands that one knows that either C or D holds. Similarly, a constructive proof of an existential statement $\exists x.\phi(x)$ requires that one can explicitly construct some term t that proves $\phi(t)$. These properties (a.k.a. *disjunction property* and *witness property*) are key characteristics of constructive theories. Furthermore, constructive logic refutes the law of the *Excluded Middle* $C \sqcup \neg C$ and the classical dualities, such that the logical operators are not expressible in terms of each other and negation. Simpson [249] postulates as a *sine qua non* of an intuitionistic analogue of K, that it is a conservative extension of propositional intuitionistic logic, and moreover, he requires that the addition of the law of the Excluded Middle collapses the theory to classical K. However, there exists no consensus on the minimal constructive analogue of the modal logic K. Instead, several different proposals for minimal *intuitionistic modal logics* (IMLs) corresponding to K exist, which on the one hand differ in the structure of their descriptive semantics, and on the other hand consider different ways of interpreting the modalities $\Box$ and $\Diamond$. This thesis establishes a minimal constructive analogue of $\mathcal{ALC}$ that can be taken as a base line from which a correspondence theory for more expressive constructive description logics (and modal logics) can be attempted.

The system $c\mathcal{ALC}$ will be derived from the constructive modal logic CK [27; 188; 272]. We approach the model theory of $c\mathcal{ALC}$ by giving an intuitionistic birelational interpretation of the standard Kripke semantics, in the sense that the intuitionistic epistemic order representing states of knowledge and the modal accessibility relations in the form of roles are relations on the same domain. In particular, this semantics ensures that any intuitionistically invalid formulæ are no longer validated. This will be addressed by providing counterexamples for principles that are classically valid, but invalid intuitionistically, and by showing that $c\mathcal{ALC}$ meets the *disjunction property.* We introduce several examples, where a constructive interpretation is essential to allow for the representation of dynamic and incomplete knowledge, as well as to explain phenomena that admit a constructive consistent explanation, while being inconsistent under the classical semantics. In contrast to standard intuitionistic modal logics, $c\mathcal{ALC}$ does not require any frame conditions, and it is non-normal in that (i) it refutes the axiom schema of *disjunctive distribution* in its binary form $\exists R.(C \sqcup D) \supset (\exists R.C \sqcup \exists R.D)$ and nullary variant $\neg\exists R.\bot$, (ii) it rejects the *interaction schema* $(\exists R.C \supset \forall R.D) \supset \forall R.(C \supset D)$, and (iii) the addition of the Excluded Middle does not collapse the theory to classical $\mathcal{ALC}$. This gives rise to a constructive DL that can be characterised by a minimal axiomatic system. Moreover, we provide a finite semantic characterisation of $c\mathcal{ALC}$, by proving the *finite model property.*

We address the question of how $c\mathcal{ALC}$ and classical DLs are related, and whether our constructive semantics provides a complexity advantage over the classical descriptive semantics. We investigate the relation of $c\mathcal{ALC}$ to classical DLs by demonstrating a faithful embedding of $c\mathcal{ALC}$ into a classical bimodal logic. This allows us on the one hand to transfer existing results from bimodal logics to $c\mathcal{ALC}$, and on the other hand gives us access to the existing reasoning technology of classical DLs. Surprisingly, it turns out that the problem of subsumption checking in the sub-Boolean $\{\sqcup, \exists\}$-fragment $\mathcal{UL}$ of $\mathcal{ALC}$ w.r.t. general TBoxes is *tractable* under the constructive semantics, while it is intractable under the classical descriptive semantics.

This thesis explores $c\mathcal{ALC}$ from a proof theoretic perspective as well. We characterise the system $c\mathcal{ALC}$ in terms of a *Hilbert-style axiomatisation* that gives a clear characterisation of the logical system in terms of its axioms and inference rules. Moreover, we establish a modal *deduction theorem* for $c\mathcal{ALC}$ that generalises the Hilbert-style axiomatisation into a system to allow for derivations from global and local assumptions. However, while Hilbert-style calculi are useful for investigating a logical system, they are not the right foundation to develop automated inference services as demanded by DL-style applications. DL-systems expect decidable reasoning problems that can be implemented with an acceptable run-time characteristic. Better suited for automated proof search are refutation-based calculi, *i.e.*, given a concept C the calculus aims at finding an interpretation that satisfies C. Consequently, if a proof for the statement *C is false* fails, then this implies that concept C is valid. We choose to provide a *Gentzen-style sequent calculus* and a *tableau-based calculus* for $c\mathcal{ALC}$ w.r.t. TBoxes. Such calculi satisfy the subformula principle, support goal-directed proof-search and allow for countermodel construction. We present a sound and complete *multi-conclusion* Gentzen-style sequent calculus for $c\mathcal{ALC}$, showing the *finite model property* and *decidability*, and that implements independent left and right introduction rules for the modalities $\exists R$ and $\forall R$. Sequent calculi are consistent with the Curry-Howard isomorphism and permit proof-extraction as well as a constructive interpretation of proofs as λ-terms. Where the application demands it, one can specialise $c\mathcal{ALC}$ back towards normal IMLs or even $\mathcal{ALC}$ by adding axioms, or by strengthening the proof systems accordingly by additional rules of inference.

Towards an implementation, we give an *effectively decidable tableau calculus* that is sound and complete, and which combines the data structure of a constraint system with methods from intuitionistic tableaux to allow for constructive reasoning. Furthermore, we analyse the problem of interpreting ABox assertions and nominals under the birelational semantics of $c\mathcal{ALC}$ and sketch an approach towards constructive ABox reasoning in $c\mathcal{ALC}$.

1.3 Synopsis of the Thesis

Chapter 2 introduces the key concepts and preliminaries of this thesis and is essentially introductory. Section 2.1 gives an introduction to the field of *description logics* by establishing the basic DL $\mathcal{ALC}$, the key concepts of reasoning w.r.t. terminological and assertional knowledge, and the standard inference problems. The section closes with a review of the most important language extensions of $\mathcal{ALC}$, and discusses the correspondence of $\mathcal{ALC}$ with the basic modal logic K on the basis of the standard translation. Constructive logic is shortly examined in Section 2.2, by defining the Kripke semantics of *intuitionistic propositional logic* (IPC) and by reviewing the structure of the semantics of *intuitionistic first-order logic* (IQC). The following Section 2.2.2 is devoted to intuitionistic modal logics, and presents the axiomatisation and Kripke-style *birelational* possible world semantics of two constructive variants of the basic modal logic K, namely *intuitionistic* K (IK) and *constructive* K (CK).

Chapter 3 presents a detailed survey of the current state of research in the field of constructive description logics.

Chapter 4 introduces our approach to constructive DLs and is dedicated to the investigation of the model theory of $c\mathcal{ALC}$. Section 4.1 justifies our choice to rely on birelational Kripke-style semantics by reviewing the existing approaches to define the semantics. In Section 4.2 we introduce the syntax and semantics of $c\mathcal{ALC}$, investigate and prove some basic model theoretic properties like the *monotonicity property*, the reasoning tasks w.r.t. TBoxes, and the *disjunction property*, and discuss several examples where the classical semantics are not adequate. The chapter finishes with the proof of the *finite model property* for $c\mathcal{ALC}$, based on the filtration method.

The proof theory of $c\mathcal{ALC}$ is investigated in Chap. 5 by introducing Hilbert and Gentzen-style deduction systems for $c\mathcal{ALC}$. Section 5.1 introduces a sound and complete Hilbert-style axiomatisation for $c\mathcal{ALC}$ and proves several meta-theorems of this calculus and a *modal deduction theorem*. Section 5.2 presents a *multi-conclusion Gentzen-style sequent calculus* for $c\mathcal{ALC}$, a proof of its *soundness and completeness* w.r.t. the birelational semantics of $c\mathcal{ALC}$, the *finite model property* as well as *decidability*. Soundness and completeness of the Hilbert system is demonstrated relative to the Gentzen sequent calculus by showing that both systems are equivalent. The third section gives an outlook on the intermediate systems between $c\mathcal{ALC}$ and $\mathcal{ALC}$ that arise from the extension of $c\mathcal{ALC}$ by several axioms.

Chapter 6 examines in Section 6.1 the relation of $c\mathcal{ALC}$ to classical DLs by demonstrating a faithful embedding of $c\mathcal{ALC}$ into a classical bimodal logic, which corresponds to a classical DL. The embedding allows us to transfer results from normal bimodal logics, and lets us obtain the *finite model property*, *decidability*, and *complexity results* for $c\mathcal{ALC}$. Section 6.2 discusses the sub-Boolean fragment $\mathcal{UL}$ of $c\mathcal{ALC}$ that turns out to be *tractable* under the constructive semantics, while it is intractable under the classical descriptive semantics.

In Chapter 7, we introduce a labelled tableau calculus for $c\mathcal{ALC}$ that is based on a constraint system with an explicit handling of the intuitionistic preorder, and present a proof of its *termination, soundness and completeness*, which gives rise to an effective decision procedure for $c\mathcal{ALC}$. The final part discusses the problem of interpreting ABox assertions under the birelational semantics and gives an outlook on how to address constructive ABox reasoning. Finally, we conclude our approach in Chapter 8 and highlight future perspectives and open problems.

1.4 Publications

Within the framework of this thesis, the following publications have appeared, which examine the constructive description logic $c\mathcal{ALC}$ w.r.t. its model theory and proof theory, as well as from a type-theoretical perspective by establishing a computational interpretation in form of a modal extension of the simply typed λ-calculus. This thesis focusses on the development of the model theory and proof theory of $c\mathcal{ALC}$, extending the results from [189–193; 195]. It does not cover the computational and type-theoretic perspective. Readers interested in the type-theoretical perspective may consult [194; 196–198].

[189] M. Mendler and S. Scheele. ‘Constructive Description Logic $c\mathcal{ALC}$ as a Type System for Semantic Streams in the Domain of Auditing’. In: *Proc. of the 1st International Workshop on Logics for Agents and Mobility (LAM 2008).* (4th–8th Aug. 2008). Ed. by B. Farwer and M. Köhler-Bußmeier. Vol. 283. Berichte des Departments Informatik. Hamburg, Germany: University of Hamburg, Aug. 2008.

[190] M. Mendler and S. Scheele. *Towards Constructive Description Logics for Abstraction and Refinement.* Bamberger Beiträge zur Wirtschaftsinformatik und Angewandten Informatik 77. Faculty of Information Systems and Applied Computer Sciences, Otto-Friedrich-University of Bamberg, Germany, Sept. 2008.

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[193] M. Mendler and S. Scheele. 'Towards a Type System for Semantic Streams'. In: *Proc. of the 1st International Workshop on Stream Reasoning (SR 2009).* Ed. by E. D. Valle, S. Ceri, D. Fensel, F. van Harmelen and R. Studer. Vol. 466. CEUR Workshop Proceedings. Heraklion, Crete, Greece: CEUR-WS.org, May 2009.

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[196] M. Mendler and S. Scheele. 'Cut-free Gentzen calculus for multimodal CK'. In: *Information and Computation* **209** (12) (Dec. 2011). Special Issue: Intuitionistic Modal Logic and Applications (IMLA 2008), pp. 1465–1490.

[197] M. Mendler and S. Scheele. *On the Computational Interpretation of* CK_n *for Contextual Information Processing – Ancillary Material –*. Bamberger Beiträge zur Wirtschaftsinformatik und Angewandten Informatik 91. Faculty of Information Systems and Applied Computer Sciences, Otto-Friedrich-University of Bamberg, Germany, May 2013.

[198] M. Mendler and S. Scheele. 'On the Computational Interpretation of CK_n for Contextual Information Processing'. In: *Fundamenta Informaticæ* **130** (1) (Jan. 2014), pp. 125–162.

CHAPTER 2

Background

This chapter gives a formal introduction to the necessary preliminaries and most of its part is a rearrangement of the existing literature. It aims to give readers unfamiliar with description logic and intuitionistic modal logic the necessary background to understand the main parts of this thesis. Readers familiar with description logic and intuitionistic modal logic can skip this chapter, or may skim through its sections to get used to the notation and terms used in this work. Section 2.1 introduces the field of DLs with the formal definition of the syntax and semantics of the basic *Attribute Language with Complement* ($\mathcal{ALC}$) covering its logical constructors and the formalism to specify assertional and terminological knowledge. Thereof, we briefly review the typical inference problems and discuss the relationship with other logical formalisms, focussing on the correspondence to modal logics. In Sec. 2.2, we give a short introduction to the role of constructive logic and its application in the field of computer science, introduce intuitionistic propositional logic and shortly revisit the semantics of intuitionistic first-order logic. Section 2.2.2 is devoted to intuitionistic modal logics (IMLs, for short). We will inspect the axiomatisation and Kripke-style possible world semantics of two representative IMLs. They will form the starting point for the development of constructive DLs, as we will see in Chap. 4.

Notation. Throughout this work we use the following notational conventions. Newly introduced terms are set in *italic* type. The symbols $\Box$, ∇ and $\blacksquare$ are used to indicate the end of each proof, definition and example (and notation, remark) respectively. $\blacksquare$

2.1 Description Logic

This section formally introduces the basic DL $\mathcal{ALC}$ and examines its close correspondence to the basic normal modal logic K.

2.1.1 The Basic Language $\mathcal{ALC}$

The basic description logic $\mathcal{ALC}$ has been introduced by Schmidt-Schauß and Smolka [247] and corresponds to the *smallest* DL which is closed under all Boolean connectives.

Definition 2.1.1 (Syntax of $\mathcal{ALC}$ [16, p. 52])**.** A DL signature is a structure $\Sigma = (N_C, N_R, N_I)$ of three denumerable and pairwise disjoint alphabets of concept names N_C, role names N_R and individual names N_I. The set of $\mathcal{ALC}$ *concept descriptions* ($\mathcal{ALC}$-concepts, for short) over signature Σ is the smallest set such that

- each atomic symbol $\perp$ (truth), $\top$ (falsehood) and $A \in N_C$ is an $\mathcal{ALC}$ concept description, and

- if C and D are $\mathcal{ALC}$ concept descriptions and $R \in N_R$ is a role then $\neg C$, $C \sqcap D$, $C \sqcup D$, $\exists R.C$ and $\forall R.C$ are $\mathcal{ALC}$ concept descriptions. ∇

The syntax of $\mathcal{ALC}$ is redundant, because in classic logic DeMorgan's law holds, *i.e.*, the logical operators and the constants $\perp, \top$ are interdefinable in terms of each other via negation, e.g., $\perp = \neg\top$, $(C \sqcap D) = \neg(\neg C \sqcup \neg D)$, $\forall R.C = \neg(\exists R.\neg C)$ and $\exists R.C = \neg(\forall R.\neg C)$. Also note that implication $C \supset D$ is equivalent to $\neg C \sqcup D$ and therefore usually omitted from the language $\mathcal{ALC}$.

Notation. Throughout this work we use the following notation: In definitions and abstract examples we use the letters A and B for *concept names*, and the letter R for *role names*. The letters C and D are used to refer to arbitrary concept descriptions. In concrete examples, the first letter of a concept name is written in uppercase, followed by the remaining letters in lowercase, e.g., Male, Female, Father, Amphibian, Toad, Treefrog. Role names begin with a lowercase letter, e.g., hasChild, hasColour, hasBinomialName and individual names are written uppercase, e.g., IGGY, KERMIT, RED, BLUE, HYLA_ARBOREA. ■

Example 2.1.1. For instance, consider the concept of a manager who manages a project in which all participants are employees and have a Masters degree or a PhD.

$$\mathsf{Male} \sqcap \exists \mathsf{doesManage}.(\mathsf{Project} \sqcap \forall \mathsf{isInvolved}.(\mathsf{Employee} \sqcap \forall \mathsf{hasDegree}.(\mathsf{MSc} \sqcup \mathsf{PhD}))).$$

Further, consider the following concept describing a Golden Poison Frog, a frog species that is categorised as likely to become extinct, whose toxic secretions belong to the group of alkaloids and is used by native Americans to poison the tips of blow-darts.

$$\begin{aligned}&\mathsf{Frog} \sqcap \exists \mathsf{hasColour}.\mathsf{Gold} \sqcap \exists \mathsf{conservationStatus}.\mathsf{EndageredSpecies}\\ &\qquad \sqcap \exists \mathsf{hasPoison}.(\mathsf{AlcaloidToxin} \sqcap \exists \mathsf{usedFor}.\mathsf{Blowdart}).\end{aligned}$$ ■

The semantics of concept descriptions is given in terms of a Tarski-style model-theoretic *interpretation* [16; 247].

Definition 2.1.2 ([16, pp. 51 f.]). An *interpretation* over signature $\Sigma = (N_C, N_R, N_I)$ is a structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where

- $\Delta^{\mathcal{I}}$, called the domain of $\mathcal{I}$ of *individuals*, is a non-empty set, and
- $\cdot^{\mathcal{I}}$ is an interpretation function mapping each atomic concept A to a subset $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and each atomic role R to a binary relation $R^{\mathcal{I}} \subseteq (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}})$.

The interpretation is lifted from atomic symbols to arbitrary concept descriptions by the following inductive definition:

$$
\begin{aligned}
\top^{\mathcal{I}} &=_{\mathit{df}} \Delta^{\mathcal{I}} && \text{(top)}\\
\bot^{\mathcal{I}} &=_{\mathit{df}} \emptyset && \text{(bottom)}\\
(\neg C)^{\mathcal{I}} &=_{\mathit{df}} \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} && \text{(negation)}\\
(C \sqcap D)^{\mathcal{I}} &=_{\mathit{df}} C^{\mathcal{I}} \cap D^{\mathcal{I}} && \text{(conjunction)}\\
(C \sqcup D)^{\mathcal{I}} &=_{\mathit{df}} C^{\mathcal{I}} \cup D^{\mathcal{I}} && \text{(disjunction)}\\
(\exists R.C)^{\mathcal{I}} &=_{\mathit{df}} \{x \mid \exists y \in \Delta^{\mathcal{I}}.\ (x,y) \in R^{\mathcal{I}}\ \ \&\ \ y \in C^{\mathcal{I}}\} && \text{(existential restriction)}\\
(\forall R.C)^{\mathcal{I}} &=_{\mathit{df}} \{x \mid \forall y \in \Delta^{\mathcal{I}}.\ (x,y) \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\} && \text{(value restriction)} \qquad \nabla
\end{aligned}
$$

We will denote the individual that corresponds to the second argument of a role by *R-successor* or *R-filler*, relative to some role name R.

2.1.2 DL Knowledge Base

The domain knowledge is represented by an *ontology*, also called DL *knowledge base*, which is given by an axiomatisation that restricts the class of admissible interpretations by imposing conditions on the frame (role axioms), on models (terminological axioms) or individuals (assertions). A knowledge base $\mathcal{K}$ (or an ontology) is defined as a pair $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, which consists of a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$. The former represents the *terminology*, or terminological axioms, *i.e.*, the vocabulary of an application domain. The latter states *assertions*, which specify facts about particular named individuals of the application domain.

Definition 2.1.3 (Syntax of $\mathcal{ALC}$ terminological axioms [16, pp. 55 ff.]). Let $R \in N_R$ be a role, $A \in N_C$ be a concept name and C, D be concept descriptions. Terminological axioms have the form of a *general concept inclusion* (GCI) $C \sqsubseteq D$, or a concept equivalence $C \equiv D$ (abbreviating $C \sqsubseteq D$ & $D \sqsubseteq C$). TBox axioms can be restricted in that their left-hand side is an atomic concept. Then, a concept equation of the form $A \equiv C$ is called *concept definition* and introduces a symbolic name for a concept description, whereas the expression $A \sqsubseteq C$ is called a *primitive concept definition.* ∇

The DL literature covers several kinds of TBoxes [16, pp. 55 ff.], which vary in their level of expressivity. The choice of a TBox formalism has a direct impact on the complexity of the various inference problems.

Definition 2.1.4 ((General, definitorial) TBox [16, pp. 56 ff.])**.** A *general TBox* (or TBox for short) is a finite set $\mathcal{T}$ of GCIs. A *definitorial* TBox is a set $\mathcal{T}$ of (primitive) concept definitions such that the following conditions hold:

- It contains only (primitive) concept definitions of the form $A \equiv C$ or $A \sqsubseteq C$.
- For each concept name $A \in N_C$, there exists at most one axiom in $\mathcal{T}$ with A as the left-hand side, and
- $\mathcal{T}$ is *acyclic*, *i.e.*, the definition of any concept A does not reference itself. Formally, an *acyclic* TBox is defined as follows: We say that $A \in N_C$ *directly uses* $B \in N_C$ if $A \equiv C \in \mathcal{T}$ or $A \sqsubseteq C \in \mathcal{T}$ and B occurs in C, and define *uses* as the transitive closure of the relation *directly uses*. Then, a TBox $\mathcal{T}$ is *acyclic* if there exists no atomic concept A in $\mathcal{T}$ that *uses* itself. ∇

In contrast to general TBoxes, the specification of definitorial TBoxes can be considered as a language of macro definitions. Reasoning w.r.t. acyclic TBoxes can be rephrased to the problem of reasoning w.r.t. definitorial TBoxes, using an expansion of the TBox [16, pp. 57 ff.].

Example 2.1.2. Consider the following TBox.

$$\begin{aligned}
\mathsf{Frog} &\equiv \mathsf{Amphibian} \sqcap \mathsf{Animal} \\
\mathsf{EuropeanTreeFrog} &\equiv \mathsf{Frog} \sqcap \exists \mathsf{liveIn}.\mathsf{EuropeanCountry} \sqcap \exists \mathsf{hasColour}.(\mathsf{Green} \sqcup \mathsf{Brown}) \\
\mathsf{PoisonDartFrog} &\equiv \mathsf{Frog} \sqcap \exists \mathsf{hasPoison}.\mathsf{Toxin} \sqcap \exists \mathsf{liveIn}.\mathsf{AmericanCountry} \\
\mathsf{GoldenPoisonFrog} &\equiv \mathsf{PoisonDartFrog} \sqcap \exists \mathsf{hasColour}.\mathsf{Gold} \\
\mathsf{GoldenPoisonFrog} &\sqsubseteq \forall \mathsf{hasPoison}.\mathsf{AlkaloidToxin} \\
\mathsf{Amphibian} \sqcap \mathsf{Mammal} &\equiv \bot
\end{aligned}$$

The first four statements introduce the concepts Frog, $\mathsf{EuropeanTreeFrog}$, $\mathsf{PoisonDartFrog}$ and $\mathsf{GoldenPoisonFrog}$ as concept definitions. The fifth statement expresses that instances of $\mathsf{GoldenPoisonFrog}$ are required to satisfy $\forall \mathsf{hasPoison}.\mathsf{AlkaloidToxin}$. The last statement expresses that the concepts $\mathsf{Amphibian}$ and Mammal must be disjoint. ■

Definition 2.1.5 (Semantics of $\mathcal{ALC}$ terminological axioms [16, pp. 56 f.])**.** An interpretation $\mathcal{I}$ *satisfies* a concept inclusion $C \sqsubseteq D$, written $\mathcal{I} \models C \sqsubseteq D$, iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

$\mathcal{I}$ *satisfies* an equality $C \equiv D$, written $\mathcal{I} \models C \equiv D$, iff $C^{\mathcal{I}} = D^{\mathcal{I}}$. An interpretation $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, written $\mathcal{I} \models \mathcal{T}$, iff it satisfies each axiom in $\mathcal{T}$. ∇

The second component of a knowledge base is represented by an ABox that specifies a concrete set of individuals as instances of concepts and roles.

Definition 2.1.6 (Assertional axioms [16, pp. 65 f.])**.** Let $\Sigma = (N_C, N_R, N_I)$ be an $\mathcal{ALC}$ signature, $a, b \in N_I$ individual names, $R \in N_R$ a role and C, D concept descriptions over Σ. *Assertional axioms* are of the form of $a : C$ (concept assertions) and $a\,R\,b$ (role assertions), also written as $(a, b) : R$. The former specifies that a is an instance of concept C, while the latter states that b is a *filler* of the role R for a. A finite set $\mathcal{A}$ of assertional axioms is called *ABox*. ∇

The set of individual names that occur in an ABox $\mathcal{A}$ is called the support of $\mathcal{A}$, written $Supp(\mathcal{A})$.

Example 2.1.3. The following ABox formalises a concrete situation based on the TBox from Ex. 2.1.2, using the individuals KERMIT, IGGY, FRANCE, LIME, COLOMBIA and BATRACHOTOXIN.

$$\begin{aligned} \mathsf{KERMIT} : \mathsf{EuropenTreeFrog}, &\qquad \mathsf{IGGY} : \mathsf{GoldenPoisonFrog}, \\ (\mathsf{KERMIT}, \mathsf{LIME}) : \mathsf{hasColour}, &\qquad (\mathsf{IGGY}, \mathsf{COLOMBIA}) : \mathsf{liveIn}, \\ (\mathsf{KERMIT}, \mathsf{FRANCE}) : \mathsf{liveIn}, &\qquad (\mathsf{IGGY}, \mathsf{BATRACHOTOXIN}) : \mathsf{hasPoison}. \end{aligned}$$

■

Definition 2.1.7 (Semantics of $\mathcal{ALC}$ assertional axioms [16, pp. 66 f.])**.** An *extended interpretation* for ABox assertions is an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where the interpretation function $\cdot^{\mathcal{I}}$ also maps each individual name $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. An extended interpretation $\mathcal{I}$ *satisfies* a concept assertion $a : C$, written $\mathcal{I} \models a : C$, if $a^{\mathcal{I}} \in C^{\mathcal{I}}$, and it *satisfies* the role assertion $a\,R\,b$, written $\mathcal{I} \models a\,R\,b$, if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$. An extended interpretation $\mathcal{I}$ is a *model* of an ABox $\mathcal{A}$, written $\mathcal{I} \models \mathcal{A}$, iff it is a model of each axiom in $\mathcal{A}$. Finally, an extended interpretation satisfies an assertion α or models an ABox $\mathcal{A}$ with respect to a TBox $\mathcal{T}$ iff $\mathcal{I} \models \alpha$ or $\mathcal{I} \models \mathcal{A}$, respectively, and additionally $\mathcal{I} \models \mathcal{T}$. ∇

The semantics of a *knowledge base* is defined as follows:

Definition 2.1.8 (DL knowledge base [16, pp. 50 f.])**.** An extended interpretation $\mathcal{I}$ is a *model* for a knowledge base $\mathcal{K}$, written $\mathcal{I} \models \mathcal{K}$, iff $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$. We also say that $\mathcal{I}$ *satisfies* $\mathcal{K}$. ∇

2.1.3 Standard Inference Problems

This section will shortly review the traditional *standard inference problems* of DLs, which can be classified into reasoning w.r.t. a TBox and an ABox.

Definition 2.1.9 (Reasoning w.r.t. a TBox [16, pp. 67 f.])**.** Let $\mathcal{T}$ be a TBox and C, D be concept descriptions.

- A concept C is called *satisfiable* w.r.t. $\mathcal{T}$ if there is a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}}$ is non-empty.
- A concept C *subsumes* a concept D w.r.t. $\mathcal{T}$, if for every model $\mathcal{I}$ of $\mathcal{T}$ it holds that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- Two concepts C and D are *equivalent* w.r.t. $\mathcal{T}$, if for every model $\mathcal{I}$ of $\mathcal{T}$ it holds that $C^{\mathcal{I}} = D^{\mathcal{I}}$.
- Two concepts C and D are *disjoint* w.r.t. $\mathcal{T}$ if for every model $\mathcal{I}$ of $\mathcal{T}$ it holds that $C^{\mathcal{I}} \cap D^{\mathcal{I}} = \emptyset$. ∇

Definition 2.1.10 (Reasoning w.r.t. an ABox [16, pp. 72 ff.])**.** Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base, $a, b \in N_I$ individual names and C, D concept descriptions.

- An ABox $\mathcal{A}$ is *consistent* w.r.t. a TBox $\mathcal{T}$ if there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{A}$ and $\mathcal{I} \models \mathcal{T}$.
- An individual a is called an *instance* of a concept C w.r.t. $\mathcal{K}$, written $\mathcal{K} \models a : C$, if it holds for all models $\mathcal{I}$ of $\mathcal{K}$ that $\mathcal{I} \models a : C$. Analogously, a tuple (a, b) is an instance of a role R w.r.t. $\mathcal{K}$, written $\mathcal{K} \models a\, R\, b$, if $\mathcal{I} \models a\, R\, b$ holds in all models $\mathcal{I}$ of $\mathcal{K}$.
- The *instance retrieval problem* for a concept C is to obtain all individuals $a \in N_I$ such that $\mathcal{K} \models a : C$. ∇

2.1.4 Language Extensions of $\mathcal{ALC}$

There exist several language extensions of $\mathcal{ALC}$ [16; 173] yielding more expressive DLs, where letters indicate the set of allowed concept and axiom constructors. For instance, $\mathcal{ALC}_{R^+}$ extends $\mathcal{ALC}$ by allowing transitive roles, for which sometimes the abbreviation $\mathcal{S}$ is used instead. $\mathcal{SHI}$ is $\mathcal{S}$ extended by role hierarchies and inverse roles. Further adding nominals yields $\mathcal{SHOI}$, which can be further extended by qualified cardinality restrictions to give $\mathcal{SHOIQ}$. One of the most expressive DLs is represented by $\mathcal{SHROIQ}$ [173], which results from the extension of $\mathcal{SHOIQ}$ by a universal role

Concept Constructors			
	Syntax	Semantics	Symbol
Transitive role	R^+	$\forall x, y, z.\ x\, R^{\mathcal{I}}\, y \wedge y\, R^{\mathcal{I}}\, z \Rightarrow x\, R^{\mathcal{I}}\, z$	$\mathcal{S}$
Inverse role	R^-	$\{(x, y) \mid (y, x) \in R^{\mathcal{I}}\}$	$\mathcal{I}$
Qualified cardinality restriction	$\leq nR.C$	$\{x \mid \lvert\{y \in C^{\mathcal{I}} \mid (x, y) \in R^{\mathcal{I}}\}\rvert \leq n\}$	$\mathcal{Q}$
	$\geq nR.C$	$\{x \mid \lvert\{y \in C^{\mathcal{I}} \mid (x, y) \in R^{\mathcal{I}}\}\rvert \geq n\}$	$\mathcal{Q}$
Nominal	$\{a\}$	$\{a^{\mathcal{I}}\}$	$\mathcal{O}$
Axioms			
Role hierarchy	$R \sqsubseteq S$	$R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$	$\mathcal{H}$
Complex role inclusion	$R_1 \circ R_2 \sqsubseteq S$	$R_1^{\mathcal{I}} \circ R_2^{\mathcal{I}} \subseteq S^{\mathcal{I}}$	$\mathcal{R}$

Figure 2.1: Language extensions – syntax and semantics [*cf.* 16, pp. 525 ff.; 173, p. 11].

U and complex role inclusion axioms. The syntax, semantics and the indicating letter of some extensions are depicted in Figure 2.1.

2.1.5 Relation to Modal Logics

Modal logic originally emerged as an extension of classical logic by adding new one-ary operators (usually called modalities) which qualify the truth of sentences. Traditionally, propositional modal logic extends the pure propositional language (CPC) by the modalities of necessity $\Box$ and possibility $\Diamond$, where $\Box\phi$ is read as '*ϕ is necessarily true*' and its dual $\Diamond\phi$ stands for '*ϕ is possibly true*'. Multimodal logics allow for more than one modal operator, usually in the form of labelled modalities like $\Box_i$ and allow to be used in a multiagent-alike environment. There, $\Box_i\phi$ can be interpreted as '*agent i knows ϕ*' or '*ϕ is true after executing action i*' [130]. The development of other modalities led to the invention of various kinds of modal logics, e.g. temporal logic, epistemic logic, deontic logic, dynamic logic, etc., which have found numerous applications in the fields of mathematics, computer science and artificial intelligence. For a comprehensive survey on the evolution of modal logic see [33, Chap. 1.7; 117], and [62; 266] giving an in-depth introduction to the field.

Description logics have a very close relationship with modal logics [33; 63; 103; 151; 229], which has already been observed two decades ago by Schild [245]. He demonstrated that several description logics are notational variants of different propositional modal and dynamic logics and exploited this correspondence to transfer results from the latter logics, such as complexity results, the finite model property, decision algorithms and axiomatisations to description logics. In particular, Schild [245] showed that the description logic $\mathcal{ALC}$ [247] is a notational variant of the multimodal logic K_m [33; 101].

Subsequent efforts [7; 112; 148; 213; 243; 246] observed similar relationships between description logics and more expressive modal logics, transferring further results from the μ-calculus [164; 165], propositional and modal dynamic logics [95; 131], hybrid logics [8; 9; 34] and modal logics including graded modalities (number restrictions) [141; 142] or frames with extended accessibility relations (reflexive, transitive, symmetric, etc.) to description logics. For a detailed overview on the connection of description logics with other formalisms see [16, Chap. 4]. In this section we will highlight the relation between description logics and modal logics, in particular the correspondence between $\mathcal{ALC}$ and the propositional modal logic K_m. We begin by introducing the syntax and semantics of the modal logic K_m.

Basic Modal Logic

The basic *m-modal propositional language* $\mathcal{ML}_m$ [33; 62; 103; 229] (for each natural number $m \geq 1$) is given by the alphabet consisting of (i) the propositional variables $p, q, \ldots$, (ii) the constant false $\bot$, (iii) the Boolean logical connectives $\wedge, \vee, \supset$, and (iv) the unary connective $\neg$, and (v) m necessity and possibility operators $\Box_1, \ldots, \Box_i$ and $\Diamond_1, \ldots, \Diamond_i$, indexed in the set I, such that $i \in I$, where $\mid I \mid \leq \omega$. Starting from the propositional variables and the logical connectives, the definition of the *well-formed formulæ* of $\mathcal{ML}_m$, and in particular of the basic modal logic K_m, is given by the following definition:

Definition 2.1.11 (Syntax of K_m [229, pp. 13 ff.; 62, pp. 1 f., 61 f.; 33, pp. 9 f.])**.** A modal signature is given by a pair (I, Var) of two fixed countable and disjoint sets consisting of the modal signature I, an index set whose elements $i \in I$ are called *labels*, and the set $Var = \{p, q, r, \ldots\}$ of propositional variables. The set of *well-formed* K_m*-formulæ* For$\mathcal{ML}_m$ over signature (I, Var) is the smallest set such that

- all propositional variables $p \in Var$ and each constant $\bot$, $\top$ are well-formed K_m-formulæ,
- if ϕ and ψ are well-formed K_m-formulæ and $i \in I$ then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \supset \psi$, $\neg\phi$, $\Diamond_i\phi$ and $\Box_i\phi$. ∇

As usual, we express equivalence $\phi \equiv \psi$ by $(\phi \supset \psi) \wedge (\psi \supset \phi)$. Note that the definition of the basic modal language K_m contains redundancy as we can express conjunction, implication and the constant $\top$ by $\phi \wedge \psi =_{df} \neg(\neg\phi \vee \neg\psi)$, $\phi \supset \psi =_{df} \neg\phi \vee \psi$ and $\top =_{df} \neg\bot$, respectively. Under the classical semantics the modalities $\Box$ and $\Diamond$ are dual connectives and interdefinable just as the existential and universal quantifiers in first-order logic. It follows from the following definition of the interpretation of modal formulæ that for each $i \in I$, the formula $\Box_i\phi$ is equivalent to $\neg\Diamond_i\neg\phi$ and vice versa.

If I is restricted to one modal label only, we simply write $\mathcal{ML}$, For$\mathcal{ML}$ and K respectively. The semantics of the basic modal language is expressed in terms of relational Kripke structures [168].

Definition 2.1.12 (Kripke model [62, pp. 64 f.; 33, pp. 16 ff.])**.** Given a signature (I, Var), a *Kripke model* is given by a pair $\mathfrak{M} = (\mathcal{F}, \mathcal{V})$, where $\mathcal{F} = (W, R_i \mid i \in I)$ is a *labelled frame* (or *m-frame*) of the modal signature I such that (i) W is a non-empty set of worlds, the *domain*, (ii) for each $i \in I$, $R_i \subseteq W \times W$ is a binary relation on W (so-called *accessibility relation* on worlds), and (iii) $\mathcal{V}$ is a *valuation* (or *truth valuation*) in frame $\mathcal{F}$ which is a mapping $\mathcal{V} : \mathit{Var} \to 2^W$ associating with each propositional variable $p \in \mathit{Var}$ a subset $\mathcal{V}(p)$ of worlds in W in which the propositional variable is true. If $x, y \in W$ and $x \, R_i \, y$, we say that y is *R_i-accessible* from x, y is an *R_i-successor* of x, or x is an *R_i-predecessor* of y. We say that the frame $\mathcal{F}$ is the *underlying* frame of $\mathfrak{M}$. ∇

The semantics of K_m-formulæ is given by the following satisfaction relation.

Definition 2.1.13 (Modal satisfaction relation [229, pp. 39 ff.; 62, pp. 64 f.; 33, pp. 16 ff.])**.** Let $\phi \in$ For$\mathcal{ML}_m$ be a well-formed K_m-formula, suppose x is a world in a model $\mathfrak{M} = (\mathcal{F}, \mathcal{V})$ and let $i \in I$ be arbitrary. We define the truth-relation $\mathfrak{M}; x \models \phi$ saying that *ϕ is true at world x in $\mathfrak{M}$* by structural recursion on ϕ as follows:

$$
\begin{array}{ll}
\mathfrak{M}; x \models p & \text{iff } x \in \mathcal{V}(p) \text{ for } p \in \mathit{Var}; \\
\mathfrak{M}; x \models \top; & \\
\mathfrak{M}; x \not\models \bot; & \\
\mathfrak{M}; x \models \neg\phi & \text{iff not } \mathfrak{M}; x \models \phi; \\
\mathfrak{M}; x \models \phi \wedge \psi & \text{iff } \mathfrak{M}; x \models \phi \text{ and } \mathfrak{M}; x \models \psi; \\
\mathfrak{M}; x \models \phi \vee \psi & \text{iff } \mathfrak{M}; x \models \phi \text{ or } \mathfrak{M}; x \models \psi; \\
\mathfrak{M}; x \models \phi \supset \psi & \text{iff } \mathfrak{M}; x \models \phi \text{ implies } \mathfrak{M}; x \models \psi; \\
\mathfrak{M}; x \models \Box_i\phi & \text{iff } \mathfrak{M}; y \models \phi \text{ for all } y \in W \text{ such that } x \, R_i \, y; \\
\mathfrak{M}; x \models \Diamond_i\phi & \text{iff } \mathfrak{M}; y \models \phi \text{ for some } y \in W \text{ such that } x \, R_i \, y.
\end{array}
$$

We say that a K_m-formula ϕ is *satisfiable* if there exists a model $\mathfrak{M}$ and a world x in its domain such that $\mathfrak{M}; x \models \phi$. This is extended in the usual way to sets Γ of formulæ, *i.e.*, $\mathfrak{M}; x \models \Gamma$ if and only if $\forall \phi \in \Gamma.\ \mathfrak{M}; x \models \phi$. $\mathfrak{M}$ is a *model* of a formula ϕ, denoted by $\mathfrak{M} \models \phi$, if and only if ϕ is satisfied at every world in $\mathfrak{M}$. A formula ϕ is *valid in a frame $\mathcal{F}$*, written $\mathcal{F} \models \phi$, iff $(\mathcal{F}, \mathcal{V}); x \models \phi$ for all valuations $\mathcal{V}$ the formula ϕ is satisfied at every world x. Finally, a formula ϕ is *valid*, denoted by $\models \phi$, if it is valid in every frame, *i.e.*, if it is true at every world of every model in all frames. ∇

Example 2.1.4 (Disjunctive distribution [33, Ex. 1.29.(i), p. 25]). In classical K_m it holds that $\models \Diamond_i(p \vee q) \supset (\Diamond_i p \vee \Diamond_i q)$, *i.e.*, the modality $\Diamond_i$ distributes over disjunction $\vee$ (axiom $\mathsf{FS4}/\mathsf{IK4}$). This can be easily observed by taking an arbitrary model $\mathfrak{M} = (\mathcal{F}, \mathcal{V})$ and a world x in its domain and showing that $\mathfrak{M}; x \models \Diamond_i(p \vee q)$ implies $\mathfrak{M}; x \models \Diamond_i p \vee \Diamond_i q$. Suppose that $\mathfrak{M}; x \models \Diamond_i(p \vee q)$. By Definition 2.1.13 there exists a world y such that $x\ R_i\ y$ and $\mathfrak{M}; y \models p \vee q$, *i.e.*, $\mathfrak{M}; y \models p$ or $\mathfrak{M}; y \models q$. Then, also $\mathfrak{M}; x \models \Diamond_i p$ or $\mathfrak{M}; x \models \Diamond_i q$. Therefore, $\mathfrak{M}; x \models \Diamond_i p \vee \Diamond_i q$. ■

We can define a semantic consequence relation by axiomatising the semantic levels of the modal satisfaction relation. The different levels of statements $\mathfrak{M}; x \models \phi$ become visible when we write $\mathcal{F}; \mathcal{V}; x \models \phi$ instead. The idea is to replace each of the elements $\mathcal{F}, \mathcal{V}, x$ by a set of formulæ Δ, Θ and Γ, which axiomatises its associated semantic level such that Δ corresponds to *frame axioms*, Θ represents *model axioms* and Γ is a set of *world axioms*.

Definition 2.1.14 (Semantic consequence relation [229; 33, p. 31 f.]). Let $\phi \in \mathrm{For}\mathcal{ML}_m$ be a well-formed K_m formula, Δ, Θ and Γ be subsets of $\mathrm{For}\mathcal{ML}_m$ and $\mathcal{F} = (W, R_i \mid i \in I)$ a frame. We say that a formula ϕ is a *local semantic consequence* of Γ, denoted by $\mathcal{F}; \emptyset; \Gamma \models \phi$, if and only if for any model $\mathfrak{M}$ based on $\mathcal{F}$ and all worlds x in its domain we have $\mathfrak{M}; x \models \phi$ whenever $\mathfrak{M}; x \models \Gamma$. A formula ϕ is a *global semantic consequence* of a set Θ of formulæ, denoted by $\mathcal{F}; \Theta; \emptyset \models \phi$, if and only if for all models $\mathfrak{M}$ based on $\mathcal{F}$, if $\mathfrak{M} \models \Theta$ then $\mathfrak{M} \models \phi$. We say that ϕ is a *semantic consequence* of Θ and Γ in $\mathcal{F}$, written $\mathcal{F}; \Theta; \Gamma \models \phi$, if and only if $\forall \mathcal{V}.(\forall \theta \in \Theta. \mathcal{F}; \mathcal{V} \models \theta) \Rightarrow \forall x.(\forall \gamma \in \Gamma. \mathcal{F}; \mathcal{V}; x \models \gamma) \Rightarrow \mathcal{F}; \mathcal{V}; x \models \phi$. Finally, a formula ϕ is a *semantic consequence* of $\Delta; \Theta; \Gamma$, denoted by $\Delta; \Theta; \Gamma \models \phi$, if and only if for all frames $\mathcal{F}$ such that $\mathcal{F} \models \delta$ for all $\delta \in \Delta$ it holds that $\mathcal{F}; \Theta; \Gamma \models \phi$. ∇

Notation. The semantic consequence relation allows us to bring together the semantic definition of a logical system, in terms of a set of formulæ valid in certain frames, with its syntactic axiomatisation in the form of a Hilbert-style calculus. For instance, K_m is determined by the class of all frames and ϕ is a theorem of K_m iff $\emptyset; \emptyset; \emptyset \models \phi$ holds, *i.e.*, $\mathcal{F} \models \phi$ for all frames $\mathcal{F}$. Similarly, $\mathsf{S4}_m; \emptyset; \emptyset \models \phi$ iff $\mathcal{F} \models \phi$ for all quasi-ordered frames $\mathcal{F}$, *i.e.*, the frames with a reflexive and transitive accessibility relation. Here, $\mathsf{S4}_m$ refers to the set of the characteristic axiom schemata of $\mathsf{S4}_m$ given by $\mathsf{T} =_{df} \Box_i \phi \supset \phi$ and $\mathsf{4} =_{df} \phi \supset \Box_i \Box_i \phi$. ■

The system K_m (named after Kripke) is defined proof-theoretically in terms of a Hilbert-style inference system.

Definition 2.1.15 (Hilbert deduction for K_m [33, Chap. 1.6, pp. 33 ff.; 62, Chap. 3.6, pp. 83 ff.]). A K_m-proof of a formula $\phi \in \mathrm{For}\mathcal{ML}_m$, written $\vdash_{\mathsf{K}_m} \phi$ is a finite sequence of formulæ ending with ϕ, each of which is a substitution instance of an axiom, or arises from earlier items through application of one of the inference rules. The axioms and inference rules of K_m (with $i \in I$) are given by:

Axioms (K_m)
All theorems of classical propositional logic (CPC), that is,
$\mathsf{CPC1}: \phi \supset (\psi \supset \phi)$
$\mathsf{CPC2}: (\phi \supset (\psi \supset \vartheta)) \supset ((\phi \supset \psi) \supset (\phi \supset \vartheta))$
$\mathsf{CPC3}: \phi \supset (\psi \supset (\phi \wedge \psi))$
$\mathsf{CPC4}: (\phi \wedge \psi) \supset \phi, \quad (\phi \wedge \psi) \supset \psi$
$\mathsf{CPC5}: \phi \supset (\psi \vee \phi), \quad \psi \supset (\psi \vee \phi)$
$\mathsf{CPC6}: (\phi \supset \vartheta) \supset ((\psi \supset \vartheta) \supset ((\phi \vee \psi) \supset \vartheta))$
$\mathsf{CPC7}: \bot \supset \phi$
$\mathsf{CPC8}: \phi \vee (\phi \supset \bot)$
and the modal axiom
$\mathsf{K}: \Box_i(\phi \supset \psi) \supset (\Box_i\phi \supset \Box_i\psi)$
Rules
$\mathsf{MP}: \phi$ and $\phi \supset \psi$ implies ψ
$\mathsf{Nec}: \phi$ implies $\Box_i\phi$

This notion can be generalised to sets of formulæ $\Delta, \Gamma \subseteq \mathrm{For}\mathcal{ML}_m$, where Δ represents *frame* hypotheses and Γ *model* hypotheses. We write $\Delta; \Gamma; \emptyset \vdash \phi$ for a Hilbert derivation of ϕ from Δ, Γ, which is a finite sequence of formulæ ending in ϕ, each of which is either a substitution instance of a frame axiom in Δ, a model hypothesis in Γ, a substitution instance of an axiom of K_m, or arises from earlier items through application of one of the inference rules. ∇

The Hilbert system of K_m is sound and complete w.r.t. its Kripke semantics, *i.e.*, the theorems of K represent exactly the valid formulæ.

Theorem 2.1.1 ([33, Chap. 4; 62, Thm. 3.53; 130]). *For all $\phi \in \mathrm{For}\mathcal{ML}_m$ we have*

$$\vdash_{\mathsf{K}_m} \phi \quad \textit{if and only if} \quad \models \phi. \qquad \nabla$$

Furthermore, K_m has the finite model property and is decidable [33, Thm. 6.8, p. 340; 62, Chap. 3.7, p. 87 f.].

A modal logic is called *normal*[3] [33, pp. 36 f., 191 f.] if it contains all substitution instances of the axioms of CPC, K, $\Diamond_i\bot \supset \bot$ and $\Diamond_i(\phi \vee \psi) \supset (\Diamond_i\phi \vee \Diamond_i\psi)$ and is closed under the rules MP, Nec and the rule $\vdash_{\mathsf{K}_m} \phi \supset \psi$ implies $\vdash_{\mathsf{K}_m} \Diamond\phi \supset \Diamond\psi$. The system

---

[3] In particular, a modal logic is called *normal* [275] if it includes the schemata $\Box(\phi \wedge \psi) \supset (\Box\phi \wedge \Box\psi)$, $\Box\top$, $\Diamond(\phi \vee \psi) \supset (\Diamond\phi \vee \Box\psi)$ and $\neg\Diamond\bot$.

K_m represents the minimal classical (multimodal) normal system. Other normal modal logics can be obtained by extending K_m through the addition of a (possibly finite) set Δ of axioms, which corresponds to certain restrictions on the accessibility relations (usually called *frame classes*) [103]. We will write $\mathsf{K}_m \oplus \Delta$ in this case, and $\mathsf{K}_m \oplus \phi$ whenever $\Delta = \{\phi\}$ is a singleton set. Some of the more popular axioms which extend K_m are T, $\mathsf{4}$, D, $\mathsf{5}$, B [33, Chap. 4, pp. 192 f.],[229, Chap. 2.3] given by:

$$\begin{aligned}
\mathsf{T} &: \Box_i\phi \supset \phi\\
\mathsf{4} &: \Box_i\phi \supset \Box_i\Box_i\phi\\
\mathsf{D} &: \Box_i\phi \supset \Diamond_i\phi\\
\mathsf{5} &: \Diamond_i\phi \supset \Box_i\Diamond_i\phi\\
\mathsf{B} &: \phi \supset \Box_i\Diamond_i\phi
\end{aligned}$$

These axioms correspond to reflexivity, transitivity, seriality, euclideaness and symmetry of the accessibility relation. For instance, $\mathsf{K4}_m = \mathsf{K}_m \oplus \mathsf{4}$, $\mathsf{S4}_m = \mathsf{K}_m \oplus \mathsf{T} \oplus \mathsf{4}$, $\mathsf{S5}_m = \mathsf{S4}_m \oplus \mathsf{5}$. The logics extending K by T, $\mathsf{4}$, D, $\mathsf{5}$ or B are sound and complete w.r.t. their corresponding frame classes (so-called *Kripke complete*) [130; 168; 169; 215].

Theorem 2.1.2. *Let Δ be any one of the following sets of formulæ: K, T, $\mathsf{4}$, D, $\mathsf{5}$, B, $\mathsf{K4_m}$, $\mathsf{S4_m}$, $\mathsf{S5_m}$. It holds for all $\phi \in For\mathcal{ML}_m$ that $\Delta; \emptyset; \emptyset \mid\!\!\!- \phi$ if and only if $\Delta; \emptyset; \emptyset \models \phi$.* ∇

Relationship between Modal and Description Logic

According to Schild [245], we can establish the correspondence between K_m and $\mathcal{ALC}$ by defining a function f mapping $\mathcal{ALC}$-concepts to K_m-formulæ assuming that $\mathcal{ALC}$ consists of m role names $R_1, R_2 \ldots, R_m$:

$$\begin{aligned}
f(A) =& A, \ A \text{ atomic};\\
f(\neg C) =& \neg f(C);\\
f(C \odot D) =& f(C) \odot f(D), \ \odot \in \{\sqcap, \sqcup\};\\
f(\exists R_i.C) =& \Diamond_i(f(C));\\
f(\forall R_i.C) =& \Box_i(f(C)).
\end{aligned}$$

The semantic connection between $\mathcal{ALC}$ and K_m is easily observable, *i.e.*, interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{ALC}$ correspond to Kripke models of K_m with $\Delta^{\mathcal{I}}$ as the set of worlds and $\cdot^{\mathcal{I}}$ expressing both the valuation of the propositional variables and the accessibility relations. In this sense, Kripke models of K_m can be viewed as $\mathcal{ALC}$ interpretations and vice versa. Hence, we obtain that $\mathcal{ALC}$ is a notational variant of K_m by showing by induction on the structure of $\mathcal{ALC}$-concepts that an $\mathcal{ALC}$-concept C is satisfiable if and only if the K_m-formula $f(C)$ is satisfiable. Note that the latter correspondence can only be established at the level of concept satisfiability, since the basic modal logic K_m does not provide the necessary expressivity to account for reasoning w.r.t. knowledge bases consisting of an ABox and a TBox. Taking into account TBoxes alone, we can say that an $\mathcal{ALC}$-concept C is satisfiable w.r.t. a TBox $\mathcal{T}$ if and only if there exists a model $\mathfrak{M}$ and a world x in its domain such that $\mathfrak{M} \models f(\mathcal{T})$ and $\mathfrak{M}; x \models f(C)$, where $f(\mathcal{T}) =_{df} \{\neg f(C) \vee f(D) \mid C \sqsubseteq D \in \mathcal{T}\}$. We can state satisfiability of a concept C w.r.t. a non-empty TBox in terms of the global consequence relation, *i.e.*, C is satisfiable w.r.t. a TBox $\mathcal{T}$ iff $\emptyset; f(\mathcal{T}); f(C) \not\models \bot$.

In contrast to TBoxes, ABoxes do not have a direct correspondence in modal logic. However, the problem of expressing ABoxes can be addressed by extending the correspondence to converse propositional dynamic logic [111; 245] or hybrid logic [7].

Remark 2.1.1. The close relationship of DLs to modal logics yields that the standard translation of modal logic into first-order logic (QC) applies for DLs as well. More precisely, $\mathcal{ALC}$ can be translated into the QC language with one free variable where unary predicates correspond to atomic concepts, and binary predicates express roles. Let x, y be first-order variables. The *standard translation* of $\mathcal{ALC}$ concepts into QC formulæ if given as follows:

$$\begin{aligned}
ST_x(A) &= P_A(x);\\
ST_x(\top) &= \top;\\
ST_x(\bot) &= \bot;\\
ST_x(\neg C) &= \neg ST_x(C);\\
ST_x(C \sqcap D) &= ST_x(C) \wedge ST_x(D);\\
ST_x(C \sqcup D) &= ST_x(C) \vee ST_x(D);\\
ST_x(\exists R.C) &= \exists y.(P_R(x, y) \wedge ST_y(C));\\
ST_x(\forall R.C) &= \forall y.(P_R(x, y) \Rightarrow ST_y(C)),
\end{aligned}$$

where y is different from x. Since this translation requires only two variables, $\mathcal{ALC}$ corresponds to the two-variable fragment of QC [180; 16, pp. 162 f.]. The translation extends to TBox and ABox statements while preserving the semantics [24, pp. 9–10]. ■

The correspondence between DLs and modal logics can be seen as the base line of our development of constructive DLs, *i.e.*, constructive $\mathcal{ALC}$ will be based on a constructive analogue of the modal logic K_m, which is introduced in the following.

2.2 Constructive Logic

Constructive (or intuitionistic) logic originated from the philosophy of constructive mathematics and has gained in importance in the last forty years, mainly influenced by the increasing interest of applying constructive methods in computer science. Simply put, constructivism can be viewed as an opposite pole to formalism and platonism, and demands that the existence of mathematical objects depends on positive evidence or proof in the form of effective constructions [265]. Classical logic is based on a platonic notion of truth, *i.e.*, it is assumed that mathematical objects exist and their truth is known independently of context, time or space under the view of an external mathematically consistent reality. The meaning of statements is determined by their truth-value, and the truth-value of a compound formula is determined by the truth-values of its components.

In contrast, the main principle of constructive logic is the notion of *constructive proof* and goes back to the philosophy of mathematics by Brouwer [53], also known under the name *Intuitionism*. Brouwer viewed mathematics as a mental activity of an idealised mathematician, where mathematical objects and their properties are determined by mental constructions. This view demands that mathematical objects are explicitly represented by positive evidence such that they (and their existence) can be verified in terms of a traceable process of construction. In particular, Brouwer criticised the axiom schema $\phi \vee \neg\phi$ (see CPC8) that is known as the principle of the *Excluded Middle* (PEM). This axiom is equivalent to the schema $\neg\neg\phi \supset \phi$ that justifies proofs by the principle of *reductio ad absurdum*, *i.e.*, one can prove the existence of an object without giving a method of how to construct it [62, pp. 2 f.]. For instance, the classical reading of the statement $\neg\forall x.\neg\phi(x) \Rightarrow \exists x.\phi(x)$ is: If we can derive a contradiction from the assumption that no object x satisfies property $\phi(x)$, then there must be an object x with property $\phi(x)$. Such *indirect* proofs are called *non-constructive* and are rejected by intuitionistic reasoning. Indeed, from a constructive perspective the PEM asserts that a mathematical problem expressed by a statement ϕ can be decided in terms of having a proof of either ϕ or $\neg\phi$. However, not every problem is decidable, which becomes obvious when we consider an open mathematical problem like $P = NP$ or Goldbach's conjecture for which we neither know a proof nor a falsification.

The first (informal) intuitionistic interpretation of logic, based on Brouwer's ideas, is the so-called proof-interpretation or *Brouwer-Heyting-Kolmogorov interpretation* (BHK-interpretation for short), invented independently around the same time by Heyting [133] and Kolmogorov [163]. Under the BHK-interpretation the meaning of a statement is explained in terms of its *proof* and the proof of a logically compound statement is determined by the proof of its components [265, p. 9; 268]. Let us recapitulate this idea for the propositional language according to Heyting's formulation, where $\neg\phi$ denotes $\phi \supset \bot$:

- A proof of $\phi \wedge \psi$ is determined by a pair (π_1, π_2) where π_1 is proof of ϕ and π_2 is a proof of ψ.
- A proof of $\phi \vee \psi$ is determined by (i, π) such that $i = 0$ and π is a proof of ϕ or $i = 1$ and π is a proof of ψ.
- A proof of $\phi \supset \psi$ is a construction t, turning any proof π of ϕ into a proof $t(\pi)$ of ψ.
- There exists no proof of $\bot$ (contradiction).
- A proof of $\neg\phi$ is a construction that transforms any proof π of ϕ into a contradiction.

It is obvious that this interpretation does not constitute a strict mathematical definition, since the notions of proof and construction are insufficiently precise [84, p. 3]. Since Heyting's interpretation several different semantics (topological, Kripke-style, algebraic, realisability semantics, etc.) [265] have been proposed to make this notion more precise. The proof interpretation by Brouwer anticipated the development of a precise meaning of the notions of construction and proof. It became notably successful in the field of computer science in the form of a computational interpretation of proofs, which identifies propositions with types, and is nowadays known as the *propositions as types* notion, *Curry-Howard isomorphism* or *proofs as programs* principle. This correspondence bridges the gap between the proof theory of constructive logic and computational calculi from the field of type theory, based on the idea that each constructive proof corresponds to a term (computer program) in a typed calculus (programming language). For instance, the inhabited types of combinatory logic coincide with the theorems of the implicational fragment of intuitionistic propositional logic, and, a similar correspondence can be shown between natural deduction or sequent calculus proofs with terms in the simply typed λ-calculus [68; 70; 248]. The Curry-Howard correspondence gave rise to an active field of research and the invention of numerous computational interpretations of constructive logic, see [29; 66; 113; 139; 140; 149; 182; 248; 252].

In the rest of this section we will focus on Kripke-style semantics and Hilbert-style proof systems. We refer the reader to [86; 134; 265] for a comprehensive introduction to the philosophy and mathematics of intuitionism and in particular intuitionistic logic. The connection of constructivism and computer science has been highlighted in [264].

2.2.1 Intuitionistic Propositional and First-Order Logic

Intuitionistic Propositional Logic

Intuitionistic propositional logic (IPC) can be viewed as a weakening of CPC where the law of the Excluded Middle is not universally valid and therefore discarded, going back to the constructive approach to mathematics by Brouwer [53; 265]. The language of IPC coincides with that of classical propositional logic.

Definition 2.2.1 (Syntax of IPC [134, pp. 97 ff.][265])**.** The language of IPC is based on a denumerable set $\mathit{Var} = \{p, q, r, \ldots\}$ of propositional variables. The set of well-formed IPC-formulæ over Var is defined inductively by the following grammar with $p \in \mathit{Var}$:

$$\phi ::= \quad p \ \mid \ \top \ \mid \ \bot \ \mid \ \neg\phi \ \mid \ \phi \wedge \phi \ \mid \ \phi \vee \phi \ \mid \phi \supset \phi \qquad \nabla$$

Notation. We recall that $\top$ is codable as $\neg\bot$, $\neg\phi$ abbreviates $\phi \supset \bot$ and $\phi \equiv \psi$ is expressed by $\phi \supset \psi \wedge \psi \supset \phi$. Contrary to classical logic, the propositional connectives are not interdefinable in IPC [72, p. 49]. We will write $\Gamma,\ \phi$ for the union $\Gamma \cup \{\phi\}$ and use the general pattern $\Gamma,\ \phi_1, \phi_2, \ldots$ to denote $\Gamma \cup \{\phi_1\} \cup \{\phi_2\} \cup \ldots$. The union of two sets of formulæ Γ and Γ' is expressed by $\Gamma,\ \Gamma'$. ■

The system IPC can be characterised in terms of a Hilbert-style axiomatisation [84; 158; 265], which is obtained from the axioms of CPC by discarding axiom CPC8 (PEM).

Definition 2.2.2 (Hilbert deduction for IPC [72, p. 9; 62, p. 45])**.** An IPC-proof of a formula ϕ, written $\vdash_{\mathsf{IPC}} \phi$, is a finite sequence of formulæ ending with ϕ, each of which is a substitution instance of an axiom, or arises from earlier items through application of one of the inference rules. The axioms and inference rules of IPC are as follows where ϕ, ψ and ϑ are IPC formulæ:

Axioms (IPC)
IPC1 : $\phi \supset (\psi \supset \phi)$
IPC2 : $(\phi \supset (\psi \supset \vartheta)) \supset ((\phi \supset \psi) \supset (\phi \supset \vartheta))$
IPC3 : $\phi \supset (\psi \supset (\phi \wedge \psi))$
IPC4 : $(\phi \wedge \psi) \supset \phi, \quad (\phi \wedge \psi) \supset \psi$
IPC5 : $\phi \supset (\psi \vee \phi), \quad \psi \supset (\psi \vee \phi)$
IPC6 : $(\phi \supset \vartheta) \supset ((\psi \supset \vartheta) \supset ((\phi \vee \psi) \supset \vartheta))$
IPC7 : $\bot \supset \phi$
Rules
MP : ϕ and $\phi \supset \psi$ implies ψ

∇

An important property of constructive logic directly connected with the BHK-proof interpretation is the *disjunction property*, which states that if the disjunction $\phi \vee \psi$ is a theorem of IPC, then either ϕ or ψ is a theorem of IPC.

Proposition 2.2.1 (Disjunction property [72, pp. 44 ff.])**.**

$$\textit{if} \ \vdash_{\mathsf{IPC}} \phi \vee \psi \quad \textit{then either} \quad \vdash_{\mathsf{IPC}} \phi \ \textit{or} \ \vdash_{\mathsf{IPC}} \psi. \qquad \nabla$$

Proof. For the proof see [72, pp. 44 ff.]. □

The notion of Hilbert derivation can be extended relative to a set of assumptions. Let IPC_{ax} denote the set of axiom schemata of IPC closed under substitution. If Γ is a set of IPC formulæ and ϕ an IPC formula, then the expression $\Gamma \vdash_{\mathsf{IPC}} \phi$ means that ϕ is derivable from the set Γ of *assumptions* with the help of the axioms and rules of Def. 2.2.2. This notion of deduction from assumptions is expressed by the following sound and complete proof system.

$$\frac{\phi \in \Gamma}{\Gamma \vdash_{\mathsf{IPC}} \phi} \qquad \frac{\phi \in \mathsf{IPC}_{ax}}{\Gamma \vdash_{\mathsf{IPC}} \phi} \qquad \frac{\Gamma \vdash_{\mathsf{IPC}} \phi \supset \psi \qquad \Gamma' \vdash_{\mathsf{IPC}} \phi}{\Gamma, \Gamma' \vdash_{\mathsf{IPC}} \psi}$$

An important meta-theorem is the *Deduction Theorem*, which states that if formula ϕ is derivable from a set of assumptions Γ conjoined with formula ψ, then the implication $\psi \supset \phi$ is derivable from Γ.

Theorem 2.2.1 (Deduction Theorem [72, p. 10])**.**

$$\Gamma, \psi \vdash_{\mathsf{IPC}} \phi \quad \textit{iff} \quad \Gamma \vdash_{\mathsf{IPC}} \psi \supset \phi \qquad \nabla$$

Proof. The ($\Rightarrow$) direction is by induction on the length of a derivation and uses the axiom schemata IPC1 and IPC2, and the fact that Hilbert derives identity $\vdash_{\mathsf{IPC}} \phi \supset \phi$. The converse direction ($\Leftarrow$) is straightforward, using monotonicity of derivations. □

The Deduction Theorem comes usually into play when proving the equivalence between a natural deduction or Gentzen-style sequent system and a Hilbert-type system. While the Deduction Theorem is an admissible rule in Hilbert-style systems obtained from the set of axioms and inference rules, it is usually part of natural deduction or sequent calculi in the form of a primitive inference rule, for instance in natural deduction this rule is known as *implication introduction.* The reader may consult the work of Porte [230] for a comprehensive retrospect on the Deduction Theorem.

Kripke Semantics

So far, a wide variety of different model-theoretic semantics [12; 265] [72, pp. 22 ff.] have been developed to give an appropriate semantic characterisation for the (informal) proof interpretation of intuitionistic logic. We will focus on Kripke-style semantics for IPC [12; 170; 265] in the following. The usual interpretation of intuitionistic logic [72, p. 25] is as a process of acquiring knowledge over stages of time from the view of an agent or mathematician who is involved in the construction of mathematical objects or statements. The agents memory is assumed to be perfect in the sense that once acquired facts become persistent over time such that knowledge increases monotonically in time. This structure induces a partially ordered set of states of knowledge, represented by possible worlds in the Kripke semantics.

Definition 2.2.3 (Kripke model [72, p. 46; 62, pp. 25 f.]). A *Kripke model* for IPC is a tuple $\mathfrak{M} = (\mathcal{F}, \mathcal{V})$, where $\mathcal{F} = (W, \preceq)$ is an *intuitionistic frame*, W is a non-empty set of worlds and $\preceq$ is a preorder (*i.e.*, reflexive and transitive) over W. The valuation $\mathcal{V}$ is mapping each propositional variable $p \in \mathit{Var}$ to a subset $\mathcal{V}(p)$ of worlds in W in which the propositional variable is true, subject to the condition that $\preceq$ is *hereditary* w.r.t. propositional variables, *i.e.*, if $x \preceq x'$ and $x \in \mathcal{V}(p)$ then $x' \in \mathcal{V}(p)$. We will write $\mathfrak{M}; x \models p$ iff $x \in \mathcal{V}(p)$ for $p \in \mathit{Var}$.
The validity relation $\mathfrak{M}; x \models \phi$ is defined for arbitrary formulæ of IPC by structural recursion on ϕ:

$$
\begin{array}{ll}
\mathfrak{M}; x \models \top; & \\
\mathfrak{M}; x \not\models \bot; & \\
\mathfrak{M}; x \models \phi \wedge \psi & \text{iff } \mathfrak{M}; x \models \phi \text{ and } \mathfrak{M}; x \models \psi; \\
\mathfrak{M}; x \models \phi \vee \psi & \text{iff } \mathfrak{M}; x \models \phi \text{ or } \mathfrak{M}; x \models \psi; \\
\mathfrak{M}; x \models \phi \supset \psi & \text{iff } \forall x' \in W.\ (x \preceq x' \ \ \&\ \ \mathfrak{M}; x' \models \phi) \Rightarrow \mathfrak{M}; x' \models \psi; \\
\mathfrak{M}; x \models \neg\phi & \text{iff } \forall x' \in W.\ x \preceq x' \Rightarrow \mathfrak{M}; x' \not\models \phi.
\end{array}
$$

A formula ϕ is *satisfiable* if there exists a model $\mathfrak{M}$ and a world x in its domain such that $\mathfrak{M}; x \models \phi$. We say that $\mathfrak{M}$ is a *model* of a formula ϕ, denoted by $\mathfrak{M} \models \phi$, if and only if for every $x \in W$, $\mathfrak{M}; x \models \phi$. A formula ϕ is *valid*, written $\models \phi$, if it is true in all models. These notions are lifted to sets of formulæ in the usual way. ∇

In intuitionistic logic once established knowledge at a certain stage becomes persistent, which is in line with the idea of accumulating certainty over *stages of information* [170; 204; 265; 267]. This is expressed by the following important lemma, which states that the heredity condition transfers to arbitrary formulæ.

Proposition 2.2.2 (Monotonicity property [265, p. 78]). *Let $\mathfrak{M} = (\mathcal{F}, \mathcal{V})$ with $\mathcal{F} = (W, \preceq)$ be an intuitionistic Kripke model and $x, x' \in W$. Then, the following holds for all* IPC *formulæ ϕ:*

$$\textit{if } x \preceq x' \quad \textit{and} \quad \mathfrak{M}; x \models \phi \quad \textit{then} \quad \mathfrak{M}; x' \models \phi. \qquad \nabla$$

Proof. By induction on the structure of ϕ [*cf.* 265, p. 78]. □

IPC is sound and complete [72, Thm. 16, p. 33], in particular, the Hilbert system of IPC is sound and complete w.r.t. the Kripke semantics for IPC [62; 170]. Furthermore, IPC has the finite model property that implies decidability [72, p. 46], with the important property that non-theorems of IPC are refuted by finite Kripke models. The complexity of the problem of determining whether a formula is intuitionistically valid in IPC has been investigated in [253; 256], and it corresponds to that of K and S4, which are PSPACE-complete.

Example 2.2.1 ([72, p. 36]). The Kripke model in Fig. 2.2 with $W = \{x_0, x_1, x_2\}$ is a counter example for the validity of the PEM $= \phi \vee \neg\phi$, its weak variant $\neg\phi \vee \neg\neg\phi$ and $(\neg\neg\phi \supset \phi) \supset (\phi \vee \neg\phi)$.

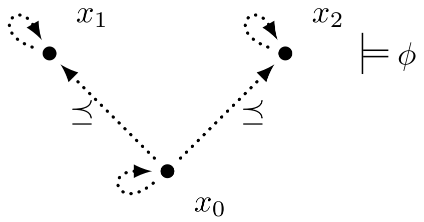

Figure 2.2: Countermodel for $\phi \vee \neg\phi$, $\neg\phi \vee \neg\neg\phi$ and $(\neg\neg\phi \supset \phi) \supset (\phi \vee \neg\phi)$. [72, p. 36]

See [72, pp. 35 ff.] for a more comprehensive exposure of counterexamples for formulæ, which are classically valid but not intuitionistically. ■

Intuitionistic First-Order Logic

Intuitionistic propositional logic can be extended to the language of first-order logic. We denote intuitionistic first-order logic by IQC, and assume that the reader is familiar with the key concepts of first-order logic. In the following, we will shortly review the intuitionistic Kripke models [170] of first-order logic, restricted to the language with predicate symbols only [see 265, pp. 80 ff.], to simplify the presentation. We assume given some first-order signature, consisting of n-ary predicate symbols $P, Q, \ldots$ and a denumerable set of individual variables $x, y, \ldots$. Terms and formulæ on the basis of the propositional connectives $\wedge, \vee, \supset, \bot$ and the quantifiers $\exists, \forall$, the concept of free and bound variables, and the notion of substitution are defined as usual. The idea is to assign a domain to each state of the Kripke model, and to extend the heredity condition such that the collection of true predicates increases monotonically along the preorder over the states of knowledge.

Definition 2.2.4 (Kripke model [265, pp. 80 ff.]). A *Kripke model* for IQC is a triple $\mathfrak{M} = (\mathcal{F}, D, \models)$, where $\mathcal{F} = (W, \preceq)$ is an *intuitionistic frame*, D is a function assigning to each state $x \in W$ a set $D(x) \neq \emptyset$ such that for all $x, y \in W.\ x \preceq y \Rightarrow D(x) \subseteq D(y)$.

Let the language be extended with constant symbols for each element of $D = \bigcup\{D(x) \mid x \in W\}$. The forcing relation $\mathfrak{M}; x \models \phi$ is defined in the extended language with constants in $D(x)$ and is given directly by $\mathfrak{M}$ such that

$$\begin{aligned}
&\mathfrak{M}; x \models P^n(d_1, \dots, d_n) \Rightarrow d_i \in D(x),\ \text{ for } 1 \leq i \leq n;\\
&\mathfrak{M}; x \models P^n(d_1, \dots, d_n) \text{ and } x \preceq x' \Rightarrow \mathfrak{M}; x' \models P^n(d_1, \dots, d_n);
\end{aligned}$$

for ground atomic predications. The propositional operators are interpreted as in Def. 2.2.3 for IPC, and the interpretation of the quantifiers is given by

$$\begin{aligned}
\mathfrak{M}; x \models \forall x.\phi(x) &\quad \text{iff } \forall x' \in W.\ (x \preceq x'\ \ \&\ \ \forall d \in D(x').\ \mathfrak{M}; x' \models \phi(d));\\
\mathfrak{M}; x \models \exists x.\phi(x) &\quad \text{iff } \mathfrak{M}; x \models \phi(d) \text{ for some } d \in D(x).
\end{aligned}$$

A formula ϕ is *satisfiable* if there exists a model $\mathfrak{M}$ and a state x such that $\mathfrak{M}; x \models \phi$. $\mathfrak{M}$ is a *model* of formula ϕ, denoted by $\mathfrak{M} \models \phi$ if and only if for every $x \in W, \mathfrak{M}; x \models \phi$. A formula ϕ is *valid*, written $\models \phi$, if it is true in all models. Again, these notions are lifted to sets of formulæ in the standard fashion. ∇

The Hilbert system for IQC is given by the axioms of IPC, extended by the following axiom schemata and rules [265, p. 72; 72, pp. 9 f.], where $\phi(x)$ stands for an arbitrary formula, t denotes a term, $\phi(t)$ denotes the result of substituting t for each free x in ϕ, and t is free for x in $\phi(x)$ in the sense that no free occurrence in t becomes bound during the substitution process.

Axioms (IQC)	**Rules** (x is not free in ϕ)
IQC$\forall$: $\forall x.\phi(x) \supset \phi(t)$	$\forall I$: $\phi \supset \psi(x)$ implies $\phi \supset \forall x.\psi(x)$
IQC$\exists$: $\phi(t) \supset \exists x.\phi(x)$	$\exists E$: $\psi(x) \supset \phi$ implies $\exists x.\psi(x) \supset \phi$

We refer the reader to [265, Chap. 2; 134] for a comprehensive presentation of first-order intuitionistic logic in terms of its semantics and proof theory.

2.2.2 Intuitionistic Modal Logic

Intuitionistic modal logic (IML) extends propositional intuitionistic logic by the modalities $\Box$ and $\Diamond$ and consequently has the same syntax as classical modal logic. In contrast to classical modal logics there is no uniform choice of *the minimal constructive modal logic* corresponding to the classical system K, but rather we are facing a '[...] plurality problem when constructivizing notions [...]' [80, p. 3] of classical mathematics. Indeed, several different versions of constructive (or intuitionistic) modal logics have been considered in the past [4; 32; 90; 96; 97; 118; 188; 196; 226; 228; 249; 276]. Technically speaking, IMLs conform in that they refute the law of the Excluded Middle and the classical double negation duality, which leads to independent $\Box$ and $\Diamond$ modalities. It is then no longer obvious how these modalities are semantically related. Whereas most IMLs agree on the interpretation of the necessity modality $\Box$, the constructive meaning of possibility $\Diamond$ is subject to controversy. This becomes clearly visible when considering the behaviour of possibility $\Diamond$ in IMLs and modal type theories [27; 28; 80; 90; 198; 210; 226; 272]. This leads to the question of what system should be considered the constructive analogue of the modal logic K. Arguably, the answer seems to be that there is no single optimal constructive reinterpretation of K, but many competing theories whose constructive reading of the modalities can serve different application domains.

We will briefly introduce two established approaches in the following and present the main characteristics of IMLs. A more detailed discussion on the semantics of these IMLs and their relation to our approach is part of Chapter 4. For a comprehensive survey on IMLs the reader is referred to [249; 276], general results on the decidability of IMLs have been discussed in [5; 275; 278].

Intuitionistic K

The traditional approach in intuitionistic modal logics is to dualise the standard algebraic characterisation of $\Box$ as a monotonic $\wedge$-preserving operator and to define $\Diamond$ as a monotonic $\vee$-preserving modality.

This approach has been realised by two equivalent axiomatisations, the system by Fischer-Servi [96], denoted by FS in [103], and IK [228; 249] by Plotkin and Stirling:

Axioms (IK)	**Axioms** (FS)
All theorems of IPC	All theorems of IPC
IK1 : $\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$	FS1 : $\Box\top$
IK2 : $\Box(\phi \supset \psi) \supset (\Diamond\phi \supset \Diamond\psi)$	FS2 : $\Box(\phi \wedge \psi) \equiv (\Box\phi \wedge \Box\psi)$
IK3 : $\neg\Diamond\bot$	FS3 : $\neg\Diamond\bot$
IK4 : $\Diamond(\phi \vee \psi) \supset (\Diamond\phi \vee \Diamond\psi)$	FS4 : $\Diamond(\phi \vee \psi) \equiv (\Diamond\phi \vee \Diamond\psi)$
IK5 : $(\Diamond\phi \supset \Box\psi) \supset \Box(\phi \supset \psi)$	FS5 : $(\Diamond\phi \supset \Box\psi) \supset \Box(\phi \supset \psi)$
	FS6 : $\Diamond(\phi \supset \psi) \supset (\Box\phi \supset \Diamond\psi)$
Rules	**Rules**
MP : ϕ and $\phi \supset \psi$ implies ψ	MP : ϕ and $\phi \supset \psi$ implies ψ
Nec : ϕ implies $\Box\phi$	Reg : $\phi \supset \psi$ implies $\Box\phi \supset \Box\psi$ and $\Diamond\phi \supset \Diamond\psi$.

One can easily show that the systems IK and FS are equivalent, by mutual simulation of the axioms and inference rules. Analogously to the Hilbert deduction relation of IPC (see Def. 2.2.2), we denote by $\vdash_{\mathsf{IK}}$ and $\vdash_{\mathsf{FS}}$ the Hilbert deduction relation of the system IK and FS, respectively.

Proposition 2.2.3. *The systems* IK *and* FS *are equivalent,* i.e., $\forall\phi \in For\mathcal{ML}$ *we have*

$$\vdash_{\mathsf{IK}} \phi \quad \textit{iff} \quad \vdash_{\mathsf{FS}} \phi. \qquad \nabla$$

Proof. For the direction IK $\Rightarrow$ FS it suffices to give an explanation for the rule Reg and the axioms FS1, FS2 and FS6, since FS3–FS5 are already covered by the axioms IK3–IK5, with the exception that for axiom FS4 we need to justify the direction $(\Diamond\phi \vee \Diamond\psi) \supset \Diamond(\phi \vee \psi)$.

Regarding rule Reg let us suppose that Hilbert derives $\phi \supset \psi$ in FS. The derivation for necessity $\Box$ is as follows:

1. $\phi \supset \psi$ Ass.;
2. $\Box(\phi \supset \psi)$ from 1 by Nec;
3. $\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$ IK1;
4. $\Box\phi \supset \Box\psi$ from 3, 2 by MP.

For possibility $\Diamond$ we proceed analogously by relying on axiom IK2:

1. $\phi \supset \psi$ Ass.;
2. $\Box(\phi \supset \psi)$ from 1 by Nec;
3. $\Box(\phi \supset \psi) \supset (\Diamond\phi \supset \Diamond\psi)$ IK2;
4. $\Diamond\phi \supset \Diamond\psi$ from 3, 2 by MP;

Axiom FS1 follows easily from the fact that IPC derives $\top$ and by an application of rule Nec. Secondly, we show that axiom FS2 is admissible in IK, which is by giving a derivation for both directions:

1. $(\phi \wedge \psi) \supset \psi$ IPC4;
2. $\Box((\phi \wedge \psi) \supset \psi)$ from 1 by Nec;
3. $\Box((\phi \wedge \psi) \supset \psi) \supset (\Box(\phi \wedge \psi) \supset \Box\psi)$ IK1;
4. $\Box(\phi \wedge \psi) \supset \Box\psi$ from 3, 2 by MP;
5. $(\phi \wedge \psi) \supset \phi$ IPC4;
6. $\Box((\phi \wedge \psi) \supset \phi)$ from 5 by Nec;
7. $\Box(\phi \wedge \psi) \supset \Box\phi$ from IK1, 6 by MP.

Then, using the abbreviations $\vartheta =_{df} \Box(\phi \wedge \psi)$, $\gamma =_{df} \Box\phi$ and $\sigma =_{df} \Box\psi$, one shows that from $\vartheta \supset \gamma$ and $\vartheta \supset \sigma$ Hilbert derives $\vartheta \supset (\gamma \wedge \sigma)$:

1. $\vartheta \supset \gamma$ Ass.;
2. $\vartheta \supset \sigma$ Ass.;
3. $\gamma \supset (\sigma \supset (\gamma \wedge \sigma))$ IPC3;
4. $\vartheta \supset (\gamma \supset (\sigma \supset (\gamma \wedge \sigma)))$ from IPC1, 3 by MP;
5. $(\vartheta \supset \gamma) \supset \big(\vartheta \supset (\sigma \supset (\gamma \wedge \sigma))\big)$ from IPC2, 4 by MP;
6. $\big(\vartheta \supset (\sigma \supset (\gamma \wedge \sigma))\big) \supset (\vartheta \supset \sigma) \supset (\vartheta \supset (\gamma \wedge \sigma))$ IPC2;
7. $(\vartheta \supset \gamma) \supset (\vartheta \supset \sigma) \supset (\vartheta \supset (\gamma \wedge \sigma))$ from 6, 5 by composition;
8. $(\vartheta \supset \sigma) \supset (\vartheta \supset (\gamma \wedge \sigma))$ from 7, 1 by MP;
9. $\vartheta \supset (\gamma \wedge \sigma)$ from 8, 2 by MP.

Note that in step 7 we use the fact that from $\phi \supset \psi$ and $\psi \supset \gamma$ Hilbert derives $\phi \supset \gamma$, which is known as '*composition*' or B-combinator [140] in combinatory logic. We will present a proof for a generalisation of the B-combinator in Sec. 5.1. In the other direction the derivation is as follows:

1. $\phi \supset (\psi \supset (\phi \wedge \psi))$ IPC3;
2. $\Box(\phi \supset (\psi \supset (\phi \wedge \psi)))$ from 1 by Nec;
3. $\Box\phi \supset \Box(\psi \supset (\phi \wedge \psi))$ from IK1, 2 by MP;
4. $\Box(\psi \supset (\phi \wedge \psi)) \supset (\Box\psi \supset \Box(\phi \wedge \psi))$ IK1;
5. $\Box\phi \supset (\Box\psi \supset \Box(\phi \wedge \psi))$ from 4, 3 by composition.

Then, $(\Box\phi \wedge \Box\psi) \supset \Box(\phi \wedge \psi)$ follows from 5 by the admissible rule of *de-currying*, which says that Hilbert derives $(\phi \wedge \psi) \supset \gamma$ from $\phi \supset (\psi \supset \gamma)$ (see Lemma 5.1.3, p. 106).

Regarding the $(\Leftarrow)$ direction of axiom FS4 the goal is to give in IK a derivation of $(\Diamond\phi \vee \Diamond\psi) \supset \Diamond(\phi \vee \psi)$ that can be obtained as follows:

1. $(\Diamond\phi \supset \Diamond(\phi \vee \psi)) \supset (\Diamond\psi \supset \Diamond(\phi \vee \psi)) \supset ((\Diamond\phi \vee \Diamond\psi) \supset \Diamond(\phi \vee \psi))$ IPC6;

2. $\phi \supset (\phi \vee \psi)$ IPC5;
3. $\psi \supset (\phi \vee \psi)$ IPC5;
4. $\Box(\phi \supset (\phi \vee \psi))$ from 2 by Nec;
5. $\Box(\psi \supset (\phi \vee \psi))$ from 3 by Nec;
6. $\Diamond\phi \supset \Diamond(\phi \vee \psi)$ from IK2, 4 by MP;
7. $\Diamond\psi \supset \Diamond(\phi \vee \psi)$ from IK2, 5 by MP;
8. $(\Diamond\phi \vee \Diamond\psi) \supset \Diamond(\phi \vee \psi)$ from (1, 6 by MP), 7 by MP.

The remaining axiom FS6 can be derived as follows: First, we show that in IPC we can derive $\phi \supset ((\phi \supset \psi) \supset \psi)$. To this end, we use the fact that Hilbert derives identity $(\phi \supset \psi) \supset (\phi \supset \psi)$ (see Lemma 5.1.2, p. 103). An application of *de-currying* together with commutativity of $\wedge$ yields $(\phi \wedge (\phi \supset \psi)) \supset \psi$. Thereof, we obtain $\phi \supset ((\phi \supset \psi) \supset \psi)$ by currying. The remaining derivation goes as follows:

1. $\phi \supset ((\phi \supset \psi) \supset \psi)$ from above;
2. $\Box(\phi \supset ((\phi \supset \psi) \supset \psi))$ from 1 by Nec;
3. $\Box\phi \supset \Box((\phi \supset \psi) \supset \psi)$ from IK1, 2 by MP;
4. $\Box((\phi \supset \psi) \supset \psi) \supset (\Diamond(\phi \supset \psi) \supset \Diamond\psi)$ IK2;
5. $\Box\phi \supset (\Diamond(\phi \supset \psi) \supset \Diamond\psi)$ from 4, 3 by composition.

Thereof, we obtain the goal $\Diamond(\phi \supset \psi) \supset (\Box\phi \supset \Diamond\psi)$ from 5 by a combination of de-currying, commutativity of $\wedge$ and currying. This completes the first part and shows that the axioms of FS are admissible in IK.

It remains to show the converse direction FS $\Rightarrow$ IK. This time, it suffices to argue admissibility of rule Nec and the axioms IK1–IK2. For rule Nec let us suppose that Hilbert derives ϕ in IK:

1. ϕ Ass.;
2. $\top \supset \phi$ from IPC1, 1 by MP;
3. $\Box\top \supset \Box\phi$ from 2 by Reg;
4. $\Box\top$ FS1;
5. $\Box\phi$ from 3, 4 by MP.

The next derivation shows how the axiom IK1 is obtained:

1. $(\phi \supset \psi) \supset (\phi \supset \psi)$ identity;
2. $((\phi \supset \psi) \wedge \phi) \supset \psi$ from 1 by de-currying;
3. $\Box((\phi \supset \psi) \wedge \phi) \supset \Box\psi$ from 2 by Reg;
4. $(\Box(\phi \supset \psi) \wedge \Box\phi) \supset \Box((\phi \supset \psi) \wedge \phi)$ FS2;
5. $(\Box(\phi \supset \psi) \wedge \Box\phi) \supset \Box\psi$ from 3, 4 by composition;
6. $\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$ from 5 by currying.

Finally, we show the derivation for axiom IK2. Similarly to the case of axiom FS6 before, we begin with $\phi \supset ((\phi \supset \psi) \supset \psi)$, which is derivable in IPC.

1. $\phi \supset ((\phi \supset \psi) \supset \psi)$ by IPC;
2. $\Diamond\phi \supset \Diamond((\phi \supset \psi) \supset \psi)$ from 1 by Reg;
3. $\Diamond((\phi \supset \psi) \supset \psi) \supset (\Box(\phi \supset \psi) \supset \Diamond\psi)$ FS6;
4. $\Diamond\phi \supset (\Box(\phi \supset \psi) \supset \Diamond\psi)$ from 3, 2 by composition.

Then, the goal $\Box(\phi \supset \psi) \supset (\Diamond\phi \supset \Diamond\psi)$ follows from 4 by a combination of de-currying, commutativity of $\wedge$ and currying. Thus, FS and IK are equivalent. □

The logic FS/IK comes with an elementary Kripke style model theory and there exist several extensions, such as IS4, IS4.3, IS5, which can be characterised in terms of appropriate frame classes [228]. The semantics of FS/IK has been given in two flavours (see [249, p. 88] and [89]):

(i) A standard Kripke semantics that separates the partial order and the accessibility relation. Indeed, it was shown by Grefe [118, pp. 45 ff.] that FS/IK can be embedded into a fragment of intuitionistic first-order logic by using the very same standard translation (*cf.* Remark 2.1.1) that embeds classical K into first-order logic.

(ii) A birelational Kripke semantics, where the intuitionistic partial order and the accessibility relation are relations on the same domain. This semantics can be observed from the embedding of FS/IK into bimodal logics [*cf.* 118, pp. 48 ff.].

We will focus on the birelational semantics in the following.

The frames $\mathcal{F} = (W, \preceq, R)$ of birelational semantics with the modal accessibility relation R extend that of classical K by adding a reflexive and transitive relation $\preceq$ that is capturing the intuitionistic notion of information increase over possible worlds. The semantics of the system FS/IK is an extension of that of IPC by the interpretation of $\Box$, $\Diamond$ as universal and existential quantifiers over accessible worlds in an intuitionistic meta-theory:

$$\mathfrak{M}; x \models \Box\phi \quad \text{iff } \forall x'.\, x \preceq x' \Rightarrow \forall y.\, x' \; R \; y \Rightarrow \mathfrak{M}; y \models \phi; \tag{2.1}$$

$$\mathfrak{M}; x \models \Diamond\phi \quad \text{iff } \exists y.\; x \; R \; y \;\;\&\;\; \mathfrak{M}; y \models \phi. \tag{2.2}$$

Intuitionistic logic demands that the semantics satisfy the usual intuitionistic heredity condition, that is, all propositions are closed under $\preceq$, *i.e.*, if $x \models \phi$ and $x \preceq x'$ then $x' \models \phi$. Obviously, this condition is built into the semantics of $\Box$ while the interpretation of $\Diamond$ coincides with its classical counterpart in K. The system FS/IK requires the two following frame conditions, whereas (F1) is necessary to achieve heredity under (2.2):

(F1) $\forall x, x', y \in W, x \preceq x' \ \&\ x \, R \, y \Rightarrow \exists y' \in W,$ such that $x' \, R \, y' \ \&\ y \preceq y'$;

(F2) $\forall x, y, y' \in W, x \, R \, y \ \&\ y \preceq y' \Rightarrow \exists x' \in W,$ such that $x \preceq x' \ \&\ x' \, R \, y'$.

Their diagrammatic representation is given by Fig. 2.3.

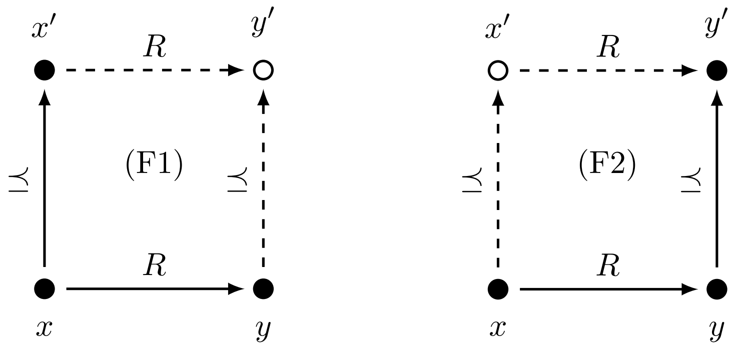

Figure 2.3: Frame conditions of FS/IK [249, p. 50].

The system FS/IK complies with several requirements [249, Chap. 3.3] one might impose on an intuitionistic modal logic: (i) FS/IK is a conservative extension of IPC, (ii) the modalities $\Box$ and $\Diamond$ are semantically independent and non-interdefinable in terms of each other, (iii) the system satisfies the disjunction property, and (iv) the addition of the law of the Excluded Middle $\phi \vee \neg\phi \equiv \top$ collapses the theory of FS/IK to give classical K.

Just like in classical K, necessity $\Box$ distributes over conjunction while possibility $\Diamond$ distributes over disjunction. According to [33; 103; 275], systems with the latter property are known under the term *normal modal logics* (*cf.* p. 23) and many results on their intuitionistic variants may be derived by exploiting the fact that there is an embedding into the classical two-dimensional modal logic S4 $\otimes$ K [103; 275; 276; 278]. More details on the model and proof theory of FS/IK can be found in [96; 118; 228; 249; 276].

Constructive K

The second representative IML is called *constructive* K (CK). Like FS/IK before, CK is a conservative extension of IPC; it satisfies the disjunction property, and the modalities $\Box$ and $\Diamond$ are independent. However, unlike FS/IK, the system CK refutes the axiom schemata FS3/IK3 – FS5/IK5, *i.e.*, CK is non-normal regarding $\Diamond$ because it does neither warrant the distribution of possibility $\Diamond$ over disjunction, nor the interaction between

$\Diamond$ and $\Box$, exhibited by axiom FS5/IK5. As argued in [196], these axiom schemata fail to have a uniform computational justification when considered in the context of computational type theories [91; 187; 205] or modal type theories [159; 209; 210; 226] that exploit the Curry-Howard isomorphism between constructive proofs and λ-terms. In particular, contextual interpretations of possibility $\Diamond$ as considered by Curry in the 50's [69] do not satisfy disjunctive distribution [92], and the first explicit refutation of FS4/IK4 was given by Wijesekera [272; 273] in the context of constructive concurrent dynamic logic.

The system that consists of the remaining schemata FS1/IK1, FS2/IK2 and FS6, which appear to be computationally justified, forms the system CK [27; 188] with the two equivalent axiomatisations:

Axioms (CK-1)	**Axioms** (CK-2)
All theorems of IPC	All theorems of IPC
IK1 : $\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$	FS1 : $\Box\top$
IK2 : $\Box(\phi \supset \psi) \supset (\Diamond\phi \supset \Diamond\psi)$	FS2 : $\Box(\phi \wedge \psi) \equiv (\Box\phi \wedge \Box\psi)$
	FS6 : $\Diamond(\phi \supset \psi) \supset (\Box\phi \supset \Diamond\psi)$
Rules	**Rules**
MP : ϕ and $\phi \supset \psi$ implies ψ	MP : ϕ and $\phi \supset \psi$ implies ψ
Nec : ϕ implies $\Box\phi$	Reg : $\phi \supset \psi$ implies $\Box\phi \supset \Box\psi$ and $\Diamond\phi \supset \Diamond\psi$.

Analogously to the Hilbert deduction relation of IPC (see Def. 2.2.2), let $\vdash_{\mathsf{CK\text{-}1}}$ and $\vdash_{\mathsf{CK\text{-}2}}$ denote the Hilbert deduction relation of the system CK-1 and CK-2, respectively. Again, one can easily show that the systems CK-1 and CK-2 are equivalent which comes as a corollary from Prop. 2.2.3.

Corollary 2.2.1. *The systems* CK-1 *and* CK-2 *are equivalent,* i.e., $\forall\phi \in \mathit{For}\mathcal{ML}$

$$\vdash_{\mathsf{CK\text{-}1}} \phi \quad \textit{iff} \quad \vdash_{\mathsf{CK\text{-}2}} \phi. \qquad \nabla$$

Constructive modal logics based on CK have adequate birelational semantics [4; 90; 188; 196; 272], topological semantics [138] and category-theoretical semantics [27]. The birelational semantics is based on (2.1) for $\Box$ and the stronger interpretation of $\Diamond$ which replaces (2.2), and was introduced in [90] in the context of propositional lax logic (PLL):

$$\mathfrak{M}; x \models \Diamond\phi \quad \text{iff } \forall x'.\, x \preceq x', \exists y.\ x'\ R\ y\ \ \&\ \ \mathfrak{M}; y \models \phi. \tag{2.3}$$

Hereby, $\Diamond\phi$ is hereditary w.r.t. $\preceq$ by definition and thus does not need the frame property (F1). This way, and by further omitting frame condition (F2) as well, the axiom schemata FS4/IK4 and FS5/IK5 turn into non-trivial frame properties generating proper extensions of CK. The axiom FS3/IK3 is refuted by the addition of fallible worlds [4; 188; 196; 269] to the birelational Kripke frame, that is worlds where every proposition is true.

In summary, the system CK is non-normal w.r.t. possibility $\Diamond$ because of the lack of disjunctive distribution. Furthermore, CK breaks with one requirement as postulated by Simpson [249], that is, the addition of the principle of the Excluded Middle $\phi \vee \neg\phi$ does not collapse the theory of CK to classical K. We refer the reader to [32; 80; 188; 196] for a more comprehensive discussion of the system CK and its model and proof theory.

CHAPTER 3

Constructive Description Logics – State of the Art

Rooted in early approaches in the field of artificial intelligence, description logics have emerged to an active and large area of research and practice with a significant impact in the field of knowledge representation and Semantic Web applications. Despite their success in various application areas, only recently the question of investigating the semantics and proof theory of constructive DLs has been addressed. A constructive interpretation of truth becomes fundamental in application scenarios where it is important to reconcile model-theoretic truth and operational behaviour of proof systems, which is very difficult or impossible under the classical semantics. However, just like there exist many different classical DL theories to suit different applications, there should be different constructive theories incorporating a choice of interpretations of constructiveness. This chapter presents an overview of the current state of research in the field of constructive description logics.

3.1 Constructive Description Logics: What, Why & How?

The work of de Paiva is one of the first proposals of '[...] possible conceptions of constructive description logics' [78, p. 1]. De Paiva puts forward the long-term goal to define a constructive 'contextual' description logic inspired by the work of Bobrow et al. [37]. The approach is motivated by philosophical, mathematical and pragmatic points of view: (i) de Paiva [78] argues from a philosophical point of view that DLs mainly concern about decidable predicates which '[...] should lead to calculi that are basically constructive' [78, p. 1]. (ii) From a mathematical perspective, the investigation of the semantics and the proof theory of constructive DLs will be useful to obtain a computational interpretation of DLs following the Curry-Howard isomorphism which will lay the ground for the development of a useful type theory based on DLs. Furthermore, constructive DLs may be used as a kind of sanity check to provide evidence for the meaningfulness of existing classical DLs. Moreover, de Paiva argues that expounding the theory of constructive DLs will allow to examine their relation to existing classical and constructive logics as well as to support their classification into the hierarchy of traditional logics [52; 78]. (iii) From a pragmatic perspective, de Paiva [78, p. 1] claims that constructive DLs would be the right foundation in application areas where one has to deal with imprecise or incomplete domain knowledge, motivated by an example from natural language processing.

The proposal puts forward three constructive variants of $\mathcal{ALC}$, following a translation-based approach. While the syntax of these constructive reinterpretations coincide with that of $\mathcal{ALC}$, only differing in including the implication operator (a.k.a. subsumption) as a concept-forming operator and replacing concept negation $\neg C$ by the abbreviation $C \supset \bot$, the adequate formulation of the semantics is subject to controversy:

(i) Firstly, $\mathcal{ALC}$ can be considered as a two-variable fragment of classical first-order logic, *i.e.*, de Paiva [78] defines the system $I\mathcal{ALC}$ by using a syntactic translation targeting intuitionistic first-order logic (IQC). This approach constructs the semantics for $I\mathcal{ALC}$ based on proper reductions of standard intuitionistic frames from IQC that arise from the translation of $\mathcal{ALC}$ into IQC. Note that *standard* intuitionistic models separate the intuitionistic partial order from the accessibility relation(s). This approach is in line with the well-known construction of standard intuitionistic Kripke-style semantics for $\mathsf{FS/IK}$ [103; 249, p. 190].

(ii) Secondly, one can exploit Schild's [245] result stating that $\mathcal{ALC}$ is a notational variant of the multimodal logic K_m, and accordingly consider a translation into a constructive version of K. However, as pointed out in [52; 78; 80; 196], there exist several non-equivalent proposals of constructive modal logics that are '[...] competing for the title [...]' [78, p.3] of *the constructive analogue* of the modal logic K. The paper proposes two IMLs as a target, firstly, the multimodal variant of the system $\mathsf{FS/IK}$ [96; 228; 249], and secondly, the system CK [27; 188; 196]. The corresponding constructive description logics are denoted by $i\mathcal{ALC}$ (based on $\mathsf{FS/IK}$) and $c\mathcal{ALC}$ (CK) respectively [78].

De Paiva claims that $c\mathcal{ALC}$ '[...] may be an adequate basis for dealing with the notion of context as used in knowledge representation, artificial intelligence and computational linguistics' [188, p. 2]. However, the proposal does not give a formal definition of the appropriate Kripke semantics for the system $c\mathcal{ALC}$. This is what will be achieved in this thesis, by investigating the model theory and proof theory of $c\mathcal{ALC}$ in detail.

3.2 Intuitionistic Semantics via Translation into IQC

3.2.1 Intuitionistic $\mathcal{ALC}$ ($I\mathcal{ALC}$)

The system $I\mathcal{ALC}$ has been investigated from a model-theoretic and a proof-theoretic perspective: Villa [270, pp. 6–11; 271] investigates the Kripke semantics of the system

$\mathcal{IALC}$, which is obtained from the translation of $\mathcal{ALC}$ into IQC and discusses why a direct translation of the Kripke semantics for intuitionistic first-order logic is not adequate for the DL domain. Villa [270; 271] proves the monotonicity and disjunction property, as well as disproves the finite model property for $\mathcal{IALC}$. The latter is argued by showing that, while each instance of the axiom schema $\mathsf{KUR} =_{df} \forall R.\neg\neg C \supset \neg\neg\forall R.C$ known as *Kuroda principle* is valid in all $\mathcal{IALC}$ models with a finite domain, one can construct a countermodel to KUR in an infinite (standard) intuitionistic frame. Indeed, Simpson [249] highlighted this fact before and showed that the finite model property for some IK logics ($\mathsf{IK}, \mathsf{IKD}, \mathsf{IKB}, \mathsf{IT}, \mathsf{IKDB}, \mathsf{IKTB}$ and $\mathsf{IS5}$) can be established relative to birelational semantics, while, w.r.t. *standard* intuitionistic Kripke models, these logics fail to possess the finite model property. A similar argumentation falsifying the finite model property for FS w.r.t. standard intuitionistic models (called FS-models in [103, p. 192]) has been presented by Gabbay et al. [103, pp. 192 f.], and they show a proof of the finite model property for FS w.r.t. non-standard (birelational) models by a filtration method [103, pp. 453 ff.].

Clément [64] investigates the proof theory of $\mathcal{IALC}$ by defining the natural deduction system $\mathrm{N}_{\mathsf{IALC}}$ inspired by Braüner and de Paiva [50] and Simpson [249], proving soundness and completeness w.r.t. $\mathcal{IALC}$ Kripke semantics, and develops a sound and complete Gentzen-style sequent calculus, denoted by $\mathrm{G}_{\mathsf{IALC}}$, for which he proves cut-admissibility. Inspired by the work on focussing[4], as introduced by Andreoli [6] in the context of classical linear logic, Clément [64, pp. 61 ff.] defines the focussing sequent calculus $\mathrm{FG}_{\mathsf{IALC}}$ for $\mathcal{IALC}$, that allows for more efficient backward proof search, and proves its correctness relative to the before-mentioned sequent calculus $\mathrm{G}_{\mathsf{IALC}}$.

3.2.2 Intuitionistic $\mathcal{ALC}$ Kuroda Logic ($\mathcal{KALC}$)

The works [39; 43; 45; 270; 271] of Bozzato et al. and Villa introduce a constructive version of K that is motivated as a refinement of $\mathcal{ALC}$, in which the classical semantics is enriched by a partial order representing states of knowledge to deal with partial or incomplete information that can increase in time [45, p. 51; 39, p. 7]. Its semantics is induced by the direct translation to intuitionistic first-order semantics which corresponds to the first proposal by de Paiva [78] for $\mathcal{IALC}$, but restricts the intuitionistic Kripke semantics such that the *Kuroda principle* $\mathsf{KUR} =_{df} \forall R.\neg\neg C \supset \neg\neg\forall R.C$ becomes an axiom. Correspondingly, this logic is called $\mathcal{KALC}$. According to [43; 270; 271] the schema KUR states that for every possible world exists a final (classical) world with perfect knowledge, that is a successor having no further successors and which

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[4] The idea of focussing is to provide a normal form for cut-free sequent calculi in which the structure of the derivations is organised by the application of invertible and non-invertible rules, in order to decrease the number of possible derivations to support efficient backward proof search.

is interpreted according to the classical semantics. Two variants of $\mathcal{KALC}$ have been introduced, namely $\mathcal{KALC}$ and $\mathcal{KALC}^{\infty}$, differing in whether the underlying poset is assumed to be finite or infinite. Bozzato, Ferrari and Villa [43] conjecture that KUR implies that $\mathcal{KALC}^{\infty}$ satisfies the finite model property, without giving a proof.

(i) The logic $\mathcal{KALC}$ [39; 45] is based on a finite Kripke semantics in the style of standard intuitionistic Kripke-semantics. The finite model property for $\mathcal{KALC}$ trivially holds by definition, since the Kripke models generating the theory $\mathcal{KALC}$ assume finiteness of the underlying poset right from the start. The notions of forcing and realisability replace classical truth, and it is shown that $\mathcal{KALC}$ enjoys the monotonicity and disjunction property. Decidability of $\mathcal{KALC}$ is demonstrated by the development of a sound, complete and terminating tableau-based decision procedure in [39, pp. 27 ff.; 45], deciding the standard DL reasoning services like concept satisfiability, subsumption and instance checking. This calculus is based on sets of signed formulæ inspired by Fitting [100] and efficiently handles duplications in the treatment of implication, building on previous work by Avellone, Ferrari and Miglioli [13] and Miglioli, Moscato and Ornaghi [200]. However, in contrast to the tableau calculus for $\mathcal{CALC}^{\mathsf{C}}$ by Odintsov and Wansing [219] as described in the following, this work does not extend the technique from [88] to handle the modalities $\exists R, \forall R$ in a duplication free way. It is exemplified that the tableau includes countermodel construction for inconsistent formulæ, explicitly constructing states of knowledge related by a partial ordering to update them. Regarding the complexity of the tableau algorithm, Bozzato [39, pp. 54–55] concludes that $\mathcal{KALC}$ realisability is PSpace-hard and conjectures that an implementation of a proof-strategy similar to the tracing technique [247; 261] would yield PSpace-completeness for $\mathcal{KALC}$. The relation of $\mathcal{KALC}$ to $\mathcal{IALC}$, $\mathcal{KALC}^{\infty}$, IQC, $\mathsf{IQC} \oplus \mathsf{KUR}$ and FS_m is studied in [39, pp. 56–65].

(ii) The logic $\mathcal{KALC}^{\infty}$ [270, pp. 13–32; 271; 43] frees $\mathcal{KALC}$ from the restriction to finite Kripke models and allows for a possibly infinite poset. In this way the semantics correspond to the standard intuitionistic Kripke semantics of $\mathcal{IALC}$, but contrary to the latter the partial order is restricted to a poset with final elements in order to force all instances of the schema KUR in $\mathcal{KALC}^{\infty}$. Hereto, the notion of final worlds is introduced, and it is required that for every possible world there exists at least a final world, which is interpreted classically. In [270, pp. 15–32; 271; 43] a sound and complete tableau calculus for $\mathcal{KALC}^{\infty}$ with an efficient handling of duplications is presented, that is inspired essentially by the calculus for Kuroda logic [200] and optimised calculi for IPC [13; 150]. Villa [270, pp. 33 ff.] introduces a natural deduction calculus, denoted by $\mathcal{ND}_{\mathrm{Kur}}$, and proves corresponding theorems for soundness and completeness w.r.t. to the forcing relation of $\mathcal{KALC}^\infty$. In particular, such natural deduction calculi are essential to study a computational interpretation and accordingly extend the Curry-Howard correspondence to $\mathcal{KALC}^\infty$.

3.3 Intuitionistic Semantics via Translation into FS/IK

De Paiva, Haeusler and Rademaker [77] present the system $i\mathcal{ALC}$ that can be considered a notational variant of Simpson's IML IK [249] and coincides with de Paiva's [78] second proposal to obtain a constructive DL. They present a birelational Kripke-style semantics for $i\mathcal{ALC}$, which is claimed to be a simplification of $c\mathcal{ALC}$ [195] in the sense that possibility distributes over nullary (FS3/IK3) and binary disjunctions (FS4/IK4). The logic $i\mathcal{ALC}$ is characterised in terms of a Hilbert-style calculus that coincides with the axiomatisation of the multimodal version of the IML IK (*cf.* p. 33). De Paiva et al. [77] state (without proof) that the Hilbert system is sound and complete w.r.t. the proposed Kripke semantics. Furthermore, they introduce a natural deduction system and a labelled sequent calculus for $i\mathcal{ALC}$ and claim (without giving a proof) that (i) the sequent calculus for $i\mathcal{ALC}$ is equivalent to the Hilbert system for $i\mathcal{ALC}$, and (ii) the sequent calculus for $i\mathcal{ALC}$ is sound and complete w.r.t. the proposed Kripke semantics for $i\mathcal{ALC}$.

In [123–126] the use of the intuitionistic DL $i\mathcal{ALC}$ is motivated as a base for representing and reasoning about legal knowledge. In this approach, the possible worlds of the domain (called legal universe in [126, p. 5]) represent individual legal statements, the intuitionistic preorder controls the precedence of legal statements and roles express the relationship between individual laws as legal connections. In [124–126] examples discussing a case of "Conflict of Laws in Space"[5] are considered, to demonstrate how intuitionistic negation supports the analysis of the coherence of legal statements. Two further sequent calculi for $i\mathcal{ALC}$ are presented:

(i) A standard label-free sequent calculus for $i\mathcal{ALC}$ is proposed in [126, p. 8], which does not have independent right- and left-introduction rules for existential ($\exists R$) and universal restriction ($\forall R$).

(ii) In [125] a sequent calculus is introduced that differentiates two kinds of rules: a) The first kind of rules uses labelled concepts in hybrid style of the form $x{:}C$ and xRy, whereas b) the remaining set of rules is free of labels. It is claimed that

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[5] The Conflict of Laws (also known as Private International Law) refers to a situation when the outcome of a legal dispute has to deal with an external *foreign* law factor which leads to a disagreement on the law (w.r.t. space, e.g. country) to be applied (cf. [185]).

> this sequent calculus is sound and complete w.r.t. the $i\mathcal{ALC}$ Kripke semantics and deciding provability and satisfiability of $i\mathcal{ALC}$ concepts is PSPACE-complete (without giving a proof).

In a recent work Haeusler and Rademaker [127] revise the sequent calculus from [125] and give a proof of its soundness and completeness. The completeness is shown relative to the axiomatisation of $i\mathcal{ALC}$ by deriving the axioms of $i\mathcal{ALC}$ in the sequent system. Soundness is proven w.r.t. the Kripke semantics, demonstrating that each sequent rule is validity preserving.

A contextual extension of $c\mathcal{ALC}$ suggesting constructive modalities as McCarthy-style contexts [186] in artificial intelligence, which has previously been discussed for CK in [188], has been introduced by de Paiva and Alechina [79]. The authors recall the syntax, Kripke semantics and proof theory of $c\mathcal{ALC}$ in terms of Hilbert and Gentzen-style sequent calculi. It is claimed that the Hilbert and Gentzen sequent calculi are sound and complete w.r.t. the $c\mathcal{ALC}$ Kripke semantics. Inspired by Wolter and Zakharyaschev [277], the system $c\mathcal{ALC}_\Box$ extends $c\mathcal{ALC}$ by the modality $\Box$, which is viewed as a constructive context operator on top of $c\mathcal{ALC}$ such that each possible world corresponds to a $c\mathcal{ALC}$ Kripke domain. A rough sketch of the proof showing decidability of the satisfiability of $c\mathcal{ALC}_\Box$ concepts is given, which relies on the results for $c\mathcal{ALC}$ by Mendler and Scheele [195]. The authors conjecture that a multimodal variant of $c\mathcal{ALC}_\Box$ may provide a system $c\mathcal{ALC}_{ctx}$ supporting several artificial intelligence contexts and suggest an application in the domain of natural language processing. However, no proofs are given and the correctness of the presented sequent calculus remains unclear.

3.4 Constructive Inconsistency-tolerant Description Logics

An early approach to define constructive DLs is by Odintsov and Wansing [220], who define three constructive inconsistency-tolerant paraconsistent description logics. In short, '[...] *paraconsistent* logics are those, which admit inconsistent but non-trivial theories, *i.e.*, the logics which allow one to make inferences in a non-trivial fashion from an inconsistent set of hypotheses' [221, p. 1]. Odintsov and Wansing's [220] proposal is motivated by applications that have to deal with incomplete knowledge, inconsistent data or negative information. The idea of their approach is to combine the intuitionistic interpretation of information as stages of knowledge, where established (true) facts monotonically increase along this order, with *strong constructive negation* that allows to constructively handle falseness of facts, for an independent treatment of *positive* and *negative* information w.r.t. the intuitionistic information order. Strong (constructive) negation, denoted by the symbol $\sim$, goes back to the paraconsistent four-valued logic N4 by Nelson [217] (cf. [221, pp. 132 ff.]) and is based on the idea that the falseness of an atomic concept can be seen directly and verified on the spot [221, pp. 1 ff.], while the falseness of complex concept descriptions is determined by the falseness of their components.

A key characteristic of Nelson's [217] logic $\mathsf{N4}$ is the *constructible falsity property* given by $\vdash \sim(C \sqcap D) \;\Rightarrow \vdash \sim C$ or $\;\vdash \sim D$, where $\vdash$ denotes the derivability relation in the system $\mathsf{N4}$ by Nelson. The logics introduced by Odintsov and Wansing [220], denoted by $\mathcal{CALC}^{\mathsf{C}}$ (*constructive classical* $\mathcal{ALC}$), $\mathcal{CALC}^{\mathsf{N4}}$ and $\mathcal{CALC}^{\mathsf{N4}d}$, extend $\mathcal{ALC}$ by intuitionistic implication and strong negation, and their semantics are induced in terms of translations:

(i) Firstly, $\mathcal{CALC}^{\mathsf{C}}$ extends the language $\mathcal{ALC}$ by adding implication as a primitive operator and replacing negation ($\neg$) by strong negation ($\sim$). Its semantics is obtained by the standard translation into classical predicate logic and uses an interpretation of the form $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \cdot^{\mathcal{I}})$ that corresponds to the usual intuitionistic birelational semantics, where $\preceq$ is the intuitionistic preorder and the objects of the domain $\Delta^{\mathcal{I}}$ represent the stages of information. The preorder $\preceq$ is hereditary w.r.t. atomic concepts A, but also w.r.t. strongly negated propositions $\sim A$. The interpretation of the operators is no surprise. Implication is interpreted intuitionistically, while the modalities $\exists R, \forall R$ are interpreted classically. To satisfy the monotonicity property w.r.t. arbitrary concepts under this interpretation, Odintsov and Wansing [220] require for all roles R the frame conditions (*cf.* Fig. 2.3) $\preceq^{-1} ; R \subseteq R ; \preceq^{-1}$ and $\preceq ; R \subseteq R ; \preceq$[6].

(ii) The logics $\mathcal{CALC}^{\mathsf{N4}}$ and $\mathcal{CALC}^{\mathsf{N4}d}$ are based on Nelson's [217] system $\mathsf{N4}$ and their semantics are induced by the translation into its first-order variant $\mathsf{QN4}$ (quantified $\mathsf{N4}$), differing in that the semantics of $\mathcal{CALC}^{\mathsf{N4}d}$ guarantees the duality of the modalities $\forall R$ and $\exists R$ w.r.t. strong negation ($\sim$). The interpretation for both systems is given in terms of standard intuitionistic Kripke-style semantics that separates the intuitionistic preorder from the modal accessibility relations. Decidability of $\mathcal{CALC}^{\mathsf{N4}d}$ is proven by establishing the correspondence between the positive fragment of $\mathcal{CALC}^{\mathsf{N4}d}$ and the positive fragment of the decidable multimodal variant of the IML FS [96; 228], building on the results by Grefe [118, pp. 24 ff.] and Simpson [249, pp. 157 ff.].

Proofs of the disjunction property and constructible falsity property are given for all three systems. Further, sound and complete tableau calculi are presented for all three systems, which use signed sets of concepts in the spirit of Fitting [100]. These allow

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[6] In [219] the authors omit the frame condition $\preceq ; R \subseteq R ; \preceq$ and replace it by the provably equivalent intuitionistic interpretation of universal restriction ($\forall R$) [cf. 249, pp. 45–46].

to express positive and negative statements of the form $+C(a)$ or $-C(a)$ relative to a possible world a, which state that C is “true” or “not true” at a respectively.

In [219], the authors enhance the tableau calculus for $\mathcal{CALC}^{\mathsf{C}}$ by providing a terminating, sound and complete tableau-based decision procedure for this logic, which is inspired by ideas from Dyckhoff [88] to avoid the duplication of formulæ and the occurrence of loops. This shows that $\mathcal{CALC}^{\mathsf{C}}$ is elementary decidable, and the authors estimate the algorithmic complexity to be 2NExpTime.

Kaneiwa [153] presents the semantics of an extension of $\mathcal{CALC}^{\mathsf{N4}}$, denoted by $\mathcal{CALC}^2_{\sim}$, which combines strong negation and Heyting (intuitionistic) negation. The author compares $\mathcal{CALC}^2_{\sim}$ with a variant of $\mathcal{ALC}$ that combines classical negation and strong negation, called $\mathcal{ALC}_{\sim}$, and proves that the semantics of $\mathcal{ALC}_{\sim}$ preserve contradictoriness and contrariness [145], while the semantics of $\mathcal{CALC}^2_{\sim}$ does not. Taking into account the deep connection between hybrid logic and description logic, the hybrid variant of $\mathsf{N4}$ defined by Braüner [51, pp. 190 ff.] is also worth mentioning, featuring a sound and complete Hilbert-style axiomatisation.

Based on the semantics of $\mathcal{CALC}^{\mathsf{C}}$ [219; 220], Villa [270, pp. 57 ff.] presents a birelational Kripke-style semantics that is obtained from the $\mathcal{CALC}^{\mathsf{C}}$ semantics by replacing strong negation with intuitionistic negation. The resulting system is denoted by $\mathcal{CALC}^{\mathsf{C}-}$. Proofs for the monotonicity and disjunction property w.r.t. $\mathcal{CALC}^{\mathsf{C}-}$ semantics are presented. Moreover, Villa [270, pp. 57 ff.] introduces a sound and complete tableau calculus for $\mathcal{CALC}^{\mathsf{C}\text{-}}$, which is similar to Odintsov and Wansing’s [220] tableau calculus for $\mathcal{CALC}^{\mathsf{C}}$, but differs in implementing the respective rules for intuitionistic negation instead of strong negation.

3.5 Computational Interpretations of Description Logics

3.5.1 Information Term Semantics for $\mathcal{ALC}$ ($\mathcal{BCDL}$)

Bozzato et al. present in [41] and [93] a constructive semantics for $\mathcal{ALC}$ that is based on *information term semantics* [201] and allows a computational reinterpretation of proofs implementing the BHK (*Brouwer–Heyting–Kolmogorov*) interpretation in the form of a realisability interpretation. This logical system is denoted by *Basic Constructive Description Logic* ($\mathcal{BCDL}$). According to [93], $\mathcal{BCDL}$ is essentially inspired by Kuroda logic, that is, IQC extended by axiom KUR. Informally, an *information term* for a formula provides a witness or explicit explanation for the truth of that formula in the form of a structured mathematical object [39; 41; 93], for instance, the proof that individual a belongs to the concept $\exists R.C$ is given by an information term associated with $\exists R.C$, which provides the witness b such that b is an R-successor of a and recursively realises concept C [93, p. 372]. The interesting feature of $\mathcal{BCDL}$ is that its semantics preserve the classical reading of the $\mathcal{ALC}$ connectives. In [93] a sound and complete natural deduction calculus, denoted by $\mathcal{ND}_c$, is presented that admits a computational interpretation of proofs. It is pointed out that in order to obtain a complete calculus it is necessary to rely on a restricted form of subsumption, *i.e.*, the subsumption $C \supset D$ is restricted by fixing the interpretation of concept C to a finite set of individual names, called *generator*. The constructive properties of $\mathcal{BCDL}$ are discussed by demonstrating proofs for the disjunction and explicit definability property [93, p. 395].

Bozzato [39, pp. 66 ff.] presents an information term semantics for the logic $\mathcal{KALC}$, which extends the language of $\mathcal{BCDL}$ to include unrestricted subsumption and implication and denotes this system by $\mathcal{BCDL}_{\mathcal{K}}$. The proof theoretic characterisation is given in terms of a natural deduction calculus $\mathcal{ND}_{\mathcal{K}}$, which is inspired by the calculus $\mathcal{ND}_c$ [41; 93] and shown to be sound w.r.t. the information term semantics of $\mathcal{BCDL}_{\mathcal{K}}$ and the Kripke-style semantics of $\mathcal{KALC}$ [39, pp. 70–79]. Based on $\mathcal{BCDL}$, a formal framework for specifying Semantic Web Services and verifying the correctness of the composition of services has been proposed in [39, pp. 66 ff.; 40; 44]. Services are viewed as processes with pre-, post-conditions and effects. Bozzato [39, p. 66] studies the question of whether the available services can be composed in order to satisfy a given composition goal with pre- and post-conditions, and defines a composition calculus that allows to extract the service implementation as a function mapping information terms for the input into information terms for the output. Such information terms are induced by the computational interpretation of the rules from the composition calculus. The calculus is a sequent-style calculus where each rule must satisfy certain applicability conditions. The correctness of a service composition is then justified by verifying that the applicability conditions of each rule are satisfied.

Bozzato's [39] approach for Web Service Composition has been extended in [137] and [136]. Hilila et al. [137] apply $\mathcal{BCDL}$ as a service composition framework for multi domain environments, which extends the work from [39; 40; 44] by adding support for flow control operators and the investigation of a methodology for service composition based on e-contracts. In [136] the authors apply this method to service composition in the domain of ambient intelligence applications and smart environments, and discuss a work-in-progress implementation of their methodology in the Isabelle/HOL theorem prover.

Villa [270] (cf. [42]) presents a DL-style action language [17] based on an information term semantics for $\mathcal{ALC}$, inspired by $\mathcal{BCDL}$. Similarly to the specification of services in [39; 40; 44] an action is seen as a process with pre- and post-conditions, and its consistency is determined by whether an information term can be generated for the output of an action application. An algorithm is presented that generates information terms for the output of an action application that operates on the notions of knowledge and state, represented in terms of an ABox and an information term respectively.

3.5.2 Type-theoretic Interpretation of $c\mathcal{ALC}$ ($\lambda\mathsf{CK}_n$)

Mendler and Scheele [198] (*cf.* [194; 196]) introduce $c\mathcal{ALC}$ (called CK_n in [198]) as a semantic type theory and introduce its simply typed contextual λ-calculus, $\lambda\mathsf{CK}_n$. The idea is to use DLs as a programming language type system [191; 206; 244]. Under this view, a TBox can be considered to be similar to classes in object oriented programming, which are specified in terms concepts (classes) and roles (properties). An ABox represents a set of entities relative and compliant to a TBox specification, and can be considered as a set of objects or instances of classes and their relationships among each other. The system is aimed as a specification formalism for programming in knowledge bases, *i.e.*, to allow for DL-typed functional programming over ABoxes as data structures, such that ABox reasoning corresponds to model checking and TBox reasoning becomes static type checking. The authors present a cut-free contextual sequent calculus and a computational interpretation for CK_n following the Curry-Howard correspondence, which allows for an interpretation of the modalities $\forall R$ and $\exists R$ as type operators with simple and independent constructors and destructors. Under the computational interpretation entities of an ABox are seen as contextual scopes, and the modalities $\forall R$ and $\exists R$ are interpreted in terms of operations to open and close, enter and leave context scopes. The flow of information is restricted, in the sense that information only flows from top to bottom. The Gentzen-style typing system presented for $\lambda\mathsf{CK}_n$ is shown to be sound and complete, and it is suitable for proof search in CK_n. The λ-calculus $\lambda\mathsf{CK}_n$ is shown to satisfy subject reduction, strong normalisation and confluence, and it also enjoys natural deduction style typing which is essential in the context of programming applications. Mendler and Scheele [198] put forward the goal to establish CK_n as a baseline for a constructive correspondence theory of constructive modal and description logics. A survey of type-theoretical interpretations of IMLs can be found in [198, pp. 27 ff.].

The calculus $\lambda\mathsf{CK}_n$ has been evaluated by means of a Haskell implementation of the typing system and β–reduction in the context of the Bachelor's thesis of Gareis [107], using maps for an efficient nameless representation of variables in λ–terms [241], that is, a binary tree is used to indicate the position of bound variables, which gives rise to a λ–calculus without need for α–conversion when reducing a term to normal form.

3.6 Minor Constructive Approaches to Description Logics

The two following approaches use a constructive interpretation of DLs primarily as a proof-theoretic tool to establish optimised decision and query answering procedures for description logics.

3.6.1 Proof-theoretic Approach by Martin Hofmann

The work by Hofmann [143] demonstrates polynomial-time decision procedures for description logics with cyclic definitions in terminologies based on a Gentzen-style proof theoretic approach using dynamic programming to obtain decision algorithms from proof systems. The focus of this work is to establish polynomial-time decision procedures for the subsumption problem for DLs under an interpretation of circular definitions as greatest fixed points and under the descriptive semantics [14; 22; 211]. Therein, the constructive interpretation of DLs plays only a supportive role as a formal approach while the main focus lies in the development and investigation of proof theoretic methods and their complexities. The author proposes sequent calculi for some language fragments of $\mathcal{ALC}$ with circular definitions and proves soundness, completeness and cut-elimination. The languages in question are the system $\mathcal{EL}$ consisting of concept intersection and existential restrictions, secondly the fragment with concept intersection and universal restrictions and finally $\mathcal{EL}$ extended by negation. These languages and their subsumption problem are investigated w.r.t. the descriptive semantics and under the greatest fixed point semantics. Hofmann [143] demonstrates that proof search for the subsumption problem in these sequent calculi is of the same complexity as previously established results from the DL literature, namely, for the language $\mathcal{EL}$ decidability of subsumption lies in polynomial time [21; 22], while it is in ExpTime for the fragment with concept intersection and universal restrictions and $\mathcal{EL}$ with negation [14].

3.6.2 Intuitionistic Approach to Query Answering in DLs

Royer and Quantz [238] present an intuitionistic, proof-theoretical characterisation of query answering in DLs with the aim to avoid the complexity that comes from the case analysis of implicit disjunctions and appears in sequent calculi in the form of right contractions and right disjunctions. Two query answering calculi are presented, a weak and a strong intuitionistic calculus, which are inspired by calculi from deductive databases. Completeness of the query answering calculi is shown relative to an axiomatic semantics based on derivability of an intuitionistic sequent calculus and a least fixed point semantics.

3.7 Our Approach

Following the proposal of de Paiva [78], we will investigate the constructive variant of $\mathcal{ALC}$ based on the system CK w.r.t. to its model theory and proof theory. Our development addresses the open questions regarding $c\mathcal{ALC}$ as raised by de Paiva's [78] proposal and confirms that $c\mathcal{ALC}$ constitutes a well-behaved constructive description logic. The constructive semantics of $c\mathcal{ALC}$ refines the classical one and hereby generates a family of theories that admit computational interpretations of proofs in line with the Curry-Howard isomorphism. Prior to this program we will examine the differences between the systems FS and CK in more detail in the first section of Chap. 4 and motivate why the semantics of CK are more desirable from a constructive point of view.

CHAPTER 4

Constructive Semantics for $\mathcal{ALC}$

The main objective of this chapter is to introduce the syntax and semantics of a constructive variant of the basic description logic $\mathcal{ALC}$, on which the following chapters will built upon. We will denote this logic by *constructive* $\mathcal{ALC}$ ($c\mathcal{ALC}$) in accordance with de Paiva's [78] proposal (as discussed in Chapter 3). Our approach exploits the close relationship between modal logics and description logics as a starting point, but is lifting it to a constructive point of view. Simply put, the logic $c\mathcal{ALC}$ will be based on the constructive modal logic CK [188]. In this way, $c\mathcal{ALC}$ is enjoying a similar relationship to intuitionistic modal logics as $\mathcal{ALC}$ can be considered a notational variant of the classical modal logic K_m (see Sec. 2.1.5).

In Sec. 4.1, we will give a short survey of the possible ways to establish a constructive Kripke-style semantics for $\mathcal{ALC}$, by discussing previous approaches from the field of intuitionistic modal logics (IML). Thereafter, we define in Sec. 4.2 the syntax and semantics of $c\mathcal{ALC}$ and thereof develop its model theory. Readers familiar with Kripke semantics of intuitionistic modal logic may skip the following section and directly continue with Sec. 4.2.

4.1 Kripke Semantics and the Choice of $c\mathcal{ALC}$

Relational semantics, which are often referred to as *Kripke semantics* have been introduced by Saul Kripke in 1959 [169] as a formal semantics for the modal logic S5. At first, its development had a focus on modal logics [168; 169; 171], later it has been extended to intuitionistic logic [170] and other non-classical systems [172]. A Kripke semantics consists usually of a domain, an accessibility relation and a valuation function. The basic idea is to assign a truth-value to a formula, relative to a specific state of affairs (or possible world) of the domain, while the accessibility relation constrains which states are reachable from a specific world of this domain. For a comprehensive historical survey of Kripke's work the reader may consult [65; 117] and [215] with a focus on Kripke completeness.

The analysis of previous approaches from the fields of intuitionistic modal logic [89; 96; 188; 228; 249], intuitionistic hybrid logic [49; 50] and the work on constructive or intuitionistic description logic [39; 64; 77; 78; 195; 270] exhibits that there exist several possible approaches on how to define Kripke-style semantics for a constructive variant of the basic DL $\mathcal{ALC}$. We will discuss these in the following section.

4.1.1 Variants of Kripke Semantics

The previous approaches[7] can mainly be differentiated into the following four categories:

- Firstly, one approach as highlighted in [161, Chap. 2.2.3] is by passing from the two-valued basis of classical DLs to a many-valued basis in the form of a finite Heyting algebra. This has been discussed in the context of IMLs in [98; 103; 161; 222] and in the setting of hybrid logics in [49, Chap. 8.1.1].

- Secondly, the semantics of a constructive DL can be obtained as an extension of intuitionistic first-order logic via the standard translation (*cf.* p. 25), considering constructive DLs as a fragment of IQC, just like classical modal logic is related to classical first-order logic. This approach relates the modalities $\exists R$ ($\Diamond$) and $\forall R$ ($\Box$) with the quantifiers of the first-order language. Kripke semantics following this direction have been studied in the context of IMLs [89; 103; 249], hybrid logic [49] and constructive DLs [39; 41; 43; 45; 64; 78; 93]. It is characterised by the separation of the intuitionistic structure (states of knowledge) from the modal possible worlds (individuals) by keeping the epistemic partial order $\preceq$ separate from the interpretation of the propositional symbols and the accessibility relation interpreting the modalities. This semantics is usually denoted by *intuitionistic Kripke semantics* or *standard intuitionistic semantics*.

- Thirdly, a constructive description logic can be obtained as a combination of IPC with the base DL $\mathcal{ALC}$ via *fibring* [56; 58; 104; 105], a very generic approach to combine and analyse logical systems. Its basic idea uses the notion of a fibring function that associates at any time models and worlds from one logical system to the other and vice versa [56; 105]. For instance, imagine we want to combine the intuitionistic implication $\supset$ from IPC with the $\mathcal{ALC}$ modality $\forall R$. Then, the evaluation of a combined formula $A_1 \supset \forall R.(A_2 \supset A_3)$ proceeds by interpreting the top-level connective $\supset$ by a pointed intuitionistic IPC model $m = (W, \preceq, a, h)$, where W is the set of possible worlds, $\preceq$ the intuitionistic preorder, $a \in W$ the actual world and h is a valuation function satisfying the heredity condition $x \in h(A)$ and $a \preceq a'$ implies $a' \in h(A)$. The statement $a \models_{m} A_1 \supset B_1$ with $B_1 = \forall R.(A_2 \supset A_3)$ holds iff for all b with $a \preceq b$ and $b \models_{m} A_1$ it follows that

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[7] A similar discussion in the context of variants of intuitionistic modal logics or hybrid logics can be found in [161, Chap. 2.2.3], [160, Chap. 2] and [49, Chap. 8].

$b \models_{m} B_1$. From the perspective of IPC the formula $B_1 = \forall R.(A_2 \supset A_3)$ is atomic, since $\forall R$ is not part of the language of IPC. The interpretation of $b \models_{m} A_1$ is clear for atomic A_1, and the idea of fibring is to get a value for the evaluation of $b \models_{m} \forall R.(A_2 \supset A_3)$. Such a value can be obtained by associating (via a fibring function) each $b \in W$ with a pointed $\mathcal{ALC}$ Kripke model $n_b = (W^b, R^b, a^b, h^b)$ with root a^b, and evaluating $\forall R.(A_2 \supset A_3)$ in the corresponding $\mathcal{ALC}$ model such that $b \models_{m} \forall R.(A_2 \supset A_3)$ iff $a^b \models_{n_b} \forall R.(A_2 \supset A_3)$. The subsequent evaluation of $A_2 \supset A_3$ is analogous and takes place at the R-successors of a^b by associating them to an intuitionistic Kripke model where the implication is evaluated relative to the intuitionistic semantics. Dov M. Gabbay identifies in [104; 105] that several intuitionistic modal logics arise as special cases of fibring, in particular, he shows that Wijesekera's system CK and Fisher Servi's system FS/IK arise from fibring IPC with K plus the addition of an interaction axiom for the system FS/IK. For a comprehensive survey on fibring logics the reader may consult [55; 58; 105].

- Finally, the classical Kripke semantics can be extended by adding an additional accessibility relation $\preceq$ to the Kripke frame, which is interpreted intuitionistically. Characteristic for this semantics is that the epistemic preorder $\preceq$ and the accessibility relation are not separated but instead relations on the same domain [90; 96; 228]. Accordingly, these semantics are usually denoted by the term *birelational semantics* [49; 103; 249] and also appeared under the term *two-frame semantics* in the context of propositional lax logic (PLL) [90]. Note that such semantics also arise by a special form of fibring logics (called *dovetailing*) where the fibring function behaves like an identity function [104; 105]. It is noteworthy to point out, that in the context of IMLs the standard Kripke semantics (as discussed above) often do not satisfy the finite model property while the birelational semantics do [49; 249]. However, the interpretation of individual names from an ABox or nominals is not clear under birelational semantics, since their classical interpretation as a singleton set violates the monotonicity property of the intuitionistic preorder [49, pp. 177 f.].

In the following we will put the attention on the last option, since on the one hand we will base c$\mathcal{ALC}$ on the IML CK for which a birelational semantics already exists [188], and on the other hand we do not require the level of generality as offered by the fibring method.

Notation. The *sequential composition* of two binary relations R and S is given by $R\,;S =_{df} \{(x,z) \mid \exists y.x \; R \; y \; S \; z\}$. ■

4.1.2 Variants of Birelational Kripke Semantics

The traditional approach in IMLs is to combine the standard intuitionistic semantics of the propositional connectives with the interpretation of $\Box, \Diamond$ as universal and existential quantifiers over possible worlds. However, the (classical) interpretation of $\Box, \Diamond$ (*cf.* Sec. 2.1.5) breaks with the intuitionistic heredity condition, *i.e.*, $x \models C$ and $x \preceq y$ implies $y \models C$. In fact, many different variants of IMLs can be found in the literature and it seems that there is a disagreement on the interpretation of the modalities and their relation to the intuitionistic accessibility relation as well as on which system represents the right intuitionistic or constructive analogue of the classical modal logic K. Since $\mathcal{ALC}$ is a notational variant of K_m (*cf.* p. 24) and forms the minimal or basic description logic, the same controversy regarding the right choice of semantics applies here, too, when devising a constructive variant of $\mathcal{ALC}$.

There exist mainly four choices to deal with this issue, which has been discussed similarly by Kojima [161, p. 8]:

- Firstly, birelational semantics also arise by *dovetailing* (fibring) [104; 105].
- The second approach as taken by Wolter and Zakharyaschev [275] is to interpret both modalities $\Box, \Diamond$ classically, where $\neg C$ is defined as $C \supset \bot$ and $\Diamond C$ as $\neg\Box\neg C$. Then, heredity is forced by imposing the frame condition $\preceq ; R = R ; \preceq = R$ [278]. In [275; 276] they introduce distinct accessibility relations $R_\Box, R_\Diamond$ as interpretations of $\Box, \Diamond$, which then require the frame conditions $\preceq ; R_\Box ; \preceq = R_\Box$ and $\preceq^{-1} ; R_\Diamond ; \preceq^{-1} = R_\Diamond$ to force the heredity condition. This system is denoted by $\mathbf{IntK}_{\Box,\Diamond}$ and its extension by the axioms $\mathsf{FS5}/\mathsf{IK5} = (\Diamond\phi \supset \Box\psi) \supset \Box(\phi \supset \psi)$ and $\mathsf{FS6} = \Diamond(\phi \supset \psi) \supset (\Box\phi \supset \Diamond\psi)$ corresponds to the logic FS/IK. Their analysis focusses on *normal IMLs* and investigates their relation to classical bimodal logics from an algebraic perspective [275; 276; 278].
- Thirdly, the standard approach in IMLs is to interpret necessity $\Box$ intuitionistically while possibility $\Diamond$ is interpreted classically:

$$x \models \Box C \quad \textit{iff} \quad \forall y.\, x \preceq y \Rightarrow \forall z.\, y\, R\, z \Rightarrow z \models C; \tag{4.1}$$

$$x \models \Diamond C \quad \textit{iff} \quad \exists z.\, x\, R\, z \;\;\&\;\; z \models C. \tag{4.2}$$

In order to force heredity under definition (4.2) it is necessary to impose conditions on the Kripke frame such that the models satisfy confluence between the partial order $\preceq$ and the accessibility relation, *i.e.*, the frame conditions $\preceq^{-1} ; R \subseteq R ; \preceq^{-1}$ and $R ; \preceq \subseteq \preceq ; R$ (*cf.* p. 38). There, $\Box$ is interpreted as a monotonic $\wedge$-preserving modality while $\Diamond$ as a monotonic $\vee$-preserving modal operator dually to $\Box$. This approach has been taken by Plotkin and Stirling [228], Fischer-Servi [96] and Simpson [249], and it corresponds to the two equivalent axiomatisations FS and IK, as introduced in Sec. 2.2.2.

- Finally, our approach is to interpret both modalities $\Box, \Diamond$ intuitionistically, using (4.1) for $\Box$ and the stronger interpretation of $\Diamond$ defined by:

  $$x \models \Diamond C \quad \textit{iff} \quad \forall y.\, x \preceq y \Rightarrow \exists z.\, y\, R\, z \;\&\; z \models C. \tag{4.3}$$

  Here, heredity of the modalities is forced by definition without any frame condition. Further, this semantics refutes the axiom schemata of disjunctive distribution FS4/IK4, *viz.* $\Diamond(C \vee D) \supset (\Diamond C \vee \Diamond D)$ and also the axiom FS5/IK5 $= (\Diamond A \supset \Box B) \supset \Box(A \supset B)$, which are tautologies in FS/IK (and also in classical modal and description logic). These axioms are problematic from a constructive point of view [90; 196; 272]. This approach has been taken first by Wijesekera [272] and is known as the constructive system CK as introduced in Sec. 2.2.2. Another theory, which follows this design, is given by the Kripke semantics of PLL [90], which can be seen as an extension of CK $\oplus$ 4. In subsequent works, the semantics of CK has been extended by Mendler and de Paiva [188][4; 196] in that they reject the axiom schema FS3/IK3, *viz.* $\neg\Diamond\bot$, for constructive reasons as well [90; 188; 196].

  Kojima [160, pp. 92 f.] stresses that this interpretation of $\Diamond$ violates the analogy between possibility $\Diamond$ and existential quantification $\exists$ in first-order logic, and in particular, a $\Diamond$ modality refuting FS4/IK4 cannot be interpreted by first-order existential quantification $\exists$, since even in intuitionistic first-order logic $\exists x.(\phi(x) \vee \psi(x)) \supset (\exists x.\phi(x) \vee \exists x.\psi(x))$ holds.

Our development of *cALC* follows the last approach discussed above and is obtained as an extension of the *constructive* modal logic CK [188; 195; 196; 272]. This decision is justified for the following reasons: (i) The third approach known as the system FS/IK seems unsatisfactory to us in that the Kripke semantics for the basic system already requires a frame condition to force heredity by connecting the intuitionistic and the modal accessibility relations. Observe, that the interpretation of $\Diamond$ (see (4.2)) not only requires a frame condition to force heredity, but it is also the cause of the axiom schemata FS4/IK4 and FS5/IK5. We will argue in the following that these are problematic from a constructive perspective. As pointed out in [196, p. 2], frame conditions of this kind indicate that their set-theoretic representation is not free and irredundant. Instead, theories such like IK/FS seem to be a special theory or extension of a system with respect to some more elementary class of interpretations. (ii) The second approach after Wolter and Zakharyaschev [275] goes even further by imposing a rather strong connection between the intuitionistic and modal dimension, which is due to the interpretation of the modalities $\Diamond, \Box$. Furthermore, its extended axiomatisation coincides with that of FS/IK, which is not in agreement with our constructive perception. (iii) The theory of fibring is a very general approach and in particular with mechanisms like *splitting* and *splicing*, it seems to be an adequate tool to analyse the correspondence between different logical systems [56; 58; 105]. However, we do not require this level of freedom for our approach.

4.1.3 The System $c\mathcal{ALC}$

In the system $c\mathcal{ALC}$, the classical principles of the Excluded Middle $C \sqcup \neg C \equiv \top$, *double negation* $\neg\neg C \equiv C$ and the *dualities* $\exists R.C \equiv \neg\forall R.\neg C$, $\forall R.C \equiv \neg\exists R.\neg C$ are no longer tautologies, which is in line with standard intuitionistic modal logics [89; 96; 228; 249]. It is well known that these principles are non-constructive and therefore need special care. In $c\mathcal{ALC}$, however, we go one step further and refute the principle of *disjunctive distribution* in its binary ($\mathsf{FS}4/\mathsf{IK}4$) and nullary variant ($\mathsf{FS}3/\mathsf{IK}3$) and also reject the interaction schema ($\mathsf{FS}5/\mathsf{IK}5$). It deserves some further explanation, why there is not a universal constructive or computational justification for the axiom schemata in question.

Let us begin with the principle of disjunctive distribution. It corresponds to the classical $\Diamond$-dual of the normality axiom $\Box(A \wedge B) \equiv \Box A \wedge \Box B$, which is commonly accepted for intuitionistic modal logics (and also classical DLs) but it is problematic from a constructive perspective [161; 188; 196; 272].

- Let us assume that $\exists R.C$ means 'C holds in some R-context'. We may be able to construct a proof, which is guaranteed to decide $C \sqcup D$ in 'some R-context'. However, this does not imply the existence of a proof of $\exists R.C \sqcup \exists R.D$, which means that we can separate the decision whether 'in some R-context C' or 'in some R-context D' holds. Suppose that role filling is an action that expenses computational resources and interacts with the environment to access or generate the data. Then, the disjunctive decision $C \sqcup D$ can only be made once the (R-filler) context to access the data has been established. This means, that the data model may well guarantee $\exists R.(C \sqcup D)$, yet it would be rather strong to assume it also satisfies $\exists R.C \sqcup \exists R.D$ where the decision is outside the scope of R and thus anticipated before the R-filler is even requested.

- Kojima [161; 162] rejects the principle of disjunctive distribution ($\mathsf{FS}4/\mathsf{IK}4$) by arguing from a type-theoretic perspective in the context of linear-time temporal

> logic (LTL). Take $\Diamond$ to represent the temporal operator *next* of LTL, where $\Diamond C$ means that 'C is true at the next instant of time' and is interpreted as a '(residual) *code* of type C' [161]. Then, a proof of FS4/IK4 corresponds to a function that produces for an input of type $\Diamond(C \vee D)$ an output value of type $\Diamond C \vee \Diamond D$. In particular, the function needs to decide whether the output value is of type $\Diamond C$ or $\Diamond D$. But, the decision if either $\Diamond C$ or $\Diamond D$ holds *now* cannot be drawn from the assumption (input) which states that 'either C or D is true at the next instant of time', since the input corresponds to some kind of quoted code which will not be executed until the next instant of time begins.

Interpretations of possibility $\Diamond$ from a contextual perspective and systems refuting the axiom schema FS4/IK4 have been considered already in early works [69; 97]. It has been argued in [92] that Curry's contextual view does not satisfy the law of disjunctive distribution. Fitch [97] developed the proof theory of an intuitionistic modal logic based on Heyting's intuitionistic logic. This axiomatisation does not contain the axiom FS4/IK4. However, it lacks any semantic justification or explanation for this reason. Presumably the first explicit rationale against the axiom FS4/IK4 has been given by Wijesekera [272] and Wijesekera and Nerode [273]. Their argumentation [272; 273] comes from the field of constructive concurrent dynamic logic (CCDL) demonstrating that the interpretation of possibility $\Diamond$ refutes the axiom schema FS4/IK4.

In a similar vein, we argue the denial of the axiom FS3/IK3. Suppose $\bot$ represents a type without value or a non-terminating or deadlocking program. Then, a program of type $\exists R.\bot$ represents a non-terminating procedure that will lock up in some R-context. However, this does not imply that the program locks up immediately without considering its context. The axiom FS3/IK3 can be rejected by extending the Kripke semantics by so-called fallible worlds [90; 188; 195; 196]. These correspond to possible worlds where every concept is true. Fallible worlds have previously been introduced in [86; 263] to provide an intuitionistic meta-theory for intuitionistic logics.

Similar problems arise in the interaction between $\Diamond$ and $\Box$ exhibited by FS5/IK5 as reported in [196], which stems from the semantic clause (4.2). Again, this axiom schema is valid in standard IMLs and classical DLs while there is no universal constructive justification for it, suggested as follows. Let $\exists R.C$ denote 'in some R-context C' and read $\forall R.C$ as 'in all R-contexts C'. The precondition $\exists R.C \supset \forall R.D$ corresponds to a procedure, which returns a proof of D for all accessible R-contexts under the assumption that C is true in some R-accessible context. However, this does not in general imply $\forall R.(C \supset D)$, *viz.* that in every R-accessible context C will imply D. Let us interpret 'R-context' as an accessible 'auditing context'. It may be possible to realise the statement '$\exists R.($*fraudulent action detected*$) \supset \forall R.($*suspect can be identified*$)$', since in the current context we have a strategy to process the evidence of fraud from some accessible audit context to obtain the identity of a suspect, which can be used in all accessible audit contexts. Contrary, it is not guaranteed that in every accessible audit context where fraud has been detected we can also identify the suspect. From the viewpoint of the local context, the source of fraud cannot be identified if there is nobody who detects fraud in the first place.

It seems to us that in constructive logics inspired by computational type theories [91; 187; 205] or modal type theories [159; 209; 210; 226], where constructive proofs turn into λ-programs, the schemata $\mathsf{FS}3/\mathsf{IK}3$ – $\mathsf{FS}5/\mathsf{IK}5$ fail to have a uniform computational justification. The schemata that do appear to be computationally justified are $\mathsf{FS}1/\mathsf{IK}1$, $\mathsf{FS}2/\mathsf{IK}2$ and $\mathsf{FS}6$. Restricting to these axioms yields the constructive system known as CK [27; 188] (*cf.* Sec. 2.2.2), which forms the starting point of our development.

This thesis is inspired by de Paiva's proposal [78] and will investigate the semantics and proof theory of the constructive description logic $c\mathcal{ALC}$. The logic $c\mathcal{ALC}$ is obtained as a multimodal extension of monomodal CK [188; 195; 196; 272], which originated from a constructive modal logic by Wijesekera [272]. As a modal logic, $c\mathcal{ALC}$ is non-normal regarding the interpretation of $\Diamond$ and thus proofs of decidability and the finite model property for standard intuitionistic modal logics (*e.g.*, for $\mathbf{IntK}_{\Box,\Diamond}$ [*cf.* 103, Chap. 10], or FS/IK [118; 249]) do not directly apply.

4.2 Syntax and Semantics of $c\mathcal{ALC}$

4.2.1 Syntax

The language of the constructive DL $c\mathcal{ALC}$ extends that of classical $\mathcal{ALC}$. As in classical $\mathcal{ALC}$, the letters A and B stand for *concept names*, the letter R for *role names*, and the propositional constants $\bot$ and $\top$ represent truth and falsehood respectively. The elementary descriptions are inductively combined into more elaborate formulæ, which are denoted by *concept descriptions*. The letters C and D are used for arbitrary concept descriptions.

Definition 4.2.1 (Syntax of $c\mathcal{ALC}$)**.** A DL signature is a structure $\Sigma = (N_C, N_R, N_I)$ of three denumerably infinite and pairwise disjoint alphabets of concept names N_C, role names N_R and individual names N_I. The set of well-formed *concept descriptions* C, D over signature Σ is defined inductively by the following grammar, where $A \in N_C$ and $R \in N_R$.

$$C, D ::= \; A \mid \top \mid \bot \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid C \supset D \mid \exists R.C \mid \forall R.C.$$

∇

Remark 4.2.1. Note that we omit the set of individual names in the following chapters, since we will not cover the ABox formalism for *c*$\mathcal{ALC}$. ■

In contrast to standard $\mathcal{ALC}$ (see Sec. 2.1.1) and affiliated classical description logics [16], this syntactical representation is more general in that it includes the implication operator $\supset$ (usually known as *subsumption* operator in classical DLs) as a concept-forming operator as in [39; 78; 219]. This allows for an arbitrary nesting of implications in concept descriptions, *e.g.*, $((D \supset C) \supset B) \supset A$. While the full power of such nested implications may not be necessary for practical DL-style applications, it will on the one hand allow us to axiomatise the full theory of *c*$\mathcal{ALC}$ conveniently in the form of a Hilbert-style calculus (see Sec. 5.1) and on the other hand it is required when passing to a computational interpretation like the Curry-Howard isomorphism [248]. Then, implication as a first-order operator of the language allows for the representation of a higher-order functional programming language in the sense that *c*$\mathcal{ALC}$ concept descriptions represent the typing of functional computations.

Like in $\mathcal{ALC}$ the universal concept $\top$ is redundant and codable as $\neg\bot$. Also, $\bot$ and $\neg$ can represent each other, *e.g.* $\bot = A \sqcap \neg A$ and $\neg C = C \supset \bot$. Otherwise, the operators of *c*$\mathcal{ALC}$ are independent and not interdefinable.

4.2.2 Semantics

The semantics of *c*$\mathcal{ALC}$ is an extension of the Kripke-style birelational semantics for CK, which has first been introduced in [188] for monomodal CK. In contrast to the Kripke-style representation in [188; 272], the semantics here are given in terms of a Tarski-style set-theoretic interpretation in the tradition of classical DLs [16]. In this sense, the interpretation of a concept corresponds to a set of individuals, while roles are interpreted as sets of pairs of individuals. The constructive interpretation is an extension of the classical semantics of $\mathcal{ALC}$. Classically, possible worlds are assumed to be atomic in the sense of that the degree of knowledge about an individual is maximally determined. In the constructive case [268], individuals are not assumed to be atomic but are abstract objects, which possess an internal structure that in general is only partially determined and thus subject to refinement. In the following *constructive* individuals are referred to as *entities*.

The constructive interpretation $\mathcal{I}$ of *c*$\mathcal{ALC}$ extends the classical models of $\mathcal{ALC}$ by (i) a preorder $\preceq^{\mathcal{I}}$ for expressing refinement and (ii) by *fallible worlds* $\bot^{\mathcal{I}}$. Accordingly, the semantics is based on two kinds of accessibility relations, namely the preorder $\preceq$ and relations R taken from a set of role names. Fallible entities are used to interpret contradictions and to invalidate the nullary distributivity law $\neg\exists R.\bot$ [4; 90; 161; 195].

Let $\preceq$ be a relation on entities and let $a \preceq b$ denote that (i) b is *more precisely determined* than a, (ii) b *refines* a or (iii) a *abstracts* b.[8] Then, each entity (possible world) constitutes a state of information and the epistemic preorder $\preceq$ models a potential increase of information or refinement of context. This contextual extension of knowledge is associated with the process of investigating stages of information (Kripke worlds) through an agent. The agent corresponds to the process of substantiating abstract entities as real individuals. In particular, $\preceq$ represents an information ordering on entities with the property that information is *robust* (or persistent) under refinement and may only potentially increase. This means, new facts may accumulate while previously established facts may never be refuted [249]. Additionally, two entities a and b may refer to the same theory when both $a \preceq b$ and $b \preceq a$ hold. Then, a and b share the same information content and thus are formally indistinguishable, yet still distinct $a \neq b$ due to existing lower-level properties. In line with the idea of refinement, we say that the interpretation of a concept C is *robust under refinement*

$$\text{if } a \in C^{\mathcal{I}} \text{ and } a \preceq^{\mathcal{I}} b \text{ then } b \in C^{\mathcal{I}}.$$

This *heredity condition* is achieved by the following definition:

Definition 4.2.2 ([195, p. 211])**.** A *constructive interpretation* or *constructive model* of $c\mathcal{ALC}$, simply called $c\mathcal{ALC}$-model, is a structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of

- a non-empty set $\Delta^{\mathcal{I}}$ of *entities*, the universe of discourse in which each entity represents a partially defined, or abstract individual;
- the *refinement* preordering $\preceq^{\mathcal{I}}$, which is a reflexive and transitive relation over $\Delta^{\mathcal{I}}$;
- a subset $\bot^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ of inconsistent worlds denoted by *fallible* entities. These are closed under refinement and role filling, and R is serial w.r.t. $\bot^{\mathcal{I}}$, *i.e.*,
  - $x \in \bot^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in \bot^{\mathcal{I}}$,
  - all fillers of a fallible entity $x \in \bot^{\mathcal{I}}$ are fallible, that is, $\forall R \in N_R.\ \forall z.\ x\, R^{\mathcal{I}}\, z \Rightarrow z \in \bot^{\mathcal{I}}$,
  - for each fallible entity $x \in \bot^{\mathcal{I}}$ exists a fallible filler, *i.e.*, $\forall R \in N_R.\ \exists z.\ x\, R^{\mathcal{I}}\, z$ & $z \in \bot^{\mathcal{I}}$; and
- a valuation function $\cdot^{\mathcal{I}}$ that defines the interpretation of role and concept names by mapping each role name $R \in N_R$ to a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and each

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[8] Alternatively, $\preceq$ may be interpreted temporally as in [227], *e.g.*, one may say that b *lies in the future of* a or b is *reachable* from a.

atomic concept $A \in N_C$ to a set $\bot^{\mathcal{I}} \subseteq A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, which is closed under refinement, *i.e.*, $x \in A^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in A^{\mathcal{I}}$.

The interpretation $\mathcal{I}$ is lifted from atomic symbols to concept descriptions, where $\Delta^{\mathcal{I}}_c =_{df} \Delta^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$ is the set of *infallible* (or *non-fallible*) elements in $\mathcal{I}$:

$$\begin{aligned}
\top^{\mathcal{I}} &=_{df} \Delta^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}_c.\ x \preceq^{\mathcal{I}} y \Rightarrow y \notin C^{\mathcal{I}}\} \\
(C \sqcap D)^{\mathcal{I}} &=_{df} C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &=_{df} C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(C \supset D)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}.\ (x \preceq^{\mathcal{I}} y \ \&\ y \in C^{\mathcal{I}}) \Rightarrow y \in D^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}.\ x \preceq^{\mathcal{I}} y \Rightarrow \exists z \in \Delta^{\mathcal{I}}.\ (y,z) \in R^{\mathcal{I}} \ \&\ z \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}.\ x \preceq^{\mathcal{I}} y \Rightarrow \forall z \in \Delta^{\mathcal{I}}.\ (y,z) \in R^{\mathcal{I}} \Rightarrow z \in C^{\mathcal{I}}\}. \qquad \nabla
\end{aligned}$$

Remark 4.2.2. The distinction to the classical descriptive semantics is the refinement relation $\preceq^{\mathcal{I}}$ and the universal quantification $\forall y \in \Delta^{\mathcal{I}}.\, x \preceq^{\mathcal{I}} y \ldots$ in the clauses of Definition 4.2.2. Entities in $\Delta^{\mathcal{I}}$ are partial descriptions representing incomplete information about individuals. Fallible entities $b \in \bot^{\mathcal{I}}$ correspond to the Kripke worlds where any concept including $\bot$ becomes true. They may be thought of as inconsistent, over-constrained tokens of information, self-contradictory objects of evidence or undefined (non-terminating) computations. They can be used to model the situation where the computation of a role-filler for an abstract entity fails. For instance, assume the computation of an R-filler for an entity a fails, *i.e.*, $\forall b.\, a\, R^{\mathcal{I}}\, b \Rightarrow b \in \bot^{\mathcal{I}}$. However, if a is an abstraction of an entity a' then a infallible role-filler $b' \in \Delta^{\mathcal{I}}_c$ may exist with $a'\, R^{\mathcal{I}}\, b'$ (*cf.* Ex. 4.2.2). ■

Observe, that the interpretation of negation $\neg$ is restricted to consider non-fallible refinements only. As mentioned before, $\neg C$ is just an abbreviation for the concept $C \supset \bot$ and it is easy to observe that its interpretation $(\neg C)^{\mathcal{I}}$ is equivalent to $(C \supset \bot)^{\mathcal{I}}$. In this sense, negation can be seen as a special case of implication $C \supset \bot$ and its interpretation is somewhat redundant. That is why we explicitly mention the case of $\neg C$ only occasionally in a proof, and omit it when its clear or trivial from the consideration of implication.

Lemma 4.2.1. *For all concepts C and interpretations $\mathcal{I}$, $(C \supset \bot)^{\mathcal{I}} = (\neg C)^{\mathcal{I}}$.* ∇

Proof. Proof by contraposition. Let $x \in \Delta^{\mathcal{I}}$ be an arbitrary entity.

($\Rightarrow$) Suppose that $x \notin (\neg C)^{\mathcal{I}}$. We need to show that $x \notin (C \supset \bot)^{\mathcal{I}}$. The assumption implies that there exists $x' \in \Delta^{\mathcal{I}}_c$ such that $x \preceq^{\mathcal{I}} x'$ and $x' \in C^{\mathcal{I}}$. Note that $x' \in \Delta^{\mathcal{I}}_c$ implies $x' \notin \bot^{\mathcal{I}}$. Therefore, $x \notin (C \supset \bot)^{\mathcal{I}}$.

($\Leftarrow$) Let us assume that $x \notin (C \supset \bot)^{\mathcal{I}}$. The goal is to demonstrate that $x \notin (\neg C)^{\mathcal{I}}$. From the assumption it follows that there exists $x' \in \Delta^{\mathcal{I}}$ such that $x \preceq^{\mathcal{I}} x'$, $x' \in C^{\mathcal{I}}$ and $x' \notin \bot^{\mathcal{I}}$. Since $x' \notin \bot^{\mathcal{I}}$, it follows that $x' \in \Delta_c^{\mathcal{I}}$. Hence, $x \notin (\neg C)^{\mathcal{I}}$. □

The extension of the classical semantics by the refinement relation $\preceq^{\mathcal{I}}$ and fallible entities $\bot^{\mathcal{I}}$ by Definition 4.2.2 allows for the desired degree of abstraction. The elements of $\Delta^{\mathcal{I}}$ are *abstract* individuals or *entities* rather than *concrete* or *atomic* individuals as in classic DLs, which is in line with our constructive view. Each entity implicitly subsumes all its refinements and truth is inherited. Specifically, one can verify the heredity condition and show that $x \in C^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in C^{\mathcal{I}}$ for arbitrary concepts C (see Prop. 4.2.2). Interestingly, by omitting fallible entities and collapsing the refinement structure of $\preceq^{\mathcal{I}}$ we arrive at the classical semantics of $\mathcal{ALC}$, as the following example illustrates.

Example 4.2.1 (Adapted from [195, p. 213, Ex. 1], with kind permission from Springer Science and Business Media)**.** The constructive semantics of $c\mathcal{ALC}$ coincide with the classical semantics of $\mathcal{ALC}$ whenever the preorder $\preceq^{\mathcal{I}}$ trivialises to an identity relation and if fallible entities are omitted. Accordingly, we call an interpretation $\mathcal{I}$ *classical* if $\preceq^{\mathcal{I}} = id_{\Delta}$ is an identity relation, *i.e.*, each entity refines itself, and if $\bot^{\mathcal{I}} = \emptyset$. In this sense, each classical $\mathcal{ALC}$ interpretation induces a trivial $c\mathcal{ALC}$ interpretation according to Def. 4.2.2 with a discrete relation $\preceq^{\mathcal{I}}$. Such interpretations validate the axioms $C \sqcup \neg C$ (PEM), $\exists R.\bot \supset \bot$ (FS3/IK3) and $\exists R.(C \sqcup D) \supset \exists R.C \sqcup \exists R.D$ (FS4/IK4) which are characteristic of classical $\mathcal{ALC}$ models. Therefore, the constructive semantics includes the classical one. ■

Fallible entities are information-wise maximal elements and therefore they are included in the interpretation of every concept.

Lemma 4.2.2 (Fallibles [195, p. 212, Lem. 1])**.** *For all $c\mathcal{ALC}$ concepts C and interpretations $\mathcal{I}$, it holds that $\bot^{\mathcal{I}} \subseteq C^{\mathcal{I}}$.* ∇

Proof. ([195, pp. 213 f.; 190, p. 35]) The proof is by induction on the structure of C. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a constructive interpretation, $x \in \Delta^{\mathcal{I}}$ and suppose that $x \in \bot^{\mathcal{I}}$. For the base case the goal is to show (i) $x \in \bot^{\mathcal{I}}$, (ii) $x \in A^{\mathcal{I}}$, and (iii) $x \in \top^{\mathcal{I}}$. Assertion (i) follows directly by assumption. It holds by Definition 4.2.2 that $\bot^{\mathcal{I}} \subseteq A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} = \top^{\mathcal{I}}$ which together with the assumption directly proves the cases (ii) and (iii). Induction step:

(**Case** $C \sqcap D$) The induction hypothesis implies $x \in C^{\mathcal{I}}$ and $x \in D^{\mathcal{I}}$. Thus, by Definition 4.2.2 it follows that $x \in (C \sqcap D)^{\mathcal{I}}$.

(**Case** $C \sqcup D$) By induction hypothesis analogously to the previous case.

(**Case** $C \supset D$) The goal is to show $x \in \bot^{\mathcal{I}} \Rightarrow x \in (C \supset D)^{\mathcal{I}}$. From the assumption $x \in \bot^{\mathcal{I}}$ we have that $\forall x'.\, x \preceq^{\mathcal{I}} x' \Rightarrow x' \in \bot^{\mathcal{I}}$. By induction hypothesis $x' \in D^{\mathcal{I}}$ and therefore $x \in (C \supset D)^{\mathcal{I}}$.

(**Case** $\exists R.C$) Let $x' \in \Delta^{\mathcal{I}}$ be arbitrary such that $x \preceq^{\mathcal{I}} x'$. Then, the assumption $x \in \bot^{\mathcal{I}}$ and Def. 4.2.2 implies $x' \in \bot^{\mathcal{I}}$ and there exists an R-filler $y \in \Delta^{\mathcal{I}}$ with $x' \, R^{\mathcal{I}} \, y$ such that $y \in \bot^{\mathcal{I}}$. The ind. hyp. implies $y \in C^{\mathcal{I}}$. Since x' was arbitrarily chosen, it follows that $x \in (\exists R.C)^{\mathcal{I}}$.

(**Case** $\forall R.C$) We need to prove $x \in \bot^{\mathcal{I}} \Rightarrow x \in (\forall R.C)^{\mathcal{I}}$. Let $x', y \in \Delta^{\mathcal{I}}$ be arbitrary entities such that $x \preceq^{\mathcal{I}} x' \, R^{\mathcal{I}} \, y$. Definition 4.2.2 implies $x' \in \bot^{\mathcal{I}}$ and $y \in \bot^{\mathcal{I}}$. Applying the induction hypothesis yields $y \in C^{\mathcal{I}}$. Since x', y were arbitrarily chosen from $\Delta^{\mathcal{I}}$, we can conclude that $x \in (\forall R.C)^{\mathcal{I}}$. □

Regarding infallible entities, one can show that their role-predecessors are infallible as well, which is due to Def. 4.2.2.

Proposition 4.2.1 (Infallibility of R-predecessors)**.** *For all interpretations $\mathcal{I}$ it holds that all R-predecessors x of an infallible entity y are infallible,* i.e., $\forall \mathcal{I}. \forall R \in N_R. \forall x, y \in \Delta^{\mathcal{I}}$. *if $y \notin \bot^{\mathcal{I}}$ and $x \, R^{\mathcal{I}} \, y$ then $x \notin \bot^{\mathcal{I}}$.* ∇

Proof. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a constructive interpretation, $R \in N_R$ and $x, y \in \Delta^{\mathcal{I}}$ be arbitrary entities. Suppose $x R^{\mathcal{I}} y$ and $y \notin \bot^{\mathcal{I}}$. The goal is to show that x is infallible, *i.e.*, $x \notin \bot^{\mathcal{I}}$. Assume to the contrary that $x \in \bot^{\mathcal{I}}$. Def. 4.2.2 implies that every fallible entity is closed under refinement and role filling, in particular this means that all R-successors of x must be fallible. Therefore, it follows that $y \in \bot^{\mathcal{I}}$. However, this contradicts the assumption, hence $x \notin \bot^{\mathcal{I}}$. □

After having introduced constructive models and investigated their properties, we define a semantic validity relation in the spirit of [33, pp. 17 f.].

Definition 4.2.3 (Validity relation [195, p. 212])**.** Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a constructive interpretation and $x \in \Delta^{\mathcal{I}}$ an entity. We say that entity x *satisfies* concept C in the interpretation $\mathcal{I}$, denoted by $\mathcal{I}; x \models C$, if and only if $x \in C^{\mathcal{I}}$. An interpretation $\mathcal{I}$ is a *model* of concept C, denoted by $\mathcal{I} \models C$, if and only if $\forall x \in \Delta^{\mathcal{I}}. \mathcal{I}; x \models C$. Finally, the notion $\models C$ holds iff $\forall \mathcal{I}. \mathcal{I} \models C$. All of these notions $\mathcal{I}; x \models \Gamma$, $\mathcal{I} \models \Gamma$ and $\models \Gamma$ are lifted to sets Γ of concepts in the usual universal fashion. ∇

Notation. When $\mathcal{I}$ is clear from the context then (i) we will identify $\mathcal{I}$ with $\cdot^{\mathcal{I}}$ and omit the superscript from $\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}$ and simply write $\Delta, \preceq, \bot$ instead, and (ii) we will omit it from a validity statement $\mathcal{I}; x \models C$ and write $x \models C$ instead. ■

The relation $\mathcal{I}; x \models C$ spreads out the validity of concept C across many $\preceq^{\mathcal{I}}$ related entities monotonically, *i.e.*, $\mathcal{I}; x \models C$ and $x \preceq^{\mathcal{I}} y$ implies $\mathcal{I}; y \models C$. The monotonicity of truth (or heredity condition) is the characteristic feature of intuitionistic semantics. The following proposition verifies the heredity condition for $c\mathcal{ALC}$.

Proposition 4.2.2 (Monotonicity property, robustness under refinement)**.** *Let* $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *be a constructive interpretation. Then, the following holds for all concepts* C *and* $x, x' \in \Delta^{\mathcal{I}}$*:*

$$\mathcal{I}; x \models C \text{ and } x \preceq^{\mathcal{I}} x' \text{ implies } \mathcal{I}; x' \models C. \qquad \nabla$$

Proof. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a constructive interpretation. Let $x, x' \in \Delta^{\mathcal{I}}$ and assume that $x \preceq^{\mathcal{I}} x'$. The proof is by structural induction on C, and is similar to the proofs for intuitionistic propositional logic [268] and constructive modal logic [272].

(**Case** atomic symbol) If C is an atomic symbol then the goal directly holds by Definition 4.2.2.

(**Case** $\neg C$) Suppose $\mathcal{I}; x \models \neg C$ and $x \preceq^{\mathcal{I}} x'$. Let x'' be such that $x' \preceq^{\mathcal{I}} x''$ and x'' is infallible. Then, by transitivity $x \preceq^{\mathcal{I}} x''$, and the assumption implies $x'' \notin C^{\mathcal{I}}$. Thus, $\mathcal{I}; x' \models \neg C$.

(**Case** $C \sqcap D$) Suppose that $\mathcal{I}; x \models C \sqcap D$ and $x \preceq^{\mathcal{I}} x'$. The goal is to show that $\mathcal{I}; x' \models C \sqcap D$. The assumption implies that $x \in (C \sqcap D)^{\mathcal{I}}$, *i.e.*, $x \in C^{\mathcal{I}}$ and $x \in D^{\mathcal{I}}$. The ind. hyp. yields $\mathcal{I}; x' \models C$ and $\mathcal{I}; x' \models D$. Hence, $\mathcal{I}; x' \models C \sqcap D$.

(**Case** $C \sqcup D$) Analogously to the previous case, by induction hypothesis.

(**Case** $C \supset D$) Suppose that $\mathcal{I}; x \models C \supset D$ and $x \preceq^{\mathcal{I}} x'$. Let $x'' \in \Delta^{\mathcal{I}}$ be an arbitrary entity such that $x' \preceq^{\mathcal{I}} x''$ and $\mathcal{I}; x'' \models C$. Then, transitivity of $\preceq^{\mathcal{I}}$ implies $x \preceq^{\mathcal{I}} x''$ and from $\mathcal{I}; x \models C \supset D$ it follows that $\mathcal{I}; x'' \models D$. Since x'' was arbitrary, we can conclude $\mathcal{I}; x' \models C \supset D$.

(**Case** $\exists R.C$) Let us suppose that $\mathcal{I}; x \models \exists R.C$ and $x \preceq^{\mathcal{I}} x'$. By assumption $x \in (\exists R.C)^{\mathcal{I}}$, *i.e.*, for all refinements of x there exists an R-successor that is contained in $C^{\mathcal{I}}$. Let $x'' \in \Delta^{\mathcal{I}}$ be arbitrary such that $x' \preceq^{\mathcal{I}} x''$. Transitivity of $\preceq^{\mathcal{I}}$ implies $x \preceq^{\mathcal{I}} x''$, which implies the existence of an entity z such that $(x'', z) \in R^{\mathcal{I}}$ and $\mathcal{I}; z \models C$. Hence, $\mathcal{I}; x' \models \exists R.C$.

(**Case** $\forall R.C$) Suppose that $\mathcal{I}; x \models \forall R.C$ and $x \preceq^{\mathcal{I}} x'$, *i.e.*, all R-successors of all refinements of x are contained in the interpretation of C. Let $x'', z \in \Delta^{\mathcal{I}}$ be arbitrary entities such that $x' \preceq^{\mathcal{I}} x''$ and $x''\ R^{\mathcal{I}}\ z$. Transitivity of $\preceq^{\mathcal{I}}$ implies $x \preceq^{\mathcal{I}} x''$. Then, the assumption $\mathcal{I}; x \models \forall R.C$ lets us conclude $\mathcal{I}; z \models C$. Hence, $\mathcal{I}; x' \models \forall R.C$. □

The following discussion is inspired by an example from [195, p. 213] in the context of database entities, and lets us reconsider the introductory example of the European tree frog (see p. 4). We will discuss in the following how the semantics of $c\mathcal{ALC}$ support the representation of dynamic behaviour and in particular exemplify the notions of *abstraction* and *refinement*. Furthermore, the example will demonstrate that the axiom schemata $\mathsf{FS4}/\mathsf{IK}4 = \exists R.(C \sqcup D) \supset (\exists R.C \supset \exists R.D)$, the law of the Excluded Middle $\mathsf{PEM} = C \sqcup \neg C$ and the duality $\neg\exists R.C \equiv \forall R.\neg C$ are refuted by the constructive semantics of $c\mathcal{ALC}$.

Example 4.2.2 (Adapted from [195, p. 213, Ex. 2], with kind permission from Springer Science and Business Media)**.** Let IGGY and POP be two instances of the concept description $\mathsf{EuropeanTreeFrog}$. Both are resident in $\mathsf{GERMANY}$, but they differ in their current location (IGGY is resident at the lily pond at the Wilhelma Zoo in Stuttgart, while POP lives on an oak tree in Coburg) and their current colouring. This means, they share the same attribute w.r.t. their residence (role $\mathsf{isResident}$), but are distinguished in the remaining attributes referenced by the roles $\mathsf{hasColour}$ and sitsOn. The instances of the concepts Green, Brown, and $\mathsf{LilyPad}$, Oak can be encoded in terms of RGB hex triplets and GPS coordinates respectively. In this sense, we use for the colours the abbreviations $\mathsf{LIME} = \#\mathsf{00FF00}$, $\mathsf{SIENNA} = \#\mathsf{A0522D}$ and use the shortcuts $\mathsf{LPAD_S} = (\mathsf{48°48'23''N}, \mathsf{9°12'21''E})$, $\mathsf{OAK_{CO}} = (\mathsf{50°15'17''N}, \mathsf{10°58'51''E})$ for the GPS coordinates. We have the following situation, where we represent the interpretation in ABox syntax:

$$\begin{aligned}
\mathsf{IGGY} : \mathsf{EuropeanTreeFrog},\ \mathsf{LIME} : \mathsf{Green},\\
\mathsf{LPAD_S} : \mathsf{LilyPad},\ (\mathsf{IGGY}, \mathsf{GERMANY}) : \mathsf{isResident},\\
(\mathsf{IGGY}, \mathsf{LPAD_S}) : \mathsf{sitsOn}, (\mathsf{IGGY}, \mathsf{LIME}) : \mathsf{hasColour},\ \text{and}
\end{aligned}$$

$$\begin{aligned}
\mathsf{POP} : \mathsf{EuropeanTreeFrog},\ \mathsf{SIENNA} : \mathsf{Brown},\\
\mathsf{OAK_{CO}} : \mathsf{Oak},\ (\mathsf{POP}, \mathsf{GERMANY}) : \mathsf{isResident},\\
(\mathsf{POP}, \mathsf{OAK_{CO}}) : \mathsf{sitsOn},\ (\mathsf{POP}, \mathsf{SIENNA}) : \mathsf{hasColour}.
\end{aligned}$$

Now, we can *abstract* from the second and third attribute (sitsOn and $\mathsf{hasColour}$) and consider the entities IGGY and POP as partially defined entities $\mathsf{IGGY}^\sharp$ and $\mathsf{POP}^\sharp$, as a way to omit or compress information. Omitting these attributes means that the abstract entities $\mathsf{IGGY}^\sharp$ and $\mathsf{POP}^\sharp$ possess the same properties and therefore can no longer be distinguished information-wise. In this sense, they can be seen as an abstraction $\mathsf{IGGYPOP}$ of a tree frog, but with internal structure. The refinement $\preceq$, which captures the degree of information of the abstract entities $\mathsf{IGGY}^\sharp$ and $\mathsf{POP}^\sharp$, is a cyclic (oscillating) refinement relationship between these entities that now refine each other, *i.e.*, $\mathsf{IGGY}^\sharp \preceq \mathsf{POP}^\sharp$ and $\mathsf{POP}^\sharp \preceq \mathsf{IGGY}^\sharp$. This cyclic relationship implies an abstract equivalence $\mathsf{IGGY}^\sharp \cong \mathsf{POP}^\sharp$, but it is not an identity $\mathsf{IGGY}^\sharp = \mathsf{POP}^\sharp$. This comes from the fact that both abstract entities $\mathsf{IGGY}^\sharp$ and $\mathsf{POP}^\sharp$ possess an incompatible realisation by $\mathsf{IGGY}^\sharp \preceq \mathsf{IGGY}$ and $\mathsf{POP}^\sharp \preceq \mathsf{POP}$ respectively.

The situation is depicted in Figure 4.1, where dotted arrows represent refinement and solid arrows correspond to the roles isResident, sitsOn and hasColour. The worlds $\mathsf{IGGY}^\sharp, \mathsf{POP}^\sharp, \mathsf{IGGY}, \mathsf{POP}, \mathsf{GERMANY}, \mathsf{OAK}_{\mathsf{CO}}, \mathsf{LPAD}_{\mathsf{S}}, \ldots$ represent the entities. We use the abbreviations $\mathsf{iR} = \mathsf{isResident}$, $\mathsf{hC} = \mathsf{hasColour}$ and $\mathsf{sO} = \mathsf{sitsOn}$ for the role names.

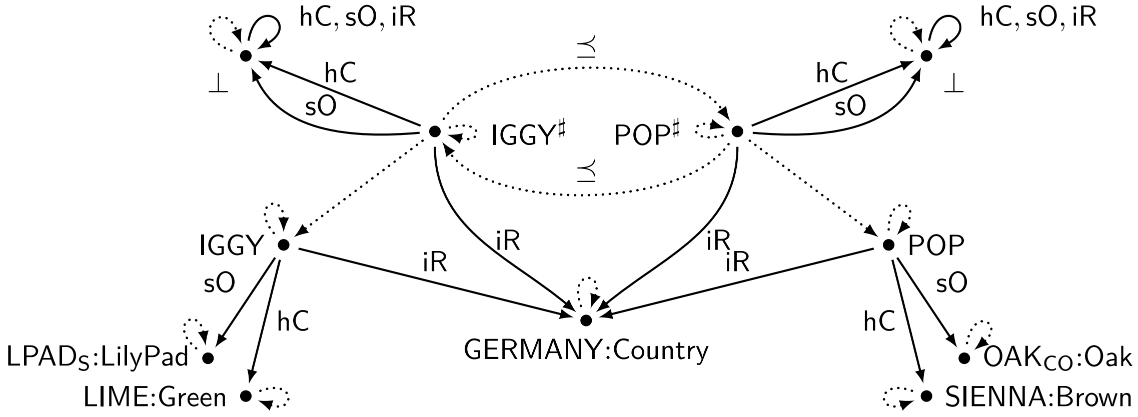

Figure 4.1: A simple data model of frogs with abstraction. Adapted from [195, p. 214, Fig. 1], with kind permission from Springer Science and Business Media.

It is important to observe that both abstract entities $\mathsf{IGGY}^\sharp, \mathsf{POP}^\sharp$ possess a fallible filler w.r.t. the roles sitsOn and hasColour. This filler corresponds to a computational deadlock when selecting one of sitsOn, hasColour for $\mathsf{IGGY}^\sharp$ or $\mathsf{POP}^\sharp$, and can be viewed as the situation where $\mathsf{IGGY}^\sharp$ or $\mathsf{POP}^\sharp$ have any location (colour) w.r.t. the role sitsOn (hasColour).

The entities $\mathsf{IGGY}^\sharp$ and $\mathsf{POP}^\sharp$ refine each other and hence are indistinguishable, *i.e.*, they share exactly the same concept descriptions. If $Th(x)$ denotes the set of concepts which entity x participates in, then $Th(\mathsf{IGGY}^\sharp) = Th(\mathsf{POP}^\sharp)$. For instance, the theory of $\mathsf{IGGY}^\sharp$ includes the concepts $\exists\mathsf{isResident.Country}, \exists\mathsf{hasColour.(Green} \sqcup \mathsf{Brown)}, \exists\mathsf{sitsOn.(LilyPad} \sqcup \mathsf{Oak)} \in Th(\mathsf{IGGY}^\sharp)$, because each refinement of $\mathsf{IGGY}^\sharp$ has GERMANY:Country as a filler for role isResident and either LIME:Green or SIENNA:Brown ($\mathsf{LPAD}_{\mathsf{S}}$:LilyPad, $\mathsf{OAK}_{\mathsf{CO}}$:Oak) as a filler for role hasColour (sitsOn).

Observe, that the disjunction $\exists\mathsf{hasColour.(Green} \sqcup \mathsf{Brown)}$ captures the choice between the two realisations of $\mathsf{IGGY}^\sharp$ as a concrete entity, *viz.* $(\mathsf{LPAD}_{\mathsf{S}}, \mathsf{LIME}, \mathsf{GERMANY})$ and $(\mathsf{OAK}_{\mathsf{CO}}, \mathsf{SIENNA}, \mathsf{GERMANY})$. But, it is not possible to resolve this choice at an abstract level as there is no single uniform choice for the filler of role hasColour. This is reflected by the fact that $\exists\mathsf{hasColour.Green} \notin Th(\mathsf{IGGY}^\sharp)$ and $\exists\mathsf{hasColour.Brown} \notin Th(\mathsf{IGGY}^\sharp)$, which implies that their disjunction $\exists\mathsf{hasColour.Green} \sqcup \exists\mathsf{hasColour.Brown}$

is not in $\mathit{Th}(\mathsf{IGGY}^\sharp)$ as well. We can analogously argue the case for the disjunction $\exists\mathsf{sitsOn}.(\mathsf{Oak} \sqcup \mathsf{LilyPad})$. Such abstractions cannot be expressed in classical DLs, where the axiom $\mathsf{FS4}/\mathsf{IK4}$ is a theorem, *i.e.*, existential restriction $\exists R$ always distributes over disjunction $\sqcup$ in $\mathcal{ALC}$, such that $\exists\mathsf{hasColour}.(\mathsf{Green} \sqcup \mathsf{Brown})$ is equivalent to $\exists\mathsf{hasColour}.\mathsf{Green} \sqcup \exists\mathsf{hasColour}.\mathsf{Brown}$.

Also note, that $\neg\exists\mathsf{hasColour}.\mathsf{Green}$ is not the same as $\forall\mathsf{hasColour}.\neg\mathsf{Green}$. The former says that it is inconsistent to assume that all refinements of $\mathsf{IGGY}^\sharp$ have a $\mathsf{hasColour}$-filler in concept Green. The latter means that no refinement has a $\mathsf{hasColour}$-filler in Green. In $\mathcal{ALC}$, this duality between $\neg\exists R.C$ and $\forall R.\neg C$ holds, but it does not in $c\mathcal{ALC}$.

Further, we can observe that the law of the Excluded Middle (PEM) is not valid. Suppose that $\mathsf{IGGY}^\sharp \models \exists\mathsf{sitsOn}.\mathsf{LilyPad} \sqcup \neg\exists\mathsf{sitsOn}.\mathsf{LilyPad}$ holds. Then, from semantic validity, which says that $\models C \sqcup D$ implies $\models C$ or $\models D$ (see Prop. 4.2.3), it follows that either $\mathsf{IGGY}^\sharp \models \exists\mathsf{sitsOn}.\mathsf{LilyPad}$ or $\mathsf{IGGY}^\sharp \models \neg\exists\mathsf{sitsOn}.\mathsf{LilyPad}$. However, the first is not possible since there is a refinement $\mathsf{IGGY}^\sharp \preceq \mathsf{POP}$ with sitsOn filler $\mathsf{OAK}_{\mathsf{CO}}$, which is not in $\mathsf{LilyPad}$. The second is refuted due to the refinement IGGY of $\mathsf{IGGY}^\sharp$, which has the sitsOn-filler $\mathsf{LPAD}_{\mathsf{S}}{:}\mathsf{LilyPad}$. ■

Another axiom schema refuted by the semantics of $c\mathcal{ALC}$ is $\mathsf{FS5}/\mathsf{IK5} = (\exists R.C \supset \forall R.D) \supset \forall R.(C \supset D)$, as the following example illustrates. Note that this axiom is actually part of Fischer-Servi's system FS/IK.

Example 4.2.3 ([196, p. 8, Ex. 2])**.** Take the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \perp^{\mathcal{I}}, \preceq^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{a_0, a_1, a_2\}$, refinement $\preceq^{\mathcal{I}} = \{(a_0, a_0), (a_1, a_1), (a_2, a_2), (a_0, a_1)\}$, $R^{\mathcal{I}} = \{(a_0, a_2)\}$, $C^{\mathcal{I}} = \{a_2\}$ and $D^{\mathcal{I}} = \emptyset$. In this interpretation it holds that $\mathcal{I}; a_0 \models \exists R.C \supset \forall R.D$, $\mathcal{I}; a_2 \models C$ and $\mathcal{I}; a_2 \not\models D$. The implication $\mathcal{I}; a_0 \models \exists R.C \supset \forall R.D$ is trivially true, since not all $\preceq$-reachable entities of a_0 satisfy $\exists R.C$. This is obvious for the entity a_1, as it does not have an R-successor at all. Regarding the entity a_0, it has the $\preceq$-successor a_1, which does not satisfy $\exists R.C$. Furthermore, it is easy to observe for entity a_0 that $\mathcal{I}; a_0 \not\models \forall R.(C \supset D)$, since $(a_0, a_2) \in R^{\mathcal{I}}$ and $\mathcal{I}; a_2 \not\models C \supset D$. In this sense, entity a_0 in this interpretation represents a witness for the refutation of the axiom schema $\mathsf{FS5}/\mathsf{IK5} = (\exists R.C \supset \forall R.D) \supset \forall R.(C \supset D)$. Hence, $\mathsf{FS5}/\mathsf{IK5}$ is not a theorem of $c\mathcal{ALC}$. The situation is depicted in Fig. 4.2. ■

Finally, let us consider $\mathsf{FS3}/\mathsf{IK3} = \exists R.\bot \supset \bot$ that is not a theorem of $c\mathcal{ALC}$. This axiom is usually assumed to be valid in classical $\mathcal{ALC}$ and Fischer-Servi's system FS/IK.

Example 4.2.4. We can easily give a countermodel falsifying axiom $\mathsf{FS3}/\mathsf{IK3}$. Take the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \perp^{\mathcal{I}}, \preceq^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{a_0, a_1, a_2\}$, $\preceq^{\mathcal{I}} = \{(a_0, a_0), (a_1, a_1), (a_2, a_2), (a_0, a_1)\}$, $R^{\mathcal{I}} = \{(a_1, a_2)\}$ and $\perp^{\mathcal{I}} = \{a_2\}$ (see Fig. 4.2). This interpretation refutes $\exists R.\bot \supset \bot$ at entity a_0. This is because there exists the refinement a_1 of a_0 which has the fallible R-successor a_2, while entity a_0 is infallible.

Let us view this axiom relative to the temporal interpretation of Kojima [161, Chap. 3; 162] where $\Diamond$ is interpreted as the temporal *next* operator. Take $\exists R$ as a contextual (possibly temporal) operator and let $\exists R.C$ denote that 'a computation of type C is taking place in the next R-context'. The concept $\bot$ represents a failing computation or a runtime error. Under this view, $\exists R.\bot$ says that there occurs a runtime error in the next R-context while $\bot$ expresses the immediate occurrence of a failing computation. Obviously, the context-dependent occurrence differs from an immediate failure. If we omit fallible entities then the axiom FS3/IK3 holds. We will discuss this matter in detail in Sec. 5.3.1. ■

Figure 4.2 summarises the discussion above by depicting the three examples of countermodels for the axiom schemata FS3/IK3 $= \exists R.\bot \supset \bot$, FS4/IK4 $= \exists R.(C \sqcup D) \supset (\exists R.C \sqcup \exists R.D)$ and FS5/IK5 $= (\exists R.C \supset \forall R.D) \supset \forall R.(C \supset D)$, where dotted arrows represent refinement $\preceq^{\mathcal{I}}$ and solid arrows stand for $R^{\mathcal{I}}$.

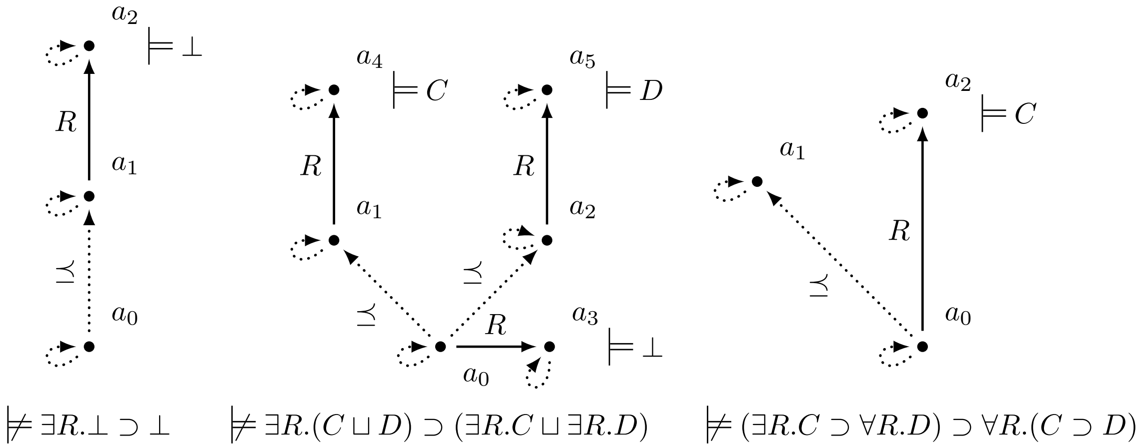

Figure 4.2: Countermodels for FS3/IK3, FS4/IK4 and FS5/IK5.

We close this section with the definition of a semantic consequence relation in the spirit of [33, pp. 31 f.], on which we will rely when defining the TBox formalism as well as in the definition of the reasoning tasks of $c\mathcal{ALC}$ w.r.t. TBoxes.

Definition 4.2.4 (Local/global semantic consequence). Let Θ and Γ be sets of $c\mathcal{ALC}$ concepts and C a single $c\mathcal{ALC}$ concept. We write $\Theta; \Gamma \models C$ if for all interpretations $\mathcal{I}$, which are models of all axioms in Θ, it is the case that every entity $x \in \Delta^{\mathcal{I}}$ which satisfies all concepts in Γ also satisfies concept C. Formally, $\forall \mathcal{I}. \forall x \in \Delta^{\mathcal{I}}. (\mathcal{I} \models \Theta \ \& \ \mathcal{I}; x \models \Gamma) \Rightarrow \mathcal{I}; x \models C$. We say that C is a *semantic consequence* of $\Theta; \Gamma$, in particular, C is a semantic consequence of *global assumptions* Θ and *local assumptions* Γ. ∇

4.2.3 Terminological Knowledge

Like classical description logics, $c\mathcal{ALC}$ provides a TBox formalism to express global (meta-level) terminological knowledge. The TBox stores the vocabulary of an application domain and is specified in terms of a set of terminological axioms in the form of concept inclusion statements. For instance, in classical DLs these are usually inclusions $C \sqsubseteq D$ or equalities $C \equiv D$, where the latter is just an abbreviation for two converse inclusions. The TBox statement that 'D subsumes C' ($C \sqsubseteq D$) can be expressed as concept equality $C \supset D \equiv \top$ as global assumption. In classical $\mathcal{ALC}$ one could translate subsumption via negation to the equation $\neg C \sqcup D \equiv \top$. In constructive logic this is no longer possible, since these operators are independent from each other.

In [191; 195] we used as notation $C \sqsubseteq D$ for both, concept descriptions as well as TBox axioms. It has been proposed to us in a personal discussion with Sattler [242] to distinguish the meta- and object-level in terms of the symbol, *i.e.*, to use $\sqsubseteq$ as a TBox operator only and to introduce a different symbol like $\supset$ as an object-level concept-forming operator. The main argument is that in classical DLs the operator $\sqsubseteq$ is used as a universally closed (meta-level) statement in the TBox only.

We agree with Sattler on using a concept forming operator different from $\sqsubseteq$ for implication, *viz.* $\supset$. However, we will not need a second operator for inclusions, since not the symbol makes the difference between meta- and object level, but rather, whether a concept like $C \supset D$ lives in the set of global or local assumptions. We prefer in this work to use the symbol $\supset$ as a concept constructor and as the symbol for expressing concept inclusions in global (terminological) axioms. This is for the following two reasons:

(i) We believe that the use of the symbol $\sqsubseteq$ restricted to the meta-level in classical DLs is by accident and due to the fact that using it as an implication at the object level would generate redundant syntax. In fact, in classical DLs it can be internalised into the object logic, expressible in terms of $\neg$ and $\sqcup$. This is not possible in constructive logic where implication $\supset$ is an independent operator.

(ii) We prefer to follow the notation of modal logic [33, pp. 31 ff.] by defining the notion of a semantic consequence relation clearly disambiguating global and local assumptions, *viz.* $\Theta; \Gamma \models C$ (see Def. 4.2.4). Here, the set Θ is taken for concepts as global axioms (TBox), while the set Γ is for concepts local w.r.t. a fixed entity. Analogously to Def. 2.1.14, this can be generalised to $\Delta; \Theta; \Gamma \models C$ relative to a set of frame axioms Δ. We will rely on this distinction of global and local context in the later development of the proof theory for $c\mathcal{ALC}$ in Chapter 5.

Inference Tasks w.r.t. a TBox

We are now ready to rephrase the main DL inference problems for $c\mathcal{ALC}$ concepts.

Definition 4.2.5 (Satisfiability, subsumption, disjointness, equivalence w.r.t. a TBox)**.** Let Θ be a TBox and C, D be concept descriptions.

- A concept description C is *satisfiable* with respect to a TBox Θ iff there exists an interpretation $\mathcal{I}$ with $\mathcal{I} \models \Theta$ and a non-fallible entity $x \in \Delta_c^{\mathcal{I}}$ such that $x \in C^{\mathcal{I}}$.
- A concept description D *subsumes* a concept description C with respect to a TBox Θ iff in all models $\mathcal{I}$ of Θ it holds that all of the entities in $C^{\mathcal{I}}$ are contained in $D^{\mathcal{I}}$.
- Two concept descriptions C and D are *disjoint* with respect to a TBox Θ if $C^{\mathcal{I}}$ and $D^{\mathcal{I}}$ do not share any non-fallible entities in all models $\mathcal{I}$ of Θ.
- Two concept descriptions C and D are *equivalent* with respect to a TBox Θ if $C^{\mathcal{I}}$ and $D^{\mathcal{I}}$ share the same non-fallible entities in all models $\mathcal{I}$ of Θ. ∇

In typical reasoning tasks the interpretation $\mathcal{I}$ and the entity x in a verification goal such as $\mathcal{I}; x \models C$ are not given directly, but are themselves axiomatised by sets of formulæ, specifically a *TBox*, *i.e.*, a set Θ of terminological axioms for $\mathcal{I}$ and a set Γ of determined formulæ for some entity $x \in \Delta^{\mathcal{I}}$.

Here is how standard concept reasoning w.r.t. a TBox is covered according to Def. 4.2.4:

Lemma 4.2.3 (Reasoning w.r.t. a TBox [195, p. 213])**.** *Let C, D be concept descriptions and Θ a TBox.*

1. $\Theta; \{C\} \not\models \bot$ *iff concept C is* satisfiable *with respect to* Θ.
2. $\Theta; \{C\} \models D$ *iff concept C is* subsumed *by concept D w.r.t.* Θ. *The same can be expressed by* $\Theta; \emptyset \models C \supset D$ *(by reflexivity of $\preceq$).*
3. $\Theta; \{C, D\} \models \bot$ *iff the concepts C and D are* disjoint *with respect to the TBox* Θ.
4. $\Theta; \{C\} \models D$ *and* $\Theta; \{D\} \models C$ *iff concepts C and D are* equivalent *w.r.t.* Θ. *The same can be expressed by* $\Theta; \emptyset \models (C \supset D) \sqcap (D \supset C)$. ∇

Proof. Let Θ be an arbitrary TBox and C, D be concept descriptions.

1. ($\Rightarrow$) Assume $\Theta; \{C\} \not\models \bot$, *i.e.*, by Def. 4.2.4 there exists an interpretation $\mathcal{I}$ that is a model of Θ and an entity $x \in \Delta^{\mathcal{I}}$ of $\mathcal{I}$ such that $\mathcal{I}; x \models C$ but $\mathcal{I}; x \not\models \bot$.

We have to show that C is satisfiable w.r.t. Θ. This follows by the assumption, namely $\mathcal{I} \models \Theta$, $\mathcal{I}; x \models C$ and x is a non-fallible entity, because $x \notin \bot^{\mathcal{I}}$. Hence, the concept C is satisfiable w.r.t. TBox Θ.

($\Leftarrow$) Assume that C is satisfiable with respect to Θ. By Def. 4.2.5 there exists an interpretation $\mathcal{I}$ with $\mathcal{I} \models \Theta$ and a non-fallible entity $x \in \Delta_c^{\mathcal{I}}$ such that $x \in C^{\mathcal{I}}$. We have to show that $\Theta; \{C\} \not\models \bot$. This follows immediately from the assumption.

2. ($\Rightarrow$) Suppose $\Theta; \{C\} \models D$. By Def. 4.2.4 this means formally that $\forall \mathcal{I}. \forall x \in \Delta^{\mathcal{I}}. (\mathcal{I} \models \Theta \ \& \ \mathcal{I}; x \models C) \Rightarrow (\mathcal{I}; x \models D)$, *i.e.*, whenever $\mathcal{I}; x \models C$ then $\mathcal{I}; x \models D$, which is nothing other than saying that C is subsumed by D.

   ($\Leftarrow$) Assume C is subsumed by D w.r.t. Θ. This implies for all models $\mathcal{I}$ of Θ that all entities x in $C^{\mathcal{I}}$ are also contained in $D^{\mathcal{I}}$. We have to show that $\Theta; \{C\} \models D$. Assume $x \in \Delta^{\mathcal{I}}$ such that $\mathcal{I}; x \models C$, *i.e.*, $x \in C^{\mathcal{I}}$. It follows by the assumption that $x \in D^{\mathcal{I}}$. Hence, $\Theta; \{C\} \models D$.

3. ($\Rightarrow$) Assume $\Theta; \{C, D\} \models \bot$. Formally this means that $\forall \mathcal{I}. \forall x \in \Delta^{\mathcal{I}}. (\mathcal{I} \models \Theta \ \& \ \mathcal{I}; x \models C \ \& \ \mathcal{I}; x \models D) \Rightarrow (\mathcal{I}; x \models \bot)$. Let $x \in \Delta^{\mathcal{I}}$ be an entity and $\mathcal{I}$ an interpretation such that $\mathcal{I} \models \Theta$, $x \in C^{\mathcal{I}}$ and $x \in D^{\mathcal{I}}$. We have to show that $x \in \bot^{\mathcal{I}}$. This follows by assumption. Hence, C and D are disjoint w.r.t. Θ.

   ($\Leftarrow$) Assume C and D are disjoint w.r.t. Θ. Suppose for the sake of contradiction, that $\Theta; \{C, D\} \not\models \bot$. This means that there exists a model $\mathcal{I}$ of Θ and an entity $x \in \Delta^{\mathcal{I}}$ such that $x \in C^{\mathcal{I}}$, $x \in D^{\mathcal{I}}$ and $x \notin \bot^{\mathcal{I}}$. But this contradicts the assumption, namely that $C^{\mathcal{I}}$ and $D^{\mathcal{I}}$ do not share any non-fallible entity. Hence $\Theta; \{C, D\} \models \bot$.

4. ($\Rightarrow$) Suppose $\Theta; \{C\} \models D$ and $\Theta; \{D\} \models C$. By part 2 above this means that D subsumes C w.r.t. Θ and vice versa. Therefore, we can conclude that C and D are equivalent.

   ($\Leftarrow$) We assume that C and D are equivalent w.r.t. Θ. Then, for all models $\mathcal{I}$ of Θ it holds that $C^{\mathcal{I}}$ and $D^{\mathcal{I}}$ share the same entities, *i.e.*, $\forall x \in \Delta^{\mathcal{I}}.(x \in C^{\mathcal{I}} \Rightarrow x \in D^{\mathcal{I}}) \ \& \ (x \in D^{\mathcal{I}} \Rightarrow x \in C^{\mathcal{I}})$. Hence, $\Theta; \{C\} \models D$ and $\Theta; \{D\} \models C$ by Def. 4.2.4. □

As demonstrated by Lem. 4.2.3, it is possible to reduce the typical DL reasoning tasks to the problem of concept subsumption $\Theta; \{C\} \models D$, which expresses that concept C is *subsumed* by concept D w.r.t. the terminology Θ, just as in classical $\mathcal{ALC}$. Under the constructive semantics, (un-)satisfiability appears as a special case of subsumption, *viz.* $\Theta; \{C\} \models \bot$. Though, contrary to classical $\mathcal{ALC}$, one cannot reduce the inference tasks to the problem of (un-)satisfiability, *i.e.*, the special form $\Theta; \{C\} \not\models \bot$. Instead, in $c\mathcal{ALC}$ we need to implement the generalised satisfiability check $\Theta; \{C\} \not\models D$ for *arbitrary* D. For instance, we cannot reduce subsumption to non-satisfiability, because $\Theta; \{C, \neg D\} \models \bot$ is not the same as $\Theta; \{C\} \models D$. For convenience, we will write in the following $\Theta; C \models D$ omitting the parentheses.

4.2.4 Representation of Dynamic and Incomplete Knowledge

In the following we will discuss two scenarios where $c\mathcal{ALC}$ can be used to represent dynamic and partial knowledge and highlight the differences between the constructive and classical semantics. Firstly, we will discuss the open world assumption from a constructive perspective and then give an example from the domain of financial auditing. The second part will illustrate an application from the domain of data streams, *i.e.*, we will consider abstract entities as streams of data and suggest that $c\mathcal{ALC}$ concepts can act as a typing specification of the static semantics of data streams.

Classic DLs allow to some degree the representation of incomplete information in ABoxes by interpreting them according to the so-called *open world assumption* (OWA) [16, Chap. 2]. In short, the OWA says that the absence of information (in an ABox) only indicates an incomplete specification or lack of knowledge w.r.t. an entity, and in this sense the OWA states that 'anything, which is absent is undefined'. This is in contrast to the *closed world assumption* (CWA), which is usually used as the interpretation of databases, Prolog and Datalog.[9] Under the CWA the absence of information (in the data) is interpreted as negative information and this means that 'anything, which is absent is false', which corresponds to the principle of negation as failure. As a consequence, an ABox w.r.t. the OWA represents many different interpretations [16, p. 74] (its models), which arise from case analysis of underspecified entities. In classical DLs like $\mathcal{ALC}$, these models assume each concept to be *static* and that the interpretation of underspecified entities has to *decide at the outset* if a concept includes an entity or not.

However, under the constructive semantics, it may be that either option is inconsistent, for instance, if the entity or the concept is not fully defined until a later stage, where lower levels of detail become available. Model-theoretically, entities in $c\mathcal{ALC}$ are not static (atomic) individuals, but they have an internal structure and their interpretation is based on *stages of information* [204; 268] so that truth is monotonically increasing.

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[9] For a comprehensive comparison between DL and Datalog based on open- and closed-world semantics see [225]. The correspondence between DLs and databases is covered by [16, Chap. 16].

From a proof-theoretical perspective, this corresponds to the idea of positive evidence and realisability [262], *i.e.*, there is no causality between the absence of entities and the presence of others, instead constructive logic insists on the existence of *computational witnesses*.

It has been discussed first in [191; 195] that the classical (*static*) OWA is not adequate in a constructive environment, since it does not support reasoning to be both correct under abstraction and sustainable under refinement. Instead, we need a constructive notion of *undefinedness* that permits concepts to evolve, *i.e.*, we require an *evolving* open world assumption (EOWA) [195]. In particular, we need to be able to express partiality [54] and incomplete knowledge beyond the standard OWA. This can be achieved by internalising the decision whether a partially defined entity participates in the interpretation of a concept or not, while sustaining the heredity condition and being closed under role-fillers.

Example 4.2.5 (Adapted from [195, p. 215, Ex. 3], with kind permission from Springer Science and Business Media.)**.** One prime example of a class of application domains that require the ability to express partial and incomplete knowledge beyond the standard open world assumption (OWA) is auditing [195]. The example is a variant of the Oedipus example from [16, p.75]. Let us consider a customer topology of companies $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}$ represented in terms of the role $\mathsf{hasCustomer}$ such that company $\mathtt{a}$ has both $\mathtt{b}$ and $\mathtt{c}$ as its customers, $\mathtt{b}$ has customer $\mathtt{c}$ which itself has a customer relation to $\mathtt{d}$. In addition, we suppose that $\mathtt{b}$ is insolvent (concept $\mathsf{Insolvent}$) and $\mathtt{d}$ is solvent ($\neg\mathsf{Insolvent}$). Nothing is known about the possible solvency status of company $\mathtt{c}$. Fig. 4.3 formalises this situation in ABox syntax.

$$(\mathtt{a},\mathtt{b}) : \mathsf{hasCustomer},\ \ (\mathtt{a},\mathtt{c}) : \mathsf{hasCustomer}, (\mathtt{b},\mathtt{c}) : \mathsf{hasCustomer}, (\mathtt{c},\mathtt{d}) : \mathsf{hasCustomer},$$
$$\mathtt{b} : \mathsf{Insolvent},\ \ \mathtt{d} : \neg\mathsf{Insolvent}$$

Figure 4.3: The Customer Topology.

We say that a company is *c̲redit-w̲orthy* if it has an insolvent customer who in turn has at least one solvent company among its customers. This is formalised by the concept

$$\mathsf{CW} \equiv \exists\mathsf{hasCustomer}.(\mathsf{Insolvent} \sqcap \exists\mathsf{hasCustomer}.\neg\mathsf{Insolvent}).$$

Suppose now that we want to infer whether company $\mathtt{a}$ is credit-worthy. This can be expressed in terms of checking, whether $\mathtt{a}$ is an instance of concept CW.

The approach of classical DLs [16, p.75] is to divide the models of an ABox into two cases, one in which company $\mathtt{c}$ is insolvent and the other where $\mathtt{c}$ is solvent. Observe that under the classical OWA it holds that $\mathsf{c} : \mathsf{Insolvent} \sqcup \neg\mathsf{Insolvent}$ regardless of c. Furthermore, this implies that company a is an instance of the concept CW. In the first case $\mathsf{c} : \mathsf{Insolvent}$ this customer of a is c, in the other case when $\mathsf{c} : \neg\mathsf{Insolvent}$ it is the company b. In classical $\mathcal{ALC}$ the $\mathsf{hasCustomer}$-filler would be unknown but fixed due to case analysis under the classical (*static*) OWA. Though, under the constructive semantics this case analysis on company c is invalid if the model arises by abstraction from a concrete taxonomy in which insolvency is a *context-dependent* defect.

The corresponding model $\mathcal{I}_{co}$ of this situation is depicted in Fig. 4.4. The relation $\mathsf{hasCustomer}$ is represented in terms of solid edges, each dotted line denotes refinement $\preceq$, and we assume that each entity refines itself. Regarding company c, there is a refinement c_1 that represents the situation when c may be insolvent under some context-dependent legal understanding of the concept $\mathsf{Insolvent}$. On the other hand, company c may be solvent w.r.t. some other legal consideration, which is represented by its refinement c_2. Note that this internalises the case analysis of the classical OWA into the model of Fig. 4.4 such that each refinement of c has company d as a customer as well.

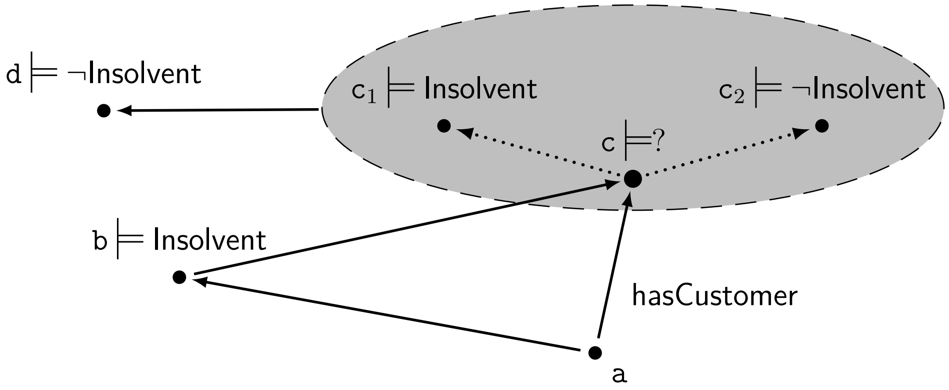

Figure 4.4: Evolving OWA model $\mathcal{I}_{co}$. Adapted from [195, p. 215, Fig. 2; 190, p. 10], with kind permission from Springer Science and Business Media.

In this model it is not possible to decide about the solvency status of company c, *i.e.*, insolvency of c is not just unknown but *undecidable*. The reason is that we cannot obtain the required $\mathsf{hasCustomer}$-filler for company a without contradicting one of the two refinements c_1, c_2 to which c may evolve. For instance, when selecting b as the $\mathsf{hasCustomer}$-filler of a then there is only c as a customer of b, but $\mathcal{I}_{co}; \mathsf{c} \not\models \neg\mathsf{Insolvent}$, because c may evolve into c_1 which is insolvent, *i.e.*, $\mathcal{I}_{co}; \mathsf{c}_1 \models \mathsf{Insolvent}$. Otherwise, when choosing company c as $\mathsf{hasCustomer}$-filler for a then $\mathcal{I}_{co}; \mathsf{c} \not\models \mathsf{Insolvent}$, since there is a refinement c_2 of c which is solvent, *i.e.*, $\mathcal{I}_{co}; \mathsf{c}_2 \models \neg\mathsf{Insolvent}$.

The EOWA is inline with the requirement that the law of the Excluded Middle (PEM) is not valid in $c\mathcal{ALC}$, in particular, observe that $\mathcal{I}_{co}; \mathsf{c} \not\models \mathsf{Insolvent} \sqcup \neg\mathsf{Insolvent}$

in Fig. 4.4. The case $\mathsf{c} : \neg\mathsf{Insolvent}$ conflicts with its refinement c_1, and the other case $\mathsf{c} : \mathsf{Insolvent}$ fails due to refinement c_2. Hence, c is neither an instance of $\mathsf{Insolvent}$ nor of $\neg\mathsf{Insolvent}$.

Under the classical *static* OWA the case analysis is performed outside the model so that fillers may depend non-uniformly on a specific decision. But, this requires a complete a-priori fixed knowledge of all possible problem parameters on which the concept of solvency may depend. In the constructive model $\mathcal{I}_{co}$ of Fig. 4.4 this decision is not fixable once and for all. Instead, under the EOWA this case analysis is internalised into the model with the requirement that the filler of a role must be robust under case analysis (evolution of concepts). Therefore, under the EOWA $\mathcal{I}_{co}; \mathsf{a} \models \mathsf{CW}$ does not hold because there does not exist a witness for the $\mathsf{hasCustomer}$-filler.

However, if we assume the PEM as a non-trivial TBox axiom $\mathsf{Insolvent} \sqcup \neg\mathsf{Insolvent}$ then $\mathcal{I}_{co}; \mathsf{a} \models \mathsf{CW}$ would be true, since the assumption rules out the *switch* between solvency and insolvency of c. ■

Example 4.2.6 (Adapted from [195, p. 215], with kind permission from Springer Science and Business Media.)**.** An important interpretation of the intuitionistic preorder $\preceq$ is the stream interpretation. Under this view, a stream of records can represent business data in terms of a linearised database table $t \in \Delta^{\mathcal{I}}$ or a time-series of financial market transactions. The key idea is to interpret streams as abstract entities and to use DL concepts to express the type of stream elements.

To illustrate this idea, let s be a *stream* of records, *i.e.*, $s = s_1 \cdot s_2 \cdot s_3 \cdot \ldots$ is a finite or infinite sequence of elements $\{s_i \mid i \geq 1\}$, and let $\mathbb{D} = \mathbb{N} \uplus \mathbb{B} \uplus (\mathbb{N} \times \mathbb{B})$ be the discrete universe of Booleans, naturals and their pairings. We will consider the domain $\Delta^{\mathcal{I}} = \mathbb{D}^{\omega} = \mathbb{D}^{*} \cup \mathbb{D}^{\infty}$ of all *streams* over $\mathbb{D}$. The refinement $\preceq^{\mathcal{I}}$ is the (inverse) suffix ordering, which is the least relation closed under the rule

$$\frac{v \in \mathbb{D}}{v \cdot s \preceq^{\mathcal{I}} s}$$

where $v \cdot s$ is the stream $s \in \mathbb{D}^{\omega}$ prefixed by value $v \in \mathbb{D}$. E.g.,

$$1 \cdot (2, \mathtt{T}) \cdot \mathtt{T} \cdot \mathtt{F} \ \preceq^{\mathcal{I}} \ (2, \mathtt{T}) \cdot \mathtt{T} \cdot \mathtt{F} \ \preceq^{\mathcal{I}} \ \mathtt{T} \cdot \mathtt{F} \ \preceq^{\mathcal{I}} \ \mathtt{F} \ \preceq^{\mathcal{I}} \ \epsilon,$$

is a stream of naturals, Booleans and their pairings, where ϵ denotes the empty stream. Under this interpretation, concepts $C^{\mathcal{I}}$, which are required to be closed under $\preceq^{\mathcal{I}}$, express *future projected behaviour* of streams.

Obviously, the empty stream ϵ has no future behaviour, but represents a computational deadlock and is interpreted as a fallible entity, *i.e.*, $\bot^{\mathcal{I}} = \{\epsilon\}$. Viewing fallible entities as stream computations means that their computational output is ϵ, *i.e.*, they do not produce any value. Type-theoretically, this means that ϵ is universally polymorphic in the sense that it is naturally contained in any type.

The access to the values of a stream is modelled in terms of a distinguished (functional) role val that relates a stream with its first data element. Each such element is considered as an infinite constant stream, if it exists, otherwise it is the empty stream, *i.e.*, $\mathsf{val}(v \cdot s, v^\infty)$ and $\mathsf{val}(\epsilon, \epsilon)$. For instance, $\mathsf{val}((2, \mathtt{T}) \cdot \mathtt{T} \cdot \mathtt{F}, (2, \mathtt{T})^\infty)$ and $\mathsf{val}(\mathtt{T} \cdot \mathtt{F}, \mathtt{T}^\infty)$.

Regarding types, let Nat and Bool represent the usual types from programming languages, which will be considered here as atomic $c\mathcal{ALC}$ concepts, *i.e.* $\mathsf{Nat}^\mathcal{I} =_{df} \mathbb{N}^\omega = \mathbb{N}^* \cup \mathbb{N}^\infty$ and $\mathsf{Bool}^\mathcal{I} =_{df} \mathbb{B}^\omega = \mathbb{B}^* \cup \mathbb{B}^\infty$. Their interpretation specifies streams of naturals and streams of Booleans respectively. Similarly, let $(\mathsf{Nat} \times \mathsf{Bool})^\mathcal{I} =_{df} (\mathbb{N} \times \mathbb{B})^\omega$ represent simple database tables as streams of data pairs. It is easy to observe that the interpretations $\mathsf{Nat}^\mathcal{I}$, $\mathsf{Bool}^\mathcal{I}$, $(\mathsf{Nat} \times \mathsf{Bool})^\mathcal{I}$ are subsets of $\Delta^\mathcal{I}$, are closed under $\preceq^\mathcal{I}$, and each contains $\perp^\mathcal{I}$.

It is not difficult to observe that in this interpretation we have the *type equivalences* $\mathsf{Nat} \equiv \forall\mathsf{val}.\mathsf{Nat} \equiv \exists\mathsf{val}.\mathsf{Nat}$ and $\mathsf{Bool} \equiv \forall\mathsf{val}.\mathsf{Bool} \equiv \exists\mathsf{val}.\mathsf{Bool}$. The fact that existential and universal restriction collapse under functional roles is not surprising, except perhaps for one thing: The existential typing $s \in (\exists\mathsf{val}.\mathsf{Nat})^\mathcal{I}$ does not imply the existence of a value $n \in \mathbb{N}$ such that $\mathsf{val}(s, n^\infty)$ as in classical logic, since the stream s could be empty due to a non-terminating or deadlocking computation. Because these properties are undecidable for useful programming languages we cannot expect the type system to express emptiness. Otherwise it would become undecidable, too.

The indistinguishability of $\forall R.C$ and $\exists R.C$ on fallible entities is but one of the constructive, *i.e.*, non-classical features of the $c\mathcal{ALC}$ type system. Another one is the rejection of PEM. For instance, one can observe that under the stream interpretation the concept $\mathsf{Bool} \sqcup \neg\mathsf{Bool}$ is not equivalent to $\top$. Consider the stream

$$s = 0 \cdot \mathtt{T} \cdot \mathtt{T} \cdot \mathtt{T} \cdots ,$$

which begins with value 0 that is followed by an infinite constant sequence of elements T of type Bool. One can verify that $s \notin \mathsf{Bool}$ and $s \notin \neg\mathsf{Bool}$. The former assertion is obvious and the latter holds, since if $s \in \neg\mathsf{Bool}$ then s must consist of non-Boolean values arbitrarily late in the stream, but this is not the case. In classic DLs the axiom PEM would imply that $\mathsf{Bool} \sqcup \neg\mathsf{Bool} \equiv \top$, which is against our computational interpretation.

Moreover, our stream interpretation rejects the axiom FS4/IK4 (disjunctive distribution) given by $\exists\mathsf{val}.(C \sqcup D) \supset \exists\mathsf{val}.C \sqcup \exists\mathsf{val}.D$. Let us illustrate this in terms of a common operation in the semantic analysis of mass data in financial auditing, *viz.* the linearisation of database tables. Let $t = (n_0, b_0) \cdot (n_1, b_1) \cdot (n_2, b_2) \cdots$ be a table of records, which can be viewed as a stream of type $\mathsf{Nat} \times \mathsf{Bool}$. The linearisation of t is given by the stream

$$t^\flat = n_0 \cdot b_0 \cdot n_1 \cdot b_1 \cdot n_2 \cdot b_2 \cdots .$$

Let us explore the type of $t^\flat$: Its type is not $\mathsf{Nat} \sqcup \mathsf{Bool}$, nor $\exists \mathsf{val}.\mathsf{Nat} \sqcup \exists \mathsf{val}.\mathsf{Bool}$, since this would globally require that all element of $t^\flat$ are either of type Nat or of type Bool. The type of $t^\flat$ is instead given by the *union type* $\mathsf{Nat} \cup \mathsf{Bool}$, which can be expressed by the concept $\exists \mathsf{val}.(\mathsf{Nat} \sqcup \mathsf{Bool})$. This concept expresses that the first element of each suffix sequence is of type Nat or Bool. Observe, that the use of existential restriction $\exists \mathsf{val}$ performs the decomposition of the stream $t^\flat$ such that the type specification $\mathsf{Nat} \sqcup \mathsf{Bool}$ is applied element-wise rather than globally. In this way, the difference between concepts $\exists \mathsf{val}.(\mathsf{Nat} \sqcup \mathsf{Bool})$ and $\exists \mathsf{val}.\mathsf{Nat} \sqcup \exists \mathsf{val}.\mathsf{Bool}$, or between $\mathsf{Nat} \cup \mathsf{Bool}$ and $\mathsf{Nat} \sqcup \mathsf{Bool}$ for that matter, permits us to distinguish between local (dynamic) and global (static) choice. In classical DLs this important distinction is collapsed.

Consider the stream $s = 0 \cdot \mathtt{T} \cdot 0 \cdot \mathtt{T} \cdot 0 \cdot \mathtt{T} \cdots$ that oscillates between elements of type Nat and Bool. This stream satisfies the concept $Osc =_{df} \neg \mathsf{Nat} \sqcap \neg \mathsf{Bool} \sqcap (\mathsf{Nat} \cup \mathsf{Bool})$ which expresses that *'s is (globally) never in* Nat *nor in* Bool *but always contained in their union* $\mathsf{Nat} \cup \mathsf{Bool}$*'*. Intuitively, the concept Osc specifies infinite streams with an oscillation between the types Nat and Bool. This is only expressible via constructive semantics, which allow to represent objects as non-atomic or non-static entities.

The flattening function $t \mapsto t^\flat$ considered above implements operationally a way of multiplexing data streams. Its type is given by the function type $\mathsf{Nat} \times \mathsf{Bool} \to \exists \mathsf{val}.(\mathsf{Nat} \sqcup \mathsf{Bool})$. Conversely, the inverse operation of de-multiplexing streams is by taking the linearised stream $t^\flat$ and transforming it back to t. The type of de-multiplexing can be specified by $\exists \mathsf{val}.(\mathsf{Nat} \sqcup \mathsf{Bool}) \to \mathsf{Nat} \times \mathsf{Bool}$. Under the Curry-Howard isomorphism [263; 268] the Cartesian product $C \times D$ corresponds to the constructive interpretation of conjunction $C \sqcap D$, while function spaces $C \to D$ express that of implication $C \supset D$. From this perspective, we can view the operations of multiplexing and de-multiplexing of data streams as different constructive realisations of the following implications:

$$(\mathsf{Nat} \sqcap \mathsf{Bool}) \supset \exists \mathsf{val}.(\mathsf{Nat} \sqcup \mathsf{Bool}), \qquad \exists \mathsf{val}.(\mathsf{Nat} \sqcup \mathsf{Bool}) \supset (\mathsf{Nat} \sqcap \mathsf{Bool}).$$

The uniform multiplexing shown above is nothing but a very particular translation program $(\cdot)^\flat$ of type $\mathsf{Nat} \times \mathsf{Bool} \supset \exists \mathsf{val}.(\mathsf{Nat} \sqcup \mathsf{Bool})$ which plays the role of a cALC TBox axiom. Moreover, observe that the fallibility of the empty stream ϵ naturally corresponds to the polymorphism of the empty list, *i.e.*, it can be used at any type. ■

4.2.5 Disjunction Property

One of the hallmarks of constructive theories is the *disjunction property* (DP), *i.e.*, the proof of a disjunction $C \sqcup D$ requires an explicit evidence for one of the disjuncts in the form of a proof of either C or D. This is contrary to classical $\mathcal{ALC}$, where it is not necessary to specify which disjunct holds. We show that $c\mathcal{ALC}$ enjoys the DP (see Prop. 4.2.3) in the sense that whenever $\models C \sqcup D$ then either $\models C$ or $\models D$. This is not a surprise, since DP is a general feature of intuitionistic logic [265]. It is well established that DP does not hold under arbitrary assumptions. Instead, one can show that DP holds under suitable hypotheses, which are restricted to the class of Harrop formulæ.

Definition 4.2.6. We define the class H of Harrop concepts by the following grammar

$$H ::= \quad A \mid \top \mid \bot \mid H \sqcap H \mid C \supset H \mid \forall R.H$$

where $A \in N_C$, $R \in N_R$ and C is an arbitrary concept of $c\mathcal{ALC}$. ∇

Then, one shows that $\emptyset;\Gamma \models C \sqcup D$ implies $\emptyset;\Gamma \models C$ or $\emptyset;\Gamma \models D$, where Γ is a set of Harrop concepts. The proof is by assuming $\emptyset;\Gamma \models C \sqcup D$ and to the contrary $\emptyset;\Gamma \not\models C$ and $\emptyset;\Gamma \not\models D$. Then, Def. 4.2.3 implies the existence of two models $\mathcal{I}_1, \mathcal{I}_2$ and entities $a_1 \in \Delta^{\mathcal{I}_1}$, $a_2 \in \Delta^{\mathcal{I}_2}$ such that $\mathcal{I}_1, \mathcal{I}_2$ satisfy all Harrop concepts in Γ but $\mathcal{I}_1; a_1 \not\models C$ and $\mathcal{I}_2; a_2 \not\models D$. First, we define the join of two models $\mathcal{I}_1, \mathcal{I}_2$, which satisfy a Harrop concept, and then show that this join satisfies the same Harrop concept as well. Secondly, we construct a model from $\mathcal{I}_1$, $\mathcal{I}_2$, which is contradictory to the assumption. In the following we will omit the superscript $\mathcal{I}$ from $\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}$ and use an index $i \in \{1,2\}$ to distinguish the components of $\mathcal{I}_1, \mathcal{I}_2$ by writing $\Delta_i, \preceq_i, \bot_i$ instead. The join model $\mathcal{I}^{\bowtie}$ of two models $\mathcal{I}_1, \mathcal{I}_2$ w.r.t. entities a_1, a_2, is defined by:

Definition 4.2.7. Let $\mathcal{I}_1 = (\Delta_1, \preceq_1, \bot_1, \cdot_1)$ and $\mathcal{I}_2 = (\Delta_2, \preceq_2, \bot_2, \cdot_2)$ with $a_1 \in \Delta_1$ and $a_2 \in \Delta_2$ be constructive models according to Def. 4.2.2 and assume without loss of generality that $\Delta_1 \cap \Delta_2 = \emptyset$. The join of $\mathcal{I}_1, \mathcal{I}_2$ w.r.t. entities a_1, a_2 in *join world a* is defined by the structure $\mathcal{I}^{\bowtie} = (\Delta, \preceq, \bot, \cdot)$, where $A \in N_C$, $R \in N_R$ and

$$\Delta =_{df} \Delta_1 \cup \Delta_2 \cup \{a\}, \text{ where } a \text{ is a fresh entity not in } \Delta_1 \cup \Delta_2;$$

$$\preceq =_{df} \preceq_1 \cup \preceq_2 \cup \{(a,x) \mid \exists i \in \{1,2\}.\ a_i \preceq_i x \text{ or } x = a\};$$

$$\bot =_{df} \{x \mid \exists i \in \{1,2\}.\ x \in \bot_i \text{ or } (x = a \text{ and } a_1 \in \bot_1 \text{ and } a_2 \in \bot_2)\};$$

and $\cdot$ given by

$$A^{\mathcal{I}^{\bowtie}} =_{df} A^{\mathcal{I}_1} \cup A^{\mathcal{I}_2} \cup \{a \mid \forall i \in \{1,2\}.a_i \in A^{\mathcal{I}_i}\};$$

$$R^{\mathcal{I}^{\bowtie}} =_{df} \{(x,y) \mid \exists i \in \{1,2\}.\ x\, R^{\mathcal{I}_i}\, y \text{ or } (a_1 \in \bot_1 \text{ and } a_2 \in \bot_2 \text{ and } x = y = a)\}. \quad \nabla$$

Before proceeding with the next step of proving that $\mathcal{I}^{\bowtie}$ is a constructive interpretation in the sense of Def. 4.2.2, we need to introduce an auxiliary lemma. We can show that the join of two models $\mathcal{I}_1, \mathcal{I}_2$ is conservative, *i.e.*, the join model $\mathcal{I}^{\bowtie}$ satisfies that the structure of the component models $\mathcal{I}_1, \mathcal{I}_2$ is preserved in a non-overlapping fashion.

Lemma 4.2.4. *Let* $\mathcal{I}_1 = (\Delta_1, \preceq_1, \perp_1, \cdot_1)$ *and* $\mathcal{I}_2 = (\Delta_2, \preceq_2, \perp_2, \cdot_2)$ *be constructive interpretations,* $\Delta_1 \cap \Delta_2 = \emptyset$ *and* $\mathcal{I}^{\bowtie} = (\Delta, \preceq, \perp, \cdot)$ *be their join according to Def. 4.2.7. Then,* $\forall x, y, z \in \Delta$, $\forall i \in \{1, 2\}$,

(i) $x \in \Delta_i$ *and* $x \preceq y \;\Rightarrow y \in \Delta_i \;\&\; x \preceq_i y$*; and*

(ii) $x \in \Delta_i$ *and* $x\, R^{\mathcal{I}^{\bowtie}}\, z \;\Rightarrow z \in \Delta_i \;\&\; x\, R^{\mathcal{I}_i}\, z$*.* ∇

Proof. Let $\mathcal{I}^{\bowtie}$ be the join model of $\mathcal{I}_1, \mathcal{I}_2$ in join world a. Moreover let $x, y, z \in \Delta$ and $x \in \Delta_i$ for $i \in \{1, 2\}$. Since $x \in \Delta_i$, it follows that $x \neq a$. For (i) suppose that $x \preceq y$. Then, we can conclude that $x \preceq_i y$ by Def. 4.2.7. Since Δ_1 and Δ_2 are assumed to be disjoint it must be that $y \in \Delta_i$. For (ii) let us assume that $x\, R^{\mathcal{I}^{\bowtie}}\, z$. From $x \neq a$ and $x\, R^{\mathcal{I}^{\bowtie}}\, z$ follows by Def. 4.2.7 that $x\, R^{\mathcal{I}_i}\, z$ and thereof $z \in \Delta_i$ as before. ☐

The next step is to show that $\mathcal{I}^{\bowtie}$ is a constructive interpretation according to Def. 4.2.2.

Lemma 4.2.5. *Let* $\mathcal{I}_1 = (\Delta_1, \preceq_1, \perp_1, \cdot_1)$ *and* $\mathcal{I}_2 = (\Delta_2, \preceq_2, \perp_2, \cdot_2)$ *be constructive models such that* $\Delta_1 \cap \Delta_2 = \emptyset$*. The join* $\mathcal{I}^{\bowtie}$ *of* $\mathcal{I}_1, \mathcal{I}_2$ *w.r.t.* $a_1 \in \Delta_1, a_2 \in \Delta_2$ *and join world* a *is a constructive interpretation w.r.t. Def. 4.2.2 if* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are.* ∇

Proof.

- The set Δ is nonempty by Def. 4.2.7.
- The relation $\preceq$ is reflexive and transitive by construction if $\preceq_1$ and $\preceq_2$ are.

  Suppose $\preceq_1, \preceq_2$ are reflexive and transitive. The goal is to show that $\forall x, y, z \in \Delta.\ x \preceq y$ and $y \preceq z \Rightarrow x \preceq x$ and $x \preceq z$. Let $x, y, z \in \Delta$ be arbitrary. Suppose $x \preceq y$ and $y \preceq z$. Then either $x = a$ or $x \neq a$.

  Case 1. If $x = a$ then reflexivity $a \preceq a$ follows directly by Def. 4.2.7. Regarding transitivity we proceed by case analysis on y. If $y = a$ then immediately $a \preceq z$. Otherwise, $y \neq a$ implies $y \in \Delta_i$ and $a_i \preceq_i y$ for some $i \in \{1, 2\}$. By Lem. 4.2.4 we conclude $y \preceq_i z$ and by transitivity of $\preceq_i$ it follows that $a_i \preceq_i z$. Thus, $a \preceq z$ by Def. 4.2.7.

  Case 2. $x \neq a$. This implies $x \in \Delta_i$ for some $i \in \{1, 2\}$. From reflexivity of $\preceq_i$ it follows that $x \preceq_i x$ and by Def. 4.2.7 one concludes $x \preceq x$. Concerning transitivity, since $x \in \Delta_i$ and $x \preceq y$ we can conclude $x \preceq_i y$ and $y \preceq_i z$ by Lem. 4.2.4. Transitivity of $\preceq_i$ implies $x \preceq_i z$ and by Def. 4.2.7 $x \preceq z$ as required.

Hence, $\preceq$ is reflexive and transitive.

- The set of fallible entities $\perp$ is closed under refinement and role-filling if $\perp_1$ and $\perp_2$ are. We have to prove $\forall x, y, z \in \Delta$ with $x \in \perp$ that :

  (i) $x \preceq y \Rightarrow y \in \perp$;

  (ii) $\forall R \in N_R.\ \exists z$ s.t. $x\, R^{\mathcal{I}^{\bowtie}}\, z$ and $z \in \perp$;

  (iii) $\forall R \in N_R.\ \forall z.\ x\, R^{\mathcal{I}^{\bowtie}}\, z \Rightarrow z \in \perp$.

  Suppose $\perp_1$ and $\perp_2$ are closed under refinement and role-filling. Let $x, y, z \in \Delta$, $R \in N_R$ be arbitrary, $i \in \{1, 2\}$ and assume that $x \in \perp$. We proceed by case analysis on x.

  Case 1. If $x = a$ then $x \in \perp$ if and only if $a_1 \in \perp_1$ and $a_2 \in \perp_2$. For (i) assume $x \preceq y$ and consider the possible cases of y. If $y = a$ then immediately $y \in \perp$. Otherwise, $a_i \preceq_i y$ and $a_i \in \Delta_i$ implies $y \in \Delta_i$. Then, by assumption $y \in \perp_i$ and we can conclude $y \in \perp$ by Def. 4.2.7. For (ii) let $z = a$, then by Def. 4.2.7 $z\, R^{\mathcal{I}^{\bowtie}}\, z$ and $z \in \perp$. For (iii) let $z \in \Delta$ be arbitrary. Assume $x\, R^{\mathcal{I}^{\bowtie}}\, z$. By Def. 4.2.7 this is the case if and only if $a_1 \in \perp$ and $a_2 \in \perp$ and $z = a$, *i.e.*, a has only itself as R-filler, thus $z \in \perp$.

  Case 2. If $x \neq a$ then $x \in \Delta_i$ and $x \in \perp_i$. In this case the goals (i), (ii) and (iii) follow immediately by assumption.

- The interpretation of atomic concepts is given by $A^{\mathcal{I}^{\bowtie}}$, which is closed under refinement if $A^{\mathcal{I}_1}$ and $A^{\mathcal{I}_2}$ are.

  Suppose $\forall A \in N_C$ that $A^{\mathcal{I}_1}$ and $A^{\mathcal{I}_2}$ are closed under refinement. Then, the goal is to show that $\forall A \in N_C.\ \forall x, y \in \Delta.\ x \preceq y$ and $x \in A^{\mathcal{I}^{\bowtie}} \Rightarrow y \in A^{\mathcal{I}^{\bowtie}}$. Let $A \in N_C$ and $x, y \in \Delta$ be arbitrary. Assume $x \preceq y$ and $x \in A^{\mathcal{I}^{\bowtie}}$. As above, we consider two cases.

  Case 1. If $x = a$ then according to Def. 4.2.7 it holds that $x \in A^{\mathcal{I}^{\bowtie}}$ if and only if $\forall i \in \{1, 2\}.\ a_i \in A^{\mathcal{I}_i}$. We consider the cases for $x \preceq y$. If $y = a$ then immediately $y \in A^{\mathcal{I}^{\bowtie}}$. Otherwise, it is the case that $a_i \preceq_i y$. Then, $a_i \in A^{\mathcal{I}_i}$ implies $y \in A^{\mathcal{I}_i}$, since by assumption $A^{\mathcal{I}_i}$ is closed under refinement. Thus, $y \in A^{\mathcal{I}^{\bowtie}}$ by Def. 4.2.7.

  Case 2. If $x \neq a$ then $x \in \Delta_i$ for some $i \in \{1, 2\}$. Lemma 4.2.4 implies $y \in \Delta_i$ and $x \preceq_i y$. Furthermore, $x \in A^{\mathcal{I}^{\bowtie}}$ if and only if $x \in A^{\mathcal{I}_i}$. Proposition 4.2.2 lets us conclude that $y \in A^{\mathcal{I}_i}$ and by Def. 4.2.7 follows $y \in A^{\mathcal{I}^{\bowtie}}$ as required.

- For the interpretation of roles $R \in N_R$ there is nothing to show. □

Furthermore, one proves that the model $\mathcal{I}^{\bowtie}$ does not alter the structure of the models $\mathcal{I}_1, \mathcal{I}_2$ while joining them, *i.e.*, the join of two models preserves satisfiability of concepts in their original models.

Lemma 4.2.6. *Let* $\mathcal{I}_1 = (\Delta_1, \preceq_1, \perp_1, \cdot_1)$ *and* $\mathcal{I}_2 = (\Delta_2, \preceq_2, \perp_2, \cdot_2)$ *be constructive models and* Δ_1, Δ_2 *be disjoint. Further, let* $\mathcal{I}^{\bowtie}$ *be the join of* $\mathcal{I}_1, \mathcal{I}_2$ *w.r.t. entities* $a_1 \in \Delta_1, a_2 \in \Delta_2$ *and join world* a *according to Def. 4.2.7. Then,* $\forall x \in \Delta_i.\ \mathcal{I}^{\bowtie}; x \models C$ *iff* $\mathcal{I}_i; x \models C$ *for all* $i \in \{1, 2\}$ *and* C *being an arbitrary concept.* ∇

Proof. The proof is by induction on the structure of C. Let i be chosen from $\{1, 2\}$ and $x \in \Delta_i$ be arbitrary. Notice that $x \in \Delta_i$ implies $x \neq a$. Therefore, as we are only interested in the entities from the set Δ_i we can simplify Def. 4.2.7 of the join model $\mathcal{I}^{\bowtie}$ by excluding the cases for the fresh entity a in the following way: The set Δ is given by $\Delta_1 \cup \Delta_2$, the preorder $\preceq$ is defined as $\preceq_1 \cup \preceq_2$, and regarding fallibles we have $\perp =_{df} \{x \mid \text{for } i \in \{1, 2\}.\ x \in \perp_i\}$. The interpretations of atomic concepts $A^{\mathcal{I}^{\bowtie}}$ and roles $R^{\mathcal{I}^{\bowtie}}$ simplify to $A^{\mathcal{I}_1} \cup A^{\mathcal{I}_2}$ and $R^{\mathcal{I}_1} \cup R^{\mathcal{I}_2}$ respectively.

(**Case** A) $\mathcal{I}^{\bowtie}; x \models A$ if and only if by Def. 4.2.7 it holds that $x \in A^{\mathcal{I}_i}$ which by Def. 4.2.2 holds iff $\mathcal{I}_i; x \models A$.

(**Case** $\perp$) By Def. 4.2.7 $\mathcal{I}^{\bowtie}; x \models \perp$ if and only if $x \in \perp_i$ which holds by Def. 4.2.2 if and only if $\mathcal{I}_i; x \models \perp$.

(**Case** $C \sqcap D$) By Def. 4.2.2 $\mathcal{I}^{\bowtie}; x \models C \sqcap D$ holds if and only if $\mathcal{I}^{\bowtie}; x \models C$ and $\mathcal{I}^{\bowtie}; x \models D$. By induction hypothesis, $\mathcal{I}_i; x \models C$ and $\mathcal{I}_i; x \models D$ and by Def. 4.2.2 we conclude that $\mathcal{I}_i; x \models C \sqcap D$.

(**Case** $C \sqcup D$) By induction hypothesis.

(**Case** $C \supset D$) Proof by contraposition. ($\Rightarrow$) Suppose that $\mathcal{I}_i; x \not\models C \supset D$. By Def. 4.2.2 there exists an entity x' with $x \preceq_i x'$ such that $\mathcal{I}_i; x' \models C$ and $\mathcal{I}_i; x' \not\models D$. By definition of $\mathcal{I}^{\bowtie}$ holds $x \preceq x'$ as well. Applying the induction hypothesis yields $\mathcal{I}^{\bowtie}; x' \models C$ and $\mathcal{I}^{\bowtie}; x' \not\models D$. Hence, $\mathcal{I}^{\bowtie}; x \not\models C \supset D$ by Def. 4.2.2.

($\Leftarrow$) Suppose that $\mathcal{I}^{\bowtie}; x \not\models C \supset D$. Again, by Def. 4.2.2 there exists an entity x' such that $x \preceq x'$, $\mathcal{I}^{\bowtie}; x' \models C$ and $\mathcal{I}^{\bowtie}; x' \not\models D$. From $x \in \Delta_i$ and $x \preceq x'$ we conclude $x' \in \Delta_i$ and $x \preceq_i x'$ by Lem. 4.2.4.(i). By the inductive hypothesis $\mathcal{I}_i; x' \models C$ and $\mathcal{I}_i; x' \not\models D$. Therefore, $\mathcal{I}_i; x \not\models C \supset D$ by Def. 4.2.2.

(**Case** $\exists R.C$) ($\Rightarrow$) Assume $\mathcal{I}^{\bowtie}; x \models \exists R.C$. Then, by Def. 4.2.2 it holds for all $\preceq$ refinements of x that there exists an $R^{\mathcal{I}^{\bowtie}}$-successor that is contained in $C^{\mathcal{I}^{\bowtie}}$. Let $x' \in \Delta_i$ be arbitrary such that $x \preceq_i x'$. Definition 4.2.7 implies $x \preceq x'$ as well, and by the assumption there exists $y \in \Delta$ such that $x' R^{\mathcal{I}^{\bowtie}} y$ and $\mathcal{I}^{\bowtie}; y \models C$. Lemma 4.2.4.(ii) implies that $y \in \Delta_i$ and $x' \, R^{\mathcal{I}_i} \, y$. Applying the inductive hypothesis yields $\mathcal{I}_i; y \models C$. Thus, $\mathcal{I}_i; x \models \exists R.C$ by Def. 4.2.2.

($\Leftarrow$) Let us suppose that $\mathcal{I}_i; x \models \exists R.C$, which is the case if for all $\preceq_i$ refinements of x there exists an $R^{\mathcal{I}_i}$-filler in $C^{\mathcal{I}_i}$. Let $x' \in \Delta$ such that $x \preceq x'$. Then, Lem. 4.2.4.(i) implies $x' \in \Delta_i$ and $x \preceq_i x'$. According to the assumption there exists $y \in \Delta_i$ such that $x' \, R^{\mathcal{I}_i} \, y$ and $\mathcal{I}_i; y \models C$. It follows from Def. 4.2.7 that $y \in \Delta$ and $x \, R^{\mathcal{I}^{\bowtie}} \, y$. We can conclude by the ind. hyp. that $\mathcal{I}^{\bowtie}; y \models C$ and by Def. 4.2.2 it follows that $\mathcal{I}^{\bowtie}; x \models \exists R.C$.

(**Case** $\forall R.C$) Proof by contraposition. ($\Rightarrow$) Let us suppose that $\mathcal{I}_i; x \not\models \forall R.C$. The goal is to show that $\mathcal{I}^{\bowtie}; x \not\models \forall R.C$. The assumption implies that there exist entities $x', y \in \Delta_i$ such that $x \preceq_i x' \, R^{\mathcal{I}_i} \, y$ and $\mathcal{I}_i; y \not\models C$. Construction of $\mathcal{I}^{\bowtie}$ (Def. 4.2.7) implies $x', y \in \Delta$, $x \preceq x'$ and $x' \, R^{\mathcal{I}^{\bowtie}} \, y$. Then, by ind. hyp. $\mathcal{I}^{\bowtie}; y \not\models C$. Thus, $\mathcal{I}^{\bowtie}; x \not\models \forall R.C$.

($\Leftarrow$) Assume that $\mathcal{I}^{\bowtie}; x \not\models \forall R.C$. The goal is $\mathcal{I}_i; x \not\models \forall R.C$. The assumption implies the existence of entities $x', y \in \Delta$ such that $x \preceq x'$, $x' \, R^{\mathcal{I}^{\bowtie}} \, y$ and $\mathcal{I}^{\bowtie}; y \not\models C$. Since $x \in \Delta_i$, we can conclude by Lemma 4.2.4 that $x', y \in \Delta_i$, and $x \preceq_i x' \, R^{\mathcal{I}_i} \, y$. Applying the ind. hyp. yields $\mathcal{I}_i; y \not\models C$. Therefore $\mathcal{I}_i; x \not\models \forall R.C$. □

Now, we can prove the following property of Harrop concepts:

Lemma 4.2.7. *Let $\mathcal{I}_1 = (\Delta_1, \preceq_1, \bot_1, \cdot_1)$ and $\mathcal{I}_2 = (\Delta_2, \preceq_2, \bot_2, \cdot_2)$ be constructive models, $\Delta_1 \cap \Delta_2 = \emptyset$ and let $\mathcal{I}^{\bowtie}$ be the join of the constructive interpretations $\mathcal{I}_1, \mathcal{I}_2$ w.r.t. entities $a_1 \in \Delta_1, a_2 \in \Delta_2$ and join world a according to Definition 4.2.7. For each Harrop concept H it holds that $\mathcal{I}^{\bowtie}; a \models H$ if and only if $\mathcal{I}_1; a_1 \models H$ and $\mathcal{I}_2; a_2 \models H$.* ∇

Proof. The proof is by induction on the structure of the Harrop concept H.

(**Case** A) $\mathcal{I}^{\bowtie}; a \models A$ if and only if for all $i \in \{1, 2\}$, $\mathcal{I}_i; a_i \models A$ by Def. 4.2.7.

(**Case** $\bot$) $\mathcal{I}^{\bowtie}; a \models \bot$ if and only if for all $i \in \{1, 2\}$, $\mathcal{I}_i; a_i \models \bot_i$ by Def. 4.2.7.

(**Case** $H_1 \sqcap H_2$) By Def. 4.2.2 it is the case that $\mathcal{I}^{\bowtie}; a \models H_1 \sqcap H_2$ iff $\mathcal{I}^{\bowtie}; a \models H_1$ and $\mathcal{I}^{\bowtie}; a \models H_2$. By induction hypothesis the latter holds iff $\mathcal{I}_i; a_i \models H_1$ and $\mathcal{I}_i; a_i \models H_2$ for all $i \in \{1, 2\}$, which by Def. 4.2.2 is the case iff $\mathcal{I}_i; a_i \models H_1 \sqcap H_2$.

(**Case** $C \supset H_1$) Proof by contraposition. ($\Rightarrow$) Let us assume that $\mathcal{I}_i; a_i \not\models C \supset H_1$ for at least one $i \in \{1, 2\}$. Definition 4.2.2 implies the existence of an entity x such that $a_i \preceq_i x$ and $\mathcal{I}_i; x \models C$, but $\mathcal{I}_i; x \not\models H_1$. Since $x \in \Delta_i$, we can conclude by

Lem. 4.2.6 that $\mathcal{I}^{\bowtie}; x \models C$ and $\mathcal{I}^{\bowtie}; x \not\models H_1$. From Def. 4.2.7 it follows that $a \preceq x$ and therefore $\mathcal{I}^{\bowtie}; a \not\models C \supset H_1$.

($\Leftarrow$) Suppose that $\mathcal{I}^{\bowtie}; a \not\models C \supset H_1$. By Def. 4.2.2 there exists an entity $x \in \Delta$ such that $a \preceq x$, $x \in C^{\mathcal{I}^{\bowtie}}$ and $x \notin H_1^{\mathcal{I}^{\bowtie}}$. We proceed by case analysis.

Case 1. If $x = a$ then the induction hypothesis implies $\mathcal{I}_i; a_i \not\models H_1$ for some $i \in \{1, 2\}$. From $a \preceq a_i$ and monotonicity (Prop. 4.2.2) follows $\mathcal{I}^{\bowtie}; a_i \models C$ and applying Lem. 4.2.6 yields $\mathcal{I}_i; a_i \models C$. Then, by Definition 4.2.2 we can conclude $\mathcal{I}_i; a_i \not\models C \supset H_1$.

Case 2. $x \neq a$ implies $a_i \preceq x$ for some $i \in \{1, 2\}$. Lemma 4.2.4 implies $x \in \Delta_i$ and $a_i \preceq_i x$. Then, by Lem. 4.2.6 we can conclude that $\mathcal{I}_i; x \models C$ and $\mathcal{I}_i; x \not\models H_1$. Therefore $\mathcal{I}_i; a_i \not\models C \supset H_1$ by Def. 4.2.2.

(**Case** $\forall R.H_1$) Proof by contraposition. ($\Rightarrow$) Suppose that $\mathcal{I}_i; a_i \not\models \forall R.H_1$ for some $i \in \{1, 2\}$. Then, there exist entities $x, y \in \Delta_i$ such that $a_i \preceq_i x \, R^{\mathcal{I}_i} \, y$ and $y \notin H_1^{\mathcal{I}_i}$. Note that $y \notin \bot_i$, which implies by Prop. 4.2.1 that $x \notin \bot_i$ and therefore $a_i \notin \bot_i$ as well. Def. 4.2.7 implies $a \preceq x \, R^{\mathcal{I}^{\bowtie}} \, y$ and by Lem. 4.2.6 we can conclude $\mathcal{I}^{\bowtie}; y \not\models H_1$. Thus, $\mathcal{I}^{\bowtie}; a \not\models \forall R.H_1$.

($\Leftarrow$) Assume that $\mathcal{I}^{\bowtie}; a \not\models \forall R.H_1$. Then there exist entities $x, y \in \Delta$ such that $a \preceq x \, R^{\mathcal{I}^{\bowtie}} \, y$ and $y \notin H_1^{\mathcal{I}^{\bowtie}}$. Analogously to the previous case, $x, y \notin \bot$ and therefore $a \notin \bot$. The case $x = a$ (Def. 4.2.7) is not possible, since then it must be that $y = a$. But this would imply by Def. 4.2.7 that $a_i \in \bot_i$ for all $i \in \{1, 2\}$, which contradicts the assumption that $a \notin \bot$. Therefore, $x \neq a$ which implies $x \in \Delta_i$. From Def. 4.2.7 follows $a_i \preceq_i x$ and Lem. 4.2.4 implies $y \in \Delta_i$ and $x \, R^{\mathcal{I}_i} \, y$. Applying Lem. 4.2.6 yields $\mathcal{I}_i; y \not\models H_1$. Hence, $\mathcal{I}_i; a_i \not\models \forall R.H_1$. ☐

Now we are ready to tackle the main proposition.

Proposition 4.2.3 (Disjunction property). *Let* Γ *be a set of Harrop concepts of c$\mathcal{ALC}$, then* $\emptyset; \Gamma \models C \sqcup D$ *implies* $\emptyset; \Gamma \models C$ *or* $\emptyset; \Gamma \models D$. ∇

Proof. Assume $\emptyset; \Gamma \models C \sqcup D$ and to the contrary $\emptyset; \Gamma \not\models C$ and $\emptyset; \Gamma \not\models D$, where Γ is a set of Harrop concepts. By Def. 4.2.4 there exist models $\mathcal{I}_1 = (\Delta_1, \preceq_1, \bot_1, \cdot_1)$ and $\mathcal{I}_2 = (\Delta_2, \preceq_2, \bot_2, \cdot_2)$ and entities $a_1 \in \Delta_1$, $a_2 \in \Delta_2$ such that

$$\begin{aligned} &\mathcal{I}_i; a_i \models \Gamma, \quad \text{for all } i \in \{1, 2\}; \text{ and} \\ &\mathcal{I}_1; a_1 \not\models C \quad \& \quad \mathcal{I}_2; a_2 \not\models D. \end{aligned}$$

We construct the join model $\mathcal{I}^{\bowtie}$ of $\mathcal{I}_1$, $\mathcal{I}_2$ w.r.t. entities a_1, a_2 in a fresh join world a according to Def. 4.2.7. By Lemma 4.2.5 it follows that the model $\mathcal{I}^{\bowtie}$ is a constructive interpretation according to Def. 4.2.2. By assumption $\mathcal{I}_1; a_1 \models \Gamma$ and $\mathcal{I}_2; a_2 \models \Gamma$. Now, let $H \in \Gamma$ be arbitrary. By Lemma 4.2.7 we can conclude $\mathcal{I}^{\bowtie}; a \models H$. Since H was arbitrary in Γ it holds that $\mathcal{I}^{\bowtie}; a \models \Gamma$. The assumptions $\mathcal{I}_1; a_1 \not\models C$, $\mathcal{I}_2; a_2 \not\models D$ and Lemma 4.2.6 imply $\mathcal{I}^{\bowtie}; a_1 \not\models C$ and $\mathcal{I}^{\bowtie}; a_2 \not\models D$. By Def. 4.2.7 $a \preceq a_1$ and $a \preceq a_2$. Therefore $\mathcal{I}^{\bowtie}; a \not\models C$ and $\mathcal{I}^{\bowtie}; a \not\models D$, which implies $\mathcal{I}; a \not\models C \sqcup D$ by Def. 4.2.2. But this contradicts our initial assumption. Hence, $\emptyset; \Gamma \models C$ or $\emptyset; \Gamma \models D$ □

The validity statement of the disjunction property can be strengthened into a disjunction property w.r.t. a TBox, *viz.* $\Theta; \Gamma \models C \sqcup D$, by restricting the axioms of Θ to inclusions of the form $C \supset H$, where H is a Harrop concept.

The standard formulation of DP comes as a corollary from Prop. 4.2.3 by assuming an empty set of concepts in the premise of the validity statement.

Corollary 4.2.1. $\models C \sqcup D$ *implies* $\models C$ *or* $\models D$. ∇

A consequence of the disjunction property (Prop. 4.2.3) is the rejection of the *principle of the Excluded Middle* in $c\mathcal{ALC}$. Note that PEM holds in classical $\mathcal{ALC}$, *i.e.*, for every concept C the statement $\models C \sqcup \neg C$ generally holds even if neither $\models C$ nor $\models \neg C$.

Example 4.2.7. The disjunction property says that if $\models C \sqcup D$, then $\models C$ or $\models D$. An important property of constructive logic is that the truth of a statement depends on the existence of a proof of it and therefore it is not possible to assume the truth of its complement if its proof fails. While the law of the Excluded Middle $C \sqcup \neg C$ is a tautology in classical logic it is not valid in constructive logic, in particular it does not belong to $c\mathcal{ALC}$. Take an interpretation $\mathcal{I}$ defined by $\Delta^{\mathcal{I}} = \{a_0, a_1\}$ with refinement $\preceq^{\mathcal{I}} = \{(a_0, a_0), (a_0, a_1), (a_1, a_1)\}$, $R^{\mathcal{I}} = \emptyset$ and valuation $C^{\mathcal{I}} = \{a_1\}$. Then, we can observe that concept C is not forced at entity a_0, *i.e.*, $\mathcal{I}; a_0 \not\models C$, nor is $\neg C$ because $\mathcal{I}; a_1 \models C$ and entity a_1 refines a_0. This lets us conclude that entity a_0 of this model is also a countermodel demonstrating that the law of the Excluded Middle $C \sqcup \neg C$ does not belong to $c\mathcal{ALC}$.

Furthermore, we can show that $c\mathcal{ALC}$ does not inherit the axiom of double negation elimination $\neg\neg C \supset C$ from classical logic. The reason for that can be observed from interpretation $\mathcal{I}$, *i.e.*, because $\neg C$ is not satisfied by any entity in $\Delta^{\mathcal{I}}$ it follows that the concept $\neg\neg C$ is satisfiable at entity a_0. But C is not forced at entity a_0, *i.e.*, $\mathcal{I}; a_0 \not\models C$. Therefore, it follows that $\mathcal{I}; a_0 \not\models \neg\neg C \supset C$.

Another axiom, which falls into this category is Peirce's law $((C \supset D) \supset C) \supset C$ that can be seen as a stronger form of the Excluded Middle. Since $\mathcal{I}; a_0 \not\models C$, it follows that $\mathcal{I}; a_0 \models (C \supset D) \supset C$ and therefore it can only be that $\mathcal{I}; a_0 \not\models C \supset D$. The latter is the case because there exists the refinement a_1 of a_0, which forces C but not D.

These laws represent equivalent formulations of classical logic in intuitionistic logic. However, this formulation of classical logic does not generalise to all intuitionistic modal logics and in particular to $c\mathcal{ALC}$ as we will see later, *i.e.*, the extension of $c\mathcal{ALC}$ by one of these axioms does not yield its classical analogue $\mathcal{ALC}$ but rather an intermediate logic, which still does not contain the principle of disjunctive distribution. ■

4.2.6 Finite Model Property

In the following section we will discuss another important property of $c\mathcal{ALC}$, that is, $c\mathcal{ALC}$ comes with a finite semantic characterisation, *i.e.*, every satisfiable concept of $c\mathcal{ALC}$ is satisfiable in a finite interpretation. This property is known as the *finite model property* and it will open the door to establish the decidability of $c\mathcal{ALC}$.

In classical modal logic, one method to obtain the finite model property of a logical system is via the filtration technique [33, pp. 77 ff.]. Roughly speaking, given a concept C and a (possibly infinite) $c\mathcal{ALC}$ Kripke-structure $\mathcal{I}$ satisfying C, one collapses those entities, which validate the same concepts including their proper subformulæ in order to produce a finite Kripke structure from $\mathcal{I}$ that satisfies C as well.

Example 4.2.8. We reconsider an example from [118, p. 25], which illustrates filtration and also allows us to give a refutation for Glivenko's Theorem [62; 114, p. 47] in $c\mathcal{ALC}$. It is well known that intuitionistic propositional logic and classical propositional logic can be related via a double-negation translation. One instance of such a translation comes from Glivenko's Theorem, which states that $\neg\neg C \in \mathsf{IPC}$ if and only if $C \in \mathsf{CPC}$. However, this result does not transfer to intuitionistic modal logics in general[10] and in particular it is not true for $c\mathcal{ALC}$. A countermodel for the concept description $\neg\neg C$ with $C = \forall R.(A \sqcup \neg A)$, is constructed in [118, p. 25], demonstrating that it does not belong to FS. By similar arguments we can show that the double-negation translation $\neg\neg C$ does not belong to $c\mathcal{ALC}$ despite obviously $C \in \mathcal{ALC}$ holds.

We construct a countermodel for the concept $\neg\neg C$. Suppose there is an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ and an entity $a_0 \in \Delta^{\mathcal{I}}$ such that $\mathcal{I}; a_0 \not\models \neg\neg\forall R.(A \sqcup \neg A)$, *i.e.*, $a_0 \notin (\neg\neg\forall R.(A \sqcup \neg A))^{\mathcal{I}}$. Then, by Def. 4.2.3 and 4.2.2 there exists a refinement $a_1 \in \Delta_c^{\mathcal{I}}$ such that $a_0 \preceq^{\mathcal{I}} a_1$ and $\mathcal{I}; a_1 \models \neg\forall R.(A \sqcup \neg A)$. Since a_1 is infallible and by reflexivity of $\preceq^{\mathcal{I}}$ we can conclude that $\mathcal{I}; a_1 \not\models \forall R.(A \sqcup \neg A)$. Then, there exist entities $a_2, b_2 \in \Delta^{\mathcal{I}}$ such that $a_1 \preceq^{\mathcal{I}} a_2 R^{\mathcal{I}} b_2$ and $\mathcal{I}; b_2 \not\models A \sqcup \neg A$. This implies in particular that $b_2 \notin (\neg A)^{\mathcal{I}}$, *i.e.*, there exists a non-fallible entity b_3 such that $b_2 \preceq b_3$ and $\mathcal{I}; b_3 \models A$. Monotonicity (Prop. 4.2.2) implies $\mathcal{I}; a_2 \models \neg\forall R.(A \sqcup \neg A)$ and therefore $\mathcal{I}; a_2 \not\models \forall R.(A \sqcup \neg A)$. Note that $a_2 \notin \bot^{\mathcal{I}}$ because b_2 is infallible and $a_2 \, R^{\mathcal{I}} \, b_2$. At this point the countermodel construction can go on ad infinitum, *i.e.*, $\mathcal{I}; a_2 \not\models \forall R.(A \sqcup \neg A)$ implies the existence of a refinement of a_2 which falsifies $\forall R.(A \sqcup \neg A)$ in the sense, that it possesses an R-successor that falsifies $A \sqcup \neg A$. If we introduce a new refinement a_3 of a_2 we get by

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[10] Extensions of Glivenko's Theorem to IMLs over Prior's MIPC [232] have been discussed in [31].

monotonicity $\mathcal{I}; a_3 \models \neg\forall R.(A \sqcup \neg A)$. From this point on we can continue by introducing a new refinement of a_3 and repeating the construction described above infinitely often. However, we can identify (filtrate) the entity a_3 and its appendant structure with a_2 and its corresponding subgraph. The finite countermodel is given in Figure 4.5. This countermodel is marginally simpler than the respective one for Fischer-Servi's system FS as shown in [118]. In order to satisfy the frame condition of Fischer-Servi's FS, *i.e.*, $x\,R\,y \wedge y \preceq z \Rightarrow \exists w.x \preceq w \wedge w\,R\,y$, it would be required to either introduce a new entity a_3 such that $a_2 \preceq a_3\,R\,b_2$, which leads to an infinite model, since it demands a witness for $\mathcal{I}; a_3 \not\models \forall R.(A \sqcup \neg A)$, or the introduction of an edge $a_2\,R\,b_3$ which yields a finite countermodel. For a characterisation of the least extension of FS satisfying a generalisation of Glivenko's Theorem see [118, Chap. 9].

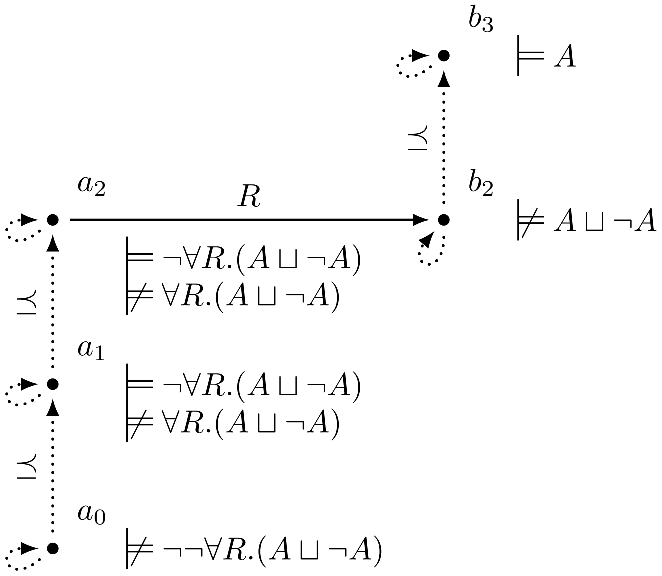

Figure 4.5: Countermodel for $\neg\neg\forall R.(A \sqcup \neg A)$.

■

The proof of the finite model property for the affiliated constructive modal logic CK has been sketched in [188]. We will demonstrate in the following that the filtration method of [188] can be extended to the multimodal case of $c\mathcal{ALC}$ (CK_n) and present the full proof.

Definition 4.2.8 (Subformula closed set)**.** A set of concepts Γ is called *subformula closed* if for all $C, D \in \Gamma$ the following holds: $\{\top, \bot\} \subseteq \Gamma$; $\forall R.\bot \in \Gamma$, for all roles R occurring in Γ; if $\neg C \in \Gamma$ then $C \in \Gamma$; if $C \odot D \in \Gamma$ then so are C and D, where $\odot \in \{\sqcap, \sqcup, \supset\}$; if $QR.C \in \Gamma$ then so is C for $Q \in \{\exists, \forall\}$. We write $Sfc(C)$ for the least subformula closed set containing all subconcepts of C. ∇

Definition 4.2.9 (Filtration model). Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a fixed but arbitrary $c\mathcal{ALC}$-model, C a $c\mathcal{ALC}$ concept, Γ be a finite subformula-closed set and let $N_R = \{R_1, R_2, \ldots, R_n\}$ be the set of roles appearing in Γ. Following the construction of [188] we need to consider two flavours of local information in order to preserve the validity of concepts in the interpretation $\mathcal{I}$ at any entity $x \in \Delta^{\mathcal{I}}$. The first component is the set $T(x)$ of all subformulæ validated at an entity x:

$$T(x) =_{df} \{D \mid D \in \Gamma \ \ \& \ \ \mathcal{I}; x \models D\}.$$

Secondly, it is necessary to preserve the set of subformulæ of C, which are falsified at each reachable R-successor of x, where $R \in N_R$:

$$F(x)_R =_{df} \{D \mid D \in \Gamma \ \ \& \ \ \forall y.x \, R^{\mathcal{I}} \, y \Rightarrow \mathcal{I}; y \not\models D\}.$$

Note that if $x \preceq^{\mathcal{I}} y$ then $T(x) \subseteq T(y)$ and if $x \, R^{\mathcal{I}} \, y$ then $\forall R^{-1} \, T(x) \subseteq T(y)$ as well as $F(x)_R \cap T(y) = \emptyset$, where $\forall R^{-1} \, \Sigma =_{df} \{C \mid \forall R.C \in \Sigma\}$ for a set Σ of concepts.

The pair $(T(x), F(x))$ with $F(x) =_{df} \{F(x)_{R_1}, F(x)_{R_2}, \ldots, F(x)_{R_n}\}$ characterises the behaviour of entity x in the interpretation $\mathcal{I}$ w.r.t. the set Γ. We denote these finite tuples as Γ-theories. In general, a Γ-*theory* in $\mathcal{I}$ is a tuple $Th_{\Gamma} =_{df} (X, Z)$ with $Z =_{df} \{Z_{R_1}, Z_{R_2}, \ldots, Z_{R_n}\}$ of subsets $X, Z_{R_i} \subseteq \Gamma$ (with $R_i \in N_R$ for $1 \leq i \leq n$), such that there exists an entity $x \in \Delta^{\mathcal{I}}$ with $X = T(x)$ and $Z_{R_i} \subseteq F(x)_{R_i}$, for all $R_i \in N_R$. The finite set of all Γ-theories in the interpretation $\mathcal{I}$ is denoted by $Th_{\mathcal{I}}(\Gamma)$.

Note that for any entity x in any $c\mathcal{ALC}$ interpretation $\mathcal{I}$ and any $c\mathcal{ALC}$ concept C with $\Gamma = Sfc(C)$, the pair $x_{\equiv} = (T(x), F(x))$ is a Γ-theory. Therefore, it follows that $Th_{\mathcal{I}}(\Gamma)$ is non-empty whatever the C and $\mathcal{I}$ are.

The filtration of $\mathcal{I}$ w.r.t. Γ is then defined by $\mathcal{I}|_{\Gamma} =_{df} (\Delta^{\mathcal{I}}|_{\Gamma}, \preceq^{\mathcal{I}}|_{\Gamma}, \bot^{\mathcal{I}}|_{\Gamma}, \cdot^{\mathcal{I}}|_{\Gamma})$, where Z' denotes the set $\{Z'_{R_1}, Z'_{R_2}, \ldots, Z'_{R_n}\}$, and

$$\begin{aligned}
\Delta^{\mathcal{I}}|_{\Gamma} \ &=_{df} Th_{\mathcal{I}}(\Gamma); \\
(X, Z) \ \preceq^{\mathcal{I}}|_{\Gamma} \ (X', Z') \ &\text{ iff } X \subseteq X'; \\
(X, Z) \in \bot^{\mathcal{I}}|_{\Gamma} \ &\text{ if } \bot \in X; \\
(X, Z) \ R^{\mathcal{I}}|_{\Gamma} \ (X', Z') \ &\text{ iff } \forall R^{-1} \, X \subseteq X' \text{ and } Z_R \cap X' = \emptyset, \text{for } Z_R \in Z; \\
(X, Z) \in A^{\mathcal{I}}|_{\Gamma} \ &\text{ if } A \in X \text{ or } \bot \in X \text{ for all } A \in N_C;
\end{aligned}$$

otherwise, we define $\cdot^{\mathcal{I}}|_{\Gamma}$ according to the inductive conditions of Def. 4.2.2. ∇

Fallible entities are maximal and therefore the filtration interpretation has only one entity (X, Z) as a placeholder for all the fallible entities where $X = \Gamma$ and the set Z is empty. Since the fallible class is maximal w.r.t. $T(x)$ and the sets Z_R are empty, any entity that does not refute a formula at its R-successors is an abstraction of this fallible entity in $\Delta^{\mathcal{I}}|_{\Gamma}$. Observe from the above definition of $R^{\mathcal{I}}|_{\Gamma}$ that a fallible entity $(X, Z) \in Th_{\mathcal{I}}(\Gamma)$ with $\bot \in X$ is $R^{\mathcal{I}}|_{\Gamma}$-connected to itself for all roles $R \in N_R$. Also note that $x \preceq^{\mathcal{I}} y$ implies $x_{\equiv} \preceq^{\mathcal{I}}|_{\Gamma}\ y_{\equiv}$ and $x\ R^{\mathcal{I}}\ y$ implies $x_{\equiv}\ R^{\mathcal{I}}|_{\Gamma}\ y_{\equiv}$.

The following lemma shows that the result of applying the filtration yields a well-defined $c\mathcal{ALC}$ interpretation.

Lemma 4.2.8 (Filtration is well-defined)**.** *Let Γ be a (finite) subformula closed set of $c\mathcal{ALC}$ concepts. For any $c\mathcal{ALC}$ interpretation $\mathcal{I}$, the filtration $\mathcal{I}|_{\Gamma}$ is a well-defined $c\mathcal{ALC}$ interpretation according to Def. 4.2.2.* ∇

Proof. Given an arbitrary but fixed $c\mathcal{ALC}$ interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$, its filtration $\mathcal{I}|_{\Gamma} = (\Delta^{\mathcal{I}}|_{\Gamma}, \preceq^{\mathcal{I}}|_{\Gamma}, \bot^{\mathcal{I}}|_{\Gamma}, \cdot^{\mathcal{I}}|_{\Gamma})$ w.r.t. Γ is a constructive interpretation according to Def. 4.2.2 due to the following facts:

- $\Delta^{\mathcal{I}}|_{\Gamma}$ is nonempty by definition.
- The relation $\preceq^{\mathcal{I}}|_{\Gamma}$ is reflexive and transitive. This follows immediately from Def. 4.2.9 of $\preceq^{\mathcal{I}}|_{\Gamma}$.
- The set $\bot^{\mathcal{I}}|_{\Gamma}$ is closed under refinement and role-filling. We need to show $\forall x, y, z \in \Delta^{\mathcal{I}}|_{\Gamma}, x \in \bot^{\mathcal{I}}|_{\Gamma}$:

  (i) $x \preceq^{\mathcal{I}}|_{\Gamma}\, y \Rightarrow y \in \bot^{\mathcal{I}}|_{\Gamma}$;

  (ii) $\forall R \in N_R.\ \exists z \in \Delta^{\mathcal{I}}|_{\Gamma}$ s.t. $x\, R^{\mathcal{I}}|_{\Gamma}\, z$ and $z \in \bot^{\mathcal{I}}|_{\Gamma}$;

  (iii) $\forall R \in N_R.\ \forall z.\ x\, R^{\mathcal{I}}|_{\Gamma}\, z \Rightarrow z \in \bot^{\mathcal{I}}|_{\Gamma}$.

  Firstly, it can be easily observed from the definition of the filtration interpretation that $\mathcal{I}|_{\Gamma}; (T(x), Z) \models \bot$ implies $\mathcal{I}|_{\Gamma}; (T(x), Z) \models A$ for all $A \in N_C$.

  Let us assume for (i) that $\mathcal{I}|_{\Gamma}; (T(x), Z) \models \bot$ and $(T(x), Z) \preceq^{\mathcal{I}}|_{\Gamma}(T(y), Z')$. We need to show that $\mathcal{I}|_{\Gamma}; (T(y), Z') \models \bot$. The assumption says that $\bot \in T(x)$. From the definition of the filtration interpretation follows that $\bot \in T(y)$ as well, since $T(x) \subseteq T(y)$. Hence, $\mathcal{I}|_{\Gamma}; (T(y), Z') \models \bot$.

  For (ii) suppose $\mathcal{I}|_{\Gamma}; (T(x), Z) \models \bot$. We need to show that there exists an entity $(T(y), Z') \in \Delta^{\mathcal{I}}|_{\Gamma}$ s.t. $(T(x), Z)\, R^{\mathcal{I}}|_{\Gamma}(T(y), Z')$ and $\mathcal{I}|_{\Gamma}; (T(y), Z') \models \bot$. The assumption means that $\bot \in T(x)$. Then, the definition of the filtration model directly implies that $(T(x), Z)\, R^{\mathcal{I}}|_{\Gamma}(T(x), Z)$.

  For (iii) let $R \in N_R$ and suppose that $\mathcal{I}|_{\Gamma}; (T(x), Z) \models \bot$, $(T(x), Z)\, R^{\mathcal{I}}|_{\Gamma}(T(y), Z')$, for arbitrary $y \in \Delta^{\mathcal{I}}$ and Z'. The goal is $\mathcal{I}|_{\Gamma}; (T(y), Z') \models \bot$. By assumption $\bot \in T(x)$, and since $\forall R.\bot \in T(x)$ (by Def. 4.2.8), it follows that $\bot$ is contained in $\forall R^{-1}\, T(x) \subseteq T(y)$. Hence, $\mathcal{I}|_{\Gamma}; (T(y), Z') \models \bot$.

- The interpretation of atomic concepts $A \in N_C$ is given by $A^{\mathcal{I}}|_{\Gamma}$, which is closed under refinement. Let us suppose that $\mathcal{I}|_{\Gamma}; (T(x), Z) \models A$ and assume that $(T(x), Z) \preceq^{\mathcal{I}}|_{\Gamma}(T(y), Z')$ for arbitrary entities and atomic concepts $A \in N_C$. The goal $\mathcal{I}|_{\Gamma}; (T(y), Z') \models A$ is a direct consequence of the definition of the filtration interpretation Def. 4.2.9, since $T(x) \subseteq T(y)$.

This shows that the filtration $\mathcal{I}|_{\Gamma}$ of an interpretation $\mathcal{I}$ is a well-defined interpretation according to Def. 4.2.2. ☐

Moreover, we can show that filtration is satisfiability preserving in the following sense.

Theorem 4.2.1 (Filtration theorem). *Let Γ be a (finite) subformula closed set of $c\mathcal{ALC}$ concepts and let $\mathcal{I}|_{\Gamma} = (\Delta^{\mathcal{I}}|_{\Gamma}, \preceq^{\mathcal{I}}|_{\Gamma}, \bot^{\mathcal{I}}|_{\Gamma}, \cdot^{\mathcal{I}}|_{\Gamma})$ be the filtration of a $c\mathcal{ALC}$ interpretation $\mathcal{I}$ w.r.t. Γ. Then, for all concepts $C \in \Gamma$, $x \in \Delta^{\mathcal{I}}$ and $(T(x), Z) \in \Delta^{\mathcal{I}}|_{\Gamma}$ where $Z = \{Z_{R_1}, Z_{R_2}, \ldots, Z_{R_n}\}$ with $Z_{R_i} \subseteq F(x)_{R_i}$ for $i \in \{1, \ldots, n\}$, we have*

$$\mathcal{I}; x \models C \quad \textit{iff} \quad \mathcal{I}|_{\Gamma}; (T(x), Z) \models C. \qquad \nabla$$

Proof. The proof is by induction on the structure of the concept C. Let $\mathcal{I}|_{\Gamma}$ be the filtration of $\mathcal{I}$ w.r.t. Γ.

(**Case** atomic symbol) Let $C = A \in \Gamma$ be an atomic concept $A \in N_C$ or $C = \bot$. ($\Rightarrow$) Suppose that $\mathcal{I}; x \models C$. Then, $C \in T(x)$ for entity $x_{\equiv}$ and by Def. 4.2.9 it follows that $\mathcal{I}|_{\Gamma}; (T(x), Z) \models C$.

($\Leftarrow$) Assume that $\mathcal{I}|_{\Gamma}; (T(x), Z) \models C$. Then, by definition of filtration $x \in C^{\mathcal{I}}$ and therefore $\mathcal{I}; x \models C$, as desired.

(**Case** $C \sqcap D$) ($\Rightarrow$) Suppose that $\mathcal{I}; x \models C \sqcap D$, *i.e.*, $x \in C^{\mathcal{I}}$ and $x \in D^{\mathcal{I}}$. As Γ is subformula closed, $C, D \in \Gamma$, thus by the induction hypothesis follows that $(T(x), Z) \in C^{\mathcal{I}}|_{\Gamma} \cap D^{\mathcal{I}}|_{\Gamma}$. Therefore, $\mathcal{I}|_{\Gamma}; (T(x), Z) \models C \sqcap D$.

($\Leftarrow$) Let us assume that $\mathcal{I}|_{\Gamma}; (T(x), Z) \models C \sqcap D$, *i.e.*, it holds that $\mathcal{I}|_{\Gamma}; (T(x), Z) \models C$ and $\mathcal{I}|_{\Gamma}; (T(x), Z) \models D$. As $C, D \in \Gamma$, it follows by the inductive hypothesis that $\mathcal{I}; x \models C$ and $\mathcal{I}; x \models D$. Hence, $\mathcal{I}; x \models C \sqcap D$.

(**Case** $C \sqcup D$) Analogously to the previous case (**Case** $C \sqcap D$) by inductive hypothesis.

(**Case** $C \supset D$) ($\Rightarrow$) Suppose that $\mathcal{I}; x \models C \supset D$. This time the goal is to show that $\mathcal{I}|_{\Gamma}; (T(x), Z) \models C \supset D$. Take an arbitrary $y \in \Delta^{\mathcal{I}}$ and arbitrary $Z' = \{Z'_{R_1}, \ldots, Z'_{R_n}\}$ such that $(T(y), Z')$ is a Γ-theory in $\mathcal{I}$, $(T(x), Z) \preceq^{\mathcal{I}}|_{\Gamma}(T(y), Z')$ and $\mathcal{I}|_{\Gamma}; (T(y), Z') \models C$. Since Γ is subformula closed, $C, D \in \Gamma$ as well. Applying the induction hypothesis yields $\mathcal{I}; y \models C$. By definition of filtration it holds that $\mathcal{I}; y \models C \supset D$, because the implication $C \supset D$ is contained in $T(x)$ and $T(x) \subseteq T(y)$. This lets us now conclude that $\mathcal{I}; y \models D$, as well. Then, the inductive hypothesis implies that $\mathcal{I}|_\Gamma; (T(y), Z') \models D$. Since $(T(y), Z')$ was an arbitrary refinement of $(T(x), Z)$, it follows that $\mathcal{I}|_\Gamma; (T(x), Z) \models C \supset D$.

($\Leftarrow$) In the converse direction let us suppose that $\mathcal{I}|_\Gamma; (T(x), Z) \models C \supset D$. This means for every refinement of $(T(x), Z)$ that it is in $D^{\mathcal{I}}|_\Gamma$ if it is in $C^{\mathcal{I}}|_\Gamma$. Take an arbitrary $y \in \Delta^{\mathcal{I}}$ such that $x \preceq^{\mathcal{I}} y$ and assume that $\mathcal{I}; y \models C$. Then, it follows from the definition of filtration that $(T(x), Z) \preceq^{\mathcal{I}}|_\Gamma\, y_\equiv$. Again, since Γ is subformula closed it follows that $C, D \in \Gamma$. Now, we can apply the induction hypothesis, which yields that $\mathcal{I}|_\Gamma; y_\equiv \models C$ and from the assumption follows that $\mathcal{I}|_\Gamma; y_\equiv \models D$. Then, we can conclude from the induction hypothesis that $\mathcal{I}; y \models D$ and, because y was an arbitrary refinement of x we can conclude that $\mathcal{I}; x \models C \supset D$.

(**Case** $\exists R.C$) ($\Rightarrow$) Let us assume that $\mathcal{I}; x \models \exists R.C$, *i.e.*, for all refinements of x exists an R-successor in $C^{\mathcal{I}}$. We need to show that $\mathcal{I}|_\Gamma; (T(x), Z) \models \exists R.C$. Take an arbitrary $y \in \Delta^{\mathcal{I}}$ such that $(T(x), Z) \preceq^{\mathcal{I}}|_\Gamma (T(y), Z')$ according to the definition of filtration. This implies in the filtrated interpretation that $\exists R.C \in T(x)$. By the definition of filtration it follows that $\exists R.C \in T(y)$, as well, due to $T(x) \subseteq T(y)$. The assumption lets us conclude the existence of an R-successor z of y such that $z \in C^{\mathcal{I}}$. This implies that $(T(y), Z')\, R^{\mathcal{I}}|_\Gamma\, z_\equiv$. As Γ is subformula closed, $C \in \Gamma$, we can apply the induction hypothesis to obtain $z_\equiv \in C^{\mathcal{I}}|_\Gamma$. Hence, $\mathcal{I}|_\Gamma; (T(x), Z) \models \exists R.C$.

($\Leftarrow$) In the other direction let us assume that $\mathcal{I}|_\Gamma; (T(x), Z) \models \exists R.C$. The goal is $\mathcal{I}; x \models \exists R.C$. Let $y \in \Delta^{\mathcal{I}}$ be arbitrary such that $x \preceq^{\mathcal{I}} y$. Then, by definition of filtration it also holds that $(T(x), Z) \preceq^{\mathcal{I}}|_\Gamma\, y_\equiv$ and according to the assumption choose an R-successor $(T(z), Z') \in \Delta^{\mathcal{I}}|_\Gamma$ of $y_\equiv$ such that $y_\equiv\, R^{\mathcal{I}}|_\Gamma (T(z), Z')$ with $\mathcal{I}|_\Gamma; (T(z), Z') \models C$. Since $\mathcal{I}|_\Gamma; (T(z), Z') \models C$, it holds that $C \in T(z)$ by induction hypothesis ($C \in \Gamma$) and because $T(z) \cap F(y)_R = \emptyset$ it follows that $C \notin F(y)_R$. We can conclude by Def. 4.2.9 that there exists an entity $z' \in \Delta^{\mathcal{I}}$ such that $y\, R\, z'$ and $\mathcal{I}; z' \models C$. Hence, $\mathcal{I}; x \models \exists R.C$.

(**Case** $\forall R.C$) ($\Rightarrow$) Suppose that $\mathcal{I}; x \models \forall R.C$, *i.e.*, for all $y \in \Delta^{\mathcal{I}}$, $x \preceq^{\mathcal{I}} y$ implies for all $z \in \Delta^{\mathcal{I}}$ that if $y\, R\, z$ then $z \in C^{\mathcal{I}}$. The goal is $\mathcal{I}|_\Gamma; (T(x), Z) \models \forall R.C$. Let $(T(y), Z'), (T(z), Z'')$ be such that $(T(x), Z) \preceq^{\mathcal{I}}|_\Gamma (T(y), Z')\, R^{\mathcal{I}}|_\Gamma (T(z), Z'')$. It holds that $T(x) \subseteq T(y)$, $\forall R.C \in T(x)$, $Z'_R \cap T(z) = \emptyset$ and $\forall R^{-1}\, T(y) \subseteq T(z)$. Then, $\forall R.C \in T(y)$ which implies that $C \in T(z)$ and by Definition 4.2.9 this means that $\mathcal{I}; z \models C$. As $C \in \Gamma$, we can infer by the inductive hypothesis that $\mathcal{I}|_\Gamma; (T(z), Z'') \models C$. Hence, $\mathcal{I}|_\Gamma; (T(x), Z) \models \forall R.C$.

($\Leftarrow$) Proof by contraposition. Assume that $\mathcal{I}; x \not\models \forall R.C$. We need to show that $\mathcal{I}|_\Gamma; (T(x), Z) \not\models \forall R.C$. The assumption implies that there exist entities $y, z \in \Delta^\mathcal{I}$ such that $x \preceq^\mathcal{I} y\, R^\mathcal{I}\, z$ and $\mathcal{I}; z \not\models C$. It follows directly from Definition 4.2.9 that $(T(x), Z) \preceq^\mathcal{I}|_\Gamma\, y_\equiv\, R^\mathcal{I}|_\Gamma\, z_\equiv$. Since Γ is subformula closed, $C \in \Gamma$ as well, and the ind. hyp. lets us conclude that $\mathcal{I}|_\Gamma; z_\equiv \not\models C$. Hence, $\mathcal{I}|_\Gamma; (T(x), Z) \not\models \forall R.C$. ☐

Proposition 4.2.4 (Finite filtration). *Let Γ be a finite subformula closed set of cALC concepts. For any cALC interpretation $\mathcal{I}$, if $\mathcal{I}|_\Gamma = (\Delta^\mathcal{I}|_\Gamma, \preceq^\mathcal{I}|_\Gamma, \perp^\mathcal{I}|_\Gamma, \cdot^\mathcal{I}|_\Gamma)$ is the filtration of $\mathcal{I}$ w.r.t. Γ, then $\Delta^\mathcal{I}|_\Gamma$ is finite.* ∇

Proof. The set $\Delta^\mathcal{I}|_\Gamma$ consists of elements of the form $(T(x), Z)$ such that $Z_R \subseteq F(x)_R$ for all $R \in N_R$. But there are at most $(2^{|\Gamma|})^{(|N_R|+1)}$ of such possible Γ-theories. ☐

Theorem 4.2.2 (Finite model property).

$$\models C \textit{ if and only if } \mathcal{I} \models C \textit{ for all finite cALC-interpretations } \mathcal{I}. \qquad \nabla$$

Proof. ($\Rightarrow$) This direction is obvious, because $\models C$ implies that C is valid in all interpretations and in particular the finite ones.

($\Leftarrow$) In the other direction let us suppose that $\not\models C$ for some concept C. This implies the existence of a countermodel $\mathcal{I} = (\Delta^\mathcal{I}, \perp^\mathcal{I}, \preceq^\mathcal{I}, \cdot^\mathcal{I})$ and an entity a in $\Delta^\mathcal{I}$ such that $\mathcal{I}; a \not\models C$. Now, take the filtration $\mathcal{I}|_\Gamma$ of $\mathcal{I}$ w.r.t. $\Gamma = Sfc(C)$ according to Def. 4.2.9. Since satisfiability is preserved under filtration (Thm. 4.2.1), it follows that $\mathcal{I}|_\Gamma; a_\equiv \not\models C$ and Prop. 4.2.4 implies finiteness of $\mathcal{I}|_\Gamma$. Hence, we found a finite model which refutes concept C. ☐

It follows that if a *cALC* concept C is satisfiable then it is satisfiable in a finite interpretation and in particular, C is satisfiable in a finite interpretation whose size is exponentially bound by the size of $Sfc(C)$ and the number of roles in N_R. It is noteworthy, that (i) the filtration given in Definition 4.2.9 is not the smallest possible filtration, and (ii) while *cALC* enjoys the finite model property, its models possibly involve cycles. We will further discuss these points in Ex. 5.2.3, in particular, the example will show that a model, which satisfies the concept $\exists R.(C \sqcup D) \sqcap \neg\exists R.C \sqcap \neg\exists R.D$ is either finite and cyclic, or acyclic and infinite. It seems that the finite models, which refute the axiom FS4/IK4 of distribution of $\exists$ over disjunction $\sqcup$ are characterised by cyclic, oscillating refinement relations, and if these cycles are collapsed, then the axiom FS4/IK4 becomes valid.

As conjectured by Mendler and de Paiva [188] the finite model property can be obtained also by embedding *cALC* into a classical bimodal logic (the fusion S4 $\otimes$ K) exploiting general results from the field of many-dimensional modal logics [103]. This question will be addressed in Chapter 6 by embedding $c\mathcal{ALC}$ into $\mathcal{ALC}$ with reflexive and transitive roles which corresponds to the bimodal logic $\mathsf{S4} \otimes \mathsf{K}_m$. It is pointed out that ‘ [...] if we require $\preceq$ to be antisymmetric, then the finite model property is lost [...]’ [188, p. 9], which can be observed from the construction of the filtration model (see also Ex. 5.2.3). We will also present a proof of the finite model property and the decidability of $c\mathcal{ALC}$ based on a sequent calculus in Sec. 5.2.2.

4.3 Summary

The aim of this work is to show that Def. 4.2.2 induces a well-behaved logic, called $c\mathcal{ALC}$, which uses the same syntactical representation as classical $\mathcal{ALC}$, but is semantically more expressive and facilitates the DL-style reasoning tasks w.r.t. TBoxes.

In this chapter we have introduced the logic $c\mathcal{ALC}$ that is related to the constructive modal logic CK [27; 188; 272] as $\mathcal{ALC}$ is related to the classical modal system K [229]. Analogously to the correspondence between $\mathcal{ALC}$ and K_m by Schild [245], one can easily show that $c\mathcal{ALC}$ is a notational variant of CK_n. The semantics of $c\mathcal{ALC}$ is birelational and derived from the monomodal constructive logic CK [188], which is based on a constructive modal logic that was firstly introduced by Wijesekera [272]. The system $c\mathcal{ALC}$ support TBox theories, and its constructive interpretation gives rise to the notion of constructive satisfiability and a strengthened form of the classical OWA, called *evolving open world assumption*. However, the birelational semantics rules out the classical interpretation of individual names from ABoxes. We want to highlight that the semantic dimension along which refinement takes place is implicit in $c\mathcal{ALC}$ and not hard-coded into the syntax, which allows several different notions of context generically in the language of the basic description logic $\mathcal{ALC}$. This context-dependency is integrated into the notion of truth rather than the terminology like in other work on special cases of context such as temporal DLs [10; 11; 38]. We demonstrated that in $c\mathcal{ALC}$ the classical principles of the Excluded Middle $C \sqcup \neg C \equiv \top$, double negation $\neg\neg C \equiv C$, the dualities $\exists R.C \equiv \neg\forall R.\neg C$, $\forall R.C \equiv \neg\exists R.\neg C$ and disjunctive distribution $\exists R.(C \sqcup D) \equiv \exists R.C \sqcup \exists R.D$ are no longer tautologies. Instead, they correspond to non-trivial TBox axioms, which can axiomatise specialised classes of application scenarios. Furthermore, the extension by fallible entities [90] rejects the nullary variant of the principle of disjunctive distribution $\neg\exists R.\bot$. We have given proofs for the monotonicity property, the disjunction property, and the finite model property relying on the filtration technique. In this chapter, several examples illustrated situations in which the classical interpretation of $\mathcal{ALC}$ is not adequate. Moreover, we exemplified the requirements of the evolving open world assumption and suggested applications in the domain of auditing, to use $c\mathcal{ALC}$ as inference mechanism as well as typing system for data streams. In summary, $c\mathcal{ALC}$ is non-normal w.r.t. the possibility modality $\exists R$ ($\Diamond$)[103; 249]. The logic $c\mathcal{ALC}$ is meant for applications where we must be robust for several implicit notions of context-dependency but do not need to reason explicitly about some specific refinement.

Notes on Related Work

Constructive DLs The Kripke semantics of two variants of intuitionistic DLs ($\mathcal{KALC}$ and $I\mathcal{ALC}$) as introduced in [39; 64] are obtained from the standard intuitionistic interpretation as proposed by de Paiva [78]. However, the system by Clément [64] does not allow $\supset$ as a concept-forming operator and focusses mainly on a proof theoretic perspective. Also, the intuitionistic DL introduced in [41] is proof-theoretic, but addresses the extraction of information terms inspired by [201].

The intuitionistic DL $\mathcal{KALC}$ [39; 43; 45] is derived from intuitionistic first-order logic via the standard translation following de Paiva's proposal for $I\mathcal{ALC}$, but it differs in that the semantics is restricted to finite domains, and by assuming the Kuroda principle $\forall R.\neg\neg C \supset \neg\neg\forall R.C$. Also, it is conjectured that the Kuroda principle implies the finite model property for $\mathcal{KALC}^{\infty}$ [39; 270], which is a variant of $\mathcal{KALC}$ not restricted to finite models. The question arises whether the restriction of $\mathcal{KALC}$ to finite posets and $\mathcal{KALC}^{\infty}$ to final worlds limits or maybe excludes the ability to express dynamic systems, which possibly never reach a final or stable state. Such oscillating processes are commonly found in mechanical or dynamic systems, for instance, just consider the cardiac cycle of the human heart, cycles between economic growth and stagnation in business, or cyclical oscillations of global or regional climate. Villa [270, pp. 6–11] stresses that $I\mathcal{ALC}$ does not provide an adequate semantics for the DL domain, since $I\mathcal{ALC}$ does not satisfy the finite model property. However, since the finite model property and decidability for $\mathcal{KALC}^{\infty}$ are still open problems, it is unclear whether the semantics of $\mathcal{KALC}$ solves the key problems of $I\mathcal{ALC}$, particularly to obtain decidable decision procedures.

Let us discuss the intuitionistic variant of $\mathcal{ALC}$ called $i\mathcal{ALC}$ by de Paiva, Haeusler and Rademaker [77; 123–126]. The authors follow the IK approach of Simpson [249]. However, the proposed semantics in [77, p. 24; 126, p. 7; 125, p. 3] for $i\mathcal{ALC}$ interprets the existential restriction intuitionistically according to the semantics of $c\mathcal{ALC}$ (and CK) and does not require the usual frame conditions from the system IK (*cf.* p. 38), *i.e.*, the interpretation of $\exists R.C$ is defined as

$$(\exists R.C)^{\mathcal{I}} \quad =_{df} \quad \{x \mid \forall y.\, x \preceq y \Rightarrow \exists z.\, y\, R^{\mathcal{I}}\, z \ \ \& \ \ z \in C^{\mathcal{I}}\}. \tag{4.4}$$

This Kripke semantics corresponds to the infallible variant of the semantics for $c\mathcal{ALC}$ and one can easily give a counterexample for axiom FS4/IK4 (*cf.* Fig. 4.2 on page 70), which contradicts their claim that the Hilbert axiomatisation for $i\mathcal{ALC}$, which is the same as that of IK including axiom FS4/IK4 (see p. 34), is sound and complete w.r.t. the proposed Kripke semantics for $i\mathcal{ALC}$. Furthermore, the authors are claiming to draw a connection between constructive hybrid logic and $i\mathcal{ALC}$ in [77]. Yet, [77] does not extend $i\mathcal{ALC}$ by nominals and hybrid logical satisfaction operators, but rather uses the term 'Hybrid-Style' to denote a labelled sequent calculus. A recent work [127] corrects the Kripke semantics for $i\mathcal{ALC}$, by interpreting $\exists R.C$ classically (*cf.* (2.2)) and requiring the frame conditions of IK (*cf.* p. 38).

IMLs The rejection of the principle of disjunctive distribution as discussed in Sec. 4.1 has been previously discussed in the literature in [4; 90; 160–162; 187; 188; 191; 194–196], in particular, the rejection of the unary distribution law FS3/IK3 = $\exists R.\bot \supset \bot$ based on the notion of explicit fallible worlds has been investigated first in [4; 90; 188]. Wijesekera [272] investigated Kripke semantics of a system similar to $c\mathcal{ALC}$, refuting the binary distribution, but satisfying the nullary one. Note that the related system CK and its extension CS4 also possess a proof-theoretic interpretation in the form of a categorical semantics[4; 27; 80]. In [4; 188] the requirement that R is serial on $\bot$, *i.e.*, if $x \in \bot$ then there exists y such that xRy and $y \in \bot$, is missing in the definition of the Kripke semantics.

Fallible and infallible Kripke semantics, and neighbourhood semantics for CK and a translation between the two semantics has been investigated in [160; 161]. Kojima introduces an intuitionistic linear-time temporal logic inspired by CK, investigating its semantics and proof theory in terms of a Hilbert and Gentzen sequent calculus [161; 162]. This system is based on intuitionistic (partially) functional[11] Kripke frames and rejects the principle of disjunctive distribution w.r.t. the temporal *next* modality, which is interpreted type-theoretically as a type of quoted code. The rejection of the distributivity law is motivated by type-theoretic means relying on the notion of *external* and *internal observers* of Kripke worlds [160–162] by stating that '[...] Kripke semantics for type-theoretically motivated modal logic [...]' necessarily needs to '[...] emulate internal observers states of knowledge in terms of possible worlds and accessibility relations[...] ' [160, p. 17]. In this view, the usual Kripke semantics of standard modal logics (and DLs) assume an ideal *external* observer with global knowledge in the sense that it observes the global state of the system (its possible worlds) from the outside. In contrast, the type-theoretic view considers observers *inside* the system, *i.e.*, each

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[11] The term *functional* Kripke frame denotes that the accessibility relation on possible worlds is a (partial) function [162].

observer's view is restricted to a fixed state of knowledge (possible world) once and for all. Then, a observer can know at state x that 'A or B is true at some state y' without having explicit knowledge whether 'A is true at y' or 'B is true at y'.

In contrast to [54], the system $c\mathcal{ALC}$ does consider only one dimension of refinement but in a more general sense. Brunet [54] presents an intuitionistic epistemic logic based on several refinement relations coding multiple *(partial) points of view.* The special feature of our refinement ordering $\preceq$ is that it may have cycles and fallible descriptions, *i.e.*, 'oscillations' and 'deadlocks' which are specific to real-world abstractions as we have seen in the examples (Ex. 4.2.2, 4.2.5 and 4.2.6) given above.

Lax Logic Propositional lax logic (PLL) [90; 91] is an interesting extension of CK with applications in the field of hardware verification. It is a mono-modal extension of intuitionistic propositional logic by the single modality $\bigcirc$ and has been studied as an extension of CS4 by the axiom $C \supset \Box C$ [4]. Regarding the semantics of PLL, it uses birelational semantics containing fallible entities and refuting the binary and nullary distribution of diamond over disjunction ($\mathsf{FS4}/\mathsf{IK4}, \mathsf{FS3}/\mathsf{IK3}$) just like $c\mathcal{ALC}$ does. Following the discussion on the relation of PLL to CS4 in [4], the system PLL can be viewed as an extension of the monomodal fragment of $c\mathcal{ALC}$ by the axioms $\mathsf{T}, \mathsf{4}$, $C \supset \forall R.C$ and the frame condition $R\,;\preceq\ \subseteq\ \preceq;R$.

Hybrid Logic As discussed in [16, Chap. 4.2.2], there is also a strong correspondence between classical DLs and hybrid logic. Intuitionistic hybrid logic was introduced in [49; 50; 52]. Its Kripke semantics is derived from standard intuitionistic logic via the standard translation [49, Chap. 8] and the separation of the *hybrid-logical machinery* from the intuitionistic partial order has been highlighted as a characteristic feature of this logic. Since the translation between hybrid logic and DLs also covers ABox assertions[12], it may be a promising candidate to investigate constructive DLs based on hybrid logics.

Many-valued DLs Our work is also to be distinguished from many-valued DLs (see [181; 224]), which are finitely valued while $c\mathcal{ALC}$ is infinitely valued and from fuzzy DLs (see [87; 144; 255]), which use a quantitative notion of approximate truth.

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[12] This is in contrast to the translation between K_m and $\mathcal{ALC}$ which can only cover the TBox machinery (see Section 2.1.5).

CHAPTER 5

Constructive Proof Systems for $c\mathcal{ALC}$

This chapter presents Hilbert and Gentzen-style deduction systems for $c\mathcal{ALC}$ that admit a direct interpretation of proofs as computations following the Curry-Howard isomorphism. In Section 5.1, we introduce a proof theoretic presentation based on a set of modal axiom schemata and rules for $c\mathcal{ALC}$ in the form of a sound and complete Hilbert axiomatisation, and discuss several meta-theorems, which are being used in its completeness proof. The section finishes with the presentation of a modal deduction theorem w.r.t. global and local premises. Section 5.2 proceeds by presenting the multi-conclusion Gentzen-style sequent calculus `G1` for $c\mathcal{ALC}$, which is sound and complete w.r.t. the Kripke semantics presented in Chap. 4, and enjoys the finite model property as well as decidability. Soundness and completeness of the Hilbert system is demonstrated by showing that the Hilbert system is equivalent to the sequent calculus `G1`. Section 5.3 discusses intermediate systems that arise between $c\mathcal{ALC}$ and $\mathcal{ALC}$ from the extension of $c\mathcal{ALC}$ by classical axioms, and provides sound and complete extensions of the Hilbert and Gentzen calculi.

5.1 Hilbert-style Axiomatisation

5.1.1 Hilbert Calculus for $c\mathcal{ALC}$

Hilbert-style proof systems as introduced by David Hilbert [135] are formal deductive systems. Conceptually, they are very simple, *i.e.*, they consist of a usually large collection of axiomatic schemata and only very few inference rules. In Hilbert systems a proof of a formula proceeds by finding its derivation, starting from appropriate substitution instantiations of axiom schemata and by applying the rules of inference to the latter. This process yields a sequence of formulæ. The following definition formalizes $c\mathcal{ALC}$ in terms of a Hilbert-style axiomatisation by defining its axioms and inference rules.

Definition 5.1.1 (Axioms [195; 196])**.** The axioms for $c\mathcal{ALC}$ (CK_m) consist of

a) all substitution instances of theorems of IPC

$$\begin{aligned}
&\mathsf{IPC1}: && C \supset (D \supset C)\\
&\mathsf{IPC2}: && (C \supset (D \supset E)) \supset ((C \supset D) \supset (C \supset E))\\
&\mathsf{IPC3}: && C \supset (D \supset (C \sqcap D))\\
&\mathsf{IPC4}: && (C \sqcap D) \supset C, \quad (C \sqcap D) \supset D\\
&\mathsf{IPC5}: && C \supset (C \sqcup D), \quad D \supset (C \sqcup D)\\
&\mathsf{IPC6}: && (C \supset E) \supset ((D \supset E) \supset ((C \sqcup D) \supset E))\\
&\mathsf{IPC7}: && \bot \supset C
\end{aligned}$$

b) and the axiom schemata

$$\begin{aligned}
&\mathsf{K}_{\forall R}: && \forall R.(C \supset D) \supset (\forall R.C \supset \forall R.D)\\
&\mathsf{K}_{\exists R}: && \forall R.(C \supset D) \supset (\exists R.C \supset \exists R.D)
\end{aligned}$$

where C and D are concept descriptions and R is a role. ∇

Definition 5.1.2 (Rules of Inference [195; 196])**.** Let C and D be concept descriptions. The system $c\mathcal{ALC}$ consists of the two rules of inference of *modus ponens* (MP) and *necessitation* (Nec) for each role $R \in N_R$:

$$\frac{C \supset D \qquad C}{D}\,\mathsf{MP} \qquad\qquad \frac{C}{\forall R.C}\,\mathsf{Nec}$$ ∇

The definition of the Hilbert calculus for $c\mathcal{ALC}$ [195; 196] is given by the usual axiomatisation of the *intuitionistic propositional calculus* (IPC) [268] (*cf.* Sec. 2.2) and the two extensionality principles $\mathsf{K}_{\forall R}, \mathsf{K}_{\exists R}$. The axioms characterise the individual logical connectives, *i.e.*, looking at the axioms of Definition 5.1.1 part a) in detail, s 1–2 cover implication $\supset$, 3–4 are for intersection $\sqcap$, 5–6 for disjunction $\sqcup$ and 7 for inconsistency $\bot$. Part b) depicts the two *extensionality principles* $\mathsf{K}_{\forall R}$, $\mathsf{K}_{\exists R}$ which handle universal and existential quantification (role filling) respectively, and are based on the modal rules of Wijesekera [272]. These stem from generalised monotonicity depicted by the rules M_1, M_2 below. An important property of the rules is that the context Γ is universally quantified in the conclusion of each rule.

$$\frac{\Gamma \vdash C \supset D}{\forall R.\Gamma \vdash \forall R.C \supset \forall R.D}\,M_1 \qquad \frac{\Gamma \vdash C \supset D}{\forall R.\Gamma \vdash \exists R.C \supset \exists R.D}\,M_2$$

The rules of *modus ponens* (MP) and *necessitation* (Nec) are given by Def. 5.1.2. Note that negation $\neg C$ is encoded in the usual way as $C \supset \bot$ and $\top$ as $\bot \supset \bot$. In the next step, we will define the notion of Hilbert derivation. When defining the notion of derivability from assumptions in an axiomatic Hilbert system, one can distinguish between global and local assumptions as pointed out by Fitting [101], Mendler and de Paiva [188] and Popkorn [229]. According to Fitting [102] the former are global assumptions that can be understood as logical truth to which the rule of necessitation applies, while the latter are local premises that correspond to contingent truth to which the necessitation rule does not apply. This notion of deduction is in line with description logics where we have global model assumptions represented by a TBox and local assumptions at the entity (or world) level.

Definition 5.1.3 (Hilbert deduction for $c\mathcal{ALC}$ [102; 195; 196; 229])**.** Let $c\mathcal{ALC}_{ax}$ be the set of axiom schemata of $c\mathcal{ALC}$ closed under substitution, Θ and Γ be sets of concepts (not schemata) and C a single concept (not a schema). We write $\Theta\,;\Gamma \models_{\mathsf{H}} C$ to denote that C is *deducible* from the set Θ of global premises and the set Γ of local premises, *i.e.*, there exists a finite sequence of concepts $C_0, C_1, \ldots, C_n$ such that $C_n = C$ which consists of a global part coming first, followed by a local part at the end. Let $G \uplus L = [0, n]$. In the global part, for all $i, k, j \in G$ one of the following holds:

- The concept C_i is either a global hypothesis, *i.e.*, a member of Θ.
- The concept C_i is a member of $c\mathcal{ALC}_{ax}$.
- The concept C_i arises from two earlier derivable concept descriptions by MP, *i.e.*, there are concepts C_j, C_k $(j, k < i)$ with $C_j = C_k \supset C_i$.
- The concept C_i is obtained by rule Nec from an earlier concept description C_j $(j < i)$ with $C_i = \forall R.C_j$, for some role $R \in N_R$.

In the local part, for all $i \in L$ the concept C_i is either a member of Γ (local hypothesis) or follows from two earlier concept descriptions C_j, C_k $(i, k \in L$ and $j, k < i)$ by rule MP (as in the global part). Rule Nec is not allowed in the local part. ∇

In other words, the rule of necessitation can be applied only to derivations with global assumptions but not local ones [128]. This restriction is important and allows us to state a deduction theorem w.r.t. global and local premises as shown later. Taking Def. 5.1.3 into account and by following the presentation of [128, p. 9], we obtain the proof system depicted below. In contrast to [128] it considers global and local premises:

Proposition 5.1.1. *Definition 5.1.3 induces the proof system, which is sound and complete, and consists of the following admissible rules:*

$$\frac{C \in \Theta \cup \Gamma}{\Theta\,;\Gamma \models_{\mathsf{H}} C} \qquad \frac{C \in c\mathcal{ALC}_{ax}}{\Theta\,;\Gamma \models_{\mathsf{H}} C}$$

$$\frac{\Theta' \,;\, \Gamma'' \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} C \supset D \qquad \Theta'' \,;\, \Gamma' \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} C}{\Theta',\, \Theta'' \,;\, \Gamma',\, \Gamma'' \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} D}\,MP \qquad \frac{\Theta \,;\, \emptyset \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} C \qquad R \in N_R}{\Theta \,;\, \Gamma \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} \forall R.C}\,Nec$$

Note that Θ *(*Θ'*,* Θ''*) is a global set of assumptions corresponding to the TBox.* ∇

Proof. By induction on the structure of a derivation. □

Proposition 5.1.2 (Monotonicity of derivations). *If* $\Theta \,;\, \Gamma \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} C$ *then* $\Theta' \,;\, \Gamma' \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} C$*, for any sets of concepts* Θ' *and* Γ' *with* $\Theta \subseteq \Theta'$ *and* $\Gamma \subseteq \Gamma'$*.* ∇

Proof. By induction on the structure of a derivation. □

Lemma 5.1.1 (Compactness). *Let* Θ *and* Γ *be sets of concepts and* C *an arbitrary concept description. If* $\Theta \,;\, \Gamma \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} C$ *then there exist finite sets* $\Theta_f \subseteq \Theta$ *and* $\Gamma_f \subseteq \Gamma$ *such that* $\Theta_f \,;\, \Gamma_f \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} C$*.* ∇

Proof. Suppose that $\Theta \,;\, \Gamma \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} C$ is canonically extended for arbitrary and possibly infinite sets of concepts Θ and Γ. By Def. 5.1.3 there is a finite sequence of concepts that ends in C. But this implies that only a finite number of global and local assumptions can be involved in the derivation of C. Therefore, there exist finite sets $\Theta_f \subseteq \Theta$ and $\Gamma_f \subseteq \Gamma$ such that $\Theta_f \,;\, \Gamma_f \mathrel{|\!\!\frac{}{\,\mathsf{H}\,}} C$. □

Notation. As usually, implication $\supset$ is right-associative, *i.e.*, $C \supset D \supset E$ denotes $C \supset (D \supset E)$. Accordingly, we will sometimes omit parentheses to achieve a better readability in Hilbert derivations.

In the following, we will adapt the list notation from [187, p. 44] to represent finite sequences of implications of the form

$$C_1 \supset C_2 \supset C_3 \supset \ldots \supset C_n \supset D.$$

Let l be a finite list of concepts. A concept expression of the form D^l is defined by

$$D^{[\,]} =_{df} D;$$
$$D^{C::l} =_{df} C \supset (D^l);$$

where $[\,]$ is the empty list, and $::$ denotes the list constructor which adds concept C to the beginning of the list l. For instance, the unfolding of $C^{[E_1,\, E_2,\, E_3]}$ yields the implication $E_1 \supset (E_2 \supset (E_3 \supset C))$. ■

There exist several admissible rules that are useful to shorten Hilbert derivations. Before going into details, let us inspect an exemplary Hilbert derivation, which considers universal and existential restrictions.

Example 5.1.1. We can show that Hilbert derives $(\forall R.C \sqcap \exists R.D) \supset \exists R.(C \sqcap D)$:

1. $C \supset (D \supset (C \sqcap D))$ IPC3;
2. $\forall R.(C \supset (D \supset (C \sqcap D)))$ from 1 by Nec;
3. $\forall R.C \supset \forall R.(D \supset (C \sqcap D))$ from $\mathsf{K}_{\forall R}$, 2 by MP;
4. $\forall R.(D \supset (C \sqcap D)) \supset (\exists R.D \supset \exists R.(C \sqcap D))$ by $\mathsf{K}_{\exists R}$;
5. $\forall R.C \supset \big(\forall R.(D \supset (C \sqcap D)) \supset (\exists R.D \supset \exists R.(C \sqcap D))\big)$ from IPC1, 4 by MP;
6. $\big(\forall R.C \supset \forall R.(D \supset (C \sqcap D))\big) \supset \big(\forall R.C \supset (\exists R.D \supset \exists R.(C \sqcap D))\big)$ from IPC2, 5 by MP;
7. $\forall R.C \supset (\exists R.D \supset \exists R.(C \sqcap D))$ from 6, 3 by MP.

Then, the goal $\vdash_{\mathsf{H}} (\forall R.C \sqcap \exists R.D) \supset \exists R.(C \sqcap D)$ follows from 7 by the admissible rule of *currying*, stating that if $\vdash_{\mathsf{H}} C_1 \supset (C_2 \supset D)$ then also $\vdash_{\mathsf{H}} (C_1 \sqcap C_2) \supset D$. The admissible rules including currying will be discussed in Lem. 5.1.2 and 5.1.3. ■

The next two lemmas state several theorems and meta-rules, which will be used later in the proof of the completeness of the Hilbert system for $c\mathcal{ALC}$ (see Prop. 5.2.1).

Lemma 5.1.2. *For all concepts C, D, E, F and $C_1, C_2, \ldots, C_n$ of $c\mathcal{ALC}$:*

(I) Identity $\vdash_{\mathsf{H}} C \supset C$ *is derivable in Hilbert. Note that identity corresponds to the closed combinator term* SKK *in combinatory logic [140] where* K *stands for* IPC1 *and* S *for* IPC2 *respectively.*

($\mathsf{B_n}$) Function composition *(possibly nested) is derivable as a closed combinator in Hilbert,* i.e., $\vdash_{\mathsf{H}} (D \supset E) \supset \big((C \supset D)^{[C_1, C_2, \ldots, C_n]}\big) \supset \big((C \supset E)^{[C_1, C_2, \ldots, C_n]}\big)$ *for* $n \geq 0$. *This theorem generalises the* B*-combinator of combinatory logic.*

($\mathsf{K_n}$) *The generalisation* $\vdash_{\mathsf{H}} D \supset D^{[C_1, C_2, \ldots, C_n]}$ *of* IPC1 *is derivable for* $n \geq 0$.

($\mathsf{S_n}$) *Hilbert derives* $\vdash_{\mathsf{H}} (E \supset (C_1 \supset C_2 \supset \ldots \supset C_n)) \supset (E \supset C_1) \supset (E \supset C_2) \supset \ldots \supset (E \supset (C)_k)$, *which generalises axiom* IPC2 *for* $2 \leq k < n$, *where* $(C)_k$ *is the subconcept in* $C =_{df} C_1 \supset C_2 \supset \ldots \supset C_n$ *at depth* k. *It is defined as follows: Let* $D =_{df} D_1 \supset D_2 \supset \ldots \supset D_j$. *The concept* $(D)_k$ *stands for the subconcept of* D *at depth* $0 \leq k < j$ *w.r.t. implication* $\supset$ *that is given by the following definition*

$$\begin{aligned} (D)_0 &=_{df} D; \\ (D \supset F)_k &=_{df} (F)_{k-1}; \\ (D)_k &=_{df} D, \text{ if } D \text{ is not an implication.} \end{aligned}$$

(FS1) *Hilbert derives* $\vdash_{\mathsf{H}} \top \equiv \forall R.\top$.

(IPC8) *Hilbert derives* $\vdash_{\mathsf{H}} (\bot \sqcup C) \supset C$.

(IPC9) *Hilbert derives* $\vdash_{\mathsf{H}} C \supset (\top \sqcap C)$. ∇

Proof. The proof is by demonstrating that each of Lem. 5.1.2 is a theorem of $c\mathcal{ALC}$. We will introduce abbreviations of complex formulæ on the fly to streamline the presentation of the more complex cases. Let C, D, E and $C_1, C_2, \ldots, C_n$ be $c\mathcal{ALC}$ concepts.

(I) The identity combinator (I) is derivable by the following sequence [*cf.* 140, p. 123]:

1. $\big(C \supset ((C \supset C) \supset C)\big) \supset (C \supset (C \supset C)) \supset (C \supset C)$ IPC2;
2. $(C \supset (C \supset C)) \supset (C \supset C)$ from 1, IPC1 by MP;
3. $C \supset C$ from 2, IPC1 by MP.

($\mathsf{B_n}$) It is well known [*cf.* 140, p. 123] that Hilbert derives $\vdash_{\mathsf{H}} (D \supset E) \supset (C \supset D) \supset C \supset E$ which is the B-combinator[13] in closed form. We can go one step further and derive the nested B_1-combinator (with an implicational prefix of depth 1) in closed form, *i.e.*, $\vdash_{\mathsf{H}} (D \supset E) \supset (F \supset C \supset D) \supset F \supset C \supset E$. Let $\varphi =_{df} (F \supset (C \supset D) \supset (C \supset E)) \supset (F \supset C \supset D) \supset (F \supset C \supset E)$, $\psi =_{df} (F \supset (C \supset D) \supset (C \supset E))$, $\vartheta =_{df} (F \supset C \supset D) \supset (F \supset C \supset E)$ and $\gamma =_{df} (D \supset E)$.

1. φ IPC2;
2. $\gamma \supset \varphi$ from IPC1, 1 by MP;
3. $(\gamma \supset (\psi \supset \vartheta)) \supset (\gamma \supset \psi) \supset (\gamma \supset \vartheta)$ IPC2;
4. $(\gamma \supset \psi) \supset (\gamma \supset \vartheta)$ from 3, 2 by MP;
5. $\gamma \supset \psi$?;
6. $\gamma \supset \vartheta$ from 4, 5 by MP.

Note that the proof of B_1 has exactly the same structure as the derivation of the standard B-combinator as shown in [140, p. 123]. The only missing piece left is to find a proof of derivation 5, *i.e.*, we need to show that Hilbert derives $\vdash_{\mathsf{H}} \gamma \supset \psi = (D \supset E) \supset (F \supset (C \supset D) \supset (C \supset E))$ which is as follows:

1. $(C \supset (D \supset E)) \supset (C \supset D) \supset (C \supset E)$ IPC2;
2. $(D \supset E) \supset (C \supset (D \supset E))$ IPC1;
3. $(D \supset E) \supset (C \supset D) \supset (C \supset E)$ from (B comb., 1 by MP), 2 by MP;
4. $((C \supset D) \supset (C \supset E)) \supset \big(F \supset (C \supset D) \supset (C \supset E)\big)$ IPC1;
5. $(D \supset E) \supset \big(F \supset (C \supset D) \supset (C \supset E)\big)$ from (B comb., 4 by MP), 3 by MP.

Now, we will generalise the discussion above by giving a derivation of the B_n-combinator with an arbitrary nesting of depth $n \geq 0$. We will show by induction on the size n of the list of concepts $[C_1,\ C_2,\ \ldots, C_n]$ that Hilbert derives

$$\vdash_{\mathsf{H}} (D \supset E) \supset \big((C \supset D)^{[C_1,C_2,\ldots,C_n]}\big) \supset \big((C \supset E)^{[C_1,C_2,\ldots,C_n]}\big). \tag{5.1}$$

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[13] The B-combinator is not to be confused with the axiom schema $\mathbf{B} = C \supset \Box\Diamond C$ after L. E. J. Brouwer.

The base case $n = 0$ is simply an instance of B. In the inductive step let $k \geq 0$. We show that if B_n holds for $n = k$ then B_n holds for $n = k+1$ as well. The structure of the proof of (5.1) equals the derivation of the B_1-combinator by taking

$$\begin{aligned}
&\varphi =_{\mathit{df}} \big(C_1 \supset ((C \supset D)^l) \supset ((C \supset E)^l)\big) \supset ((C \supset D)^{C_1::l}) \supset ((C \supset E)^{C_1::l}),\\
&\psi =_{\mathit{df}} \big(C_1 \supset ((C \supset D)^l) \supset ((C \supset E)^l)\big),\\
&\vartheta =_{\mathit{df}} ((C \supset D)^{C_1::l}) \supset ((C \supset E)^{C_1::l}),\\
&\gamma =_{\mathit{df}} (D \supset E),
\end{aligned}$$

where $l =_{\mathit{df}} [C_2,\ \ldots,\ C_{k+1}]$ is a list of concepts of length k. As for B_1, we need to give a proof of derivation 5, *i.e.*, this time we demonstrate how to obtain

$$\begin{aligned}
&\vdash_{\mathsf{H}} \gamma \supset \psi\\
&\quad = (D \supset E) \supset \big(C_1 \supset ((C \supset D)^l) \supset ((C \supset E)^l)\big).
\end{aligned}$$

This is the interesting part where the induction hypothesis comes into play:

1. $\gamma \supset \big(((C \supset D)^l) \supset ((C \supset E)^l)\big)$ by ind. hyp. (B_k);
2. $\big(((C \supset D)^l) \supset ((C \supset E)^l)\big) \supset \big(C_1 \supset ((C \supset D)^l) \supset ((C \supset E)^l)\big)$ by $\mathsf{IPC1}$;
   $= \big(((C \supset D)^l) \supset ((C \supset E)^l)\big) \supset \psi$
3. $\gamma \supset \psi$ from (B, 2 by MP), 1 by MP.

This completes the proof of ($\mathsf{B_n}$).

($\mathsf{K_n}$) One shows by induction on the size n of the list $[C_1,\ C_2,\ \ldots,\ C_n]$ that Hilbert derives $\vdash_{\mathsf{H}} D \supset D^{[C_1,\, C_2,\, \ldots,\, C_n]}$. The base case $n = 0$ is just an instance of the identity combinator (I). In the inductive case let $k \geq 0$. We need to show that ($\mathsf{K_n}$) for $n = k$ implies ($\mathsf{K_n}$) for $n = k+1$. Let $l =_{\mathit{df}} [C_2,\ \ldots,\ C_{k+1}]$:

1. $D \supset D^l$ by ind. hyp.;
2. $(D^l) \supset \big(C_1 \supset (D^l)\big)$ $\mathsf{IPC1}$;
3. $\big((D^l) \supset C_1 \supset (D^l)\big) \supset (D \supset D^l) \supset (D \supset C_1 \supset D^l)$ ($\mathsf{B_n}$).
4. $D \supset C_1 \supset (D^l) = D \supset D^{C_1::l}$ from (3, 2 by MP), 1 by MP.

($\mathsf{S_n}$) Next, we demonstrate that ($\mathsf{S_n}$) is admissible in Hilbert. This is proved by induction on $n \geq 2$. For $k = 2$ and $n \geq 2$ we have to show $\vdash_{\mathsf{H}} (E \supset (C_1 \supset \ldots \supset C_n)) \supset (E \supset C_1) \supset (E \supset (C_2 \supset \ldots \supset C_n))$ which is just an instance of $\mathsf{IPC2}$. In the inductive case $2 < k \leq n+1$ the goal is $\vdash_{\mathsf{H}} (E \supset (C_1 \supset C_2 \supset \ldots \supset C_{n+1})) \supset (E \supset C_1) \supset (E \supset C_2) \supset \ldots \supset (E \supset (C)_k)$ where $C =_{\mathit{df}} (E \supset (C_1 \supset C_2 \supset \ldots \supset C_{n+1}))$.

The latter goal is given by the following derivation where $l =_{df} [C_2, \dots, C_n]$:

1. $(E \supset (C_2 \supset \dots \supset C_{n+1})) \supset (E \supset C_2) \supset \dots \supset (E \supset (C)_k)$ by ind. hyp.;
$= (C_{n+1}^{E::l}) \supset \big((E \supset (C)_k)^{[(E \supset C_2), \dots, (E \supset C_{k-1})]}\big)$
2. $\big(E \supset (C_1 \supset (C_2 \supset \dots \supset C_{n+1}))\big) \supset \big((E \supset C_1) \supset (E \supset (C_2 \supset \dots \supset C_{n+1}))\big)$
$= (C_{n+1}^{E::(C_1::l)}) \supset \big((E \supset C_1) \supset (C_{n+1}^{E::l})\big)$ IPC2;
3. $(C_{n+1}^{E::(C_1::l)}) \supset \big((E \supset C_1) \supset ((E \supset (C)_k)^{[(E \supset C_2), \dots, (E \supset C_{k-1})]})\big)$
from (($\mathsf{B_n}$), 1 by MP), 2 by MP.

$$= (E \supset (C_1 \supset C_2 \supset \dots \supset C_{n+1})) \supset (E \supset C_1) \supset (E \supset C_2) \supset \dots \supset (E \supset (C)_k)$$

(FS1) The Hilbert derivation of $\vdash_{\mathsf{H}} \forall R.\top \supset \top$ follows from the fact that Hilbert derives $\top$ by IPC7 and from IPC1 by the rule MP. We remind the reader that $\top$ is an abbreviation for $\bot \supset \bot$.

The derivation of $\vdash_{\mathsf{H}} \top \supset \forall R.\top$ is as follows:

1. $\forall R.\top \supset (\top \supset \forall R.\top)$ by IPC1;
2. $\forall R.\top$ from IPC7 by Nec;
3. $\top \supset \forall R.\top$ from 1, 2 by MP.

(IPC8) The Hilbert derivation of $\vdash_{\mathsf{H}} (\bot \sqcup C) \supset C$ is as follows:

1. $\bot \supset C$ IPC7;
2. $C \supset C$ (I);
3. $(\bot \supset C) \supset (C \supset C) \supset ((\bot \sqcup C) \supset C)$ IPC6;
4. $(\bot \sqcup C) \supset C$ from (3, 1 by MP), 2 by MP.

(IPC9) The Hilbert derivation of $\vdash_{\mathsf{H}} C \supset (\top \sqcap C)$ is as follows:

1. $\top$ IPC7;
2. $\top \supset (C \supset (\top \sqcap C))$ IPC3;
3. $C \supset (\top \sqcap C)$ from 2, 1 by MP.

□

Lemma 5.1.3. *The following rules are admissible in the Hilbert system of c$\mathcal{ALC}$:*

(ARB) *If Hilbert derives $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D^{[C_1, C_2, \dots, C_n]}$ and $\Theta\,;\Gamma \vdash_{\mathsf{H}} D \supset E$ then also $\vdash_{\mathsf{H}} C \supset E^{[C_1, C_2, \dots, C_n]}$ for $n \geq 0$. This rules is denoted by* composition.

(ARK) *If $\Theta\,;\Gamma \vdash_{\mathsf{H}} C$ then also $\Theta\,;\Gamma \vdash_{\mathsf{H}} C^{[E_1, E_2, \dots, E_n]}$ for $n \geq 0$, which is* ($\mathsf{K_n}$) *as admissible rule. We will write* $(\mathsf{ARK})_{[E_1, E_2, \dots, E_n]}$ *to denote the application of this rule to a derivation w.r.t. a list of concepts $[E_1, E_2, \dots, E_n]$.*

(ARS) *If Hilbert derives* $\Theta\,;\Gamma \vdash_{\mathsf{H}} C_1 \supset C_2 \supset \ldots \supset C_n$ *then there is a derivation for* $\Theta\,;\Gamma \vdash_{\mathsf{H}} (C_1^{[E_1,\,E_2,\,\ldots,\,E_m]}) \supset (C_2^{[E_1,\,E_2,\,\ldots,\,E_m]}) \supset \ldots \supset (C_k^{[E_1,\,E_2,\,\ldots,\,E_m]})$ *with* $2 \leq k < n$ *and* $m \geq 0$ *as well, which is* $(\mathsf{S_n})$ *as admissible rule. We will write* $(\mathsf{ARS})_{[E_1,E_2,\ldots,E_m]}$ *to denote the application of this rule to a derivation w.r.t. the list* $[E_1, E_2, \ldots, E_m]$*.*

(ARC) *If* $\Theta\,;\Gamma \vdash_{\mathsf{H}} (C_1 \sqcap C_2) \supset D$ *then also* $\Theta\,;\Gamma \vdash_{\mathsf{H}} C_1 \supset (C_2 \supset D)$*. This rule is known as* currying [70]*, whereas its inverse direction* $(\mathsf{ARC})^{-1}$ *is called* de-currying*.*

(ARW) *If* $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D$ *then also* $\Theta\,;\Gamma \vdash_{\mathsf{H}} (C_1 \sqcap C \sqcap C_2) \supset D$ *and* $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset (D_1 \sqcup D \sqcup D_2)$*. This is known as* weakening *where the first case denotes* left-weakening *and the second case is called* right-weakening*.*

(ARCW) *If* $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D$ *then* $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D^{[E_1,E_2,\ldots,E_n]}$*, denoted by* weakening-currying*. The application of rule* (ARCW) *to a Hilbert derivation w.r.t. a list of concepts* $[E_1, E_2, \ldots, E_n]$ *will be denoted by* $(\mathsf{ARCW})_{[E_1,E_2,\ldots,E_n]}$*.*

(ARM) *If* $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D$ *then by* monotonicity *also* $\Theta\,;\Gamma \vdash_{\mathsf{H}} (E_1 \sqcap C \sqcap E_2) \supset (E_1 \sqcap D \sqcap E_2)$*.*

(ARE) Elimination of neutral elements*: If* $\Theta\,;\Gamma \vdash_{\mathsf{H}} (\top \sqcap C) \supset D$ *then also* $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D$*, and if* $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset (\bot \sqcup D)$ *then also* $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D$*.* ∇

Proof. The proof is by demonstrating that each rule of Lem. 5.1.3 is admissible in the Hilbert system of $c\mathcal{ALC}$. Note that we will omit $\Theta\,;\Gamma$ from the presentation of the derivation of each rule.

(ARB) The proof of the admissible rule (ARB) (*composition*) is by showing that from $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D^{[C_1,C_2,\ldots,C_n]}$ and $\Theta\,;\Gamma \vdash_{\mathsf{H}} D \supset E$ we can find a derivation of $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset E^{[C_1,C_2,\ldots,C_n]}$. The goal can be derived by starting from an appropriate instance of $(\mathsf{B_n})$ w.r.t. n and using rule MP with the latter and the assumption.

(ARK) Suppose that $\Theta\,;\Gamma \vdash_{\mathsf{H}} C$. We must shows how to derive $\Theta\,;\Gamma \vdash_{\mathsf{H}} C^{[E_1,\,E_2,\,\ldots,\,E_n]}$. We can take an instance of $(\mathsf{K_n})$ which by Prop. 5.1.2 also holds under extended assumptions, *i.e.*, $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset C^{[E_1,\,E_2,\,\ldots,\,E_n]}$. Then, the goal follows from the latter and the assumption by rule MP.

(ARS) Let us suppose that $\Theta\,;\Gamma \vdash_{\mathsf{H}} C_1 \supset C_2 \supset \ldots \supset C_n$. The goal is to derive $\Theta\,;\Gamma \vdash_{\mathsf{H}} (C_1^{[E_1,\,E_2,\,\ldots,\,E_m]}) \supset (C_2^{[E_1,\,E_2,\,\ldots,\,E_m]}) \supset \ldots \supset ((C)_k^{[E_1,\,E_2,\,\ldots,\,E_m]})$ for $2 \leq k < n$, where $(C)_k$ is the kth subconcept of the assumption $C =_{df} C_1 \supset C_2 \supset \ldots \supset C_n$ w.r.t. implication (see $(\mathsf{S_n})$ in Lem. 5.1.2). The proof is by induction on the size m

of the list $[E_1,\ E_2,\ \ldots,\ E_m]$. For the base case $m = 0$ the goal follows immediately by assumption. For the inductive step let $i \geq 0$. We have to show that (ARS) for $m = i$ implies (ARS) for $m = i + 1$. This is given by the following derivation w.r.t. $\Theta\,;\Gamma$, where $l =_{df} [E_2, E_3, \ldots, E_{i+1}]$ is a list of concepts of length i:

1. $(C_1^l) \supset \ldots \supset ((C)_k^l)$ by ind. hyp.;
2. $E_1 \supset \big((C_1^l) \supset \ldots \supset ((C)_k^l)\big)$ from 1 by $(\mathsf{ARK})_{[E_1]}$;
3. $\big(E_1 \supset ((C_1^l) \supset \ldots \supset ((C)_k^l))\big) \supset (E_1 \supset (C_1^l)) \supset \ldots \supset (E_1 \supset ((C)_k^l))$ IPC2;
4. $(E_1 \supset (C_1^l)) \supset \ldots \supset (E_1 \supset ((C)_k^l))$ from 3, 2 by MP.
   $= (C_1^{E_1::l}) \supset \ldots \supset ((C)_k^{E_1::l})$

(ARC) For the rule of currying (ARC) the Hilbert derivation is as follows where we use the abbreviation $\varphi =_{df} (C_1 \sqcap C_2)$:

1. $\vartheta =_{df} (C_1 \sqcap C_2) \supset D$ Ass.;
   $= \varphi \supset D$
2. $(C_2 \supset \vartheta) \supset (C_2 \supset \varphi) \supset (C_2 \supset D)$ IPC2;
3. $(C_1 \supset (C_2 \supset \vartheta)) \supset (C_1 \supset (C_2 \supset \varphi)) \supset C_1 \supset (C_2 \supset D)$ from 2 by $(\mathsf{ARS})_{[C_1]}$;
4. $C_1 \supset (C_2 \supset \vartheta)$ from 1 by $(\mathsf{ARK})_{[C_1,C_2]}$;
5. $(C_1 \supset (C_2 \supset \varphi)) \supset (C_1 \supset (C_2 \supset D))$ from 3, 4 by MP;
6. $C_1 \supset (C_2 \supset D)$ from 5, IPC3 by MP.

For the derivation of de-currying $(\mathsf{ARC})^{-1}$ let $\varphi =_{df} (C_1 \sqcap C_2)$:

1. $C_1 \supset (C_2 \supset D)$ Ass.;
2. $(C_1 \sqcap C_2) \supset C_1$ IPC4;
   $= \varphi \supset C_1$
3. $\varphi \supset (C_2 \supset D)$ from 2, 1 by (ARB);
4. $(\varphi \supset (C_2 \supset D)) \supset (\varphi \supset C_2) \supset (\varphi \supset D)$ IPC2;
5. $(\varphi \supset C_2) \supset (\varphi \supset D)$ from 4, 3 by MP;
6. $(C_1 \sqcap C_2) \supset C_2$ IPC4;
   $= \varphi \supset C_2$
7. $(C_1 \sqcap C_2) \supset D$ from 5, 6 by MP.

(ARW) The admissibility of the rule of left- and right-weakening (ARW) is shown by the next two derivations. The first case demonstrates left-weakening while the second argues right-weakening. In the following derivation let $\varphi =_{df} (C_1 \sqcap (C \sqcap C_2))$:

1. $C \supset D$ Ass.;
2. $(C \sqcap C_2) \supset C$ IPC4;
3. $(C_1 \sqcap (C \sqcap C_2)) \supset (C \sqcap C_2)$ IPC4;
   $= \varphi \supset (C \sqcap C_2)$
4. $\big(\varphi \supset ((C \sqcap C_2) \supset C)\big) \supset (\varphi \supset (C \sqcap C_2)) \supset (\varphi \supset C)$ IPC2;
5. $\varphi \supset ((C \sqcap C_2) \supset C)$ from 2 by (ARK)$_{[\varphi]}$;
6. $(\varphi \supset (C \sqcap C_2)) \supset (\varphi \supset C)$ from 4, 5 by MP;
7. $\varphi \supset C$ from 6, 3 by MP;
   $= (C_1 \sqcap (C \sqcap C_2)) \supset C$
8. $(C_1 \sqcap (C \sqcap C_2)) \supset D$ from 7, 1 by (ARB).

The derivation of right-weakening goes analogously to the latter case but it is relying on the constructor IPC5 for disjunction. In the following let $\varphi =_{df} ((D_1 \sqcup D) \sqcup D_2)$.

1. $C \supset D$ Ass.;
2. $D \supset (D_1 \sqcup D)$ IPC5;
3. $(D_1 \sqcup D) \supset ((D_1 \sqcup D) \sqcup D_2)$ IPC5;
   $= (D_1 \sqcup D) \supset \varphi$
4. $\big(D \supset ((D_1 \sqcup D) \supset \varphi)\big) \supset (D \supset (D_1 \sqcup D)) \supset (D \supset \varphi)$ IPC2;
5. $((D_1 \sqcup D) \supset \varphi) \supset \big(D \supset ((D_1 \sqcup D) \supset \varphi)\big)$ IPC1;
6. $D \supset ((D_1 \sqcup D) \supset \varphi)$ from 5, 3 by MP;
7. $(D \supset (D_1 \sqcup D)) \supset (D \supset \varphi)$ from 4, 6 by MP;
8. $D \supset \varphi$ from 7, 2 by MP;
   $= D \supset ((D_1 \sqcup D) \sqcup D_2)$
9. $C \supset ((D_1 \sqcup D) \sqcup D_2)$ from 1, 8 by (ARB).

(ARCW) Admissible rule (ARCW) is derivable from a simplified form of left-weakening and de-currying (ARC)$^{-1}$. Suppose that $\Theta ; \Gamma \vdash_{\mathsf{H}} C \supset D$. The goal is a Hilbert derivation of $\Theta ; \Gamma \vdash_{\mathsf{H}} C \supset D^{[E_1, E_2, \ldots, E_n]}$. We proceed by induction on n. The base case $n = 0$ is trivial. In the inductive step let $k \geq 0$. Suppose that (ARCW) holds for $n = k$. We need to show that (ARCW) holds for $n = k + 1$ as well. We have the following derivation where $l =_{df} [E_2, E_3, \ldots, E_{k+1}]$ is a list of length k:

1. $C \supset (D^l)$ by ind. hyp.;
2. $(C \sqcap E_1) \supset C$ IPC4;
3. $(C \sqcap E_1) \supset (D^l)$ from 2, 1 by (ARB);
4. $C \supset (E_1 \supset (D^l))$ from 3 by (ARC).
   $= C \supset (D^{E_1 :: l})$

(ARM) For the rule of *monotonicity* we have to show that Hilbert derives $\Theta\,;\,\Gamma \vdash_{\mathsf{H}} (E_1 \sqcap (C \sqcap E_2)) \supset (E_1 \sqcap (D \sqcap E_2))$ from the assumption $\Theta\,;\,\Gamma \vdash_{\mathsf{H}} C \supset D$. In this proof we need the general result of commutativity of $\sqcap$, *i.e.*, at first we will demonstrate that Hilbert derives

$$\vdash_{\mathsf{H}} (C \sqcap D) \supset (D \sqcap C). \tag{5.2}$$

1. $\big((C\sqcap D) \supset (C \supset (D\sqcap C))\big) \supset ((C\sqcap D) \supset C) \supset ((C\sqcap D) \supset (D\sqcap C))$ IPC2;
2. $(C \sqcap D) \supset (C \supset (D \sqcap C))$ from IPC4, IPC3 by (ARB);
3. $((C\sqcap D) \supset C) \supset ((C\sqcap D) \supset (D\sqcap C))$ from 1, 2 by MP;
4. $(C \sqcap D) \supset (D \sqcap C)$ from 3, IPC4 by MP.

Secondly, we show that from a derivation of $\Theta\,;\,\Gamma \vdash_{\mathsf{H}} C \supset D$ we can derive the weaker form $\Theta\,;\,\Gamma \vdash_{\mathsf{H}} (C \sqcap E) \supset (D \sqcap E)$ of monotonicity, denoted by (ARM$_\mathsf{w}$):

1. $C \supset D$ Ass.;
2. $\big((C\sqcap E) \supset (E \supset (D\sqcap E))\big) \supset ((C\sqcap E) \supset E) \supset ((C\sqcap E) \supset (D\sqcap E))$ IPC2;
3. $(C\sqcap E) \supset (E \supset (D\sqcap E))$ from (IPC4, 1 by (ARB)), IPC3 by (ARB);
4. $((C \sqcap E) \supset E) \supset ((C \sqcap E) \supset (D \sqcap E))$ from 2, 3 by MP;
5. $(C \sqcap E) \supset (D \sqcap E)$ from 4, IPC4 by MP.

Finally, we derive the primary goal $\Theta\,;\,\Gamma \vdash_{\mathsf{H}} (E_1 \sqcap (C \sqcap E_2)) \supset (E_1 \sqcap (D \sqcap E_2))$:

1. $C \supset D$ Ass.;
2. $((C \sqcap E_2) \sqcap E_1) \supset ((D \sqcap E_2) \sqcap E_1)$ from 1 by (ARM$_\mathsf{w}$) done twice;
3. $(E_1 \sqcap (C \sqcap E_2)) \supset ((C \sqcap E_2) \sqcap E_1)$ by (5.2);
4. $((D \sqcap E_2) \sqcap E_1) \supset (E_1 \sqcap (D \sqcap E_2))$ by (5.2);
5. $(E_1 \sqcap (C \sqcap E_2)) \supset ((D \sqcap E_2) \sqcap E_1)$ from 3, 2 by (ARB);
6. $(E_1 \sqcap (C \sqcap E_2)) \supset (E_1 \sqcap (D \sqcap E_2)$ from 5, 4 by (ARB).

(ARE) Elimination of neutral elements by the rule (ARE) is given as follows:

1. $(\top \sqcap C) \supset D$ Ass.;
2. $C \supset (\top \sqcap C)$ (IPC9);
3. $C \supset D$ from 2, 1 by (ARB).

1. $C \supset (\bot \sqcup D)$ Ass.;
2. $(\bot \sqcup D) \supset D$ (IPC8);
3. $C \supset D$ from 2, 1 by MP.

□

5.1.2 Modal Deduction Theorem

The *deduction theorem* as introduced in [128] is a meta-theorem stating that if there is a deduction $\Gamma,\, C \vdash_{\mathsf{H}} D$ of a concept D from a set of concepts Γ extended by a concept C, then one can derive $\Gamma \vdash_{\mathsf{H}} C \supset D$.

It has been stated in the literature several times that the deduction theorem does not hold for modal logics in its general form, in particular, Mendler and de Paiva [188] point out that the common form of unrestricted Hilbert deduction $\Theta\,;\,\Gamma \vdash_{\mathsf{H}} C$ does not enjoy the deduction theorem in the system CK. The problem arises when considering assumptions as axioms. In this case it would follow by rule Nec that $\Theta\,;\,C \vdash_{\mathsf{H}} \forall R.C$, while soundness of $\vdash_{\mathsf{H}}$ will give $\Theta\,;\, \not\vdash_{\mathsf{H}} C \supset \forall R.C$.

Hakli and Negri [128] argue that the problem arises from the definition of the notion of derivability from assumptions in an axiomatic Hilbert-style system and review several solutions to the problem. They present a solution in [128], which is by restricting the rule of necessitation Nec to be applicable only to derivations which are independent from assumptions, *i.e.*, the rule Nec restricted to apply to theorems only. However, the representation of Hakli and Negri [128] only covers derivations from local premises.

The work of Fitting [101] gives a notion of Hilbert deduction which differentiates between global and local assumptions. Global assumptions represent fixed axiom instances and can be thought of as TBox axioms in the DL-context, while local premises live at the entity (world) level. Accordingly, the rule of necessitation is then restricted to be applicable only to global premises, but is banned for local ones. Fitting's definition of Hilbert deduction accommodates the two kinds of premises by dividing a derivation into a global and a local part respectively, such that rule Nec is not applicable in the latter.

Our definition of Hilbert deduction Def. 5.1.3 implements Fitting's idea and allows us to state the deduction theorem w.r.t. global and local premises. For a comprehensive survey on the deduction theorem in modal logics see [128]. Having global and local hypotheses saves the deduction theorem, but as Fitting [101] points out, we need two versions of it, namely, a local and a global Deduction Theorem:

Theorem 5.1.1 (Deduction Theorem [101; 102]). *For all sets of concepts Θ and Γ, and concepts C, D the following holds:*

(i) $\Theta\,;\,\Gamma,\, C \vdash_{\mathsf{H}} D$ *iff* $\Theta\,;\,\Gamma \vdash_{\mathsf{H}} C \supset D$; *(local deduction)*

(ii) $\Theta,\, C\,;\,\Gamma \vdash_{\mathsf{H}} D$ *iff* $\Theta\,;\,\Gamma \cup \forall^{*}C \vdash_{\mathsf{H}} D$, *(global deduction)*

where $\forall^{*}C$ *is the least set containing concept* C *and* $\forall R.E$, $\forall R \in N_R$ *and* $E \in \forall^{*}C$. ∇

Proof. Let us begin with the proof of the local Deduction Theorem 5.1.1.(i).

($\Rightarrow$) The proof is by induction on the structure of a derivation and follows the overall structure of the proof as given in [128, Thm. 2], but differs from the latter which only covers the classical modal logic K while we treat c$\mathcal{ALC}$ and therefore will rely on a different argumentation of the individual cases.

Suppose that $\Theta\,;\Gamma,\, C \vdash_{\mathsf{H}} D$, *i.e.*, there is a Hilbert derivation $D_1, D_2, \ldots, D_n$ of D from $\Theta\,;\Gamma \cup \{C\}$. To prove $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D$ we demonstrate the statement $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D_i$ for any D_i in the proof of D with $1 \leq i \leq n$. We proceed by induction on i:

- In the base case $i = 1$ we have two possibilities, namely that D_1 is an assumption itself or an axiom.
  (i) If D_1 is an assumption and $D_1 = C$, then the goal $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D_1$ follows by a derivation of $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset C$. The latter holds by monotonicity Prop. 5.1.2 and the fact that $\vdash_{\mathsf{H}} C \supset C$ (aka *identity*) is derivable in Hilbert, where the latter derivation follows by Lem. 5.1.2.

  (ii) Otherwise, D_1 is an assumption and $D_1 \in \Gamma$ or D_1 is a substitution instance of an axiom according to Def. 5.1.1. Then, in either case the derivation of the goal is as follows:

  1. $\Theta\,;\Gamma \vdash_{\mathsf{H}} D_1$ Def. 5.1.3;
  2. $\Theta\,;\Gamma \vdash_{\mathsf{H}} D_1 \supset (C \supset D_1)$ IPC1 and Prop. 5.1.2;
  3. $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D_1$ from 2, 1 by MP.

- In the inductive step let us suppose that $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D_j$ for $j < i$. The goal is to obtain a derivation of $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D_i$. We proceed by case analysis:
  (i) If concept D_j is an axiom according to Def. 5.1.1 or an assumption in Γ, then the goal follows similarly to the base case by taking $D_1 = D_j$. Otherwise, D_j is obtained by the application of an inference rule.

  (ii) If the last rule applied is Nec then we have $D_i = \forall R.E$ in the conclusion, *i.e.*, the derivation is of the form

  $$\frac{\Theta\,;\emptyset \vdash_{\mathsf{H}} E \qquad R \in N_R}{\Theta\,;\Gamma,\, C \vdash_{\mathsf{H}} \forall R.E}$$

  By rule Nec Hilbert derives $\Theta\,;\Gamma \vdash_{\mathsf{H}} \forall R.E$ as well. The derivation of the goal is as follows:

  1. $\Theta\,;\Gamma \vdash_{\mathsf{H}} \forall R.E$ by Nec;
  2. $\Theta\,;\Gamma \vdash_{\mathsf{H}} \forall R.E \supset (C \supset \forall R.E)$ IPC1 and Prop. 5.1.2;
  3. $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset \forall R.E$ from 2, 1 by MP.

(iii) If the last rule applied is MP we have to consider two cases. Note that Γ is partitioned into Γ' and Γ'', while Θ is global.

Case 1. The left premise has not been derived by rule Nec. Then C is either part of the assumptions in Γ' of the left premise or the right premise Γ'' of rule MP, respectively.

Case 1.1. In the first case, $C \in \Gamma'$ and the last rule application looks like

$$\frac{\Theta\,;\,\Gamma',\, C \vdash_{\mathsf{H}} D_k \qquad \Theta\,;\,\Gamma'' \vdash_{\mathsf{H}} \overbrace{D_k \supset D_i}^{D_j}}{\Theta\,;\,\Gamma',\,\Gamma'',\, C \vdash_{\mathsf{H}} D_i}$$

with $j, k < i$. We need to show that Hilbert derives $\Theta\,;\,\Gamma',\,\Gamma'' \vdash_{\mathsf{H}} C \supset D_i$. Applying the induction hypothesis to the left premise gives us the derivation of $\Theta\,;\,\Gamma' \vdash_{\mathsf{H}} C \supset D_k$. Then, the derivation of the goal is as follows:

1. $\Theta\,;\,\Gamma' \vdash_{\mathsf{H}} C \supset D_k$ by ind. hyp. left prem.;
2. $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (C \supset (D_k \supset D_i)) \supset (C \supset D_k) \supset (C \supset D_i)$ $\mathsf{IPC2}$ and Prop. 5.1.2;
3. $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (D_k \supset D_i) \supset (C \supset (D_k \supset D_i))$ $\mathsf{IPC1}$ and Prop. 5.1.2;
4. $\Theta\,;\,\Gamma'' \vdash_{\mathsf{H}} (C \supset (D_k \supset D_i))$ from right. prem., 3 by MP;
5. $\Theta\,;\,\Gamma'' \vdash_{\mathsf{H}} (C \supset D_k) \supset (C \supset D_i)$ from 2, 4 by MP;
6. $\Theta\,;\,\Gamma',\,\Gamma'' \vdash_{\mathsf{H}} C \supset D_i$ from 5, 1 by MP.

Case 1.2. In the second case, $C \in \Gamma''$ and the last rule application looks like

$$\frac{\Theta\,;\,\Gamma' \vdash_{\mathsf{H}} D_k \qquad \Theta\,;\,\Gamma'',\, C \vdash_{\mathsf{H}} \overbrace{D_k \supset D_i}^{D_j}}{\Theta\,;\,\Gamma',\,\Gamma'',\, C \vdash_{\mathsf{H}} D_i}$$

with $j, k < i$. Applying the induction hypothesis to the right premise yields the derivation of $\Theta\,;\,\Gamma'' \vdash_{\mathsf{H}} C \supset (D_k \supset D_i)$. We proceed as follows:

1. $\Theta\,;\,\Gamma'' \vdash_{\mathsf{H}} C \supset (D_k \supset D_i)$ by ind. hyp. right prem.;
2. $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (C \supset (D_k \supset D_i)) \supset (C \supset D_k) \supset (C \supset D_i)$ $\mathsf{IPC2}$ and Prop. 5.1.2;
3. $\Theta\,;\,\Gamma'' \vdash_{\mathsf{H}} (C \supset D_k) \supset (C \supset D_i)$ from 2, 1 by MP;
4. $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} D_k \supset (C \supset D_k)$ $\mathsf{IPC1}$ and Prop. 5.1.2.
5. $\Theta\,;\,\Gamma' \vdash_{\mathsf{H}} C \supset D_k$ from 4, left. prem. by MP;
6. $\Theta\,;\,\Gamma',\,\Gamma'' \vdash_{\mathsf{H}} C \supset D_i$ from 3, 5 by MP.

Case 2. If rule Nec is involved in the deduction of one of the premises of rule MP then it must be in the left premise. There are two cases depending on whether C is part of the assumptions of the left or right premise.

Case 2.1. Suppose that C lives in the left premise, then the derivation is

$$\dfrac{\dfrac{\Theta\,;\emptyset \vdash_{\mathsf{H}} E \qquad R \in N_R}{\Theta\,;\Gamma',\, C \vdash_{\mathsf{H}} \forall R.E} \qquad \Theta\,;\Gamma'' \vdash_{\mathsf{H}} \overbrace{\forall R.E \supset D_i}^{D_j}}{\Theta\,;\Gamma',\, C,\, \Gamma'' \vdash_{\mathsf{H}} D_i}$$

where $D_k = \forall R.E$ and $j, k < i$. Rule Nec derives $\Theta\,;\Gamma' \vdash_{\mathsf{H}} \forall R.E$ as well, such that we obtain via MP a derivation of $\Theta\,;\Gamma',\, \Gamma'' \vdash_{\mathsf{H}} D_i$. Now, taking the instance $\vdash_{\mathsf{H}} D_i \supset (C \supset D_i)$ of IPC1 and applying MP to the former derivation gives us the goal $\Theta\,;\Gamma',\, \Gamma'' \vdash_{\mathsf{H}} C \supset D_i$.

Case 2.2. Otherwise, if C is part of the right premise, the situation is

$$\dfrac{\dfrac{\Theta\,;\emptyset \vdash_{\mathsf{H}} E \qquad R \in N_R}{\Theta\,;\Gamma' \vdash_{\mathsf{H}} \forall R.E} \qquad \Theta\,;\Gamma'',\, C \vdash_{\mathsf{H}} \overbrace{\forall R.E \supset D_i}^{D_j}}{\Theta\,;\Gamma',\, \Gamma'',\, C \vdash_{\mathsf{H}} D_i}$$

which is argued similarly like Case 1.2.

($\Leftarrow$) The inverse direction is easily shown. Let us assume that $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \supset D$. It follows by monotonicity Prop. 5.1.2 that Hilbert derives $\Theta\,;\Gamma,\, C \vdash_{\mathsf{H}} C \supset D$. Taking the fact $\Theta\,;\Gamma,\, C \vdash_{\mathsf{H}} C$ and applying rule MP to the former gives $\Theta\,;\Gamma,\, C \vdash_{\mathsf{H}} D$ as desired.

This finishes the proof of the local Deduction Theorem. We postpone the proof of the global Deduction Theorem 5.1.1.(ii) since it relies on another auxiliary lemma which will be introduced below. □

Lemma 5.1.4. *For all sets of concepts Θ and Γ, and concept C we have*

(i) $\Theta\,;\Gamma \vdash_{\mathsf{H}} C \Rightarrow \Theta\,;\forall R\,\Gamma \vdash_{\mathsf{H}} \forall R.C$, *where* $\forall R\,\Gamma =_{\mathit{df}} \{\forall R.D \mid D \in \Gamma\}$*;*

(ii) $\Theta,\, C\,;\emptyset \vdash_{\mathsf{H}} D \Rightarrow \Theta,\, C\,;\emptyset \vdash_{\mathsf{H}} \mathbb{Q}$*, where* $\mathbb{Q}$ *is a concept of the form* $\forall R_1.\forall R_2.\ldots.\forall R_k.D$ *prefixed by a quantifier sequence of length* $k \geq 0$ *and each* R_i $(0 < i \leq k)$ *is some* $R_i \in N_R$*.* ∇

Proof. Lem. 5.1.4.(i) is shown by induction on the size of Γ. Suppose that $\Theta\,;\Gamma \vdash_{\mathsf{H}} C$.

- In the base case $\Gamma = \emptyset$ the goal $\Theta\,;\emptyset \vdash_{\mathsf{H}} \forall R.C$ is an immediate consequence of applying rule Nec to the assumption.

- In the inductive step let us suppose that $\Theta\,;\Gamma,\, D \vdash_{\mathsf{H}} C$. We need to show that $\Theta\,;\forall R\,(\Gamma \cup \{D\}) \vdash_{\mathsf{H}} \forall R.C$. It follows from the local Deduction Theorem 5.1.1 that $\Theta\,;\Gamma \vdash_{\mathsf{H}} D \supset C$. By applying the ind. hyp. we obtain the derivation $\Theta\,;\forall R\,\Gamma \vdash_{\mathsf{H}} \forall R.(D \supset C)$. Taking an instance of axiom $\mathsf{K}_{\forall R}$ gives $\vdash_{\mathsf{H}} \forall R.(D \supset C) \supset (\forall R.D \supset \forall R.C)$, which by Prop. 5.1.2 has a derivation under weakening as well, *i.e.*, $\Theta\,;\,\forall R\,\Gamma \vdash_{\mathsf{H}} \forall R.(D \supset C) \supset (\forall R.D \supset \forall R.C)$. Then, an application of rule MP to the former derivations yields $\Theta\,;\,\forall R\,\Gamma \vdash_{\mathsf{H}} \forall R.D \supset \forall R.C$. Now, we apply the local Deduction Theorem (Thm. 5.1.1.(i)) in the inverse direction to obtain the derivation of the goal $\Theta\,;\,\forall R\,\Gamma,\ \forall R.D \vdash_{\mathsf{H}} \forall R.C$.

Part (ii) of Lem. 5.1.4 is shown by induction on the depth of the quantifier prefix of $\mathcal{Q}$.

- In the base case suppose that $\Theta,\ C\,;\,\emptyset \vdash_{\mathsf{H}} D$. Since $\mathcal{Q} = D$, the goal follows by assumption.

- In the inductive step suppose that $\Theta,\ C\,;\,\emptyset \vdash_{\mathsf{H}} D$. The goal is to find a derivation of $\Theta,\ C\,;\,\emptyset \vdash_{\mathsf{H}} \mathcal{Q}$. The ind. hyp. yields a derivation of $\Theta,\ C\,;\,\emptyset \vdash_{\mathsf{H}} \mathcal{Q}'$ with $\mathcal{Q} = \forall R.\mathcal{Q}'$ and $\mathcal{Q}'$ having a quantifier depth of $k \geq 0$. Applying rule Nec w.r.t. some $R \in N_R$ to the latter derivation gives us a Hilbert proof of $\Theta, C\,;\,\emptyset \vdash_{\mathsf{H}} \forall R.\mathcal{Q}'$ which was to be shown, observing that $\mathcal{Q} = \forall R.\mathcal{Q}'$. □

We are now ready to tackle the global Deduction Theorem.

Proof of Thm. 5.1.1.(ii). Let us suppose that $\Theta,\ C\,;\,\Gamma \vdash_{\mathsf{H}} D$, *i.e.*, there is a derivation $D_1, D_2, \ldots, D_n$ of D from $\Theta,\ C\,;\,\Gamma$. The proof is by induction on the length of a Hilbert derivation. We prove $\Theta\,;\,\Gamma \cup \forall^* C \vdash_{\mathsf{H}} D$ by demonstrating the statement $\Theta\,;\,\Gamma \cup \forall^* C \vdash_{\mathsf{H}} D_i$ for any D_i in the proof of D with $1 \leq i \leq n$.

($\Rightarrow$) Base case: $i = 1$. Let us assume that $\Theta,\ C\,;\,\Gamma \vdash_{\mathsf{H}} D_1$. The goal is to show a derivation of $\Theta\,;\,\Gamma \cup \forall^* C \vdash_{\mathsf{H}} D_1$. We have the following cases:

(i) If D_1 is an instance of an axiom then $\vdash_{\mathsf{H}} D_1$ and by monotonicity Prop. 5.1.2 immediately $\Theta\,;\,\Gamma \cup \forall^* C \vdash_{\mathsf{H}} D_1$.

(ii) If D_1 is an assumption and $D_1 \in \Theta \cup \Gamma$ then the goal follows by monotonicity Prop. 5.1.2;

(iii) If $C = D_1$ then $\Theta\,;\,C \vdash_{\mathsf{H}} D_1$ as well. By monotonicity Prop. 5.1.2 it follows that $\Theta\,;\,\Gamma,\ \forall^* C \vdash_{\mathsf{H}} D_1$ which shows the goal, where $C \in \forall^* C$ by definition of $\forall^* C$.

In the inductive step we proceed by case analysis on the last rule applied:

(i) If the last rule is Nec then we have the following situation

$$\frac{\Theta,\ C\,;\,\emptyset \vdash_{\mathsf{H}} D_k \qquad R \in N_R}{\Theta,\ C\,;\,\Gamma \vdash_{\mathsf{H}} \forall R.D_k}$$

where $D_i = \forall R.D_k$. By the induction hypothesis applied to the premise we obtain $\Theta\,;\,\forall^* C \vdash_{\mathsf{H}} D_k$. By compactness Lem. 5.1.1 there exists a finite set $\forall^* C_f \subseteq \forall^* C$ such that $\Theta\,;\,\forall^* C_f \vdash_{\mathsf{H}} D_k$. Then, we can use Lem. 5.1.4.(i) to obtain a derivation of $\Theta\,;\,\forall R\,\forall^* C_f \vdash_{\mathsf{H}} \forall R.D_k$. Now, observe that $\forall^* C$ is a superset of $\forall R\,\forall^* C_f$. The goal $\Theta\,;\,\forall^* C \vdash_{\mathsf{H}} \forall R.D_k$ follows by exploiting monotonicity Prop. 5.1.2.

(ii) If the last rule applied is MP then we have

$$\frac{\Theta,\, C\,;\, \Gamma' \vdash_{\mathsf{H}} D_k \qquad \Theta,\, C\,;\, \Gamma'' \vdash_{\mathsf{H}} \overbrace{D_k \supset D_i}^{D_j}}{\Theta,\, C\,;\, \Gamma',\, \Gamma'' \vdash_{\mathsf{H}} D_i}$$

with $j, k < i$. Applying the ind. hyp. to the premises yields the derivations $\Theta\,;\, \Gamma',\, \forall^* C \vdash_{\mathsf{H}} D_k$ and $\Theta\,;\, \Gamma'',\, \forall^* C \vdash_{\mathsf{H}} D_k \supset D_i$. The goal follows by an application of rule MP to give $\Theta\,;\, \Gamma',\, \Gamma'',\, \forall^* C \vdash_{\mathsf{H}} D_i$.

($\Leftarrow$) Suppose that $\Theta\,;\, \Gamma \cup \forall^* C \vdash_{\mathsf{H}} D$. By compactness Lem. 5.1.1 there exist finite sets $\Theta_f \subseteq \Theta$, $\Gamma_f \subseteq \Gamma$, $\forall^* C_f \subseteq \forall^* C$ such that $\Theta_f\,;\, \Gamma_f \cup \forall^* C_f \vdash_{\mathsf{H}} D$. We proceed by induction on the size of the set $\forall^* C_f$: The base case is trivial. In the inductive step we have a derivation of $\Theta_f\,;\, \Gamma_f \cup \forall^* C_f \cup \{\rlap{$\forall$}Q\} \vdash_{\mathsf{H}} D$ where $\rlap{$\forall$}Q$ is of the form $\forall R_1.\forall R_2.\ldots\forall R_k.C$ having a quantifier prefix with depth $k \geq 0$. An application of the local Deduction Theorem 5.1.1 yields $\Theta_f\,;\, \Gamma_f \cup \forall^* C_f \vdash_{\mathsf{H}} \rlap{$\forall$}Q \supset D$. Applying the ind. hyp. to the latter derivation gives us

$$\Theta_f,\, C\,;\, \Gamma_f \vdash_{\mathsf{H}} \rlap{$\forall$}Q \supset D. \tag{5.3}$$

Taking the fact $\Theta_f,\, C\,;\, \emptyset \vdash_{\mathsf{H}} C$, we can conclude by Lem. 5.1.4.(ii) and Prop. 5.1.2 (monotonicity) that Hilbert derives $\Theta_f,\, C\,;\, \Gamma_f \vdash_{\mathsf{H}} \rlap{$\forall$}Q$. Finally, from the latter derivation and (5.3) follows by an application of rule MP that Hilbert derives $\Theta_f,\, C\,;\, \Gamma_f \vdash_{\mathsf{H}} D$ from which we obtain the goal by weakening Prop. 5.1.2. □

5.1.3 Soundness and Completeness

Theorem 5.1.2 (Hilbert Soundness and Completeness). *For all concepts C and sets of concepts Θ and Γ we have $\Theta\,;\, \Gamma \models C$ if and only if $\Theta\,;\, \Gamma \vdash_{\mathsf{H}} C$.* ∇

Proof. Soundness and completeness of the Hilbert system follow from soundness and completeness of the associated Gentzen sequent calculus $\mathsf{G1}$ (see Theorem 5.2.1), which will be introduced in Sec. 5.2. The proof, as presented in Sec. 5.2.3, is by showing that every derivation in the Hilbert system can be translated into a derivation in the Gentzen sequent calculus $\mathsf{G1}$ and vice versa (see Proposition 5.2.1). □

The Hilbert calculus implements reasoning w.r.t. TBoxes in the following sense. It decides the semantic relationship $\Theta; \emptyset \models C$, which expresses that concept C is a *universal* concept, *i.e.*, it holds for *all* entities in all models of a TBox Θ.

Example 5.1.2 (Adapted from [195, pp. 219 ff.], with kind permission from Springer Science and Business Media.)**.** This example[14] is inspired by the classical one of Brachman et al. [46] and its variation as reported by Bozzato et al. [41]. It describes the scenario of a *Food&Wine Recommendation System* that relates different types of foods with suitable kinds of wines. The procedure of recommending a wine for a given food can be operationalised by seeking the implementation of a function for

$$\Theta\,;\,\emptyset \mid_{\overline{\mathsf{H}}} \mathsf{Food} \supset \exists \mathsf{goesWith}.(\mathsf{Colour} \sqcap \exists \mathsf{isColourOf}.\mathsf{Wine}). \tag{5.4}$$

in the sense that *every input of type food yields as output a colour that is the colour of a wine*, under the following global assumptions given in terms of the TBox

$$\Theta = \{Ax_1 =_{df} \ \mathsf{Food} \supset \exists \mathsf{goesWith}.\mathsf{Colour} \text{ and } Ax_2 =_{df} \ \mathsf{Colour} \supset \exists \mathsf{isColourOf}.\mathsf{Wine}\}.$$

The Curry-Howard isomorphism can be adapted to understand any Hilbert-proof of (5.4) as a program construction, *i.e.*, its proof corresponds to a $c\mathcal{ALC}$ type-directed construction of a data-base program w.r.t. a Food&Wine knowledge base. Under this view, the TBox axiom Ax_1 represents a function ax_1 which translates an input entity f of type Food into an entity c of type Colour which is located within a $\mathsf{goesWith}$ context so that $(f, c) : \mathsf{goesWith}$. Likewise, the axiom Ax_2 is the type of a function ax_2 which accepts an entity c of type Colour as input and outputs a wine w of type Wine in some $\mathsf{isColourOf}$ context such that $(c, w) : \mathsf{isColourOf}$ holds. Since the axioms Ax_1 and Ax_2 are global assumptions represented through the TBox Θ, they are type-able in every context, *i.e.*, the rule of necessitation Nec can be applied to these without any restriction, for instance we can derive $\Theta\,;\,\emptyset \mid_{\overline{\mathsf{H}}} \forall \mathsf{goesWith}.Ax_2$ which represents the type of the term $\mathsf{Nec}\, ax_2$, *i.e.*, applying the rule Nec to the function ax_2 lifts it up w.r.t. a $\mathsf{goesWith}$ context.

The Hilbert proof of (5.4) is given as follows where we use the abbreviations $C =_{df}$ Colour, $F =_{df}$ Food, $W =_{df}$ Wine, $ico =_{df}$ $\mathsf{isColourOf}$ and $gw =_{df}$ $\mathsf{goesWith}$. First, the idea is to construct a derivation of

$$\Theta\,;\,\emptyset \mid_{\overline{\mathsf{H}}} C \supset (C \sqcap \exists ico.W), \tag{5.5}$$

corresponding to a function which *for each input of type colour returns a pair of the colour and its ico-associated wine.* Its derivation is given as follows.

---

[14] This presentation extends and corrects the example as given in [195]. It amends the representation of the basic combinators IPC2, IPC3, $\mathsf{K}_{\exists R}$ and the rules MP, Nec and (ARB). Furthermore, it shows a corrected version of the proof term for (5.4).

1. $C \supset (\exists ico.W \supset (C \sqcap \exists ico.W))$ IPC3;
2. $(C \supset (\exists ico.W \supset (C \sqcap \exists ico.W))) \supset (C \supset \exists ico.W) \supset (C \supset (C \sqcap \exists ico.W))$ IPC2;
3. $(C \supset \exists ico.W) \supset (C \supset (C \sqcap \exists ico.W))$ from 2, 1 by MP;
4. $(C \supset (C \sqcap \exists ico.W)$ from 3, Ax_2 by MP.

Secondly, we want to obtain from (5.5) a derivation of

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} \exists gw.C \supset \exists gw.(C \sqcap \exists ico.W), \tag{5.6}$$

that corresponds to a function that *for every colour in a gw context as input, it returns the colour and its associated wine within a gw context.*. This can be derived by lifting (5.5) up to a goesWith context by a combination of rule Nec and axiom $\mathsf{K}_{\exists R}$ using the role *gw*:

1. $\forall gw.\big(C \supset (C \sqcap \exists ico.W)\big)$ from (5.5) by Nec;
2. $\forall gw.\big(C \supset (C \sqcap \exists ico.W)\big) \supset \big(\exists gw.C \supset \exists gw.(C \sqcap \exists ico.W)\big)$ $\mathsf{K}_{\exists gw}$;
3. $\exists gw.C \supset \exists gw.(C \sqcap \exists ico.W)$ from 2, 1 by MP.

Finally, the derivation of (5.4) can be obtained from (5.6) and Ax_1 by the rule (ARB) (*function composition*). This completes the construction of a uniform function from Food f to pairs (c, w) of Colour c and Wine w with goesWith(f, c) and isColourOf(c, w). The proof of (5.4) gives a combinator expression of the form

$$(\mathsf{ARB})\ \Big(\mathsf{MP}\ \mathsf{K}_{\exists gw}\ \big(\mathsf{Nec}\ (\mathsf{MP}\ (\mathsf{MP}\ \mathsf{IPC2}\ \mathsf{IPC3})\ Ax_2)\big)\Big)\ Ax_1. \tag{5.7}$$

One way to assign a computational meaning to proofs for constructive $\mathcal{ALC}$ has been introduced by Bozzato et al. [41] in the form of a constructive semantics based on *information terms* [201] which are according to [93] a kind of *valuation form semantics* [175] strongly related to the BHK interpretation. Intuitively, an information term for a formula provides a witness or explicit explanation for the truth of that formula in the form of a mathematical object [39; 41; 93]. In the following we recall and extend their constructions. With each concept C we associate a set of *realisers* or *information terms* $\mathrm{IT}(C)$. These realisers are taken as extra ABox parameters so that instead of $\mathcal{I}; x \models C$ we declare what it means that $\mathcal{I}; x$ *realises* $\langle\alpha\rangle C$ for a particular realiser $\alpha \in \mathrm{IT}(C)$, written as $\mathcal{I}; x \rhd \langle\alpha\rangle C$. This so-called *realisability predicate* gives additional constructive semantics to our concepts in the sense that $\mathcal{I}; x \rhd \langle\alpha\rangle C$ implies $\mathcal{I}; x \models C$ while $\mathcal{I}; x \models C$ does not mean $\mathcal{I}; x \rhd \langle\alpha\rangle C$ for all but only for specific α if at all. We define the information terms $\mathrm{IT}(C)$ and refined concepts $\langle\alpha\rangle C$ by induction on C as follows, listing only the information terms which are required by this example.

$$
\begin{aligned}
\mathrm{IT}(A) &=_{\mathit{df}} \{\mathrm{tt}\}, \quad \text{if } A \text{ is an atomic concept;}\\
\mathrm{IT}(C \sqcap D) &=_{\mathit{df}} \mathrm{IT}(C) \times \mathrm{IT}(D);\\
\mathrm{IT}(C \supset D) &=_{\mathit{df}} \mathrm{IT}(C) \to \mathrm{IT}(D);\\
\mathrm{IT}(\exists R.C) &=_{\mathit{df}} \Delta^{\mathcal{I}} \times \mathrm{IT}(C);\\
\mathrm{IT}(\forall R.C) &=_{\mathit{df}} \Delta^{\mathcal{I}} \to \mathrm{IT}(C).
\end{aligned}
$$

Let C be a $c\mathcal{ALC}$ concept, $\mathcal{I}$ a constructive interpretation, $x \in \Delta^{\mathcal{I}}$ and $\eta \in \mathrm{IT}(C)$. The *realisability relation* $\mathcal{I}; x \rhd \langle\eta\rangle C$ is defined by induction on the structure of C:

$$
\begin{aligned}
&\mathcal{I}; x \rhd \langle \mathrm{tt}\rangle A \text{ iff } x \in A^{\mathcal{I}};\\
&\mathcal{I}; x \rhd \langle \alpha, \beta\rangle (C \sqcap D) \text{ iff } \mathcal{I}; x \rhd \langle\alpha\rangle C \text{ and } \mathcal{I}; x \rhd \langle\beta\rangle C;\\
&\mathcal{I}; x \rhd \langle f\rangle (C \supset D) \text{ iff } \forall \alpha \in \mathrm{IT}(C).\ \mathcal{I}; x \rhd \langle\alpha\rangle C \Rightarrow \mathcal{I}; x \rhd \langle f\alpha\rangle D;\\
&\mathcal{I}; x \rhd \langle a, \alpha\rangle (\exists R.C) \text{ iff } (x, a) \in R^{\mathcal{I}} \text{ and } \mathcal{I}; a \rhd \langle\alpha\rangle C;\\
&\mathcal{I}; x \rhd \langle\alpha\rangle (\forall R.C) \text{ iff } \forall a \in \Delta^{\mathcal{I}}.\ (x, a) \in R^{\mathcal{I}} \Rightarrow \mathcal{I}; a \rhd \langle\alpha\, a\rangle C.
\end{aligned}
$$

These realisers are interpreted *locally* w.r.t. an entity $x \in \Delta^{\mathcal{I}}$. Remark, that the realisability relation resembles the classical semantics of $\mathcal{ALC}$ and is compatible with the constructive semantics of $c\mathcal{ALC}$, *i.e.*, $\mathcal{I}; x \rhd \langle\eta\rangle C$ implies $\mathcal{I}; x \models C$ which can be shown by induction on the structure of concept C.

Then, one demonstrates that every Hilbert proof of C generates for any interpretation $\mathcal{I}$ a function $f : \Delta^{\mathcal{I}} \to \mathrm{IT}(C)$ such that $\forall u \in \Delta^{\mathcal{I}}.\mathcal{I}; u \rhd \langle fu\rangle C$. Such Hilbert proofs of C relative to a (possibly non-empty) TBox Θ derive globally valid formulæ. This global behaviour has to be respected by the generated function f. Thus, it takes as first argument an element from $\Delta^{\mathcal{I}}$ denoting the entity relative to which f returns an information term for C. We will call a function $f : \Delta^{\mathcal{I}} \to \mathrm{IT}(C)$ a *global realiser* of a concept C, representing the computational behaviour of a Hilbert proof of C and will use the notation $\langle\!\langle f\rangle\!\rangle C$ in the following. In this regard, we can give a function for each of the Hilbert axioms involved in the proof of (5.7) represented in terms of an expression in the simply-typed λ-calculus extended by products and sums:

$$
\begin{aligned}
&\mathit{ipc}2 : \Delta^{\mathcal{I}} \to \mathrm{IT}\big((C \supset (D \supset E)) \supset (C \supset D) \supset C \supset E\big)\\
&\mathit{ipc}2 =_{\mathit{df}} \lambda u.\lambda x.\lambda y.\lambda z.\ (x\, z)\, (y\, z);\\
&\mathit{ipc}3 : \Delta^{\mathcal{I}} \to \mathrm{IT}\big(C \supset D \supset (C \sqcap D)\big)\\
&\mathit{ipc}3 =_{\mathit{df}} \lambda u.\lambda x.\lambda y.\ (x, y);\\
&k_{\exists R} : \Delta^{\mathcal{I}} \to \mathrm{IT}\big(\forall R.(C \supset D) \supset \exists R.C \supset \exists R.D\big)\\
&k_{\exists R} =_{\mathit{df}} \lambda u.\lambda x.\lambda y.(\pi_1\, y, x(\pi_1\, y)\, (\pi_2\, y)).
\end{aligned}
$$

Furthermore, we refine the rules of MP, Nec and (ARB) as follows:

If $\langle\!\langle\alpha\rangle\!\rangle C$ and $\langle\!\langle\beta\rangle\!\rangle(C \supset D)$ then $\langle\!\langle\lambda u.(\beta\, u)(\alpha\, u)\rangle\!\rangle D$;
If $\langle\!\langle\alpha\rangle\!\rangle C$ then $\langle\!\langle\lambda u.\lambda x.\, \alpha\, x\rangle\!\rangle(\forall R.C)$;
If $\langle\!\langle\alpha\rangle\!\rangle(C \supset D)$ and $\langle\!\langle\beta\rangle\!\rangle(D \supset E)$ then $\langle\!\langle\lambda u.\lambda x.\,(\beta\, u)((\alpha\, u)\; x)\rangle\!\rangle(C \supset E)$.

Under this view, we can give a function f for (5.5), which is the Hilbert combinator (MP (MP IPC2 IPC3) Ax_2) represented by the λ-term

$$f = \; \lambda u.\lambda x.(x, (ax_2\; u)\; x),$$

for which $\langle\!\langle f\rangle\!\rangle(C \supset (C \sqcap \exists ico.W))$ holds. Next, a function g of (5.6) can be generated by rule MP from $k_{\exists R}$, ax_2 and the result of applying Nec to f yields a term for MP $\mathsf{K}_{\exists gw}$ $\big($Nec (MP (MP IPC2 IPC3) Ax_2)$\big)$. The latter application Nec f produces a function $f' : \Delta^{\mathcal{I}} \to \mathrm{IT}\big(\forall gw.(C \supset (C \sqcap \exists ico.W))\big)$ given by

$$f' = \; \lambda u.\lambda v.\lambda x.(x, (ax_2\; v)\; x),$$

and the former then generates the function g such that $\langle\!\langle g\rangle\!\rangle(\exists gw.C \supset \exists gw.(C \sqcap \exists ico.W))$, which corresponds to the following term up to reductions and α-conversion in the λ-calculus:

$$g = \; \lambda u.\lambda x.\big(\pi_1\; x,\; \big(\pi_2\; x, (ax_2\; (\pi_1\; x))\; (\pi_2\; x)\big)\big).$$

Finally, the derivation of (5.4) follows by rule (ARB) from g and ax_1 and yields the term *prf* up to reductions and α-conversion

$$prf = \lambda u.\lambda x.\big(\pi_1\; ((ax_1\; u)\; x),\; \big(\pi_2\; ((ax_1\; u)\; x),\; ax_2\; (\pi_1\; ((ax_1\; u)\; x))(\pi_2\; ((ax_1\; u)\; x))\big)\big)$$

such that for all $u \in \Delta^{\mathcal{I}}$ the realiser $\langle prf\, u\rangle$ represents an information term so that

$$\forall u.\; \mathcal{I}; u \rhd \langle prf u\rangle(\mathsf{Food} \supset \exists \mathsf{goesWith}.(\mathsf{Colour} \sqcap \exists \mathsf{isColourOf}.\mathsf{Wine}))$$

assuming that $\forall u.\, \mathcal{I}; u \rhd \langle ax_1\; u\rangle Ax_1$ and $\forall u.\, \mathcal{I}; u \rhd \langle ax_2\; u\rangle Ax_2$. Such global realisers ax_1, ax_2 either arise as proof terms themselves, or they are determined by a semantic specification in terms of an ABox, which has previously been demonstrated in [39; 41; 93]. Following their example, let the ABox $\mathcal{A}$ with $N_I =_{df} \{$`BAROLO`, `CHARDONNAY`, `RED`, `WHITE`, `FISH`, `MEAT`, `FF`$\}$ be described by the following facts.

$$\begin{array}{rrr}
\texttt{BAROLO} : \mathsf{Wine}, & \texttt{RED} : \mathsf{Colour}, & \texttt{FISH} : \mathsf{Food}, \\
\texttt{CHARDONNAY} : \mathsf{Wine}, & \texttt{WHITE} : \mathsf{Colour}, & \texttt{MEAT} : \mathsf{Food}, \\
(\texttt{WHITE}, \texttt{CHARDONNAY}) : \mathsf{isColourOf}, & (\texttt{RED}, \texttt{BAROLO}) : \mathsf{isColourOf}, & \texttt{FF} : \bot, \\
(\texttt{MEAT}, \texttt{RED}) : \mathsf{goesWith}, & (\texttt{FISH}, \texttt{WHITE}) : \mathsf{goesWith}, &
\end{array}$$

where FF specifies a fallible entity. We can give an interpretation $\mathcal{I}$ for the ABox $\mathcal{A}$ by

$$\begin{aligned}
\Delta^{\mathcal{I}} &=_{df} \{\texttt{BAROLO}, \texttt{CHARDONNAY}, \texttt{RED}, \texttt{WHITE}, \texttt{FISH}, \texttt{MEAT}, \texttt{FF}\}, \\
\bot^{\mathcal{I}} &=_{df} \{\texttt{FF}\}, \\
\preceq^{\mathcal{I}} &=_{df} id_{\Delta^{\mathcal{I}}}, \\
\mathsf{Wine}^{\mathcal{I}} &=_{df} \{\texttt{BAROLO}, \texttt{CHARDONNAY}, \texttt{FF}\}, \\
\mathsf{Colour}^{\mathcal{I}} &=_{df} \{\texttt{RED}, \texttt{WHITE}, \texttt{FF}\}, \\
\mathsf{Food}^{\mathcal{I}} &=_{df} \{\texttt{FISH}, \texttt{MEAT}, \texttt{FF}\}, \\
\mathsf{isColourOf}^{\mathcal{I}} &=_{df} \{(\texttt{RED}, \texttt{BAROLO}), (\texttt{WHITE}, \texttt{CHARDONNAY}), (\texttt{FF}, \texttt{FF})\}, \\
\mathsf{goesWith}^{\mathcal{I}} &=_{df} \{(\texttt{MEAT}, \texttt{RED}), (\texttt{FISH}, \texttt{WHITE}), (\texttt{FF}, \texttt{FF})\}, \\
\forall n \in N_I . n^{\mathcal{I}} &=_{df} n,
\end{aligned}$$

where $id_{\Delta^{\mathcal{I}}}$ is the identity relation on the set $\Delta^{\mathcal{I}}$. Under the interpretation $\mathcal{I}$ the realisers ax_1, ax_2 can be chosen as

$$\begin{aligned}
ax_1 &=_{df} \lambda u.\lambda x.\mathsf{case}\, u \,\mathsf{of}\, [\texttt{MEAT} \to (\texttt{RED}, \mathrm{tt}) \,|\, \texttt{FISH} \to (\texttt{WHITE}, \mathrm{tt}) \,|\, \mathrm{otherwise} \to (\texttt{FF}, \mathrm{tt})], \\
ax_2 &=_{df} \lambda u.\lambda x.\mathsf{case}\, u \,\mathsf{of}\, [\texttt{RED} \to (\texttt{BAROLO}, \mathrm{tt}) \,|\, \texttt{WHITE} \to (\texttt{CHARDONNAY}, \mathrm{tt}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \,|\, \mathrm{otherwise} \to (\texttt{FF}, \mathrm{tt})],
\end{aligned}$$

such that it holds that

$$\begin{aligned}
&ax_1 : \Delta^{\mathcal{I}} \to \mathrm{IT}(\mathsf{Food} \supset \exists \mathsf{goesWith}.\mathsf{Colour}), \text{ and} \\
&ax_2 : \Delta^{\mathcal{I}} \to \mathrm{IT}(\mathsf{Colour} \supset \exists \mathsf{isColourOf}.\mathsf{Wine}).
\end{aligned}$$

Note that the fallible entity FF is used here as an output that specifies a state of failure for the case when the first input argument u of ax_1 and ax_2 is not an entity in $\mathsf{Food}^{\mathcal{I}}$ or $\mathsf{Colour}^{\mathcal{I}}$ respectively. The realisers ax_1 and ax_2 express the constructive content of Ax_1, Ax_2 in $\mathcal{I}$. Then, for instance, the reduction of the λ-term $prf\, \texttt{MEAT}\ \mathrm{tt}$ (with tt being a realizer for Food) yields the information term (RED, (tt, (BAROLO, tt))) such that

$$\mathcal{I}; \texttt{MEAT} \rhd \langle(\texttt{RED},\ (\mathrm{tt},\ (\texttt{BAROLO},\ \mathrm{tt})))\rangle(\exists \mathsf{goesWith}.(\mathsf{Colour} \sqcap \exists \mathsf{isColourOf}.\mathsf{Wine})). \qquad \blacksquare$$

5.2 Gentzen Sequent Calculus G1 for $c\mathcal{ALC}$

While Hilbert systems have a clear structure and are conceptually very simple, their approach to find the proof of a formula is rather tedious. The process is purely syntactic and gives a finite sequence of formulæ in which each formula is either an axiom or the result of applying an inference rule to previous derivations. However, it is an awkward process to locate the required substitution of formulæ for an appropriate instantiation of an axiom, since there are infinitely many possibilities. The Hilbert calculus fails to provide an algorithm for generating proofs, and therefore is inappropriate as an efficient method for goal-directed and automated proof search. Better suited for this task are refutation based calculi like Gentzen-style sequent [109] or semantic tableau calculi [25; 30; 71]. These calculi combine both goal-directed proof-search as well as countermodel construction and are suitable for automated theorem proving.

This section introduces the Gentzen sequent calculus G1. The system G1 does not require explicit world labels, and is a *multi-succedent* (or *multi-conclusion*) sequent calculus in the spirit of G3im [214, Chap. 5.3], [257] and Dragalin's GHPC [84, Chap. 1]. Moreover, it can be considered as a multi-sequent system in the style of Masini [183; 184] whose introduction and elimination rules involve sets rather than individual formulæ.

Definition 5.2.1 (G1-sequent [195])**.** G1 uses hypothetical judgements, called *sequents*, of the form $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$, where Θ, Γ, Φ are sets of concepts that are not necessarily finite. Σ and Ψ are functions mapping role names $R \in N_R$ to sets of concepts $\Sigma(R)$, $\Psi(R)$, which may be infinite as well. In contrast to [195, p. 221], the mappings Σ and Ψ are not partial functions, but defined for all roles in N_R, and we do not assume that the domains of these functions are finite and identical. Moreover, let $dom(\Sigma)$ and $dom(\Psi)$ denote the non-empty *domain* of the functions Σ and Ψ, respectively, which is defined by $dom(\Sigma) =_{df} \{R \in N_R \mid \Sigma(R) \neq \emptyset\}$ and $dom(\Psi) =_{df} \{R \in N_R \mid \Psi(R) \neq \emptyset\}$. ∇

Notation. We write $\Theta\,;\,\Sigma\,;\,\Gamma \vdash_{\mathsf{G1}} \Phi\,;\,\Psi$ to refer to a G1 sequent and analogously index the turnstile when we are referring to variations of G1. The symbol G1 will be omitted whenever the sequent system in use is clear from the context. ■

The structure of a sequent complies with the Kripke semantics of $c\mathcal{ALC}$, *i.e.*, a sequent $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ formally refines the semantic consequence relation $\Theta;\Gamma \models \Phi$ (see Def. 4.2.4) generalised to a set of concepts Φ, by the additional constraints Σ, Ψ as follows: Θ is a set of concepts representing model assumptions and can be considered as the TBox. The sets Σ, Γ, Φ, Ψ of a sequent specify constraints and encode information about individual entities relative to Θ. We denote Γ and Φ as the *local* sequent (hypothesis and conclusion respectively) while Σ and Ψ are the *global* sequent. The local sequent specifies an entity locally while the global sequent constrains its R-reachable successors. Informally, the sets Σ and Γ specify what an entity shall satisfy and the sets Φ, Ψ what an entity must *not* satisfy. This kind of entity specification [195] in terms of positive and negative constraints is the novel constructive aspect of the following definition.

Definition 5.2.2 (Constructive satisfiability [190; 191; 195]). Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a constructive interpretation and $x \in \Delta^{\mathcal{I}}$ an entity. The pair $(\mathcal{I}, x)$ *satisfies* the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \mid\!- \Phi\,;\,\Psi$ if $\mathcal{I} \models \Theta$ and for all $R \in N_R$ the following holds:

(i) $\forall x'.\, \forall y.\, (x \preceq^{\mathcal{I}} x' \;\;\&\;\; x' R^{\mathcal{I}} y) \Rightarrow \mathcal{I}; y \models \Sigma(R)$, *i.e.*, all R-fillers of x and of its refinements x' are part of all concepts of $\Sigma(R)$;

(ii) $\mathcal{I}; x \models \Gamma$, *i.e.*, x and all its refinements are part of all concepts of Γ;

(iii) $\mathcal{I}; x \not\models \Phi$, *i.e.*, x is not contained in any of the concepts in Φ;

(iv) $\forall y.\, x\, R^{\mathcal{I}}\, y \Rightarrow \mathcal{I}; y \not\models \Psi(R)$, *i.e.*, none of the R-fillers y of x is contained in any concept of $\Psi(R)$.

A sequent $\Theta\,;\,\Sigma\,;\,\Gamma \mid\!- \Phi\,;\,\Psi$ is called *(constructively) satisfiable iff* there exists an interpretation $\mathcal{I}$ and entity $a \in \Delta^{\mathcal{I}}$ such that $(\mathcal{I}, a)$ satisfies the sequent. We write $\Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi$ to denote that the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \mid\!- \Phi\,;\,\Psi$ is satisfiable. ∇

The purpose of a *sequent* or *refutation* proof is to establish that an entity specification presented as a sequent is not satisfiable. In the other case, if no closed sequent derivation can be found and the calculus is complete, then the failed proof search implies the existence of a satisfying entity. The sequent calculus G1 is given by the rules in Fig. 5.1.

Notation. ([195, p. 222]) The hypotheses Θ, $\Sigma(R)$, Γ and conclusions Φ, $\Psi(R)$ in each rule of Fig. 5.1 are handled as sets rather than lists. For instance, the hypothesis $\Gamma, C \supset D$ of rule $\supset\! L$ denotes the set $\Gamma \cup \{C \supset D\}$. Therefore, if $C \supset D \in \Gamma$ then Γ in the premise of rule $\supset\! L$ is identical to $\Gamma, C \supset D$ in the conclusion of the rule. Furthermore, we write Γ, Γ' for $\Gamma \cup \Gamma'$ and we will also use the notation $\Phi \cup \Psi$ to denote $\Phi \cup (\bigcup_{R \in N_R} \Psi(R))$. The symbol $\emptyset$ is used both as the empty set and the constant function $\emptyset(R) = \emptyset$. The mapping $[R \mapsto C]$ is the finite function with domain $\{R\}$ mapping R to the singleton set $\{C\}$ and $\Sigma \cup [R \mapsto C]$ represents the union of functions with domain $dom(\Sigma) \cup \{R\}$ such that $(\Sigma \cup [R \mapsto C])(S) = \Sigma(S)$ for $S \neq R$ and $(\Sigma \cup [R \mapsto C])(R) = \Sigma(R) \cup \{C\}$, otherwise. We write Σ, Σ' for the union of two functions Σ and Σ' w.r.t. all roles. We assume implicit duplication, contraction and permutation (structural rules). ■

$$\dfrac{}{\Theta ; \Sigma ; \Gamma, C \vdash \Phi, C ; \Psi}\,\mathit{Ax} \qquad \dfrac{|\Phi \cup \Psi| \geq 1}{\Theta ; \Sigma ; \Gamma, \bot \vdash \Phi ; \Psi}\,\bot L$$

$$\dfrac{\Theta ; \Sigma ; \Gamma, C, D \vdash \Phi ; \Psi}{\Theta ; \Sigma ; \Gamma, C \sqcap D \vdash \Phi ; \Psi}\,\sqcap L \quad \dfrac{\Theta ; \Sigma ; \Gamma \vdash \Phi, C ; \Psi \qquad \Theta ; \Sigma ; \Gamma \vdash \Phi, D ; \Psi}{\Theta ; \Sigma ; \Gamma \vdash \Phi, C \sqcap D ; \Psi}\,\sqcap R$$

$$\dfrac{\Theta ; \Sigma ; \Gamma \vdash \Phi, C, D ; \Psi}{\Theta ; \Sigma ; \Gamma \vdash \Phi, C \sqcup D ; \Psi}\,\sqcup R \quad \dfrac{\Theta ; \Sigma ; \Gamma, C \vdash \Phi ; \Psi \qquad \Theta ; \Sigma ; \Gamma, D \vdash \Phi ; \Psi}{\Theta ; \Sigma ; \Gamma, C \sqcup D \vdash \Phi ; \Psi}\,\sqcup L$$

$$\dfrac{\Theta ; \Sigma ; \Gamma \vdash \Phi, C ; \Psi \qquad \Theta ; \Sigma ; \Gamma, D \vdash \Phi ; \Psi}{\Theta ; \Sigma ; \Gamma, C \supset D \vdash \Phi ; \Psi}\,\supset L \quad \dfrac{\Theta ; \Sigma ; \Gamma, C \vdash D ; \emptyset}{\Theta ; \Sigma ; \Gamma \vdash \Phi, C \supset D ; \Psi}\,\supset R$$

$$\dfrac{\Theta ; \Sigma ; \Gamma \vdash \emptyset ; [R \mapsto C]}{\Theta ; \Sigma ; \Gamma \vdash \Phi, \exists R.C ; \Psi}\,\exists R \qquad \dfrac{\Theta ; \emptyset ; \Sigma(R), C \vdash \Psi(R) ; \emptyset}{\Theta ; \Sigma ; \Gamma, \exists R.C \vdash \Phi ; \Psi}\,\exists L$$

$$\dfrac{\Theta ; \Sigma \cup [R \mapsto C] ; \Gamma \vdash \Phi ; \Psi}{\Theta ; \Sigma ; \Gamma, \forall R.C \vdash \Phi ; \Psi}\,\forall L \qquad \dfrac{\Theta ; \emptyset ; \Sigma(R) \vdash C ; \emptyset}{\Theta ; \Sigma ; \Gamma \vdash \Phi, \forall R.C ; \Psi}\,\forall R$$

$$\dfrac{\Theta ; \Sigma \cup [R \mapsto C] ; \Gamma \vdash \Phi ; \Psi \qquad R \in N_R}{\Theta, C ; \Sigma ; \Gamma \vdash \Phi ; \Psi}\,\mathit{Hyp}_1 \qquad \dfrac{\Theta ; \Sigma ; \Gamma, C \vdash \Phi ; \Psi}{\Theta, C ; \Sigma ; \Gamma \vdash \Phi ; \Psi}\,\mathit{Hyp}_2$$

Figure 5.1: Gentzen rules for $c\mathcal{ALC}$. Adapted from [195, p. 222, Fig. 4], with kind permission from Springer Science and Business Media.

Definition 5.2.3 (Tableau, constructive consistency [195, p. 222])**.** A *tableau* for a sequent $\Theta ; \Sigma ; \Gamma \vdash \Phi ; \Psi$ is a finite and closed derivation tree T with a single root $\Theta ; \Sigma ; \Gamma \vdash \Phi ; \Psi$ that is built using instances of the rules in Fig. 5.1, in which every leaf is closed in the sense that it ends in Ax or $\bot L$. A sequent is *(constructively) consistent*, written $\Theta ; \Sigma ; \Gamma \nvdash \Phi ; \Psi$, if no tableau exists for it. ∇

Example 5.2.1 ([195, p. 225, Fig. 7].)**.** Derivation of $\mathsf{K}_{\forall R}$, $\mathsf{K}_{\exists R}$ in $\mathsf{G1}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset ; \emptyset ; A \vdash B, A ; \emptyset}\,\mathit{Ax} \qquad \dfrac{}{\emptyset ; \emptyset ; A, B \vdash B ; \emptyset}\,\mathit{Ax}}{\emptyset ; \emptyset ; A \supset B, A \vdash B ; \emptyset}\,\supset L}{\emptyset ; [R \mapsto A \supset B, A] ; \emptyset \vdash \forall R.B ; \emptyset}\,\forall R}{\emptyset ; [R \mapsto A \supset B] ; \forall R.A \vdash \forall R.B ; \emptyset}\,\forall L}{\emptyset ; \emptyset ; \forall R.(A \supset B), \forall R.A \vdash \forall R.B ; \emptyset}\,\forall L}{\emptyset ; \emptyset ; \forall R.(A \supset B) \vdash \forall R.A \supset \forall R.B ; \emptyset}\,\supset R}{\emptyset ; \emptyset ; \emptyset \vdash \forall R.(A \supset B) \supset (\forall R.A \supset \forall R.B) ; \emptyset}\,\supset R.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset\,;\emptyset\,;A \vdash B,\,A\,;\emptyset}\,Ax \qquad \dfrac{}{\emptyset\,;\emptyset\,;A,\,B \vdash B\,;\emptyset}\,Ax}{\emptyset\,;\emptyset\,;A\supset B, A \vdash B\,;\emptyset}\,\supset\! L}{\emptyset\,;[R\mapsto A\supset B]\,;\exists R.A \vdash \emptyset\,;[R\mapsto B]}\,\exists L}{\emptyset\,;[R\mapsto A\supset B]\,;\exists R.A \vdash \exists R.B\,;\emptyset}\,\exists R}{\emptyset\,;\emptyset\,;\forall R.(A\supset B),\,\exists R.A \vdash \exists R.B\,;\emptyset}\,\forall L}{\emptyset\,;\emptyset\,;\forall R.(A\supset B) \vdash \exists R.A\supset\exists R.B\,;\emptyset}\,\supset\! R}{\emptyset\,;\emptyset\,;\emptyset \vdash \forall R.(A\supset B)\supset(\exists R.A\supset\exists R.B)\,;\emptyset}\,\supset\! R.$$

■

The G1 calculus is formulated in the spirit of Gentzen. The rules are divided into *left* introduction rules $\sqcap L$, $\sqcup L$, $\supset L$, $\forall L$, $\exists L$ and *right* introduction rules $\sqcap R$, $\sqcup R$, $\supset R$, $\forall R$, $\exists R$ for each logical connective. Besides their interpretation as sequent-style refutation steps it is also possible to give the rules a computational meaning [194; 198]. The Gentzen style presentation also lends itself to a game-theoretic interpretation [254]. These features are distinct advantages over natural deduction systems [27; 41; 64; 77].

Remark 5.2.1. One can observe, that the rules of the propositional part of G1 depicted by the upper four lines of Fig. 5.1 do not manipulate the model assumptions and the global sequent, *i.e.*, the sets Θ, Σ and Ψ. These rules correspond to the original Gentzen calculus LJ [109] of intuitionistic logic. This can be easily seen by removing the sets Θ, Σ, Ψ from these rules. Example 5.2.2 illustrates the latter by showing the proof of axiom IPC2 (cf. Def. 5.1.1). The modal rules $\forall L$, $\exists L$, $\forall R$, $\exists R$ involve the global sequent, *i.e.*, they explain the meaning of universal ($\forall R.C$) and existential ($\exists R.C$) role filling by externalising the quantifiers in terms of the Σ and Ψ components of a sequent. The two remaining rules Hyp_1, Hyp_2 are left introduction rules and introduce hypotheses from the set Θ of global model assumptions which correspond to a TBox. Formally, the latter rules express that each assumption in the TBox Θ can be used as an additional assumption for the current (local) entity inside Γ (Hyp_2) as well as in all its R-accessible fillers inside $\Sigma(R)$ (Hyp_1).

Furthermore, note that there is no right introduction rule $\bot R$, because it is not needed. One shows for the rules in Fig. 5.1 that if there is a tableau for $\Theta\,;\Sigma\,;\Gamma \vdash \Phi\,;\Psi$ then the weakened sequent $\Theta\,;\Sigma\,;\Gamma \vdash \Phi, \bot\,;\Psi$ is derivable as well which corresponds basically to $\bot R$. It is possible to directly treat negated concepts. This is achieved by the following left and right introduction rules $\neg L$ resp. $\neg R$ [195, p. 223]

$$\dfrac{\Theta\,;\Sigma\,;\Gamma \vdash \Phi,\,C\,;\Psi \qquad |\Phi\cup\Psi|\geq 1}{\Theta\,;\Sigma\,;\Gamma,\,\neg C \vdash \Phi\,;\Psi}\,\neg L \qquad \dfrac{\Theta\,;\Sigma\,;\Gamma,\,C \vdash \bot\,;\emptyset}{\Theta\,;\Sigma\,;\Gamma \vdash \Phi,\,\neg C\,;\Psi}\,\neg R$$

which are admissible from the rules in Fig. 5.1. The rule $\neg L$ is simply a combination of $\supset L$ and $\bot L$, whereas rule $\neg R$ is an instance of $\supset R$. ■

Example 5.2.2. Derivation of axiom IPC2 (omitting the components Θ, Σ, Ψ).

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$C \supset D, C \vdash C$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$C, D \supset E \vdash C$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$C, D \vdash D$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$C, D, E \vdash E$}
\RightLabel{$\supset L$}
\BinaryInfC{$C, D \supset E, D \vdash E$}
\RightLabel{$\supset L$}
\BinaryInfC{$C \supset D, C, D \supset E \vdash E$}
\RightLabel{$\supset L$}
\BinaryInfC{$C \supset (D \supset E), C \supset D, C \vdash E$}
\RightLabel{$\supset R$}
\UnaryInfC{$C \supset (D \supset E), C \supset D \vdash C \supset E$}
\RightLabel{$\supset R$}
\UnaryInfC{$C \supset (D \supset E) \vdash (C \supset D) \supset (C \supset E)$}
\RightLabel{$\supset R$}
\UnaryInfC{$\emptyset \vdash (C \supset (D \supset E)) \supset ((C \supset D) \supset (C \supset E))$}
\end{prooftree}
$$

■

Example 5.2.3 (Adapted from [195, p. 225, Ex. 8], with kind permission from Springer Science and Business Media)**.** This example shows that (i) the axiom of disjunctive distribution FS4/IK4 $= \exists R.(C \sqcup D) \supset (\exists R.C \sqcup \exists R.D)$ is not a theorem of $c\mathcal{ALC}$, and (ii) there are finite models involving cycles for which no finite cycle-free equivalent models exist. It is based on Grimm's fairy tale '*The hare and the hedgehog*', which can be considered as a classical instance of fraud. A hedgehog takes a walk and runs into a hare which he wishes a good morning. The hare, however, jokes about the hedgehog's short bowlegs. The hedgehog becomes angry and challenges the hare for a race. He persuades his wife to play a joke on the hare, by positioning his wife at the far finish of the racetrack. When the race starts and the victory-certain hare near-storms the wife of the hedgehog rises up and says: 'I am already here!' The two hedgehogs look so much alike that the hare is not able to distinguish them and so he looses the race. The hare does not accept its defeat and they repeat the race for another 73 times, each ending with the same result. At the 74th run the hare breaks down exhaustedly and dies.

This situation is modelled by the interpretation given in Fig. 5.2. There, the entity PEP belongs to the concept Himhog representing a male hedgehog such that $\mathsf{Himhog}^{\mathcal{I}} = \{\mathsf{PEP}\}$, and PIA represents his wife, belonging to the concept of a female hedgehog defined by $\mathsf{Herhog}^{\mathcal{I}} = \{\mathsf{PIA}\}$. Both entities, PEP and PIA are contained in the concept Hedgehog. The position of PEP and PIA is determined in terms of a role-filler over relation hasPosition to GPS-coordinates that belong to the concept Start and Finish respectively. These are subsumed by the concept Position. Since PEP and PIA are not distinguishable, there is a cyclic refinement between them, *i.e.*, they refine each other.

The hare corresponds to an internal agent [161] with restricted knowledge about the model, *i.e.*, he only sees the information content of the current location (Start or Finish). From this perspective, the hare puts the situation into question, how the hedgehog is able to be '*coherently at Start or Finish*', and '*not to be coherently at Start* and *not coherently at Finish*' at the same time. Note that the hare assumes

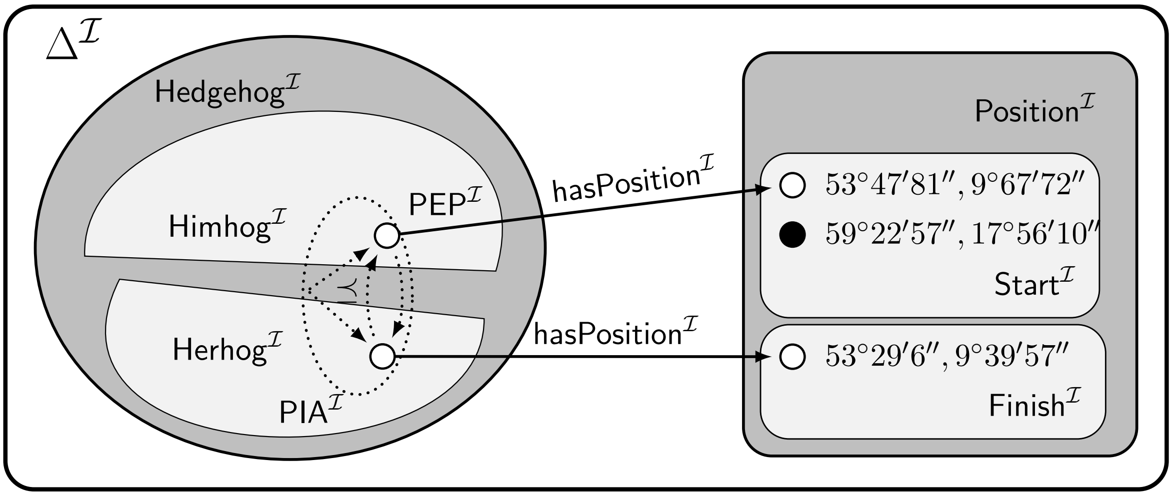

Figure 5.2: Model of the hedgehogs. Adapted from [191, p. 226, Fig. 8], with kind permission from Springer Science and Business Media.

the theorems of classical $\mathcal{ALC}$, in particular, he assumes that $\exists R$ distributes over $\sqcup$ (FS4/IK4). We will use in the following the abbreviations $F = \mathsf{Finish}$, $S = \mathsf{Start}$ and $R = \mathsf{hasPosition}$. Then, the above puzzle can be formalised by the sequent $\emptyset; \emptyset; \exists R.(S \sqcup F), \neg\exists R.S, \neg\exists R.F \vdash \bot ; \emptyset$. This sequent expresses the hare's assumption, *viz.* that for the hedgehog there exists either one R-filler to Start or one R-filler to Finish, expressed by $\exists R.(S \sqcup F)$. Further, the hare assumes that the hedgehog is neither sitting at the location Start nor at Finish, which is formulated by $\neg\exists R.S$, $\neg\exists R.F$.

First, we will discuss claim (i). If the existential quantifier would distribute over $\sqcup$ as it does in classical $\mathcal{ALC}$ then this would imply that we can find a derivation for $\emptyset ; \emptyset ; \exists R.(S \sqcup F), \neg(\exists R.S \sqcup \exists R.F) \vdash \bot ; \emptyset$. If the first hypothesis $\exists R.(S \sqcup F)$ of the sequent implies $\exists R.S \sqcup \exists R.F$, then this contradicts the second hypothesis $\neg(\exists R.S \sqcup \exists R.F)$ which implies $\bot$. Now, we show that the sequent cannot be derived in $c\mathcal{ALC}$. First, note that $\neg(\exists R.S \sqcup \exists R.F)$ is equivalent to $\neg\exists R.S \sqcap \neg\exists R.F$. Hence we have to show that there does not exist a closed tableau for the sequent $\emptyset ; \emptyset ; \exists R.(S \sqcup F), \neg\exists R.S, \neg\exists R.F \vdash \bot ; \emptyset$.

Because of constructiveness and the fact that Himhog and Herhog are indistinguishable, the hedgehog is able to be in both positions at the same time dependent upon choice of refinement. Fig. 5.3 shows the constructed countermodel for the sequent, where $\Gamma =_{df} \exists R.(S \sqcup F), \neg\exists.S, \neg\exists.F$. Dashed edges represent the application of a Gentzen rule indicated by its name, dotted and solid arrows are for refinement $\preceq$ and for the role R, and × indicates the closed leaf of a tableau. Note that the sets Θ and Σ in the sequents are omitted, since they are not needed in the proof.

We obtain a cyclic model with two clusters of equivalent individuals which represent Himhog and Herhog, and that can be refined to each other. The countermodel of Fig. 5.3

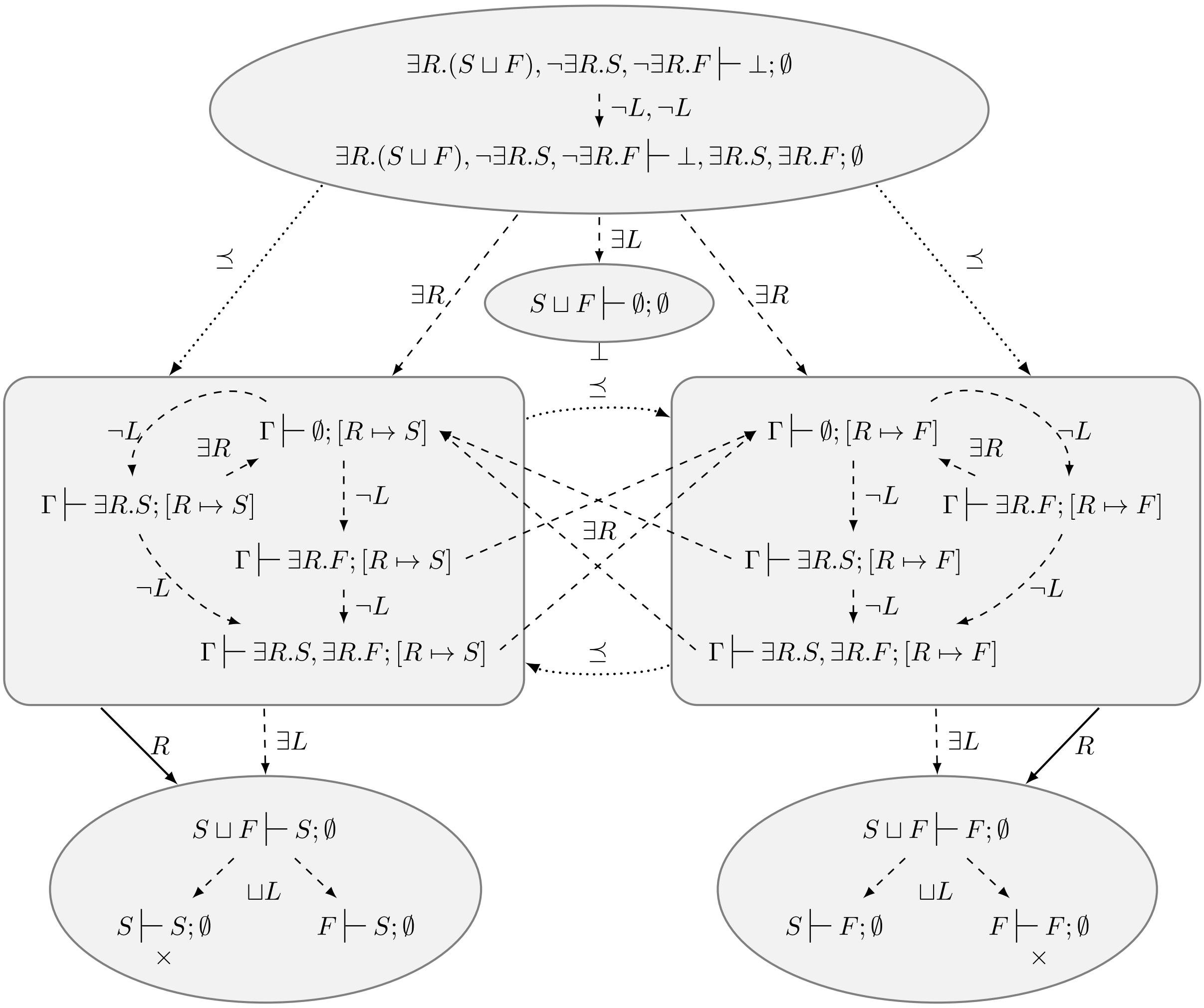

Figure 5.3: Proof attempt for the sequent $\exists R.(S \sqcup F), \neg\exists R.S, \neg\exists R.F \vdash \bot \; ; \; \emptyset$. Adapted from [195, p. 227, Fig. 9], with kind permission from Springer Science and Business Media.

can be simplified to the model given in Fig. 5.4 that represents the situation already shown in Fig. 5.2. We use the notation $[R \mapsto C]$ relative to an entity in the following figures to denote that in all its accessible R-successors concept C is refuted.

Secondly, let us look at claim (ii), *viz.* that any cycle-free model satisfying the concept $\exists R.(S \sqcup F) \sqcap \neg\exists R.S \sqcap \neg\exists R.F$ is infinite. It is easy to observe from Fig. 5.3 that if we require the model to be acyclic then it is necessary to introduce a new entity each time when the rule $\exists R$ is fired. This leads to the construction of an infinite tree w.r.t. refinement $\preceq$. Indeed, this corresponds to the cycle-free unfolding of the finite model of Fig. 5.4. This situation is depicted in Fig. 5.5. Observe, that the same happens if we require $\preceq$ to be anti-symmetric, in particular, this means that the finite model property is lost. Note that Fig. 5.5 could also be depicted as a linear infinite path in which every consecutive pair of entities oscillates between $[R \mapsto S]$ and $[R \mapsto F]$ respectively.

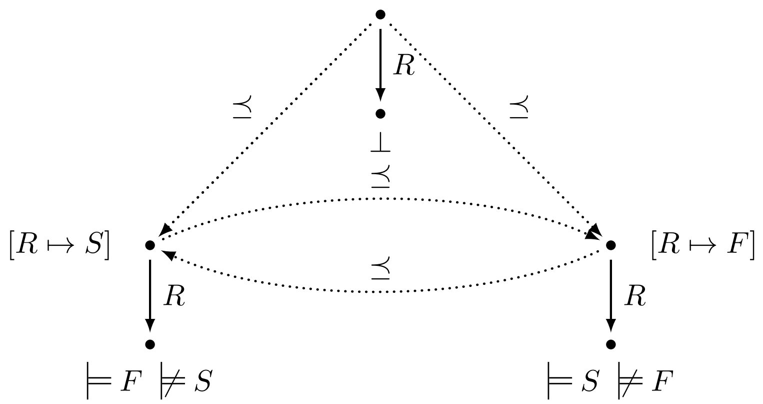

Figure 5.4: Simplified, finite and cyclic model of Fig. 5.3. Adapted from [195, p. 227, Fig. 10], with kind permission from Springer Science and Business Media.

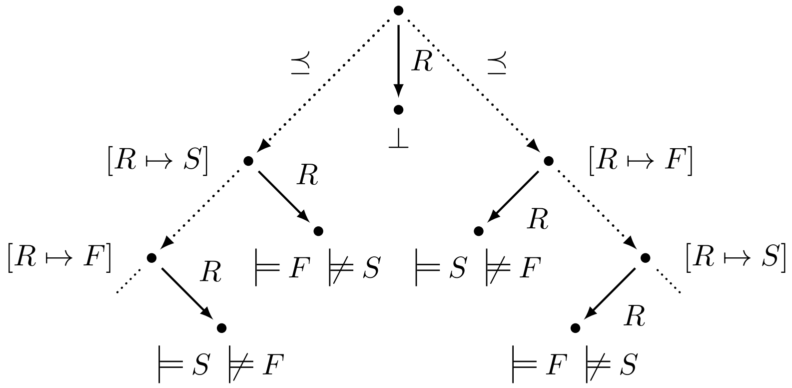

Figure 5.5: Infinite cycle-free unfolding of Fig. 5.4.

When we apply filtration (Def. 4.2.9) to the model of Fig. 5.4 then we will obtain the countermodel as depicted in Fig. 5.3 with two additional $\preceq$ edges, from the left and right $\preceq$-successors to their common $\preceq$-predecessor, *i.e.*, filtration may add additional structure to the model. However, observe that even Fig. 5.3 is not the smallest possible countermodel. This can be obtained by removing the upper entity such that we have a model with two entities, which refine each other, and their respective R-successors. The former corresponds to the instances PEP and PIA and the latter ones to the location in Start and Finish of Fig. 5.2 respectively. ■

Example 5.2.4 (Based on [41, p. 3, Ex. 1]). We revisit the *Food&Wine Recommendation System* of Ex. 5.1.2 to illustrate TBox reasoning in G1. We want to prove the formula $\mathsf{Food} \supset \exists \mathsf{goesWith}.(\mathsf{Colour} \sqcap \exists \mathsf{isColourOf}.\mathsf{Wine})$ in the system G1 w.r.t. the TBox $\Theta = \{Ax_1, Ax_2\}$ with

$$Ax_1 =_{df} \mathsf{Food} \supset \exists \mathsf{goesWith}.\mathsf{Colour} \text{ and}$$
$$Ax_2 =_{df} \mathsf{Colour} \supset \exists \mathsf{isColourOf}.\mathsf{Wine}.$$

The goal is to find a closed tableau for the sequent

$$\Theta\,;\,\emptyset\,;\,\emptyset \vdash \mathsf{Food} \supset \exists \mathsf{goesWith}.(\mathsf{Colour} \sqcap \exists \mathsf{isColourOf}.\mathsf{Wine})\,;\,\emptyset. \tag{5.8}$$

As before we use the following abbreviations: $C =_{df}$ Colour, $F =_{df}$ Food, $W =_{df}$ Wine, $ico =_{df}$ isColourOf and $gw =_{df}$ goesWith. The proof of (5.8) in G1 is as follows.

$$\dfrac{\dfrac{}{\Theta\,;\,\emptyset\,;\,C,\,\exists ico.W \vdash C\,;\,\emptyset}\,Ax \qquad \dfrac{}{\Theta\,;\,\emptyset\,;\,C,\,\exists ico.W \vdash \exists ico.W\,;\,\emptyset}\,Ax}{\cdots\cdots\cdots\cdots\cdots\cdots}\,\sqcap R$$

$$\dfrac{\dfrac{\dfrac{}{\Theta\,;\,\emptyset\,;\,F \vdash F,\,\ldots\,;\,\emptyset}\,Ax \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Theta\,;\,\emptyset\,;\,C \vdash C,\,\ldots\,;\,\emptyset}\,Ax \qquad \dfrac{\cdots\cdots\cdots\cdots\cdots\cdots}{\Theta\,;\,\emptyset\,;\,C,\,\exists ico.W \vdash C \sqcap \exists ico.W\,;\,\emptyset}}{\Theta\,;\,\emptyset\,;\,C \supset \exists ico.W,\,C \vdash C \sqcap \exists ico.W\,;\,\emptyset}\,\supset L}{\Theta\,;\,[gw \mapsto C \supset \exists ico.W]\,;\,F,\,\exists gw.C \vdash \emptyset\,;\,[gw \mapsto C \sqcap \exists ico.W]}\,\exists L}{\Theta\,;\,\emptyset\,;\,F,\,\exists gw.C \vdash \emptyset\,;\,[gw \mapsto C \sqcap \exists ico.W]}\,Hyp_1}{\Theta\,;\,\emptyset\,;\,F,\,\exists gw.C \vdash \exists gw.(C \sqcap \exists ico.W)\,;\,\emptyset}\,\exists R}{\Theta\,;\,\emptyset\,;\,F,\,F \supset \exists gw.C \vdash \exists gw.(C \sqcap \exists ico.W)\,;\,\emptyset}\,\supset L}{\dfrac{\Theta\,;\,\emptyset\,;\,F \vdash \exists gw.(C \sqcap \exists ico.W)\,;\,\emptyset}{\Theta\,;\,\emptyset\,;\,\emptyset \vdash F \supset \exists gw.(C \sqcap \exists ico.W)\,;\,\emptyset}\,\supset R.}\,Hyp_2$$

■

5.2.1 Soundness and Completeness

This section is devoted to the soundness and completeness of the Gentzen sequent calculus G1. The following theorem provides a bridge between syntax and derivability on the one side, and semantics and validity on the other side. In the first step, we prove the soundness of G1 by showing that every inference rule in Fig. 5.1 is satisfiability preserving. Thereafter, we show that G1 is *strongly complete.* The proof is based on a usual canonical model construction, demonstrating that there exists a satisfying canonical model for each consistent sequent.

Theorem 5.2.1 (Strong soundness and completeness). *A sequent is satisfiable iff it is consistent,* i.e., $\Theta; \Sigma; \Gamma \not\models \Phi; \Psi \Leftrightarrow \Theta; \Sigma; \Gamma \not\vdash \Phi; \Psi$. ∇

Soundness

Theorem 5.2.2 (Soundness of G1 [190; 195]). *Every satisfiable sequent is consistent,* i.e., $\Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi$ *implies* $\Theta\,;\,\Sigma\,;\,\Gamma \not\vdash \Phi\,;\,\Psi$. ∇

Proof. Parts of this proof have been previously published in [190; 195], where the cases for the rules $\supset R$, $\supset L$ and $\forall R$ have been omitted. Here, we will present the full proof.

For soundness we show for each derivation rule in Fig. 5.1 that if the conclusion is satisfiable then *at least one* of its premises is satisfiable as well. Starting from the axioms it follows by induction on the size of the derivation that if a sequent is inconsistent then it is not satisfiable. This is done by assuming satisfiability of the conclusion sequent and showing for one of its premises of the form $\Theta\,;\,\Sigma\,;\,\Gamma \mathrel{|\!\!-} \Phi\,;\,\Psi$ that there exists a pair $(\mathcal{I}, x)$ such that the following conditions are fulfilled:

$$\mathcal{I} \models \Theta; \tag{5.9}$$

for all $R \in N_R,\ L \in \Sigma(R),\ M \in \Gamma,\ N \in \Phi,\ K \in \Psi(R)$:

$$\forall x'.\, \forall y.\, (x \preceq^{\mathcal{I}} x' \ \ \&\ \ x'\, R^{\mathcal{I}}\, y) \Rightarrow \mathcal{I}; y \models L; \tag{5.10}$$

$$\mathcal{I}; x \models M; \tag{5.11}$$

$$\mathcal{I}; x \not\models N; \tag{5.12}$$

$$\forall y.\, x\, R^{\mathcal{I}}\, y \Rightarrow \mathcal{I}; y \not\models K. \tag{5.13}$$

Note that $\mathcal{I} \models \Theta$ holds by assumption in all cases.

(**Case** Ax) For the base case axiom Ax nothing needs to be shown, since the conclusion sequent $\Theta\,;\,\Sigma\,;\,\Gamma,\ C \mathrel{|\!\!-} \Phi,\ C\,;\,\Psi$ is already not satisfiable, *i.e.*, it is not possible to satisfy and not satisfy concept C at the same time.

(**Case** $\bot L$) The conclusion $\Theta\,;\,\Sigma\,;\,\Gamma,\ \bot \mathrel{|\!\!-} \Phi\,;\,\Psi$ of axiom $\bot L$ is not satisfiable, since every fallible entity satisfies all concepts. It is important to point out that the succedent $\Phi; \Psi$ is constrained to be non-empty. Contrary, a fallible entity satisfies a sequent with an empty succedent, *e.g.*, $\Theta\,;\,\Sigma\,;\,\Gamma,\ \bot \mathrel{|\!\!-} \emptyset\,;\,\emptyset$.

(**Case** $\sqcup L$) Suppose that the conclusion $s_c =_{df} \Theta\,;\,\Sigma\,;\,\Gamma,\ C \sqcup D \mathrel{|\!\!-} \Phi\,;\,\Psi$ is satisfiable, *i.e.*, $\Theta\,;\,\Sigma\,;\,\Gamma,\ C \sqcup D \not\models \Phi\,;\,\Psi$. By Def. 5.2.2 there is a pair $(\mathcal{I}, a)$ that *satisfies* the sequent s_c. In particular, it holds that $\mathcal{I}; a \models C \sqcup D$. The goal is to show that at least one of the premises $s_{p_1} =_{df} \Theta\,;\,\Sigma\,;\,\Gamma,\ C \mathrel{|\!\!-} \Phi\,;\,\Psi$ or $s_{p_2} =_{df} \Theta\,;\,\Sigma\,;\,\Gamma,\ D \mathrel{|\!\!-} \Phi\,;\,\Psi$ is satisfiable. We claim that $(\mathcal{I}, a)$ satisfies one of s_{p_1}, s_{p_2}:

- Regarding the conditions (5.10), (5.12), (5.13), nothing needs to be shown here, since the sets Σ, Φ and Ψ are equal in s_c and both s_{p_1} and s_{p_2}.
- Regarding (5.11), by assumption $\mathcal{I}; a \models \Gamma$ for both s_{p_1} and s_{p_2}. Moreover, from the assumption $\mathcal{I}; a \models C \sqcup D$ follows by Def. 4.2.2 that $\mathcal{I}; a \models C$ or $\mathcal{I}; a \models D$. In the first case $(\mathcal{I}, a)$ satisfies s_{p_1} and in the second case s_{p_2}.

(**Case** $\sqcup R$) Assume that the conclusion sequent $s_c =_{df} \Theta\,;\, \Sigma\,;\, \Gamma \mid\!\!- \Phi,\, C \sqcup D\,;\, \Psi$ is satisfiable. The goal is to show that its premise $s_p =_{df} \Theta\,;\, \Sigma\,;\, \Gamma \mid\!\!- \Phi,\, C,\, D\,;\, \Psi$ is satisfiable as well. From the assumption follows by Def. 5.2.2 that there exists a pair $(\mathcal{I}, a)$ that *satisfies* the sequent s_c. In particular, we have $\mathcal{I}; a \not\models C \sqcup D$. We claim that $(\mathcal{I}, a)$ satisfies the premise sequent s_p:

- The conditions (5.10), (5.11), (5.12), (5.13), for the sets Σ, Γ, Φ and Ψ follow by assumption, because, they are equal in s_c and s_p.
- The assumption $\mathcal{I}; a \not\models C \sqcup D$ implies by Def. 4.2.2 that $\mathcal{I}; a \not\models C$ and $\mathcal{I}; a \not\models D$, which proves the satisfiability of the premise s_p.

(**Case** $\sqcap L$) Assume that $\Theta\,;\, \Sigma\,;\, \Gamma,\, C \sqcap D \not\models \Phi\,;\, \Psi$, *i.e.*, there is a pair $(\mathcal{I}, a)$ that satisfies the conclusion sequent, in particular $\mathcal{I}; a \models C \sqcap D$. We claim that $(\mathcal{I}, a)$ satisfies the premise sequent $\Theta\,;\, \Sigma\,;\, \Gamma,\, C,\, D \mid\!\!- \Phi\,;\, \Psi$. We only have to analyse the conjunction $C \sqcap D$, since the conditions (5.10), (5.11), (5.12), (5.13), for Σ, Γ, Φ and Ψ directly follow by assumption. From the assumption $\mathcal{I}; a \models C \sqcap D$ it follows by Def. 4.2.2 that $\mathcal{I}; a \models C$ and $\mathcal{I}; a \models D$. Hence, $(\mathcal{I}, a)$ satisfies the premise.

(**Case** $\sqcap R$) Suppose that the conclusion sequent $s_c =_{df} \Theta\,;\, \Sigma\,;\, \Gamma \mid\!\!- \Phi,\, C \sqcap D\,;\, \Psi$ is satisfiable, *i.e.*, $\Theta\,;\, \Sigma\,;\, \Gamma \not\models \Phi,\, C \sqcap D\,;\, \Psi$. Then, Def. 5.2.2 implies that there exists a pair $(\mathcal{I}, a)$ that satisfies s_c, notably, it holds that $\mathcal{I}; a \not\models C \sqcap D$. The goal is to show that one of the premise sequents $s_{p_1} =_{df} \Theta;\ \Sigma;\ \Gamma \vdash \Phi, C;\ \Psi$ or $s_{p_2} =_{df} \Theta;\ \Sigma;\ \Gamma \vdash \Phi, D;\ \Psi$ is satisfiable.

- As before, nothing needs to be shown regarding the conditions for Σ, Γ, Φ and Ψ, because these sets are equal in s_c, s_{p_1} and s_{p_2}.
- The assumption $\mathcal{I}; a \not\models C \sqcap D$ implies by Def. 4.2.2 that either $\mathcal{I}; a \not\models C$ or $\mathcal{I}; a \not\models D$. The first case proves satisfiability of s_{p_1}, the second that of s_{p_2}.

(**Case** $\supset L$) Assume that the conclusion sequent $s_c =_{df} \Theta\,;\, \Sigma\,;\, \Gamma,\, C \supset D \mid\!\!- \Phi\,;\, \Psi$ is satisfiable, *i.e.*, $\Theta\,;\, \Sigma\,;\, \Gamma,\, C \supset D \not\models \Phi\,;\, \Psi$. We claim that one of the premise sequents $s_{p_1} =_{df} \Theta\,;\, \Sigma\,;\, \Gamma \mid\!\!- \Phi,\, C\,;\, \Psi$ or $s_{p_2} =_{df} \Theta\,;\, \Sigma\,;\, \Gamma,\, D \mid\!\!- \Phi\,;\, \Psi$ is satisfiable as well. Def. 5.2.2 implies that there exists a pair $(\mathcal{I}, a)$ that satisfies the sequent s_c. Particularly, it is the case that $\mathcal{I}; a \models C \supset D$. Therefore, using $a \preceq^{\mathcal{I}} a$ (by reflexivity of $\preceq^{\mathcal{I}}$) it holds that $\mathcal{I}; a \not\models C$ or $\mathcal{I}; a \models D$. In the former case $(\mathcal{I}, a)$ satisfies the sequent s_{p_1} and in the latter case s_{p_2}, using the fact that the conditions (5.10), (5.11), (5.12), (5.13) for Σ, Γ, Φ and Ψ directly follow by assumption.

(**Case** $\supset R$) Suppose that the conclusion sequent $s_c =_{df} \Theta\,;\, \Sigma\,;\, \Gamma \mid\!\!- \Phi,\, C \supset D\,;\, \Psi$ is satisfiable, *i.e.*, $\Theta\,;\, \Sigma\,;\, \Gamma \not\models \Phi,\, C \supset D\,;\, \Psi$. The goal is to show that the premise sequent $s_p =_{df} \Theta\,;\, \Sigma\,;\, \Gamma,\, C \mid\!\!- D\,;\, \emptyset$ is satisfiable, too. From the assumption and Def. 5.2.2 follows the existence of a pair $(\mathcal{I}, a)$ that satisfies the sequent s_c. The assumption $\mathcal{I}; a \not\models C \supset D$ lets us conclude that there exists an entity a' with $a \preceq^{\mathcal{I}} a'$ such that $\mathcal{I}; a' \models C$ and $\mathcal{I}; a' \not\models D$. Observe that the latter implies that a' is infallible. We claim that $(\mathcal{I}, a')$ satisfies the sequent s_p:

- The condition (5.10) for Σ follows from transitivity of $\preceq^{\mathcal{I}}$ and the assumption.
- Condition (5.11) for Γ, C follows by the assumption and monotonicity of refinement.
- By assumption $\mathcal{I}; a' \not\models D$ which satisfies condition (5.12).
- Nothing needs to be shown for (5.13), because $\Psi_{s_p} = \emptyset$.

Hence, $(\mathcal{I}, a')$ satisfies the premise s_p.

(**Case** $\exists L$) Suppose that the conclusion sequent $s_c =_{df} \Theta\,;\,\Sigma\,;\,\Gamma,\ \exists R.C \vdash \Phi\,;\,\Psi$ is satisfiable. Def. 5.2.2 implies that there is a pair $(\mathcal{I}, a)$ that satisfies the sequent s_c, in particular a is contained in the interpretation of each concept in $\Gamma, \exists R.C$.

The assumption $\mathcal{I}; a \models \exists R.C$ implies for all refinements of a that there exists an R-successor which lies in the interpretation of C. Then, it follows by reflexivity of $\preceq^{\mathcal{I}}$, *i.e.*, $a \preceq^{\mathcal{I}} a$, that there exists an entity b such that $a\, R^{\mathcal{I}}\, b$ and $\mathcal{I}; b \models C$. We claim that $(\mathcal{I}, b)$ satisfies the premise sequent $s_p =_{df} \Theta\,;\,\emptyset\,;\,\Sigma(R),\ C \vdash \Psi(R)\,;\,\emptyset$.

- Regarding (5.10) nothing needs to be shown, since the set of role mappings to a set of concepts is empty in s_p.
- The goal $\mathcal{I}; b \models \Sigma(R) \cup \{C\}$, condition (5.11), follows by construction.
- For (5.12) we need to show for all $N \in \Psi(R)$ that $\mathcal{I}; b \not\models N$. By the assumption this is the case for all R-successors of a, in particular for b.
- For condition (5.13) nothing needs to be shown, since $\Psi(R) = \emptyset$ in s_p.

(**Case** $\exists R$) Suppose for the conclusion sequent that $\Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi,\ \exists R.C\,;\,\Psi$. We need to show that $\Theta\,;\,\Sigma\,;\,\Gamma \not\models \emptyset\,;\,[R \mapsto C]$. Def. 5.2.2 implies that there exists a pair $(\mathcal{I}, a)$ that satisfies the conclusion sequent, in particular $\mathcal{I}; a \not\models \exists R.C$. The latter implies that there is an entity a' with $a \preceq^{\mathcal{I}} a'$ such that none of its R-fillers is contained in $C^{\mathcal{I}}$. We claim that $(\mathcal{I}, a')$ satisfies the premise sequent.

- Condition (5.10) follows from transitivity of $\preceq^{\mathcal{I}}$ and the assumption.
- (5.11) follows by the assumption and monotonicity of refinement.
- Nothing needs to be shown for (5.12).
- Finally, condition (5.13), follows from construction of a'.

Therefore, $(\mathcal{I}, a')$ satisfies s_p.

(**Case** $\forall L$) Suppose that the conclusion $s_c =_{df} \Theta\,;\,\Sigma\,;\,\Gamma,\ \forall R.C \vdash \Phi\,;\,\Psi$ is satisfiable. We will show that its premise $s_p =_{df} \Theta\,;\,\Sigma \cup [R \mapsto C]\,;\,\Gamma \vdash \Phi\,;\,\Psi$ is satisfiable, too. By assumption $\Theta\,;\,\Sigma\,;\,\Gamma,\,\forall R.C \not\models \Phi\,;\,\Psi$ that implies that there is a pair $(\mathcal{I}, a)$ satisfying s_c, in particular $\mathcal{I}; a \models \forall R.C$, which means that all R-successors of all refinements of a are in $C^{\mathcal{I}}$. The pair $(\mathcal{I}, a)$ satisfies the premise s_p:

- Condition (5.10) for $\bigcup_{R' \neq R} \Sigma(R') \cup \Sigma(R) \cup \{C\}$ follows directly by assumption.
- (5.11) follows by assumption, because $\Gamma \subseteq \Gamma \cup \{\forall R.C\}$.
- Nothing needs to be shown for (5.12) and (5.13) for Φ and Ψ, because, these sets are equal in s_c and s_p.

Therefore, $(\mathcal{I}, a)$ satisfies s_p.

(**Case** $\forall R$) Suppose for the conclusion that $\Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi,\,\forall R.C\,;\,\Psi$. The goal is to show for the premise that By assumption it holds that $\mathcal{I}; a \not\models \forall R.C$, *i.e.*, there exists an entity a' with $a \preceq^{\mathcal{I}} a'$ and an R-successor b with $a'\,R^{\mathcal{I}}\,b$ such that $\mathcal{I}; b \not\models C$. We claim that $(\mathcal{I}, b)$ satisfies the premise sequent:

- For the conditions (5.10) and (5.13) nothing needs to be shown, since the respective sets are empty in s_p.
- Condition (5.11), *i.e.*, $\mathcal{I}; b \models \Sigma(R)$ follows by assumption.
- The assumption $\mathcal{I}; b \not\models C$ proves condition (5.12).

Therefore, $(\mathcal{I}, b)$ satisfies the premise sequent.

(**Case** Hyp_1, Hyp_2) Finally, to show soundness of Hyp_1 and Hyp_2 let us assume their conclusion sequent $s_c =_{df} \Theta,\,C\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ is satisfiable. The goal is to demonstrate that the premise sequent $s_{p_1} =_{df} \Theta\,;\,\Sigma \cup [R \mapsto C]\,;\,\Gamma \vdash \Phi\,;\,\Psi$ of rule Hyp_1 and $s_{p_2} =_{df} \Theta\,;\,\Sigma\,;\,\Gamma,\,C \vdash \Phi\,;\,\Psi$ of rule Hyp_2 is satisfiable as well.

The assumption $\Theta,\,C\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi$ implies by Def. 5.2.2 that there is a pair $(\mathcal{I}, a)$ that satisfies the conclusion s_c, *i.e.*, $\mathcal{I} \models \Theta \cup \{C\}$, *i.e.*, C holds at every entity in $\mathcal{I}$. We claim that $(\mathcal{I}, a)$ satisfies s_{p_1} and s_{p_2}.

- For premise s_{p_1}, condition (5.10) directly follows by assumption, *i.e.*, we have $\forall a', b.(a \preceq^{\mathcal{I}} a' \;\&\; a'\,R^{\mathcal{I}}\,b) \Rightarrow \mathcal{I}; b \models C$ and in particular $\forall b.a\,R^{\mathcal{I}}\,b \Rightarrow \mathcal{I}; b \models C$. Then, $(\mathcal{I}, a)$ satisfies the premise s_{p_1} taking into account that the remaining conditions (5.11), (5.12), (5.13) follow by assumption.
- Regarding the premise s_{p_2}, it follows from the assumption that $\mathcal{I}; a \models C$ and furthermore this holds for all refinements $a \preceq^{\mathcal{I}} a'$. The conditions (5.9), (5.10), (5.11), (5.12), (5.13) for Σ, Γ, Φ and Ψ follow by assumption. Hence, s_{p_2} is satisfied by $(\mathcal{I}, a)$. □

Completeness

This section shows that the Gentzen sequent calculus G1 is *strongly complete*, *i.e.*, complete w.r.t. the semantic consequence relation, in the sense that every sequent is consistent if and only if it is satisfiable. This section extends the completeness proof that has been published in [190; 195] by giving proofs for all auxiliary lemmata, by generalising the saturation of consistent sequents, and by using a more compact notation. In order to prove the completeness direction of Thm. 5.2.1 we first need some technical definitions and auxiliary lemmas.

Notation. It is technically convenient that we have assumed that the functions Σ, Ψ are defined for all role names. We may then lift set operations to the functions Σ, Ψ from role names into sets of concepts in the standard way. E.g., for every $R \in N_R$ we put $(\Sigma_1 \cup \Sigma_2)(R) =_{df} \Sigma_1(R) \cup \Sigma_2(R)$. Further, $\Sigma_1 \subseteq \Sigma_2$ holds iff for all $R \in N_R$, $\Sigma_1(R) \subseteq \Sigma_2(R)$. In this spirit we identify the empty set $\emptyset$ with the empty function $\emptyset(R) = \emptyset$. Furthermore, we assume that Θ is a fixed TBox. ■

Lemma 5.2.1 ([190; 195]). *Let $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ be an inconsistent sequent,* i.e.*, there exists a closed derivation for it based on the rules in Fig. 5.1. Then, the following holds:*

(i) $|\Phi \cup (\bigcup_{R \in N_R} \Psi(R))| \geq 1$*;*

(ii) For every weakening $\Theta \subseteq \Theta'$, $\Sigma \subseteq \Sigma'$, $\Gamma \subseteq \Gamma'$, $\Phi \subseteq \Phi'$ and $\Psi \subseteq \Psi'$ the sequent $\Theta'; \Sigma'; \Gamma' \vdash \Phi'; \Psi'$ is inconsistent as well, without increase in derivation height. ∇

Proof. Let $s = \Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ be an inconsistent sequent, *i.e.*, there exists a closed derivation for s. The proof is by induction on the height of the derivation. We only give an indication below to show what is involved and then cover the cases $\sqcap L$, $\supset R$, $\exists L$, $\exists R$ and $\forall R$.

For the base case, let $n = 0$. Then, the last rule applied can only be Ax or $\bot L$ and either C is a concept in $\Gamma \cap \Phi$ or $\bot \in \Gamma$ and $|\Phi \cup \Psi| \geq 1$ which shows condition (i). Regarding (ii), in either case, the weakening $\Theta'; \Sigma'; \Gamma' \vdash \Phi'; \Psi'$ of s is an axiom Ax or concluded by $\bot L$, since either $C \in \Gamma' \cap \Phi'$ or $\bot \in \Gamma'$ and $|\Phi' \cup \Psi'| \geq 1$.

Suppose now that the conditions (i) and (ii) are admissible up to derivations of height n and let s be derived in $n + 1$ steps. Note that the condition (i) follows easily by induction hypothesis in all cases of left rules and Hyp_1, Hyp_2, and is obvious by construction in all cases of right rules. The proof is by case analysis on the last rule applied to the derivation and by utilising the induction hypothesis:

(**Case** $\sqcap L$) If the last rule applied is $\sqcap L$, then $\Gamma = \Gamma_1,\ C_1 \sqcap C_2$ and the last step is

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma_1,\ C_1,\ C_2 \mathrel{|\!\!-} \Phi\,;\,\Psi}}{\Theta\,;\,\Sigma\,;\,\Gamma_1,\ C_1 \sqcap C_2 \mathrel{|\!\!-} \Phi\,;\,\Psi}\,\sqcap L.$$

The premise $s_p =_{df} \Theta\,;\,\Sigma\,;\,\Gamma_1, C_1, C_2 \mathrel{|\!\!-} \Phi\,;\,\Psi$ is derivable in $\leq n$ steps. By ind. hyp. the conditions (i) and (ii) hold for the premise, *i.e.*, $|\Phi \cup (\bigcup_{R\in N_R} \Psi(R))| \geq 1$ and the weakening $s_{p'} =_{df} \Theta'\,;\,\Sigma'\,;\,\Gamma_1',\ C_1,\ C_2 \mathrel{|\!\!-} \Phi'\,;\,\Psi'$ is derivable in $\leq n$ steps as well. Then, an application of rule $\sqcap L$ gives for s_p a derivation of the conclusion sequent in $\leq n+1$ steps where (i) holds, and, for $s_{p'}$ we obtain a derivation of

$$\Theta'\,;\,\Sigma'\,;\,\Gamma_1',\ C_1 \sqcap C_2 \mathrel{|\!\!-} \Phi'\,;\,\Psi'$$

in $\leq n+1$ steps where the condition (ii) is satisfied as well.

(**Case** $\supset R$) Suppose the sequent s is derived by rule $\supset R$. Then, $\Phi = \Phi_1,\ C \supset D$ and the last rule application looks like

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma,\ C \mathrel{|\!\!-} D\,;\,\emptyset}}{\Theta\,;\,\Sigma\,;\,\Gamma \mathrel{|\!\!-} \Phi_1,\ C \supset D\,;\,\Psi}\,\supset R.$$

Applying the induction hypothesis to the premise implies that the conditions (i) and (ii) hold, in particular this means that the sequent $\Theta'\,;\,\Sigma'\,;\,\Gamma',\ C \mathrel{|\!\!-} D\,;\,\emptyset$ is derivable in $\leq n$ steps. It follows by application of rule $\supset R$ that the conclusion sequent and its weakening $\Theta'\,;\,\Sigma'\,;\,\Gamma' \mathrel{|\!\!-} \Phi_1',\ C \supset D\,;\,\Psi'$ are derivable in $\leq n+1$ steps where the conditions (i) and (ii) apply, too.

The remaining propositional cases follow by similar arguments. Next, we will consider the modal cases $\exists L$, $\exists R$ and $\forall R$:

(**Case** $\exists L$) Suppose the sequent s is derived by rule $\exists L$. Then, $\Gamma = \Gamma_1,\ \exists R.C$ and the last step is

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\emptyset\,;\,\Sigma(R),\ C \mathrel{|\!\!-} \Psi(R)\,;\,\emptyset}}{\Theta\,;\,\Sigma\,;\,\Gamma_1,\ \exists R.C \mathrel{|\!\!-} \Phi\,;\,\Psi}\,\exists L.$$

By ind. hyp. applied to the premise it holds that $|\Psi(R)| \geq 1$ and the weakened sequent $\Theta'\,;\,\emptyset\,;\,\Sigma'(R),\ C \mathrel{|\!\!-} \Psi'(R)\,;\,\emptyset$ is derivable in $\leq n$ steps. Then, by application of rule $\exists R$ the conclusion sequent and its weakening $\Theta'\,;\,\Sigma'\,;\,\Gamma_1',\ \exists R.C \mathrel{|\!\!-} \Phi'\,;\,\Psi'$ are derivable in $\leq n+1$ steps. Obviously, $|\Phi \cup \Psi| \geq |\Psi(R)| \geq 1$. This shows that conditions (i) and (ii) are satisfied.

(**Case** $\exists R$) Assume that the sequent s is derived by rule $\exists R$. This means that $\Phi = \Phi_1,\ \exists R.C$ and the last derivation step looks like

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\ \emptyset\,;\,[R \mapsto C]}}{\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\ \Phi_1,\ \exists R.C\,;\,\Psi}\ \exists R.$$

The ind. hyp. lets us conclude the conditions (i) and (ii) for the premise sequent, and, in particular that the weakened sequent $\Theta'\,;\,\Sigma'\,;\,\Gamma' \mid\!\!-\ \emptyset\,;\,[R \mapsto C]$ is derivable in $\leq n$ steps. The conclusion and its weakened variant $\Theta'\,;\,\Sigma'\,;\,\Gamma' \mid\!\!-\ \Phi'_1,\ \exists R.C\,;\,\Psi'$ are derivable in $\leq n+1$ steps by rule $\exists R$ which shows that the conditions (i) and (ii) hold, too.

(**Case** $\forall R$) Let us assume that s is derived by rule $\forall R$, *i.e.*, $\Phi = \Phi_1,\ \forall R.C$ and the last step is

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\emptyset\,;\,\Sigma(R) \mid\!\!-\ C\,;\,\emptyset}}{\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\ \Phi_1,\ \forall R.C\,;\,\Psi}\ \forall R.$$

By ind. hyp. applied to the premise s_p, condition (i) holds, and the weakened sequent $s_{p'} =_{df} \Theta'\,;\,\emptyset\,;\,\Sigma(R)' \mid\!\!-\ C\,;\,\emptyset$ is derivable in $\leq n$ steps. Applying rule $\exists R$ yields that the conclusion sequent and its weakening $\Theta'\,;\,\Sigma'\,;\,\Gamma' \mid\!\!-\ \Phi'_1,\ \forall R.C\,;\,\Psi'$ are derivable in $\leq n+1$ steps, which immediately shows conditions (i) and (ii).

The remaining cases $\forall L$, *Hyp*$_1$ and *Hyp*$_2$ follow by induction hypothesis. □

From the first condition (i) expressed by Lem. 5.2.1 it can be concluded that all sequents with an empty succedent are necessarily consistent, *i.e.*, sequents of the form $\Theta;\Sigma;\Gamma \mid\!\!-\ \emptyset;\emptyset$. In fact, such sequents are satisfiable in any interpretation with a fallible entity. In what follows we will introduce the construction of consistent and saturated sequents. Later, we will rely on such sequents when building the canonical model in the sense that every entity of the canonical model is associated with a consistent and saturated sequent.

Notation. In the following, when we refer to a sequent s we assume that it has the form $\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\ \Phi\,;\,\Psi$ if not specified explicitly. For an explicitly defined sequent s as an entity of a domain Δ, we write $\langle\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\ \Phi\,;\,\Psi\rangle$. We use an index $i \geq 0$ and write s_i to denote the sequent $\Theta_i\,;\,\Sigma_i\,;\,\Gamma_i \mid\!\!-\ \Phi_i\,;\,\Psi_i$ that is component-wise indexed by i. In the other direction, when we refer to one of the components $\Theta, \Sigma, \Gamma, \Phi$ or Ψ of a sequent s_i with $i \geq 0$, we write $\Theta_i, \Sigma_i, \Gamma_i, \Phi_i$ and Ψ_i. If the sequent is named without an index like $s, s', s'', \ldots$, then we write $\Theta_s, \Sigma_s, \Gamma_s, \Phi_s$ and Ψ_s instead. ■

Definition 5.2.4 (Subsequent). Let s and s' be sequents. We say that s is a *subsequent* of s' written $s \subseteq s'$ iff $\Theta_s \subseteq \Theta_{s'}$, $\Sigma_s \subseteq \Sigma_{s'}$, $\Gamma_s \subseteq \Gamma_{s'}$, $\Phi_s \subseteq \Phi_{s'}$ and $\Psi_s \subseteq \Psi_{s'}$. ∇

Definition 5.2.5 (Finite sequent). A sequent s is called *finite* if every component $\Theta_s, \Sigma_s(R), \Gamma_s, \Phi_s$ and $\Psi_s(R)$ with $R \in N_R$ is finite, and $\Sigma_s(R), \Psi_s(R) = \emptyset$ for all but a finite number of roles $R \in N_R$. Otherwise we say that s is *infinite.* ∇

Definition 5.2.6 (Union of sequents). Given two sequents s and s', the *union* of s and s' written $s \cup s'$ is given by their component-wise union by taking the sequent

$$\Theta_s \cup \Theta_{s'}\,;\, \Sigma_s \cup \Sigma_{s'}\,;\, \Gamma_s \cup \Gamma_{s'} \vdash \Phi_s \cup \Phi_{s'}\,;\, \Psi_s \cup \Psi_{s'}. \qquad \nabla$$

Definition 5.2.7 (Basic notions [73; 1]). A *partially ordered set* (or poset for short) is a pair $(X, \leqslant)$, consisting of a set X together with a binary relation $\leqslant\, \subseteq X \times X$ that is reflexive, transitive and antisymmetric. Given a poset $(X, \leqslant)$ and a subset $Y \subseteq X$, $x \in X$ is an *upper bound* of Y if $\forall y \in Y.\, y \leq x$; moreover, $x \in X$ is called the *least upper bound* of Y if for all upper bounds y of Y, $x \leqslant y$. A subset $Y \subseteq X$ of a poset $(X, \leqslant)$ is a *chain* if every pair of elements of Y is comparable, that is, for all $x, y \in Y$ either $x \leqslant y$ or $y \leqslant x$. If Y is a chain then $(Y, \leqslant)$ is called a *total order.* For a poset $(X, \leqslant)$ the interval between two elements $x, y \in X$ is defined by $[x, y] =_{df} \{z \in X \mid x \leqslant z \leqslant y\}$. A poset is called *locally finite* if every interval of it is finite. An *ω-chain* of a poset $(X, \leqslant)$ is a sequence of elements $\{x_i \mid i \in \mathbb{N}_0\}$ such that for all $i \in \mathbb{N}_0$, $x_i \leqslant x_{i+1}$, *i.e.*, $x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots$, that is, an ω–chain is isomorphic to the natural numbers. $(X, \leqslant)$ is called a *complete partial order* (cpo for short) if it has a least element, $\bot \in X$, and the least upper bound $\bigsqcup Y$ exists for all chains Y of X. A poset $(X, \leqslant)$ in which every ω–chain has a least upper bound is called an *ω-complete poset* (ω–cpo). A function $f : X \to X$ on X is *monotonic* if $\forall x, x' \in X.\, x \leqslant x' \Rightarrow f(x) \leqslant f(x')$, and f is *increasing* if $x \leqslant f(x)$ for all $x \in X$. ∇

Lemma 5.2.2 (ω-cpo of consistent sequents). *Let Δ^* be the set of all consistent sequents of the form $\langle \Theta\,;\, \Sigma\,;\, \Gamma \vdash \Phi\,;\, \Psi \rangle$. The partial order $(\Delta^*, \subseteq)$ is an ω-cpo over Δ^*,* i.e., *the empty sequent $s_\bot =_{df} \langle \emptyset\,;\, \emptyset\,;\, \emptyset \vdash \emptyset\,;\, \emptyset \rangle$ is the bottom element and every ω-chain $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots \in \Delta^*$ has the least upper bound $\bigcup_{i \geq 0} s_i \in \Delta^*$.* ∇

Proof. First, we show that $\bigcup_{i \geq 0} s_i \in \Delta^*$ is the least upper bound of $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots \in \Delta^*$. Clearly, it holds for all s_i ($i \geq 0$) that $s_i \subseteq \bigcup_{i \geq 0} s_i \in \Delta^*$. Suppose that the consistent sequent s_u is an upper bound such that $\forall i \geq 0. s_i \subseteq s_u$. Then, $\bigcup_{i \geq 0} s_i \subseteq s_u$, whence $\bigcup_{i \geq 0} s_i \in \Delta^*$ is the least upper bound of the chain $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots \in \Delta^*$. Secondly, we show that Δ^* is closed under union, *i.e.*, the least upper bound of a chain of consistent sequents is consistent. Let $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots \in \Delta^*$ be a chain of consistent sequents and suppose that $\bigcup_{i \geq 0} s_i$ is inconsistent. It follows from compactness, saying that the derivation of a sequent s can be restricted to a finite subsequent of s, that there already exists a finite sequent s_k in the chain at position k that is inconsistent. However, this contradicts the assumption that the chain is consistent. Therefore, the least upper bound is consistent. $\square$

Next, we define the notion of an extension rule over the set Δ^* of consistent sequents.

Definition 5.2.8 (Extension rule). An *extension rule* ν is a map $\nu : \Delta^* \to \Delta^*$ on sequents satisfying the conditions below, where we write $s \to_\nu s'$ to denote that $\nu(s) = s'$.

(i) if $s \to_\nu s'$ then $s \subseteq s'$;

(ii) for all $s, s', s'', s''' \in \Delta^*$ if $s \to_\nu s'$ and $s \subseteq s'' \to_\nu s'''$ then $s' \subseteq s'''$;

(iii) $\nu(s) \neq s \Rightarrow \exists$ finite $s' \subseteq s$ such that $\forall s'' \supseteq s'.\ \nu(s'') = \nu(s') \cup s''$. ∇

Note that Def. 5.2.8 requires that an extension rule is consistency preserving, that is, if a sequent s is consistent and $s \to_\nu s'$ then s' is consistent as well.

Definition 5.2.9 (Single extension step). Let $X_\mathcal{L}$ be a well-ordered, locally finite set of extension rules over Δ^* and $s \in \Delta^*$ be a sequent. A *single extension step* is the map $\hat{X}_\mathcal{L} : \Delta^* \to \Delta^*$ such that $\hat{X}_\mathcal{L}(s) = \nu(s)$ where ν is the smallest element in $X_\mathcal{L}$ such that $s \subset \nu(s)$ if it exists, or otherwise $\hat{X}_\mathcal{L}(s) = s$ if for all $\nu \in X_\mathcal{L}.\nu(s) = s$. ∇

Lemma 5.2.3. *Let $X_\mathcal{L}$ be a well-ordered, locally finite set of extension rules. The single step extension $\hat{X}_\mathcal{L}$ w.r.t. $X_\mathcal{L}$ satisfies the properties below:*

(i) $\hat{X}_\mathcal{L}$ is monotonic, i.e., *for all consistent sequents $s, s' \in \Delta^*$ with $s \subseteq s'$ it holds that $\hat{X}_\mathcal{L}(s) \subseteq \hat{X}_\mathcal{L}(s')$;*

(ii) $\hat{X}_\mathcal{L}$ is increasing, i.e., *it satisfies $s \subseteq \hat{X}_\mathcal{L}(s)$ for all $s \in \Delta^*$;*

(iii) $\hat{X}_\mathcal{L}$ is ω-continuous, i.e., *it holds that $\hat{X}_\mathcal{L}\big(\bigcup_{i\geq 0} s_i\big) = \bigcup_{i\geq 0} \hat{X}_\mathcal{L}(s_i)$, where $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots \in \Delta^*$ is an ω-chain.* ∇

Proof. Monotonicity (i) of $\hat{X}_\mathcal{L}$ follows immediately from the fact that all rules $\nu \in X_\mathcal{L}$ are monotonic by Def. 5.2.8.(ii). Claim (ii) follows immediately from Def. 5.2.9. Regarding property (iii), we show that $\hat{X}_\mathcal{L}\big(\bigcup_{i\geq 0} s_i\big) = \bigcup_{i\geq 0} \hat{X}_\mathcal{L}(s_i)$.

($\Rightarrow$) The goal is to demonstrate that $\hat{X}_\mathcal{L}\big(\bigcup_{i\geq 0} s_i\big) \subseteq \bigcup_{i\geq 0} \hat{X}_\mathcal{L}(s_i)$. Consider the sequent $\hat{X}_\mathcal{L}\big(\bigcup_{i\geq 0} s_i\big)$. By Def. 5.2.9 there are two possible cases for $\hat{X}_\mathcal{L}$, namely

- $\hat{X}_\mathcal{L}\big(\bigcup_{i\geq 0} s_i\big) = \nu\big(\bigcup_{i\geq 0} s_i\big)$ for some rule $\nu \in X_\mathcal{L}$ such that $\bigcup_{i\geq 0} s_i \subset \nu\big(\bigcup_{i\geq 0} s_i\big)$, or
- $\hat{X}_\mathcal{L}\big(\bigcup_{i\geq 0} s_i\big) = \bigcup_{i\geq 0} s_i$.

We only cover the first case, since the second is included in the first. From Def. 5.2.8.(iii) follows that there exists a finite sequent s' with $s' \subseteq \bigcup_{i\geq 0} s_i$ such that $\forall s'' \supseteq s'$. $\nu(s'') = \nu(s') \cup s''$. In particular $s' \subseteq \bigcup_{i\geq 0} s_i$ and therefore $\nu(\bigcup_{i\geq 0} s_i) = \nu(s') \cup \bigcup_{i\geq 0} s_i$. We proceed by case analysis:

Case 1. The goal is to show that $\nu(s') \subseteq \bigcup_{i\geq 0} \hat{X}_{\mathcal{L}}(s_i)$. The assumption $s' \subseteq \bigcup_{i\geq 0} s_i$ implies $s' \subseteq s_j$ for some $0 \leq j$ in the chain, and by Def. 5.2.8.(iii) $\nu(s_j) = \nu(s') \cup s_j$. We proceed by case analysis on ν:

- Case 1.1 If ν is the minimal applicable rule of $X_{\mathcal{L}}$ w.r.t. s_j then $\nu(s_j) = \hat{X}_{\mathcal{L}}(s_j) \subseteq \hat{X}_{\mathcal{L}}(s_j) \cup \bigcup_{i\geq 0} \hat{X}_{\mathcal{L}}(s_i) = \bigcup_{i\geq 0} \hat{X}_{\mathcal{L}}(s_i)$.

- Case 1.2 Otherwise, ν is not minimal w.r.t. s_j.

  Let $s_j = s_{j_1}$. Because ν is not minimal w.r.t. s_{j_1} there must exist a rule $\nu_1 < \nu$ that is minimal w.r.t. s_{j_1} such that $s_{j_1} \subset \nu_1(s_{j_1})$. Observe that $\nu_1(s_{j_1})$ is already contained in $\bigcup_{i\geq 0} s_i$, because minimality of ν w.r.t. $\bigcup_{i\geq 0} s_i$ implies that ν_1 does not add anything new to $\bigcup_{i\geq 0} s_i$, *i.e.*, $\nu_1(s_j) \subseteq \nu_1(\bigcup_{i\geq 0} s_i) = \bigcup_{i\geq 0} s_i$, because otherwise ν would not be minimal w.r.t. $\bigcup_{i\geq 0} s_i$ contradictory to the assumption.

  Now, let us consider the sequent s_{j_2} such that $s_{j_1} \subset \nu_1(s_{j_1}) \subseteq s_{j_2}$. If ν is not minimal w.r.t. s_{j_2} then there must exist a rule ν_2 in $X_{\mathcal{L}}$ which is minimal w.r.t. s_{j_2} such that $\nu_2 \neq \nu_1$ and $\nu_2 < \nu$ and $s_{j_2} \subset \nu_2(s_{j_2}) \subseteq \bigcup_{i\geq 0} s_i$.

  Due to the fact that ν is fixed and $X_{\mathcal{L}}$ a well-ordered and locally finite set of rules there are only finitely many rules ν_k with $k \geq 0$ such that $\nu_k < \nu$, *i.e.*, there cannot be an infinite chain of rule applications until we reach the case that ν becomes minimal w.r.t. a sequent $s_{j^*} \subseteq \bigcup_{i\geq 0} s_i$. Hence, at some point we reach the case that ν is minimal w.r.t. a sequent s_{j^*} of a chain $s_{j_1} \subseteq s_{j_2} \subseteq \ldots \subseteq s_{j^*}$ that is constructed as described above. Then, $\nu(s') \subseteq \nu(s_{j^*})$ holds by monotonicity Def. 5.2.8.(ii) and $\nu(s_{j^*}) \subseteq \bigcup_{i\geq 0} \hat{X}_{\mathcal{L}}(s_i)$ is argued analogously to Case 1.1 above.

Case 2. The goal is to demonstrate that $\bigcup_{i\geq 0} s_i \subseteq \bigcup_{i\geq 0} \hat{X}_{\mathcal{L}}(s_i)$. This is a direct consequence of Lem. 5.2.3.(ii) (*increasing*), *i.e.*, it holds that $s_i \subseteq \hat{X}_{\mathcal{L}}(s_i) \subseteq \bigcup_{i\geq 0} \hat{X}_{\mathcal{L}}(s_i)$ which was to be shown.

($\Leftarrow$) We need to show that $\bigcup_{i\geq 0} \hat{X}_{\mathcal{L}}(s_i) \subseteq \hat{X}_{\mathcal{L}}\big(\bigcup_{i\geq 0} s_i\big)$ holds. This follows from $s_i \subseteq \bigcup_{i\geq 0} s_i$ and monotonicity of $\hat{X}_{\mathcal{L}}$ by Lem. 5.2.3.(i). $\square$

Definition 5.2.10 (Saturation). Let $s =_{df} \langle \Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi \,;\, \Psi \rangle$ be a sequent and $X_{\mathcal{L}}$ a well-ordered and locally finite set of extension rules. A sequent s is called *$X_{\mathcal{L}}$-saturated* if $\hat{X}_{\mathcal{L}}(s) = s$, *i.e.*, it holds for all rules $\nu \in X_{\mathcal{L}}$ that $\nu(s) = s$. ∇

Lemma 5.2.4 (Consistent and saturated extension)**.** *Every consistent sequent has a consistent and $X_\mathcal{L}$-saturated extension w.r.t. a well-ordered, locally finite set $X_\mathcal{L}$ of extension rules.* ∇

Proof. Let $s =_{df} \langle \Theta\,;\, \Sigma\,;\, \Gamma \vdash \Phi\,;\, \Psi\rangle \in \Delta^*$ be a consistent sequent and $X_\mathcal{L}$ a set of extension rules according to Def. 5.2.8. We construct monotonically increasing sequences

$$s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots \subseteq s_n \subseteq \ldots$$

of sequents s_n according to the extension rules $X_\mathcal{L}$ by taking $s_{i+1} = \hat{X}_\mathcal{L}(s_i)$ and by starting from $s_0 = s$. The application of the extension rules from $X_\mathcal{L}$ generates a chain $\{s_i \mid i \geq 0\}$. The construction continues until saturation is reached. Note that $s_0 \subseteq s_1$ by property (ii) of Lem. 5.2.3 and $s_i \subseteq s_{i+1}$ for $i \geq 1$ because of Lem. 5.2.3.(i). From Lem. 5.2.2 it follows that that $s^* =_{df} \bigcup_{n<\omega} s_n \in \Delta^*$ is the least upper bound for the chain $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots \subseteq s_n \subseteq \ldots$ such that $s \subseteq s^*$. Finally, taking the ω-complete partial order $(\Delta^*, \subseteq)$ we can conclude from continuity of $\hat{X}_\mathcal{L}$ by Lem. 5.2.3.(iii) and the fixed-point theorem of Kleene that the least fixed-point w.r.t. $\hat{X}_\mathcal{L}$ is the supremum s^* of the ascending chain such that $\hat{X}_\mathcal{L}(s^*) = s^*$. ☐

Definition 5.2.11 (Extension rules for c$\mathcal{ALC}$)**.** Let C, D be c$\mathcal{ALC}$ concepts, $R \in N_R$ a role and the sequent $s =_{df} \langle \Theta\,;\, \Sigma\,;\, \Gamma \vdash \Phi\,;\, \Psi\rangle \in \Delta^*$ be consistent. The extension rules for c$\mathcal{ALC}$ are a countable set built up from the following extension rules, which are parametrised w.r.t. concepts and roles. For instance, we will write E$i_{C,R}$ to denote the family of rules that are parametrised in the index $i \in \mathbb{N}_0$, the concept C and role R. We will just write Ei instead of E$i_{C,R}$ when the parameters are clear from the context. These rules are syntactically ordered by their index and the concepts and roles from the alphabets of lexicographically well-ordered and countable sets of concepts and roles respectively. Then, the extension s' of s is according to the following extension rules if the respective precondition holds, otherwise $s' = s$.

E$0_{C,R}$ $s \rightarrow_{\mathrm{E0}} s' = \langle \Theta\,;\, \Sigma \cup [R \mapsto C]\,;\, \Gamma \cup \{C\} \vdash \Phi\,;\, \Psi\rangle$,
if $C \in \Theta$ and $C \notin \Sigma(R)$ for some $R \in N_R$ or $C \notin \Gamma$.

E$1_{C,D}$ $s \rightarrow_{\mathrm{E1}} s' = \langle \Theta\,;\, \Sigma\,;\, \Gamma \cup \{C,\ D\} \vdash \Phi\,;\, \Psi\rangle$,
if there is $C \sqcap D \in \Gamma$ but $C \notin \Gamma$ or $D \notin \Gamma$.

E$2_{C,D}$ $s \rightarrow_{\mathrm{E2}} s'$ such that

- either $s' = \langle \Theta\,;\, \Sigma\,;\, \Gamma \cup \{C\} \vdash \Phi\,;\, \Psi\rangle$, if it is consistent,
- or otherwise $s' = \langle \Theta\,;\, \Sigma\,;\, \Gamma \cup \{D\} \vdash \Phi\,;\, \Psi\rangle$, if it is consistent,

if there is $C \sqcup D \in \Gamma$ but $C \notin \Gamma$ and $D \notin \Gamma$.

E3$_{C,D}$ $s \to_{\text{E3}} s'$ such that

- either $s' = \langle \Theta\,;\,\Sigma\,;\,\Gamma \cup \{D\} \mid\!\!- \Phi\,;\,\Psi\rangle$, if it is consistent,
- or otherwise $s' = \langle \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!- \Phi \cup \{C\}\,;\,\Psi\rangle$, if it is consistent,

if there exists $C \supset D \in \Gamma$ but $C \notin \Phi$ and $D \notin \Gamma$.

E4$_{C,R}$ $s \to_{\text{E4}} s' = \langle \Theta\,;\,\Sigma \cup [R \mapsto C]\,;\,\Gamma \mid\!\!- \Phi\,;\,\Psi\rangle$,
if $\forall R.C \in \Gamma$ but $C \notin \Sigma(R)$.

E5$_{C,D}$ $s \to_{\text{E5}} s' = \langle \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!- \Phi \cup \{C, D\}\,;\,\Psi\rangle$,
if there is $C \sqcup D \in \Phi$ but $C \notin \Phi$ or $D \notin \Phi$.

E6$_{C,D}$ $s \to_{\text{E6}} s'$ such that

- either $s' = \langle \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!- \Phi \cup \{C\}\,;\,\Psi\rangle$, if it is consistent,
- or otherwise $s' = \langle \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!- \Phi \cup \{D\}\,;\,\Psi\rangle$, if it is consistent,

if $C \sqcap D \in \Phi$ but $C \notin \Phi$ and $D \notin \Phi$.

E7$_R$ $s \to_{\text{E7}} s' = \langle \Theta\,;\,\Sigma \cup [R \mapsto \bot]\,;\,\Gamma \mid\!\!- \Phi\,;\,\Psi\rangle$,
if $\bot \in \Gamma$ but $\bot \notin \Sigma(R)$ for some R. ∇

Note that the rule E0$_{C,R}$ only needs to be applied at most once for each concept $C \in \Theta$. Furthermore, all rules of Def. 5.2.11 only extend the components Σ, Γ and Φ.

Notation. In the following $X_{c\mathcal{ALC}}$ denotes the set of extension rules of Def. 5.2.11. ■

Lemma 5.2.5. *The extension rules $X_{c\mathcal{ALC}}$ of Def. 5.2.11 are according to Def. 5.2.8.* ∇

Proof. Let $s =_{df} \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!- \Phi\,;\,\Psi$ be a consistent sequent. We need to show for all rules $\nu \in X_{c\mathcal{ALC}}$ with $s \to_\nu s'$ that the following conditions are met:

(i) if $s \to_\nu s'$ then $s \subseteq s'$;

(ii) for all $s, s', s'', s''' \in \Delta^*$ if $s \to_\nu s'$ and $s \subseteq s'' \to_\nu s'''$ then $s' \subseteq s'''$;

(iii) $\nu(s) \neq s \Rightarrow \exists$ finite $s' \subseteq s$ such that $\forall s'' \supseteq s'.\ \nu(s'') = \nu(s') \cup s''$;

(iv) ν is consistency preserving, *i.e.*, the sequent s' is consistent.

These conditions can be verified directly by inspection of the rules of Def. 5.2.11. Below, we only give an indication to show what is involved and cover the rules E0, E3, E4 and E7.

E0$_{C,R}$ Suppose that the rule applied is E0$_{C,R}$, *i.e.*, $s \to_{\text{E0}} s'$ such that either $s = s'$ or $s' = \Theta\,;\,\Sigma \cup [R \mapsto C]\,;\,\Gamma \cup \{C\} \mid\!\!- \Phi\,;\,\Psi$, for some $C \in \Theta$ and $R \in N_R$.

- Condition (i) holds trivially by inspection of the rules.
- For monotonicity (ii) let s'', s''' be sequents and suppose that $s \rightarrow_{\mathrm{E0}} s'$ and $s \subseteq s'' \rightarrow_{\mathrm{E0}} s'''$. The goal is to show that $s' \subseteq s'''$. There are two cases: Case 1: If $\mathrm{E0}(s) = s'$ then immediately $s' \subseteq s'''$.

  Case 2: Otherwise, $s \subset s'$. Then, it is either the case that the precondition is not satisfied for $s'' \rightarrow_{\mathrm{E0}} s'''$, *i.e.*, $[R \mapsto C]$ and C are already included in Σ'' and Γ'' respectively, and therefore it holds that $s' \subseteq s'''$. Otherwise, it holds that $s'' \subset s'''$ which together with $s \subseteq s''$ let us conclude that $s' \subseteq s'''$.
- Regarding condition (iii), suppose that $\nu(s) \neq s$. We can easily find a finite and consistent subsequent s_f of s by taking $s_f = C\,;\,\emptyset\,;\,\emptyset \mid\!\!- \emptyset\,;\,\emptyset$ such that for all $s' \supseteq s_f$. $\mathrm{E0}(s') = \mathrm{E0}(s_f) \cup s'$.
- Finally, we show that rule E0 is consistency preserving. If $\mathrm{E0}(s) = s'$ then s' is trivially consistent. Otherwise, suppose that the sequent s' is inconsistent. This implies the existence of a closed derivation for s'. But, by rule Hyp_1 and Hyp_2 of Fig. 5.1 this would yield a closed derivation for the sequent s. This contradicts the assumption that the sequent s is consistent. Hence, the sequent s' is consistent as well.

$\mathrm{E3}_{C,D}$ If the last rule which gets applied is $\mathrm{E3}_{C,D}$ then $s \rightarrow_{\mathrm{E3}} s'$ such that $s = s'$ or, either $s' = \Theta\,;\,\Sigma\,;\,\Gamma \cup \{D\} \mid\!\!- \Phi\,;\,\Psi$ or $s' = \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!- \Phi \cup \{C\}\,;\,\Psi$.

- The conditions (i), (ii) can be argued as above.
- Regarding condition (iii), assume that $\nu(s) \neq s$. We take the finite and consistent sequent $s_f = \emptyset\,;\,C \supset D\,;\,\emptyset \mid\!\!- \emptyset\,;\,\emptyset$ such that for all $s' \supseteq s_f$. $\mathrm{E3}(s') = \mathrm{E3}(s_f) \cup s'$.
- Consistency of E3 is argued as follows: Let us suppose that s' is inconsistent. This implies the existence of a closed derivation for the two latter cases of s'. However, by rule $\supset L$ from Fig. 5.1 this would yield a closed derivation for the sequent s. This contradicts the assumption that s is consistent. Therefore, the sequent s' is consistent as well.

$\mathrm{E4}_{C,R}$ Let us suppose that that the sequent s' has been derived by rule $\mathrm{E4}_{C,R}$, *i.e.*, $s \rightarrow_{\mathrm{E4}} s'$ such that either $s = s'$ or $s' = \Theta\,;\,\Sigma \cup [R \mapsto C]\,;\,\Gamma \mid\!\!- \Phi\,;\,\Psi$.

- The conditions (i), (ii) and (iii) are argued similarly as before.
- Assume that the sequent s' is inconsistent, *i.e.*, there exists a closed derivation for $\Theta\,;\,\Sigma \cup [R \mapsto C]\,;\,\Gamma \mid\!\!- \Phi\,;\,\Psi$. By rule $\forall L$ we obtain

$$\cfrac{\cfrac{\vdots}{\Theta\,;\,\Sigma \cup [R \mapsto C]\,;\,\Gamma \mathrel{|\!\!-} \Phi\,;\,\Psi}}{\Theta\,;\,\Sigma\,;\,\Gamma,\,\forall R.C \mathrel{|\!\!-} \Phi\,;\,\Psi}\;\forall L$$

which would give for $\Theta\,;\,\Sigma\,;\,\Gamma \cup \{\forall R.C\} \mathrel{|\!\!-} \Phi\,;\,\Psi$ a closed derivation. However, this would lead to the inconsistency of the sequent s which is a contradiction. Hence, s' has to be consistent.

E7_R Assume that the sequent s' has been derived by rule E7_R, *i.e.*, $s \to_{\mathrm{E7}} s'$ such that either $s = s'$ or $s' = \Theta\,;\,\Sigma \cup [R \mapsto \bot]\,;\,\Gamma \mathrel{|\!\!-} \Phi\,;\,\Psi$ and $\bot \in \Gamma$.

- The conditions (i), (ii) and (iii) are argued similarly as before.
- Regarding condition (iv) suppose that s' is inconsistent, *i.e.*, there is a derivation of s'. We can conclude by Lem. 5.2.1 that $|\Phi \cup \Psi| \geq 1$. In particular, since $\bot \in \Gamma$ there exists a derivation

$$\frac{|\Phi \cup \Psi| \geq 1}{\Theta\,;\,\Sigma\,;\,\Gamma \mathrel{|\!\!-} \Phi\,;\,\Psi}\;\bot L$$

which contradicts the consistency of the sequent s. Therefore, the sequent s' is consistent.

All other rules can be argued similarly. ☐

Remark 5.2.2. One can show for a finite and consistent sequent s that the process of generating its saturation s^* terminates, which is due to the following facts: Finiteness of s implies that the set of subformulæ contained in s is finite, and this implies that the set of extension rules can be made finite by restricting the family of extension rules to the set of subformulæ of s. The strategy of applying the rules is fair, *i.e.*, each rule fires at some point. Thus, by monotonicity, increasing and the fact that only subformulæ are added, the saturation of s terminates after a finite number of rule applications.

The fixed point construction of the set of saturated and consistent sequents can be further generalised (Lüttgen [179, personal communication]) by using a generalisation of the *Chaotic Fixed Point Iteration Theorem* [110], which states that every *fair* chaotic iteration computes the least fixed point of a complete partial order w.r.t. a family of continuous functions, given a fair strategy in the sense that every function fires at some point of the construction. ■

The remaining section will focus on the construction of a canonical model out of the set of all consistent and $X_{c\mathcal{ALC}}$-saturated sequents.

Definition 5.2.12 (Canonical Interpretation [195]). Let Θ be a fixed TBox and Δ^* be the set of all $X_{c\mathcal{ALC}}$-saturated and consistent sequents of the form $\langle\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!- \Phi\,;\,\Psi\rangle$. All these sequents have Θ as their first component but may have different Σ, Γ, Φ, Ψ. The canonical interpretation $\mathcal{I}^* =_{df} (\Delta^{\mathcal{I}^*}, \preceq^{\mathcal{I}^*}, \perp^{\mathcal{I}^*}, \cdot^{\mathcal{I}^*})$ is defined by

$$\Delta^{\mathcal{I}^*} =_{df} \Delta^*; \tag{5.14}$$

$$\preceq^{\mathcal{I}^*} =_{df} \{(s, s') \in \Delta^{\mathcal{I}^*} \times \Delta^{\mathcal{I}^*} \mid \Sigma_s \subseteq \Sigma_{s'} \quad \& \quad \Gamma_s \subseteq \Gamma_{s'}\}; \tag{5.15}$$

$$R^{\mathcal{I}^*} =_{df} \{(s, s') \in \Delta^{\mathcal{I}^*} \times \Delta^{\mathcal{I}^*} \mid \Sigma_s(R) \subseteq \Gamma_{s'} \quad \& \quad \Psi_s(R) \subseteq \Phi_{s'}\}; \tag{5.16}$$

$$\perp^{\mathcal{I}^*} =_{df} \{s \in \Delta^{\mathcal{I}^*} \mid \perp \in \Gamma_s\}; \tag{5.17}$$

$$A^{\mathcal{I}^*} =_{df} \{s \in \Delta^{\mathcal{I}^*} \mid A \in \Gamma_s \text{ or } \perp \in \Gamma_s\}, \tag{5.18}$$

for all $R \in N_R$ and $A \in N_C$. ∇

Lemma 5.2.6 ([195]). *The canonical interpretation* $\mathcal{I}^* =_{df} (\Delta^{\mathcal{I}^*}, \preceq^{\mathcal{I}^*}, \perp^{\mathcal{I}^*}, \cdot^{\mathcal{I}^*})$ *is a constructive model according to Def. 4.2.2.* ∇

Proof. The proof appeared in [195]. We show that the canonical interpretation $\mathcal{I}^*$ is a constructive interpretation in line with Def. 4.2.2:

- The set $\Delta^{\mathcal{I}^*}$ is obviously non-empty by Def. 5.2.12, for instance consider the consistent sequent $\langle\Theta\,;\,\emptyset\,;\,A \mid\!\!- \emptyset\,;\,\emptyset\rangle$ for any $A \in N_C$.
- The relation $\preceq^{\mathcal{I}^*}$ is reflexive and transitive by construction by (5.15). However, note that $\preceq^{\mathcal{I}^*}$ is not in general antisymmetric.
- Regarding fallible entities we have to show that $\perp^{\mathcal{I}^*}$ is closed under refinement and role-filling.

  By definition $\perp^{\mathcal{I}^*} \subseteq A^{\mathcal{I}^*}$ for all $A \in N_C$. Furthermore, it is obvious for two sequents $s, s' \in \Delta^{\mathcal{I}^*}$ that if $s \preceq^{\mathcal{I}^*} s'$ and s is fallible then s' is fallible, hence $\perp^{\mathcal{I}^*}$ is closed under $\preceq^{\mathcal{I}^*}$.

  Let $s \in \Delta^{\mathcal{I}^*}$ be a consistent and $X_{c\mathcal{ALC}}$-saturated sequent. If $s \in \perp^{\mathcal{I}^*}$ then it holds by (5.17) that $\perp \in \Gamma_s$. First, we show that $R^{\mathcal{I}^*}$ is serial w.r.t. $\perp^{\mathcal{I}^*}$, that is, there exists a consistent and $X_{c\mathcal{ALC}}$-saturated sequent s' such that $\perp \in \Gamma_{s'}$ and $s\ R^{\mathcal{I}^*}\ s'$. Consider the sequent $s' =_{df} \langle\Theta_s\,;\,\emptyset\,;\,\Sigma_s(R) \cup \{\perp\} \mid\!\!- \Psi_s(R)\,;\,\emptyset\rangle$. We claim that the sequent s' is consistent. Suppose, by contradiction that s' is derivable. This implies by Lemma 5.2.1 that $\Psi_s(R) \neq \emptyset$. But, by rule $\perp L$ this contradicts already the consistency of the sequent s. Lemma 5.2.4 lets us conclude that there exists a consistent and $X_{c\mathcal{ALC}}$-saturated extension s^* of s' such that $\Psi_s(R) \subseteq \Phi_{s^*}$ and $\Sigma_s(R) \subseteq \Sigma_s(R) \cup \{\perp\} \subseteq \Gamma_{s^*}$. Hence $s\ R^{\mathcal{I}^*}\ s^*$ and $s^* \in \perp^{\mathcal{I}^*}$. Secondly, we demonstrate that all R-successors s' of s are in

$\perp^{\mathcal{I}^*}$ as well. By assumption $\perp \in \Gamma_s$. Lemma 5.2.4 w.r.t. $X_{c\mathcal{ALC}}$ (rule $E7_R$) and definition of saturation lets us conclude that $\perp \in \Sigma_s(R)$ for all $R \in N_R$. Let s' be an arbitrary R-successor of s such that $\perp \in \Sigma_s(R) \subseteq \Gamma_{s'}$ by (5.16) and therefore $s' \in \perp^{\mathcal{I}^*}$. Since s' was an arbitrary R-successor of s, it follows that all R-successors of s are in $\perp^{\mathcal{I}^*}$ as claimed.

- Finally, we show for all $A \in N_C$ that the interpretation $A^{\mathcal{I}^*}$ is closed under $\preceq^{\mathcal{I}^*}$. Let $s, s' \in \Delta^{\mathcal{I}^*}$ and assume that $s \in A^{\mathcal{I}^*}$ and $s \preceq^{\mathcal{I}^*} s'$. (5.15) implies that $\Gamma_s \subseteq \Gamma_{s'}$, *i.e.*, whenever $s \in A^{\mathcal{I}^*}$ then also $s' \in A^{\mathcal{I}^*}$. Thus, $A^{\mathcal{I}^*}$ is closed under refinement $\preceq^{\mathcal{I}^*}$.

Hence, $\mathcal{I}^*$ is a constructive interpretation as claimed. Note that for all fallible sequents $s \in \Delta^{\mathcal{I}^*}$ consistency implies that $\Phi_s = \Psi_s = \emptyset$, otherwise there would exist a closed derivation by rule $\perp L$. Consequently, whenever Φ_s or $\Psi_s(R)$ (w.r.t. some $R \in N_R$) are nonempty, then we know that $s \in \Delta_c^{\mathcal{I}^*} = \Delta^{\mathcal{I}^*} \setminus \perp^{\mathcal{I}^*}$. □

Having established the canonical model, the remaining agenda is to prove *selfsatisfaction* in the sense that the canonical interpretation $\mathcal{I}^*$ is a constructive model under the terms of Def. 4.2.2 such that for every sequent $s \in \Delta^{\mathcal{I}^*}$ the pair $(\mathcal{I}^*, s)$ satisfies the sequent s according to Def. 5.2.2.

Lemma 5.2.7 (Mendler and Scheele [195]). *Let Θ be a fixed TBox, Δ^* the set of all $X_{c\mathcal{ALC}}$-saturated and consistent sequents. The canonical interpretation $\mathcal{I}^* =_{df} (\Delta^{\mathcal{I}^*}, \preceq^{\mathcal{I}^*}, \perp^{\mathcal{I}^*}, \cdot^{\mathcal{I}^*})$ is a constructive model so that for all $X_{c\mathcal{ALC}}$-saturated and consistent sequents $s \in \Delta^*$ the pair $(\mathcal{I}^*, s)$ satisfies s in the sense of Def. 5.2.2. In particular, $\mathcal{I}^* \models \Theta$ and if $s = \langle \Theta\,;\, \Sigma\,;\, \Gamma \vdash \Phi\,;\, \Psi \rangle$ then for all $R \in N_R$ it holds that*

$$\forall s', s'' \in \Delta^{\mathcal{I}^*}.\, s \preceq^{\mathcal{I}^*} s'\ R^{\mathcal{I}^*}\ s'' \Rightarrow \mathcal{I}^*; s'' \models \Sigma(R); \tag{5.19}$$

$$\mathcal{I}^*; s \models \Gamma; \tag{5.20}$$

$$\mathcal{I}^*; s \not\models \Phi; \tag{5.21}$$

$$\forall s' \in \Delta^{\mathcal{I}^*}.\, s\ R^{\mathcal{I}^*}\ s' \Rightarrow \mathcal{I}^*; s' \not\models \Psi(R). \tag{5.22}$$

∇

Proof. The proof is similar to the one in [195] but uses a more compact notation, relies on a more general form of saturation and argues the case $\forall R.C$ differently. We claim that $\mathcal{I}^* \models \Theta$. This follows from condition (5.20), which is proven below, and the fact that by Def. 5.2.11 and Lem. 5.2.4 all sequents $s \in \Delta^*$ are $X_{c\mathcal{ALC}}$-saturated such that $\Theta_s = \Theta \subseteq \Gamma_s$.

Let $s \in \Delta^{\mathcal{I}^*}$ be an arbitrary but $X_{c\mathcal{ALC}}$-saturated and consistent sequent. To show *self-satisfaction* of the sequent s we will demonstrate the truth conditions (5.19)–(5.22). Notice that the conditions (5.19) and (5.22) follow directly from (5.20) and (5.21) respectively, relying on Def. 5.2.12 of $\preceq^{\mathcal{I}^*}$ and $R^{\mathcal{I}^*}$:

- Let us deal with condition (5.22) first. Let $s' \in \Delta^{\mathcal{I}^*}$ be an arbitrary R-successor of s such that $s\ R^{\mathcal{I}^*}\ s'$. The construction of $R^{\mathcal{I}^*}$ implies by (5.16) that $\Psi_s(R) \subseteq \Phi_{s'}$ and therefore $\mathcal{I}^*; s' \not\models \Psi_s(R)$ by condition (5.21).
- Regarding (5.19) let $s', s'' \in \Delta^{\mathcal{I}^*}$ such that $s \preceq^{\mathcal{I}^*} s'\ R^{\mathcal{I}^*}\ s''$. The definitions of $\preceq^{\mathcal{I}^*}$ and $R^{\mathcal{I}^*}$ (Def. 5.2.12) imply that $\Sigma_s(R) \subseteq \Sigma_{s'}(R) \subseteq \Gamma_{s''}$. Hence, we get $\mathcal{I}^*; s'' \models \Sigma_s(R)$ by (5.20).

In the remaining proof we verify the conditions (5.20) and (5.21), and lift the '*membership=truth*' condition to arbitrary concepts. More precisely, we show simultaneously by induction on the structure of concept C that

$$C \in \Gamma_s \Rightarrow \mathcal{I}^*; s \models C;$$
$$C \in \Phi_s \Rightarrow \mathcal{I}^*; s \not\models C.$$

(**Case** atomic symbol) For the base case let $C = A \in N_C$ or $C = \bot$.

- If $C \in \Gamma_s$ then $\mathcal{I}^*; s \models C$ holds trivially by Def. 5.2.12 of $\mathcal{I}^*$.
- Analogously, if $C \in \Phi_s$ then $\mathcal{I}^*; s \not\models C$, since otherwise we would have that $C \in \Gamma_s$ by Def. 5.2.12 of $\mathcal{I}^*$ which would contradict the consistency of the sequent s by rule *Ax*.

(**Case** $C \sqcap D$)

- Assume that $C \sqcap D \in \Gamma_s$. The fact that the sequent s is $X_{c\mathcal{ALC}}$-saturated implies $C, D \in \Gamma_s$ by rule E1 of Def. 5.2.11, *i.e.*, $\mathcal{I}^*; s \models C$ and $\mathcal{I}^*; s \models D$ by induction hypothesis. Hence, $\mathcal{I}^*; s \models C \sqcap D$.
- Suppose that $C \sqcap D \in \Phi_s$. Saturation of s implies that either $C \in \Phi_s$ or $D \in \Phi_s$ by rule E6 of Def. 5.2.11. The induction hypothesis lets us conclude that $\mathcal{I}^*; s \not\models C$ or $\mathcal{I}^*; s \not\models D$, *i.e.*, $s \notin C^{\mathcal{I}^*} \cap D^{\mathcal{I}^*}$, which implies that $\mathcal{I}^*; s \not\models C \sqcap D$.

(**Case** $C \sqcup D$)

- Suppose that $C \sqcup D \in \Gamma_s$. Because of $X_{c\mathcal{ALC}}$-saturation of s we have $C \in \Gamma_s$ or $D \in \Gamma_s$ by rule E2 of Def. 5.2.11, which means $\mathcal{I}^*; s \models C$ or $\mathcal{I}^*; s \models D$ by induction hypothesis. Hence, $\mathcal{I}^*; s \models C \sqcup D$.

- If $C \sqcup D \in \Phi_s$ then by saturation of s follows $C, D \in \Phi_s$ by rule E5 of Def. 5.2.11. The ind. hyp. implies $\mathcal{I}^*; s \not\models C$ and $\mathcal{I}^*; s \not\models D$. Thus, $\mathcal{I}^*; s \not\models C \sqcup D$.

(**Case** $C \supset D$)

- Suppose that $C \supset D \in \Gamma_s$. The goal is to show that $\mathcal{I}^*; s \models C \supset D$. Let $s' \in \Delta^{\mathcal{I}^*}$ be arbitrary such that $s \preceq^{\mathcal{I}^*} s'$ and $\mathcal{I}^*; s' \models C$. The construction of $\mathcal{I}^*$ implies by Def. 5.2.12 that $C \supset D \in \Gamma_s \subseteq \Gamma_{s'}$. Also, $C \notin \Phi_{s'}$ by (5.21), for otherwise if $C \in \Phi_{s'}$ (5.21) implies $\mathcal{I}^*; s' \not\models D$ which contradicts the assumption. Since the sequent s' is $X_{c\mathcal{ALC}}$-saturated, it follows from rule E3 of Def. 5.2.11 that $D \in \Gamma_{s'}$. The induction hypothesis lets us conclude that $\mathcal{I}^*; s' \models D$. Hence, $\mathcal{I}^*; s \models C \supset D$.
- Conversely, suppose $C \supset D \in \Phi_s$. The goal is to show that there exists an infallible refinement of s that lies in $C^{\mathcal{I}^*}$ but not in $D^{\mathcal{I}^*}$. Let us consider the sequent $s' =_{df} \langle \Theta\,;\, \Sigma_s\,;\, \Gamma_s,\, C \vdash D\,;\, \emptyset \rangle$ which must be consistent. Suppose to the contrary that s' is inconsistent. Then there must exist a closed tableau for s', however, this would contradict the consistency of the sequent s by rule $\supset R$ of Fig. 5.1. Since s' is consistent, we obtain from Lem. 5.2.4 a $X_{c\mathcal{ALC}}$-saturated and consistent extension s^* of s'. Note that $D \in \Phi_{s^*}$ directly implies that s^* is infallible and therefore $s^* \in \Delta_c^{\mathcal{I}^*}$. Now, we observe that $\Sigma_s \subseteq \Sigma_{s^*}$ and $\Gamma_s \subseteq \Gamma_s \cup \{C\} \subseteq \Gamma_{s^*}$ from which we can conclude by construction of $\mathcal{I}^*$ from Def. 5.2.12 that $s \preceq^{\mathcal{I}^*} s^*$. Also, from $C \in \Gamma_{s^*}$ and $D \in \Phi_{s^*}$ we can infer $\mathcal{I}^*; s^* \models C$ and $\mathcal{I}^*; s^* \not\models D$ by ind. hyp. Hence, $\mathcal{I}^*; s \not\models C \supset D$ as desired.

(**Case** $\exists R.C$)

- Assume that $\exists R.C \in \Gamma_s$. We have to show that for all refinements of s there exists an R-successor in the interpretation of C. Let $s' \in \Delta^{\mathcal{I}^*}$ such that $s \preceq^{\mathcal{I}^*} s'$. Definition 5.2.12 of $\preceq^{\mathcal{I}^*}$ implies that $\exists R.C \in \Gamma_{s'}$. Now, let us consider the sequent $s'' =_{df} \langle \Theta\,;\, \emptyset\,;\, \Sigma_{s'}(R),\, C \vdash \Psi_{s'}(R)\,;\, \emptyset \rangle$ which is obviously consistent. Otherwise, there would exist a closed derivation for the sequent s'', but this would contradict consistency of s' by rule $\exists L$ of Fig. 5.1. Thus, s'' is consistent and an application of Lem. 5.2.4 yields a $X_{c\mathcal{ALC}}$-saturated and consistent extension s^* of s''. Now, observe that by construction we have $\Sigma_{s'}(R) \subseteq \Sigma_{s'}(R) \cup \{C\} \subseteq \Gamma_{s^*}$ and $\Psi_{s'}(R) \subseteq \Phi_{s^*}$ such that $s'\ R^{\mathcal{I}^*}\ s^*$ by Def. 5.2.12. Since $C \in \Gamma_{s^*}$, we can conclude that $\mathcal{I}^*; s^* \models C$ by the induction hypothesis which proves $\mathcal{I}^*; s \models \exists R.C$.
- Vice versa, let us assume that $\exists R.C \in \Phi_s$. We have to show that there exists a refinement of s such that all its R-successors are not in $C^{\mathcal{I}^*}$. Consider the sequent $\langle \Theta\,;\, \Sigma_s\,;\, \Gamma_s \vdash \emptyset\,;\, [R \mapsto C] \rangle$ which must be consistent, since otherwise

there would exists a closed tableau for the sequent which by rule $\exists R$ from Fig. 5.1 would contradict the consistency of s. Again, Lem. 5.2.4 yields a $X_{c\mathcal{ALC}}$-saturated and consistent extension $s' \in \Delta_c^{\mathcal{I}^*}$ of $\langle \Theta\,;\,\Sigma_s\,;\,\Gamma_s \mid\!\!- \emptyset\,;\,[R \mapsto C]\rangle$, which is infallible because of $C \in \Psi_{s'}(R)$. By construction holds that $\Sigma_s \subseteq \Sigma_{s'}$ and $\Gamma_s \subseteq \Gamma_{s'}$ and therefore $s \preceq^{\mathcal{I}^*} s'$. Now, let $s'' \in \Delta^{\mathcal{I}}$ be an arbitrary R-successor such that $s'\ R^{\mathcal{I}^*}\ s''$, *i.e.*, $C \in \Psi_{s'}(R) \subseteq \Phi_{s''}$ by definition of $R^{\mathcal{I}^*}$ (Def. 5.2.12). By ind. hyp. $\mathcal{I}^*; s'' \not\models C$ and therefore $\mathcal{I}^*; s \not\models \exists R.C$.

(**Case** $\forall R.D$)

- Proof by contraposition. Suppose that $\mathcal{I}^*; s \not\models \forall R.C$. We have to show that $\forall R.C \notin \Gamma_s$. The assumption implies that there exist $s', s'' \in \Delta_c^{\mathcal{I}^*}$ such that $s \preceq^{\mathcal{I}^*} s'\ R^{\mathcal{I}^*}\ s''$ and $\mathcal{I}^*; s'' \not\models C$. The induction hypothesis implies that $C \notin \Gamma_{s''}$. Furthermore, we know that $\Sigma_s(R) \subseteq \Sigma_{s'}(R) \subseteq \Gamma_{s''}$ from Def. 5.2.12. Now, observe that $\forall R.C \notin \Gamma_s$, since otherwise saturation would give us $C \in \Sigma(R) \subseteq \Sigma_{s'}(R) \subseteq \Gamma_{s''}$ by Lem. 5.2.4 (rule E4) which contradicts the assumption. Hence, $\forall R.C \notin \Gamma_s$.
- Finally, let us assume that $\forall R.C \in \Phi_s$. The goal is to show that there exists a refinement of s with an R-successor which is not in $C^{\mathcal{I}^*}$. We observe that the sequent $s' =_{df} \langle \Theta\,;\,\Sigma_s\,;\,\Gamma_s \mid\!\!- \Phi_s\,;\,\emptyset\rangle$, in which the last constraint is omitted, is consistent and saturated as well, *i.e.*, $s' \in \Delta^{\mathcal{I}^*}$. The former is a consequence of weakening by Lem. 5.2.1 while the latter is obvious, since the component Ψ_s is not affected by any saturation rule of $X_{c\mathcal{ALC}}$. It follows from Def. 5.2.12 that $s \preceq^{\mathcal{I}^*} s'$ and furthermore $\forall R.C \in \Phi_s \neq \emptyset$ implies $s' \in \Delta_c^{\mathcal{I}^*}$. Now, let us consider the sequent $s'' =_{df} \langle \Theta\,;\,\emptyset\,;\,\Sigma_s(R) \mid\!\!- C\,;\,\emptyset\rangle$. The sequent s'' is obviously infallible and consistent, since otherwise if there would exist a closed tableau for s'' then we could obtain a closed tableau for the sequent s' by rule $\forall R$ of Fig. 5.1 contradictory to the assumption. Lemma 5.2.4 yields a consistent and $X_{c\mathcal{ALC}}$-saturated extension s^* of s'' such that $\Sigma_s(R) \subseteq \Gamma_{s^*}$ and $C \in \Phi_{s^*}$, which implies $s'\ R^{\mathcal{I}^*}\ s^*$. The induction hypothesis lets us conclude that $\mathcal{I}^*; s^* \not\models C$. Therefore, $\mathcal{I}^*; s \not\models \forall R.C$. □

Finally, we are ready to argue completeness.

Theorem 5.2.3 (Completeness [195]). *Every consistent sequent is satisfiable,* i.e.,

$$\Theta\,;\,\Sigma\,;\,\Gamma \not\vdash \Phi\,;\,\Psi \Rightarrow \Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi. \qquad \nabla$$

Proof. Let $s = \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!- \Phi\,;\,\Psi$ be a consistent sequent. The goal is to demonstrate that s is satisfiable according to Def. 5.2.2, *i.e.*, there is a constructive model $\mathcal{I}$ and an entity a in $\Delta^{\mathcal{I}}$ such that the pair $(\mathcal{I}, a)$ satisfies the sequent s. An application of Lem. 5.2.4 yields a $X_{c\mathcal{ALC}}$-saturated and consistent extension s^* of s such that $\Sigma_s \subseteq \Sigma_{s^*}$, $\Gamma_s \subseteq \Gamma_{s^*}$ and $\Phi_s \subseteq \Phi_{s^*}$. We can use Def. 5.2.12 to construct a canonical model $\mathcal{I}^* =_{df} (\Delta^{\mathcal{I}^*}, \preceq^{\mathcal{I}^*}, \bot^{\mathcal{I}^*}, \cdot^{\mathcal{I}^*})$ using Θ as a fixed TBox. Finally, Lemma 5.2.7 implies that the pair $(\mathcal{I}^*, s^*)$ satisfies the sequent s^* and particularly satisfies the sequent s, which was to be shown. □

Following the lines of [84, p. 99] and [132] we want to point out that Theorem 5.2.1, *i.e.*, proving soundness and completeness of the cut-free calculus `G1`, allows us to semantically prove the admissibility of the rule *Cut*:

$$\frac{\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\ \Phi,\ C\,;\,\Psi \qquad \Theta\,;\,\Sigma'\,;\,\Gamma',\ C \mid\!\!-\ \Phi'\,;\,\Psi'}{\Theta\,;\,\Sigma,\ \Sigma'\,;\,\Gamma,\ \Gamma' \mid\!\!-\ \Phi,\ \Phi'\,;\,\Psi,\ \Psi'}\ \mathit{Cut}$$

Corollary 5.2.1. *The rule Cut is admissible in the sequent calculus* `G1`*.* ∇

Proof. Let Θ be a fixed but arbitrary TBox and suppose that the two premise sequents $s_{p_1} =_{df} \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!- \Phi,\ C\,;\,\Psi$ and $s_{p_2} =_{df} \Theta\,;\,\Sigma\,;\,\Gamma,\ C \mid\!\!- \Phi\,;\,\Psi$ are derivable, but that the conclusion sequent $s_c =_{df} \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!- \Phi\,;\,\Psi$ is not, *i.e.*, $\Theta\,;\,\Sigma\,;\,\Gamma \not\mid\!\!- \Phi\,;\,\Psi$. Then, completeness of Thm. 5.2.1 implies $\Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi$, *i.e.*, there exists a pair $(\mathcal{I}, a)$ that satisfies s_c. The assumption implies by the soundness direction of Thm. 5.2.1 that $\Theta\,;\,\Sigma\,;\,\Gamma \models \Phi,\ C\,;\,\Psi$ and $\Theta\,;\,\Sigma\,;\,\Gamma,\ C \models \Phi\,;\,\Psi$, which contradicts the choice of the pair $(\mathcal{I}, a)$. □

5.2.2 Decidability of G1

A sequent $\Theta; \Sigma; \Gamma \mid\!\!- \Phi; \Psi$ is *finite* if it has a finite domain and for all $R \in N_R$ the sets $\Sigma(R)$, $\Psi(R)$ as well as Θ, Γ, Φ are finite as well. The tableau rules in Fig. 5.1 induce a decidable deduction system for finite sequents. In fact, the proof of Thm. 5.2.1 shows that finite countermodels can be obtained essentially by unfolding unprovable finite end-sequents.

Theorem 5.2.4 (Finite model property & decidability (Mendler and Scheele [195]))**.** *A finite sequent is satisfiable iff it is satisfiable in a finite interpretation. Consistency of finite sequents is decidable.* ∇

Proof. For the full proof see [195, p. 224]. Decidability is obtained by the simple fact that the tableau rules in Fig. 5.1 have the *subformula* property: All formulæ in the premises of a rule are (not necessarily proper) subformulæ of formulæ in the conclusion.

Also, the domain of a premise sequent is updated at most by a role appearing in the concepts of the conclusion sequent (as in rules $\exists R$, $\forall L$ or already being part of the domains). In rule *Hyp*$_1$ a role already existing in the domain of the conclusion sequent is updated. Thus, the sizes of the domains and formula sets in a tableau are bounded by the root sequent. More specifically, if we are searching for a closed tableau of a finite sequent $\langle \Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi \rangle$ then we only need to consider tableaux with nodes formed from those (sub-)concepts and roles contained in $\langle \Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi \rangle$. Since there are only a finite number of such nodes and the tableau rules are finitely branching, there are only a finite number of possible tableaux with finite root sequent $\langle \Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi \rangle$. These can be enumerated and checked effectively in bounded time.

The finite model property is a consequence of the completeness direction of Thm. 5.2.1, which can be extended by proving that the canonical model that satisfies a given finite sequent is finite as well (see [195, p. 241] for details). □

5.2.3 Equivalence of Gentzen and Hilbert systems

The Hilbert system H and the Gentzen sequent system G1 for the language $c\mathcal{ALC}$ have been introduced in Section 5.1.1 and 5.2 respectively. Theorem 5.2.1 demonstrates soundness and completeness of the Gentzen sequent system G1 w.r.t. the Kripke semantics as introduced in Chapter 4. Now, we want to discuss the connection between these two systems by showing that any $c\mathcal{ALC}$ concept which is provable in G1 is also provable in the Hilbert system H and vice versa. Establishing this equivalence between the latter calculi yields soundness and completeness of the Hilbert system H. The following Prop. 5.2.1 states the equivalence of the Hilbert system to the Gentzen calculus G1.

Proposition 5.2.1 ([190; 195]). *The Hilbert and the Gentzen* G1*-sequent calculus are equivalent. For any set of concepts* Θ *(TBox) and concept* C *we have* $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} C$ *iff the sequent* $\Theta\,;\,\emptyset\,;\,\emptyset \vdash_{\mathsf{G1}} C\,;\,\emptyset$ *is inconsistent.* ∇

Proof. A sketch of the proof has been given in [190], covering the axioms IPC2, IPC3 and the rule of necessitation Nec for the soundness direction, and the rules $\exists R, \exists L, \forall R, \forall L$ and *Hyp*$_1$, *Hyp*$_2$ for the completeness direction. The following proof will cover all cases in both directions. For the soundness direction, the proof is by demonstrating that the Gentzen sequent calculus can simulate the Hilbert deductions in the sense that if $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} C$ then there exists a closed derivation for the sequent $\Theta\,;\,\emptyset\,;\,\emptyset \vdash C\,;\,\emptyset$, *i.e.*, we show that the axioms of the Hilbert calculus for $c\mathcal{ALC}$ are theorems of the Gentzen calculus G1 and verify that the inference rules MP and Nec can be emulated by valid instances of rules in the sequent calculus. This direction from Hilbert proofs to Gentzen

G1 proofs is the easy part. Thereof, we obtain soundness of Hilbert from soundness of the Gentzen calculus as demonstrated in Thm. 5.2.1.

(⇒) We begin by showing that the inference rules Nec and MP are admissible in the Gentzen sequent calculus G1 for $c\mathcal{ALC}$. The rule of necessitation Nec is admissible, simply by an application of the sequent rule $\forall R$:

$$\frac{\Theta\,;\,\emptyset\,;\,\emptyset \vdash C\,;\,\emptyset}{\Theta\,;\,\emptyset\,;\,\emptyset \vdash \forall R.C\,;\,\emptyset}\,\forall R$$

Following the lines of Negri, Plato and Ranta [214, pp. 42 ff.] we can treat rule modus ponens MP: In order to translate derivations of the Hilbert system into G1 we can introduce a sequent calculus rule version of modus ponens and show that this rule is admissible in G1 using rule *Cut*.

$$\frac{\Theta\,;\,\Sigma_1\,;\,\Gamma_1 \vdash \Phi_1,\, D \supset C\,;\,\Psi_1 \qquad \Theta\,;\,\Sigma_2\,;\,\Gamma_2 \vdash \Phi_2,\, D\,;\,\Psi_2}{\Theta\,;\,\Sigma_1,\,\Sigma_2\,;\,\Gamma_1,\,\Gamma_2 \vdash \Phi_1,\,\Phi_2,\,C\,;\,\Psi_1,\,\Psi_2}\,\mathsf{MP}$$

Then, we replace each application of the rule modus ponens in a Hilbert proof of $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} C$ by its sequent calculus version and demonstrate that the conclusion sequent $\Theta\,;\,\emptyset\,;\,\emptyset \vdash C\,;\,\emptyset$ is derivable from the premise sequents $\Theta\,;\,\emptyset\,;\,\emptyset \vdash D \supset C\,;\,\emptyset$ and $\Theta\,;\,\emptyset\,;\,\emptyset \vdash D\,;\,\emptyset$. This derivation starts from the sequent $\Theta\,;\,\emptyset\,;\,D \supset C,\,D \vdash C\,;\,\emptyset$ on the right side which is a consequence of the rule $\supset L$. It proceeds with an application of the rule *Cut* done twice to yield the goal sequent. The following derivation depicts this situation in a more general form.

$$\frac{\Theta\,;\,\Sigma_2\,;\,\Gamma_2 \vdash \Phi_2,\,D\,;\,\Psi_2 \qquad \dfrac{\Theta\,;\,\Sigma_1\,;\,\Gamma_1 \vdash \Phi_1,\,D \supset C\,;\,\Psi_1 \qquad \Theta\,;\,\emptyset\,;\,D \supset C,\,D \vdash C\,;\,\emptyset}{\Theta\,;\,\Sigma_1\,;\,\Gamma_1,\,D \vdash \Phi_1,\,C\,;\,\Psi_1}\,\mathit{Cut}}{\Theta\,;\,\Sigma_1,\,\Sigma_2\,;\,\Gamma_1,\,\Gamma_2 \vdash \Phi_1,\,\Phi_2,\,C\,;\,\Psi_1,\,\Psi_2}\,\mathit{Cut}$$

Then, it follows by the admissibility of the *Cut* rule from Cor. 5.2.1 that the sequent $\Theta\,;\,\emptyset\,;\,\emptyset \vdash C\,;\,\emptyset$ is derivable in G1 without using rule *Cut*.

The axioms of $c\mathcal{ALC}$ are proved in G1 as follows:

(**Case** IPC1)

$$\dfrac{\dfrac{\dfrac{}{\emptyset\,;\,\emptyset\,;\,C, D \vdash C\,;\,\emptyset}\,Ax}{\emptyset\,;\,\emptyset\,;\,C \vdash D \supset C\,;\,\emptyset}\,\supset R}{\emptyset\,;\,\emptyset\,;\,\emptyset \vdash C \supset (D \supset C)\,;\,\emptyset}\,\supset R$$

(**Case** IPC2)

$$\dfrac{\dfrac{}{\emptyset;\emptyset;C,\,D\supset E\vdash E,C;\emptyset}\,Ax \qquad \dfrac{\dfrac{}{\emptyset;\emptyset;C,\,D\vdash E,D;\emptyset}\,Ax \quad \dfrac{}{\emptyset;\emptyset;C,\,D,\,E\vdash E;\emptyset}\,Ax}{\emptyset;\emptyset;C,\,D\supset E,\,D\vdash E;\emptyset}\supset L}{}\supset L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset;\emptyset;C\supset D,\,C\vdash E,C;\emptyset}\,Ax \qquad \begin{array}{c}\cdots\cdots\cdots\cdots\cdots\cdots\\ \emptyset;\emptyset;C\supset D,\,C,\,D\supset E\vdash E;\emptyset\end{array}}{\emptyset;\emptyset;C\supset(D\supset E),\,C\supset D,\,C\vdash E;\emptyset}\supset L}{\emptyset;\emptyset;C\supset(D\supset E),\,C\supset D\vdash C\supset E;\emptyset}\supset R}{\emptyset;\emptyset;C\supset(D\supset E)\vdash(C\supset D)\supset(C\supset E);\emptyset}\supset R}{\emptyset;\emptyset;\emptyset\vdash(C\supset(D\supset E))\supset((C\supset D)\supset(C\supset E));\emptyset}\supset R$$

(**Case** IPC3)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset;\emptyset;C,\,D\vdash C;\emptyset}\,Ax \qquad \dfrac{}{\emptyset;\emptyset;C,\,D\vdash D;\emptyset}\,Ax}{\emptyset;\emptyset;C,\,D\vdash C\sqcap D;\emptyset}\sqcap R}{\emptyset;\emptyset;C\vdash D\supset(C\sqcap D);\emptyset}\supset R}{\emptyset;\emptyset;\emptyset\vdash C\supset(D\supset(C\sqcap D));\emptyset}\supset R$$

(**Case** IPC4) For axiom IPC4 we have the two similar cases:

$$\dfrac{\dfrac{\dfrac{}{\emptyset;\emptyset;C,\,D\vdash C;\emptyset}\,Ax}{\emptyset;\emptyset;C\sqcap D\vdash C;\emptyset}\sqcap L}{\emptyset;\emptyset;\emptyset\vdash(C\sqcap D)\supset C;\emptyset}\supset R \qquad \dfrac{\dfrac{\dfrac{}{\emptyset;\emptyset;C,\,D\vdash D;\emptyset}\,Ax}{\emptyset;\emptyset;C\sqcap D\vdash D;\emptyset}\sqcap L}{\emptyset;\emptyset;\emptyset\vdash(C\sqcap D)\supset D;\emptyset}\supset R$$

(**Case** IPC5) For axiom IPC5 there are two analogous cases:

$$\dfrac{\dfrac{\dfrac{}{\emptyset;\emptyset;C\vdash C,\,D;\emptyset}\,Ax}{\emptyset;\emptyset;C\vdash C\sqcup D;\emptyset}\sqcup R}{\emptyset;\emptyset;\emptyset\vdash C\supset(C\sqcup D);\emptyset}\supset R \qquad \dfrac{\dfrac{\dfrac{}{\emptyset;\emptyset;D\vdash C,\,D;\emptyset}\,Ax}{\emptyset;\emptyset;D\vdash C\sqcup D;\emptyset}\sqcup R}{\emptyset;\emptyset;\emptyset\vdash D\supset(C\sqcup D);\emptyset}\supset R$$

(**Case** IPC6) For axiom IPC6, the derivation is as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1 \qquad\qquad \pi_2}{\emptyset;\emptyset;C\supset E,\,D\supset E,\,C\sqcup D\vdash E;\emptyset}\sqcup L}{\emptyset;\emptyset;C\supset E,\,D\supset E\vdash(C\sqcup D)\supset E;\emptyset}\supset R}{\emptyset;\emptyset;C\supset E\vdash(D\supset E)\supset((C\sqcup D)\supset E);\emptyset}\supset R}{\emptyset;\emptyset;\emptyset\vdash(C\supset E)\supset((D\supset E)\supset((C\sqcup D)\supset E));\emptyset}\supset R$$

where derivation π_1 is given by

$$\dfrac{\dfrac{}{\emptyset;\emptyset;D\supset E,\,C\vdash E,\,C;\emptyset}\,Ax \qquad \dfrac{}{\emptyset;\emptyset;D\supset E,\,C,\,E\vdash E;\emptyset}\,Ax}{\emptyset;\emptyset;C\supset E,\,D\supset E,\,C\vdash E;\emptyset}\supset L$$

and derivation π_2 is

$$\dfrac{\dfrac{}{\emptyset\,;\,\emptyset\,;\,C \supset E,\; D \vdash E,\; D\,;\,\emptyset}\,Ax \qquad \dfrac{}{\emptyset\,;\,\emptyset\,;\,C \supset D,\; D,\; E \vdash E\,;\,\emptyset}\,Ax}{\emptyset\,;\,\emptyset\,;\,C \supset E,\; D \supset E,\; D \vdash E\,;\,\emptyset}\,\supset\! L$$

(**Case** $\mathsf{IPC7}$) Axiom $\mathsf{IPC7}$ is derived by:

$$\dfrac{\dfrac{}{\emptyset\,;\,\emptyset\,;\,\bot \vdash C\,;\,\emptyset}\,\bot L}{\emptyset\,;\,\emptyset\,;\,\emptyset \vdash \bot \supset C\,;\,\emptyset}\,\supset\! R$$

(**Case** $\mathsf{K}_{\forall R}$, $\mathsf{K}_{\exists R}$) For the $\mathsf{G1}$-proof of the axioms $\mathsf{K}_{\forall R}, \mathsf{K}_{\exists R}$ see Ex. 5.2.1.

($\Leftarrow$) In the converse direction let us assume that there is a closed derivation for the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$. Because of compactness we may assume without loss of generality that the sequent is finite according to Def. 5.2.5. Also note that $|\Phi \cup \Psi| \geq 1$ by Lem. 5.2.1. The following proof will show that each such finite sequent can be derived in the Hilbert system in closed form as an implication using the following translation

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \;\sqcap\; \wedge\Gamma) \;\supset\; (\vee\Phi \;\sqcup\; \vee\Psi),$$

where $dom(\Sigma)$, $dom(\Psi) \subseteq N_R$ denote the non-empty domain w.r.t. Σ and Ψ according to Def. 5.2.1. The subformulæ $\wedge\Sigma, \wedge\Gamma, \vee\Phi$ and $\vee\Psi$ are defined as follows:

- $\wedge\Sigma =_{df} \bigwedge_{R \in dom(\Sigma)} \forall R.\, \wedge\Sigma(R) = \bigwedge_{R \in dom(\Sigma)} \forall R.\, \bigwedge_{K \in \Sigma(R)} K$. If $dom(\Sigma) = \emptyset$ then $\wedge\Sigma = \top$.

  We will need the decomposition of $\wedge\Sigma$ and $\vee\Psi$ in the following proof. The decomposition of $\wedge\Sigma$ with $dom(\Sigma) = \{R_1, \ldots, R_n\}$ and $n \geq 1$ is given by

$$\begin{aligned} \wedge\Sigma &= \bigwedge_{1 \leq i \leq n} \forall R_i.\wedge\Sigma(R_i) \\ &= \forall R_1.\wedge\Sigma(R_1) \sqcap \Big(\forall R_2.\wedge\Sigma(R_2) \sqcap \big(\ldots \sqcap \forall R_n.\wedge\Sigma(R_n) \ldots\big)\Big) \\ &=^{*} \forall R_k.\wedge\Sigma(R_k) \sqcap \underbrace{\Big(\bigwedge_{1 \leq i \leq n,\; i \neq k} \forall R_i.\wedge\Sigma(R_i)\Big)}_{=\,\top \text{ in the degenerated case if } n=1}\,, \end{aligned} \tag{5.23}$$

  where $=^{*}$ means *up to provability* and k is a choice of $\{1 \ldots n\}$.

- $\wedge\Gamma =_{df} \bigwedge_{L \in \Gamma} L$, where $\bigwedge$ is the intersection $\sqcap$ over a set of concepts, e.g., if $\Gamma = \{L_1, L_2, \ldots, L_n\}$ then $\wedge\Gamma = L_1 \sqcap L_2 \sqcap \ldots \sqcap L_n$. In the special case where $\Gamma = \emptyset$ we put $\wedge\emptyset =_{df} \top$;

- $\vee\Phi =_{df} \bigvee_{M \in \Phi} M$, where $\bigvee$ is the disjunction $\sqcup$ over a set of concepts, e.g., if

$\Phi = \{M_1, M_2, \ldots, M_n\}$ then $\vee\Phi = M_1 \sqcup M_2 \sqcup \ldots \sqcup M_n$. In the special case where $\Phi = \emptyset$ we put $\vee\emptyset =_{df} \bot$;

- $\vee\Psi =_{df} \bigvee_{R\in dom(\Psi)} \exists R. \vee\Psi(R) = \bigvee_{R\in dom(\Psi)} \exists R. \bigvee_{N\in\Psi(R)} N$. If $dom(\Psi) = \emptyset$ then $\vee\Psi = \bot$.

  Analogously to $\wedge\Sigma$, the decomposition of $\vee\Psi$ with $dom(\Psi) = \{R_1, \ldots, R_n\}$ and $n \geq 1$ is defined by

$$\begin{aligned}\vee\Psi &= \bigvee_{1\leq i\leq n} \exists R_i.\vee\Psi(R_i)\\ &= \exists R_1.\vee\Psi(R_1) \sqcup \Big(\exists R_2.\vee\Psi(R_2) \sqcup \big(\ldots \sqcup \exists R_n.\vee\Psi(R_n)\ldots\big)\Big)\\ &=^* \exists R_k.\vee\Psi(R_k) \sqcup \underbrace{\Big(\bigvee_{1\leq i\leq n,\, i\neq k} \exists R_i.\vee\Psi(R_i)\Big)}_{=\,\bot \text{ in the degenerated case if } n=1},\end{aligned} \tag{5.24}$$

  where $=^*$ means up to provability and k is a choice of $\{1\ldots n\}$.

The proof is by induction on the structure of a closed derivation for $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ showing that all instances of sequent calculus rules can be turned into closed implications, which are derivable in the Hilbert system. All derivations assume associativity, commutativity, idempotence of $\sqcap, \sqcup$ and that eliminating neutral elements $\top, \bot$ (see admissible rule (ARE) in Lem. 5.1.3) is for free.

(**Case** Ax) Suppose $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ is derived by the rule Ax, *i.e.*, $\Gamma = \Gamma', C$ and $\Phi = \Phi', C$. We must show how to derive

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \ \sqcap\ \wedge\Gamma' \sqcap C) \ \supset\ (\vee\Phi' \sqcup C \ \sqcup\ \vee\Psi). \tag{5.25}$$

The derivation of (5.25) is given by the following proof (using commutativity of $\sqcap$):

1. $C \supset C$ by (I);
2. $(\wedge\Sigma \sqcap C \sqcap \wedge\Gamma') \supset C$ from 1 by (ARW);
3. $C \supset (\vee\Phi' \sqcup C \sqcup \wedge\Psi)$ from 1 by (ARW);
4. $(\wedge\Sigma \sqcap C \sqcap \wedge\Gamma') \supset (\vee\Phi' \sqcup C \sqcup \wedge\Psi)$ from 2, 3 by (ARB).

(**Case** $\bot L$) Assume that $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ is derived by rule $\bot L$. Then it holds that $\Gamma = \Gamma', \bot$ and $|\Phi \cup \Psi| \geq 1$. The goal is to show the Hilbert derivation

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \ \sqcap\ \bot \sqcap \wedge\Gamma') \ \supset\ (\vee\Phi \ \sqcup\ \vee\Psi), \tag{5.26}$$

which is a direct consequence of Hilbert axiom IPC7 and weakening (ARW).

(**Case** $\sqcap L$) Suppose the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\, \Phi\,;\,\Psi$ is derived by rule $\sqcap L$, *i.e.*, $\Gamma = \Gamma', C \sqcap D$ and the last rule application looks like this

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma',\, C,\, D \mid\!\!-\, \Phi\,;\,\Psi}}{\Theta\,;\,\Sigma\,;\,\Gamma',\, C \sqcap D \mid\!\!-\, \Phi\,;\,\Psi}\,\sqcap L$$

The goal is to find a derivation for

$$\Theta\,;\,\emptyset \mid\!\!\frac{\;\;}{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap C \sqcap D) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.27}$$

Applying the induction hypothesis to the premise of the sequent yields the Hilbert derivation $\Theta\,;\,\emptyset \mid\!\!\frac{\;\;}{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap C \sqcap D) \supset (\vee\Phi \sqcup \vee\Psi)$ which was to be shown.

(**Case** $\sqcap R$) Assume the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\, \Phi\,;\,\Psi$ is derived by rule $\sqcap R$, *i.e.*, $\Phi = \Phi', C \sqcap D$ and the last rule application is

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\, \Phi',\, C\,;\,\Psi} \qquad \dfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\, \Phi',\, D\,;\,\Psi}}{\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!-\, \Phi',\, C \sqcap D\,;\,\Psi}\,\sqcap R$$

We must find a Hilbert derivation for

$$\Theta\,;\,\emptyset \mid\!\!\frac{\;\;}{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi' \sqcup (C \sqcap D) \sqcup \vee\Psi). \tag{5.28}$$

Applying the ind. hyp. to the premises of the sequent gives us the derivations

$$\Theta\,;\,\emptyset \mid\!\!\frac{\;\;}{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi' \sqcup C \sqcup \vee\Psi), \tag{5.29}$$

$$\Theta\,;\,\emptyset \mid\!\!\frac{\;\;}{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi' \sqcup D \sqcup \vee\Psi). \tag{5.30}$$

Using the abbreviations $\varphi =_{df} (\wedge\Sigma \sqcap \wedge\Gamma)$ and $\psi =_{df} (\vee\Phi' \sqcap \vee\Psi)$ we will show that Hilbert derives

$$\Theta\,;\,\emptyset \mid\!\!\frac{\;\;}{\mathsf{H}} (\varphi \supset (\psi \sqcup C)) \supset (\varphi \supset (\psi \sqcup D)) \supset \big(\varphi \supset (\psi \sqcup (C \sqcap D))\big). \tag{5.31}$$

The idea behind the proof of (5.31) is to show the distribution law of disjunction $\sqcup$ over conjunction $\sqcap$, *i.e.*, $\mid\!\!\frac{\;\;}{\mathsf{H}} \big((\psi \sqcup C) \sqcap (\psi \sqcup D)\big) \supset \big(\psi \sqcup (C \sqcap D)\big)$. Let $\vartheta =_{df} (\psi \sqcup (C \sqcap D))$ and $\sigma =_{df} (\psi \sqcup D)$:

1. $C \supset (D \supset (C \sqcap D))$ IPC3;
2. $(\psi \supset (\sigma \supset \vartheta)) \supset (C \supset (\sigma \supset \vartheta)) \supset ((\psi \sqcup C) \supset (\sigma \supset \vartheta))$ IPC6;

3. $\psi \supset (\psi \sqcup (C \sqcap D))$ IPC5;
4. $\psi \supset (\psi \sqcup D) \supset (\psi \sqcup (C \sqcap D))$ from 3 by $(\mathsf{ARCW})_{[(\psi \sqcup D)]}$;
$= \psi \supset (\sigma \supset \vartheta)$
5. $(C \sqcap D) \supset (\psi \sqcup (C \sqcap D))$ IPC5;
$= (C \sqcap D) \supset \vartheta$
6. $(\psi \supset \vartheta) \supset (D \supset \vartheta) \supset ((\psi \sqcup D) \supset \vartheta)$ IPC6;
$= (\psi \supset \vartheta) \supset (D \supset \vartheta) \supset (\sigma \supset \vartheta)$
7. $(D \supset \vartheta) \supset ((\psi \sqcup D) \supset \vartheta)$ from 6, IPC5 by MP;
8. $C \supset (D \supset \vartheta)$ from 1, 5 by (ARB);
9. $C \supset ((\psi \sqcup D) \supset \vartheta)$ from 8,7 by (ARB);
$= C \supset (\sigma \supset \vartheta)$
10. $(\psi \sqcup C) \supset (\sigma \supset \vartheta)$ from (2, 4 by MP) and 9 by MP.
$= (\psi \sqcup C) \supset (\psi \sqcup D) \supset (\psi \sqcup (C \sqcap D))$

Then, the subgoal (5.31) is a consequence of applying the admissible rule $(\mathsf{ARS})_{[\varphi]}$ to the derivation 10 above. Finally, the goal (5.28) follows by the rule MP from (5.31) and the assumptions (5.29) and (5.30).

(**Case** $\sqcup L$) Assume that the sequent $\Theta ; \Sigma ; \Gamma \vdash \Phi ; \Psi$ is derived by rule $\sqcup L$, *i.e.*, $\Gamma = \Gamma', C \sqcup D$ and the last rule application is

$$\dfrac{\dfrac{\vdots}{\Theta ; \Sigma ; \Gamma',\ C \vdash \Phi ; \Psi} \qquad \dfrac{\vdots}{\Theta ; \Sigma ; \Gamma',\ D \vdash \Phi ; \Psi}}{\Theta ; \Sigma ; \Gamma',\ C \sqcup D \vdash \Phi ; \Psi} \sqcup L$$

The goal is to find the derivation for

$$\Theta ; \emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap (C \sqcup D)) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.32}$$

The induction hypothesis yields the Hilbert derivations

$$\Theta ; \emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap C) \supset (\vee\Phi \sqcup \vee\Psi), \tag{5.33}$$

$$\Theta ; \emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap D) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.34}$$

The goal (5.32) follows from the following Hilbert derivation where we use the abbreviations $\varphi =_{df} \wedge\Sigma \sqcap \wedge\Gamma'$ and $\psi =_{df} \vee\Phi \sqcup \vee\Psi$:

1. $(C \supset \psi) \supset (D \supset \psi) \supset ((C \sqcup D) \supset \psi)$ IPC6;
2. $(\varphi \supset C \supset \psi) \supset (\varphi \supset D \supset \psi) \supset (\varphi \supset (C \sqcup D) \supset \psi)$ from 1 $(\mathsf{ARS})_{[\varphi]}$;
3. $\varphi \supset C \supset \psi$ from (5.33) by (ARC);

4. $(\varphi \supset D \supset \psi) \supset (\varphi \supset (C \sqcup D) \supset \psi)$ from 2, 3 by MP;
5. $\varphi \supset (C \sqcup D) \supset \psi$ from 4, ((ARC) with (5.34)) by MP;
6. $(\varphi \sqcap (C \sqcup D)) \supset \psi$ from 5 by $(\mathsf{ARC})^{-1}$,

which was to be shown for goal (5.32).

(**Case** $\sqcup R$) Assume the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ is derived by rule $\sqcup R$, *i.e.*, $\Phi = \Phi', C \sqcup D$ and the last rule application is

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi',\, C,\, D\,;\,\Psi}}{\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi',\, C \sqcup D\,;\,\Psi}\,\sqcup R$$

The goal is to obtain a Hilbert derivation for

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi' \sqcup (C \sqcup D) \sqcup \vee\Psi). \tag{5.35}$$

The goal (5.35) follows immediately by ind. hyp. (similarly to **Case** $\sqcap L$).

(**Case** $\supset L$) Suppose that the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ is derived by rule $\supset L$, *i.e.*, $\Gamma = \Gamma', C \supset D$ and the last rule application looks like this

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi,\, C\,;\,\Psi} \qquad \dfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma,\, D \vdash \Phi\,;\,\Psi}}{\Theta\,;\,\Sigma\,;\,\Gamma',\, C \supset D \vdash \Phi\,;\,\Psi}\,\supset L$$

The goal is to find a Hilbert derivation for

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap (C \supset D)) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.36}$$

The induction hypothesis yields the derivations $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup C \sqcup \Psi)$ and $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma \sqcap D) \supset (\vee\Phi \sqcup \vee\Psi)$. Using the abbreviations $\varphi =_{df} \wedge\Sigma \sqcap \wedge\Gamma$ and $\psi =_{df} \vee\Phi \sqcup \vee\Psi$, the objective is to find a derivation $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\varphi \sqcap (C \supset D)) \supset \psi$, taking into account the derivations from the induction hypothesis

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} \varphi \supset (\psi \sqcup C), \tag{5.37}$$

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\varphi \sqcap D) \supset \psi. \tag{5.38}$$

In order to find this derivation one shows first that Hilbert derives the nested implication from the assumptions to the goal in closed form.

$$\vdash_{\mathsf{H}} (\varphi \supset (\psi \sqcup C)) \supset ((\varphi \sqcap D) \supset \psi) \supset ((\varphi \sqcap (C \supset D)) \supset \psi). \tag{5.39}$$

In other words, by using the abbreviations γ, σ and ϑ defined by

$$\begin{aligned} \gamma &=_{df} (\varphi \supset (\psi \sqcup C)), \\ \sigma &=_{df} ((\varphi \sqcap D) \supset \psi), \\ \vartheta &=_{df} (\varphi \sqcap (C \supset D)), \end{aligned}$$

we want to find a derivation for

$$\vdash_{\mathsf{H}} \gamma \supset \sigma \supset \vartheta \supset \psi. \tag{5.40}$$

The first step to obtain the subgoal (5.40) is by applying rule $(\mathsf{ARS})_{[\gamma,\sigma,\vartheta]}$ to the instance $\vdash_{\mathsf{H}} (\psi \supset \psi) \supset (C \supset \psi) \supset (\psi \sqcup C) \supset \psi$ of axiom IPC6 which yields

$$\vdash_{\mathsf{H}} (\gamma \supset \sigma \supset \vartheta \supset \psi \supset \psi) \supset (\gamma \supset \sigma \supset \vartheta \supset C \supset \psi) \supset \\ (\gamma \supset \sigma \supset \vartheta \supset (\psi \sqcup C)) \supset (\gamma \supset \sigma \supset \vartheta \supset \psi). \tag{5.41}$$

Then, to get (5.40) from (5.41) via rule MP one must find a derivation for each of

$$\vdash_{\mathsf{H}} \gamma \supset \sigma \supset \vartheta \supset \psi \supset \psi, \tag{5.42}$$

$$\vdash_{\mathsf{H}} \gamma \supset \sigma \supset \vartheta \supset C \supset \psi, \tag{5.43}$$

$$\vdash_{\mathsf{H}} \gamma \supset \sigma \supset \vartheta \supset (\psi \sqcup C). \tag{5.44}$$

Let us start with (5.42). This can be derived simply by starting from the instance $\vdash_{\mathsf{H}} \psi \supset \psi$ of identity (I) and applying rule $(\mathsf{ARK})_{[\gamma,\sigma,\vartheta]}$ to the latter.

Next, we have to find a derivation for subgoal (5.43). We proceed by taking an instance of IPC2, namely $\big(C \supset ((\varphi \sqcap D) \supset \psi)\big) \supset \big(C \supset (\varphi \sqcap D)\big) \supset \big(C \supset \psi\big)$, and an application of $(\mathsf{ARS})_{[\gamma,\sigma,\vartheta]}$ to the latter yields

$$\big(\gamma \supset \sigma \supset \vartheta \supset C \supset ((\varphi \sqcap D) \supset \psi)\big) \supset \\ \big(\gamma \supset \sigma \supset \vartheta \supset C \supset (\varphi \sqcap D)\big) \supset \big(\gamma \supset \sigma \supset \vartheta \supset C \supset \psi\big). \tag{5.45}$$

In order to obtain subgoal (5.43) from derivation (5.45) above via MP, we need to find the two derivations

$$\vdash_{\mathsf{H}} \gamma \supset \sigma \supset \vartheta \supset C \supset ((\varphi \sqcap D) \supset \psi), \tag{5.46}$$

$$\vdash_{\mathsf{H}} \gamma \supset \sigma \supset \vartheta \supset C \supset (\varphi \sqcap D). \tag{5.47}$$

First, observe that (5.46) is nothing but $\vdash_{\mathsf{H}} \gamma \supset \sigma \supset \vartheta \supset C \supset \sigma$. Its derivation is as follows:

1. $\sigma \supset \sigma$ (I);
2. $\sigma \supset \vartheta \supset C \supset \sigma$ from 1 by $(\mathsf{ARCW})_{[\vartheta,C]}$;
3. $\gamma \supset \sigma \supset \vartheta \supset C \supset \sigma$ from 2 by $(\mathsf{ARK})_{[\gamma]}$.

Regarding the derivation of (5.47) we begin with an instance of axiom IPC3, namely, $\vdash_{\mathsf{H}} \varphi \supset (D \supset (\varphi \sqcap D))$. An application of rule $(\mathsf{ARS})_{[\sigma,\vartheta,C]}$ to the latter yields

$$\vdash_{\mathsf{H}} (\sigma \supset \vartheta \supset C \supset \varphi) \supset (\sigma \supset \vartheta \supset C \supset D) \supset (\sigma \supset \vartheta \supset C \supset (\varphi \sqcap D)), \tag{5.48}$$

i.e., we can obtain (5.47) if we find derivations of

$$\vdash_{\mathsf{H}} \sigma \supset \vartheta \supset C \supset \varphi, \tag{5.49}$$

$$\vdash_{\mathsf{H}} \sigma \supset \vartheta \supset C \supset D. \tag{5.50}$$

These can be derived as follows. We begin with (5.49):

1. $(\varphi \sqcap (C \supset D)) \supset \varphi$ IPC4;
   $= \vartheta \supset \varphi$
2. $\vartheta \supset C \supset \varphi$ from 1 by $(\mathsf{ARCW})_{[C]}$;
3. $\sigma \supset \vartheta \supset C \supset \varphi$ from 2 by $(\mathsf{ARK})_{[\sigma]}$;

and for (5.50)

1. $(\varphi \sqcap (C \supset D)) \supset (C \supset D)$ IPC4;
   $= \vartheta \supset C \supset D$
2. $\sigma \supset \vartheta \supset C \supset D$ from 1 by $(\mathsf{ARK})_{[\sigma]}$.

Then, using rule MP with (5.48), (5.49) and (5.50) gives us $\vdash_{\mathsf{H}} \sigma \supset \vartheta \supset C \supset (\varphi \sqcap D)$, and an application of $(\mathsf{ARK})_{[\gamma]}$ to the latter gives us (5.47) as desired. This completes the proof of the subgoal (5.43).

The final step is to find a derivation for (5.44):

1. $(\vartheta \supset (\varphi \supset (\psi \sqcup C))) \supset (\vartheta \supset \varphi) \supset (\vartheta \supset (\psi \sqcup C))$ IPC2;
   $= (\vartheta \supset \gamma) \supset (\vartheta \supset \varphi) \supset (\vartheta \supset (\psi \sqcup C))$
2. $(\gamma \supset \sigma \supset \vartheta \supset \gamma) \supset (\gamma \supset \sigma \supset \vartheta \supset \varphi) \supset (\gamma \supset \sigma \supset \vartheta \supset (\psi \sqcup C))$ from 1 by $(\mathsf{ARS})_{[\gamma,\sigma]}$;
3. $(\varphi \sqcap (C \supset D)) \supset \varphi$ IPC4;
   $= \vartheta \supset \varphi$
4. $\gamma \supset \sigma \supset \vartheta \supset \varphi$ from 3 by $(\mathsf{ARK})_{[\gamma,\sigma]}$;
5. $\gamma \supset \gamma$ (I);
6. $\gamma \supset \sigma \supset \vartheta \supset \gamma$ from 5 by $(\mathsf{ARCW})_{[\sigma,\vartheta]}$;
7. $(\gamma \supset \sigma \supset \vartheta \supset \varphi) \supset (\gamma \supset \sigma \supset \vartheta \supset (\psi \sqcup C))$ from 2, 6 by MP;
8. $\gamma \supset \sigma \supset \vartheta \supset (\psi \sqcup C)$ from 7, 4 by MP.

The subgoal (5.40) then follows from (5.41), (5.42), (5.43) and (5.44) by MP. Finally, the goal (5.36) is obtained from the derivations of the induction hypothesis and (5.40) by rule MP, and monotonicity Prop. 5.1.2.

(**Case** $\supset R$) Suppose that the sequent $\Theta\,;\Sigma\,;\Gamma \vdash \Phi\,;\Psi$ is derived by rule $\supset R$, *i.e.*, $\Phi = \Phi', C \supset D$ with the last rule application being

$$\dfrac{\dfrac{\vdots}{\Theta\,;\Sigma\,;\Gamma,\, C \vdash D\,;\emptyset}}{\Theta\,;\Sigma\,;\Gamma \vdash \Phi',\, C \supset D\,;\Psi}\supset R$$

The goal is to find a derivation for

$$\Theta\,;\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi' \sqcup (C \supset D) \sqcup \vee\Psi). \tag{5.51}$$

An application of the induction hypothesis to the premise of the sequent yields the derivation $\Theta\,;\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma \sqcap C) \supset (D \sqcup \bot)$, and by the rule (ARE) follows $\Theta\,;\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma \sqcap C) \supset D$. By the admissible rule (ARC) (*currying*) one obtains the derivation $\Theta\,;\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (C \supset D)$. Then, the goal (5.51) follows by right-weakening (ARW).

(**Case** $\exists L$) If the sequent $\Theta\,;\Sigma\,;\Gamma \vdash \Phi\,;\Psi$ is derived by the rule $\exists L$ then $\Gamma = \Gamma', \exists R.C$ and the last rule application is

$$\dfrac{\dfrac{\vdots}{\Theta\,;\emptyset\,;\Sigma(R),\, C \vdash \Psi(R)\,;\emptyset}}{\Theta\,;\Sigma\,;\Gamma',\, \exists R.C \vdash \Phi\,;\Psi}\exists L$$

We must find a derivation

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap \exists R.C) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.52}$$

By the induction hypothesis and the rule (ARE) we obtain the Hilbert derivation $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma(R) \sqcap C) \supset \vee\Psi(R)$. Note that $\Psi(R) \neq \emptyset$ by Lem. 5.2.1. By applying Nec to the latter derivation it follows that $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} \forall R.\big((\wedge\Sigma(R) \sqcap C) \supset \vee\Psi(R)\big)$. Taking the appropriate instance of axiom $\mathsf{K}_{\exists R}$ and by an application of the rule MP this generates the derivation

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} \exists R.(\wedge\Sigma(R) \sqcap C) \supset \exists R.\vee\Psi(R). \tag{5.53}$$

One can construct a Hilbert derivation for

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\forall R.\wedge\Sigma(R) \sqcap \exists R.C) \supset \exists R.(\wedge\Sigma(R) \sqcap C), \tag{5.54}$$

which has already been demonstrated in Ex. 5.1.1 in Sec. 5.1.

Then, by applying admissible rule (ARB) (*"composition"*)

to (5.54) and (5.53) we obtain

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\forall R.\wedge\Sigma(R) \sqcap \exists R.C) \supset \exists R.\vee\Psi(R). \tag{5.55}$$

At this point, one observes that Hilbert derives

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} \wedge\Sigma \supset \forall R.\wedge\Sigma(R), \tag{5.56}$$

whether either $\Sigma(R) = \emptyset$ or $\Sigma(R) \neq \emptyset$.

Case 1. If $\Sigma(R) = \emptyset$ then the goal is to show that $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} \wedge\Sigma \supset \forall R.\top$. Observe that

$$\wedge\Sigma = \bigwedge_{R' \in dom(\Sigma), R' \neq R} \forall R'.\wedge\Sigma(R').$$

The derivation of (5.56) is as follows:

1. $\wedge\Sigma \supset (\top \sqcap \wedge\Sigma)$ ($\mathsf{IPC9}$);
2. $\top \supset \forall R.\top$ ($\mathsf{FS1}$);
3. $(\top \sqcap \wedge\Sigma) \supset \top$ $\mathsf{IPC4}$;
4. $\wedge\Sigma \supset \forall R.\top$ from (3, 1 by (ARB)), 2 by (ARB).

Case 2. If $\Sigma(R) \neq \emptyset$ then one has to show that $\Theta\,;\,\emptyset \mid_{\overline{\mathsf{H}}} \wedge\Sigma \supset \forall R.\wedge\Sigma(R)$, where

$$\wedge\Sigma = \bigwedge_{R' \in dom(\Sigma)\setminus\{R\}} \forall R'.\wedge\Sigma(R') \sqcap \forall R.\wedge\Sigma(R).$$

A derivation of (5.56) can be obtained by applying the admissible rule (ARW) (*left-weakening*) to the instance $\Theta\,;\,\emptyset \mid_{\overline{\mathsf{H}}} \forall R.\wedge\Sigma(R) \supset \forall R.\wedge\Sigma(R)$ of identity (I).

Thus, Hilbert derives (5.56), whether $\Sigma(R) = \emptyset$ or not.

By monotonicity (rule (ARM)) we obtain from (5.56) a Hilbert derivation of $\Theta\,;\,\emptyset \mid_{\overline{\mathsf{H}}} (\wedge\Sigma \sqcap \exists R.C) \supset (\forall R.\wedge\Sigma(R) \sqcap \exists R.C)$, and an application of the rule (ARW) (*left-weakening*) yields $\Theta\,;\,\emptyset \mid_{\overline{\mathsf{H}}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap \exists R.C) \supset (\forall R.\wedge\Sigma(R) \sqcap \exists R.C)$. Now, one can apply the rule (ARB) to the latter and (5.55) to obtain a derivation of

$$\Theta\,;\,\emptyset \mid_{\overline{\mathsf{H}}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap \exists R.C) \supset \exists R.\vee\Psi(R). \tag{5.57}$$

At this point one observes that $\exists R.\vee\Psi(R)$ is a disjunctive part of $\vee\Psi$, since in the premise sequent $\vee\Psi(R) \neq \emptyset$ by Lem. 5.2.1, otherwise there would not exist a closed derivation for it. Obviously, if this restriction is not valid and $R \notin dom(\Psi)$, then we need axiom $\mid_{\overline{\mathsf{H}}} \exists R.\bot \supset \bot$ to derive (5.52) (as argued later).

Hence, the goal (5.52) follows by weakening (rule (ARW)) from (5.57).

(**Case** $\exists R$) Assume that the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ is derived by rule $\exists R$, *i.e.*, $\Phi = \Phi', \exists R.C$ and the last rule application looks like this

$$\cfrac{\cfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma \vdash \emptyset\,;\,[R \mapsto C]}}{\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi',\,\exists R.C\,;\,\Psi}\,\exists R$$

The induction hypothesis applied to the premise of the sequent and the admissible rule (ARE) lets us conclude that $\Theta\,;\,\emptyset \mid_{\overline{\mathsf{H}}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset \exists R.C$. By the admissible rule of weakening (ARW) this implies $\Theta\,;\,\emptyset \mid_{\overline{\mathsf{H}}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi' \sqcup \exists R.C \sqcup \vee\Psi)$ as desired.

(**Case** $\forall L$) Next, let the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ be derived by rule $\forall L$, *i.e.*, $\Gamma = \Gamma', \forall R.C$ and the last rule application is as follows

$$\cfrac{\cfrac{\vdots}{\Theta\,;\,\Sigma \cup [R \mapsto C]\,;\,\Gamma' \vdash \Phi\,;\,\Psi}}{\Theta\,;\,\Sigma\,;\,\Gamma',\,\forall R.C \vdash \Phi\,;\,\Psi}\,\forall L$$

The goal is to find a Hilbert derivation for

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap \forall R.C) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.58}$$

The induction hypothesis for the premise yields the Hilbert derivation

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} ((\bigwedge_{R' \in dom(\Sigma)\setminus\{R\}} \forall R'.\wedge\Sigma(R')) \sqcap \forall R.(\wedge\Sigma(R) \sqcap C) \sqcap \wedge\Gamma') \supset (\vee\Phi \sqcup \vee\Psi).$$

Let us abbreviate this by

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma' \sqcap \forall R.(\varphi \sqcap C) \sqcap \wedge\Gamma') \supset (\vee\Phi \sqcup \vee\Psi), \tag{5.59}$$

where $\wedge\Sigma' =_{df} (\bigwedge_{R' \in dom(\Sigma)\setminus\{R\}} \forall R'.\wedge\Sigma(R'))$ and $\varphi =_{df} \wedge\Sigma(R)$.

The plan is to find a derivation for

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma' \sqcap \forall R.\varphi \sqcap \forall R.C \sqcap \wedge\Gamma') \supset (\wedge\Sigma' \sqcap \forall R.(\varphi \sqcap C) \wedge \Gamma'). \tag{5.60}$$

Case 1. If $\Sigma(R) = \emptyset$ then the subgoal (5.60) follows by (I).

Case 2. Otherwise if $\Sigma(R) \neq \emptyset$ then the subgoal (5.60) can be shown by distributing $\forall R$ over $\sqcap$, *i.e.*, we need to find a derivation for

$$\vdash_{\mathsf{H}} (\forall R.\varphi \sqcap \forall R.C) \supset \forall R.(\varphi \sqcap C). \tag{5.61}$$

First, we demonstrate how to obtain (5.61):

1. $\varphi \supset (C \supset (\varphi \sqcap C))$	IPC3;
2. $\forall R.(\varphi \supset (C \supset (\varphi \sqcap C)))$	from 1 by Nec;
3. $\forall R.\varphi \supset \forall R.(C \supset (\varphi \sqcap C))$	from $\mathsf{K}_{\forall R}$, 2 by MP;
4. $\forall R.(C \supset (\varphi \sqcap C)) \supset (\forall R.C \supset \forall R.(\varphi \sqcap C))$	by $\mathsf{K}_{\forall R}$;
5. $\forall R.\varphi \supset (\forall R.C \supset \forall R.(\varphi \sqcap C))$	from 4, 3 by (ARB);
6. $(\forall R.\varphi \sqcap \forall R.C) \supset \forall R.(\varphi \sqcap C)$	from 5 by $(\mathsf{ARC})^{-1}$.

Then, the application of rule (ARM) (*'monotonicity'*) to derivation (5.61) and Prop. 5.1.2 yields the subgoal (5.60). Using rule (ARB) (*'composition'*) with (5.60) and (5.59) yields

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma' \sqcap \forall R.\varphi \sqcap \forall R.C \sqcap \wedge\Gamma') \supset (\vee\Phi \sqcup \vee\Psi), \tag{5.62}$$

if $\Sigma(R) \neq \emptyset$, or otherwise $\forall R.\varphi$ is not part of the left-hand side conjunction of (5.62). Now, observe that by unfolding $\wedge\Sigma'$, and $\forall R.\varphi$ (if it exists) in (5.62) we obtain a derivation of

$$\Theta\,;\,\emptyset \mathrel{|\!\!\!\underline{\;\;}_{\mathsf{H}}} \big(\big(\bigwedge_{R' \in dom(\Sigma)\setminus\{R\}} \forall R'.\wedge\Sigma(R')\big) \sqcap \forall R.\wedge\Sigma(R) \sqcap \forall R.C \sqcap \wedge\Gamma'\big) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.63}$$

Considering the decomposition of $\wedge\Sigma$ (5.23) (see p. 154) we obtain from (5.63) our goal (5.58) as desired (using commutativity & associativity of $\sqcap$).

(**Case** $\forall R$) Let us suppose that the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ is derived by rule $\forall R$, *i.e.*, $\Phi = \Phi', \forall R.C$ and the last rule application is

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\emptyset\,;\,\Sigma(R) \vdash C\,;\,\emptyset}}{\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi',\,\forall R.C\,;\,\Psi}\,\forall R$$

This time the goal is to obtain a derivation for

$$\Theta\,;\,\emptyset \mathrel{|\!\!\!\underline{\;\;}_{\mathsf{H}}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi' \sqcup \forall R.C \sqcup \vee\Psi). \tag{5.64}$$

The induction hypothesis and rule (ARE) let us conclude that $\Theta\,;\,\emptyset \mathrel{|\!\!\!\underline{\;\;}_{\mathsf{H}}} \wedge\Sigma(R) \supset C$. An application of rule Nec gives us the derivation $\Theta\,;\,\emptyset \mathrel{|\!\!\!\underline{\;\;}_{\mathsf{H}}} \forall R.(\wedge\Sigma(R) \supset C)$. From the latter and the instance $\Theta\,;\,\emptyset \mathrel{|\!\!\!\underline{\;\;}_{\mathsf{H}}} \forall R.(\wedge\Sigma(R) \supset C) \supset (\forall R.\wedge\Sigma(R) \supset \forall R.C)$ of axiom $\mathsf{K}_{\forall R}$ it follows by rule MP that

$$\Theta\,;\,\emptyset \mathrel{|\!\!\!\underline{\;\;}_{\mathsf{H}}} \forall R.\wedge\Sigma(R) \supset \forall R.C. \tag{5.65}$$

At this point, we use the fact (see p. 162) that Hilbert derives

$$\Theta\,;\,\emptyset \mathrel{|\!\!\!\underline{\;\;}_{\mathsf{H}}} \wedge\Sigma \supset \forall R.\wedge\Sigma(R), \tag{5.66}$$

whether $\Sigma(R) = \emptyset$ or not, and obtain by admissible rule (ARB) from (5.65) and (5.66) the derivation

$$\Theta\,;\,\emptyset \mathrel{|\!\!\!\underline{\;\;}_{\mathsf{H}}} \wedge\Sigma \supset \forall R.C. \tag{5.67}$$

Using weakening (rule (ARW)) with (5.67) yields the goal (5.64).

(**Case** *Hyp*$_1$) Suppose the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ is derived by rule *Hyp*$_1$, *i.e.*, $\Theta = \Theta', C$. Then the last rule application is

$$\dfrac{\dfrac{\vdots}{\Theta'\,;\,\Sigma \cup [R \mapsto C]\,;\,\Gamma \vdash \Phi\,;\,\Psi \quad R \in N_R}}{\Theta',\, C\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi}\; Hyp_1$$

We must find a Hilbert derivation for

$$\Theta',\, C\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.68}$$

The induction hypothesis for the premise implies the existence of the derivation

$$\Theta'\,;\,\emptyset \vdash_{\mathsf{H}} ((\bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')) \sqcap \forall R.(\wedge\Sigma(R) \sqcap C) \sqcap \wedge\Gamma\big) \supset (\vee\Phi \sqcup \vee\Psi),$$

which by Prop. 5.1.2 also holds under extended assumptions, *i.e.*,

$$\Theta', C\,;\,\emptyset \vdash_{\mathsf{H}} ((\bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')) \sqcap \forall R.(\wedge\Sigma(R) \sqcap C) \sqcap \wedge\Gamma\big) \supset (\vee\Phi \sqcup \vee\Psi).$$

To see what is going on let us abbreviate this as

$$\Theta', C\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma' \sqcap \forall R.(\varphi \sqcap C) \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \vee\Psi), \tag{5.69}$$

where $\wedge\Sigma' =_{df} \bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')$ and $\varphi =_{df} \wedge\Sigma(R)$.

We have to consider two cases: Case 1. If $\Sigma(R) \neq \emptyset$ then we need to consider the distribution of $\forall R$ over $\sqcap$ as shown in the proof of the rule $\forall L$ on page 164. Thereof we obtain a derivation for

$$\Theta', C\,;\,\emptyset \vdash_{\mathsf{H}} (\forall R.\varphi \sqcap \forall R.C) \supset \forall R.(\varphi \sqcap C). \tag{5.70}$$

Then, by an application of rule (ARM) (*'monotonicity'*) we obtain from (5.70)

$$\Theta', C\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma' \sqcap \forall R.\varphi \sqcap \forall R.C \sqcap \wedge\Gamma) \supset (\wedge\Sigma' \sqcap \forall R.(\varphi \sqcap C) \sqcap \wedge\Gamma). \tag{5.71}$$

An application of rule (ARB) (*'composition'*) to (5.71) and (5.69) lets us conclude that $\Theta', C\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma' \sqcap \forall R.\varphi \sqcap \forall R.C \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \vee\Psi)$. Furthermore, by rule (ARC) (*currying*) and commutativity of $\sqcap$ follows the derivation for

$$\Theta', C\,;\,\emptyset \vdash_{\mathsf{H}} \forall R.C \supset ((\wedge\Sigma' \sqcap \forall R.\varphi \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \vee\Psi)). \tag{5.72}$$

The global assumption Θ', C lets us easily conclude that $\Theta', C\,;\,\emptyset \mathrel{|\!\!\frac{}{\mathsf{H}}} C$ and by rule Nec it follows that $\Theta', C\,;\,\emptyset \mathrel{|\!\!\frac{}{\mathsf{H}}} \forall R.C$. Then, the application of rule MP to the latter derivation and (5.72) generates

$$\Theta', C\,;\,\emptyset \mathrel{|\!\!\frac{}{\mathsf{H}}} (\wedge\Sigma' \sqcap \forall R.\varphi \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.73}$$

Unfolding φ and $\wedge\Sigma'$ according to (5.23) we obtain the derivation (5.73)

$$\Theta', C\,;\,\emptyset \mathrel{|\!\!\frac{}{\mathsf{H}}} \Big(\Big(\bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')\Big) \sqcap \forall R.\wedge\Sigma(R) \sqcap \wedge\Gamma\Big) \supset (\vee\Phi \sqcup \vee\Psi),$$

which represents the goal derivation (5.68).

Case 2. If $\Sigma(R) = \emptyset$ then $\Theta', C\,;\,\emptyset \mathrel{|\!\!\frac{}{\mathsf{H}}} (\wedge\Sigma' \sqcap \forall R.C \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \vee\Psi)$. This case is argued analogously to the first case, differing only in that $\forall R.\wedge\Sigma(R)$ is not part of the conjunctions of each corresponding derivation step above.

(**Case** *Hyp*$_2$) Finally, suppose the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ is derived by rule *Hyp*$_2$, *i.e.*, $\Theta = \Theta', C$ and the last rule application looks like

$$\dfrac{\dfrac{\vdots}{\Theta'\,;\,\Sigma\,;\,\Gamma,\ C \vdash \Phi\,;\,\Psi}}{\Theta',\ C\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi}\ Hyp_2$$

The goal is to find a derivation for

$$\Theta', C\,;\,\emptyset \mathrel{|\!\!\frac{}{\mathsf{H}}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.74}$$

The ind. hyp. for the premise yields $\Theta'\,;\,\emptyset \mathrel{|\!\!\frac{}{\mathsf{H}}} (\wedge\Sigma\sqcap\wedge\Gamma\sqcap C) \supset (\vee\Phi\sqcup\vee\Psi)$ which holds under extended assumptions as well, *i.e.*, $\Theta', C\,;\,\emptyset \mathrel{|\!\!\frac{}{\mathsf{H}}} (\wedge\Sigma \sqcap \wedge\Gamma \sqcap C) \supset (\vee\Phi \sqcup \vee\Psi)$. By applying admissible rule (ARC) (*'currying'*) and commutativity of $\sqcap$ we obtain the derivation

$$\Theta', C\,;\,\emptyset \mathrel{|\!\!\frac{}{\mathsf{H}}} C \supset ((\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \vee\Psi)). \tag{5.75}$$

The assumption Θ', C lets us obtain

$$\Theta', C\,;\,\emptyset \mathrel{|\!\!\frac{}{\mathsf{H}}} C. \tag{5.76}$$

Then, the goal (5.74) follows immediately from (5.76) and (5.75) by MP. □

Finally, we will tackle the *stronger* form of the equivalence statement between Hilbert and Gentzen, which does consider local assumptions as well.

Proposition 5.2.2. *For any sets of concepts* Θ*,* Γ *and concept* C *we have* $\Theta\,;\,\Gamma \mathrel{|\!\!\frac{}{\mathsf{H}}} C$ *iff the sequent* $\Theta\,;\,\emptyset\,;\,\Gamma \vdash C\,;\,\emptyset$ *is inconsistent.* ∇

Proof. The proof is by induction on the size of the set Γ.

($\Rightarrow$) In the base case $\Gamma = \emptyset$ the goal follows immediately by Prop. 5.2.1.

In the inductive step, let us suppose that we have a Hilbert derivation for $\Theta\,;\,\Gamma,\,D \mathrel{|\!\!\frac{}{\mathsf{H}}} C$. We have to show that the sequent $\Theta\,;\,\emptyset\,;\,\Gamma,\,D \vdash C\,;\,\emptyset$ is inconsistent. By compactness Prop. 5.1.1 there exist finite sets $\Theta_f \subseteq \Theta$ and $\Gamma_f \subseteq \Gamma$ such that $\Theta_f\,;\,\Gamma_f,\,D \mathrel{|\!\!\frac{}{\mathsf{H}}} C$ holds. Applying the local Deduction Theorem 5.1.1 yields $\Theta_f\,;\,\Gamma_f \mathrel{|\!\!\frac{}{\mathsf{H}}} D \supset C$. Now, we can use the inductive hypothesis to obtain that the sequent $\Theta_f\,;\,\emptyset\,;\,\Gamma_f \vdash D \supset C\,;\,\emptyset$ is inconsistent as well. By the general property of weakening Lem. 5.2.1 this also holds for the sequent $\Theta_f\,;\,\emptyset\,;\,\Gamma_f,\,D \vdash D \supset C\,;\,\emptyset$. Taking the inconsistent sequent $\Theta_f\,;\,\emptyset\,;\,\Gamma_f,\,D \vdash D\,;\,\emptyset$ we can apply the admissible sequent rule MP which has been introduced in the proof of Prop. 5.2.1 as follows:

$$\frac{\Theta_f\,;\,\emptyset\,;\,\Gamma_f,\,D \vdash D \supset C\,;\,\emptyset \qquad \Theta_f\,;\,\emptyset\,;\,\Gamma_f,\,D \vdash D\,;\,\emptyset}{\Theta_f\,;\,\emptyset\,;\,\Gamma_f,\,D \vdash C\,;\,\emptyset}\,\mathsf{MP}$$

Inconsistency of the sequent $\Theta\,;\,\emptyset\,;\,\Gamma,\,D \vdash C\,;\,\emptyset$ follows by weakening Lem. 5.2.1.

($\Leftarrow$) In the base case $\Gamma = \emptyset$ the goal follows directly by Prop. 5.2.1.

In the inductive step, assume that the sequent $\Theta\,;\,\emptyset\,;\,\Gamma,\,D \vdash C\,;\,\emptyset$ is inconsistent. We need to show that there is a Hilbert derivation for $\Theta\,;\,\Gamma,\,D \mathrel{|\!\!\frac{}{\mathsf{H}}} C$. An application of rule $\supset\! R$ gives us the following situation

$$\frac{\Theta\,;\,\emptyset\,;\,\Gamma,\,D \vdash C\,;\,\emptyset}{\Theta\,;\,\emptyset\,;\,\Gamma \vdash D \supset C\,;\,\emptyset}\,{\supset}R$$

Applying the ind. hyp. to the conclusion of the sequent yields the Hilbert derivation $\Theta\,;\,\Gamma \mathrel{|\!\!\frac{}{\mathsf{H}}} D \supset C$. The goal $\Theta\,;\,\Gamma,\,D \mathrel{|\!\!\frac{}{\mathsf{H}}} C$ follows by the ($\Leftarrow$ direction) of the local Deduction Theorem 5.1.1. $\square$

5.3 Towards Intermediate Logics between $c\mathcal{ALC}$ and $\mathcal{ALC}$

There are at least four natural dimensions in which $c\mathcal{ALC}$ is a constructive weakening of $\mathcal{ALC}$ corresponding to the axiom schemata:

(i) $\mathsf{FS3}/\mathsf{IK3} =_{df} \neg\exists R.\bot$ (*Infallible Fillers*),

(ii) $\mathsf{FS4}/\mathsf{IK4} =_{df} \exists R.(C \sqcup D) \supset (\exists R.C \sqcup \exists R.D)$ (*Disjunctive Distribution*),

(iii) $\mathsf{FS5}/\mathsf{IK5} =_{df} (\exists R.C \supset \forall R.D) \supset \forall R.(C \supset D)$ (*Interaction Scheme*),

(iv) $\mathsf{PEM} =_{df} C \sqcup \neg C$ (*Excluded Middle*).

Each of them is associated with a specific semantic restriction of interpretations. In this section we will discuss FS3/IK3, FS4/IK4 and PEM, and show how these axioms can be captured by modifying the $c\mathcal{ALC}$ sequent calculus. Restricting $c\mathcal{ALC}$ in this way yields several possible combinations of non-classical DLs between $c\mathcal{ALC}$ and $\mathcal{ALC}$. Note that we do not cover axiom schema FS5/IK5, since on the one hand we were not able to semantically characterise this axiom by an appropriate frame condition, and on the other hand the structure of the sequent calculus G1 seems to be insufficiently expressive to capture this axiom in terms of a rule.

The system $c\mathcal{ALC}$ can be extended in a uniform (global) fashion by forcing one of the above axioms for all roles $R \in N_R$, or relative to specific roles $R \in N_R$ to express in a multimodal (local) fashion that the axiom in question holds for some specific roles only. We will focus on the global view in the following and only discuss axiom FS3/IK3 from the local view. Note that the investigation of the completeness of each extension w.r.t. the local view is related to the question whether the fusion of abnormal IMLs preserves Kripke completeness. However, this is a general open problem, which we will not address in the following.

5.3.1 Infallible Kripke Semantics – Axiom FS3/IK3

From a proof-theoretic perspective we can axiomatise the non-fallible semantics of $c\mathcal{ALC}$ by adding the axiom schema $\neg\exists R.\bot$. This is possible in a global fashion by forcing axiom $\neg\exists R.\bot$ for all roles $R \in N_R$ meaning that '*there are no inconsistent entities at all*' [161, p.15] or relative to specific roles $R \in N_R$ to say that '*there are no inconsistent entities reachable by role R*'.

At first, we will discuss the global view and show a corresponding extension of the sequent calculus G1. This viewpoint corresponds to a uniform view in which the axiom $\neg\exists R.\bot$ is validated for all $R \in N_R$, *i.e.*, by restricting to infallible interpretations.

Global Extension

Interpretations without fallible elements, denoted by *infallible interpretations*, where $\bot^{\mathcal{I}} = \emptyset$, can be axiomatised by the schema $\neg\exists R.\bot$ to say that 'any entity can always be refined to become fully defined for all roles $R \in N_R$'. Particularly, this means that all R-fillers of an entity are non-fallible. In fact, the absence of axiom $\neg\exists R.\bot$ is the only effect of fallibility. It indicates the existence of entities all of whose refinements have fallible R-fillers.

Proposition 5.3.1. *$\mathcal{I} \models \neg\exists R.\bot$ for all infallible c$\mathcal{ALC}$ interpretations $\mathcal{I}$.* ∇

Proof. The statement is vacuously true. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \emptyset, \cdot^{\mathcal{I}})$ be an arbitrary but infallible c$\mathcal{ALC}$ interpretation, $a \in \Delta^{\mathcal{I}}$ and suppose that $\mathcal{I}; a \models \exists R.\bot$. The assumption implies that there exists $b \in \Delta^{\mathcal{I}}$ with $a\, R^{\mathcal{I}}\, b$ and $b \in \bot^{\mathcal{I}} = \emptyset$ which is always false and therefore $\mathcal{I}; a \models \bot$. Hence, $\mathcal{I} \models \neg\exists R.\bot$. $\square$

One can show that if an interpretation $\mathcal{I}$ validates $\neg\exists R.\bot$ globally, *i.e.*, $\mathcal{I} \models \neg\exists R.\bot$ for all $R \in N_R$, then the set $\bot^{\mathcal{I}}$ is redundant in that we can find a stripped interpretation $\mathcal{I}_s$ so that for all concepts C we have $C^{\mathcal{I}_s} = C^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$. The bottom line is that as long as we are only interested in non-fallible entities, the interpretations $\mathcal{I}$ and $\mathcal{I}_s$ are identical.

Proposition 5.3.2. *For all interpretations $\mathcal{I}$ with $\mathcal{I} \models \neg\exists R.\bot$ exists a stripped interpretation $\mathcal{I}_s$ with $\bot^{\mathcal{I}_s} = \emptyset$ such that for all concepts C we have $C^{\mathcal{I}_s} = C^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$.* ∇

Proof. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a constructive interpretation such that $\mathcal{I} \models \neg\exists R.\bot$. We define its stripped interpretation as follows by $\mathcal{I}_s =_{df} (\Delta^{\mathcal{I}_s}, \preceq^{\mathcal{I}_s}, \bot^{\mathcal{I}_s}, \cdot^{\mathcal{I}_s})$ with

$$\begin{aligned}
&\Delta^{\mathcal{I}_s} =_{df} \Delta^{\mathcal{I}} \setminus \bot^{\mathcal{I}};\\
&\preceq^{\mathcal{I}_s} =_{df} \ \preceq^{\mathcal{I}} \cap (\Delta^{\mathcal{I}_s} \times \Delta^{\mathcal{I}_s});\\
&\bot^{\mathcal{I}_s} =_{df} \emptyset;
\end{aligned}$$

and define $\cdot^{\mathcal{I}_s}$ by taking

$$\begin{aligned}
&A^{\mathcal{I}_s} =_{df} A^{\mathcal{I}} \setminus \bot^{\mathcal{I}}, \text{ for } A \in N_C;\\
&R^{\mathcal{I}_s} =_{df} \{(x,y) \in \Delta^{\mathcal{I}_s} \times \Delta^{\mathcal{I}_s} \mid x\, R^{\mathcal{I}}\, y \text{ or } \exists y', x'.\, x\, R^{\mathcal{I}} y' \in \bot^{\mathcal{I}} \ \&\ x \preceq^{\mathcal{I}} x'\, R^{\mathcal{I}}\, y, \},\\
&\qquad\qquad \text{for all } R \in N_R.
\end{aligned}$$

We proceed by induction on the structure of concept C showing that $C^{\mathcal{I}_s} = C^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$.

(**Case** A) The base case $A^{\mathcal{I}_s} = A^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$ holds directly by definition of the stripped interpretation.

(**Case** $C \sqcap D$),(**Case** $C \sqcup D$) By induction hypothesis.

(**Case** $C \supset D$) ($\Rightarrow$) Let us assume that $x \in (C \supset D)^{\mathcal{I}_s}$. We need to show that $x \in (C \supset D)^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$. By definition of $\mathcal{I}_s$ it follows that $x \in \Delta^{\mathcal{I}}_c$. Let $y \in \Delta^{\mathcal{I}}$ such that $x \preceq^{\mathcal{I}} y$. We proceed by case analysis:

Case 1. If $y \in \bot^{\mathcal{I}}$ then the goal follows trivially.

Case 2. If $y \in C^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$ then $y \in \Delta^{\mathcal{I}}_c = \Delta^{\mathcal{I}_s}$. It follows by the induction hypothesis that $y \in C^{\mathcal{I}_s}$ and the assumption lets us conclude that $y \in D^{\mathcal{I}_s}$. Applying the induction hypothesis gives us $y \in D^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$ as desired. Hence, $x \in (C \supset D)^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$.

($\Leftarrow$) Suppose that $x \in (C \supset D)^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$, *i.e.*, $x \notin \bot^{\mathcal{I}}$ and therefore $x \in \Delta^{\mathcal{I}}_c = \Delta^{\mathcal{I}_s}$. The goal is to show $x \in (C \supset D)^{\mathcal{I}_s}$. Take an arbitrary $y \in \Delta^{\mathcal{I}_s}$ such that $x \preceq^{\mathcal{I}_s} y$ and suppose that $y \in C^{\mathcal{I}_s}$. The definition of $\mathcal{I}_s$ implies that $x \preceq^{\mathcal{I}} y$ as well. It follows by the induction hypothesis that $y \in C^{\mathcal{I}}$ and the assumption lets us conclude $y \in D^{\mathcal{I}}$. A further application of the ind. hyp. yields $y \in D^{\mathcal{I}_s}$. Thus, $x \in (C \supset D)^{\mathcal{I}_s}$.

(**Case** $\exists R.C$) ($\Rightarrow$) Suppose that $x \in (\exists R.C)^{\mathcal{I}_s}$. The goal is $x \in (\exists R.C)^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$. Definition 4.2.2 implies that for all x' with $x \preceq^{\mathcal{I}_s} x'$ there exists an entity $y \in \Delta^{\mathcal{I}_s}$ such that $x' \, R^{\mathcal{I}_s} \, y$ and $y \in C^{\mathcal{I}_s}$. In particular $x \in \Delta^{\mathcal{I}}$ and $x \notin \bot^{\mathcal{I}}$ by definition of $\mathcal{I}_s$.

Let $x' \in \Delta^{\mathcal{I}}_c$ and suppose that $x \preceq^{\mathcal{I}} x'$. Then, by def. of $\mathcal{I}_s$ we have $x \preceq^{\mathcal{I}_s} x'$ as well and by assumption there is $y \in \Delta^{\mathcal{I}_s} = \Delta^{\mathcal{I}}_c$ such that $x' \, R^{\mathcal{I}_s} \, y$ and $y \in C^{\mathcal{I}_s}$. By def. of $R^{\mathcal{I}_s}$ it follows that

- either $x' \, R^{\mathcal{I}} \, y$,
- or $\exists y', x''$ s.t. $x' \, R^{\mathcal{I}} \, y' \in \bot^{\mathcal{I}}$ and $x' \preceq^{\mathcal{I}} x'' \, R^{\mathcal{I}} \, y$.

In the first case $y \in C^{\mathcal{I}}$ by induction hypothesis, in the second case $y' \in C^{\mathcal{I}}$. Hence, $x \in (\exists R.C)^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$.

($\Leftarrow$) Let us assume that $x \in (\exists R.C)^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$. The goal is $x \in (\exists R.C)^{\mathcal{I}_s}$. By assumption $x \in \Delta^{\mathcal{I}}_c = \Delta^{\mathcal{I}_s}$. Now, let $x' \in \Delta^{\mathcal{I}_s}$ such that $x \preceq^{\mathcal{I}_s} x'$. Then, $x' \in \Delta^{\mathcal{I}}_c$ and $x \preceq^{\mathcal{I}} x'$ as well. The assumption lets us conclude that there is $y \in \Delta^{\mathcal{I}}$ with $x' \, R^{\mathcal{I}} \, y$ and $y \in C^{\mathcal{I}}$.

Case 1. If $y \notin \bot^{\mathcal{I}}$ then $x \, R^{\mathcal{I}_s} \, y$ and by ind. hyp. follows that $y \in C^{\mathcal{I}_s}$.

Case 2. Else if $y \in \bot^{\mathcal{I}}$ then the fact $\mathcal{I} \models \neg\exists R.\bot$ implies that there exists $x'' \in \Delta^{\mathcal{I}}_c$ with $x' \preceq^{\mathcal{I}} x''$ such that all its R-successors are infallible. By monotonicity there exists $y' \in \Delta_c^{\mathcal{I}}$ with $x'' \, R^{\mathcal{I}} \, y'$ and $y' \in C^{\mathcal{I}}$, but $y' \notin \bot^{\mathcal{I}}$. The definition of $\mathcal{I}_s$ implies $x' \, R^{\mathcal{I}_s} \, y'$ as well and by ind. hyp. follows that $y' \in C^{\mathcal{I}_s}$. Therefore, $x \in (\exists R.C)^{\mathcal{I}_s}$.

(**Case** $\forall R.C$) ($\Rightarrow$) By contraposition. Assume that $x \notin (\forall R.C)^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$, *i.e.*, either $x \in \bot^{\mathcal{I}}$ or $x \notin (\forall R.C)^{\mathcal{I}}$. In the first case obviously $x \notin \Delta^{\mathcal{I}_s}$ and therefore $x \notin (\forall R.C)^{\mathcal{I}_s}$. Otherwise, there exist $x', y \in \Delta^{\mathcal{I}}$ such that $x \preceq^{\mathcal{I}} x' \, R^{\mathcal{I}} \, y$ and $y \notin C^{\mathcal{I}}$. The latter implies $y \notin \bot^{\mathcal{I}}$ and $x' \notin \bot^{\mathcal{I}}$ follows by Proposition 4.2.1. Then, $x', y \in \Delta^{\mathcal{I}_s}$ and $x \preceq^{\mathcal{I}_s} x' \, R^{\mathcal{I}_s} \, y$ as well, and by induction hypothesis $y \notin C^{\mathcal{I}_s}$ which was to be shown.

($\Leftarrow$) By contraposition. Suppose that $x \notin (\forall R.C)^{\mathcal{I}_s}$, *i.e.*, there exist $x', y \in \Delta^{\mathcal{I}_s} = \Delta_c^{\mathcal{I}}$ such that $x \preceq^{\mathcal{I}_s} x' \, R^{\mathcal{I}_s} \, y$ and $y \notin C^{\mathcal{I}_s}$. The goal is to show that $x \notin (\forall R.C)^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$. We claim that $x \notin (\forall R.C)^{\mathcal{I}}$. The definition of $\mathcal{I}_s$ implies $x \preceq^{\mathcal{I}} x'$ and either

- $x' \, R^{\mathcal{I}} \, y$, or
- $\exists x'', y' \in \Delta^{\mathcal{I}}$ s.t. $x' \, R^{\mathcal{I}} \, y' \in \bot^{\mathcal{I}}$ and $x' \preceq^{\mathcal{I}} x'' \, R^{\mathcal{I}} \, y$.

It follows by ind. hyp. that $y \notin C^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$. Therefore, $x \notin (\forall R.C)^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$. □

The exclusion of fallibility from the sequent calculus `G1` for $c\mathcal{ALC}$ is by dropping the side-condition $|\Phi \cup \Psi| \geq 1$ from the premise of rule $\bot L$. This modification implements the axiom schema $\neg\exists R.\bot$ globally for all roles $R \in N_R$. We will call the stronger rule without the side-condition $\bot L^+$ [195, p. 228]:

$$\frac{}{\Theta\,;\,\Sigma\,;\,\Gamma,\ \bot \vdash \Phi\,;\,\Psi}\ \bot L^+$$

In the presence of rule $\bot L^+$, the set of fallibles can be identified with an empty succedent such that we obtain the usual right and left introduction rules $\neg R$ and $\neg L$ for intuitionistic negation as shown below [195, p. 228]:

$$\frac{\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi,\ C\,;\,\Psi}{\Theta\,;\,\Sigma\,;\,\Gamma,\ \neg C \vdash \Phi\,;\,\Psi}\ \neg L \qquad\qquad \frac{\Theta\,;\,\Sigma\,;\,\Gamma,\ C \vdash \emptyset\,;\,\emptyset}{\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi,\ \neg C\,;\,\Psi}\ \neg R$$

While rule $\neg R$ is admissible already in $c\mathcal{ALC}$ being a variant of rule $\supset R$, rule $\neg L$ is not. Figure 5.6 depicts the derivations for the axiom $\neg\exists R.\bot$ and the rule $\neg L$ based on the rule $\bot L^+$, where $\neg C$ abbreviates $C \supset \bot$.

Notation. We will denote the infallible sequent calculus by $\mathtt{G1}_F$ and state analogously to Def. 5.2.3 that a $\mathtt{G1}_F$-sequent is consistent if no tableau exists for it. Furthermore, let us write $c\mathcal{ALC}^F =_{df} c\mathcal{ALC} \oplus \{\neg\exists R.\bot \mid R \in N_R\}$ for the globally R-infallible Hilbert system of $c\mathcal{ALC}$, and $c\mathcal{ALC}^F\,;\,\Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi$ to express that the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \vdash_{\mathtt{G1}_F} \Phi\,;\,\Psi$ is satisfiable at some entity of an infallible interpretation. ■

$$\dfrac{\dfrac{\dfrac{}{\emptyset\,;\,\emptyset\,;\,\bot \vdash \emptyset\,;\,\emptyset}\,\bot L^{+}}{\emptyset\,;\,\emptyset\,;\,\exists R.\bot \vdash \bot\,;\,\emptyset}\,\exists L}{\emptyset\,;\,\emptyset\,;\,\emptyset \vdash \neg\exists R.\bot\,;\,\emptyset}\,\supset R \qquad\qquad \dfrac{\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Psi,\, C\,;\,\Psi \qquad \dfrac{}{\Theta\,;\,\Sigma\,;\,\Gamma,\,\bot \vdash \Phi\,;\,\Psi}\,\bot L^{+}}{\Theta\,;\,\Sigma\,;\,\Gamma,\,\neg C \vdash \Phi\,;\,\Psi}\,\supset L$$

Figure 5.6: Tableau proofs for $\neg\exists R.\bot$ and $\neg L$. Adapted from [195, p. 228, Fig. 11], with kind permission from Springer Science and Business Media.

In the next step we will show that the extended, infallible sequent calculus $\mathsf{G1}_F$ is sound and complete for the class of constructive interpretations without fallible entities.

Theorem 5.3.1. $\mathsf{G1}_F$ *is sound and complete, formally*

$$c\mathcal{ALC}^F\,;\,\Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi \quad \textit{iff} \quad \Theta\,;\,\Sigma\,;\,\Gamma \nvdash_{\mathsf{G1}_F} \Phi\,;\,\Psi. \qquad \nabla$$

Proof. ($\Rightarrow$) Soundness of $\mathsf{G1}_F$ is obvious and argued as before by Thm. 5.2.2, only differing in the fact that for rule $\bot L^+$ the succedent is not constrained to be non-empty anymore, which is unproblematic in the absence of fallible entities.

($\Leftarrow$) The proof of completeness is a specialisation of the proof of Thm. 5.2.1. The construction of the canonical model needs to be refined as follows: (i) The process of saturating a consistent sequent is restricted to rely on the set $X_{c\mathcal{ALC}^F} =_{df} X_{c\mathcal{ALC}} \setminus \{\mathrm{E7}_R\}$ of extension rules. Then, the application of Lem. 5.2.4 yields a $X_{c\mathcal{ALC}^F}$-saturated and consistent extension for all consistent sequents. (ii) Secondly, we need to change the definition of the canonical model. Let Δ^F be the set of all consistent and $X_{c\mathcal{ALC}^F}$-saturated sequents. The canonical model is given by $\mathcal{I}^F =_{df} (\Delta^F, \preceq^F, \bot^F, \cdot^F)$ following the lines of Def. 5.2.12, except that now $\bot^F =_{df} \emptyset$ and $A^F =_{df} \{s \in \Delta^F \mid A \in \Gamma_s\}$. Obviously, this model is still a constructive model in line with the proofs of Lem. 5.2.6 and 5.2.7. Then, the proof of Thm. 5.2.1 applies here as before. In particular, let $s =_{df} \Theta\,;\,\Sigma\,;\,\Gamma \vdash_{\mathsf{G1}_F} \Phi\,;\,\Psi$ be a consistent sequent. By Lem. 5.2.4 we obtain a $X_{c\mathcal{ALC}^F}$-saturated and consistent extension $s^* \in \Delta^F$ of s. We can construct an infallible canonical model $\mathcal{I}^F$ as described before such that $(\mathcal{I}^F, s^*)$ satisfies the sequent s^* by repeating the proof of Lem. 5.2.6 and 5.2.7, and particularly satisfies the sequent s. □

Remark 5.3.1. Alternatively we can make use of Prop. 5.3.2 to prove the ($\Leftarrow$) direction of Thm. 5.3.1. Suppose that the TBox Θ includes axiom $\neg\exists R.\bot$ for all roles in N_R and that $\Theta\,;\,\Sigma\,;\,\Gamma \nvdash_{\mathsf{G1}} \Phi\,;\,\Psi$, *i.e.*, $\{\neg\exists R.\bot \mid R \in N_R\} \subseteq \Theta$ and there exists no closed tableau for the sequent $s = \Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ in $\mathsf{G1}$. We claim that every $\mathsf{G1}$ proof with $\{\neg\exists R.\bot \mid R \in N_R\} \subseteq \Theta$ can be transformed into a $\mathsf{G1}_F$ proof where Θ does not include any instance of axiom $\neg\exists R.\bot$. We obtain by Lem. 5.2.4 a $X_{c\mathcal{ALC}}$-saturated and consistent extension $s^* \in \Delta^*$ of s and construct a canonical model $\mathcal{I}^*$ according to Def. 5.2.12. By satisfiability of Θ this means that the axiom schema $\neg\exists R.\bot$ holds at every world of Δ^*. Hence, $\mathcal{I}^*$ is an interpretation such that $\mathcal{I}^* \models \neg\exists R.\bot$. Thereof, we obtain by Prop. 5.3.2 a stripped interpretation $\mathcal{I}^*_s$ of $\mathcal{I}^*$ such that $(\mathcal{I}^*_s, s^*)$ satisfies the sequent s. ■

Equivalence of the Infallible System to the Hilbert System

It remains to show that the Hilbert system $c\mathcal{ALC}^F$ is sound and complete by showing that the Hilbert system for $c\mathcal{ALC}^F$ is equivalent to the sequent system $\mathsf{G1}_F$.

Proposition 5.3.3. *Let* $c\mathcal{ALC}^F =_{df} c\mathcal{ALC} \oplus \{\neg\exists R.\bot \mid R \in N_R\}$ *be the globally R-infallible axiomatisation. The Hilbert system* $c\mathcal{ALC}^F$ *is sound and complete,* i.e., *for every concept* C *and set of concepts* Θ *we have*

$$c\mathcal{ALC}^F\,;\,\Theta\,;\,\emptyset \models C \textit{ in all infallible interpretations} \quad \textit{iff} \quad c\mathcal{ALC}^F\,;\,\Theta\,;\,\emptyset \mid\!\frac{}{\;\mathsf{H}\;} C. \qquad \nabla$$

Proof. The proof is by showing that every derivation of the extended system $c\mathcal{ALC}^F$ can be translated into a derivation of the Gentzen sequent calculus $\mathsf{G1}_F$ and vice versa. This is by extending the proof of Prop. 5.2.1 by showing that axiom $\neg\exists R.\bot$ is derivable in $\mathsf{G1}_F$ and in the other direction we assume the premise of the rules $\bot L^+$ and $\exists L$ to be admissible and then give a Hilbert derivation of its conclusion.

($\Rightarrow$) For soundness of the Hilbert system $c\mathcal{ALC}^F$ it suffices to show that the axiom $\neg\exists R.\bot$ is derivable in $\mathsf{G1}_F$ as depicted in Fig. 5.6.

($\Leftarrow$) In the other direction we assume the premise of the rules $\bot L^+$ and $\exists L$ to be admissible and then give a Hilbert derivation of its conclusion.

(**Case** $\bot L^F$) Let us assume that $\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\frac{}{\;\mathsf{G1}_F\;} \Phi\,;\,\Psi$ is derived by rule $\bot L^+$, *i.e.*, $\Gamma = \Gamma', \bot$. The goal is to show that Hilbert derives

$$\Theta\,;\,\emptyset \mid\!\frac{}{\;\mathsf{H}\;} (\wedge\Sigma \;\sqcap\; \bot \sqcap \wedge\Gamma') \;\supset\; (\vee\Phi \;\sqcup\; \vee\Psi),$$

which is a consequence of IPC7 (Def. 5.1.1) and weakening (ARW). Note that the special case with $\Phi = \emptyset$ is handled already by its translation $\vee\Phi$ to give $\vee\emptyset = \bot$.

(**Case** $\exists L$) Let us revisit rule $\exists L$ (see p. 161). If the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\frac{}{\;\mathsf{G1}_F\;} \Phi\,;\,\Psi$ is derived by rule $\exists L$ then $\Gamma = \Gamma', \exists R.C$ and the last rule application is

$$\dfrac{\dfrac{\vdots}{\Theta\,;\,\emptyset\,;\,\Sigma(R),\, C \mid\!\frac{}{\;\mathsf{G1}_F\;} \Psi(R)\,;\,\emptyset}}{\Theta\,;\,\Sigma\,;\,\Gamma',\, \exists R.C \mid\!\frac{}{\;\mathsf{G1}_F\;} \Phi\,;\,\Psi}\;\exists L$$

Now, we have to consider the case where $\Psi(R) = \emptyset$. The task is to derive

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap \exists R.C) \supset (\vee\Phi \sqcup \bigvee_{R \neq R'} \exists R'.\vee\Psi(R')). \tag{5.77}$$

From the induction hypothesis and the admissible rule (ARE) follows the Hilbert derivation $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma(R) \sqcap C) \supset \bot$. Applying rule Nec to the latter yields $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} \forall R.((\wedge\Sigma(R) \sqcap C) \supset \bot)$. Taking the appropriate instance of axiom $\mathsf{K}_{\exists R}$ and by an application of rule MP this generates the derivation

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} \exists R.(\wedge\Sigma(R) \sqcap C) \supset \exists R.\bot. \tag{5.78}$$

As before on p. 161, we can construct a Hilbert derivation for

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\forall R.\wedge\Sigma(R) \sqcap \exists R.C) \supset \exists R.(\wedge\Sigma(R) \sqcap C). \tag{5.79}$$

Applying admissible rule (ARB) (*"composition"*) to (5.79) and (5.78) yields

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\forall R.\wedge\Sigma(R) \sqcap \exists R.C) \supset \exists R.\bot. \tag{5.80}$$

At this point, axiom $\exists R.\bot \supset \bot$ comes into play such that by rule (ARB) applied to the latter and (5.80) we obtain

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\forall R.\wedge\Sigma(R) \sqcap \exists R.C) \supset \bot. \tag{5.81}$$

Then, we use the fact (see p. 162) that Hilbert derives

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} \wedge\Sigma \supset \forall R.\wedge\Sigma(R), \tag{5.82}$$

whether $\Sigma(R) = \emptyset$ or $\Sigma(R) \neq \emptyset$, and thereof obtain by rule (ARM) a derivation of

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \exists R.C) \supset (\forall R.\wedge\Sigma(R) \sqcap \exists R.C). \tag{5.83}$$

From an appropriate instance of axiom IPC7, (5.83) and (5.81) we obtain by the rule (ARB) a derivation of

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \exists R.C) \supset (\vee\Phi \sqcup \bigvee_{R' \in dom(\Psi)\setminus\{R\}} \exists R'.\vee\Psi(R')). \tag{5.84}$$

Then, the goal (5.77) follows by weakening (rule (ARW)) from (5.84). ☐

Multimodal Extension

The exclusion of fallible entities in the previous section does not allow mixed systems in the spirit of DLs, where only some roles come with a particular restriction. An alternative view is to allow the restriction of specific roles R to reach infallible entities only, which is in line with range restrictions on roles as used in the description logic literature. Taking this route allows us to restrict a specific role $R \in N_R$ to reach only non-fallible entities and to define mixed systems in the tradition of DLs.

Thereto we restrict the interpretations by a set of roles N_{R^F}, which represents the infallible roles of a specific domain. Intuitively, an infallible role has only infallible R-fillers.

Definition 5.3.1 (Infallible role)**.** Let $N_{R^F} \subseteq N_R$ be the set of *infallible roles.* ∇

Definition 5.3.2 (R-infallible interpretation)**.** We call an interpretation $\mathcal{I} =_{df} (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ to be N_{R^F}*-infallible* if it holds for all $R \in N_{R^F}$ that

$$\forall x \in \Delta^{\mathcal{I}}_c, y \in \Delta^{\mathcal{I}}.\ xRy \Rightarrow y \notin \bot^{\mathcal{I}}.$$

Given a role $R \in N_{R^F}$ we also say that an interpretation $\mathcal{I}$ is R*-infallible.* ∇

Remark 5.3.2. Note that the condition of Def. 5.3.2 takes only infallible entities into account and is in line with the semantics of c$\mathcal{ALC}$ (see Def. 4.2.2). However, an alternative view is to require this condition also for fallible entities, but this has some severe consequences and imposes a new class of interpretations with a different philosophical interpretation of fallibility. In this case the interpretation of fallible entities (*cf.* Def. 4.2.2) needs to be weakened such that non-fallible roles are excluded from their interpretation. This is by relaxing Def. 4.2.2 as follows: An N_{R^F}-infallible interpretation is a structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}$ and $\cdot^{\mathcal{I}}$ are defined as in Def. 4.2.2, and the set $\bot^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ of *fallible* entities is closed under refinement and role filling as follows:

- $x \in \bot^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in \bot^{\mathcal{I}}$,
- $x \in \bot^{\mathcal{I}}$ and $\forall R \in N_R \setminus N_{R^F}.\ \forall z.\ x\, R^{\mathcal{I}}\, z \Rightarrow z \in \bot^{\mathcal{I}}$,
- $x \in \bot^{\mathcal{I}}$ and $\forall R \in N_R \setminus N_{R^F}.\ \exists z.\ x\, R^{\mathcal{I}}\, z\ \&\ z \in \bot^{\mathcal{I}}$.

Intuitively, this interpretation would allow a fallible entity to recover from its state of fallibility after doing for some infallible role $R \in N_{R^F}$ one R-step to an infallible successor. We leave the investigation of this semantics for future work. ■

Proposition 5.3.4. *For all interpretations $\mathcal{I}$ and roles $R \in N_{R^F}$, if $\mathcal{I}$ is R-infallible then $\mathcal{I} \models \neg\exists R.\bot$.* ∇

Proof. Let $R \in N_{R^F}$ and suppose that the interpretation $\mathcal{I} =_{df} (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is R-infallible. Let $a, a' \in \Delta^{\mathcal{I}}_c$ such that $a \preceq^{\mathcal{I}} a'$ and $\mathcal{I}; a' \not\models \exists R.\bot$. The latter holds if there is an infallible refinement a'' with $a' \preceq^{\mathcal{I}} a''$ such that for all R-successors of a'' it holds that $\mathcal{I}; a'' \not\models \bot$. This is a consequence of Def. 5.3.2. Hence, $\mathcal{I} \models \neg\exists R.\bot$. □

Proposition 5.3.5. *For all N_{R^F}-infallible interpretations $\mathcal{I}$, if $N_R = N_{R^F}$ then there exists a stripped interpretation $\mathcal{I}_s$ with $\bot^{\mathcal{I}_s} = \emptyset$ such that for all concepts C we have $C^{\mathcal{I}_s} = C^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$.* ∇

Proof. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a N_{R^F}-infallible interpretation and $N_R = N_{R^F}$. We define its stripped interpretation as follows by $\mathcal{I}_s =_{df} (\Delta^{\mathcal{I}_s}, \preceq^{\mathcal{I}_s}, \bot^{\mathcal{I}_s}, \cdot^{\mathcal{I}_s})$ with

$$\begin{aligned}
&\Delta^{\mathcal{I}_s} =_{df} \Delta^{\mathcal{I}} \setminus \bot^{\mathcal{I}};\\
&\preceq^{\mathcal{I}_s} =_{df} \preceq^{\mathcal{I}} \cap (\Delta^{\mathcal{I}_s} \times \Delta^{\mathcal{I}_s});\\
&\bot^{\mathcal{I}_s} =_{df} \emptyset;
\end{aligned}$$

and define $\cdot^{\mathcal{I}_s}$ by taking

$$\begin{aligned}
&A^{\mathcal{I}_s} =_{df} A^{\mathcal{I}} \setminus \bot^{\mathcal{I}}, \text{ for } A \in N_C;\\
&R^{\mathcal{I}_s} =_{df} R^{\mathcal{I}} \cap (\Delta^{\mathcal{I}_s} \times \Delta^{\mathcal{I}_s}), \text{for all } R \in N_R.
\end{aligned}$$

Then, one shows by induction on the structure of concept C that $C^{\mathcal{I}_s} = C^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$. □

The definition of satisfiability of a sequent is strengthened to refer to an infallible entity.

Definition 5.3.3. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a constructive interpretation and $x \in \Delta^{\mathcal{I}}_c$ an infallible entity. The pair $(\mathcal{I}, x)$ *infallibly satisfies* (*i-satisfies* for short) the sequent $s =_{df} \Theta\,;\Sigma\,;\Gamma \vdash \Phi\,;\Psi$ if $(\mathcal{I}, x)$ satisfies s according to Def. 5.2.2 and $x \notin \bot^{\mathcal{I}}$. ∇

We can enforce $\exists R.\bot \supset \bot$ for particular roles $R \in N_{R^F}$ in terms of the following right rule $\bot R^F$ that constrains sequents to be infallible.

$$\frac{\Theta\,;\Sigma\,;\Gamma \vdash \Phi\,;\Psi \cup [R \mapsto \bot] \quad R \in N_{R^F}}{\Theta\,;\Sigma\,;\Gamma \vdash \Phi\,;\Psi}\,\bot R^F$$

Furthermore, we need to strengthen rule $\exists L$ and $\bot L$ accordingly to accommodate Def. 5.3.3. Semantically, rule $\exists L^F$ is restricted to generate only infallible successor sequents and rule $\bot L$ does not rely on the precondition $|\Phi \cup \Psi| \geq 1$ anymore.

$$\frac{\Theta\,;\emptyset\,;\Sigma(R),\, C \vdash \Psi(R)\,;\emptyset \quad \Psi(R) \neq \emptyset}{\Theta\,;\Sigma\,;\Gamma,\, \exists R.C \vdash \Phi\,;\Psi}\,\exists L^F \qquad \frac{}{\Theta\,;\Sigma\,;\Gamma,\, \bot \vdash \Phi\,;\Psi}\,\bot L^F$$

The proof of axiom FS3/IK3 in the calculus $\mathsf{G1}_{R^F}$ is shown in Fig. 5.7.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset\,;\,\emptyset\,;\,\bot \vdash \bot\,;\,\emptyset}\,\bot L}{\emptyset\,;\,\emptyset\,;\,\exists R.\bot \vdash \bot\,;\,[R \mapsto \bot]}\,\exists L^F}{\emptyset\,;\,\emptyset\,;\,\exists R.\bot \vdash \bot\,;\,\emptyset}\,\bot R^F}{\emptyset\,;\,\emptyset\,;\,\emptyset \vdash \neg\exists R.\bot\,;\,\emptyset}\,\supset R \qquad \dfrac{\begin{array}{c}\hdashline \emptyset\,;\,\emptyset\,;\,\exists S.\bot \vdash \bot\,;\,\emptyset\end{array}\,?}{\emptyset\,;\,\emptyset\,;\,\emptyset \vdash \neg\exists S.\bot\,;\,\emptyset}\,\supset R$$

Figure 5.7: Proof of axiom FS3/IK3 with $R \in N_{R^F}$ and $S \in N_R \setminus N_{R^F}$ in $\mathsf{G1}_{R^F}$.

Notation. Let $\mathsf{G1}_{R^F}$ denote the R-infallible sequent calculus consisting of the rules $\bot R^F$, $\exists L^F$ and $\bot L^F$, and the remaining ones from $\mathsf{G1}$. We say that a $\mathsf{G1}_{R^F}$-sequent is consistent if no tableau exists for it (*cf.* Def. 5.2.3). We write $c\mathcal{ALC}^{R^F} =_{df} c\mathcal{ALC} \oplus \{\neg\exists R.\bot \mid R \in N_{R^F}\}$ for the R-infallible Hilbert system, and $c\mathcal{ALC}^{R^F}; \Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi$ to say that $\Theta\,;\,\Sigma\,;\,\Gamma \vdash_{\mathsf{G1}_{R^F}} \Phi\,;\,\Psi$ is i-satisfiable at some infallible entity of a N_{R^F}-infallible interpretation. When $\mathsf{G1}_{R^F}$ is clear from the context we simply write $\vdash$. ■

Theorem 5.3.2 (Soundness of $\mathsf{G1}_{R^F}$). *Every $\mathsf{G1}_{R^F}$-sequent is not i-satisfiable in a N_{R^F}-infallible interpretation,* i.e.,

$$c\mathcal{ALC}^{R^F}; \Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi \quad \textit{implies} \quad \Theta\,;\,\Sigma\,;\,\Gamma \not\vdash_{\mathsf{G1}_{R^F}} \Phi\,;\,\Psi. \qquad \nabla$$

Proof. Analogously to the proof of Thm. 5.2.2, but relying on Def. 5.3.3 for satisfiability of a $\mathsf{G1}_{R^F}$-sequent, relative to N_{R^F}-infallible interpretations according to Def. 5.3.2.

We show by induction for each rule of $\mathsf{G1}_{R^F}$ that if the conclusion is i-satisfiable then *at least one* of its premises is i-satisfiable as well. Suppose the conclusion sequent is i-satisfiable. We show for one of its premises of the form $\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$ that there exists a pair $(\mathcal{I}, x)$ with $\mathcal{I}$ being N_{R^F}-infallible and $x \in \Delta^{\mathcal{I}}_c$ such that:

$$\mathcal{I} \models \Theta; \tag{5.85}$$

and for all $R \in N_R,\ L \in \Sigma(R),\ M \in \Gamma,\ N \in \Phi,\ K \in \Psi(R)$:

$$\forall x'.\,\forall y.\,(x \preceq^{\mathcal{I}} x' \ \ \&\ \ x' \, R^{\mathcal{I}} \, y) \Rightarrow \mathcal{I}; y \models L; \tag{5.86}$$

$$\mathcal{I}; x \models M; \tag{5.87}$$

$$\mathcal{I}; x \not\models N; \tag{5.88}$$

$$\forall y.\, x \, R^{\mathcal{I}} \, y \Rightarrow \mathcal{I}; y \not\models K. \tag{5.89}$$

Again, condition (5.85) follows by assumption. We will only cover the rules $\bot L^F$, $\exists L^F$ and $\bot R^F$ and omit the remaining rules, since they can be argued as before.

(**Case** $\bot L$) The conclusion $\Theta\,;\,\Sigma\,;\,\Gamma,\ \bot \mid\!\!\!- \Phi\,;\,\Psi$ of $\bot L^F$ cannot be i-satisfied.

(**Case** $\exists L^F$) Suppose that the conclusion sequent $s_c =_{df} \Theta\,;\,\Sigma\,;\,\Gamma,\ \exists R.C \mid\!\!\!- \Phi\,;\,\Psi$ is i-satisfiable. Definition 5.3.3 implies that there exists a pair $(\mathcal{I}, a)$ with $a \in \Delta_c^{\mathcal{I}}$ that satisfies the sequent s_c, in particular a is contained in the interpretation of each concept in $\Gamma \cup \{\exists R.C\}$. Note that $\Psi(R)$ is constrained to be non-empty.

The assumption $\mathcal{I}; a \models \exists R.C$ implies for all refinements of a that there exists an R-successor which lies in the interpretation of C. Then, it follows by reflexivity of $\preceq$, *i.e.*, $a \preceq^{\mathcal{I}} a$, that there exists an entity b such that $a\ R^{\mathcal{I}}\ b$ and $\mathcal{I}; b \models C$. The condition $\Psi(R) \neq \emptyset$ enforces b to be infallible as well.

We claim that $(\mathcal{I}, b)$ i-satisfies the premise sequent $s_p =_{df} \Theta\,;\,\emptyset\,;\,\Sigma(R), C \mid\!\!\!- \Psi(R)\,;\,\emptyset$.

- Regarding (5.86) and (5.89) nothing needs to be shown.
- The goal $\mathcal{I}; b \models \Sigma(R) \cup \{C\}$, condition (5.87), follows by assumption and reflexivity of $\preceq^{\mathcal{I}}$.
- For (5.88) we need to show for all $N \in \Psi(R)$ that $\mathcal{I}; b \not\models N$. By assumption this is the case for all R-successors of a, in particular for b.

(**Case** $\bot R^F$) Suppose the conclusion sequent $s_c =_{df} \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!\!- \Phi\,;\,\Psi$ is i-satisfiable. We need to show for all $R \in N_{R^F}$ that its premise $s_p =_{df} \Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!\!- \Phi\,;\,\Psi \cup [R \mapsto \bot]$ is i-satisfiable as well, *i.e.*, $c\mathcal{ALC}^{R^F}; \Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi \cup [R \mapsto \bot]$.

Let $R \in N_{R^F}$ be an arbitrary R-infallible role. Definition 5.3.3 implies that there exists a pair $(\mathcal{I}, a)$ that i-satisfies s_c, with $\mathcal{I}$ being N_{R^F}-infallible and $a \in \Delta_c^{\mathcal{I}}$.

We claim that $(\mathcal{I}, a)$ i-satisfies the premise sequent.

- The conditions (5.86), (5.87) and (5.88) follow directly by assumption.
- Condition (5.89) holds by assumption and $\forall b.a\ R^{\mathcal{I}}\ b \Rightarrow \mathcal{I}, b \not\models \bot$ follows from R-infallibility of $\mathcal{I}$ by Def. 5.3.2. ☐

We leave the completeness of $\mathsf{G1}_{R^F}$ as an open problem.

Equivalence of the N_{R^F}-Infallible System to the Hilbert System

We conclude this section by showing that the Hilbert system $c\mathcal{ALC}^{R^F}$ can be translated into the sequent system $\mathsf{G1}_{R^F}$ and vice versa.

Proposition 5.3.6. *Let $c\mathcal{ALC}^{R^F} =_{df} c\mathcal{ALC} \oplus \{\neg\exists R.\bot \mid R \in N_{R^F}\}$ be the N_{R^F}-infallible system where $N_{R^F} \subseteq N_R$ is a set of infallible roles. For every concept C and set of concepts Θ we have*

$$\Theta\,;\,\emptyset\,;\,\emptyset \mid\!\!\!\frac{\quad}{\mathsf{G1}_{R^F}} C\,;\,\emptyset \text{ is inconsistent} \quad \text{iff} \quad c\mathcal{ALC}^{R^F}; \Theta\,;\,\emptyset \mid\!\!\!\frac{\quad}{\mathsf{H}} C. \qquad \nabla$$

Proof. The proof extends Prop. 5.2.1 and is by giving a translation between the Hilbert system $c\mathcal{ALC}^{R^F}$ and the Gentzen sequent calculus $\mathsf{G1}_{R^F}$ and vice versa. It suffices to argue axiom $\neg\exists R.\bot$ and the rules $\bot L^F$, $\bot R^F$ and $\exists L^F$.

($\Rightarrow$) We have to show that for all $R \in N_{R^F}$ the axiom $\neg\exists R.\bot$ is derivable in $\mathsf{G1}_{R^F}$. The derivation is depicted in Fig. 5.7.

($\Leftarrow$) In the other direction we assume the premise of the rules $\bot L^F$, $\bot R^F$ and $\exists L^F$ to be admissible and then give a Hilbert derivation of its conclusion.

(**Case** $\bot L^F$) Suppose that $\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!\frac{\quad}{\mathsf{G1}_{R^F}} \Phi\,;\,\Psi$ is derived by rule $\bot L^F$, *i.e.*, $\Gamma = \Gamma', \bot$.

The goal is to demonstrate that Hilbert derives

$$\Theta\,;\,\emptyset \mid\!\!\frac{\quad}{\mathsf{H}} (\wedge\Sigma \;\sqcap\; \bot \sqcap \wedge\Gamma') \;\supset\; (\vee\Phi \;\sqcup\; \vee\Psi).$$

This is a direct consequence of Hilbert axiom IPC7 (Def. 5.1.1) and weakening (ARW).

(**Case** $\bot R^F$) If the sequent $\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!\frac{\quad}{\mathsf{G1}_{R^F}} \Phi\,;\,\Psi$ is derived by rule $\bot R^F$ then $R \in N_{R^F}$ and the last rule application looks like

$$\cfrac{\cfrac{\vdots}{\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!\frac{\quad}{\mathsf{G1}_{R^F}} \Phi\,;\,\Psi \cup [R \mapsto \bot]}}{\Theta\,;\,\Sigma\,;\,\Gamma \mid\!\!\frac{\quad}{\mathsf{G1}_{R^F}} \Phi\,;\,\Psi}\;\bot R^F$$

The goal is a Hilbert derivation of

$$\Theta\,;\,\emptyset \mid\!\!\frac{\quad}{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \vee\Psi). \tag{5.90}$$

Applying the induction hypothesis to the premise yields the derivation

$$\Theta\,;\,\emptyset \mid\!\!\frac{\quad}{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \bigvee_{R' \neq R, R' \in dom(\Sigma)} \exists R'.\Psi(R') \sqcup \exists R.(\vee\Psi(R) \sqcup \bot)), \tag{5.91}$$

where $\vee\Psi$ is decomposed according to (5.24).

We use the following abbreviations $\varphi =_{df} (\wedge\Sigma \sqcap \wedge\Gamma)$, $\psi =_{df} \vee\Phi \sqcup \bigvee_{R' \neq R} \exists R'.\vee\Psi(R')$ and $\gamma =_{df} \vee\Psi(R)$. For $\vee\Psi(R)$ there are two cases, namely: The goal is to show that Hilbert derives from the assumption $\varphi \supset (\psi \sqcup \exists R.(\gamma \sqcup \bot))$ either the goal

$$\Theta\,;\,\emptyset \mid\!\!\frac{\quad}{\mathsf{H}} \varphi \supset (\psi \sqcup \exists R.\gamma) \quad \text{if } \Psi(R) \neq \emptyset, \text{ or} \tag{5.92}$$

$$\Theta\,;\,\emptyset \mid\!\!\frac{\quad}{\mathsf{H}} \varphi \supset \psi \quad \text{if } \Psi(R) = \emptyset. \tag{5.93}$$

Since Hilbert derives $\bot \supset \exists R.\bot$ and $\exists R.\bot \supset \bot$, it holds that $\exists R.\bot \equiv \bot$.

The proof relies on the following admissible rule proven below:

$$\vdash_{\mathsf{H}} C \supset D \text{ implies } \vdash_{\mathsf{H}} (E \sqcup C) \supset (E \sqcup D) \tag{5.94}$$

1. $C \supset D$ Ass.;
2. $(C \supset (E \sqcup D)) \supset (E \supset (E \sqcup D)) \supset ((E \sqcup C) \supset (E \sqcup D))$ IPC6;
3. $E \supset (E \sqcup D)$ IPC5;
4. $C \supset (E \sqcup D)$ from 1 by (ARW);
5. $(E \sqcup C) \supset (E \sqcup D)$ from (2, 3 by MP), 4 by MP.

The case (5.92) can be derived as follows:

1. $\varphi \supset (\psi \sqcup \exists R.(\gamma \sqcup \bot))$ Ass.;
2. $(\gamma \sqcup \bot) \supset \gamma$ (IPC8);
3. $\forall R.((\gamma \sqcup \bot) \supset \gamma)$ from 2 by Nec;
4. $\forall R.((\gamma \sqcup \bot) \supset \gamma) \supset (\exists R.(\gamma \sqcup \bot) \supset \exists R.\gamma)$ $\mathsf{K}_{\exists R}$;
5. $\exists R.(\gamma \sqcup \bot) \supset \exists R.\gamma$ from 4, 3 by MP;
6. $(\psi \sqcup \exists R.(\gamma \sqcup \bot)) \supset (\psi \sqcup \exists R.\gamma)$ from 5 by (5.94);
7. $\varphi \supset (\psi \sqcup \exists R.\gamma)$ from 6, 1 by (ARB).

Unfolding φ, ψ and γ, we obtain the derivation

$$\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset \big(\vee\Phi \sqcup \bigvee_{R' \neq R, R' \in dom(\Psi)} \exists R'.\Psi(R') \sqcup \exists R.\vee\Psi(R)\big),$$

which by definition of $\vee\Psi$ (5.24) is nothing but $\Theta\,;\,\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi \sqcup \vee\Psi)$.

The second case (5.93) is argued as follows:

1. $\varphi \supset (\psi \sqcup \exists R.\bot)$ Ass.;
2. $\exists R.\bot \supset \bot$ FS3/IK3;
3. $(\psi \sqcup \exists R.\bot) \supset (\psi \sqcup \bot)$ from 2 by (5.94);
4. $\varphi \supset (\psi \sqcup \bot)$ from 3, 1 by (ARB);
5. $\varphi \supset \psi$ from 4 by (ARE),

which was to be shown.

(**Case** $\exists L^F$) Analogously to the proof of (**Case** $\exists L$), see p. 161. □

5.3.2 The Principle of Disjunctive Distribution – Axiom FS4/IK4

The principle of *disjunctive distribution* – $\exists R$ distributes over $\sqcup$ – is valid in classical $\mathcal{ALC}$ and *normal* classical and intuitionistic modal logics [33, p. 191 ff.],[103, p. 7 ff.]. This principle is expressed by the axiom

$$\mathsf{FS4/IK4} =_{df} \exists R.(C \sqcup D) \supset (\exists R.C \sqcup \exists R.D),$$

which is part of the IML FS/IK as discussed in Sec. 2.2.2. However, as argued in Sec. 4.1, this axiom is problematic from a constructive, type-theoretic point of view and therefore it is not part of the system $c\mathcal{ALC}$.

Remark 5.3.3. In [195] has been claimed that $c\mathcal{ALC} \oplus \mathsf{FS4/IK4}$ corresponds to saying that '*role filling is confluent with refinement*' in the sense that filling and refinement become orthogonal concepts. It is known for PLL [90] that the axiom FS4/IK4 completely captures the frame condition $\preceq^{-1} ; R \subseteq R ; \preceq^{-1}$, which is due to the fact that in PLL[15] necessity $\Box$ is trivialised by the axiom $C \supset \Box C$. ■

The following section is devoted to the investigation of the semantic properties of confluent interpretations w.r.t. the axiom FS4/IK4. Thereafter, we will introduce a sound and complete extension of G1 in order to accommodate axiom FS4/IK4.

Definition 5.3.4 (Confluent $c\mathcal{ALC}$ interpretation)**.** A $c\mathcal{ALC}$ interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \perp^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is called *confluent* if for all $R \in N_R$, $x, y, z \in \Delta^{\mathcal{I}}$, $x\ R^{\mathcal{I}}\ y$ and $x \preceq^{\mathcal{I}} z$ imply that there exists an entity $w \in \Delta^{\mathcal{I}}$ such that $y \preceq^{\mathcal{I}} w$ and $z\ R^{\mathcal{I}}\ w$. This property can also be expressed in terms of the frame condition $\preceq^{-1}; R \subseteq R; \preceq^{-1}$, globally for all $R \in N_R$. ∇

First, we show that the axiom FS4/IK4 is valid in every confluent $c\mathcal{ALC}$ interpretation.

Proposition 5.3.7. *For all confluent $c\mathcal{ALC}$ interpretations $\mathcal{I}$, concepts C, D and roles $R \in N_R$ it holds that*

$$\mathcal{I} \models \exists R.(C \sqcup D) \supset (\exists R.C \sqcup \exists R.D) \qquad \nabla$$

Proof. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \perp^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an arbitrary but confluent $c\mathcal{ALC}$ interpretation and $a_0 \in \Delta^{\mathcal{I}}$ be arbitrary. Let us suppose that $\mathcal{I}; a_0 \models \exists R.(C \sqcup D)$ and to the contrary that $\mathcal{I}; a_0 \not\models \exists R.C \sqcup \exists R.D$. The latter means $\mathcal{I}; a_0 \not\models \exists R.C$ and $\mathcal{I}; a_0 \not\models \exists R.D$ and therefore implies that there exist $a_1, a_2 \in \Delta^{\mathcal{I}}_c$ such that $a_0 \preceq a_1, a_0 \preceq a_2$ and for all $y \in \Delta^{\mathcal{I}}$, if $a_1\ R\ y$ then $\mathcal{I}; y \not\models C$, and if $a_2\ R\ y$ then $\mathcal{I}; y \not\models D$. From the assumption $\mathcal{I}; a_0 \models \exists R.(C \sqcup D)$ follows that there exists $b_0 \in \Delta^{\mathcal{I}}$ such that $a_0\ R\ b_0$

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[15] Usually, PLL is presented with the single modality $\bigcirc$. However, it has been shown in [4] that PLL naturally arises as an extension of CS4 by adding the axiom $C \supset \Box C$.

and $\mathcal{I}; b_0 \models C \sqcup D$. Now, confluence of $\mathcal{I}$ lets us conclude that there are entities $b_1, b_2 \in \Delta^{\mathcal{I}}$ with $a_1 \ R \ b_1 \preceq^{-1} b_0$ and $a_2 \ R \ b_2 \preceq^{-1} b_0$. The assumption implies that $\mathcal{I}; b_1 \not\models C$ and $\mathcal{I}; b_2 \not\models D$. However, this is contradictory to $\mathcal{I}; b_0 \models C \sqcup D$ and the assumption $\mathcal{I}; a_0 \models \exists R.(C \sqcup D)$. Hence, $\mathcal{I}; a_0 \models \exists R.C \sqcup \exists R.D$. The situation is depicted in Fig. 5.8, where the dashed cones below the entities a_1, a_2 denote that for all R-successors of a_1 and a_2 it holds that $\not\models C$ and $\not\models D$ respectively. The place of contradiction is indicated by underlining. □

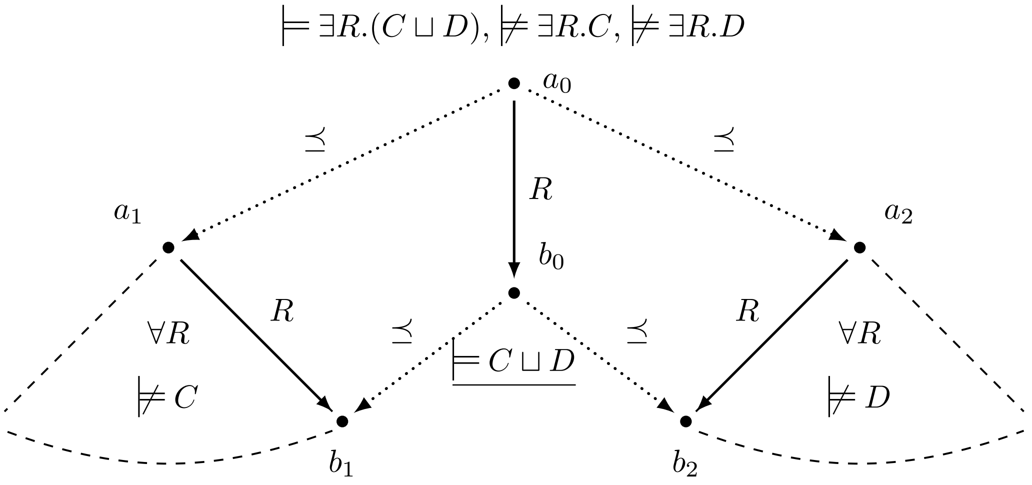

Figure 5.8: Validity of axiom FS4/IK4 in confluent interpretations.

In contrast to normal IMLs (see Chap. 4.1), c$\mathcal{ALC}$ uses the strengthened interpretation of $\exists R$ given by

$$(\exists R.C)^{\mathcal{I}} =_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}.\ x \preceq^{\mathcal{I}} y \Rightarrow \exists z \in \Delta^{\mathcal{I}}.\ (y, z) \in R^{\mathcal{I}} \ \& \ z \in C^{\mathcal{I}}\},$$

which forces $\exists R.C$ to be hereditary for refinement $\preceq$ without a frame property. However, in confluent interpretations, *i.e.*, in the presence of the frame condition $\preceq^{-1}; R \subseteq R; \preceq^{-1}$, the universal quantification over $\preceq$ becomes redundant. Observe from the proof of Prop. 5.3.7 that confluence simplifies validity of $\exists R.C$ in the sense that it forces the heredity of role filling w.r.t. refinement $\preceq$. This frame condition lets us simplify the definition of the interpretation of $\exists R.C$ by omitting the universal quantification over $\preceq$ as follows.

Lemma 5.3.1. *Let $\mathcal{I}$ be a confluent c$\mathcal{ALC}$ interpretation and $x \in \Delta^{\mathcal{I}}$. Then,*

$$\mathcal{I}; x \models \exists R.C \text{ iff } \exists y \in \Delta^{\mathcal{I}}.\ (x, y) \in R^{\mathcal{I}} \ \& \ \mathcal{I}; y \models C. \qquad \nabla$$

Proof. Let $a \in \Delta^{\mathcal{I}}$ be arbitrary. ($\Rightarrow$) Suppose that $\mathcal{I}; a \models \exists R.C$. We have to show that $a \in \{x \mid \exists y \in \Delta^{\mathcal{I}}.\ (x, y) \in R^{\mathcal{I}} \ \& \ y \in C^{\mathcal{I}}\}$. The assumption, *viz.* $a \in (\exists R.C)^{\mathcal{I}}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset\,;\emptyset\,;C \vdash C,D\,;\emptyset}\,Ax \qquad \dfrac{}{\emptyset\,;\emptyset\,;D \vdash C,D\,;\emptyset}\,Ax}{\emptyset\,;\emptyset\,;C\sqcup D \vdash C,D\,;\emptyset}\,\sqcup L}{\emptyset\,;\emptyset\,;\exists R.(C\sqcup D) \vdash \emptyset\,;[R\mapsto C,D]}\,\exists L}{\emptyset\,;\emptyset\,;\exists R.(C\sqcup D) \vdash \exists R.D\,;[R\mapsto C]}\,\exists R^+}{\emptyset\,;\exists R.(C\sqcup D)\,;\emptyset \vdash \exists R.C,\,\exists R.D\,;\emptyset}\,\exists R^+}{\emptyset\,;\exists R.(C\sqcup D)\,;\emptyset \vdash \exists R.C\sqcup\exists R.D\,;\emptyset}\,\sqcup R}{\emptyset\,;\emptyset\,;\emptyset \vdash \exists R.(C\sqcup D)\supset(\exists R.C\sqcup\exists R.D)\,;\emptyset}\,\supset R$$

Figure 5.9: Sequent proof for axiom FS4/IK4 based on $\exists R^+$. Adapted from [195, p. 229, Fig. 12], with kind permission from Springer Science and Business Media.

says that for all refinements of a there exists an R-successor which is contained in the interpretation of C. From reflexivity of $\preceq^{\mathcal{I}}$ it follows that $a \preceq^{\mathcal{I}} a$ and by the assumption there exists an entity $b \in \Delta^{\mathcal{I}}$ such that $a\ R^{\mathcal{I}}\ b$ and $\mathcal{I}; b \models C$. Hence, $a \in \{x \mid \exists z \in \Delta^{\mathcal{I}}.\ (y,z) \in R^{\mathcal{I}}\ \ \&\ \ z \in C^{\mathcal{I}}\}$.

($\Leftarrow$) In the other direction let us assume that $a \in \{x \mid \exists y \in \Delta^{\mathcal{I}}.\ (x,y) \in R^{\mathcal{I}}\ \ \&\ \ y \in C^{\mathcal{I}}\}$. We have to show that $a \in (\exists R.C)^{\mathcal{I}}$, *i.e.*, $a \in \{x \mid \forall y \in \Delta^{\mathcal{I}}.\ x \preceq^{\mathcal{I}} y \Rightarrow \exists z \in \Delta^{\mathcal{I}}.\ (y,z) \in R^{\mathcal{I}}\ \ \&\ \ z \in C^{\mathcal{I}}\}$. Take an arbitrary refinement $a' \in \Delta^{\mathcal{I}}$ of a. The assumption lets us choose an entity $b \in \Delta^{\mathcal{I}}$ such that $a\ R^{\mathcal{I}}\ b$ and $b \in C^{\mathcal{I}}$. Since $\mathcal{I}$ is confluent, it follows by Def. 5.3.4 that there is an entity $c \in \Delta^{\mathcal{I}}$ with $a'\ R^{\mathcal{I}}\ c$ and $b \preceq^{\mathcal{I}} c$. By monotonicity Prop. 4.2.2 follows that $b \in C^{\mathcal{I}}$. Because a' was an arbitrary refinement of a for which there exists the R-successor b in $C^{\mathcal{I}}$, we can conclude that $\mathcal{I}; a \models \exists R.C$. □

The schema FS4/IK4 can be accommodated in the sequent calculus G1 by strengthening the right rule $\exists R$ to $\exists R^+$ [195, p. 229]:

$$\dfrac{\Theta\,;\Sigma\,;\Gamma \vdash \Phi\,;\Psi \cup [R\mapsto C]}{\Theta\,;\Sigma\,;\Gamma \vdash \Phi,\,\exists R.C\,;\Psi}\,\exists R^+$$

Observe that due to this modification the rule $\exists R^+$ becomes dual to rule $\forall L$. The proof of axiom FS4/IK4 based on rule $\exists R^+$ is depicted in Fig. 5.9.

Notation. We will call this strengthened sequent calculus $\mathsf{G1}_D$ in the following. Furthermore, let us write $c\mathcal{ALC}^D =_{df} c\mathcal{ALC} \oplus \{\exists R.(C \sqcup D) \supset (\exists R.C \sqcup \exists R.D) \mid R \in N_R\}$ for the globally restricted system of $c\mathcal{ALC}$, and $c\mathcal{ALC}^D; \Theta; \Sigma; \Gamma \not\models \Phi; \Psi$ for the statement that the sequent $\Theta\,;\Sigma\,;\Gamma \vdash_{\mathsf{G1}_D} \Phi\,;\Psi$ is satisfiable at some entity of a confluent interpretation. If $\mathsf{G1}_D$ is clear from the context then we use the plain turnstile. ■

Theorem 5.3.3 (Soundness of $\mathsf{G1}_D$). *Every satisfiable* $\mathsf{G1}_D$*-sequent is consistent,* i.e.,

$$c\mathcal{ALC}^D; \Theta ; \Sigma ; \Gamma \not\models \Phi ; \Psi \quad \textit{implies} \quad \Theta ; \Sigma ; \Gamma \not\vdash_{\mathsf{G1}_D} \Phi ; \Psi. \qquad \nabla$$

Proof. ($\Rightarrow$) Soundness of $\mathsf{G1}_D$ is argued as before by Thm. 5.2.2. It is sufficient to check soundness of the rule $\exists R^+$. Suppose that the sequent $s_c =_{df} \Theta ; \Sigma ; \Gamma \vdash \Phi, \exists R.C ; \Psi$ is satisfiable. The goal is to show that its premise $s_p =_{df} \Theta ; \Sigma ; \Gamma \vdash \Phi ; \Psi, [R \mapsto C]$ is satisfiable as well. The assumption implies by Definition 5.2.2 that there exists a confluent interpretation $\mathcal{I}$ and an entity $x \in \Delta^{\mathcal{I}}$ such that $(\mathcal{I}, a)$ satisfies the sequent s_c, *i.e.*, $\mathcal{I} \models \Theta$ and for all $R \in N_R$ it holds that

$$\forall x', y \in \Delta^{\mathcal{I}}.\, x \preceq^{\mathcal{I}} x' \; R^{\mathcal{I}} \; y \Rightarrow \mathcal{I}; y \models \Sigma(R); \tag{5.95}$$

$$\mathcal{I}; x \models \Gamma; \tag{5.96}$$

$$\mathcal{I}; x \not\models \Phi; \tag{5.97}$$

$$\forall y \in \Delta^{\mathcal{I}}.\, x \; R^{\mathcal{I}} \; y \Rightarrow \mathcal{I}; y \not\models \Psi(R). \tag{5.98}$$

We claim that $(\mathcal{I}, a)$ satisfies the sequent s_p.

- The conditions (5.95), (5.96) and (5.97) follow directly by assumption.
- For condition (5.98), let $b \in \Delta^{\mathcal{I}}$ such that $a \; R^{\mathcal{I}} \; b$. From $\mathcal{I}; a \not\models \exists R.C$ and Lem. 5.3.1 follows $\mathcal{I}; b \not\models C$, which was to be shown. ☐

Remark 5.3.4. Regarding completeness we need to show that

$$\Theta ; \Sigma ; \Gamma \not\vdash_{\mathsf{G1}_D} \Phi ; \Psi \Rightarrow c\mathcal{ALC}^D ; \Theta ; \Sigma ; \Gamma \not\models \Phi ; \Psi.$$

The proof is based on the following observations.

(i) The rules $\exists R^+$ and $\forall L$ are dual to each other, *i.e.*, when applying rule $\exists R^+$ we do not loose the Ψ component, analogously to $\forall L$ not loosing the Σ component. Semantically, this means that the rule does not introduce a new $\preceq$ successor, but rather acts on the spot. This is because the theory $\Phi \cup \{\exists R.C \mid C \in \Psi(R) \text{ for } R \in N_R\}$ of the conclusion of rule $\exists R^+$ is equivalent to the corresponding theory of its premise.

(ii) The components Σ and Ψ are redundant. This can be observed as follows: Let $\exists R^{-1}\,\Phi =_{df} \{C \mid \exists R.C \in \Phi\}$, $\forall R^{-1}\,\Gamma =_{df} \{C \mid \forall R.C \in \Gamma\}$, and Δ^D be the set of all consistent sequents w.r.t. the $\mathsf{G1}_D$ sequent calculus. Remember that the set of extension rules $X_{c\mathcal{ALC}}$ contains the following rule for universal restrictions in Γ (see Def. 5.2.11):

$\mathrm{E4}_{C,R}$ $s \to_{\mathrm{E4}} s' = \langle \Theta \,;\, \Sigma \cup [R \mapsto C] \,;\, \Gamma \vdash \Phi \,;\, \Psi \rangle$,
if $\forall R.C \in \Gamma$ but $C \notin \Sigma(R)$.

Analogously one can extend $X_{c\mathcal{ALC}}$ by one rule to handle existential restrictions in Φ, and furthermore add two rules for the converse direction. Let us denote this set of extension rules by $X_{c\mathcal{ALC}^D}$:

$\mathrm{ED1}_{C,R}$ $s \to_{\mathrm{ED1}} s' = \Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi \,;\, \Psi \cup [R \mapsto C]$,
if $\exists R.C \in \Phi$ but $C \notin \Psi(R)$.

$\mathrm{ED2}_{C,R}$ $s \to_{\mathrm{ED2}} s' = \Theta \,;\, \Sigma \,;\, \Gamma \cup \{\forall R.C\} \vdash \Phi \,;\, \Psi$,
if $C \in \Sigma(R)$ and $\forall R.C \notin \Gamma$.

$\mathrm{ED3}_{C,R}$ $s \to_{\mathrm{ED3}} s' = \Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi \cup \{\exists R.C\} \,;\, \Psi$,
if $C \in \Psi(R)$ and $\exists R.C \notin \Phi$.

The additional saturation rules $\mathrm{ED2}_{C,R}$, and $\mathrm{ED3}_{C,R}$ are consistency preserving, which can be shown by proving that the corresponding sequent rules $\forall L$ and $\exists R^+$ are invertible, *i.e.*, if $\Theta \,;\, \Sigma \,;\, \Gamma,\, \forall R.C \vdash \Phi \,;\, \Psi$ then $\Theta \,;\, \Sigma \cup [R \mapsto C] \,;\, \Gamma \vdash \Phi \,;\, \Psi$, and if $\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi,\, \exists R.C \,;\, \Psi$ then $\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi \,;\, \Psi \cup [R \mapsto C]$. Then, by Lem. 5.2.4 we obtain for each consistent $\mathsf{G1}_D$ sequent a $X_{c\mathcal{ALC}^D}$-saturated and consistent sequent, where it holds that

- $\forall R \in \mathit{dom}(\Sigma).\ \forall R^{-1}\,\Gamma = \Sigma(R)$, and
- $\forall R \in \mathit{dom}(\Psi).\ \exists R^{-1}\,\Phi = \Psi(R)$.

(iii) Since the components Σ and Ψ are redundant and can be derived from Γ and Φ, we could obtain a suitable canonical $c\mathcal{ALC}^D$ model from the consistent and $X_{c\mathcal{ALC}^D}$-saturated sequents restricted to the form $\langle \Theta \,;\, \Gamma \vdash \Phi \rangle$.

However, we rely on a Lindenbaum construction to argue the completeness of $\mathsf{G1}_D$, since we require maximally consistent sequents where the components Γ and Φ are complementary, that is, sequents where for all concepts C either $C \in \Gamma$ or $C \in \Phi$ holds. ■

Theorem 5.3.4 (Completeness of $\mathsf{G1}_D$). *Every consistent $\mathsf{G1}_D$-sequent is satisfiable in a confluent interpretation, assuming that $\mathsf{G1}_D$ is cut-free,* i.e.,

$$\Theta \,;\, \Sigma \,;\, \Gamma \nvdash_{\mathsf{G1}_D} \Phi \,;\, \Psi \Rightarrow c\mathcal{ALC}^D \,;\, \Theta \,;\, \Sigma \,;\, \Gamma \not\models \Phi \,;\, \Psi. \qquad \nabla$$

Proof-sketch. Let $s = \Theta\,;\Sigma\,;\Gamma \mathrel{|\!\!-} \Phi\,;\Psi$ be a consistent $\mathtt{G1}_D$ sequent. The proof-technique is standard and uses a canonical model construction based on a generalisation of the Lindenbaum Theorem [33, p. 197], showing that (i) there exists a constructive $c\mathcal{ALC}^D$ countermodel $\mathcal{I}$ and an entity a in $\Delta^{\mathcal{I}}$ such that $(\mathcal{I}, a)$ satisfies the sequent s, and (ii) the canonical $c\mathcal{ALC}^D$ interpretation is confluent.

Regarding (i), the construction of the canonical $c\mathcal{ALC}^D$ interpretation relies on the set of maximally consistent $\mathtt{G1}_D$ sequents, denoted by Δ^*, which is generated by the following Lindenbaum construction: Let $s = \langle \Theta\,;\Sigma\,;\Gamma \mathrel{|\!\!-} \Phi\,;\Psi\rangle$ be a consistent $\mathtt{G1}_D$ sequent and $C_0, C_1, C_2, \ldots$ be an enumeration of all $c\mathcal{ALC}$ concepts with infinite repetition. Define a sequence of consistent sequents $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots \subseteq s_n \subseteq s_{n+1} \subseteq \ldots$ by taking $s_0 = s$ and, for $n \geq 0$,

$$s_{n+1} =_{df} \begin{cases} \langle \Theta\,;\Sigma_{n+1}\,;\Gamma_{s_n} \cup \{C\} \mathrel{|\!\!-} \Phi_{s_n}\,;\Psi_{n+1}\rangle & \text{if it is } \mathtt{G1}_D \text{ consistent;} \\ \langle \Theta\,;\Sigma_{n+1}\,;\Gamma_{s_n} \mathrel{|\!\!-} \Phi_{s_n} \cup \{C\}\,;\Psi_{n+1}\rangle & \text{otherwise,} \end{cases}$$

where

$$\Sigma_{n+1}(R) =_{df} \Sigma_{s_n}(R) \cup \forall R^{-1}\,\Gamma_{s_{n+1}}, \text{ and}$$
$$\Psi_{n+1}(R) =_{df} \Psi_{s_n}(R) \cup \exists R^{-1}\,\Phi_{s_{n+1}}.$$

Then, one shows that consistency of s_n entails consistency of s_{n+1}, *i.e.*, each step in the construction preserves consistency. This can be shown by proving cut-admissibility for $\mathtt{G1}_D$, which we leave as an open problem. The maximally consistent sequent is given by $s^* =_{df} \langle \Theta\,;\Sigma^*\,;\Gamma^* \mathrel{|\!\!-} \Phi^*\,;\Psi^*\rangle$ where

$$\Sigma^* =_{df} \bigcup_{n<\omega} \Sigma_n, \quad \Gamma^* =_{df} \bigcup_{n<\omega} \Gamma_n, \quad \Phi^* =_{df} \bigcup_{n<\omega} \Phi_n, \quad \Psi^* =_{df} \bigcup_{n<\omega} \Psi_n.$$

This shows that every consistent sequent s has a maximally consistent extension s^* which is $X_{c\mathcal{ALC}^D}$-saturated. The canonical interpretation $\mathcal{I}^*$ is built from maximally consistent $\mathtt{G1}_D$ sequents in Δ^* according to Def. 5.2.12. We conjecture that the canonical $c\mathcal{ALC}^D$ interpretation $\mathcal{I}^*$ is a constructive $c\mathcal{ALC}^D$ model, and for all maximally consistent sequents $s^* \in \Delta^*$ the pair $(\mathcal{I}^*, s^*)$ satisfies s^*.

Secondly, we will tackle (ii) and argue that the canonical $c\mathcal{ALC}^D$ interpretation is confluent. Let s_0, s_1, s_2 be maximally consistent sequents from Δ^* and suppose that $s_0 \preceq^* s_2$ and $s_0\, R^*\, s_1$, that is, $\Gamma_{s_0} \subseteq \Gamma_{s_2}$, $\Sigma_{s_0} \subseteq \Sigma_{s_2}$, $\Sigma_{s_0}(R) \subseteq \Gamma_{s_1}$ and $\Psi_{s_0}(R) \subseteq \Phi_{s_1}$. Let us consider the sequent

$$s_3 =_{df} \langle \Theta\,;\Sigma_{s_1}\,;\Gamma_{s_1} \cup \Sigma_{s_2}(R) \mathrel{|\!\!-} \Psi_{s_2}(R)\,;\emptyset\rangle.$$

We claim that the sequent s_3 is consistent. Suppose by contradiction that there are finite $\Sigma'_{s_1} \subseteq \Sigma_{s_1}, \Gamma'_{s_1} \subseteq \Gamma_{s_1}, \Sigma'_{s_2}(R) \subseteq \Sigma_{s_2}(R), \Psi'_{s_2}(R) \subseteq \Psi_{s_2}(R)$ such that for all $R \in N_R$ we have $\forall R^{-1}\,\Gamma'_{s_1} = \Sigma'_{s_1}(R)$ and

$$s'_3 =_{df} \langle \Theta\,;\, \Sigma'_{s_1}\,;\, \Gamma'_{s_1} \cup \Sigma'_{s_2}(R) \vdash \Psi'_{s_2}(R)\,;\, \emptyset \rangle$$

is inconsistent. Then, there exists a closed tableau for s'_3. Taking into account $\forall S \in N_R.\ \forall S^{-1}\Gamma_{s_1} = \Sigma_{s_1}(S)$, we can successively use the tableau rule $\forall L$ as follows to obtain the sequent s_4, where the second component Σ_{s_4} is empty. This is followed by successive applications of rule $\sqcap L$ to give the sequent s_5, where the third component contains the conjunction $\wedge\Gamma'_{s_1}$, which is the intersection over concepts of the set Γ'_{s_1}, *i.e.*, if $\Gamma = \{C_1, C_2, \ldots, C_n\}$ then $\wedge\Gamma = C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n$:

$$\cfrac{\cfrac{\cfrac{\vdots}{s'_3 = \Theta\,;\, \Sigma'_{s_1}\,;\, \Gamma'_{s_1},\, \Sigma'_{s_2}(R) \vdash \Psi'_{s_2}(R)\,;\, \emptyset}}{s_4 = \Theta\,;\, \emptyset\,;\, \Gamma'_{s_1},\, \Sigma'_{s_2}(R) \vdash \Psi'_{s_2}(R)\,;\, \emptyset}\,\forall L^*}{s_5 = \Theta\,;\, \emptyset\,;\, \wedge\Gamma'_{s_1},\, \Sigma'_{s_2}(R) \vdash \Psi'_{s_2}(R)\,;\, \emptyset}\,\sqcap L^*$$

Now, let us consider any concept $C \in \Gamma_{s_1}$. From consistency of s_1 it follows that $C \notin \Phi_{s_1}$, and in particular $\wedge\Gamma'_{s_1} \notin \Phi_{s_1}$. Moreover, since $\exists R^{-1}\,\Phi_{s_0} = \Psi_{s_0}(R)$ it follows that $\exists R.C \notin \Phi_{s_0}$, particularly $\exists R.\wedge\Gamma'_{s_1} \notin \Phi_{s_0}$. Therefore, it must be that $\exists R.C \in \Gamma_{s_0} \subseteq \Gamma_{s_2}$, and also $\exists R.\wedge\Gamma'_{s_1} \in \Gamma_{s_0} \subseteq \Gamma_{s_2}$. But this means that we can apply rule $\exists L$ as follows

$$\cfrac{\cfrac{\vdots}{s_5 = \Theta\,;\, \emptyset\,;\, \wedge\Gamma'_{s_1},\, \Sigma'_{s_2}(R) \vdash \Psi'_{s_2}(R)\,;\, \emptyset}}{s_6 = \Theta\,;\, \Sigma_{s_6}\,;\, \exists R. \wedge \Gamma'_{s_1},\, \Gamma_{s_2} \vdash \Phi_{s_2}\,;\, \Psi_{s_6}}\,\exists L,$$

where $\Sigma_{s_6} = [R \mapsto \Sigma'_{s_2}(R)] \cup [S \mapsto \Sigma_{s_2}(S) \mid S \neq R]$ and $\Psi_{s_6} = [R \mapsto \Psi'_{s_2}(R)] \cup [S \mapsto \Psi_{s_2}(S) \mid S \neq R]$. Now, one observes that $s_6 \subseteq s_2$, which would imply that the sequent s_2 is inconsistent, contradictory to the assumption. Hence, the sequent s_3 is consistent.

Finally, take a maximally consistent extension s^* of s_3. It follows from maximality that $\Sigma_{s_2}(R) = \forall R^{-1}\,\Gamma_{s_2} \subseteq \Gamma_{s^*}$ and $\Psi_{s_2}(R) = \exists R^{-1}\,\Phi_{s_2} \subseteq \Phi_{s^*}$. Hence, $s_2\ R^*\ s^*$ by Def. 5.2.12. Moreover, it holds that $\Gamma_{s_1} \subseteq \Gamma_{s^*}$ and $\Sigma_{s_1} = \forall R^{-1}\,\Gamma_{s_1} \subseteq \Sigma_{s^*}$, and due to the fact that Γ_{s_1} is the complement of Φ_{s_1} we have $\Phi_{s^*} \subseteq \Gamma_{s_1}$ and $\Psi_{s^*}(R) \subseteq \Psi_{s_1}(R)$. Concluding, it holds that $s_1 \preceq^{\mathcal{I}^*} s^*$, and this shows that the canonical $c\mathcal{ALC}^D$ interpretation is confluent. □

Equivalence of the Confluent System to the Hilbert System

It remains to show that the Hilbert system $c\mathcal{ALC}^D$ is sound and complete.

Proposition 5.3.8. *For every concept C and set of concepts Θ we have*

$$c\mathcal{ALC}^D\,;\Theta\,;\emptyset \models C \text{ in all confluent interpretations} \quad \text{iff} \quad c\mathcal{ALC}^D;\Theta \vdash_{\mathsf{H}} C. \qquad \nabla$$

Proof. The proof extends that of Prop. 5.2.1, showing that every derivation of the extended system $c\mathcal{ALC}^D$ can be translated into a derivation of the extended Gentzen sequent calculus $\mathsf{G1}_D$ and vice versa. Note that we need the rule *Cut* to emulate the rule MP in the ($\Rightarrow$)-direction.

($\Rightarrow$) Axiom FS4/IK4 is derivable in $\mathsf{G1}_D$, as depicted by Fig. 5.9.

($\Leftarrow$) In the other direction, let us suppose that the sequent $\Theta\,;\Sigma\,;\Gamma \vdash_{\mathsf{G1}_D} \Phi\,;\Psi$ is derived by rule $\exists R^+$, *i.e.*, $\Phi = \Phi', \exists R.C$ and the last rule application looks like this

$$\cfrac{\cfrac{\vdots}{\Theta\,;\Sigma\,;\Gamma \vdash_{\mathsf{G1}_D} \Phi'\,;\Psi \cup [R \mapsto C]}}{\Theta\,;\Sigma\,;\Gamma \vdash_{\mathsf{G1}_D} \Phi',\,\exists R.C\,;\Psi}\,\exists R^+.$$

We have to find a derivation of

$$\Theta\,;\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi' \sqcup \exists R.C \sqcup \vee\Psi). \tag{5.99}$$

The induction hypothesis applied to the premise yields the derivation

$$\Theta\,;\emptyset \vdash_{\mathsf{H}} (\wedge\Sigma \sqcap \wedge\Gamma) \supset (\vee\Phi' \sqcup \bigvee_{R' \in dom(\Psi)\setminus\{R\}} \exists R'.\Psi(R') \sqcup \exists R.(\vee\Psi(R) \sqcup C)), \tag{5.100}$$

where $\vee\Psi$ is decomposed according to (5.24). Note that if $\Psi(R) = \emptyset$ then the disjunct $\vee\Psi(R)$ is not part of the right-hand side of (5.100), and the goal (5.99) follows trivially.

Otherwise, if $\Psi(R) \neq \emptyset$ then we proceed as follows: Using the abbreviations $\varphi =_{df} (\wedge\Sigma \sqcap \wedge\Gamma)$, $\psi =_{df} \vee\Phi' \sqcup \bigvee_{R' \neq R} \exists R'.\Psi(R')$ and $\gamma =_{df} \vee\Psi(R)$, the goal is to show that Hilbert derives from the assumption $\varphi \supset (\psi \sqcup \exists R.(\gamma \sqcup C))$ the goal $\varphi \supset (\psi \sqcup \exists R.\gamma \sqcup \exists R.C)$. The proof uses (5.94) (see p. 181): $\vdash_{\mathsf{H}} C \supset D$ implies $\vdash_{\mathsf{H}} (E \sqcup C) \supset (E \sqcup D)$.

Then, the derivation of (5.99) is as follows:

1. $\varphi \supset (\psi \sqcup \exists R.(\gamma \sqcup C))$	Ass.;
2. $\exists R.(\gamma \sqcup C) \supset (\exists R.\gamma \sqcup \exists R.C)$	FS4/IK4;
3. $(\psi \sqcup \exists R.(\gamma \sqcup C)) \supset (\psi \sqcup (\exists R.\gamma \sqcup \exists R.C))$	from 2 by (5.94);
4. $\varphi \supset (\psi \sqcup (\exists R.\gamma \sqcup \exists R.C))$	from 3, 1 by (ARB).

□

5.3.3 The Principle of the Excluded Middle

The addition of axiom $\mathsf{PEM} =_{df}= C \sqcup \neg C$ to $c\mathcal{ALC}$ corresponds to assuming that $\preceq$ is an equivalence relation, that is, it becomes *symmetric* such that $x \preceq y$ implies $y \preceq x$ [195, p. 229]. This implies that there are no proper refinements of entities but only *clusters of indistinguishables*. However, note that we cannot identify two equivalent entities, since they may have incompatible realisations w.r.t. the accessibility relation of roles.

Let $c\mathcal{ALC}^{\mathsf{PEM}}$ denote the theory $c\mathcal{ALC} \oplus \mathsf{PEM}$. To incorporate the above, one can modify the semantics of a $c\mathcal{ALC}$ interpretation $\mathcal{I}$ modulo an equivalence relation $\sim$ to replace $\preceq$.

Lemma 5.3.2. *$\mathcal{I} \models C \sqcup \neg C$ in all $c\mathcal{ALC}$ interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\preceq^{\mathcal{I}}$ is an equivalence relation.* ∇

Proof. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a $c\mathcal{ALC}$ interpretation such that $\preceq^{\mathcal{I}}$ is an equivalence relation, $x \in \Delta^{\mathcal{I}}$ and suppose to the contrary that $\mathcal{I}; x \not\models C \sqcup \neg C$, *i.e.*, $\mathcal{I}; x \not\models C$ and $\mathcal{I}; x \not\models \neg C$. The latter implies that there exists $x' \in \Delta_c^{\mathcal{I}}$ such that $x \preceq^{\mathcal{I}} x'$ and $\mathcal{I}; x' \models C$. It follows from symmetry of $\preceq^{\mathcal{I}}$ that $x' \preceq^{\mathcal{I}} x$, and by monotonicity follows $\mathcal{I}; x \models C$ contradictory to the assumption. Hence, $\mathcal{I}; x \models C \sqcup \neg C$ □

Definition 5.3.5 (Degenerated interpretation)**.** A $c\mathcal{ALC}^{\mathsf{PEM}}$ interpretation, called *degenerated interpretation*, is a constructive interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$, in which $\preceq^{\mathcal{I}}$ is an equivalence relation. We will write $\sim^{\mathcal{I}}$ instead of $\preceq^{\mathcal{I}}$. ∇

Lemma 5.3.3. *Let $\mathcal{I}$ be a degenerated interpretation and $x \in \Delta^{\mathcal{I}}$, then*

$$\mathcal{I}; x \models C \supset D \quad \textit{iff} \quad \mathcal{I}; x \not\models C \ \textit{or}\ \mathcal{I}; x \models D. \qquad \nabla$$

Proof. Let $\mathcal{I}$ be an arbitrary but degenerated interpretation and $a \in \Delta^{\mathcal{I}}$.

($\Rightarrow$) Proof by contraposition. Suppose that $\mathcal{I}; a \models C$ and $\mathcal{I}; a \not\models D$. We have to show that there exists a refinement a' of a such that $\mathcal{I}; a' \models C$ and $\mathcal{I}; a' \not\models D$. By reflexivity of $\preceq^{\mathcal{I}}$ follows $a \preceq^{\mathcal{I}} a$. Thus, $\mathcal{I}; a \not\models C \supset D$.

($\Leftarrow$) Proof by contraposition. Suppose that $\mathcal{I}; a \not\models C \supset D$, *i.e.*, there exists $a' \in \Delta^{\mathcal{I}}$ such that $a \preceq^{\mathcal{I}} a'$ with $\mathcal{I}; a' \models C$ and $\mathcal{I}; a' \not\models D$. We have to show that $\mathcal{I}; a \models C$ and $\mathcal{I}; a \not\models D$. Since $\preceq^{\mathcal{I}}$ is an equivalence relation, it holds that $a' \preceq^{\mathcal{I}} a$ such that by monotonicity Prop. 4.2.2 it follows that $\mathcal{I}; a \models C$. Moreover, $\mathcal{I}; a \not\models D$, since otherwise if $\mathcal{I}; a \models D$ then this would contradict the assumption by monotonicity of $\preceq^{\mathcal{I}}$. Hence, $\mathcal{I}; a \models C$ and $\mathcal{I}; a \not\models D$. □

Proposition 5.3.9 (Monotonicity). *Let C be an arbitrary $c\mathcal{ALC}$ concept and $\mathcal{I}$ be a degenerated interpretation, then $\forall x, y \in \Delta^{\mathcal{I}}$ with $x \sim^{\mathcal{I}} y$ it holds that*

$$\mathcal{I}; x \models C \text{ iff } \mathcal{I}; y \models C. \qquad \nabla$$

Proof. The proof is trivial, since $x \preceq^{\mathcal{I}} y$ implies $y \preceq^{\mathcal{I}} x$. □

Example 5.3.1. In $c\mathcal{ALC}^{\mathsf{PEM}}$ the modalities $\exists R$ and $\forall R$ are not dual to each other. For instance, let us suppose that $\mathcal{I}; a_0 \models \neg\forall R.\neg C$ and to the contrary that $\mathcal{I}; a_0 \not\models \exists R.C$. Fig. 5.10 below shows a corresponding countermodel.

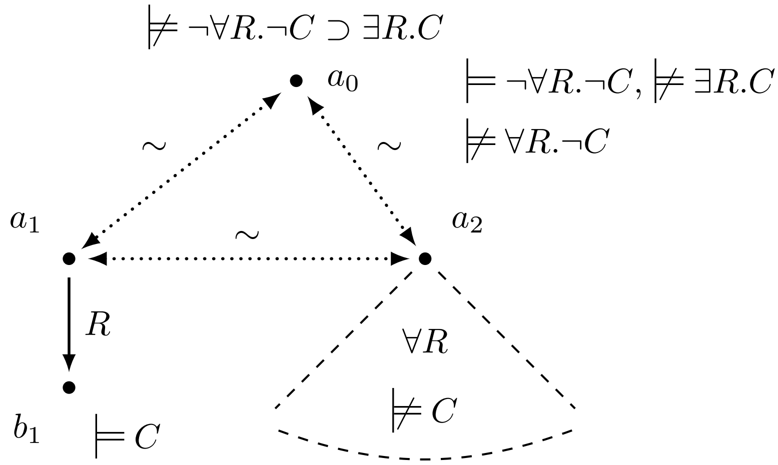

Figure 5.10: Countermodel for the duality of $\exists R$ and $\forall R$.

■

In the sequent calculus one can enforce axiom PEM by replacing the intuitionistic rule for implication $\supset R$ from Fig. 5.1 with the classical one $\supset R^+$ [195, p. 229]:

$$\frac{\Theta ; \Sigma ; \Gamma, C \vdash \Phi, D ; \Psi}{\Theta ; \Sigma ; \Gamma \vdash \Phi, C \supset D ; \Psi} \supset R^+$$

Rule $\supset R^+$ is different from $\supset R$ in that the components Φ, Ψ are preserved when applying $\supset R^+$ backwards, which corresponds to the classical interpretation of implication $C \supset D$. Figure 5.11 shows the proof of the Excluded Middle (PEM) utilising rule $\supset R^+$.

Notation. We will denote $\mathsf{G1}$ extended by rule $\supset R^+$ by $\mathsf{G1}_{\mathsf{PEM}}$ and say analogously to Def. 5.2.3 that a $\mathsf{G1}_{\mathsf{PEM}}$-sequent is consistent if no tableau exists for it. Furthermore, let $c\mathcal{ALC}^{\mathsf{PEM}} =_{df} c\mathcal{ALC} \oplus C \sqcup \neg C$, and let $c\mathcal{ALC}^{\mathsf{PEM}} ; \Theta ; \Sigma ; \Gamma \not\models \Phi ; \Psi$ denote that $\Theta ; \Sigma ; \Gamma \vdash_{\mathsf{G1}_{\mathsf{PEM}}} \Phi ; \Psi$ is satisfiable at some entity of a degenerated interpretation.

$$\dfrac{\dfrac{\dfrac{}{\emptyset\,;\,\emptyset\,;\,C \vdash C,\ \bot\,;\,\emptyset}\;Ax}{\emptyset\,;\,\emptyset\,;\,\emptyset \vdash C,\ \neg C\,;\,\emptyset}\;{\supset}R^{+}}{\emptyset\,;\,\emptyset\,;\,\emptyset \vdash C \sqcup \neg C\,;\,\emptyset}\;{\sqcup}R$$

Figure 5.11: Rule $\supset R^+$ implements PEM. Adapted from [195, p. 229, Fig. 13], with kind permission from Springer Science and Business Media.

Moreover, we say that the pair $(\mathcal{I}, x)$ consisting of a $c\mathcal{ALC}^{\mathsf{PEM}}$ (degenerated) interpretation $\mathcal{I}$ and an entity $x \in \Delta^{\mathcal{I}}$ satisfies a sequent $s = \Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi$, if $\mathcal{I} \models \Theta$ and for all $R \in N_R$ it holds that

$$\forall x', y \in \Delta^{\mathcal{I}}.\, x \sim^{\mathcal{I}} x'\, R^{\mathcal{I}}\, y \Rightarrow \mathcal{I}; y \models \Sigma(R); \tag{5.101}$$

$$\mathcal{I}; x \models \Gamma; \tag{5.102}$$

$$\mathcal{I}; x \not\models \Phi; \tag{5.103}$$

$$\forall y \in \Delta^{\mathcal{I}}.\, x\, R^{\mathcal{I}}\, y \Rightarrow \mathcal{I}; y \not\models \Psi(R). \tag{5.104}$$

■

Theorem 5.3.5 (Soundness of $\mathsf{G1}_{\mathsf{PEM}}$)**.** *Every satisfiable* $\mathsf{G1}_{\mathsf{PEM}}$*-sequent is consistent,* i.e.*,* $c\mathcal{ALC}^{\mathsf{PEM}}$; $\Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi$ *implies* $\Theta\,;\,\Sigma\,;\,\Gamma \nvdash_{\mathsf{G1}_{\mathsf{PEM}}} \Phi\,;\,\Psi$. ∇

Proof. ($\Rightarrow$) The proof is analogously to the one of Thm. 5.2.2. We will only consider rule $\supset R^+$ here. Suppose that the sequent $s_c =_{df} \Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi,\, C \supset D\,;\,\Psi$ is satisfiable. The goal is to show that its premise $s_p =_{df} \Theta\,;\,\Sigma\,;\,\Gamma,\ C \vdash \Phi,\ D\,;\,\Psi$ is satisfiable as well. The assumption implies that there exists a pair $(\mathcal{I}, a)$ that satisfies s_c.
Claim: $(\mathcal{I}, a)$ satisfies the sequent s_p. By assumption $\mathcal{I}; a \not\models C \supset D$, *i.e.*, by Lem. 5.3.3 $\mathcal{I}; a \not\models C$ or $\mathcal{I}; a \models D$. The conditions (5.101)–(5.104) for Σ, Γ, Φ and Ψ follow by assumption. Hence, $(\mathcal{I}, a)$ satisfies s_p. □

The canonical models for $c\mathcal{ALC}^{\mathsf{PEM}}$ are constructed as follows: We obtain a suitable set of $\mathsf{G1}_{\mathsf{PEM}}$ consistent and saturated sequents Δ^* by Lem. 5.2.4, using the set of saturation rules $X_{c\mathcal{ALC}^{\mathsf{PEM}}}$, which corresponds to the extension of $X_{c\mathcal{ALC}}$ (see Def. 5.2.11) by the following rule to treat implication:

$\mathrm{EEM1}_{C,D}$ $\quad s \rightarrow_{\mathrm{EEM1}} s' = \langle \Theta\,;\,\Sigma\,;\,\Gamma \cup \{C\} \vdash \Phi \cup \{D\}\,;\,\Psi \rangle$,
if $C \supset D \in \Phi$ but $C \notin \Gamma$ and $D \notin \Phi$.

Definition 5.3.6 (Canonical $c\mathcal{ALC}^{\mathsf{PEM}}$ interpretation)**.** Let Θ be a fixed TBox and Δ^* be the set of all $X_{c\mathcal{ALC}^{\mathsf{PEM}}}$-saturated and consistent sequents of the form $\langle \Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi \rangle$. All these sequents have Θ as their first component but may have different Σ, Γ, Φ, Ψ. The canonical $c\mathcal{ALC}^{\mathsf{PEM}}$ interpretation $\mathcal{I}^* =_{df} (\Delta^{\mathcal{I}^*}, \sim^{\mathcal{I}^*}, \bot^{\mathcal{I}^*}, \cdot^{\mathcal{I}^*})$ is defined by

$$\begin{aligned}
\Delta^{\mathcal{I}^*} &=_{df} \Delta^*;\\
\sim^{\mathcal{I}^*} &=_{df} \{(s,s') \in \Delta^{\mathcal{I}^*} \times \Delta^{\mathcal{I}^*} \mid \Sigma_s = \Sigma_{s'} \quad \& \quad \Gamma_s = \Gamma_{s'}\};\\
R^{\mathcal{I}^*} &=_{df} \{(s,s') \in \Delta^{\mathcal{I}^*} \times \Delta^{\mathcal{I}^*} \mid \Sigma_s(R) \subseteq \Gamma_{s'} \quad \& \quad \Psi_s(R) \subseteq \Phi_{s'}\};\\
\bot^{\mathcal{I}^*} &=_{df} \{s \in \Delta^{\mathcal{I}^*} \mid \bot \in \Gamma_s\};\\
A^{\mathcal{I}^*} &=_{df} \{s \in \Delta^{\mathcal{I}^*} \mid A \in \Gamma_s \text{ or } \bot \in \Gamma_s\},
\end{aligned}$$

for all $R \in N_R$ and $A \in N_C$. ∇

Lemma 5.3.4. *The canonical $c\mathcal{ALC}^{\mathsf{PEM}}$ interpretation $\mathcal{I}^* =_{df} (\Delta^{\mathcal{I}^*}, \sim^{\mathcal{I}^*}, \bot^{\mathcal{I}^*}, \cdot^{\mathcal{I}^*})$ is a degenerated model according to Def. 5.3.5.* ∇

Proof. The proof is analogously to the proof of Lem. 5.2.6, only differing in that the relation $\sim^{\mathcal{I}^*}$ is an equivalence relation by construction of $\mathcal{I}^*$. □

Lemma 5.3.5. *Let Θ be a fixed TBox, Δ^* the set of all $X_{c\mathcal{ALC}^{\mathsf{PEM}}}$-saturated and consistent sequents of the form $\langle\Theta\,;\,\Sigma\,;\,\Gamma \vdash \Phi\,;\,\Psi\rangle$. The canonical interpretation $\mathcal{I}^* =_{df} (\Delta^{\mathcal{I}^*}, \sim^{\mathcal{I}^*}, \bot^{\mathcal{I}^*}, \cdot^{\mathcal{I}^*})$ is a degenerated model such that for all $X_{c\mathcal{ALC}^{\mathsf{PEM}}}$-saturated and consistent sequents $s \in \Delta^*$ the pair $(\mathcal{I}^*, s)$ satisfies s in the sense of Def. 5.3.5.* ∇

Proof. Analogously to the proof of Lem. 5.2.7. □

Theorem 5.3.6 (Completeness of $\mathsf{G1}_{\mathsf{PEM}}$)**.** *Every consistent $\mathsf{G1}_{\mathsf{PEM}}$-sequent is satisfiable in a degenerated interpretation,* i.e.,

$$\Theta\,;\,\Sigma\,;\,\Gamma \nvdash_{\mathsf{G1}_{\mathsf{PEM}}} \Phi\,;\,\Psi \Rightarrow c\mathcal{ALC}^{\mathsf{PEM}}\,;\,\Theta\,;\,\Sigma\,;\,\Gamma \not\models \Phi\,;\,\Psi. \qquad \nabla$$

Proof. Analogously to the proof of Thm. 5.2.3. □

Equivalence of $\mathsf{G1}_{\mathsf{PEM}}$ to the Hilbert System

Finally, we show that the Hilbert system $c\mathcal{ALC}^{\mathsf{PEM}}$ can be translated into the sequent system $\mathsf{G1}_{\mathsf{PEM}}$ and vice versa.

Proposition 5.3.10. *For every concept C and set of concepts Θ we have*

$$c\mathcal{ALC}^{\mathsf{PEM}}\,;\,\Theta\,;\,\emptyset \models C \text{ in all degenerated interpretations iff } c\mathcal{ALC}^{\mathsf{PEM}}\,;\,\Theta\,;\,\emptyset \vdash_{\mathsf{H}} C.$$

∇

Proof. The proof extends Prop. 5.2.1, *i.e.*, we give a translation between the Hilbert system $c\mathcal{ALC}^{\mathsf{PEM}}$ and the Gentzen sequent calculus $\mathsf{G1}_{\mathsf{PEM}}$, and argue axiom PEM and the rule $\supset R^+$ only. ($\Rightarrow$) The derivation of PEM is shown in Fig. 5.11.

($\Leftarrow$) In the other direction suppose that $\Theta\,;\Sigma\,;\Gamma \mid\!\frac{}{\mathsf{G1}_{\mathsf{PEM}}} \Phi\,;\Psi$ is derived by rule $\supset R^+$, *i.e.*, $\Phi = \Phi', C \supset D$, and the situation looks like

$$\cfrac{\cfrac{\vdots}{\Theta\,;\Sigma\,;\Gamma,\,C \mid\!\frac{}{\mathsf{G1}_{\mathsf{PEM}}} \Phi,\,D\,;\Psi}}{\Theta\,;\Sigma\,;\Gamma \mid\!\frac{}{\mathsf{G1}_{\mathsf{PEM}}} \Phi,\,(C \supset D)\,;\Psi}\supset R^+$$

The goal is to demonstrate that Hilbert derives

$$\Theta\,;\emptyset \mid\!\frac{}{\mathsf{H}} (\wedge\Sigma\ \sqcap\ \wedge\Gamma)\ \supset\ (\vee\Phi\ \sqcup\ (C \supset D)\ \sqcup\ \vee\Psi). \tag{5.105}$$

Applying the ind. hyp. to the premise of the sequent yields the Hilbert derivation

$$\Theta\,;\emptyset \mid\!\frac{}{\mathsf{H}} (\wedge\Sigma\ \sqcap\ \wedge\Gamma\ \sqcap\ C)\ \supset\ (\vee\Phi\ \sqcup\ D\ \sqcup\ \vee\Psi). \tag{5.106}$$

Since Hilbert derives PEM, DeMorgan's laws and the dualities from classical propositional logic CPC hold. Let $\varphi =_{df} \wedge\Sigma \sqcap \wedge\Gamma$ and $\psi =_{df} \vee\Phi \sqcup \vee\Psi$. We start from the following instance of axiom IPC2

$$\begin{aligned}\mid\!\frac{}{\mathsf{H}} (\varphi \supset (C \supset (\psi \sqcup D)) \supset (\psi \sqcup (C \supset D))) \supset (\varphi \supset C \supset (\psi \sqcup D)) \supset\\ (\varphi \supset (\psi \sqcup (C \supset D))),\end{aligned} \tag{5.107}$$

i.e., in order to derive (5.105) we need to find derivations for

$$\mid\!\frac{}{\mathsf{H}} \varphi \supset ((C \supset (\psi \sqcup D)) \supset (\psi \sqcup (C \supset D))), \tag{5.108}$$

$$\mid\!\frac{}{\mathsf{H}} \varphi \supset C \supset (\psi \sqcup D). \tag{5.109}$$

Subgoal (5.108) follows from the fact that Hilbert derives $(C \supset (\psi \sqcup D)) \supset (\psi \sqcup (C \supset D))$ in classical logic. One observes that the latter holds in CPC due to the provable equivalence $C \supset D \equiv \neg C \sqcup D$ and the law of permutation (commutativity of $\supset$). Thus, in the presence of axiom PEM it is true in $c\mathcal{ALC}^{\mathsf{PEM}}$ as well, and we obtain (5.108) from the latter derivation by $(\mathsf{ARK})_{[\varphi]}$.

Subgoal (5.109) is a consequence of applying rule (ARC) (*currying*) to (5.106). Hence, the goal (5.105) follows by rule MP from (5.107), (5.108) and (5.109). ☐

5.3.4 Obtaining classical $\mathcal{ALC}$

Note that $c\mathcal{ALC}^{\mathsf{PEM}}$ is properly more expressive than $\mathcal{ALC}$, *i.e.*, $c\mathcal{ALC}$ is not the intuitionistic analogue of $\mathcal{ALC}$ according to Simpson [249], because $c\mathcal{ALC} \oplus \mathsf{PEM}$ does not yield classical $\mathcal{ALC}$. The extension $c\mathcal{ALC}^F = c\mathcal{ALC} \oplus \mathsf{FS3/IK3}$ without fallible fillers corresponds to the multimodal version of Wijesekera's [272] constructive modal logic. Moreover, $c\mathcal{ALC}^F \oplus \mathsf{FS4/IK4}$ yields the multimodal version $I\mathcal{ALC}$ of the normal intuitionistic modal logic $\mathsf{FS/IK}$ [96; 103; 249]. In Ex. 4.2.1, we argued that if the preorder $\preceq$ is the identity relation and there are no fallible entities, then the interpretation becomes classical $\mathcal{ALC}$. This corresponds to extending $c\mathcal{ALC}$ by the schemata $\mathsf{FS4/IK4}$, $\mathsf{FS3/IK3}$, and PEM. It can be observed that the classical $\mathcal{ALC}$ interpretations arise from $c\mathcal{ALC}^{\mathsf{PEM}}$ interpretations, if fallible entities are omitted and for all x, y, it holds that $x \sim y$ iff $x = y$.

5.4 Summary

This chapter investigated the proof theory of $c\mathcal{ALC}$ by characterising it in terms of a sound and complete Hilbert-style axiomatisation and a decidable multi-conclusion Gentzen sequent calculus. The Hilbert deduction differentiates between local and global (TBox) hypotheses, and we proved a modal deduction theorem that admits derivability from local and global assumptions. Both calculi are sound and complete w.r.t. the Kripke semantics of $c\mathcal{ALC}$. Soundness and completeness of the Hilbert calculus follows from that of the Gentzen sequent calculus, and has been argued by showing that any deduction in either system can be translated into the other. We presented that both, the Hilbert and the Gentzen sequent calculus, admit the standard reasoning services of DLs w.r.t. TBoxes. In the final section, we discussed the intermediate systems between $c\mathcal{ALC}$ and $\mathcal{ALC}$ that arise from the extension of $c\mathcal{ALC}$ by the axiom schemata $\mathsf{FS3/IK3}$, $\mathsf{FS4/IK4}$, and PEM.

Notes on Related Work

A possible world semantics for CK was first described in [188] and it was proved that the Hilbert axiomatisation of monomodal CK is sound and complete w.r.t. this semantics. A single conclusion Gentzen-style sequent calculus for CK_n was presented in Mendler and Scheele [196]. This sequent calculus is more expressive than $\mathsf{G1}$ in that it preserves the contextual R-structure w.r.t. roles from N_R as a path encoded within a sequent. Sound extensions of this sequent calculus towards IK, $\mathsf{CS4}$, PLL, or Masini's deontic system [184] have been demonstrated in [196, p. 5], in particular an extension was shown to accommodate axiom $\mathsf{FS5/IK5}$. This calculus has been turned into a typing system for an extension of the simply typed lambda calculus λ-CK in [198] (see Chap. 3).

Other constructive variants of $\mathcal{ALC}$ have been investigated proof-theoretically in terms of Hilbert systems, Gentzen-style sequent calculi and natural deduction systems (see Chap. 3 for a detailed survey).

Kojima [162; 161] investigated a temporal (LTL) variant of CK. He defines a Kripke semantics and characterises the system in terms of a sound and complete Hilbert axiomatisation and the cut-free Gentzen-style sequent calculus $LJ^{\bigcirc}$. In contrast to the sequent calculus $\mathsf{G1}$, Kojima and Igarashi's [162] sequent calculus is for linear-time temporal logic, includes only the temporal next operator $\bigcirc$ (corresponding to $\exists R/\Diamond$ in $c\mathcal{ALC}/\mathsf{CK}$), uses formulæ C^n, D^m annotated by time stamps $n, m \in \mathbb{N}$ and sequents of the form $\Gamma \Rightarrow F$, where Γ is a set of annotated formulæ and F an annotated formula. The time stamp label is elegantly used to prevent the distribution of the next operator $\bigcirc$ over disjunction (axiom $\mathsf{FS4}$/ $\mathsf{IK4}$), in contrast to $\mathsf{G1}$ where the contextual R-structure of the component Ψ is used to omit the latter axiom. Then, the $LJ^{\bigcirc}$ proof of $\bigcirc(C \vee D)^n \Rightarrow (\bigcirc C \vee \bigcirc D)^n$ fails, since the rule for the next operator is (temporally) restricted such that if it is applied backwards then it yields the sequent $C \vee D^{n+1} \Rightarrow \bigcirc C \vee \bigcirc D^n$, *i.e.*, the disjunction $C \vee D^{n+1}$ of the antecedent is decided at the next instant of time $n + 1$ rather than at the current instant of time n where the disjunction $\bigcirc C \vee \bigcirc D^n$ is determined. It would be interesting to see whether this approach can be generalised to support both types of modalities $\forall R$ and $\exists R$ w.r.t. a set of roles N_R, and to annotate a formula with an encoding of the contextual path information.

CHAPTER 6

The Relation of $c\mathcal{ALC}$ to Classical Description Logics

The first part of this chapter is dedicated to the relation of $c\mathcal{ALC}$ to classical DLs by demonstrating a faithful translation of $c\mathcal{ALC}$ into a classical DL, which is naturally obtained from the birelational Kripke semantics of $c\mathcal{ALC}$. Decidability of satisfiability and subsumption of $c\mathcal{ALC}$ concepts is not surprising, since $c\mathcal{ALC}$ can be embedded into the fusion of $\mathsf{S4}$ and K_m ($\mathsf{S4} \otimes \mathsf{K}_m$) [103, Chap. 3]. Note that one $\mathsf{S4}$ modality suffices for the embedding of $c\mathcal{ALC}$ into ($\mathsf{S4} \otimes \mathsf{K}_m$). From a more general perspective, the fusion ($\mathsf{S4}_n \otimes \mathsf{K}_m$) corresponds to the description logic $\mathcal{ALC}$ extended by roles which are reflexive and transitive, which we will denote by the term $\mathcal{ALC}_{R^*}$ accordingly. We will show that the PSPACE-complexity of ($\mathsf{S4} \otimes \mathsf{K}_m$) [103, p. 218] forms an upper bound for satisfiability of $c\mathcal{ALC}$-concepts. By relying on the result from Statman [253] stating that IPC is PSPACE-complete, we will demonstrate PSPACE-hardness of satisfiability of $c\mathcal{ALC}$ concepts and thereof obtain PSPACE-completeness of $c\mathcal{ALC}$. By exploiting the fusion mechanism further results like the finite model property and decidability can be transferred to $c\mathcal{ALC}$.

The second part of this chapter is devoted to the $\{\sqcup, \exists\}$-fragment $\mathcal{UL}$ [192] of $c\mathcal{ALC}$ w.r.t. general TBoxes, which turns out to be tractable under the constructive semantics while it is intractable under the classical descriptive semantics. We will summarise the result from our publication [192] demonstrating that the problem of subsumption checking in $\mathcal{UL}$ w.r.t. general TBoxes under the constructive semantics lies in PTIME while it increases to EXPTIME under the classical descriptive semantics.

6.1 Embedding $c\mathcal{ALC}$ into classical Description Logics

It is well known that some intuitionistic modal logics can be embedded into classical ones by using the Gödel translation [103; 115]. As Gabbay et al. [103, p. 443] recall, 'one of the main reasons for introducing modal logics, in particular $\mathsf{S4}$, was the desire to find a classical interpretation of intuitionistic logic'. Accordingly, Gödel [115] defined an embedding of intuitionistic propositional logic IPC into the modal logic $\mathsf{S4}$ and it has been shown that the latter translation can be extended to capture more expressive intuitionistic modal logics, for instance $IntK_{\Box}$ in [103]. The idea of the Gödel translation

[115] is to prefix all subformulæ of a given constructive formula with a distinguished box modality $\Box_I$ that captures the intuitionistic structure [103]. There exist different variations of the Gödel translation, for instance, some translations only insert $\Box_I$ before intuitionistically interpreted subformulæ, typically leaving conjunction and disjunction untouched, while others like [103; 275] prefix every subformula with $\Box_I$. All such variants of translations are provably equivalent in $\mathsf{S4}$. For a deeper understanding the reader may consult the work by Wolter and Zakharyaschev [278]. They established a relationship between intuitionistic modal logics and classical bimodal logics in order to demonstrate that results from the latter can be transferred to IMLs.

Classical bimodal logics arise from the combination of two or more modal logics. One general mechanism for combining logics is by *fusions* (also known as *independent joins*) which have been introduced by Thomason [260]. According to Gabbay et al. [103, Chap. 3.1], and Blackburn, van Benthem and Wolter [35], the fusion $L_1 \otimes L_2$ of two modal logics L_1 and L_2 formulated in the languages $\mathcal{ML}_n$ and $\mathcal{ML}_m$, both sharing the same classical signature, but with distinct modal operators (say, $\Box_1, \ldots, \Box_n$ and $\Box_{n+1}, \ldots, \Box_{n+m}$ respectively), is the smallest $(n+m)$-modal logic containing both L_1 and L_2. If each logic L_i (where $i \in \{1,2\}$) has an axiomatisation in the form of a set of axioms Ax_i, then $L_1 \otimes L_2$ is axiomatised by $Ax_1 \cup Ax_2$ [35, Chap. 15]. Also, note that the modal operators of the component logics remain independent in their fusion. Regarding the semantic interpretation of fusions, it has been shown that Kripke completeness of the component logics transfers to their fusion. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the characterising frame classes of the modal logics L_1 and L_2 respectively, such that $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed under disjoint unions and isomorphic copies. One can show that their fusion $\mathcal{C}_1 \otimes \mathcal{C}_2$ forms the class of $(n+m)$-frames [35, Chap. 15, Thm. 3]

$$\mathcal{C}_1 \otimes \mathcal{C}_2 = \{(W, R_1, \ldots, R_n, S_1, \ldots, S_m) \mid (W, R_1, \ldots, R_n) \in \mathcal{C}_1 \ \& \ (W, S_1, \ldots, S_m) \in \mathcal{C}_2\},$$

which characterises the fusion $L_1 \otimes L_2$. Note that $\mathcal{C}_1 \otimes \mathcal{C}_2$ share the same set of possible worlds. Results of many logical systems are preserved under their fusions, for instance, soundness, Kripke completeness, the finite model property, decidability, and complexity results [35; 103; 166; 167]. This mechanism will be exploited in the following to argue the complexity, finite model property and decidability of $c\mathcal{ALC}$. In particular, we show that $c\mathcal{ALC}$ can be embedded into $\mathcal{ALC}$ with reflexive and transitive roles (denoted by $\mathcal{ALC}_{R^*}$), which corresponds to the fusion $\mathsf{S4}_n \otimes \mathsf{K}_m$. Hereby, it suffices to embed $c\mathcal{ALC}$ into $\mathsf{S4} \otimes \mathsf{K}_m$, which includes exactly one reflexive and transitive role in contrast to n-modal $\mathsf{S4}_n \otimes \mathsf{K}_m$. Our translation extends the typical Gödel translation in the following ways: (i) We prefix all subformulæ except conjunction and disjunction by $\forall\preceq$ and extend the classical signature by a reflexive and transitive role $\preceq$. (ii) Regarding fallible entities, we need to introduce a representative concept in their translation, which will be denoted by the distinguished propositional constant F. Since fallible entities validate any atomic concept, it is necessary to include the concept F in the translation of atomic concepts. Furthermore, we have to consider fallible entities in the translation of each concept description.

Notation. Regarding the interpretation of formulæ relative to the *classical semantics* (including that of $\mathcal{ALC}_{R^*}$), we will index the interpretation and relation symbols by cs. Accordingly, we will use the symbol $\models_{cs}$ for the validity relation of formulæ interpreted under the classical semantics, and $\mathcal{I}_{cs} = (\Delta^{\mathcal{I}_{cs}}, \cdot^{\mathcal{I}_{cs}})$ for a classical interpretation. As introduced earlier, we will write $\mathcal{I}_{cs}; x \models_{cs} C$ (or equivalently $x \in C^{\mathcal{I}_{cs}}$) to express that entity x *satisfies* concept C in the interpretation $\mathcal{I}_{cs}$. ■

Before detailing the embedding, let us characterise the semantics of $\mathcal{ALC}_{R^*}$.

Definition 6.1.1 ($\mathcal{ALC}_{R^*}$)**.** Let N_C be a set of concept names, N_R a set of atomic role names, and $N_{R^*} \subseteq N_R$ a set of reflexive and transitive role names. $\mathcal{ALC}_{R^*}$ uses the same syntax as $\mathcal{ALC}$ (see Definition 2.1.1), but extends it by allowing the use of reflexive and transitive roles in concept descriptions. The interpretation of $\mathcal{ALC}_{R^*}$ concepts is by extending Definition 2.1.2 by an interpretation of reflexive and transitive roles $R \in N_{R^*} \subseteq N_R$ such that for all $x, y, z \in \Delta^{\mathcal{I}_{cs}}$

$$x \, R^{\mathcal{I}_{cs}} \, x, \text{ and}$$
$$\text{if } x \, R^{\mathcal{I}_{cs}} \, y \text{ and } y \, R^{\mathcal{I}_{cs}} \, z, \text{ then } x \, R^{\mathcal{I}_{cs}} \, z. \qquad \nabla$$

Taking into account the syntactic translation f by Schild [245] (*cf.* Sec. 2.1.5), it is easy to show that $\mathcal{ALC}_{R^*}$ corresponds to the fusion $\mathsf{S4}_n \otimes \mathsf{K}_m$. This fusion is characterised by frames $\mathcal{F}$ of the form $(W, R_1, \dots, R_n, R_{n+1}, \dots, R_{n+m})$ where $R_1, \dots, R_n$ are the modalities from $\mathsf{S4}_n$ and the modal operators $R_{n+1}, \dots, R_m$ come from K_m. Now, observe that every $\mathcal{ALC}_{R^*}$ interpretation $\mathcal{I}_{cs} = (\Delta^{\mathcal{I}_{cs}}, \cdot^{\mathcal{I}_{cs}})$ relative to a set of concepts names N_C, and roles $N_{R^*} = \{R_1, \dots, R_n\}$ and $N_R = \{R_{n+1}, \dots, R_{n+m}\} \cup N_{R^*}$ can be transformed into a $\mathsf{S4}_n \otimes \mathsf{K}_m$-model and vice versa, *viz.*, by taking $\mathfrak{M}_{\mathcal{I}_{cs}} = (\mathcal{F}, \mathcal{V})$ with $\mathcal{F} = (W, R_1, \dots, R_n, R_{n+1}, \dots, R_{n+m})$ where $W = \Delta^{\mathcal{I}_{cs}}$, $R_i = R_i^{\mathcal{I}_{cs}}$ (for $1 \le i \le n + m$), and $\mathcal{V}(A) = A^{\mathcal{I}_{cs}}$ for all concept names A. Then, for all concepts C, models $\mathcal{I}_{cs}$ and individuals $a \in \Delta^{\mathcal{I}_{cs}}$ it holds that

$$\mathcal{I}_{cs}; a \models C \quad \text{iff} \quad \mathfrak{M}_{\mathcal{I}_{cs}}; a \models_{cs} f(C),$$

by extending f (see p. 24) to translate all reflexive and transitive roles into the corresponding $\mathsf{S4}$ modalities and all other roles into K_m modalities. When considering an $\mathcal{ALC}_{R^*}$ TBox Θ and a set Γ of $\mathcal{ALC}_{R^*}$ concepts we get $\mathcal{ALC}_{R^*};\Theta;\Gamma \models_{cs} C$ iff $\mathsf{S4}_n \otimes \mathsf{K}_m; f(\Theta); f(\Gamma) \models_{cs} f(C)$, where $f(\Theta)$ and $f(\Gamma)$ are the translations of the TBox axioms in Θ and concepts in Γ respectively.

6.1.1 Translation of $c\mathcal{ALC}$ into $\mathcal{ALC}_{R^*}$

The following definition gives a syntactic translation τ from concepts in $c\mathcal{ALC}$ to the language of $\mathcal{ALC}_{R^*}$.

Definition 6.1.2 (Translation of concepts)**.** Let F be a distinguished atomic concept. The translation τ of a $c\mathcal{ALC}$ concept C into $\mathcal{ALC}_{R^*}$ is inductively defined by

$$\begin{aligned}
\tau(A) &= F \sqcup \forall{\preceq}.A;\\
\tau(\bot) &= F;\\
\tau(C \supset D) &= F \sqcup \forall{\preceq}.(\neg\tau(C) \sqcup \tau(D));\\
\tau(\neg C) &= \tau(C \supset \bot);\\
\tau(C \odot D) &= \tau(C) \odot \tau(D), \text{ where } \odot \in \{\sqcap, \sqcup\};\\
\tau(\exists R.C) &= F \sqcup \forall{\preceq}.(F \sqcup \exists R.\tau(C));\\
\tau(\forall R.C) &= F \sqcup \forall{\preceq}.(F \sqcup \forall R.\tau(C)). \qquad \nabla
\end{aligned}$$

One can observe that this translation corresponds to the embedding of IPC into S4 as presented by Gödel [115] if we restrict the translation τ to the language of IPC, *i.e.*, by omitting the modalities, the fallible atom F and by prefixing conjunction and disjunction by $\forall\preceq$ as well. We claim that the translation τ embeds $c\mathcal{ALC}$ into $\mathcal{ALC}_{R^*}$. In the first step we will restrict our attention to the semantics and define an operation which transforms a constructive interpretation into a classical one and vice versa.

Definition 6.1.3 (Constructive translation τ)**.** Given a $c\mathcal{ALC}$ interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ over signature $\Sigma = (N_C, N_R)$, the translation $\tau(\mathcal{I})$ is defined by the structure $(\Delta^{\mathcal{I}_{cs}}, \cdot^{\mathcal{I}_{cs}})$ over extended signature $\Sigma' = (N_C \cup \{F\}, N_R \cup \{\preceq\})$, where F is a distinguished atomic concept such that $F \notin N_C$, $\preceq \notin N_R$ is a fresh relational symbol (denoting a reflexive and transitive role), by taking

$$\Delta^{\mathcal{I}_{cs}} =_{df} \Delta^{\mathcal{I}}; \qquad \preceq^{\mathcal{I}_{cs}} =_{df} \preceq^{\mathcal{I}}; \qquad F^{\mathcal{I}_{cs}} =_{df} \bot^{\mathcal{I}};$$

and $\forall A \in N_C, \forall R \in N_R$,

$$A^{\mathcal{I}_{cs}} =_{df} A^{\mathcal{I}}; \qquad R^{\mathcal{I}_{cs}} =_{df} R^{\mathcal{I}}. \qquad \nabla$$

For the inverse direction let us consider the following translation.

Definition 6.1.4 (Classic translation τ')**.** Given an $\mathcal{ALC}_{R^*}$ interpretation $\mathcal{I}_{cs} = (\Delta^{\mathcal{I}_{cs}}, \cdot^{\mathcal{I}_{cs}})$ over signature $\Sigma' = (N_C \cup \{F\}, N_R \cup \{\preceq\})$ with the reflexive and transitive role $\preceq$. The translation $\tau'(\mathcal{I}_{cs})$ is given by the structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ over signature $\Sigma = (N_C, N_R)$ by taking

$$\begin{aligned}
&\Delta^{\mathcal{I}} =_{df} \Delta^{\mathcal{I}_{cs}};\\
&\preceq^{\mathcal{I}} =_{df} \{(x, y) \mid (x \in F^{\mathcal{I}_{cs}} \text{ and } x = y) \text{ or } (x \notin F^{\mathcal{I}_{cs}} \text{ and } x \preceq^{\mathcal{I}_{cs}} y)\};\\
&\bot^{\mathcal{I}} =_{df} F^{\mathcal{I}_{cs}};
\end{aligned}$$

and $\forall A \in N_C \setminus \{F\}, \forall R \in N_R \setminus \{\preceq\}$,

$$\begin{aligned}
&A^{\mathcal{I}} =_{df} \{x \mid x \in F^{\mathcal{I}_{cs}} \text{ or } \forall y.x \preceq^{\mathcal{I}_{cs}} y \Rightarrow y \in A^{\mathcal{I}_{cs}}\};\\
&R^{\mathcal{I}} =_{df} \{(x, y) \mid (x \notin F^{\mathcal{I}_{cs}} \text{ and } x\, R^{\mathcal{I}_{cs}}\, y) \text{ or } (x \in F^{\mathcal{I}_{cs}} \text{ and } x = y)\}. \qquad \nabla
\end{aligned}$$

Note that Definition 6.1.4 forces fallible entities to be connected only to themselves via roles $R \in N_R$ and refinement $\preceq^{\mathcal{I}}$.

Lemma 6.1.1. *Let the constructive interpretation $\mathcal{I}$ in c$\mathcal{ALC}$ and the classic interpretation $\mathcal{I}_{cs}$ in $\mathcal{ALC}_{R^*}$ be arbitrary. The following holds:*

(i) $\tau(\mathcal{I})$ is a standard $\mathcal{ALC}_{R^*}$ interpretation;

(ii) $\tau'(\mathcal{I}_{cs})$ is a constructive interpretation according to Definition 4.2.2. ∇

Proof. (i) One can easily observe that $\tau(\mathcal{I})$ yields a classic interpretation with the reflexive and transitive role $\preceq$.

For (ii), we argue that the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ resulting from the translation $\tau'(\mathcal{I}_{cs})$ is a constructive interpretation according to Definition 4.2.2.

- The set $\Delta^{\mathcal{I}}$ is nonempty by Definition 6.1.4, since $\Delta^{\mathcal{I}_{cs}}$ of the classical interpretation is non-empty by definition.
- By definition the role $\preceq$ is reflexive and transitive. Note that fallible entities only refine themselves.
- The set of fallible entities $\bot^{\mathcal{I}}$ is closed under refinement and role-filling. This is a direct consequence of Def. 6.1.4, because fallible entities in $\Delta^{\mathcal{I}}$ are connected only to themselves via roles $R \in N_R$ and refinement $\preceq^{\mathcal{I}}$, and for each role $R \in N_R$ there exists exactly one cycle from each fallible entity to itself.

- The interpretation of atomic concepts is given by $A^{\mathcal{I}}$, which is closed under refinement by Def. 6.1.4.
- For the interpretation of roles $R \in N_R$ there is nothing to show. □

The main theorem states that $c\mathcal{ALC}$ can be embedded into $\mathcal{ALC}_{R^*}$. First, we will show the embedding without considering TBoxes, *i.e.*, a $c\mathcal{ALC}$ concept C is valid if and only if its translation (Def. 6.1.2) is valid in $\mathcal{ALC}_{R^*}$.

Proposition 6.1.1. *For every $c\mathcal{ALC}$ concept C not containing concept F,*

$$\emptyset; \emptyset \models C \quad \textit{if and only if} \quad \mathcal{ALC}_{R^*}; \emptyset; \emptyset \models_{cs} \tau(C). \qquad \nabla$$

To prove Proposition 6.1.1 we need to demonstrate (i) for all classical $\mathcal{ALC}_{R^*}$ interpretations $\mathcal{I}_{cs}$ and for all entities $x \in \Delta^{\mathcal{I}_{cs}}$ it holds that $\mathcal{I}_{cs}; x \models_{cs} \tau(C)$ from the assumption $\emptyset; \emptyset \models C$, and (ii) for all constructive interpretations $\mathcal{I}$ and for all entities $y \in \Delta^{\mathcal{I}}$ it holds that $\mathcal{I}; y \models C$ under the assumption that $\mathcal{ALC}_{R^*}; \emptyset; \emptyset \models_{cs} \tau(C)$ is true. This is formulated by the following auxiliary lemma.

Lemma 6.1.2. *For every $c\mathcal{ALC}$ concept C, not containing the atomic concept F,*

$$\forall \mathcal{I}_{cs}, \forall x \in \Delta^{\mathcal{I}_{cs}}.\ \tau'(\mathcal{I}_{cs}); x \models C \quad \textit{if and only if} \quad \mathcal{I}_{cs}; x \models_{cs} \tau(C), \textit{ and} \tag{6.1}$$

$$\forall \mathcal{I}, \forall y \in \Delta^{\mathcal{I}}.\ \mathcal{I}; y \models C \quad \textit{if and only if} \quad \tau(\mathcal{I}); y \models_{cs} \tau(C). \tag{6.2}$$

∇

Proof. For (6.1), let $\mathcal{I}_{cs}$ be a classical $\mathcal{ALC}_{R^*}$ interpretation and $a \in \Delta^{\mathcal{I}_{cs}}$ be arbitrarily chosen. We claim that

$$\tau'(\mathcal{I}_{cs}); a \models C \quad \text{iff} \quad \mathcal{I}_{cs}; a \models_{cs} \tau(C).$$

The proof is by induction on the structure of C:

(**Case** A) (⇒) Suppose that $\tau'(\mathcal{I}_{cs}); a \models A$ holds. The task is to show that $\mathcal{I}_{cs}; a \models_{cs} F \sqcup \forall\preceq.A$. By Def. 4.2.3 $a \in A^{\tau'(\mathcal{I}_{cs})}$ and Def. 6.1.4 implies $a \in F^{\mathcal{I}_{cs}}$ or $\forall y \in \Delta^{\mathcal{I}_{cs}}.a \preceq^{\mathcal{I}_{cs}} y \Rightarrow y \in A^{\mathcal{I}_{cs}}$. Case 1. If $a \in F^{\mathcal{I}_{cs}}$ then the goal follows immediately.

Case 2. Otherwise, if $a \notin F^{\mathcal{I}_{cs}}$, let $b \in \Delta^{\mathcal{I}_{cs}}$ be arbitrary such that $a \preceq^{\mathcal{I}_{cs}} b$. By applying the assumption it follows that $b \in A^{\mathcal{I}_{cs}}$, *i.e.*, $\mathcal{I}_{cs}; b \models_{cs} A$. Since b was an arbitrary $\preceq^{\mathcal{I}_{cs}}$-successor of a we can conclude $\mathcal{I}_{cs}; a \models_{cs} \forall\preceq.A$.

Thus, $\mathcal{I}_{cs}; a \models_{cs} \tau(A)$.

(⇐) Assume that $\mathcal{I}_{cs}; a \models_{cs} F \sqcup \forall\preceq.A$. This means that $a \in F^{\mathcal{I}_{cs}}$ or $\forall y.a \preceq^{\mathcal{I}_{cs}} y \Rightarrow y \in A^{\mathcal{I}_{cs}}$. Then, it follows straight from Def. 6.1.4 that $\tau'(\mathcal{I}_{cs}); a \models A$.

(**Case** $\bot$) $(\Rightarrow)$ Suppose that $\tau'(\mathcal{I}_{cs}); a \models \bot$, *i.e.*, $a \in \bot^{\tau'(\mathcal{I}_{cs})}$ and since $\bot^{\tau'(\mathcal{I}_{cs})} = F^{\mathcal{I}_{cs}}$ by Def. 6.1.4 we can conclude that $\mathcal{I}_{cs}; a \models_{cs} \tau(\bot) = F$.

$(\Leftarrow)$ Assume that $\mathcal{I}_{cs}; a \models_{cs} \tau(\bot) = F$. From $a \in F^{\mathcal{I}_{cs}}$ it it follows directly that $a \in \bot^{\tau'(\mathcal{I}_{cs})}$ by Def. 6.1.4. Therefore, $\tau'(\mathcal{I}_{cs}); a \models \bot$.

(**Case** $C \supset D$) $(\Rightarrow)$ Let us suppose that $\tau'(\mathcal{I}_{cs}); a \models C \supset D$. Then, by Def. 4.2.2 it holds for all $\preceq^{\tau'(\mathcal{I}_{cs})}$ refinements x of a that $x \in C^{\tau'(\mathcal{I}_{cs})}$ implies $x \in D^{\tau'(\mathcal{I}_{cs})}$. We need to prove $\mathcal{I}_{cs}; a \models_{cs} F \sqcup \forall\preceq.(\neg\tau(C) \sqcup \tau(D))$.

Case 1. If a is fallible, *i.e.*, $a \in \bot^{\tau'(\mathcal{I}_{cs})} = F^{\mathcal{I}_{cs}}$, then immediately $\mathcal{I}_{cs}; a \models_{cs} F \sqcup \forall\preceq.(\neg\tau(C) \sqcup \tau(D))$.

Case 2. Otherwise, $a \notin \bot^{\tau'(\mathcal{I}_{cs})}$ and let $b \in \Delta^{\mathcal{I}_{cs}}$ be arbitrary such that $a \preceq^{\mathcal{I}_{cs}} b$. It follows by definition of τ' that $a \preceq^{\tau'(\mathcal{I}_{cs})} b$. We proceed by case analysis:

Case 2.1. $b \notin C^{\tau'(\mathcal{I}_{cs})}$. By induction hypothesis (direction $\Leftarrow$) it follows that $b \notin (\tau(C))^{\mathcal{I}_{cs}}$ and thus $b \in (\neg\tau(C))^{\mathcal{I}_{cs}}$.

Case 2.2. If $b \in C^{\tau'(\mathcal{I}_{cs})}$ then it follows from the assumption that $b \in D^{\tau'(\mathcal{I}_{cs})}$. Applying the induction hypothesis (direction $\Rightarrow$) yields $b \in (\tau(D))^{\mathcal{I}_{cs}}$.

Since b was an arbitrary $\preceq^{\mathcal{I}_{cs}}$-successor of a, the goal $\mathcal{I}_{cs}; a \models_{cs} F \sqcup \forall\preceq.(\neg\tau(C) \sqcup \tau(D))$ holds.

$(\Leftarrow)$ Proof by contraposition. Suppose that $\tau'(\mathcal{I}_{cs}); a \not\models C \supset D$. Then, there exists an entity $b \in \Delta^{\tau'(\mathcal{I}_{cs})}$ such that $a \preceq^{\tau'(\mathcal{I}_{cs})} b$ and $b \in C^{\tau'(\mathcal{I}_{cs})}$ but $b \notin D^{\tau'(\mathcal{I}_{cs})}$. We need to prove that $\mathcal{I}_{cs}; a \not\models_{cs} F \sqcup \forall\preceq.(\neg\tau(C) \sqcup \tau(D))$. From the assumption it follows that a is an infallible entity, *i.e.*, $a \notin \bot^{\tau'(\mathcal{I}_{cs})}$, and by Def. 6.1.4 it holds that $a \preceq^{\mathcal{I}_{cs}} b$. The ind. hyp. implies that $b \in \tau(C)^{\mathcal{I}_{cs}}$ and $b \notin \tau(D)^{\mathcal{I}_{cs}}$ which proves $\mathcal{I}_{cs}; a \not\models_{cs} F \sqcup \forall\preceq.(\neg\tau(C) \sqcup \tau(D))$.

(**Case** $C \sqcap D$) By Def. 4.2.3, $\tau'(\mathcal{I}_{cs}); a \models C \sqcap D$ holds if and only if $\tau'(\mathcal{I}_{cs}); a \models C$ and $\tau'(\mathcal{I}_{cs}); a \models D$. Using the induction hypothesis this holds if and only if $\mathcal{I}_{cs}; a \models_{cs} \tau(C)$ and $\mathcal{I}_{cs}; a \models_{cs} \tau(D)$ and by Def. 6.1.1 $\mathcal{I}_{cs}; a \models_{cs} \tau(C \sqcap D)$.

(**Case** $C \sqcup D$) By ind. hyp. similarly to (**Case** $C \sqcap D$).

(**Case** $\exists R.C$) $(\Rightarrow)$ Suppose that $\tau'(\mathcal{I}_{cs}); a \models \exists R.C$. The task is to prove that $\mathcal{I}_{cs}; a \models_{cs} F \sqcup \forall\preceq.(F \sqcup \exists R.\tau(C))$. By Def. 4.2.2 it holds for all $\preceq^{\tau'(\mathcal{I}_{cs})}$ refinements of a that there exists an R-filler contained in $C^{\tau'(\mathcal{I}_{cs})}$, in particular by reflexivity of $\preceq^{\tau'(\mathcal{I}_{cs})}$ for a itself.

Case 1. If a is fallible then $a \in F^{\mathcal{I}_{cs}}$ by Definition 6.1.4 and the goal $\mathcal{I}_{cs}; a \models_{cs} F \sqcup \forall\preceq.(F \sqcup \exists R.\tau(C))$ follows immediately.

Case 2. Otherwise, suppose that a is not fallible, *i.e.*, $a \notin F^{\mathcal{I}_{cs}}$. Now, let $b \in \Delta^{\mathcal{I}_{cs}}$ be arbitrary such that $a \preceq^{\mathcal{I}_{cs}} b$ and $b \notin F^{\mathcal{I}_{cs}}$. It follows from Def. 6.1.4 that $a \preceq^{\tau'(\mathcal{I}_{cs})} b$. By the assumption there exists $d \in \Delta^{\tau'(\mathcal{I}_{cs})}$ such that $b\, R^{\tau'(\mathcal{I}_{cs})}\, d$ and $d \in C^{\tau'(\mathcal{I}_{cs})}$. Since $b \notin F^{\mathcal{I}_{cs}}$, we have $b\, R^{\mathcal{I}_{cs}}\, d$ and it follows by the induction hypothesis that $d \in \tau(C)^{\mathcal{I}_{cs}}$. Therefore, $\mathcal{I}_{cs}; a \models_{cs} F \sqcup \forall\preceq.(F \sqcup \exists R.\tau(C))$.

($\Leftarrow$) Proof by contraposition. Suppose that $\tau'(\mathcal{I}_{cs}); a \not\models \exists R.C$. The task is to demonstrate that $\mathcal{I}_{cs}; a \not\models_{cs} F \sqcup \forall\preceq.(F \sqcup \exists R.\tau(C))$. From the assumption it follows that $a \notin (\exists R.C)^{\tau'(\mathcal{I}_{cs})}$, *i.e.*, there exists an entity b in $\Delta^{\tau'(\mathcal{I}_{cs})}$ which is a $\preceq^{\tau'(\mathcal{I}_{cs})}$ refinement of a such that all of its R-successors are not in the interpretation of $C^{\tau'(\mathcal{I}_{cs})}$. The assumption implies that $a \notin \bot^{\tau'(\mathcal{I}_{cs})}$ and by Def. 6.1.4 it holds that $a \notin F^{\mathcal{I}_{cs}}$, therefore, $\mathcal{I}_{cs}; a \not\models_{cs} F$. It remains to show that $\mathcal{I}_{cs}; a \not\models_{cs} \forall\preceq.(F \sqcup \exists R.\tau(C))$. Definition 6.1.4 implies $a \preceq^{\mathcal{I}_{cs}} b$. Now, let $c \in \Delta^{\mathcal{I}_{cs}}$ be arbitrary and assume that $b\, R^{\mathcal{I}_{cs}}\, c$. Observe that b is infallible, *i.e.*, $b \notin \bot^{\tau'(\mathcal{I}_{cs})} = F^{\mathcal{I}_{cs}}$. Otherwise, if b would be fallible then Def. 6.1.4 would imply that $b\, R^{\tau'(\mathcal{I}_{cs})}\, b$, but this would contradict the assumption that $b \notin C^{\tau'(\mathcal{I}_{cs})}$. We can conclude from Def. 6.1.4 that $c \in \Delta^{\tau'(\mathcal{I}_{cs})}$ and $b\, R^{\tau'(\mathcal{I}_{cs})}\, c$. The assumption yields $c \notin C^{\tau'(\mathcal{I}_{cs})}$ and the ind. hyp. yields $c \notin \tau(C)^{\mathcal{I}_{cs}}$, which proves $\mathcal{I}_{cs}; a \not\models_{cs} \forall\preceq.(F \sqcup \exists R.\tau(C))$.

(**Case** $\forall R.C$) ($\Rightarrow$) Suppose that $\tau'(\mathcal{I}_{cs}); a \models \forall R.C$, we have to show that $\mathcal{I}_{cs}; a \models_{cs} \tau(\forall R.C)$. The assumption implies by Def. 4.2.2 that all R-fillers of all refinements of a are contained in $C^{\tau'(\mathcal{I}_{cs})}$, in particular by reflexivity of $\preceq^{\tau'(\mathcal{I}_{cs})}$ this holds for a itself.

Case 1. $a \in F^{\mathcal{I}_{cs}}$ implies $\mathcal{I}_{cs}; a \models_{cs} \tau(\forall R.C) = F \sqcup \forall\preceq.(F \sqcup \forall R.\tau(C))$.

Case 2. $a \notin F^{\mathcal{I}_{cs}}$. Let $b \in \Delta^{\mathcal{I}_{cs}}$ be arbitrary and suppose that $a \preceq^{\mathcal{I}_{cs}} b$ and $b \notin F^{\mathcal{I}_{cs}}$. Moreover, let $c \in \Delta^{\mathcal{I}_{cs}}$ and suppose that $b\, R^{\mathcal{I}_{cs}}\, c$. Definition 6.1.4 implies that $b, c \in \Delta^{\tau'(\mathcal{I}_{cs})}$, $a \preceq^{\tau'(\mathcal{I}_{cs})} b$ and $b\, R^{\tau'(\mathcal{I}_{cs})}\, c$. The assumption lets us conclude that $c \in C^{\tau'(\mathcal{I}_{cs})}$. Applying the induction hypothesis yields $c \in \tau(C)^{\mathcal{I}_{cs}}$.

Therefore, $\mathcal{I}_{cs}; a \models_{cs} F \sqcup \forall\preceq.(F \sqcup \forall R.\tau(C))$.

($\Leftarrow$) Proof by contraposition. Suppose that $\tau'(\mathcal{I}_{cs}); a \not\models \forall R.C$, *i.e.*, there exist $b, c \in \Delta^{\tau'(\mathcal{I}_{cs})}$ such that $a \preceq^{\tau'(\mathcal{I}_{cs})} b\, R^{\tau'(\mathcal{I}_{cs})}\, c$ and $c \notin C^{\tau'(\mathcal{I}_{cs})}$. It is necessary to prove that $\mathcal{I}_{cs}; a \not\models_{cs} F \sqcup \forall\preceq.(F \sqcup \forall R.\tau(C))$. The assumption implies that a is not fallible, *i.e.*, $a \notin F^{\mathcal{I}_{cs}}$. By Definition 6.1.4 $b, c \in \Delta^{\mathcal{I}_{cs}}$. Now, observe that b, c are infallible. Infallibility of c follows from the assumption and b is infallible, since infallible R-successors cannot have a fallible R-predecessor (see Proposition 4.2.1). Then, it holds by Definition 6.1.4 that $a \preceq^{\mathcal{I}_{cs}} b\, R^{\mathcal{I}_{cs}}\, c$. The induction hypothesis lets us conclude that $c \notin \tau(C)^{\mathcal{I}_{cs}}$, which completes the proof of $\mathcal{I}_{cs}; a \not\models_{cs} \tau(\forall R.C)$.

For the proof of (6.2), let $\mathcal{I}$ be a constructive interpretation and $a \in \Delta^{\mathcal{I}}$ be arbitrarily chosen. We show by induction on the structure of C that

$$\mathcal{I}; a \models C \quad \textit{iff} \quad \tau(\mathcal{I}); a \models_{cs} \tau(C),$$

(**Case** A) ($\Rightarrow$) Suppose $\mathcal{I}; a \models A$, *i.e.*, $a \in A^{\mathcal{I}}$. Because $A^{\mathcal{I}}$ is closed under refinement $\preceq^{\mathcal{I}}$, it follows directly that $\tau(\mathcal{I}); a \models_{cs} \forall\preceq.A$. Thus, $\tau(\mathcal{I}); a \models_{cs} F \sqcup \forall\preceq.A$ as desired.

($\Leftarrow$) In the other direction suppose that $\tau(\mathcal{I}); a \models_{cs} \tau(A)$, *i.e.*, $a \in (F \sqcup \forall\preceq.A)^{\tau(\mathcal{I})}$. If $a \in F^{\tau(\mathcal{I})}$ then $a \in \bot^{\mathcal{I}}$ and because of $\bot^{\mathcal{I}} \subseteq A^{\mathcal{I}}$ this implies $a \in A^{\mathcal{I}}$. Otherwise, if $a \notin F^{\tau(\mathcal{I})}$ then it follows by Def. 6.1.3 that all $\preceq^{\tau(\mathcal{I})}$-successors of a are contained in $A^{\tau(\mathcal{I})}$, in particular by reflexivity of $\preceq^{\tau(\mathcal{I})}$ entity a itself. Therefore by Def. 6.1.3 $\mathcal{I}; a \models A$.

(**Case** $\bot$) ($\Rightarrow$) Suppose that $\mathcal{I}; a \models \bot$, *i.e.*, $a \in \bot^{\mathcal{I}}$. From Def. 6.1.3 it follows that $a \in F^{\tau(\mathcal{I})}$ and therefore $\tau(\mathcal{I}); a \models_{cs} F$.

($\Leftarrow$) In the other direction suppose $\tau(\mathcal{I}); a \models_{cs} \tau(\bot)$, *i.e.*, $a \in F^{\mathcal{I}_{cs}}$. From the assumption and Def. 6.1.3 we can conclude $a \in \bot^{\mathcal{I}}$. Hence, $\mathcal{I}; a \models \bot$.

(**Case** $C \supset D$) ($\Rightarrow$) Assume that $\mathcal{I}; a \models C \supset D$. By Def. 4.2.2, it holds for all refinements x of a that $x \in C^{\mathcal{I}}$ implies $x \in D^{\mathcal{I}}$. We need to prove $\tau(\mathcal{I}); a \models_{cs} F \sqcup \forall\preceq.(\neg\tau(C) \sqcup \tau(D))$. Let $b \in \Delta^{\tau(\mathcal{I})}$ be arbitrary such that $a \preceq^{\tau(\mathcal{I})} b$. Assume that $b \in (\tau(C)^{\tau(\mathcal{I})})$. From Def. 6.1.3 it follows that $b \in \Delta^{\mathcal{I}}$ and $a \preceq^{\mathcal{I}} b$. The induction hypothesis lets us conclude that $b \in C^{\mathcal{I}}$. Then, the assumption implies that $b \in D^{\mathcal{I}}$ and applying the induction hypothesis returns $b \in (\tau(D)^{\tau(\mathcal{I})})$ as desired. Therefore, $\tau(\mathcal{I}); a \models_{cs} \forall\preceq.(\neg\tau(C) \sqcup \tau(D))$ which implies $\tau(\mathcal{I}); a \models_{cs} F \sqcup \forall\preceq.(\neg\tau(C) \sqcup \tau(D))$.

($\Leftarrow$) Proof by contraposition. In the other direction suppose that $\mathcal{I}; a \not\models C \supset D$, *i.e.*, there exists an entity $b \in \Delta^{\mathcal{I}}$ such that $a \preceq^{\mathcal{I}} b$ and $b \in C^{\mathcal{I}}$ but $b \notin D^{\mathcal{I}}$. The goal is $\tau(\mathcal{I}); a \not\models_{cs} F \sqcup \forall\preceq.(\neg\tau(C) \sqcup \tau(D))$. From the assumption it follows that $a \notin \bot^{\mathcal{I}} = F^{\tau(\mathcal{I})}$. Def. 6.1.3 implies $b \in \Delta^{\tau(\mathcal{I})}$ and $a \preceq^{\tau(\mathcal{I})} b$ as well. Applying the induction hypothesis yields $\tau(\mathcal{I}); b \models_{cs} \tau(C)$ and $\tau(\mathcal{I}); b \not\models_{cs} \tau(D)$. Hence, $\tau(\mathcal{I}); a \not\models_{cs} \tau(C \supset D)$.

(**Case** $C \sqcap D$) Suppose that $\mathcal{I}; a \models C \sqcap D$. By Definition 4.2.2 this holds if and only if $\mathcal{I}; a \models C$ and $\mathcal{I}; a \models D$. Using the induction hypothesis the latter holds if and only if $\tau(\mathcal{I}); a \models_{cs} \tau(C)$ and $\tau(\mathcal{I}); a \models_{cs} \tau(D)$ which is the case if and only if $\tau(\mathcal{I}); a \models_{cs} \tau(C \sqcap D)$.

(**Case** $C \sqcup D$) Argued similarly as the previous case (**Case** $C \sqcap D$) by ind. hyp.

(**Case** $\exists R.C$) ($\Rightarrow$) Assume that $\mathcal{I}; a \models \exists R.C$. By Def. 4.2.2 it holds for all $\preceq^{\mathcal{I}}$ refinements of a that there exists an R-filler contained in $C^{\mathcal{I}}$. We need to show $\tau(\mathcal{I}); a \models_{cs} F \sqcup \forall{\preceq}.(F \sqcup \exists R.\tau(C))$. Let $b \in \Delta^{\tau(\mathcal{I})}$ be arbitrary such that $a \preceq^{\tau(\mathcal{I})} b$. Definition 6.1.3 implies $b \in \Delta^{\mathcal{I}}$ and $a \preceq^{\mathcal{I}} b$ in its original interpretation. According to the assumption, let $c \in \Delta^{\mathcal{I}}$ be such that $b\, R^{\mathcal{I}}\, c$ and $c \in C^{\mathcal{I}}$. From Def. 6.1.3 it follows that $c \in \Delta^{\tau(\mathcal{I})}$ and $b\, R^{\tau(\mathcal{I})}\, c$. Applying the induction hypothesis lets us conclude that $c \in (\tau(C))^{\tau(\mathcal{I})}$. Since b was arbitrary it follows that $\tau(\mathcal{I}); a \models_{cs} \forall{\preceq}.(\exists R.\tau(C))$ which implies $\tau(\mathcal{I}); a \models_{cs} F \sqcup \forall{\preceq}.(F \sqcup \exists R.\tau(C))$.

($\Leftarrow$) Proof by contraposition. Suppose that $\mathcal{I}; a \not\models \exists R.C$, *i.e.*, there exists an entity b in $\Delta^{\mathcal{I}}$ which is a refinement of a such that all its R-successors are not in the interpretation of C. We need to prove that $\tau(\mathcal{I}); a \not\models_{cs} F \sqcup \forall{\preceq}.(F \sqcup \exists R.\tau(C))$. From the assumption it follows that a is infallible, therefore $a \notin F^{\tau(\mathcal{I})}$ and $\tau(\mathcal{I}); a \not\models_{cs} F$. By Def. 6.1.3 it holds that $a, b \in \Delta^{\tau(\mathcal{I})}$ and $a \preceq^{\tau(\mathcal{I})} b$. Now, let $c \in \Delta^{\tau(\mathcal{I})}$ and suppose that $b\, R^{\tau(\mathcal{I})}\, c$. From Definition 6.1.3 it follows that $b\, R^{\mathcal{I}}\, c$ as well and by assumption $c \notin C^{\mathcal{I}}$. Using the induction hypothesis yields $c \notin (\tau(C))^{\tau(\mathcal{I})}$. The latter also implies that c is infallible, *i.e.*, $c \notin \bot^{\mathcal{I}} = F^{\tau(\mathcal{I})}$. Proposition 4.2.1 implies infallibility of b, *i.e.*, $b \notin \bot^{\mathcal{I}} = F^{\tau(\mathcal{I})}$. Thus, $\tau(\mathcal{I}); a \not\models_{cs} F \sqcup \forall{\preceq}.(F \sqcup \exists R.\tau(C))$.

(**Case** $\forall R.C$) ($\Rightarrow$) Assume that $\mathcal{I}; a \models \forall R.C$. By Def. 4.2.2 it holds that all R-fillers of all $\preceq^{\mathcal{I}}$ refinements of a are contained in $C^{\mathcal{I}}$. The goal is $\tau(\mathcal{I}); a \models_{cs} F \sqcup \forall{\preceq}.(F \sqcup \forall R.\tau(C))$. Suppose that $a \notin \bot^{\mathcal{I}}$, *i.e.*, $a \notin F^{\tau(\mathcal{I})}$, and let $b, c \in \Delta^{\tau(\mathcal{I})}$ be arbitrary such that $a \preceq^{\tau(\mathcal{I})} b\; R^{\tau(\mathcal{I})}\; c$. We can conclude from Def. 6.1.3 that $b, c \in \Delta^{\mathcal{I}}$ and $a \preceq^{\mathcal{I}} b\; R^{\mathcal{I}}\; c$ hold in the original interpretation. Accordingly, the assumption implies $c \in C^{\mathcal{I}}$. Applying the ind. hyp. lets us conclude that $c \in (\tau(C))^{\tau(\mathcal{I})}$. Since b and c were arbitrary, it follows that $\tau(\mathcal{I}); a \models_{cs} \forall{\preceq}.(\forall R.\tau(C))$ which implies $\tau(\mathcal{I}); a \models_{cs} F \sqcup \forall{\preceq}.(F \sqcup \forall R.\tau(C))$.

($\Leftarrow$) Proof by contraposition. Suppose that $\mathcal{I}; a \not\models \forall R.C$, *i.e.*, there exist entities b, c in $\Delta^{\mathcal{I}}$ such that $a \preceq^{\mathcal{I}} b\; R^{\mathcal{I}}\; c$ and $c \notin C^{\mathcal{I}}$. The task is to show that $\tau(\mathcal{I}); a \not\models_{cs} F \sqcup \forall{\preceq}.(F \sqcup \forall R.\tau(C))$. The assumption implies that a is infallible. By Def. 6.1.3 it follows that $b, c \in \Delta^{\tau(\mathcal{I})}$ such that $a \preceq^{\tau(\mathcal{I})} b\; R^{\tau(\mathcal{I})}\; c$ as well. The ind. hyp. lets us conclude that $c \notin (\tau(C))^{\tau(\mathcal{I})}$, which also implies that c is infallible and by Prop. 4.2.1 entity b is infallible as well. Hence, $\tau(\mathcal{I}); a \not\models_{cs} F \sqcup \forall{\preceq}.(F \sqcup \forall R.\tau(C))$.

□

Now, we are ready to tackle Proposition 6.1.1.

Proof of Proposition 6.1.1. The proposition is a consequence of the Kripke completeness of the involved logics and the natural translation of their models into each other, while preserving the validity of concepts.

($\Rightarrow$) Assume that $\emptyset; \emptyset \models C$, let $\mathcal{I}_{cs}$ be a classical $\mathcal{ALC}_{R^*}$ interpretation and $a \in \Delta^{\mathcal{I}_{cs}}$ arbitrarily chosen. Lemma 6.1.1 implies that the translated interpretation $\tau'(\mathcal{I}_{cs})$ is a constructive interpretation and by the assumption it holds in particular that $\tau'(\mathcal{I}_{cs}); a \models C$. Lemma 6.1.2 (6.1) lets us conclude that $\mathcal{I}_{cs}; a \models \tau(C)$. Since $\mathcal{I}_{cs}$ and a were arbitrary, we have $\mathcal{ALC}_{R^*}; \emptyset; \emptyset \models_{cs} \tau(C)$.

($\Leftarrow$) In the other direction let us suppose that $\mathcal{ALC}_{R^*}; \emptyset; \emptyset \models_{cs} \tau(C)$ holds. Let $\mathcal{I}$ be a constructive interpretation and $a \in \Delta^{\mathcal{I}}$ be arbitrarily chosen. The assumption implies in particular that $\tau(\mathcal{I}); a \models_{cs} \tau(C)$ holds. Lemma 6.1.1 implies that $\tau(\mathcal{I})$ is an $\mathcal{ALC}_{R^*}$ interpretation. We can conclude by Lemma 6.1.2 (6.2) that $\mathcal{I}; a \models C$. Since $\mathcal{I}$ and a were arbitrarily chosen, it holds that $\emptyset; \emptyset \models C$. □

Example 6.1.1. Let us consider the translation of the three $c\mathcal{ALC}$ concepts

$$\begin{aligned}
\mathsf{K}_{\forall R}\colon &\quad \forall R.(C \supset D) \supset (\forall R.C \supset \forall R.D),\\
\mathsf{K}_{\exists R}\colon &\quad \forall R.(C \supset D) \supset (\exists R.C \supset \exists R.D),\\
\mathsf{FS4/IK4}\colon &\quad \exists R.(C \sqcup D) \supset (\exists R.C \sqcup \exists R.D).
\end{aligned}$$

The translation of $\mathsf{K}_{\forall R}$ yields the following $\mathcal{ALC}_{R^*}$ formula, where we use the indexed markers $\llcorner \cdots \lrcorner_{\mathrm{i}}$ with $i \geq 1$ to highlight subformulæ in the original $c\mathcal{ALC}$ formula and in its corresponding translation into $\mathcal{ALC}_{R^*}$:

$$\begin{aligned}
&\tau(\mathsf{K}_{\forall R})\\
&= \tau(\llcorner \forall R.(\llcorner C \supset D \lrcorner_2) \lrcorner_1 \supset \llcorner (\llcorner \forall R.C \lrcorner_4 \supset \llcorner \forall R.D \lrcorner_5) \lrcorner_3)\\
&= F \sqcup \forall{\preceq}.\Big(\neg(\llcorner F \sqcup \forall{\preceq}.(F \sqcup \forall R.(\llcorner F \sqcup \forall{\preceq}.(\neg(F \sqcup \forall{\preceq}.C) \sqcup (F \sqcup \forall{\preceq}.D)) \lrcorner_2)) \lrcorner_1)\Big)\\
&\sqcup \Big(\llcorner F \sqcup \forall{\preceq}.(\neg(\llcorner F \sqcup \forall{\preceq}.(F \sqcup \forall R.(F \sqcup \forall{\preceq}.C)) \lrcorner_4)) \sqcup (\llcorner F \sqcup \forall{\preceq}.(F \sqcup \forall R.(F \sqcup \forall{\preceq}.D)) \lrcorner_5) \lrcorner_3\Big)
\end{aligned}$$

The translation of the two other concepts $\mathsf{K}_{\exists R}$ and $\mathsf{FS4/IK4}$ is done analogously. We can use the translation and exploit an existing classical DL reasoner to decide validity or satisfiability of concepts. The translation of the above concepts can be represented as concept definitions in the KRSS-style (Knowledge Representation System Specification) language of the classical reasoner Racer [119; 120] using the following TBox `calc`, where the role `Bi` represents the reflexive and transitive (intuitionistic) accessibility relation $\preceq$ and `F` the distinguished concept to represent $\perp$ (fallible) (see Def. 6.1.2).

TBox

```
(init-tbox calc)
;; intuitionistic accessibility relation
(define-primitive-role Bi)
(transitive Bi)
(reflexive Bi)
 ;; normal accessibility relation
(define-primitive-role R)
```

Then, the above concepts are defined in Racer by `boxK` $=_{df}$ $\tau(\mathsf{K}_{\forall R})$, `diaK` $=_{df}$ $\tau(\mathsf{K}_{\exists R})$ and `fs4ik4` $=_{df}$ $\tau(\mathsf{FS4}/\mathsf{IK4})$ as follows:

Concept Definitions

```
;; translation of axiom boxK
(define-concept boxK
   (or F (all Bi (or
      (not (or F (all Bi (or F (all R (or F (all Bi (or
         (not (or F (all Bi C)))
         (or F (all Bi D))))))))))
      (or F (all Bi (or
         (not (or F (all Bi (or F (all R (or F (all Bi C)))))))
         (or F (all Bi (or F (all R (or F (all Bi D)))))))))))))

;; translation of axiom diaK
(define-concept diaK
   (or F (all Bi (or
      (not (or F (all Bi (or F (all R (or F (all Bi (or
         (not (or F (all Bi C)))
         (or F (all Bi D))))))))))
      (or F (all Bi (or
         (not (or F (all Bi (or F (some R (or F (all Bi C)))))))
         (or F (all Bi (or F (some R (or F (all Bi D)))))))))))))

;; translation of axiom fs4ik4
(define-concept fs4ik4
   (or F (all Bi (or
      (not (or F (all Bi (or F (some R (or
         (or F (all Bi C))
         (or F (all Bi D))))))))
      (or   (or F (all Bi (or F (some R (or F (all Bi C))))))
            (or F (all Bi (or F (some R (or F (all Bi D)))))))))))
```

Validity of these formulæ can be checked in terms of whether the concept $\top$ (`*top*` in Racer) is subsumed by `boxK`, `diaK` and `fs4ik4` respectively. This can be expressed in Racer by the query `(concept-subsumes? D C)` where `D` is the subsumer and `C` the subsumee. This query returns the value `T` if the subsumption holds, and the return value `NIL` expresses non-subsumption. The corresponding output of these queries in Racer is as follows:

Checking Validity

```
? (concept-subsumes? diaK *top*)
> T

? (concept-subsumes? boxK *top*)
> T

? (concept-subsumes? fs4ik4 *top*)
> NIL
```

■

Remark 6.1.1. Proposition 6.1.1 shows that we can obtain a faithful translation of *c*$\mathcal{ALC}$ into a classical description (modal) logic, which is derived from the birelational Kripke semantics of *c*$\mathcal{ALC}$ by a variant of the Gödel translation. However, as also noted by Fairtlough and Mendler [90, p. 20], it is important to express falsity $\bot$ by a distinguished propositional constant F, since otherwise we could construct formulæ which are invalid in *c*$\mathcal{ALC}$, but their translation into $\mathcal{ALC}_{R^*}$ becomes valid. For instance, when taking the naive translation $\tau(\bot) = \bot$ and $\bot$ instead of F in all other cases of Def. 6.1.2, then the formula $\neg\exists R.\bot$ which is not valid in *c*$\mathcal{ALC}$ becomes translated to $\tau(\neg\exists R.\bot) = \bot \sqcup \forall\preceq .(\neg(\bot \sqcup \forall\preceq .(\bot \sqcup \exists R.\bot)) \sqcup \bot)$ which is a theorem of $\mathcal{ALC}_{R^*}$.

Moreover, as also reported in [90, p. 20], it is necessary to restrict Proposition 6.1.1 such that the distinguished propositional constant F is not contained in the *c*$\mathcal{ALC}$ concept C to be translated, since (i) for instance $C =_{df} F \sqcup \neg F$ which is the PEM is not valid in *c*$\mathcal{ALC}$, but its translation $\tau(F \sqcup \neg F) = (F \sqcup \forall\preceq.F) \sqcup (F \sqcup \forall\preceq.(\neg(F \sqcup \forall\preceq F) \sqcup F))$ is valid in $\mathcal{ALC}_{R^*}$; (ii) if $C =_{df} F \supset D$ (as reported in [90, p. 20]) then the translation yields $\tau(F \supset D) = F \sqcup (\forall\preceq.(\neg(F \sqcup \forall\preceq.F) \sqcup (F \sqcup \forall\preceq.D)))$ which is a theorem of $\mathcal{ALC}_{R^*}$.

■

6.1.2 The Complexity of *c*$\mathcal{ALC}$

Considering the complexity, the embedding of *c*$\mathcal{ALC}$ into $\mathcal{ALC}_{R^*}$ shows that deciding satisfiability or subsumption of concepts in *c*$\mathcal{ALC}$ is not worse than deciding the same problems in $\mathcal{ALC}_{R^*}$. Clearly, the translation $\tau(C)$ of a concept C can be obtained in polynomial space.

Let us first focus on the complexity of $\mathcal{ALC}_{R^*}$ ($\mathsf{S4}_n \otimes \mathsf{K}_m$) which is the target for our embedding of $c\mathcal{ALC}$. It has been shown by Halpern and Moses [130] that the complexity of K_m, K_4 and $\mathsf{S4}_n$ is PSPACE-complete. Note that the accessibility relation of K_4 and S4 is transitive, and reflexive and transitive respectively. Consequently, by transferring these results to description logics it has been demonstrated by Sattler [243] that deciding the satisfiability of concepts in $\mathcal{ALC}$ with transitive roles (so-called $\mathcal{ALC}_{R^+}$) is of the same complexity as $\mathcal{ALC}$, namely PSPACE-complete [16]. $\mathcal{ALC}$ with reflexive and transitive roles is an extension of $\mathcal{ALC}_{R^+}$ and corresponds to the fusion of $\mathsf{S4}_n \otimes \mathsf{K}_m$ [103] denoted here by $\mathcal{ALC}_{R^*}$. It is well known that the complexity of K and S4 is preserved under their fusions [103, Thm. 4.19, p. 218] which lets us conclude the PSPACE-completeness of $\mathcal{ALC}_{R^*}$ by Prop. 6.1.1. Therefore, the PSPACE-complexity of $\mathcal{ALC}_{R^*}$ forms an upper bound for the satisfiability of $c\mathcal{ALC}$ concepts.

Corollary 6.1.1 (Upper bound). *Satisfiability and subsumption of concepts in cALC can be decided using polynomial space.* ∇

There are several possible ways to establish a lower bound for the decision problem of $c\mathcal{ALC}$. We use the result by Statman [253] who proved that the problem of deciding whether an intuitionistic propositional formula is valid is PSPACE-complete. It remains to show that $c\mathcal{ALC}$ is a conservative extension of IPC.

Lemma 6.1.3 (Conservativity). *$c\mathcal{ALC}$ is a conservative extension of* IPC. ∇

Proof. The proof is by showing that any consequence of $c\mathcal{ALC}$, which only uses symbols from IPC (*i.e.*, it is free of modalities) and is derived by a sound and complete sequent calculus for $c\mathcal{ALC}$ (G1), is a consequence of IPC as well. This means that every such consequence can be derived without the modal rules of the sequent calculus for $c\mathcal{ALC}$. First, by inspecting the rules of the single conclusion sequent calculus in [196] for $c\mathcal{ALC}$ one observes that the propositional rules correspond to a sequent calculus for IPC like for instance the Gentzen calculus LJ or G3ip [214, Chap. 2]. The same can be observed for the multi conclusion sequent calculus G1, relative to the calculus by Švejdar [257] or Dragalin's GHPC [84, Chap. 1]. Taking into account the sequent calculus G1 for $c\mathcal{ALC}$ one can easily observe its similarity to LJ or the propositional part of GHPC by omitting the sets Θ, Σ, Ψ. Then, one shows by induction on the structure of a derivation and case analysis on the non-modal sequent rules that the premise of a modal-free consequence is free of modalities as well. Hence, a modal-free formula is derivable in G1 iff it is derivable in IPC. □

Since $c\mathcal{ALC}$ is a conservative extension of IPC, we obtain PSPACE-hardness from PSPACE-completeness of IPC.

Theorem 6.1.1 (Complexity w/o TBoxes)**.** *Satisfiability and subsumption of concepts in $c\mathcal{ALC}$ without TBoxes is* PSPACE*-complete.* ∇

Proof. PSPACE-hardness follows from PSPACE-completeness of IPC [253; 256] whereas the upper bound follows by Corollary 6.1.1. □

We conjecture that another possibility to establish the PSPACE-hardness of $c\mathcal{ALC}$ goes by demonstrating that concepts in a special, constructive (double) negation normal form coincide in $\mathcal{ALC}$ and $c\mathcal{ALC}$. Double-negation translation is one approach to embed classical logic into intuitionistic logic, and in the case of CPC it holds that $\models_{\mathsf{CPC}} C$ iff $\models_{\mathsf{IPC}} \neg\neg C$, which is known as Glivenko's Theorem [62; 114, p. 47]. However, the simple double negation translation is not sufficient here since Glivenko's Theorem does not extend to IQC and in particular it does not hold in general for $c\mathcal{ALC}$. For instance $C =_{df} \forall R.(D \sqcup \neg D)$ is a theorem of $\mathcal{ALC}$, but its double negation $\neg\neg C$ is not a theorem of $c\mathcal{ALC}$ (see Example 4.2.8). There exist several *negative* translations of classical logic into intuitionistic logic which also extend to the first-order case. The most well-known ones are due to Kolmogorov, Gödel & Gentzen, Kuroda and Krivine [94; 265, pp. 56 ff.]. We conjecture that we can modify one of these negative translations (cNNF for short) for the case of $\mathcal{ALC}$ and show by induction on the structure of a concept C that it is valid in $\mathcal{ALC}$ if and only if its translation to cNNF is valid in $c\mathcal{ALC}$. For instance, one can translate each $\mathcal{ALC}$ concept with the Gödel–Gentzen translation that puts double negation in front of each atomic concept, disjunction and existential restriction. Further, one can show that all $\mathcal{ALC}$-concepts can be transformed into an equivalent concept in cNNF in linear time and thereof obtain PSPACE-hardness of satisfiability in $c\mathcal{ALC}$ from PSPACE-completeness of $\mathcal{ALC}$. Note that this cNNF normal form is also different from the usual NNF in classic DLs where negation only occurs immediately before atomic concepts.

The general formulation of the embedding of $c\mathcal{ALC}$ into $\mathcal{ALC}_{R^*}$ with respect to TBoxes comes as a consequence from Proposition 6.1.1.

Theorem 6.1.2 (Classic embedding)**.** *Let Θ be a constructive TBox and Γ be a finite set of $c\mathcal{ALC}$-concepts. For every $c\mathcal{ALC}$ concept C,*

$$\Theta; \Gamma \models C \text{ if and only if } \mathcal{ALC}_{R^*}; \Theta'; \Gamma' \models_{cs} \tau(C),$$

where $\Theta' =_{df} \{\tau(D) \mid D \in \Theta\}$ and $\Gamma' =_{df} \{\tau(D) \mid D \in \Gamma\}$. ∇

Proof. Proof by contraposition. ($\Leftarrow$) Suppose that $\Theta; \Gamma \not\models C$, *i.e.*, there exists an interpretation $\mathcal{I}$ and an entity $a \in \Delta^{\mathcal{I}}$ such that $\mathcal{I} \models \Theta, \mathcal{I}; a \models \Gamma$ but $\mathcal{I}; a \not\models C$. Then, Lemma 6.1.2 (6.2) lets us conclude $\tau(\mathcal{I}) \models_{cs} \tau(\Theta)$, $\tau(\mathcal{I}); a \models_{cs} \tau(\Gamma)$ and $\tau(\mathcal{I}); a \not\models_{cs} \tau(C)$ in $\mathcal{ALC}_{R^*}$.

($\Rightarrow$) Analogously to the former direction by Lemma 6.1.2 (6.1). □

Regarding TBoxes, it has been shown for $\mathcal{ALC}$ that subsumption without a TBox as well as with acyclic TBoxes is PSPACE-complete while it increases to EXPTIME-completeness for general TBoxes [16].

EXPTIME-hardness of $c\mathcal{ALC}$ will be established as follows, showing that all valid $\mathcal{ALC}$ concepts C hold in $c\mathcal{ALC}$ as well if we assume PEM and the duality of $\forall R$ and $\exists R$ for the subformulæ of C.

Definition 6.1.5. Let Γ be a finite set of concepts. The *classical forcing* of Γ is denoted by CL(Γ) by taking

$$\begin{aligned}\mathrm{CL}(\Gamma) =_{df} \ & \{D \sqcup \neg D \mid D \in sub(\Gamma)\} \\ & \cup \{\exists R.D \sqcup \forall R.\neg D, \neg\exists R.\bot \mid \exists R.D \in sub(\Gamma)\},\end{aligned}$$

where $sub(\Gamma)$ denotes the set of subformulæ of all concepts in Γ. ∇

Lemma 6.1.4. *For every $\mathcal{ALC}$ concept C and finite set of concepts Γ,*

$$\forall \mathcal{I}_{cs}, \forall x \in \Delta^{\mathcal{I}_{cs}}.\ \mathcal{I}_{cs}; x \models_{cs} C \text{ if and only if } \tau^{\dagger}(\mathcal{I}_{cs}); x \models C, \text{ and} \tag{6.3}$$

$$\begin{aligned}&\forall D \in sub(\Gamma), \forall \mathcal{I}, \forall y \in \Delta^{\mathcal{I}}_{c}. \\ &\quad \mathcal{I}; y \models \mathrm{CL}(\Gamma) \Rightarrow \Big(\mathcal{I}; y \models D \text{ if and only if } \tau^{\ddagger}(\mathcal{I}); y \models_{cs} D \Big),\end{aligned} \tag{6.4}$$

using the translations $\tau^{\dagger}$ and $\tau^{\ddagger}$ by taking

$$\begin{aligned}\tau^{\dagger}(\mathcal{I}_{cs}) &=_{df} (\Delta^{\mathcal{I}_{cs}}, id_{\Delta^{\mathcal{I}_{cs}}}, \emptyset, \cdot^{\mathcal{I}_{cs}}); \\ \tau^{\ddagger}(\mathcal{I}) &=_{df} (\Delta^{\mathcal{I}}_{c}, \cdot^{\mathcal{I}^{\ddagger}}),\end{aligned}$$

where $id_{\Delta^{\mathcal{I}_{cs}}}$ denotes the identity relation over $\Delta^{\mathcal{I}_{cs}}$ and $\cdot^{\mathcal{I}^{\ddagger}}$ is given by

$$\begin{aligned}A^{\ddagger} &=_{df} A^{\mathcal{I}}; \text{ and} \\ R^{\ddagger} &=_{df} \{(x, z) \mid \exists y, z \in \Delta^{\mathcal{I}}_{c}.\ x \preceq^{\mathcal{I}} y R^{\mathcal{I}} z\}.\end{aligned}$$ ∇

Proof. First, one shows analogously to the proof of Lem. 6.1.1 that the translation $\tau^{\dagger}(\mathcal{I}_{cs})$ of a classical model gives a constructive interpretation according to Def. 4.2.2, and that $\tau^{\ddagger}(\mathcal{I})$ yields a classical $\mathcal{ALC}$ interpretation.

The proof of (6.3) is straightforward by induction on the structure of concept C. For (6.4) let $D \in sub(\Gamma)$, $\mathcal{I}$ and $a \in \Delta^{\mathcal{I}}_c$ be arbitrary, and $\mathcal{I}^{\ddagger} =_{df} \tau^{\ddagger}(\mathcal{I})$. Suppose that $\mathcal{I}; a \models \mathrm{CL}(\Gamma)$. We proceed by induction on the structure of concept D. Note that the proof requires the classical forcing of Γ to argue the cases for $\supset, \exists R$ and $\forall R$.

(**Case** A) $\mathcal{I}; a \models A$, *i.e.*, $a \in A^{\mathcal{I}}$ holds iff by the definition of $\tau^{\ddagger}$ it holds that $a \in A^{\mathcal{I}^{\ddagger}}$ iff $\mathcal{I}^{\ddagger}; a \models_{cs} A$.

(**Case** $\bot$) ($\Rightarrow$) $\mathcal{I}; a \models \bot$ does not hold, since by assumption $a \notin \bot^{\mathcal{I}}$.

($\Leftarrow$) $\mathcal{I}^{\ddagger}; a \models_{cs} \bot$ does not hold classically.

(**Case** $C \sqcap D$) $\mathcal{I}; a \models C \sqcap D$ holds iff $\mathcal{I}; a \models C$ and $\mathcal{I}; a \models D$, which by the ind. hyp. hold iff $\mathcal{I}^{\ddagger}; a \models_{cs} C$ and $\mathcal{I}^{\ddagger}; a \models_{cs} D$, and by Def. 4.2.2 iff $\mathcal{I}^{\ddagger}; a \models_{cs} C \sqcap D$.

(**Case** $C \sqcup D$) By ind. hyp. similarly to (**Case** $C \sqcap D$).

(**Case** $C \supset D$) ($\Rightarrow$) Assume that $\mathcal{I}; a \models C \supset D$, *i.e.*, for all $\preceq^{\mathcal{I}}$-refinements a' of a if $a' \in C^{\mathcal{I}}$ then $a' \in D^{\mathcal{I}}$. We have to prove that $\mathcal{I}^{\ddagger}; a \models_{cs} C \supset D$. Suppose that $\mathcal{I}^{\ddagger}; a \models_{cs} C$. It follows from the ind. hyp. that $\mathcal{I}; a \models C$ from which the assumption implies that $\mathcal{I}; a \models D$. Another application of the ind. hyp. yields $\mathcal{I}^{\ddagger}; a \models_{cs} D$. Thus, $\mathcal{I}^{\ddagger}; a \models_{cs} C \supset D$.

($\Leftarrow$) Suppose that $\mathcal{I}^{\ddagger}; a \models_{cs} C \supset D$, *i.e.*, $a \notin C^{\mathcal{I}^{\ddagger}}$ or $a \in D^{\mathcal{I}^{\ddagger}}$. Let $a' \in \Delta^{\mathcal{I}}$ such that $a \preceq^{\mathcal{I}} a'$ and assume that $\mathcal{I}; a' \models C$.

Case 1. If $a' \in \bot^{\mathcal{I}}$ then immediately $a' \in D^{\mathcal{I}}$.

Case 2. Otherwise, $a' \notin \bot^{\mathcal{I}}$. Now, observe that $C \sqcup \neg C \in \mathrm{CL}(\Gamma)$ and by the assumption it holds that $\mathcal{I}; a \models C \sqcup \neg C$. We can conclude from the latter and the assumption $\mathcal{I}; a' \models C$ that $\mathcal{I}; a \models C$, since otherwise $\mathcal{I}; a \models \neg C$ contradicts our assumption. The ind. hyp. lets us conclude that $\mathcal{I}^{\ddagger}; a \models_{cs} C$. Then, it follows by the assumption that $\mathcal{I}^{\ddagger}; a \models_{cs} D$ and the ind. hyp. yields $\mathcal{I}; a \models D$. Now, monotonicity of $\preceq^{\mathcal{I}}$ (Prop. 4.2.2) implies that $\mathcal{I}; a' \models D$.

Hence, $\mathcal{I}; a \models C \supset D$.

(**Case** $\exists R.C$) ($\Rightarrow$) Assume that $\mathcal{I}; a \models \exists R.C$. Moreover, observe that $\neg\exists R.\bot \in \mathrm{CL}(\Gamma)$ by Def. 6.1.5 and $\mathcal{I}; a \models \neg\exists R.\bot$ by assumption. This means that there exists $a' \in \Delta^{\mathcal{I}}$ with $a \preceq^{\mathcal{I}} a'$ such that all its $R^{\mathcal{I}}$-successors are infallible. Monotonicity of refinement (Prop. 4.2.2) implies $\mathcal{I}; a' \models \exists R.C$, *i.e.*, by reflexivity of $\preceq^{\mathcal{I}}$ holds $a' \preceq^{\mathcal{I}} a'$ and there exists $b \in \Delta^{\mathcal{I}}$ with $a' \, R^{\mathcal{I}} \, b$ such that $b \in C^{\mathcal{I}}$. Now, observe that $b \notin \bot^{\mathcal{I}}$ by assumption and $a' \notin \bot^{\mathcal{I}}$ by Prop. 4.2.1. Then, the definition of $\tau^{\ddagger}$ implies that $a \, R^{\mathcal{I}^{\ddagger}} \, b$. At this point we can apply the ind. hyp. to obtain $\mathcal{I}^{\ddagger}; b \models_{cs} C$. Hence, $\mathcal{I}^{\ddagger}; a \models_{cs} \exists R.C$.

($\Leftarrow$) Assume that $\mathcal{I}^{\ddagger}; a \models_{cs} \exists R.C$, *i.e.*, $\exists b \in \Delta^{\mathcal{I}^{\ddagger}}$ such that $a\, R^{\mathcal{I}^{\ddagger}}\, b$ and $b \in C^{\mathcal{I}^{\ddagger}}$. By the definition of $\tau^{\ddagger}$ it holds that there exist $a', b \in \Delta_c^{\mathcal{I}}$ such that $a \preceq^{\mathcal{I}} a'\, R^{\mathcal{I}}\, b$ and the ind. hyp. implies $b \in C^{\mathcal{I}}$. Since, $\exists R.C \sqcup \forall R.\neg C \in \mathrm{CL}(\Gamma)$ by Def. 6.1.5 it holds by assumption that $\mathcal{I}; a \models \exists R.C \sqcup \forall R.\neg C$. If $\mathcal{I}; a \models \forall R.\neg C$ then this contradicts the assumption that $b \in C^{\mathcal{I}}$. Thus, $\mathcal{I}; a \models \exists R.C$.

(**Case** $\forall R.C$) ($\Rightarrow$) Suppose $\mathcal{I}; a \models \forall R.C$, *i.e.*, $\forall a', b \in \Delta^{\mathcal{I}}.\ a \preceq^{\mathcal{I}} a' R^{\mathcal{I}} b$ implies $b \in C^{\mathcal{I}}$. Let $b \in \mathcal{I}^{\ddagger}$ such that $a\, R^{\mathcal{I}^{\ddagger}}\, b$. The definition of $\tau^{\ddagger}$ implies that $b \in \Delta_c^{\mathcal{I}}$ and there exists $a' \in \Delta_c^{\mathcal{I}}$ such that $a \preceq^{\mathcal{I}} a'\, R^{\mathcal{I}}\, b$, and the assumption implies that $b \in C^{\mathcal{I}}$. Now, the ind. hyp. lets us conclude that $b \in C^{\mathcal{I}^{\ddagger}}$. Hence, $\mathcal{I}^{\ddagger}; a \models_{cs} \forall R.C$.

($\Leftarrow$) Proof by contraposition. Let us assume that $\mathcal{I}; a \not\models \forall R.C$, *i.e.*, there exist $a', b \in \Delta_c^{\mathcal{I}}$ such that $a \preceq^{\mathcal{I}} a'\, R^{\mathcal{I}}\, b$ and $\mathcal{I}; b \not\models C$. The definition of $\tau^{\ddagger}$ implies $a\, R^{\mathcal{I}^{\ddagger}}\, b$ and the ind. hyp. lets us conclude that $\mathcal{I}^{\ddagger}; b \models_{cs} C$. Hence, $\mathcal{I}^{\ddagger}; a \not\models_{cs} \forall R.C$. ☐

We obtain from (6.3) the following:

Corollary 6.1.2. *For all finite sets of concepts* Γ,

$$\forall \mathcal{I}_{cs}, \forall x \in \Delta^{\mathcal{I}_{cs}}.\ \tau^{\dagger}(\mathcal{I}_{cs}); x \models \mathrm{CL}(\Gamma). \qquad \nabla$$

Proposition 6.1.2. *For every* $\mathcal{ALC}$ *concept* C,

$$\mathcal{ALC}; \emptyset; \emptyset \models_{cs} C \quad \textit{if and only if} \quad \mathrm{CL}(\{C\}); \emptyset \models C. \qquad \nabla$$

Proof. ($\Rightarrow$) Suppose that $\mathcal{ALC}; \emptyset; \emptyset \models_{cs} C$. Let $\mathcal{I}$ be a constructive interpretation, $a \in \Delta_c^{\mathcal{I}}$ and suppose that $\mathcal{I} \models \mathrm{CL}(\{C\})$. The translation $\tau^{\ddagger}(\mathcal{I})$ yields a classical interpretation. The assumption implies $\tau^{\ddagger}(\mathcal{I}); a \models_{cs} C$. Then, Lem. 6.1.4 (6.4) yields $\mathcal{I}; a \models C$. Thus, $\mathrm{CL}(C); \emptyset \models C$.

($\Leftarrow$) Assume that $\mathrm{CL}(\{C\}); \emptyset \models C$. Let $\mathcal{I}_{cs}$ be an $\mathcal{ALC}$ interpretation and $a \in \Delta^{\mathcal{I}_{cs}}$. The translation $\tau^{\dagger}(\mathcal{I}_{cs})$ yields a constructive interpretation. By Cor. 6.1.2 follows that $\tau^{\dagger}(\mathcal{I}_{cs}); a \models \mathrm{CL}(\{C\})$. Then, the assumption implies $\tau^{\dagger}(\mathcal{I}_{cs}); a \models C$. Now we can apply Lem. 6.1.4 (6.3) to obtain $\mathcal{I}_{cs}; a \models_{cs} C$ as desired. Hence, $\mathcal{ALC}; \emptyset; \emptyset \models_{cs} C$. ☐

Theorem 6.1.3. *Let* Θ *be a* $\mathcal{ALC}$ *TBox and* Γ *be a finite set of* $\mathcal{ALC}$*-concepts. For every* $\mathcal{ALC}$ *concept* C,

$$\mathcal{ALC}; \Theta; \Gamma \models_{cs} C \textit{ if and only if } \Theta \cup \mathrm{CL}(\Theta \cup \Gamma \cup \{C\}); \Gamma \models C. \qquad \nabla$$

Proof. Analogously to the proof of Thm. 6.1.2, but relying on Lem. 6.1.4. ☐

Theorem 6.1.4 (Complexity w.r.t. general TBoxes)**.** *Satisfiability and subsumption of concepts in $c\mathcal{ALC}$ w.r.t. general TBoxes is* ExpTime-complete. ∇

Proof. ExpTime-completeness is a consequence of Thm. 6.1.3 and Thm. 6.1.2. $\square$

While ExpTime-completeness w.r.t. general TBoxes transfers to $c\mathcal{ALC}$ as a consequence of Thm. 6.1.2 and Thm. 6.1.3, the PSpace-completeness w.r.t. simple TBoxes does not carry over relatively to the embedding of $c\mathcal{ALC}$ into $\mathcal{ALC}_{R^*}$. This is due to the translation of a constructive TBox into an $\mathcal{ALC}_{R^*}$ TBox. In particular, the application of the translation defined by Def. 6.1.2 to a simple $c\mathcal{ALC}$ TBox does not yield a simple $\mathcal{ALC}_{R^*}$ TBox but rather a special form of a general TBox. We conjecture that we can alter the translation of Def. 6.1.2 to

$$\begin{aligned}
\tau(A) &= \forall{\preceq}.(F \sqcup A);\\
\tau(\bot) &= \forall{\preceq}.F;\\
\tau(C \supset D) &= \forall{\preceq}.(\tau(C) \supset \tau(D));\\
\tau(\neg C) &= \tau(C \supset \bot);\\
\tau(C \odot D) &= \tau(C) \odot \tau(D), \text{ where } \odot \in \{\sqcap, \sqcup\};\\
\tau(\exists R.C) &= \forall{\preceq}.(\exists R.\tau(C));\\
\tau(\forall R.C) &= \forall{\preceq}.(\forall R.\tau(C));
\end{aligned}$$

while still preserving validity w.r.t. the embedding into $\mathcal{ALC}_{R^*}$. The translation above corresponds to an extension of the embedding of PLL into $\mathsf{S4} \otimes \mathsf{S4}$ as reported by Fairtlough and Mendler [90, p. 19]. This translation uses the non-trivial translation of fallible $\bot$ into $\forall{\preceq}.F$. Under this view and the fact that TBox axioms hold at all worlds, the translation of a simple TBox axiom $A \supset C$ is as follows:

$$\tau(A \supset C) = \tau(A) \supset \tau(C).$$

This means, that a simple (constructive) TBox Θ containing only simple axioms of the form $A_1 \supset C_1, \ldots, A_n \supset C_n$ is translated into a classical TBox containing axioms where the left side is boxed by $\forall{\preceq}$, *i.e.*, $\forall{\preceq}.(F \sqcup A_1) \supset \tau(C_1), \ldots, \forall{\preceq}.(F \sqcup A_n) \supset \tau(C_n)$. We leave it as an open problem whether reasoning relative to this special form of a *left-universal restricted simple* TBox remains PSpace-complete for classical $\mathcal{ALC}$.

6.1.3 Decidability and Finite Model Property

The embedding of $c\mathcal{ALC}$ into the fusion ($\mathsf{S4}_n \otimes \mathsf{K}_m$) allows to transfer further results to $c\mathcal{ALC}$. Gabbay et al. [103, Chapter 4] show that Kripke completeness, the finite model property (fmp), decidability and the interpolation property are preserved under fusions of modal logics. We exhibit this result to show that the embedding of $c\mathcal{ALC}$ into $\mathcal{ALC}_{R^*}$ yields a further opportunity to establish decidability and the finite model property for $c\mathcal{ALC}$. The decidability of $c\mathcal{ALC}$ is a direct consequence of Thm. 6.1.2.

Theorem 6.1.5 (Decidability). *$c\mathcal{ALC}$ is decidable.* ∇

Proof. From decidability of K_m and $\mathsf{S4}_n$ [130], and the result by Gabbay et al. [103, Thm. 4.12], saying that decidability is preserved under fusions of multimodal logics, we can conclude the decidability of $\mathcal{ALC}_{R^*} = (\mathsf{S4}_n \otimes \mathsf{K}_m)$. Then, decidability of $c\mathcal{ALC}$ follows from the embedding into $\mathcal{ALC}_{R^*}$ by Theorem 6.1.2. $\square$

Theorem 6.1.6 (Finite model property). *$c\mathcal{ALC}$ has the* finite model property*, i.e., $\models C$ if and only if $\models C$ in all finite interpretations.* ∇

Proof. According to Gabbay et al. [103, Thms. 1.21, 1.25, 1.26] the logics K and $\mathsf{S4}$ have the finite model property and it is preserved under their fusion [103, Thm. 4.2]. Therefore, we can conclude that K_m and $\mathsf{S4}_n$ have the fmp, where K_m is the m-fusion of K and $\mathsf{S4}_n$ the n-fusion of $\mathsf{S4}$ respectively. According to this construction, the fusion $(\mathsf{S4}_n \otimes \mathsf{K}_m) = \mathcal{ALC}_{R^*}$ has the fmp as well. We proceed by contraposition.

($\Rightarrow$) Suppose $\not\models C$, *i.e.*, there exists a pair $(\mathcal{I}, a)$ with $a \in \Delta^{\mathcal{I}}$ such that $\mathcal{I}; a \not\models C$. From Lem. 6.1.2 it follows that $\tau(\mathcal{I}); a \not\models_{cs} \tau(C)$ in $\mathcal{ALC}_{R^*}$. Since $\mathcal{ALC}_{R^*}$ has the fmp, the former also holds in a finite model $\mathcal{I}_{cs}$. Def. 6.1.4 lets us translate the finite model $\mathcal{I}_{cs}$ to a finite $c\mathcal{ALC}$ model $\tau'(\mathcal{I}_{cs})$. Applying Lem. 6.1.2 yields $\not\models C$ in the finite interpretation $\tau'(\mathcal{I}_{cs})$, observing that $\tau'(\mathcal{I}_{cs})$ preserves finiteness.

($\Leftarrow$) If $\mathcal{I} \not\models C$ holds in some finite interpretation $\mathcal{I}$ then obviously $\not\models C$. $\square$

The fmp w.r.t. TBoxes corresponds to the fmp w.r.t. the global consequence relation. It can be obtained as an extension of Thm. 6.1.6 by utilising the existing results for the systems K and $\mathsf{S4}$ [103, page 34 ff.], which that state that their global consequence relation is determined by finite frames and the internalisation of their global consequence relation via universal modalities $\boxed{\forall}$, $\Diamond\!\!\!\!\exists$ has the fmp as well, where $\boxed{\forall}$ denotes '[. . .] everywhere in the model [and $\Diamond\!\!\!\!\exists$ stands for] somewhere in the model [. . .]' [103, p. 37]. For a n-modal logic L this is by taking the $n+1$-modal logic $L_U =_{df} L \oplus \{$axioms of $\mathsf{S5}$ for $\boxed{\forall}, \Diamond\!\!\!\!\exists\} \oplus \{\boxed{\forall} C \supset \Box_i C \mid 1 \leq i \leq n\}$, according to [103, p. 38].

Then, one shows that $c\mathcal{ALC}$ with a TBox Θ can be embedded into the bimodal system $(\mathsf{S4} \otimes \mathsf{K}_m)_U$ where the axioms of the TBox are translated and conjoined [16, pp. 335 f.] to a single axiom $\hat{\Theta} =_{df} \bigwedge_{D \in \Theta} \tau(D)$ and internalised via the universal modalities of $(\mathsf{S4} \otimes \mathsf{K}_m)_U$ to demonstrate that $(\mathsf{S4} \otimes \mathsf{K}_m)_U; \emptyset; \emptyset \models \boxed{\forall} \hat{\Theta} \supset \tau(C)$ iff $c\mathcal{ALC}; \Theta; \emptyset \models C$.

6.2 The Fragment $\mathcal{UL}$

This section summarises our result from [192] showing that the complexity of subsumption checking in $\mathcal{UL}$ is more efficient (tractable) if the constructive semantics are adopted.

6.2.1 Introduction to Tractable DLs

In recent years, besides the research on very expressive DLs, there has been an increasing interest in lightweight DLs with limited expressive power. Several useful fragments with tractable reasoning problems, even in the presence of general TBoxes, have been identified [14; 15; 18–20; 22; 47; 48; 121; 122]. Lightweight DLs have applications in large scale ontologies, in life sciences like SNOMED CT [67; 152] and bio-medical domains [18]. The key reasoning task in DLs is to decide whether a concept is subsumed by another concept w.r.t. a TBox. It is well-known [16] that the complexity of deciding the latter inference problem depends on (i) the expressivity of the language, (ii) the structure of the TBox formalism used, *i.e.*, whether the TBox allows for acyclic, cyclic or even general concept inclusion axioms (GCIs), and (iii) the semantic interpretation of the logical connectives. Because some applications which are dealing with large scale ontologies do not require the expressivity of $\mathcal{ALC}$ or even more expressive DLs like OWL DL, but focus on efficiency of reasoning and therefore require tractable decision problems, restricted languages below $\mathcal{ALC}$ have been considered. The research in DLs focussed in the past mainly on two strands among these so-called *sub-Boolean* formalisms, that is,

(i) the family of $\mathcal{FL}$-type languages which starts from the fragment $\mathcal{FL}_0$ consisting of $\{\forall, \sqcap\}$, and

(ii) the family of $\mathcal{EL}$-type languages which is based on the operator set $\{\exists, \sqcap\}$.

For $\mathcal{FL}_0$ it is known that subsumption checking is PTime for empty TBoxes [174], it becomes coNP-complete for acyclic TBoxes [212], while for cyclic TBoxes it is PSpace-complete under the descriptive semantics [154] as well as greatest and least fixed point semantics [20]. However, it increases to ExpTime-complete in the presence of GCI's under the descriptive semantics [14; 143].

Example 6.2.1 ([112]). Even restricted languages like $\mathcal{FL}_0$ are capable of expressing interesting and useful concept definitions involving cycles, *e.g.*, one can represent in $\mathcal{FL}_0$ the concept of a finite acyclic directed graph by the following expression [112]:

$$\mathsf{Dag} \equiv \mathsf{Node} \sqcap \forall \mathsf{arc}.\mathsf{Dag}. \qquad \blacksquare$$

On the other side, it has been shown for the language $\mathcal{EL}$ that the subsumption checking problem w.r.t. both cyclic and acyclic TBoxes is tractable and remains PTIME under the descriptive semantics as well as greatest and least fixed point semantics [22]. Moreover, this even holds under the descriptive semantics if general TBoxes are admitted [47; 143]. In general, the $\mathcal{EL}$-family can be characterised as the fragment that is behaving more efficiently complexity-wise, and this is even preserved for a larger set of language extension, in contrast to the $\mathcal{FL}$-family. The tractability of $\mathcal{EL}$ and of its extensions has been investigated in detail in [14; 15; 47] and in particular there exist several extensions of $\mathcal{EL}$, which remain tractable, such as adding the constants $\bot$, $\top$, nominals, concrete domains and more [15]. The addition of disjunction $\sqcup$ to $\mathcal{EL}$ known as $\mathcal{ELU}$, which brings back Boolean expressiveness, increases the complexity of subsumption to coNP-hardness [47] for empty TBoxes, PSPACE for acyclic and EXPTIME-completeness for cyclic and general TBoxes [14]. See [122] for an overview of the $\mathcal{EL}$ family and [18] for a discussion on the practicability of $\mathcal{EL}$ and of its extension $\mathcal{EL}^+$ in the context of bio-medical applications.

The investigation of tractable fragments below $\mathcal{ALC}$ seems to be reasonable, in particular for applications that demand for large-scale ontologies, involve mass data or interact with real-time sensor data. But also from a theoretical viewpoint it is expedient to further examine sub-Boolean languages below $\mathcal{ALC}$, since there is still some undiscovered territory left to be explored, which has been motivated by Mendler and Scheele [192] as follows:

> ‘ On the one hand, the existing fragments $\mathcal{FL}$ and $\mathcal{EL}$ represent only two of the four corners of the Aristotelian classification square [145, Chap. 1]: $\mathcal{FL}_0$ with $\{\forall, \sqcap\}$ permits us to make general statements of the form *“all S are P”* while $\mathcal{EL}$ with $\{\exists, \sqcap\}$ corresponds[16] to *“some S are P”*. Less attention has been given to the so-called contraries *“no S is P”* and *“not all S are P”* which correspond to fragments $\{\forall, \sqcup\}$ and $\{\exists, \sqcup\}$. Are these also useful as a basis in specific applications and if so what are their complexities? On the other hand, there is the semantics issue: The standard descriptive semantics which follows a classical Tarskian model-theory is not the only reasonable way of interpreting concept description languages. There is the Scottian least fixed or greatest fixed point view for cyclic TBoxes introduced by Nebel [211] or the automata-theoretic interpretation of Baader [20]. Also, the concept algebras introduced by Dionne et al. [81; 82] provide alternative ways of giving intensional semantics to concept descriptions and TBoxes.

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[16] To see this consider conjunction $\sqcap$ as representing generic affirmative statements with $\top$ as nullary case and disjunction $\sqcup$ as a generic refutative statement with $\bot$ as the degenerated case. Of course, the refutation about $\sqcup$ consists in giving choices, thus avoiding commitment.[192, p. 2]

Depending on application and language fragment some of these may be more appropriate than the classical descriptive semantics. The semantics issue, too, leaves room for further systematic investigations. ' [192, p. 2]

6.2.2 The Language $\mathcal{UL}$

The following section is devoted to the $\{\exists, \sqcup\}$ fragment of $\mathcal{ALC}$ which is called $\mathcal{UL}$. It is inspired by Hofmann's approach [143] who rephrases Baader's [21] and Brandt's [47] results on PTime for $\mathcal{EL}$ in terms of Gentzen proof systems.

Concept descriptions of this class are formed according to the following definition.

Definition 6.2.1 (Constructive $\mathcal{UL}$ [192, p. 5]). The set of well-formed *concept descriptions* C, D over signature $\Sigma = (N_C, N_R)$ consisting of two denumerably infinite and pairwise disjoint alphabets of concept names N_C and role names N_R, is defined inductively by the following grammar, where $A \in N_C$ and $R \in N_R$.

$$C, D ::= \quad A \mid \top \mid \bot \mid C \sqcup D \mid \exists R.C. \qquad \nabla$$

We remark, that in this fragment it is possible to include unconditional negative and positive axioms $\neg C$ and C in TBoxes just like in the classical setting. This can be implemented by using TBox axioms of the form $C \supset \bot$ and $\top \supset C$. For the remaining section we will consider TBoxes Θ that only include subsumptions of the form $C \supset D$ where both C and D are $\supset$-free. The PTime result will hold for the *guarded* fragment of $\mathcal{UL}$, which is according to the following definition.

Definition 6.2.2 (Existentially guarded concept [192, p. 5]). A concept E is *existentially guarded* if it is generated by the following grammar over signature $\Sigma = (N_C, N_R)$:

$$E ::= \quad A \mid \top \mid \bot \mid \exists R.C,$$

where $A \in N_C$, $R \in N_R$ and C is an $\mathcal{UL}$ concept. A general $\mathcal{UL}$ TBox Θ is *existentially guarded* if the conclusion E of all axioms $C \supset E \in \Theta$ is existentially guarded. ∇

Intuitively, an existentially guarded concept E is restricted such that all disjunctions appearing in E are guarded behind existential quantifiers.

6.2.3 Existentially Guarded $\mathcal{UL}_0$ is ExpTime-hard for Classical Descriptive Semantics

It has been proven by Mendler and Scheele [192] that the subsumption checking problem in $\mathcal{FL}_0$ w.r.t. the classical semantics can be reduced in linear time to the problem of subsumption checking in $\mathcal{UL}_0$, where $\mathcal{UL}_0$ denotes the fragment of $\mathcal{UL}$ without the constants $\top, \bot$ and relative to guarded $\mathcal{UL}_0$ TBoxes. The language $\mathcal{FL}_0$ consists only of the operators $\{\forall, \sqcap\}$ and allows for expressing concept conjunction and universal restriction. $\mathcal{FL}_0$ concepts C over signature $\Sigma = (N_C, N_R)$ are generated by the grammar

$$C ::= \quad A \mid C \sqcap D \mid \forall R.C,$$

where $A \in N_C$ and $R \in N_R$. The following proof is based on the observation that $\mathcal{FL}_0$ can be seen as the dual of $\mathcal{UL}_0$.

Theorem 6.2.1 ([192, Thm. 1]). *$\mathcal{UL}_0$ subsumption checking under the classical semantics and relative to existentially guarded (general) TBoxes is* ExpTime-hard. ∇

Proof. (Mendler and Scheele [192, p. 5 f.]) The proof reduces the problem of subsumption checking in $\mathcal{FL}_0$ to that in $\mathcal{UL}_0$ by using a dualisation. This is by taking the dual $d(C)$ of $\mathcal{FL}_0$ concepts C and TBox axioms, and a (linear) expansion $exp(\Theta)$ of $\mathcal{FL}_0$ TBoxes Θ. The former is by replacing $\sqcap \mapsto \sqcup$ and $\forall \mapsto \exists$ and by swapping left and right-hand sides of subsumptions. This means that all TBox axioms are existentially guarded behind an additional role. One can show that for all $\mathcal{FL}_0$ concepts C, D and general TBoxes Θ of $\mathcal{FL}_0$ it holds that $\Theta; C \models_{cs} D$ iff $exp(d(\Theta)); d(D) \models_{cs} d(C)$. Then, ExpTime-hardness of subsumption checking in $\mathcal{UL}_0$ relative to existentially guarded TBoxes under the classical semantics follows from the fact that subsumption checking in $\mathcal{FL}_0$ w.r.t. general TBoxes and relative to the classical semantics is ExpTime-hard [14; 143]. $\square$

6.2.4 Existentially Guarded $\mathcal{UL}$ is in PTime for Constructive Descriptive Semantics

In [192] it was demonstrated that subsumption checking in $\mathcal{UL}$ under the constructive semantics and relative to existentially guarded TBoxes is in PTime. A PTime decision procedure for $\mathcal{UL}$ can be obtained by pruning the constructive Gentzen-style sequent calculus for $c\mathcal{ALC}$ (see Chap. 5.2) to fit the $\mathcal{UL}$ fragment. In particular, for existentially guarded $\mathcal{UL}$ the sequent calculus $\mathsf{G1}$ for $c\mathcal{ALC}$ can be restricted to simple sequents of the form $\Theta; C \vdash D$ without losing completeness. Such simple sequents correspond to $\mathsf{G1}$ sequents $\Theta\,;\,\Sigma\,;\,\Gamma \vdash_{\mathsf{G1}} \Phi\,;\,\Psi$, where $\bigcup_{R\in N_R}\Sigma(R)=\emptyset$ and $|\Gamma| = |\Phi\cup\Psi| = 1$.

The restricted calculus $\mathsf{G1}_{\mathcal{UL}}$ is presented by the rules in Fig. 6.1. Like the sequent system $\mathsf{G1}$ in Chap. 5.2, the calculus $\mathsf{G1}_{\mathcal{UL}}$ is formulated in the style of Gentzen with left introduction rules $\sqcup L$, $\supset L$, $\exists LR$, $\bot L$ and right introduction rules $\sqcup R_1$, $\sqcup R_2$, $\exists LR$, $\top R$ for each logical connective of $\mathcal{UL}$.

$$\dfrac{}{\Theta\,;\,C \vdash C}\,Ax \qquad \dfrac{}{\Theta\,;\,\bot \vdash C}\,\bot L \qquad \dfrac{}{\Theta\,;\,C \vdash \top}\,\top R$$

$$\dfrac{\Theta\,;\,E \vdash C}{\Theta\,;\,E \vdash C\sqcup D}\,\sqcup R_1 \qquad \dfrac{\Theta\,;\,E \vdash D}{\Theta\,;\,E \vdash C\sqcup D}\,\sqcup R_2 \qquad \dfrac{\Theta\,;\,C \vdash E \qquad \Theta\,;\,D \vdash E}{\Theta\,;\,C\sqcup D \vdash E}\,\sqcup L$$

$$\dfrac{\Theta\,;\,E \vdash C \qquad \Theta\,;\,D \vdash F}{\Theta,\,C\supset D\,;\,E \vdash F}\,\supset L \qquad \dfrac{\Theta\,;\,C \vdash D}{\Theta\,;\,\exists R.C \vdash \exists R.D}\,\exists LR$$

Figure 6.1: Gentzen sequent rules for $\mathcal{UL}$ [192, p. 7].

Note that the rule $\exists LR$ combines the left and right introduction rules for $\exists R$ from $\mathsf{G1}$. This is possible because of the following properties which can easily be observed from the simple representation of the rules $\exists L$ and $\exists R$ as depicted by Fig. 6.2: (i) The mappings of the form $[R \mapsto C]$ are never present at the left-hand side of a sequent, since due to the absence of universal restriction (operator $\forall R$) in $\mathcal{UL}$ there is no corresponding introduction rule. (ii) On the right-hand side, a mapping of the form $[R \mapsto C]$ can only be introduced by the (simple) $\mathsf{G1}$ rules $\bot L$ and $\exists L$, and only be used by rule $\exists R$.

$$\dfrac{\Theta\,;\,\emptyset\,;\,C \vdash_{\mathsf{G1}} D\,;\,\emptyset}{\Theta\,;\,\emptyset\,;\,\exists R.C \vdash_{\mathsf{G1}} \emptyset\,;\,[R\mapsto D]}\,\exists L \qquad \dfrac{\Theta\,;\,\emptyset\,;\,C \vdash_{\mathsf{G1}} \emptyset\,;\,[R\mapsto D]}{\Theta\,;\,\emptyset\,;\,C \vdash_{\mathsf{G1}} \exists R.D\,;\,\emptyset}\,\exists R$$

Figure 6.2: $\exists L$ and $\exists R$ as simple rules.

The following definition restricts the notion of a *satisfiable sequent* (Def. 5.2.2) to (simple) $\mathcal{UL}$ sequents.

Definition 6.2.3 (Constructive satisfiability [192, p. 6] of $\mathcal{UL}$ sequents)**.** Satisfiability of $\mathcal{UL}$ sequents is phrased in terms of Def. 5.2.2 by viewing an $\mathcal{UL}$ sequent $\Theta; C \vdash_{\mathsf{G1}_{\mathcal{UL}}} D$ as an abbreviation for the $\mathsf{G1}$ sequent $\Theta\,;\,\emptyset\,;\,\{C\} \vdash_{\mathsf{G1}} \{D\}\,;\,\emptyset$. Remember, that $\Theta; C \models D$ expresses that concept C is subsumed by concept D w.r.t. the TBox Θ (see Lem. 4.2.3). Then, the statement that the pair $(\mathcal{I}, a)$ satisfies the sequent $\Theta; C \vdash D$ is equivalent to expressing non-subsumption, *i.e.*, $\Theta; C \not\models D$. Analogously to Def. 5.2.2, we will write $\Theta; C \not\models D$ to denote that the sequent $\Theta; C \vdash D$ is satisfiable. ∇

Proposition 6.2.1 (Cut admissibility [192, p. 7])**.** *In the proof system of Fig. 6.1 the cut rule is admissible,* i.e.*, if* $\Theta; C \vdash D$ *and* $\Theta; D \vdash E$ *then* $\Theta; C \vdash E$*.* ∇

Proof. The proof was not included in [192]. Given derivations π_1 of $\Theta; C \vdash D$ and π_2 of $\Theta; D \vdash E$ we show how to transform π_1 and π_2 to a derivation π_3 of $\Theta; C \vdash E$. The proof is by induction on the structure of the cut formula D and the derivations π_1 and π_2 and uses the following left weakening rules (global and local), which are easily proven to be admissible by induction on derivations:

$$\frac{\Theta\,;\,C \vdash D}{\Theta,\,E\,;\,C \vdash D}\,wL_g \qquad \frac{\Theta\,;\,\top \vdash D}{\Theta\,;\,C \vdash D}\,wL_l$$

The argument is standard and the more interesting cases are the principal cuts, *i.e.*, the cases where the cut formula D is concluded by a right rule for one of the operators $\top$, $\bot$, $\sqcup$, $\exists$ in π_1 and used by a left rule in π_2 for the corresponding operator.

Case 1. π_1 is either an axiom, a conclusion of $\bot L$ or ends in $\top R$.

Subcase 1.1. If π_1 is an axiom then $D = C$.

$$\pi_1 = \frac{}{\Theta\,;\,C \vdash C}\,Ax$$

Then, $\pi_2 = \Theta; C \vdash E$.

Subcase 1.2. π_1 is ends in $\bot L$.

$$\pi_1 = \frac{}{\Theta\,;\,\bot \vdash D}\,\bot L$$

The derivation π_3 can be concluded by $\bot L$.

Subcase 1.3. If π_1 ends in $\top R$ then $D = \top$.

$$\pi_1 = \frac{}{\Theta\,;\,C \vdash \top}\,\bot L$$

This implies that $\pi_2 = \Theta; \top \vdash E$ and we obtain the desired derivation $\Theta; C \vdash E$ from π_2 by left-weakening wL_l.

Case 2. π_2 is an axiom, a conclusion of $\bot L$ or ends in $\top R$.

Subcase 2.1. π_2 is an axiom, *i.e.*, $D = E$:

$$\pi_2 = \frac{}{\Theta\,;\,E \vdash E}\,Ax$$

In this case, $\pi_1 = \Theta; C \vdash E$.

Subcase 2.2. π_2 ends in $\bot L$, *i.e.*, $D = \bot$:

$$\pi_2 = \frac{}{\Theta\,;\,\bot \vdash E}\,\bot L$$

Either π_1 is an axiom and $\Theta;\bot\vdash E$ follows by $\bot L$, or $\pi_1 = \Theta;C\vdash\bot$ is derived by one of the left rules $\bot L$, $\sqcup L$ or $\supset L$. Note that the three cases with $D = \bot$ correspond to special cases of the transformations given below.

(i) If the left rule is $\bot L$ then $\pi_1 = \Theta;\bot\vdash\bot$ and the derivation of π_3 is an immediate consequence by rule $\bot L$, just like before.

(ii) If the left rule is $\sqcup L$ then π_1 looks like

$$\pi_1 = \frac{\overset{\pi_{11}}{\Theta\,;\,C_1 \vdash \bot} \qquad \overset{\pi_{12}}{\Theta\,;\,C_2 \vdash \bot}}{\Theta\,;\,C_1 \sqcup C_2 \vdash \bot}\,\sqcup L$$

and the induction hypothesis lets us cut π_{11} and π_{12} with π_2 to get $\Theta;C_1\vdash E$ and $\Theta;C_2\vdash E$ from which we obtain $\Theta;C_1 \sqcup C_2\vdash E$ by rule $\sqcup L$.

(iii) If the left rule is $\supset L$ then π_1 looks like

$$\pi_1 = \frac{\overset{\pi_{11}}{\Theta\,;\,C \vdash F} \qquad \overset{\pi_{12}}{\Theta\,;\,G \vdash \bot}}{\Theta,\,F \supset G\,;\,C \vdash \bot}\,\supset L$$

By induction hypothesis we can cut π_{12} with π_2 to get $\Theta;G\vdash E$, and then obtain π_3 from the latter and π_{11} by rule $\supset L$.

Subcase 2.3. π_2 ends in $\top R$ and $E = \top$.

$$\pi_2 = \frac{}{\Theta\,;\,D \vdash \top}\,\bot L$$

Then, the desired derivation $\Theta;C\vdash\top$ is an immediate conclusion of $\top R$.

Neither premise π_1, π_2 is an axiom, a conclusion of $\bot L$ nor ends in $\top R$:

Case 3. The cut formula D is *principal* in π_1 and π_2. These are the interesting cases where π_1 is concluded by a right rule and used by a left rule in π_2. We have the following subcases:

Subcase 3.1. $D = D_1 \sqcup D_2$, π_1 ends in $\sqcup R_1$ and π_2 in $\sqcup L$.

$$\pi_1 = \frac{\overset{\pi_{11}}{\Theta\,;\,C \vdash D_1}}{\Theta\,;\,C \vdash D_1 \sqcup D_2}\,\sqcup R_1 \qquad \pi_2 = \frac{\overset{\pi_{21}}{\Theta\,;\,D_1 \vdash E} \qquad \overset{\pi_{22}}{\Theta\,;\,D_2 \vdash E}}{\Theta\,;\,D_1 \sqcup D_2 \vdash E}\,\sqcup L$$

By the induction hypothesis, we can cut π_{11} with π_{21} to obtain $\Theta;C\vdash E$. The argument runs analogously if π_1 ends in $\sqcup R_2$ and π_2 in $\sqcup L$, where we can cut by induction hypothesis π_{11} with π_{22} to get derivation $\Theta;C\vdash E$.

Subcase 3.2. $D = \exists R.D_1$, and π_1 ends in $\exists LR$, *i.e.*, $C = \exists R.C_1$:

$$\pi_1 = \cfrac{\Theta\,;\, \overset{\pi_{11}}{C_1 \vdash D_1}}{\Theta\,;\, \exists R.C_1 \vdash \exists R.D_1}\,\exists LR$$

Then, we only have to consider the case that π_2 is a conclusion of rule $\exists LR$, *i.e.*, $E = \exists R.E_1$ and

$$\pi_2 = \cfrac{\Theta\,;\, \overset{\pi_{21}}{D_1 \vdash E_1}}{\Theta\,;\, \exists R.D_1 \vdash \exists R.E_1}\,\exists LR$$

The induction hypothesis lets us cut π_{11} with π_{21} to get $\Theta; C_1 \vdash E_1$, from which we obtain $\Theta; \exists R.C_1 \vdash \exists R.E_1$ by rule $\exists LR$.

Case 4. The cut formula D is not principal in π_1, *i.e.*, it is not derived by a right rule. We have to consider two cases:

(i) π_1 is derived by rule $\sqcup L$, *i.e.*,

$$\pi_1 = \cfrac{\Theta\,;\, \overset{\pi_{11}}{C_1 \vdash D} \qquad \Theta\,;\, \overset{\pi_{12}}{C_2 \vdash D}}{\Theta\,;\, C_1 \sqcup C_2 \vdash D}\,\sqcup L$$

Then, by induction hypothesis we can cut π_{11} and π_{12} with π_2 which yields $\Theta; C_1 \vdash E$ and $\Theta; C_2 \vdash E$, and thereof obtain $\Theta; C_1 \sqcup C_2 \vdash E_1$ by rule $\sqcup L$.

(ii) If the last rule in π_1 is $\supset L$ we have the following situation:

$$\pi_1 = \cfrac{\Theta\,;\, \overset{\pi_{11}}{C \vdash F} \qquad \Theta\,;\, \overset{\pi_{12}}{G \vdash D}}{\Theta,\, F \supset G\,;\, C \vdash D}\,\supset L$$

and π_2 is $\Theta,\ F \supset G; D \vdash E$. Weakening of π_{12} by rule wL_g gives $\Theta,\ F \supset G; G \vdash D$ which we can cut by ind. hyp. with π_2 to obtain $\Theta,\ F \supset G; G \vdash E$. Thereof and from π_{11} we can conclude $\Theta,\ F \supset G; C \vdash E$ by rule $\supset L$.

Case 5. The cut formula D is not principal in π_2.

Subcase 5.1. π_2 ends in $\sqcup R_1$, *i.e.*, $E = E_1 \sqcup E_2$ and we have for π_2 the following:

$$\pi_2 = \cfrac{\Theta\,;\, \overset{\pi_{21}}{D \vdash E_1}}{\Theta\,;\, D \vdash E_1 \sqcup E_2}\,\sqcup R_1$$

By the induction hypothesis we can cut π_1 with π_{21} which yields $\Theta; C \mid\!\!\!- E_1$ and thereof obtain $\Theta; C \mid\!\!\!- E_1 \sqcup E_2$ by rule $\sqcup R_1$. If π_2 ends in $\sqcup R_2$ the argument is symmetrically, *i.e.*, the ind. hyp. lets us cut π_1 with π_{22} to get the derivation $\Theta; C \mid\!\!\!- E_2$ and thereof obtain the goal by rule $\sqcup R_2$.

Subcase 5.2. If π_2 ends in $\supset L$ then we have the following situation:

$$\pi_2 = \frac{\overset{\pi_{21}}{\Theta\,;\, D \mid\!\!\!- F} \qquad \overset{\pi_{22}}{\Theta\,;\, G \mid\!\!\!- E}}{\Theta,\ F \supset G\,;\, D \mid\!\!\!- E} \supset L$$

and π_1 is $\Theta,\ F \supset G; C \mid\!\!\!- D$. We can apply left global weakening wL_g to π_{21} which yields $\Theta,\ F \supset G; D \mid\!\!\!- F$. Then, by the ind. hyp. we can cut π_1 with the latter derivation to obtain $\Theta,\ F \supset G; C \mid\!\!\!- F$. An application of rule $\supset L$ to the latter derivation and π_{22} gives us $\Theta,\ F \supset G; C \mid\!\!\!- E$ as desired. ☐

Under Def. 6.2.3, completeness of the $\mathcal{UL}$ calculus of Fig. 6.1 means that the sequent $\Theta; C \mid\!\!\!- D$ is satisfiable if no proof for it can be found. Soundness expresses that if the sequent $\Theta; C \mid\!\!\!- D$ is derivable then $\Theta; C \models D$, *i.e.*, C is subsumed by D. Here, we only give an indication of what is involved, and refer the reader to [192] for the full proof of soundness and completeness of the calculus $\mathsf{G1}_{\mathcal{UL}}$.

Proposition 6.2.2 ([192, Prop. 4]). *The rules in Fig. 6.1 are sound for constructive subsumption in $\mathcal{UL}$ and complete relative to existentially guarded (general) TBoxes Θ.* ∇

Proof. (Mendler and Scheele [192]) The soundness direction, *viz.* that $\Theta; C \mid\!\!\!- D$ implies $\Theta; C \models D$, is no surprise and follows immediately from soundness of the calculus $\mathsf{G1}$. Like before, it can be proven by induction on derivations, analogously to the proof of Thm. 5.2.2.

The proof of the completeness direction is by a canonical model construction and relies on admissibility of the cut rule in $\mathsf{G1}_{\mathcal{UL}}$ (see [192] for the details). ☐

Theorem 6.2.2 ([192, Thm. 2]). *Subsumption checking in $\mathcal{UL}$ under the constructive semantics and relative to existentially guarded (general) TBoxes is in* PTime. ∇

Proof. (Mendler and Scheele [192]) Let Θ be an existentially guarded TBox and C, D concept descriptions in $\mathcal{UL}$. A subsumption $C \supset D$ relative to Θ can be decided thanks to Prop. 6.2.2 by proof search using the rules from Fig. 6.1.

Because of the subformula property it is the case that each possible node in a sequent derivation tree is determined by a pair of concept descriptions in $sub(\Theta \cup \{C, D\})$. Therefore, the number of possible nodes in a derivation tree is at worst quadratic in the size of the TBox Θ. One can systematically tabulate all pairs $(X, Y) \in sub(\Theta \cup \{C, D\})^2$ such that $\Theta; X \vdash_{\mathsf{G1}_{\mathcal{UL}}} Y$ using dynamic programming memorisation techniques and hereby decide any fixed subsumption in polynomial time. $\square$

In [192], it was conjectured that the PTime result extends directly to $\mathcal{UL}^-$, which is $\mathcal{UL}$ plus limited universal quantification $\forall R.\bot$. In this case, one can exploit the fact from constructive logic that modalities are not interdefinable such that $\forall\neg$ is not the same as $\neg\exists$, which can be of an advantage complexity-wise. Another, and even more interesting extension is by adding conjunction $\sqcap$ to $\mathcal{UL}$ which yields a constructive version of $\mathcal{ELU}$. This is of particular interest as the fragment $\mathcal{EL}$ has turned out to be of practical usefulness in bio-medical applications [18]. However, the problem of checking subsumption in $\mathcal{ELU}$ has turned out to be coNP-complete w.r.t. empty TBoxes, PSpace-complete w.r.t. acyclic TBoxes, and it increases to ExpTime-complete in the presence of cyclic and general TBoxes [48; 121; 122]. In the case of $\mathcal{ELU}$ we expect coNP-hardness, but hopefully not more, since the Boolean reasoning should remain safely contained between the levels of existential restrictions, due to the lack of disjunctive distribution under the constructive semantics, so that the Boolean combinatorics does not spill over quantifiers. It is an open question whether $\mathcal{ELU}$ under the constructive semantics and relative to existentially guarded (cyclic or general) TBoxes behaves better complexity-wise than under the classical descriptive semantics.

6.3 Summary

In this chapter we have defined an embedding of $c\mathcal{ALC}$ into the fusion $\mathsf{S4}_n \otimes \mathsf{K}_m$, which corresponds to the DL $\mathcal{ALC}_{R^*}$. The embedding allows to transfer results from classical bimodal logics to $c\mathcal{ALC}$ and we demonstrated that the problem of checking satisfiability or subsumption of a $c\mathcal{ALC}$ concept without TBoxes is PSpace-complete while it increases to ExpTime-complete w.r.t. to general TBoxes. However, the PSpace-complexity of reasoning w.r.t. simple TBoxes in $\mathcal{ALC}_{R^*}$ does not trivially transfer to $c\mathcal{ALC}$ which is due to our translation of formulæ, that is, the translation of a simple $c\mathcal{ALC}$ TBox does not yield a simple $\mathcal{ALC}_{R^*}$ TBox. We leave it for future work whether the PSpace result can be established for the mentioned reasoning problems w.r.t. simple $c\mathcal{ALC}$ TBoxes under a modified translation. Further results transferred via the embedding include the finite model property and decidability of $c\mathcal{ALC}$. The embedding has been positively evaluated using the classical reasoner Racer[17] [119]. The section on the fragment $\mathcal{UL}$ highlights that the choice of semantic interpretation can cause an advantage regarding the complexity of reasoning in a DL. $\mathcal{UL}$ represents a

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[17] Racer is available from `http://racer.sts.tuhh.de/`.

special fragment of $c\mathcal{ALC}$ that allows only for disjunction and existential restrictions, and requires that TBox axioms are restricted to include only existentially guarded disjunctions in the right-hand side of each axiom. The complexity of subsumption checking in $\mathcal{UL}$ with respect to general TBoxes reduces from ExpTime to PTime if the constructive semantics is adopted [192]. However, as long as there is no application domain identifiable for the $\mathcal{UL}$ fragment these results are only of theoretical interest. It remains an open question whether this approach can be extended to more expressive DLs in directions that may lead to practical applications.

Notes on Related Work

IPC The proof of PSpace-hardness is usually done via a polynomial-time reduction to the well-known quantified Boolean formula problem (QBF), which is PSpace-complete [108; 223]. It has been shown firstly by Statman [253] that the problem of deciding validity of an intuitionistic propositional formula is PSpace-complete, using a natural reduction from QBF to IPC via proof-theoretic methods. Later, Švejdar [256] presented a simplified proof that the decision problem of IPC is PSpace-complete. His method is not relying on particular properties of IPC (like soundness, completeness, etc.) but only concentrates on the reduction of QBF and argues purely semantically.

Classical DLs Schmidt-Schauß and Smolka [247] prove for $\mathcal{ALC}$ that deciding satisfiability (called coherence in [247]) and subsumption is PSpace-complete, establishing the PSpace upper bound by using a tracing technique. For the lower bound they use a reduction from QBF to $\mathcal{ALC}$ to show the PSpace-hardness of the satisfiability problem for $\mathcal{ALC}$. In summary, the complexity of deciding subsumption for $\mathcal{ALC}$ relative to the descriptive semantics without a TBox as well as w.r.t. simple TBoxes is PSpace-complete, while in the presence of GCI's it becomes ExpTime-complete [16].

PSpace-hardness of $\mathcal{ALC}_{R^+}$ as shown by Sattler [243, Thm. 9] comes from the PSpace-completeness of $\mathcal{ALC}$ and the fact that $\mathcal{ALC}$ is a proper sub-language of $\mathcal{ALC}_{R^+}$, and PSpace-completeness is argued by means of using the tracing technique [247; 261] for the tableau construction algorithm. For a complete survey of different notions of transitive roles for DLs see [243]. A detailed study of the complexity of tableau algorithms for description logics including transitive roles and further extension like inverse roles, role hierarchies, cardinality restrictions (graded modalities), etc. can be found in [16; 261].

Note that it is essential to use $\mathcal{ALC}_{R^*}$ as the target language of our embedding. The decision problem of $\mathcal{ALC}_{R^*}$ is known to be PSpace-complete [103, p. 218] by adapting the proof by Halpern and Moses [130]. For $c\mathcal{ALC}$, the reflexivity of $\preceq$ is the key to obtain a translation which is polynomial in the size of a concept and therefore gives us the PSpace upper bound. There are other possible targets for an embedding of $c\mathcal{ALC}$ into a classical description logic as well, for instance a $c\mathcal{ALC}$ concept C can be embedded into $\mathcal{ALC}_{R^+}$. In this case, one has to include reflexivity in the translation of a concept description which leads to an exponential blow-up of the size of the translation $\tau(C)$ of the concept C. It is also possible to represent fallibility of entities by a TBox axiom $F \supset \forall \preceq .F \sqcap \forall R.F \sqcap \exists R.F$ for all $R \in N_R$, which is problematic due to the cyclic definition. In the latter case one can internalise the TBox using a universal role [16], inverse roles and role hierarchies. However, then the decision problem is potentially in ExpTime.

Constructive DLs Regarding the complexity of the constructive DL $\mathcal{KALC}$, Bozzato [39, Chap. 3.5] shows that $\mathcal{KALC}$ realisability is PSpace-hard using a translation between IPC and $\mathcal{KALC}$ and the well-known result by Statman [253]. He conjectures that the trace-technique [261], which is a proof strategy for deciding $\mathcal{ALC}$ satisfiability, can be adopted for $\mathcal{KALC}$ to yield PSpace-completeness. While $\mathcal{KALC}$ assumes a finite domain, it is unknown whether the system $\mathcal{KALC}^\infty$, *i.e.*, a variant of $\mathcal{KALC}$ that admits an infinite domain, has the finite model property. Furthermore, Bozzato [39, Chap. 4] investigates the relation of $\mathcal{KALC}$ to $\mathcal{KALC}^\infty$, $\mathcal{IALC}^\infty$, IQC and the multimodal variant of the IML FS/IK, where $\mathcal{IALC}^\infty$ corresponds to the DL $\mathcal{IALC}$ as proposed by de Paiva [78], but based on a different notation. It has been shown that the axiom schema $\mathsf{KUR} =_{df} \forall R.\neg\neg C \supset \neg\neg\forall R.C$ known as the *Kuroda principle* is valid in $\mathcal{KALC}$ and $\mathcal{KALC}^\infty$, but does not hold in $\mathcal{IALC}^\infty$ and the IML FS/IK. The latter fact is used to argue that $\mathcal{IALC}^\infty$ does not possess the finite model property. The relation of $\mathcal{KALC}^\infty$ relative to IQC, extended by the first-order variant of the axiom KUR, and FS/IK is investigated by giving a faithful translation that preserves validity of the axiom KUR. For future work it would be interesting to see how $c\mathcal{ALC}$ and $\mathcal{KALC}$ are related.

In [77, p. 28] the authors put forward as future work the goal to prove the finite model property for $i\mathcal{ALC}$. However, considering that $i\mathcal{ALC}$ is a notational variant of multimodal IK, existing results for FS/IK should apply to $i\mathcal{ALC}$ as well. In particular, the finite model property for $i\mathcal{ALC}$ should follow from the existing proofs for the system IK [118; 249] extended to the multimodal case.

Lax Logic A similar embedding to ours can be found in [90, pp. 18–20], where Fairtlough and Mendler demonstrate that propositional lax logic (PLL) can be embedded into the bimodal theory $\mathsf{S4} \otimes \mathsf{S4}$ with the modalities $\Box_i$ and $\Box_m$, extended by the axiom schema $\Box_i C \supset \Box_m C$, where the intuitionistic accessibility relation is translated into

$\Box_i$ and the lax modality $\bigcirc$ into $\Box_m$ respectively. The embedding of PLL[18] into this extension of S4 $\otimes$ S4 differs from ours in that (i) they include the fallible symbol F only in the translation of atomic formulæ and the constant $\perp$ itself, *i.e.*, they translate $\perp$ itself into $\Box_i f$, but omit it from the translation of $\supset$ and $\bigcirc$, and (ii) translate atomic A into $\Box_i(A \sqcup f)$.

IMLs The finite model property of the IML FS/IK has been demonstrated in [118; 249]. General results on the finite model property, decidability and Kripke-completeness for normal IMLs and for several of their extensions have been proved in [275; 278], by using an embedding into classical bimodal logics. Alechina and Shkatov [5] present another general method to prove decidability of IMLs which uses a translation into the two variable monadic guarded fragment of first-order logic. In particular, their method also applies for non-normal IMLs, where $\exists R$ ($\Diamond$) does not distribute over disjunction, like for instance $c\mathcal{ALC}$ (CK) without fallibility.

Ranalter [235] investigates the relation between CK and FS/IK as a first step to harmonise their differing proof theory. He shows that CK is a fragment of FS/IK by using a proof-theoretic embedding based on the natural deduction systems for CK [27] and IK [249] denoted by NCK and NIK respectively. While NIK comes with proper introduction and elimination rules for the modal operators, NCK lacks this feature. Ranalter shows that the modal rules of NCK can be considered as derived rules in NIK, in the sense that NCK proofs can be embedded into NIK proofs, and he extends this result to the natural deduction systems of CS4 and IS4. He argues that his method might also be taken as a modular approach for the intermediate systems between CK and FS/IK which arise by extending CK with one or more axioms of IK3–IK5.

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[18] The authors of [90] use f for the distinguished concept F and write *false* for $\perp$.

CHAPTER 7

Tableau-based Calculus for $c\mathcal{ALC}$

This chapter presents a tableau-based calculus for $c\mathcal{ALC}$, denoted by $\mathfrak{T}_{c\mathcal{ALC}}$, which is inspired by the single conclusion sequent calculus for multimodal CK [196], and follows the style of labelled tableau calculi [16; 25; 216]. According to Negri [216], such tableau calculi '[...] provide a conservative extension of the syntax and permit a methodologically uniform investigation of a vast class of non-classical logics [...]', and that such systems can be obtained '[...] by the addition of rules that correspond to frame properties[...]'. In particular, tableau algorithms have turned out to be well-suited for implementation and optimisation [16, pp. 329 ff.], and have found numerous practical application in DLs [147; 16, pp. 375 ff.]. Hence, the investigation of tableau for $c\mathcal{ALC}$ is important to lay the foundations for future implementations of the $c\mathcal{ALC}$ reasoning services, which can be practically used in application domains that demand for constructive reasoning. The tableau calculus for $c\mathcal{ALC}$ tries to prove the satisfiability of a concept or a subsumption relation between two concepts by explicitly constructing a structure which induces the existence of a (prototypical) model, called *pre-model* in the following. It will be shown that every pre-model can be transformed into an appropriate model. Note that we will not cover ABox reasoning in detail. However, in the final part we will sketch how *constructive reasoning* with ABoxes can be implemented by means of an intuitive example.

7.1 Constraint System

Before we can give a formal definition of the calculus we have to agree on an appropriate data structure to represent constraints of the form:

- 'it is true that entity a belongs to concept C',
- 'it is false that entity a belongs to concept C',
- 'entity a' is a refinement of entity a', or
- 'entity b is an R-filler of entity a'.

For instance, the first statement is represented in classical $\mathcal{ALC}$ by a constraint of the form $a : C$ where a is a *constraint variable* (or entity) and C a *conceptual assertion* (in $\mathcal{ALC}$ simply a concept) that is bound to a.

We follow the idea of [247; 25; 16; 83] by manipulating a special kind of data structure – a constraint system – by applying a set of satisfiability preserving expansion rules. The notion of a constraint system to represent the (possibly partial) structure of models was introduced first by Schmidt-Schauß and Smolka [247] in the context of DLs. Such a constraint system contains in the usual case a number of entities, for which we assert their membership to binary relations (roles) and to the extension of concepts. This is similar to the style of representation of an ABox by Baader and Sattler [25]. In classical DLs, a constraint can be viewed as an ABox assertion and a set of constraints corresponds to an ABox. However, this is not the case anymore in constructive logic, which is based on birelational semantics, as we shall see later.

In contrast to the definition of a classical $\mathcal{ALC}$ ABox, our data structure of a constraint system has to take into account the constructive semantics of c$\mathcal{ALC}$. Consequently, in our constraint system we need to add the notion of *polarity* (also called *signed formulæ*) to constraints using the arithmetic symbols + and $-$ to sign concepts. Intuitively, $+C$ expresses that concept C is true (in some model), while $-C$ asserts that C is false. This notation embodies the intuitionistic structure of the sequent rules of `G1`, by taking the formulæ of the antecedent as + signed concepts and the formulæ of the succedent with the $-$ sign respectively. Tableau systems with signed formulæ $(+, -)$ were introduced first by Lis [176, p. 41] for classical logic, and independently investigated by Smullyan [251, pp. 15 ff.] for classical first-order logic and Fitting [99, pp. 28 ff.] for IPC, but using the signs `T` and `F` instead. In the context of constructive DLs this notion of polarity has been used by Odintsov and Wansing [219], following the notation by Lis. The Fitting-style tableau systems for $\mathcal{KALC}$ by Bozzato, Ferrari and Villa [43],[39; 45; 270], are utilising the notation by Smullyan. For a comprehensive introduction to tableau systems and a historical survey see [71], and [16; 25] for DL-specific tableaux.

Furthermore, we include a special role name to explicitly represent the reflexive and transitive refinement relation $\preceq$ between entities, and introduce a special set of entities to represent the focus of construction. The latter set, called *active set*, represents the entities that have been realized in the pre-model at a specific state of the tableau construction process.

Notation. In the following we denote *constraints* by the letters $\mathfrak{c}, \mathfrak{d}$, *constraint assertions* by r, s, and u, v denote *variables* that occur in a constraint, *i.e.*, these are variables to which a constraint assertion is bound. ■

Definition 7.1.1 (Constraint, constraint system). Let V_E be a set of variable symbols together with a well-order $\prec$ on V_E. The variables (or entities) of V_E are denoted by the letters u, v and their variants $u', u'', \ldots, v', v'', \ldots$. A *constraint* is a syntactic object of the following form

$$u{:}{+}C, \quad u{:}{-}C, \quad u{:}{-}_{\forall R}C, \quad u{:}{-}_{\exists R}C, \quad u\,R\,v, \quad u \preceq u'.$$

The symbols C, D stand for arbitrary concept descriptions and $R \in N_R$ is a role name. A *constraint system* is a pair $\mathfrak{S} = (\mathfrak{C}\,;\,\mathfrak{A})$ where $\mathfrak{C}$ is a finite, non-empty set of *constraints* and the second component $\mathfrak{A} \subseteq V_E$ is a set of variables, called the *active set* of $\mathfrak{S}$, such that every element of $\mathfrak{A}$ occurs in at least one of the constraints of $\mathfrak{C}$. The set of variables occurring in $\mathfrak{C}$ is called the *support* of $\mathfrak{S}$, written $Supp(\mathfrak{S})$. Note that $\mathfrak{A} \subseteq Supp(\mathfrak{S}) \neq \emptyset$. ∇

Notation. For constraints of the form $u{:}{+}C$, $u{:}{-}C$, $u{:}{-}_{\forall R}C$ or $u{:}{-}_{\exists R}C$ we will use the term *conceptual constraints* and denote $u\,R\,v$ and $u \preceq u'$ by *relational constraints*. We write $u{:}{\pm}C$ to denote one of $\{u{:}{+}C, u{:}{-}C, u{:}{-}_{\exists R}C, u{:}{-}_{\forall R}C\}$. Let $\mathfrak{S} = \{\mathfrak{C}, \mathfrak{A}\}$ and $\mathfrak{S}' = \{\mathfrak{C}', \mathfrak{A}'\}$ be constraint systems. We write $\mathfrak{S}' \subseteq \mathfrak{S}$ iff $\mathfrak{C}' \subseteq \mathfrak{C}$ and $\mathfrak{A}' \subseteq \mathfrak{A}$. ■

The first component $\mathfrak{C}$ represents a set of assertions which can be viewed as the realisation of an ABox. Conceptual constraints include polarity $(+, -)$ to express if either a concept is true at a given entity or false in the other case. For instance, the constraint $u{:}{+}C$ expresses that the concept C is true at the variable u. A negative constraint $u{:}{-}C$ means that 'C is *false* at u', $u{:}{-}_{\forall R}C$ formulates that 'C is false in all constructible (accessible) R-successors of u' and $u{:}{-}_{\exists R}C$ denotes that 'there exists an R-successor of u in which C is false'. The remaining (relational) constraints represent accessibility and refinement, *i.e.*, $u\,R\,v$ specifies that the 'entity v is accessible from entity u via the role R', whereas $u \preceq u'$ says that 'u' refines u'.

The second component $\mathfrak{A}$ is the *active set* of variables. Accordingly, the elements of $Supp(\mathfrak{S}) \setminus \mathfrak{A}$ are called *inactive*. The applicability of the tableau rules is restricted to constraints which are bound to an active variable, that is, a variable which is contained in $\mathfrak{A}$. We will denote them as *active constraints* in the following. Active constraints can be thought of as those formulæ which are realized by the tableau procedure and model-theoretically represented by (accessible) entities being spotlighted in the active scope compartment of construction, at which completion rules are applicable. All the other constraints outside the active set can be seen as hypotheses.

Each conceptual constraint binds a conceptual assertion to a variable, *e.g.*, $x{:}{+}C$ binds the conceptual assertion $+C$ to the variable x. The following definition defines the set of conceptual constraints of a constraint variable.

Definition 7.1.2 (Constraints of a constraint entity)**.** Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a constraint system and u a variable in $Supp(\mathfrak{S})$. The set of conceptual constraint assertions bound to variable u in $\mathfrak{C}$ is defined by

$$CA(u, \mathfrak{S}) = \{\mathsf{r} \mid \mathsf{r} \text{ is a conceptual assertion, which is bound to variable } u \text{ in } \mathfrak{C}\}.$$

If $\mathfrak{S}$ is clear from the context we will just write $CA(u)$ instead. If $CA(u, \mathfrak{S})$ contains only $+$ signed constraint assertions of the form $u{:}{+}C$ then we call variable u *optimistic*. ∇

Definition 7.1.3 (R-successor, $\preceq$-successor, $\preceq^*$-successor, reachable)**.** Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a constraint system, $u, v, u' \in Supp(\mathfrak{S})$ and $R \in N_R$. We call v an R-successor of u in $\mathfrak{S}$ if $u\,R\,v \in \mathfrak{C}$. Entity u is called an R-predecessor of v if v is an R-successor of u. Similarly we say that u' is a $\preceq$-successor (or refinement) of u in $\mathfrak{S}$ if $u \preceq u' \in \mathfrak{C}$. u' is called a $\preceq^*$-successor if there exist $u_1, u_2, u_3, \ldots, u_n$ in $Supp(\mathfrak{S})$ such that $u_i \preceq u_{i+1} \in \mathfrak{C}$ for $i = 1, \ldots, n-1$ and $u_1 = u, u_n = u'$. Note that u is always a $\preceq^*$-successor of itself if $n = 1$. We call u' a $\preceq^+$-successor of u if u' is a $\preceq^*$-successor of u and $u \neq u'$. A variable v is called *reachable* from u in $\mathfrak{S}$ if there exists a arbitrarily interleaved path via $\preceq$ and R from u to v in $\mathfrak{S}$. Moreover, u is called an ancestor of v iff v is reachable from u. ∇

Note that in general the relation ancestor can be symmetric. However, in the case of the tableau for $c\mathcal{ALC}$ it is not symmetric, *i.e.*, there are no cycles in R and $\preceq$.

Definition 7.1.4 (Constraint satisfiability)**.** Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a constraint system, $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ a constructive interpretation, $R \in N_R$ and C a $c\mathcal{ALC}$ concept. An *$\mathcal{I}$-assignment* is a valuation function α mapping each variable symbol $u \in Supp(\mathfrak{S}) \subseteq V_E$ to an element of $\Delta^{\mathcal{I}}$. We say that α *satisfies* a constraint $\mathfrak{c} \in \mathfrak{C}$ in the interpretation $\mathcal{I}$, written $\mathcal{I}; \alpha \models \mathfrak{c}$, according to the following rules:

$$\mathcal{I}; \alpha \models u{:}{+}C \quad \text{if } \alpha(u) \in C^{\mathcal{I}}, \tag{7.1}$$

$$\mathcal{I}; \alpha \models u{:}{-}C \quad \text{if } \alpha(u) \notin C^{\mathcal{I}}, \tag{7.2}$$

$$\mathcal{I}; \alpha \models u\,R\,v \quad \text{if } (\alpha(u), \alpha(v)) \in R^{\mathcal{I}}, \tag{7.3}$$

$$\mathcal{I}; \alpha \models u \preceq u' \quad \text{if } (\alpha(u), \alpha(u')) \in \preceq^{\mathcal{I}}, \tag{7.4}$$

$$\mathcal{I}; \alpha \models u{:}{-_{\forall R}}C \quad \text{if } \forall y \in \Delta^{\mathcal{I}}.(\alpha(u), y) \in R^{\mathcal{I}} \Rightarrow y \notin C^{\mathcal{I}}. \tag{7.5}$$

$$\mathcal{I}; \alpha \models u{:}{-_{\exists R}}C \quad \text{if } \exists y \in \Delta^{\mathcal{I}}.(\alpha(u), y) \in R^{\mathcal{I}} \;\&\; y \notin C^{\mathcal{I}}. \tag{7.6}$$

A constraint system $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ is *satisfied* by an interpretation $\mathcal{I}$ and an $\mathcal{I}$-assignment α, written $\mathcal{I}; \alpha \models \mathfrak{S}$, if for all $\mathfrak{c} \in \mathfrak{C}$ it holds that $\mathcal{I}; \alpha \models \mathfrak{c}$ and for all variables $u \in \mathfrak{A}$ the assignment $\alpha(u)$ is infallible, *i.e.*, $\alpha(u) \notin \bot^{\mathcal{I}}$. We call the pair $(\mathcal{I}, \alpha)$ a *model* of $\mathfrak{S}$.

A constraint system $\mathfrak{S}$ is satisfiable if it has a model. We say that a constraint system $\mathfrak{S}$ is satisfiable w.r.t. a TBox Θ iff it has a model $(\mathcal{I}, \alpha)$ and $\mathcal{I} \models \Theta$ in the sense of Def. 4.2.5. ∇

Proposition 7.1.1 below lifts the key reasoning tasks w.r.t. a TBox to constraint systems.

Proposition 7.1.1 (Reasoning tasks). *Let concepts C, D and TBox Θ be arbitrary.*

(i) Concept C is satisfiable (w.r.t. Θ) iff the constraint system $(\{u{:}{+}C, \}, \{u\})$ is satisfiable (w.r.t. Θ).

(ii) Concept C is subsumed by concept D (w.r.t. Θ) if and only if the constraint system $(\{u{:}{+}C, u{:}{-}D\}, \{u\})$ is not satisfiable (w.r.t. Θ).

(iii) Two concepts C and D are disjoint (w.r.t. Θ) if and only if the constraint system $(\{u{:}{+}C, u{:}{+}D, \}, \{u\})$ is not satisfiable (w.r.t. Θ).

(iv) Concepts C and D are equivalent (w.r.t. Θ) if and only if the constraint systems $(\{u{:}{+}C, u{:}{-}D\}, \{u\})$ and $(\{u{:}{-}C, u{:}{+}D\}, \{u\})$ are not satisfiable (w.r.t. Θ). ∇

Proof. (i) ($\Rightarrow$) Assume that C is satisfiable w.r.t. TBox Θ. We need to show that the constraint system $(\{u{:}{+}C, \}, \{u\})$ is satisfiable w.r.t. Θ. By Def. 4.2.5 there exists a pair $(\mathcal{I}, a)$ which models C and $\mathcal{I}$ is a model of the TBox Θ, *i.e.*, formally $a \in C^{\mathcal{I}}$, $a \notin \bot^{\mathcal{I}}$ and $\mathcal{I} \models \Theta$. By assumption, $\mathcal{I} \models \Theta$ follows directly. We claim that there exists an $\mathcal{I}$-assignment α such that the pair $(\mathcal{I}, \alpha)$ satisfies the constraint $u{:}{+}C$, and $\alpha(u)$ is infallible. According to Def. 7.1.4, an $\mathcal{I}$-assignment α satisfies a constraint $u{:}{+}C$ in the interpretation $\mathcal{I}$ if $\alpha(u) \in C^{\mathcal{I}}$. Let $\alpha(u) =_{df} a$. This implies $\mathcal{I}; \alpha(u) \models C$. Furthermore $\alpha(u)$ is infallible, since by assumption $\alpha(u) = a \notin \bot^{\mathcal{I}}$.

($\Leftarrow$) Suppose that the constraint system $(\{u{:}{+}C, \}, \{u\})$ is satisfiable w.r.t. TBox Θ. By Definition 7.1.4 there exists an interpretation $\mathcal{I}$ together with an $\mathcal{I}$-assignment α such that $\mathcal{I}; \alpha \models u{:}{+}C$, *i.e.*, $\alpha(u)$ is infallible and $\mathcal{I} \models \Theta$. This means $\alpha(u) \in C^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$. We have to show that there exists an interpretation $\mathcal{I}$ and an entity $a \in \Delta^{\mathcal{I}}_c = \Delta^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$ such that $a \in C^{\mathcal{I}}$. The assumption implies that $\alpha(u)$ is such an entity. Thus, C is satisfiable w.r.t. Θ.

(ii) ($\Rightarrow$) The proof is by contraposition. Suppose that $(\{u{:}{+}C, u{:}{-}D\}, \{u\})$ is satisfiable w.r.t. TBox Θ. From Definition 7.1.4 it follows that there exists a pair $(\mathcal{I}, \alpha)$ such that $\mathcal{I}; \alpha \models u{:}{+}C$ and $\mathcal{I}; \alpha \models u{:}{-}D$ with $\alpha(u)$ being an infallible entity. It is necessary to show that C is not subsumed by D w.r.t. TBox Θ. By Definition 4.2.5 this is the case if there exists an interpretation and entity $a \in \Delta^{\mathcal{I}}_c$ such that $a \in C^{\mathcal{I}}$

and $a \notin D^{\mathcal{I}}$. Choosing $a =_{df} \alpha(u)$ implies that a is infallible and contained in $C^{\mathcal{I}}$ but not in $D^{\mathcal{I}}$. Thus, C is not subsumed by D w.r.t. Θ.

($\Leftarrow$) Proof by contraposition. Suppose C is not subsumed by D w.r.t. TBox Θ. From the assumption it follows by Def. 4.2.5 that there exists an interpretation $\mathcal{I}$ and an entity $a \in \Delta_c^{\mathcal{I}}$ such that $a \in C^{\mathcal{I}}$, $a \notin D^{\mathcal{I}}$ and $\mathcal{I} \models \Theta$. We have to show that the constraint system $\mathfrak{S} = (\{u{:}{+}C, u{:}{-}D\}, \{u\})$ is satisfiable w.r.t Θ. This is the case if we can find a pair $(\mathcal{I}, \alpha)$ such that $\mathcal{I}; \alpha \models u{:}{+}C$ and $\mathcal{I}; \alpha \models u{:}{-}D$ with $\alpha(u)$ being an infallible entity. Let $\alpha(u) =_{df} a$. By Definition 7.1.4 it holds that $\mathcal{I}; \alpha \models u{:}{+}C$ and $\mathcal{I}; \alpha \models u{:}{-}D$ and $\alpha(u)$ is infallible. Thus $\mathfrak{S}$ is satisfiable w.r.t. Θ.

(iii) ($\Rightarrow$) Suppose the concepts C and D are disjoint w.r.t. TBox Θ. By Def. 4.2.5 this means that for all interpretations $\mathcal{I}$ it holds that $C^{\mathcal{I}} \cap D^{\mathcal{I}} = \emptyset$. Let $\mathcal{I}, \alpha$, $a \in \Delta^{\mathcal{I}}$ be arbitrary and suppose that $\alpha(u) = a$. We proceed by case analysis.

Case 1. If $a \in C^{\mathcal{I}}$ and $a \notin D^{\mathcal{I}}$ then this implies by Def. 7.1.4 that $\mathcal{I}; \alpha \models u{:}{+}C$ but $\mathcal{I}; \alpha \not\models u{:}{+}D$.

Case 2. Otherwise, from $a \notin C^{\mathcal{I}}$ and $a \in D^{\mathcal{I}}$ we conclude by Def. 7.1.4 that $\mathcal{I}; \alpha \models u{:}{+}D$ but $\mathcal{I}; \alpha \not\models u{:}{+}C$.

Case 3. If $a \notin C^{\mathcal{I}}$ and $a \notin D^{\mathcal{I}}$ then $\mathcal{I}; \alpha \not\models u{:}{+}C$ and $\mathcal{I}; \alpha \not\models u{:}{+}D$.

In all three cases it follows that $a \notin \bot^{\mathcal{I}}$. Hence, $\mathfrak{S} = (\{u{:}{+}C, u{:}{+}D, \}, \{u\})$ is not satisfiable.

($\Leftarrow$) Proof by contraposition. Assume that C and D are not disjoint w.r.t. Θ. From Definition 4.2.5 it follows that there exists an interpretation $\mathcal{I}$ and an entity $a \in \Delta_c^{\mathcal{I}}$ such that $a \in C^{\mathcal{I}}$ and $a \in D^{\mathcal{I}}$. We need to show that the constraint system $\mathfrak{S} = (\{u{:}{+}C, u{:}{+}D, \}, \{u\})$ is satisfiable w.r.t. Θ. Let $\alpha(u) =_{df} a$. Then, according to Definition 7.1.4 the pair $(\mathcal{I}, \alpha)$ satisfies the constraint system $\mathfrak{S}$ w.r.t. Θ.

(iv) According to Definition 4.2.5, two concepts C and D are equivalent if they share the same infallible entities in all models of Θ. This holds if C subsumes D and vice versa w.r.t. Θ. By Proposition 7.1.1.(iv) this is the case if and only if the constraint systems $(\{u{:}{+}C, u{:}{-}D\}, \{u\})$ and $(\{u{:}{-}C, u{:}{+}D\}, \{u\})$ are not satisfiable w.r.t. TBox Θ. $\square$

7.2 Tableau Rules

The tableau calculus $\mathfrak{T}_{c\mathcal{ALC}}$ decides concept satisfiability and concept subsumption. The algorithm receives a pair $(\{\mathfrak{S}_0\}, \Theta)$ as starting point, where $\mathfrak{S}_0$ is the initial constraint system for which the consistency problem is to be decided w.r.t. the TBox Θ. It applies satisfiability preserving completion rules which modify the constraint system until either an obvious contradiction (so-called *clash*) occurs, or no further rule applies. In the latter case, the generated pre-model is complete. The initial constraint system

$\mathfrak{S}_0$ is satisfiable w.r.t. Θ iff the final result is a non-contradictory (*clash-free*) constraint system. We consider a finite set of constraint systems – known as *generalised knowledge base* – in the spirit of [16, pp. 88 ff.; 23, pp. 202 ff.].

Definition 7.2.1 (Generalised knowledge base (GKB))**.** A *generalised knowledge base* $\mathcal{K}$ is a pair $(\mathcal{M}, \Theta)$ where $\mathcal{M}$ is a finite set of constraint systems and Θ a TBox. ∇

Note that a normal or *standard* knowledge base in the sense of DLs [16] (see Chapter 2.1) is a GKB $\mathcal{K} = (\mathcal{M}, \Theta)$ with cardinality $|\mathcal{M}| = 1$. The following definition establishes the notion of satisfiability for a GKB $\mathcal{M}$.

Definition 7.2.2 (Satisfiability of a GKB)**.** A GKB $\mathcal{K} = (\mathcal{M}, \Theta)$ with $\mathcal{M} = \{\mathfrak{S}_1, \ldots, \mathfrak{S}_l\}$ and $l \geq 1$ is satisfied by an interpretation $\mathcal{I}$ and an $\mathcal{I}$-assignment α if and only if there exists some $i, 1 \leq i \leq l$ such that $\mathcal{I}; \alpha \models \mathfrak{S}_i$ w.r.t. TBox Θ. ∇

Furthermore, we require a restriction on constraint systems such that only active entities may have refinement relations.

Definition 7.2.3 (Non-speculative constraint system)**.** A constraint system $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ is called *non-speculative* iff for each constraint $u \preceq u' \in \mathfrak{C}$ or $u\, R\, v \in \mathfrak{C}$ it holds that $u, u' \in \mathfrak{A}$, and $\forall v \notin \mathfrak{A}$ it holds that v is optimistic, *i.e.*, it occurs only in positive constraints. ∇

We will define the completion rules as a relation on generalised knowledge bases.

Definition 7.2.4 (Tableau rule)**.** A *tableau rule* ξ is a relation on knowledge bases. We write $\mathcal{K} \rightarrow_\xi \mathcal{K}'$ to denote that ξ relates generalised knowledge base $\mathcal{K}$ to $\mathcal{K}'$. ∇

Note that Def. 7.2.4 is a very generic definition of a tableau rule and enables a rule ξ to depend either on the full global state of a knowledge base $\mathcal{K}$ or on the choice of some individual constraint system in $\mathcal{M}$. Furthermore, such a tableau rule can possibly transform the TBox Θ as well. We do not require this degree of freedom. We consider only the case where a tableau rule ξ depends only on at most one constraint system in $\mathcal{M}$ and the TBox is assumed to be static. This leads to the notion of locality of a tableau rule.

Definition 7.2.5 (Local tableau rule)**.** Let $\mathcal{K} = (\mathcal{M}, \Theta)$ be a generalised knowledge base. A tableau rule ξ is called *local* if its application to $\mathcal{K}$ does not change the TBox Θ and it only depends on the structure of a single constraint system in $\mathcal{M}$. Formally, we require $(\mathcal{M}, \Theta) \rightarrow_\xi (\mathcal{M}', \Theta')$ if and only if

- $\Theta' = \Theta$, and
- there exists a constraint system $\mathfrak{S} \in \mathcal{M}$ such that $(\{\mathfrak{S}\}, \Theta) \rightarrow_\xi (\mathcal{M}'', \Theta)$ for some $\mathcal{M}''$ and $\mathcal{M}' = \mathcal{M} \setminus \{\mathfrak{S}\} \cup \mathcal{M}''$.

For a local tableau rule we write $\mathfrak{S} \overset{\Theta}{\to}_\xi \mathcal{M}$ to abbreviate $(\{\mathfrak{S}\}, \Theta) \to_\xi (\mathcal{M}, \Theta)$. ∇

The application of local tableau rules can be described as follows: Let $\mathcal{K}_i = (\mathcal{M}_i, \Theta)$ be a generalised knowledge base and let ξ be a local tableau rule. Then, $\mathcal{K}_i \to_\xi \mathcal{K}_{i+1}$ if and only if there exists an $\mathfrak{S} \in \mathcal{M}_i$ such that $\mathfrak{S} \overset{\Theta}{\to}_\xi \mathcal{M}'_i$ and $\mathcal{K}_{i+1} = (\mathcal{M}_{i+1}, \Theta)$ where $\mathcal{M}_{i+1} = \mathcal{M}_i \setminus \{\mathfrak{S}\} \cup \mathcal{M}'_i$.

Definition 7.2.6 (Regular tableau rule)**.** Let $\mathfrak{S}$ be a constraint system, Θ be an arbitrary TBox and $\mathcal{M}$ be the result of the rule application of $\mathfrak{S} \overset{\Theta}{\to}_\xi \mathcal{M}$. We call a local tableau rule ξ *regular* if the following holds: For all interpretations $\mathcal{I}$ there exists a valuation α such that $\mathcal{I}; \alpha \models \mathfrak{S}$ if and only if there exists a constraint system $\mathfrak{S}'$ in $\mathcal{M}$ and α' such that $\mathcal{I}; \alpha' \models \mathfrak{S}'$. In the direction ($\Rightarrow$) the $\mathcal{I}$-assignment α' is typically an extension of the $\mathcal{I}$-assignment α, in the sense that the map α on $Supp(\mathfrak{S})$ coincides on its domain with α' such that $\forall u \in Supp(\mathfrak{S}).\alpha(u) = \alpha'(u)$, and the domain of α' extends that of α, *i.e.*, $Supp(\mathfrak{S}') \supseteq Supp(\mathfrak{S})$. ∇

7.2.1 Tableau Rules of $\mathfrak{T}_{c\mathcal{ALC}}$

The following definition establishes the concrete set of tableau rules for $\mathfrak{T}_{c\mathcal{ALC}}$.

Definition 7.2.7 ($c\mathcal{ALC}$ tableau rules)**.** Let $\mathcal{K} = (\mathcal{M}, \Theta)$ be a GKB and let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a constraint system in $\mathcal{M}$. For each $c\mathcal{ALC}$ tableau rule ξ, we will write

$$\mathfrak{S} \to_\xi \mathfrak{S}' \qquad \text{and}$$
$$\mathfrak{S} \to_\xi \mathfrak{S}',\ \mathfrak{S}''$$

to denote that $\mathfrak{S} \overset{\Theta}{\to}_\xi \mathcal{M}'$ and either $\mathcal{M}' = \{\mathfrak{S}'\}$ in the case of a deterministic tableau rule ξ or in the non-deterministic case $\mathcal{M}' = \{\mathfrak{S}', \mathfrak{S}''\}$. The $c\mathcal{ALC}$ tableau rules are depicted in Figure 7.1. Observe that the TBox Θ is left implicit in the specification of the $c\mathcal{ALC}$ tableau rules, as it is assumed to be static and clear from the context. ∇

Remark 7.2.1. Note that the rules ($\to_{\sqcap-}$), ($\to_{\sqcup+}$) and ($\to_{\supset+}$) are *non-deterministic*, *i.e.*, there are two possibilities to continue the construction of a tableau. According to the terminology used in [261, p. 21], the rules ($\to_{\supset-}$), ($\to_{\forall-}$), ($\to_{\exists-}$) and ($\to_{R\exists-}$), which generate new successors via $\preceq$ or R will be referred to as *generating rules*, whereas all other rules are called *non-generating rules.*

We can make the following observations from inspection of the tableau rules of Fig. 7.1:

(i) A tableau rule is only applied if it extends a constraint system by a *new constraint.*

(ii) Only non-deterministic rules replace one constraint system with two constraint systems.

$(\rightarrow_{\sqcap +})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\sqcap +}\; \mathfrak{S}' = (\{u{:}{+}C, u{:}{+}D\} \cup \mathfrak{C}, \mathfrak{A})$
if for some $u \in \mathfrak{A}$, $u{:}{+}C \sqcap D \in \mathfrak{C}$, and $\{u{:}{+}C, u{:}{+}D\} \not\subseteq \mathfrak{C}$.

$(\rightarrow_{\sqcap -})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\sqcap -}\; \mathfrak{S}' = (\{u{:}{-}C\} \cup \mathfrak{C}, \mathfrak{A}),\ \mathfrak{S}'' = (\{u{:}{-}D\} \cup \mathfrak{C}, \mathfrak{A})$
if for some $u \in \mathfrak{A}$, $u{:}{-}C \sqcap D$ is in $\mathfrak{C}$ and neither $u{:}{-}C$ nor $u{:}{-}D$ in $\mathfrak{C}$.

$(\rightarrow_{\sqcup +})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\sqcup +}\; \mathfrak{S}' = (\{u{:}{+}C\} \cup \mathfrak{C}, \mathfrak{A}),\ \mathfrak{S}'' = (\{u{:}{+}D\} \cup \mathfrak{C}, \mathfrak{A})$
if for some $u \in \mathfrak{A}$, $u{:}{+}C \sqcup D$ is in $\mathfrak{C}$ and neither $u{:}{+}C$ nor $u{:}{+}D$ is in $\mathfrak{C}$.

$(\rightarrow_{\sqcup -})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\sqcup -}\; \mathfrak{S}' = (\{u{:}{-}C, u{:}{-}D\} \cup \mathfrak{C}, \mathfrak{A})$
if for some $u \in \mathfrak{A}$, $u{:}{-}C \sqcup D$ is in $\mathfrak{C}$ and $u{:}{-}C$, $u{:}{-}D$ are not both in $\mathfrak{C}$.

$(\rightarrow_{\supset +})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\supset +}\; \mathfrak{S}' = (\{u{:}{-}C\} \cup \mathfrak{C}, \mathfrak{A}),\ \mathfrak{S}'' = (\{u{:}{+}D\} \cup \mathfrak{C}, \mathfrak{A})$
if for some $u \in \mathfrak{A}$, $u{:}{+}C \supset D$ is in $\mathfrak{C}$, and neither $u{:}{-}C$ nor $u{:}{+}D$ is in $\mathfrak{C}$.

$(\rightarrow_{\supset -})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\supset -}\; \mathfrak{S}' = (\{u \preceq u', u'{:}{+}C, u'{:}{-}D\} \cup \mathfrak{C}, \mathfrak{A} \cup \{u'\})$
if for some $u \in \mathfrak{A}$, $u{:}{-}C \supset D$ is in $\mathfrak{C}$, u' is a new variable and there exists no $\preceq^*$-successor u'' of u in $\mathfrak{S}$, with $u''{:}{+}C, u''{:}{-}D$.

$(\rightarrow_{\forall +})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\forall +}\; \mathfrak{S}' = (\{v{:}{+}C\} \cup \mathfrak{C}, \mathfrak{A})$
if for some $u \in \mathfrak{A}$, $u{:}{+}\forall R.C$ is in $\mathfrak{C}$, and there exists an R-successor v of u in $\mathfrak{S}$ such that $v{:}{+}C$ is not in $\mathfrak{C}$.

$(\rightarrow_{\forall -})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\forall -}\; \mathfrak{S}' = (\{u \preceq u', u'{:}{-}_{\exists R}C\} \cup \mathfrak{C}, \mathfrak{A} \cup \{u'\})$
if for some $u \in \mathfrak{A}$, $u{:}{-}\forall R.D$ is in $\mathfrak{C}$, u' is a fresh variable and there is no $\preceq^*$-successor u'' of u in $\mathfrak{S}$ such that $u''{:}{-}_{\exists R}C$ is in $\mathfrak{C}$.

$(\rightarrow_{\exists +})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\exists +}\; \mathfrak{S}' = (\{u\, R\, v, v{:}+C\} \cup \mathfrak{C}, \mathfrak{A})$
if for some $u \in \mathfrak{A}$, $u{:}{+}\exists R.C$ is in $\mathfrak{C}$, v is a new variable and there is no R-successor w of u in $\mathfrak{S}$ such that $w{:}{+}C$ is in $\mathfrak{C}$.

$(\rightarrow_{\exists -})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\exists -}\; \mathfrak{S}' = (\{u \preceq u', u'{:}{-}_{\forall R}C\} \cup \mathfrak{C}, \mathfrak{A} \cup \{u'\})$
if for some $u \in \mathfrak{A}$, $u{:}{-}\exists R.C$ is in $\mathfrak{C}$, u' is a new variable and there exists no $\preceq^*$-successor u'' of u in $\mathfrak{S}$ such that $u''{:}{-}_{\forall R}C$ is in $\mathfrak{C}$.

$(\rightarrow_{R_\forall -})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{R_\forall -}\; \mathfrak{S}' = (\{v{:}{-}C\} \cup \mathfrak{C}, \mathfrak{A} \cup \{v\})$
if for some $u \in \mathfrak{A}$, $u{:}{-}_{\forall R}C$ is in $\mathfrak{C}$, there exists an R-successor v of u in $\mathfrak{S}$ such that $v{:}{-}C$ is not in $\mathfrak{C}$.

$(\rightarrow_{R_\exists -})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{R_\exists -}\; \mathfrak{S}' = (\{u\, R\, v, v{:}{-}C\} \cup \mathfrak{C}, \mathfrak{A} \cup \{v\})$
if for some $u \in \mathfrak{A}$, $u{:}{-}_{\exists R}C$ is in $\mathfrak{C}$, v is a new variable and there exists no R-successor w of u in $\mathfrak{S}$ such that $w{:}{-}C$ is in $\mathfrak{C}$.

$(\rightarrow_{\preceq +})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\preceq +}\; \mathfrak{S}' = (\{u'{:}{+}C\} \cup \mathfrak{C}, \mathfrak{A})$
if for some $u \in \mathfrak{A}$, $u{:}{+}C$ is in $\mathfrak{C}$, u' is a $\preceq^+$-successor of u in $\mathfrak{S}$ and $u'{:}{+}C \notin \mathfrak{C}$.

$(\rightarrow_{\mathtt{ax}})$ $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A}) \;\rightarrow_{\mathtt{ax}}\; \mathfrak{S}' = (\{u{:}{+}C\} \cup \mathfrak{C}, \mathfrak{A}\})$
if for some $u \in \mathfrak{A}$, $u{:}{+}C \notin \mathfrak{C}$ for some $C \in \Theta$.

Figure 7.1: Completion rules of $\mathfrak{T}_{c\mathcal{ALC}}$.

(iii) If a constraint system $\mathfrak{S}$ is replaced with $\mathfrak{S}'$, then it holds that $\mathfrak{S} \subseteq \mathfrak{S}'$.

(iv) A rule can only be applied to constraints with an active variable. This means in particular that $\preceq$-successors are only generated for active constraints and they become active as well. Moreover, an inactive entity cannot have any successor, neither an R-successor nor a $\preceq$-successor, which can be observed by inspection of the generating rules of Fig. 7.1.

(v) Rule $(\rightarrow_{\exists+})$ can be applied to active constraints to generate a new R-successor, which is inactive. Such inactive R-successors can only become active by rule $(\rightarrow_{R_{\forall}-})$ applied to a constraint of the form $-_{\forall R}C$, which can be introduced by rule $(\rightarrow_{\exists-})$. In other words, new inactive entities only contain positive information.

(vi) If the tableau calculus starts with an initial constraint system of the form $\mathfrak{S}_0 = (\{u{:}\mathsf{r}\}, \{u\})$, where r is a conceptual constraint assertion, then it holds for all $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ derived from $\mathfrak{S}_0$ by the tableau rules of Fig. 7.1 and all $v \in \mathit{Supp}(\mathfrak{S})$ that if $v{:}-C \in \mathfrak{C}$, $v{:}-_{\exists R}C \in \mathfrak{C}$ or $v{:}-_{\forall R}C \in \mathfrak{C}$ then $v \in \mathfrak{A}$.

(vii) The tableau rules preserve cycle-freeness, *i.e.*, if $\mathfrak{S}$ is cycle-free then all $\mathfrak{S}'$, which are derived from $\mathfrak{S}$ by the tableau rules of Fig. 7.1, are cycle-free.

■

Proposition 7.2.1 (Non-speculativity). *Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a non-speculative constraint system and let $\mathfrak{S}' = (\mathfrak{C}', \mathfrak{A}')$ be derived from $\mathfrak{S}$ by an application of a c$\mathcal{ALC}$ tableau rule ξ (see Fig. 7.1) w.r.t. a TBox Θ,* i.e., *$\mathfrak{S} \overset{\Theta}{\rightarrow}_{\xi} \mathcal{M}$ and $\mathfrak{S}' \in \mathcal{M}$. Then, $\mathfrak{S}'$ is non-speculative.* ∇

Proof. Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a non-speculative constraint system, ξ a completion rule of c$\mathcal{ALC}$, and suppose that $\mathfrak{S}' = (\mathfrak{C}', \mathfrak{A}')$ is derived from $\mathfrak{S}$ by rule ξ. We need to show for all $u, u', v \in \mathit{Supp}(\mathfrak{S})$ that

(i) if $u \preceq u' \in \mathfrak{C}'$ then $u, u' \in \mathfrak{A}'$;

(ii) if $u\, R\, v \in \mathfrak{C}'$ then $u \in \mathfrak{A}'$;

(iii) if $v \notin \mathfrak{A}'$ then v is optimistic.

We distinguish generating and non-generating rules:
Case 1. If ξ is a non-generating rule then the conditions (i) and (ii) follow by assumption. Since all rules can only be applied to an active entity, it is not possible to introduce a negative constraint for an inactive entity, *i.e.* all inactive entities remain optimistic. Moreover, rule $(\rightarrow_{\forall+})$ only propagates positive assertions to R-successors. Hence, condition (iii) is satisfied as well.

Case 2. Otherwise, ξ is a generating rule. Considering Remark 7.2.1.(iv), one observes by inspection of the generating rules of Fig. 7.1 that all $\preceq$-generating rules – this are the rules $(\rightarrow_{\supset-}), (\rightarrow_{\forall-}), (\rightarrow_{\exists-})$ – are only applicable to an active entity and the newly created $\preceq$-successor is added to $\mathfrak{A}$. Regarding the R-generating rules $(\rightarrow_{\exists+}), (\rightarrow_{R_\exists-})$ it holds that the rule $(\rightarrow_{R_\exists-})$ adds the fresh R-successor to the active set, while $(\rightarrow_{\exists+})$ introduces an inactive R-successor that is optimistic. Hence, (i), (ii) and (iii) hold. Therefore, $\mathfrak{S}'$ is non-speculative. □

The following definitions establish the notion of saturation and clash for knowledge bases and constraint systems.

Definition 7.2.8 (Saturation)**.** A generalised knowledge base $\mathcal{K} = (\mathcal{M}, \Theta)$ is called *saturated* if no tableau rule is applicable to it. Analogously, we say that a constraint system $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ is saturated w.r.t. a TBox Θ (in short, $\mathfrak{S}$ is Θ-saturated) if no tableau rule is applicable to $(\{\mathfrak{S}\}, \Theta)$. A saturated constraint system $\mathfrak{S}^* \supseteq \mathfrak{S}$ is called a *saturation* of $\mathfrak{S}$. ∇

Definition 7.2.9 (Clash)**.** A constraint system $\mathfrak{S}$ contains a *clash* (is *clashed*) if for some entity $u \in Supp(\mathfrak{C})$, arbitrary concept description C and $R \in N_R$, one of the following conditions holds:

(i) $u{:}{+}C$ and $u{:}{-}C$ is in $\mathfrak{C}$;

(ii) $u \in \mathfrak{A}$ and $u{:}{+}\bot$ is in $\mathfrak{C}$;

(iii) $u{:}{+}\bot$ and $u{:}{-}C$ or $u{:}{-}_{\forall R}C$ or $u{:}{-}_{\exists R}C$ is in $\mathfrak{C}$.

If $\mathfrak{S}$ contains no clash it is called *clash-free*. We call a generalised knowledge base $\mathcal{K} = (\mathcal{M}, \Theta)$ *clashed* if every constraint system in $\mathcal{M}$ contains a clash. ∇

Note for (ii), that in a satisfiable constraint system all $u \in \mathfrak{A}$ represent infallible worlds.

Principle of the Tableau Calculus $\mathfrak{T}_{c\mathcal{ALC}}$ Let us recall the principle of the tableau calculus. The tableau calculus applies satisfiability preserving rules to a constraint system $\mathfrak{S}$, an element of $\mathcal{M}$ in a GKB $(\mathcal{M}, \Theta)$, as given in Figure 7.1, and replaces it either by one or two enriched constraint systems. The TBox Θ is static and remains unchanged. Intuitively, this process generates a tree Υ of constraint systems, bearing in mind the history of constraint systems created by the execution of the tableau calculus. The structure of a constraint system can be viewed as an 'and-structure', while a generalised knowledge base corresponds to an 'or-structure', *i.e.*, Υ represents an or-tree of and-structures. The successive constraint systems are constructed by enriching the parent constraint system with at least one additional constraint. Some rules add new entities (refinements and/or role fillers) to the structure as well. In particular, the tableau calculus starts with a GKB $\mathcal{K} = (\mathcal{M}, \Theta)$ with $\mathcal{M} = \mathfrak{S}_0$ where $\mathfrak{S}_0$ is the initial constraint system, and applies transformation rules as long as possible. The rules are applied to $\mathcal{K}$ in the following sense: A constraint system $\mathfrak{S}$ is arbitrarily chosen from $\mathcal{M}$. Then, by application of a rule, the constraint system $\mathfrak{S}$ is replaced either by a constraint system $\mathfrak{S}'$, or in the non-deterministic case by two constraint systems $\mathfrak{S}'$ and $\mathfrak{S}''$. This process is iterated until we reach a state of saturation, *i.e.*, until either one clash-free constraint system is created to which no more rule is applicable, or all non-deterministic choices, that is, all the leaves of Υ contain an obvious contradiction (clash). In the former case, the constraint system is satisfiable and makes up a finite pre-model for the initial constraint system $\mathfrak{S}_0$, otherwise it is unsatisfiable.

All rules of Fig. 7.1 are sound in terms of that a constraint system $\mathfrak{S}$ is satisfiable w.r.t. a TBox Θ if and only if one of the constraint systems which replace $\mathfrak{S}$ is satisfiable w.r.t. Θ. Hence, if $(\mathcal{M}, \Theta)$ is the result of applying a sequence of transformation rules to $(\{\mathfrak{S}_0\}, \Theta)$ then $(\mathfrak{S}_0, \Theta)$ is satisfiable if and only if $(\mathcal{M}, \Theta)$ is satisfiable.

The calculus for c$\mathcal{ALC}$ is non-deterministic due to the rules $(\rightarrow_{\sqcup+}), (\rightarrow_{\sqcap-}), (\rightarrow_{\supset+})$. Furthermore, the algorithm chooses non-deterministically a constraint system from a generalised knowledge base and applies a tableau rule to it. Note that we did not define an order or precedence on the tableau rules that specifies which tableau rule will be applied if more than one is applicable. According to Tobies [261] the correctness proof of showing that such a non-deterministic tableau algorithm is a proper decision procedure has to demonstrate that (i) the algorithm terminates, *i.e.*, every sequence of rule applications is finite; (ii) the tableau rules are sound, *i.e.*, if the algorithm constructs a saturated and clash-free GKB for an initial constraint system $\mathfrak{S}_0$ then $\mathfrak{S}_0$ is satisfiable; (iii) and completeness, *i.e.*, starting from a satisfiable constraint system $\mathfrak{S}_0$ there exists a finite sequence of rule applications that ends in a saturated and clash-free GKB.

In contrast to reasoning in standard $\mathcal{ALC}$ without TBoxes we have to cover the problem that the tableau procedure does no longer terminate due to two reasons:

(i) The explicit handling of the refinement relation as a transitive role in the constraint system may lead to the duplication of positive constraints along a $\preceq$-path, which is comparable to the non-termination problem of the tableau system for $\mathcal{ALC}_{R^+}$ [243] or $\mathsf{S4}$ [130]. For instance, consider the problem in $\mathcal{ALC}_{R^+}$ where a conceptual constraint of the form $\exists R.C \sqcap \forall R.\exists R.C$ with a transitive roles R leads to an infinite R-path [*cf.* 243, pp. 5 ff.].

Similarly, in our system we may have constraints of the form $u{:}{+}\neg E$, where E is one of $\{C \supset D, \exists R.C, \forall R.C\}$. An application of the rule $(\rightarrow_{\supset+})$ to such a constraint leads to the creation of a negative constraint $u{:}-E$. A further application of the appropriate rule ($(\rightarrow_{\supset -})$, $(\rightarrow_{\exists -})$ or $(\rightarrow_{\forall -})$) will introduce a new variable u' and the relational constraint $u \preceq u'$. Then, the tableau can proceed with rule $(\rightarrow_{\preceq +})$ that propagates the positive conceptual constraints, in particular $u{:}{+}\neg E$, to variable u', and the same process can be iterated again for u', yielding an infinite $\preceq$-path.

Observe, that due to our constructive semantics and the reflexivity and transitivity of the refinement relation $\preceq$ it is possible to introduce cyclical or oscillating refinement chains in a model. Technically speaking this is one feature of our semantics, that is, the possibility to express cyclical relational structures which sustain the finite model property.

(ii) In the presence of general axioms, *i.e.*, general TBoxes, the depth of an R-path is no longer bounded by the maximum quantifier depth that occurs in the initial constraint system [25, pp. 17 f.], because it cannot become smaller than the maximum quantifier depth that occurs in the TBox. In particular, rule $(\rightarrow_{\mathtt{ax}})$ may lead to duplications of constraints taken from the TBox along an $R/\preceq$-path. For instance, consider we have a TBox axiom $\top \supset \exists R.C$, and suppose that by application of the tableau rules a new active R-successor is introduced. By rule $(\rightarrow_{\mathtt{ax}})$ this axiom is propagated, downwards to the new entity, which by itself may lead again to the creation of a new active R-successor. This process leads to an infinite chain and does not terminate.

The following example Ex. 7.2.1 illustrates the case of non-termination due to the duplication of constraints in the handling of the $\preceq$-relation.

Example 7.2.1 (Looping tableau)**.** Consider the following initial constraint system $\mathfrak{S}_0 = (\mathfrak{C}_0, \mathfrak{A}_0)$ where $\mathfrak{C}_0 = \{u_0{:}{+}\exists R.(C \sqcup D), u_0{:}{+}\neg\exists R.C, u_0{:}{+}\neg\exists R.D\}$ and $\mathfrak{A}_0 = \{u_0\}$. Remember that negation $\neg C$ is an abbreviation for $C \supset \bot$.

Applying rule $(\rightarrow_{\supset +})$ twice yields $(\mathfrak{C}_0 \cup \{u_0{:}-\exists R.C, u_0{:}-\exists R.D\}, \mathfrak{A}_0)$. One can proceed by applying rule $(\rightarrow_{\exists -})$ to both of the negative constraints above which yields two new $\preceq$-successors of u_0. In the following, let us focus on one branch and suppose that we expand the constraint $u_0{:}-\exists R.C$ only. This leads to the creation of the $\preceq$-successor u_1 such that $(\mathfrak{C}_1 = \mathfrak{C}_0 \cup \{u_0 \preceq u_1, u_1{:}-\forall_R C\}, \mathfrak{A}_0 \cup \{u_1\})$. Rule $(\rightarrow_{\preceq +})$ allows to propagate positive constraints along the $\preceq$-path and results in the constraint system

$$(\mathfrak{C}_2 = \mathfrak{C}_1 \cup \{u_1{:}{+}\exists R.(C \sqcup D), u_1{:}{+}\neg\exists R.C, u_1{:}{+}\neg\exists R.D\}, \mathfrak{A}_0 \cup \{u_1\}). \qquad (7.7)$$

At this point, an application of $(\rightarrow_{\exists +})$ and $(\rightarrow_{R\forall -})$ introduces an R-successor v_1 of u_1 and yields the constraint system $(\mathfrak{C}_3 = \mathfrak{C}_2 \cup \{u_1 \; R \; v_1, v_1{:}{+}C \sqcup D, v_1{:}-C\}, \mathfrak{A}_1 =$

$\mathfrak{A}_0 \cup \{u_1, v_1\})$. Note that the application of $(\rightarrow_{R_\forall -})$ adds variable v_1 to the set of active variables.

Analogously, one can proceed by applying rule $(\rightarrow_{\supset +})$, which yields the constraint system $(\mathfrak{C}_4 = \mathfrak{C}_3 \cup \{u_1{:}{-}\exists R.C, u_1{:}{-}\exists R.D\}, \mathfrak{A}_1)$. An application of rule $(\rightarrow_{\exists -})$ to one of $u_1{:}{-}\exists R.C, u_1{:}{-}\exists R.D$ yields a new $\preceq$-successor u_2 with either $u_2{:}{-}_{\forall R}C$ or $u_2{:}{-}_{\forall R}D$. Suppose that we continue with the expansion of $u_1{:}{-}\exists R.D$. This yields the constraint system $(\mathfrak{C}_5 = \mathfrak{C}_4 \cup \{u_1 \preceq u_2, u_2{:}{-}_{\forall R}D\}, \mathfrak{A}_1 \cup \{u_2\})$. A further application of rule $(\rightarrow_{\preceq +})$ forces that all positive constraints from u_1 are added to u_2, and we obtain the following constraint system

$$(\mathfrak{C}_6 = \mathfrak{C}_5 \cup \{u_2{:}{+}\exists R.(C \sqcup D), u_2{:}{+}\neg\exists R.C, u_2{:}{+}\neg\exists R.D\}, \mathfrak{A}_1 \cup \{u_2\}). \qquad (7.8)$$

We can reiterate the above expansion for the constraint $u_2{:}{+}\neg\exists R.C$ by using u_3 of u_2 with $u_3{:}{-}_{\forall R}C$. Again, rule $(\rightarrow_{\preceq +})$ propagates the positive conceptual constraints downwards to u_3 and we obtain the constraint system

$$\begin{aligned}(\mathfrak{C}_7 = \mathfrak{C}_6 \cup \{&u_2 \preceq u_3, u_3{:}{-}_{\forall R}C, u_3{:}{+}\exists R.(C \sqcup D),\\ &u_3{:}{+}\neg\exists R.C, u_3{:}{+}\neg\exists R.D\}, \mathfrak{A}_1 \cup \{u_2, u_3\}). \qquad (7.9)\end{aligned}$$

Now, observe that variable u_1 in (7.7) and u_3 in (7.9) share the same conceptual constraints, *i.e.*, $\mathfrak{C}_7(u_3) = \mathfrak{C}_2(u_1)$.

We can continue by following the above construction scheme for the newly created refinement u_3. If we repeat the above construction in an alternating order to the constraints $-\exists R.C$ and $-\exists R.D$, and continue the construction analogously as before, we obtain the following infinite tableau, see Fig. 7.2. Note that $\mathfrak{C}^+(u_i)$, $i \geq 0$ represents the set of positive conceptual constraints w.r.t. entity u_i; $\mathfrak{C}^+(u_i), \mathsf{r}$ denotes $\mathfrak{C}^+(u_i) \cup \{\mathsf{r}\}$, and dotted lines represent the refinement relation $\preceq$.

The problem of non-termination arises from the heredity condition imposed on positive constraints, implemented by rule $(\rightarrow_{\preceq +})$, and the interaction with implication (negation) in positive constraints. Observe, that $\mathfrak{C}^+(u_{i+1}) = \mathfrak{C}^+(u_0)$, for $i \geq 0$; and similarly v_1 coincides with v_3 and v_2 with v_4 in the set of positive conceptual constraints. Moreover, in Figure 7.2 the entities u_1 and u_3 share the same theory, *i.e.*, $CA(u_1) = CA(u_3)$. The same holds for the entities u_2 and u_4. This is where the technique of *blocking* (also known as loop-checking) comes into play, *i.e.*, one can stop the further expansion of the constraints of an entity if there exists already an earlier introduced entity in the constraint system, which has the same theory (or a superset of it). Considering Fig. 7.2 this means that entity u_1 blocks the further expansion of u_3, and the same holds for entity u_2 which blocks u_4. The infinite branch of Figure 7.2

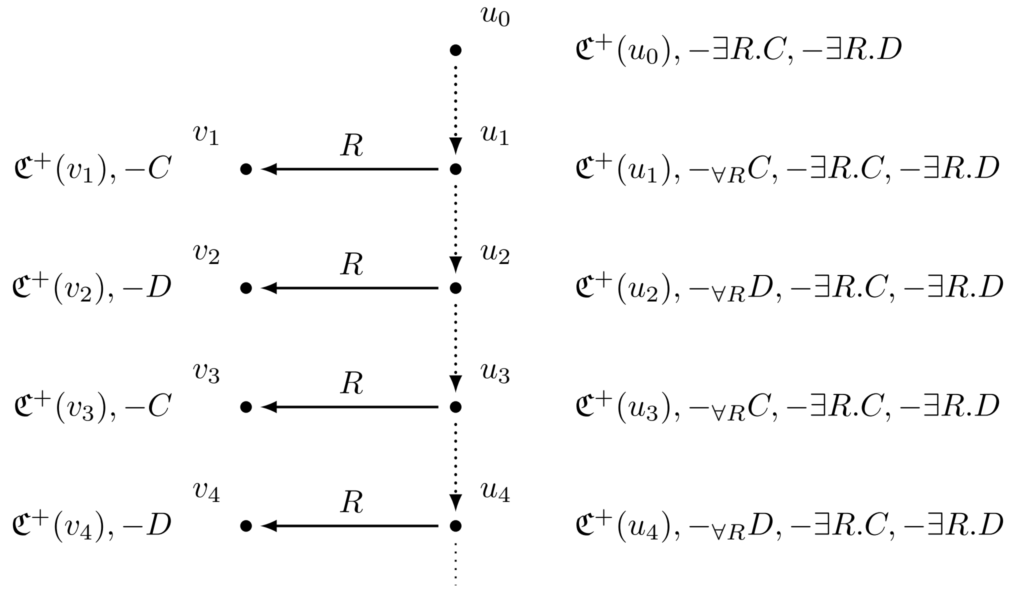

Figure 7.2: Infinite pre-model.

can be represented by a finite one as shown in Figure 7.3. Now, the entities sharing the same theory refine each other.

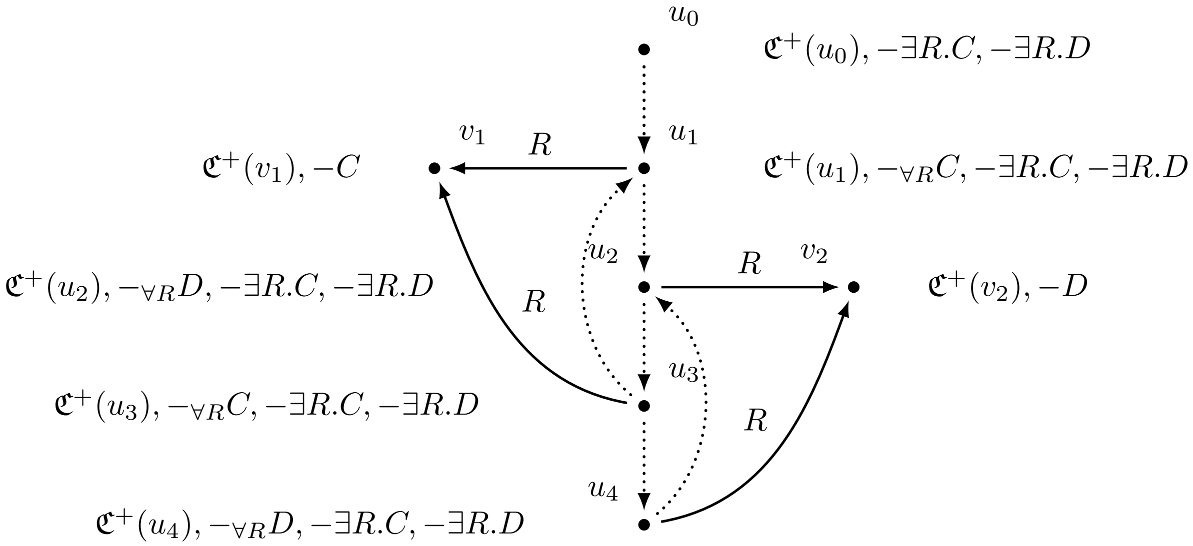

Figure 7.3: Finite representation of Figure 7.2.

The model of Fig. 7.3 can be minimised further by omitting the blocked nodes u_3 and u_4, and by adding an arc to the blocking node immediately, see Fig. 7.4.

■

As demonstrated by Ex. 7.2.1, it is necessary to detect cyclic computations, *i.e.*, looping constructions in the tableau in order to guarantee the termination of the tableau

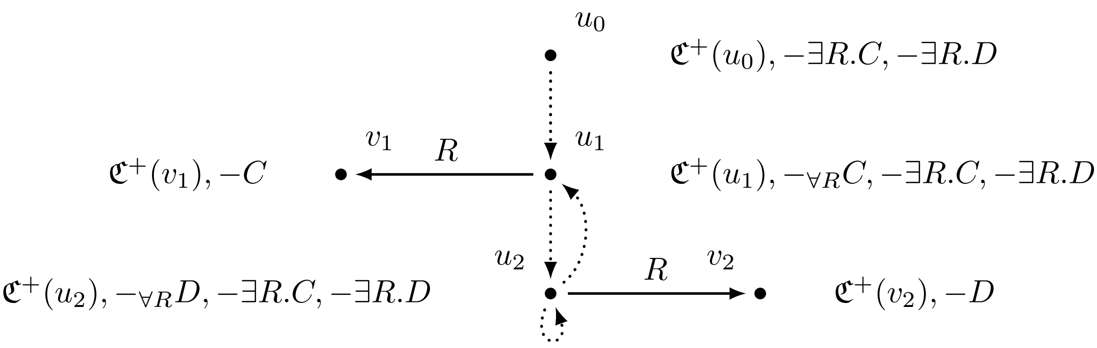

Figure 7.4: Minimised representation of Figure 7.3.

calculus. Following the scheme of [16; 23; 83; 146] the applicability of the generating rules of $c\mathcal{ALC}$ is restricted by the *blocking technique*. We assume that new variables in a constraint system $\mathfrak{S}$ are introduced according to the enumeration (order) $\prec$. This means that if $\mathfrak{S} \rightarrow_\xi \mathfrak{S}'$ and v is a new variable in $\mathfrak{S}'$ then v appears later in the order $\prec$ than all variables in $Supp(\mathfrak{S})$, *i.e.*, $\forall u \in Supp(\mathfrak{S}).\, \forall v \in Supp(\mathfrak{S}') \setminus Supp(\mathfrak{S}).\, u \prec v$.

Definition 7.2.10 (Blocking (BC))**.** Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a constraint system. An entity $u \in \mathfrak{A}$ is *blocked* by an entity $w \in \mathfrak{A}$ iff w is the least element such that $w \prec u$ and $CA(u) \subseteq CA(w)$. ∇

Remark 7.2.2. Note that we restrict blocking to active entities only. This is possible, because the rules of $c\mathcal{ALC}$ can only be applied to active entities anyway, *i.e.*, an infinite sequence of rule applications on inactive entities is not possible. ■

We attach the following blocking condition as a precondition on the considered entity of each generating tableau rule of $c\mathcal{ALC}$ (see Fig. 7.1 on p. 239) by requiring:

> ‘A generating rule of $c\mathcal{ALC}$ can be applied to an active variable u only if u is not blocked by some w.’

From here on we assume that the saturation of a GKB or a constraint system (Def. 7.2.8) is w.r.t. the extended notion of the applicability of a tableau rule including the blocking precondition above.

This will assure the termination of the tableau calculus for $c\mathcal{ALC}$. The blocking condition corresponds to dynamic subset blocking, and is also known as *anywhere blocking*, firstly introduced by Baader, Buchheit and Hollander [23]. Note that we do not assume a prioritisation, strategy or order in which the tableau rules are applied. Moreover, observe that once established blocks can be broken and re-established again [*cf.* 146, pp. 4 f.]. Static blocking can be obtained by utilising a strategy that delays the application of generating rules to an entity u until no more non-generating rule can be applied to u [*cf.* 25, p. 18].

Lemma 7.2.1. *Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a constraint system and $u, w \in \mathfrak{A}$. An entity w that blocks an entity u cannot be blocked by any other entity in $\mathfrak{S}$.* ∇

Proof. Given $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ and $u, v, w \in \mathfrak{A}$, suppose that u is blocked by w, w is blocked by v and w is the least element w.r.t. the well-order $\prec$ that blocks u. This means that $CA(u) \subseteq CA(w)$ and $CA(w) \subseteq CA(v)$. Thus, $CA(u) \subseteq CA(v)$. It follows from the assumption that w is blocked by v that $v \prec w$. However, since w is the least element w.r.t. $\prec$ that blocks u it must be that $w \prec v$ which contradicts the assumption that $v \prec w$. $\square$

7.2.2 Fitting-style Representation of $\mathfrak{T}_{c\mathcal{ALC}}$

Before we continue with the correctness proof, we show that the tableau rules can be represented equivalently as rules in the spirit of Fitting [100] and we will use this notation in some examples. The rules are of the form

$$\frac{\mathfrak{C}\,;\,\mathfrak{A}}{\mathfrak{C}_1\,;\,\mathfrak{A}_1 \qquad \ldots \qquad \mathfrak{C}_n\,;\,\mathfrak{A}_n}\;\Theta\;(\xi)$$

where $\mathfrak{C}\,;\,\mathfrak{A}$ is the premise, Θ is a TBox, and $\mathfrak{C}_1\,;\,\mathfrak{A}_1 \ldots \mathfrak{C}_n\,;\,\mathfrak{A}_n$ are the consequences of rule ξ. The TBox Θ is assumed to be clear from the context and omitted accordingly from the presentation, except for the rule $(\rightarrow_{\texttt{ax}})$ that directly uses it. These rules must satisfy the respective pre-conditions w.r.t. rule ξ and only add a constraint in a conclusion if it is not already included in $\mathfrak{C}$. The presentation of the rules as depicted in Fig. 7.5 mimics that of the rules of Figure 7.1. We will write $\mathfrak{C}, \mathfrak{c}$ as a shortcut for $\mathfrak{C} \cup \{\mathfrak{c}\}$, and in the same sense we will write $\mathfrak{A}, u$ expressing $\mathfrak{A} \cup \{u\}$ for the active set.

The tableau rules of Fig. 7.5 construct a downward branching tree Υ where each node is a pair that consists of a set of constraints and a set of active variables. Like mentioned before, such a tree can be considered as the representation of the or-structure of its leaves. From here on, we will denote such a tree simply by *tableau*. In contrast to a generalised knowledge base, a tableau depicts the history of the constraint systems generated by the calculus, *i.e.*, the intermediate stages of construction become explicit in a tableau. A tableau is called *saturated* if no more rule is applicable to any branch of it. A branch of a tableau is *closed* if it contains a clash, and a tableau is *closed* if all its branches are closed. We add a superscript $i > 0$ and write $(\xi)^i$ to denote that rule ξ is applied i–times in a row. In the following examples we will underline clashing constraints and use the symbol ⨳ as an abbreviation for a clashed tableau branch.

$$\dfrac{\mathfrak{C}, u{:}{+}C \sqcap D \,;\, \mathfrak{A}}{\mathfrak{C}, u{:}{+}C, u{:}{+}D \,;\, \mathfrak{A}}\,(\to_{\sqcap+}) \qquad \dfrac{\mathfrak{C}, u{:}{-}C \sqcap D \,;\, \mathfrak{A}}{\mathfrak{C}, u{:}{-}C \,;\, \mathfrak{A} \qquad \mathfrak{C}, u{:}{-}D \,;\, \mathfrak{A}}\,(\to_{\sqcap-})$$

$$\dfrac{\mathfrak{C}, u{:}{+}C \sqcup D \,;\, \mathfrak{A}}{\mathfrak{C}, u{:}{+}C \,;\, \mathfrak{A} \qquad \mathfrak{C}, u{:}{+}D \,;\, \mathfrak{A}}\,(\to_{\sqcup+}) \qquad \dfrac{\mathfrak{C}, u{:}{-}C \sqcup D \,;\, \mathfrak{A}}{\mathfrak{C}, u{:}{-}C, u{:}{-}D \,;\, \mathfrak{A}}\,(\to_{\sqcup-})$$

$$\dfrac{\mathfrak{C}, u{:}{+}C \supset D \,;\, \mathfrak{A}}{\mathfrak{C}, u{:}{-}C \,;\, \mathfrak{A} \qquad \mathfrak{C}, u{:}{+}D \,;\, \mathfrak{A}}\,(\to_{\supset+}) \qquad \dfrac{\mathfrak{C}, u{:}{-}C \supset D \,;\, \mathfrak{A}}{\mathfrak{C}, u \preceq u', u'{:}{+}C, u'{:}{-}D \,;\, \mathfrak{A}, u'}\,(\to_{\supset-})$$

$$\dfrac{\mathfrak{C}, u\,R\,v, u{:}{+}\forall R.C \,;\, \mathfrak{A}}{\mathfrak{C}, v{:}{+}C \,;\, \mathfrak{A}}\,(\to_{\forall+}) \qquad \dfrac{\mathfrak{C}, u{:}{-}\forall R.C \,;\, \mathfrak{A}}{\mathfrak{C}, u \preceq u', u'{:}{-}_{\exists R}C \,;\, \mathfrak{A}, u'}\,(\to_{\forall-})$$

$$\dfrac{\mathfrak{C}, u{:}{+}\exists R.C \,;\, \mathfrak{A}}{\mathfrak{C}, u\,R\,v, v{:}{+}C \,;\, \mathfrak{A}}\,(\to_{\exists+}) \qquad \dfrac{\mathfrak{C}, u{:}{-}\exists R.C \,;\, \mathfrak{A}}{\mathfrak{C}, u \preceq u', u'{:}{-}_{\forall R}C \,;\, \mathfrak{A}, u'}\,(\to_{\exists-})$$

$$\dfrac{\mathfrak{C}, u\,R\,v, u{:}{-}_{\forall R}C \,;\, \mathfrak{A}}{\mathfrak{C}, v{:}{-}C \,;\, \mathfrak{A}, v}\,(\to_{R_{\forall}-}) \qquad \dfrac{\mathfrak{C}, u{:}{-}_{\exists R}C \,;\, \mathfrak{A}}{\mathfrak{C}, u\,R\,v, v{:}{-}C \,;\, \mathfrak{A}, v}\,(\to_{R_{\exists}-})$$

$$\dfrac{\mathfrak{C}, u \preceq u', u{:}{+}C \,;\, \mathfrak{A}}{\mathfrak{C}, u'{:}{+}C \,;\, \mathfrak{A}}\,(\to_{\preceq+}) \qquad \dfrac{\mathfrak{C} \,;\, \mathfrak{A}}{\mathfrak{C}, u{:}{+}C \,;\, \mathfrak{A}}\,C \in \Theta\ (\to_{\mathtt{ax}})$$

In the premise of all rules variable u must occur in $\mathfrak{A}$, and in the rules $(\to_{\supset-}), (\to_{\forall-}), (\to_{\exists+}), (\to_{\exists-})$ the variables u' and v are fresh.

Figure 7.5: Tableau rules for $c\mathcal{ALC}$.

Example 7.2.2. Validity of axiom $\mathsf{K}_{\exists R}$ is proved by the following closed tableau, starting from the initial constraint system $\mathfrak{S}_0 =_{df} (\{u_0{:}{-}\forall R.(C \supset D) \supset (\exists R.C \supset \exists R.D)\}, \{u_0\})$.

$$\begin{array}{ccl}
\multicolumn{2}{c}{\mathfrak{C}_0 = u_0{:}{-}\{\forall R.(C \supset D) \supset (\exists R.C \supset \exists R.D) \,;\, u_0} & \\ \cline{1-2}
\multicolumn{2}{c}{\mathfrak{C}_1 = \mathfrak{C}_0, u_0 \preceq u_1, u_1{:}{+}\forall R.(C \supset D), u_1{:}{-}\exists R.C \supset \exists R.D \,;\, u_0, u_1} & (\to_{\supset-}) \\ \cline{1-2}
\multicolumn{2}{c}{\mathfrak{C}_2 = \mathfrak{C}_1, u_1 \preceq u_2, u_2{:}{+}\exists R.C, u_2{:}{-}\exists R.D \,;\, u_0, u_1, u_2} & (\to_{\supset-}) \\ \cline{1-2}
\multicolumn{2}{c}{\mathfrak{C}_3 = \mathfrak{C}_2, u_2{:}{+}\forall R.(C \supset D) \,;\, \mathfrak{A}_1 = u_0, u_1, u_2} & (\to_{\preceq+}) \\ \cline{1-2}
\multicolumn{2}{c}{\mathfrak{C}_4 = \mathfrak{C}_3, u_2 \preceq u_3, u_3{:}{-}_{\forall R}D, u_3{:}{+}\exists R.C, u_3{:}{+}\forall R.(C \supset D) \,;\, \mathfrak{A}_2 = \mathfrak{A}_1, u_3} & (\to_{\exists-}), (\to_{\preceq+})^2 \\ \cline{1-2}
\multicolumn{2}{c}{\mathfrak{C}_4, u_3\,R\,v_1, v_1{:}{+}C \,;\, \mathfrak{A}_2} & (\to_{\exists+}) \\ \cline{1-2}
\multicolumn{2}{c}{\mathfrak{C}_5 = \mathfrak{C}_4, u_3\,R\,v_1, v_1{:}{+}C, v_1{:}{-}D, v_1{:}{+}C \supset D \,;\, \mathfrak{A}_3 = \mathfrak{A}_2, v_1} & (\to_{R_{\forall}-}), (\to_{\forall R+}) \\ \cline{1-2}
\mathfrak{C}_5, \underline{v_1{:}{+}C}, \underline{v_1{:}{-}C} \,;\, \mathfrak{A}_3 & \mathfrak{C}_5, \underline{v_1{:}{-}D}, \underline{v_1{:}{+}D} \,;\, \mathfrak{A}_3 & (\to_{\supset+}) \\ \cline{1-1} \cline{2-2}
\text{※} & \text{※} &
\end{array}$$

■

Example 7.2.3 (Countermodel construction)**.** Validity of axiom $\mathsf{FS4}/\mathsf{IK4}$ is refuted by the following saturated tableau, which is satisfiable, and starts from the initial constraint system $\mathfrak{S}_0 =_{df} (\{u_0{:}{-}\exists R.(C \sqcup D) \supset (\exists R.C \sqcup \exists R.D)\}, \{u_0\})$.

$$
\begin{array}{cl}
\mathfrak{C}_0 = u_0{:}-\{\exists R.(C \sqcup D) \supset (\exists R.C \sqcup \exists R.D) \,;\, u_0 & \\ \hline
\mathfrak{C}_0, u_0 \preceq u_1, u_1{:}+\exists R.(C \sqcup D), u_1{:}-\exists R.C \sqcup \exists R.D \,;\, u_0, u_1 & (\to_{\supset -}) \\ \hline
\mathfrak{C}_1 = \mathfrak{C}_0, u_0 \preceq u_1, u_1{:}+\exists R.(C \sqcup D), u_1{:}-\exists R.C \sqcup \exists R.D, u_1 \, R \, v_0, v_0{:}+C \sqcup D \,;\, u_0, u_1 & (\to_{\exists +}), (\to_{\sqcup +}) \\ \hline
\mathfrak{C}_2 = \mathfrak{C}_1, u_0 \preceq u_1, u_1{:}+\exists R.(C \sqcup D), u_1{:}-\exists R.C, u_1{:}-\exists R.D \,;\, u_0, u_1 & (\to_{\sqcup -}) \\ \hline
\mathfrak{C}_3 = \mathfrak{C}_2, u_1 \preceq u_2, u_2{:}-_{\forall R}C, u_1 \preceq u_3, u_3{:}-_{\forall R}D \,;\, \mathfrak{A}_0 = u_0, u_1, u_2, u_3 & (\to_{\exists -}), (\to_{\exists -}) \\ \hline
\mathfrak{C}_4 = \mathfrak{C}_3, u_2{:}+\exists R.(C \sqcup D), u_3{:}+\exists R.(C \sqcup D) \,;\, \mathfrak{A}_0 & (\to_{\preceq +})^2 \\ \hline
\mathfrak{C}_4, u_2 \, R \, v_1, v_1{:}+C \sqcup D, u_3 \, R \, v_2, v_2{:}+C \sqcup D \,;\, \mathfrak{A}_0 & (\to_{\exists +})^2 \\ \hline
\mathfrak{C}_5 = \mathfrak{C}_4, u_2 \, R \, v_1, v_1{:}+C \sqcup D, u_3 \, R \, v_2, v_2{:}+C \sqcup D, v_1{:}-C, v_2{:}-D \,;\, \mathfrak{A}_1 = \mathfrak{A}_0, v_1, v_2 & (\to_{R_\forall -})^2 \\ \hline
\begin{array}{cc}
\begin{array}{c}
\mathfrak{C}_5, \underline{v_1{:}+C}, \underline{v_1{:}-C} \,;\, \mathfrak{A}_1 \\ \hline
\text{※}
\end{array}
&
\begin{array}{cl}
\mathfrak{C}_5, v_1{:}+D, v_1{:}-C, v_2{:}+C \sqcup D, v_2{:}-D \,;\, \mathfrak{A}_1 & \\ \hline
\begin{array}{ccc}
\begin{array}{c}
\mathfrak{C}_6 = \mathfrak{C}_5, v_2{:}+C \,;\, \mathfrak{A}_1 \\ \hline
\mathfrak{C}_6, v_0{:}+D \,;\, \mathfrak{A}_1 \quad \mathfrak{C}_6, v_0{:}+C \,;\, \mathfrak{A}_1
\end{array}
& (\to_{\sqcup +}) &
\begin{array}{c}
\mathfrak{C}_5, \underline{v_2{:}+D}, \underline{v_2{:}-D} \,;\, \mathfrak{A}_1 \\ \hline
\text{※}
\end{array}
\end{array}
& (\to_{\sqcup +})
\end{array}
\end{array}
& (\to_{\sqcup +})
\end{array}
$$

We point out for the above derivation that $v_0{:}+C \sqcup D \in \mathfrak{C}_5$ and the decision of the disjunction $v_0{:}+C \sqcup D$ is postponed until the last step. One can easily extract a countermodel for axiom FS4/IK4 from the tableau, which is depicted by Fig. 7.6.

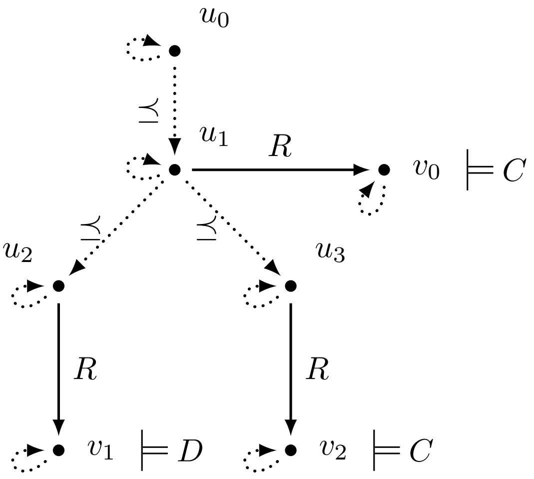

Figure 7.6: Countermodel for IK4/FS4.

■

7.3 Proof of Correctness

7.3.1 Termination

Definition 7.3.1. The set of roles that occur in a generalised knowledge base $\mathcal{K} = (\mathcal{M}, \Theta)$ is denoted by $R_{\mathcal{K}}$, and defined by

$$
\begin{aligned}
R_{\mathcal{K}} =_{df} \{R \mid\ & R \text{ occurs in a constraint } \mathsf{r} \text{ in some } \mathfrak{S} \in \mathcal{M} \\
& \text{or } R \text{ occurs in some axiom of } \Theta\}.
\end{aligned}
\qquad \nabla
$$

Definition 7.3.2 (Set of subconcepts)**.** Let C be a concept description, we define the set $sub(C)$ of subconcepts of C by

$$\begin{aligned} sub(A) &= \{A\}, \text{ for } A \in N_C \text{ or } A \in \{\bot, \top\}; \\ sub(C \odot D) &= \{C \odot D\} \cup sub(C) \cup sub(D), \text{ where } \odot \in \{\sqcap, \sqcup, \supset\}; \\ sub(QR.C) &= \{QR.C\} \cup sub(C), \text{ where } Q \in \{\exists, \forall\}. \end{aligned}$$

This is lifted to a finite set of concepts Γ in the usual way by taking $sub(\Gamma) =_{df} \bigcup_{C \in \Gamma} sub(C)$. We extend this to conceptual constraints by

$$\begin{aligned} sub(u{:}\pm C) &= sub(C); \\ sub(u{:}-_{\forall R} C) &= sub(C); \\ sub(u{:}-_{\exists R} C) &= sub(C). \end{aligned}$$

Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a finite constraint system. The set $sub(\mathfrak{S})$ is defined by

$$sub(\mathfrak{S}) = sub(\mathfrak{C}) = \bigcup_{\mathfrak{c} \in \mathfrak{C}} sub(\mathfrak{c}).$$

and analogously for a $\mathcal{K} = (\mathcal{M}, \Theta)$ one takes

$$sub(\mathcal{K}) = \bigcup_{\mathfrak{S} \in \mathcal{M}} sub(\mathfrak{S}) \cup sub(\Theta).$$

∇

According to [25, pp. 17 f.], instead of considering a finite set of axioms $C_1, C_2, \ldots, C_n$ of a TBox Θ in the tableau procedure, it is sufficient to take into account only the single axiom $\hat{C} \equiv \top$, defined by

$$\hat{C} =_{df} \bigwedge_{C \in \Theta} C,$$

where $\bigwedge$ is the intersection $\sqcap$ over a set of concepts, such that rule $(\rightarrow_{\mathsf{ax}})$ adds $u{:}+\hat{C}$ for some $u \in \mathfrak{A}$ to a constraint system. This means that rule $(\rightarrow_{\mathsf{ax}})$ only fires once for each active constraint variable.

Lemma 7.3.1. *The tableau rules of c$\mathcal{ALC}$ are monotonic and satisfy the subformula property:*

(i) If $\mathfrak{S} \rightarrow_{\xi} \mathfrak{S}'$ *and* $u \in Supp(\mathfrak{S})$ *then* $CA(u, \mathfrak{S}) \subseteq CA(u, \mathfrak{S}')$*.*

(ii) Let $\mathcal{K}_0 =_{df} (\mathfrak{S}_0, \Theta)$ *with* $\mathfrak{S}_0 = (\mathfrak{C}_0, \mathfrak{A}_0)$ *be a finite GKB and let us suppose that*

$\mathcal{K}_1 = (\mathcal{M}, \Theta)$ is obtained from $\mathcal{K}_0$ by a finite run of the c$\mathcal{ALC}$ tableau rules and let $\mathfrak{S} \in \mathcal{M}$. Then, for all conceptual constraints $u{:}\pm C$, $u{:}-_{\forall R}C$ or $u{:}-_{\exists R}C$ in $\mathfrak{S}$ it holds that $C \in sub(\mathcal{K}_0)$.

(iii) *Let $\mathcal{K}$ be a finite GKB, and $m =_{df} 2 * n + 2 * |R_{\mathcal{K}}| * n$ where $n =_{df} |sub(\mathcal{K})|$. The measure m is the maximal number of possible conceptual constraints (w.r.t. the three different kinds of conceptual constraints) bound to a variable in any $\mathcal{K}'$ reachable from $\mathcal{K}$ through application of the tableau rules.* ∇

Proof. The first two properties (i) and (ii) are easily observable by inspection of the tableau rules of Fig. 7.1, and (iii) is a consequence of (ii). □

Proposition 7.3.1 (Termination). *Let C be a c$\mathcal{ALC}$ concept and $\mathcal{K}_0 = (\{\mathfrak{S}_0\}, \Theta)$ a finite generalised knowledge base where the structure of $\mathfrak{S}_0$ is according to the key reasoning tasks of Prop. 7.1.1. Then, there is an upper bound on the length of any sequence of rule applications $\mathfrak{S}_0 \overset{\Theta}{\to}{}^*_{\xi} \mathcal{M}$ starting with $\mathcal{K}_0$.* ∇

Proof. The proof of termination follows the argumentation of Baader, Buchheit and Hollander [23] and is a consequence of the following facts: Let m be the bound as defined by Lem. 7.3.1.(iii).

(i) In each rule application step a constraint system is either replaced by one or in the non-deterministic case by at most two constraint systems. The non-deterministic case is triggered by the rules $(\to_{\sqcup+})$, $(\to_{\sqcap-})$ and $(\to_{\supset+})$. Hence, for each variable u of a constraint system the number of generated constraint systems is bounded by m, since there can be at most m constraints of the form $u{:}+C \sqcup D$, $u{:}-C \sqcap D$ and $u{:}+C \supset D$.

(ii) The tableau rules never remove constraints from a constraint system.

(iii) For all constraint systems $\mathfrak{S}$ and all variables $u \in Supp(\mathfrak{S})$ the size of $CA(u, \mathfrak{S})$ is bounded by m.

Suppose to the contrary that there is an unbounded sequence of rule applications that gives the GKBs $(\mathfrak{S}_0, \Theta), (\mathcal{M}_1, \Theta), (\mathcal{M}_2, \Theta), \ldots$. Since branching of a constraint system is finite by (i), König's-Lemma implies that there is an infinite sequence of constraint systems $\mathfrak{S}_0 \to \mathfrak{S}_1 \to \mathfrak{S}_2 \to \ldots$. For any entity u occurring in a constraint system in this sequence the size of $CA(u)$ is bound by m considering fact (iii) above and Lem. 7.3.1.(iii). Hence, there must be infinitely many entities generated to obtain an infinite sequence of rule applications, *i.e.*, the generating rules must have been applied infinitely often. The corresponding rules are $(\to_{\supset-}), (\to_{\exists-}), (\to_{\forall-}), (\to_{\exists+}), (\to_{R\exists-})$.

It is sufficient to consider the former three $\preceq$-generating rules: Let us consider rule $(\rightarrow_{\exists+})$ first. Any new entity generated by rule $(\rightarrow_{\exists+})$ is inactive at first. Remember that the tableau rules do not apply to inactive entities, and in particular they do not generate R- or $\preceq$-successors for inactive entities. Such an inactive entity can only become active in combination with rule $(\rightarrow_{\exists-})$ which introduces a $\preceq$-successor. Therefore, there cannot be infinitely many applications of rule $(\rightarrow_{\exists+})$ generating an infinite R-path without creating a new $\preceq$-successor by rule $(\rightarrow_{\exists-})$ each time, *i.e.*, the creation of infinitely many R-successors depends on the possibility to introduce infinitely many $\preceq$-successors. The argument is similarly for rule $(\rightarrow_{R_{\exists}-})$, *i.e.*, it introduces at each application step one new R-successor by expanding a constraint of the form $u{:}-_{\exists R}C$. Such a constraint needs to be introduced first by rule $(\rightarrow_{\forall-})$, *i.e.*, infinitely many R-successors can only be created by introducing infinitely many $\preceq$-successors. Thus, it suffices to consider the $\preceq$-generating rules only.

Observe that the generating rules cannot be applied to a fixed entity u infinitely often, since the number of constraints bound to u, for which one of the generating rules fires, is bound by m. Therefore, there must be an infinite chain $u_1, u_2, u_3, \ldots$ to which the generating rules were applied. Because $\prec$ is well-founded it holds for any entity u_i, $i \geq 0$ that there is no infinite decreasing chain from u_i, and therefore there can only be finitely many smaller entities. Therefore, we can assume without loss of generality that $u_1 \prec u_2 \prec u_3 \prec \ldots$, and assume that all these entities are new, *i.e.*, new entities introduced by a generating rule.

The argument is the same for the three $\preceq$-generating rules. Let *gen* denote one of $(\rightarrow_{\supset-}), (\rightarrow_{\exists-}), (\rightarrow_{\forall-})$. For all i, let $\mathfrak{S}_{j_i} \rightarrow_{gen} \mathfrak{S}_{j_i+1}$ be the step at which rule $(\rightarrow_{gen})$ is applied to entity u_i. Now let us consider the sets $CA(u_i, \mathfrak{S}_{j_i})$ and observe that there can only exist finitely many such sets (their number is bounded by m). Therefore, there must exist indices $k < l$ such that $u_k \prec u_l$ and $CA(u_k, \mathfrak{S}_{j_k}) = CA(u_l, \mathfrak{S}_{j_l})$. In particular, it holds that $CA(u_l, \mathfrak{S}_{j_l}) = CA(u_k, \mathfrak{S}_{j_k}) \subseteq CA(u_k, \mathfrak{S}_{j_l})$, since $j_k \prec j_l$, *i.e.*, in $\mathfrak{S}_{j_l}$ the number of constraints in which u_k occurs may have been increased. Since $u_k \prec u_l$ and u_l is new, it follows that entity u_l should have been blocked in the constraint system $\mathfrak{S}_{j_l}$. However, this contradicts our assumption that the rule $(\rightarrow_{gen})$ has been applied to u_l in $\mathfrak{S}_{j_l}$. But this contradicts the assumption that infinitely many entities are created. Hence, any sequence of rule applications terminates. $\square$

7.3.2 Soundness and Completeness

To prove the soundness of the tableau rules it is necessary to show that each rule is locally sound in the sense that its application preserves the satisfiability of the constraint system. The proof is by showing that the rules of $c\mathcal{ALC}$ are regular according to Definition 7.2.6, which corresponds to the local correctness of the $c\mathcal{ALC}$ tableau rules.

Proposition 7.3.2 (Regularity of $c\mathcal{ALC}$ tableau rules). *Each tableau rule ξ of $c\mathcal{ALC}$ is regular.* ∇

Proof. Let $\mathfrak{S}$, $\mathfrak{S}'$ be a constraint system, let Θ be an arbitrary but fixed TBox and ξ a tableau rule of $c\mathcal{ALC}$. Note that we consider the TBox Θ implicitly and omit it in the following.

The goal is to show for all interpretations $\mathcal{I}$ that there exists a valuation α such that $\mathcal{I}, \alpha \models \mathfrak{S}$ if and only if there exists a constraint system $\mathfrak{S}'$ in $\mathcal{M}$ and an (possibly) extended valuation α' such that $\mathcal{I}, \alpha' \models \mathfrak{S}'$.

($\Leftarrow$) This direction can be shown in general for all rules by monotonicity. Let $\mathfrak{S}' = (\mathfrak{C}', \mathfrak{A}')$ be derived from $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ by an application of a tableau rule ξ of $c\mathcal{ALC}$, *i.e.*, $\mathfrak{S} \rightarrow_{\xi} \mathfrak{S}'$. Assume that $\mathfrak{S}'$ is satisfiable, then there exists a model $(\mathcal{I}, \alpha)$ that satisfies $\mathfrak{S}'$. The derived constraint system $\mathfrak{S}'$ differs from $\mathfrak{S}$ by having an extension of both, the set of constraints $\mathfrak{C}$ and the active set $\mathfrak{A}$. According to the rules in Figure 7.1 the following holds: Both, the set of constraints and the set of active entities in $\mathfrak{S}$ are included in its extensions, namely $\mathfrak{C} \subseteq \mathfrak{C}'$ with $\mathfrak{C}' = \mathfrak{C} \cup \mathfrak{C}_e$ and $\mathfrak{A} \subseteq \mathfrak{A}'$ with $\mathfrak{A}' = \mathfrak{A} \cup \mathfrak{A}_e$ where the index e labels the respective extension. Therefore, the former assumption $\mathcal{I}; \alpha \models \mathfrak{S}'$ yields $\mathcal{I}, \alpha \models \mathfrak{S}$. Hence $\mathcal{I}; \alpha \models \mathfrak{S}$, *i.e.*, $\mathfrak{S}$ is satisfiable. The non-deterministic case of $\mathfrak{S} \rightarrow_{\xi} \mathfrak{S}', \mathfrak{S}''$ is argued analogously.

($\Rightarrow$) Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a constraint system and $\mathcal{I}$ be arbitrarily chosen, and suppose there exists an α such that $\mathcal{I}, \alpha \models \mathfrak{S}$ holds. We have to show for each completion rule ξ separately that under the assumption that $\mathfrak{S}$ is satisfiable we can give an extension of its model that satisfies $\mathfrak{S}'$ as well. This is proven for each completion rule separately.

As a reminder note that a given constraint system $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ is satisfiable if and only if there exists a pair $(\mathcal{I}, \alpha)$ that satisfies $\mathfrak{S}$, *i.e.*, for all constraints $\mathfrak{c} \in \mathfrak{C}$ it holds that $\mathcal{I}; \alpha \models \mathfrak{c}$ and for all $u \in \mathfrak{A}$ the assignment $\alpha(u)$ is infallible.

($\rightarrow_{\sqcap+}$) Assume that $\mathfrak{S}$ contains a constraint $u{:}{+}C \sqcap D$ with $u \in \mathfrak{A}$. Suppose that the rule $\rightarrow_{\sqcap+}$ has been applied to $\mathfrak{S}$ yielding a constraint system $\mathfrak{S}' = (\mathfrak{C}', \mathfrak{A})$ which differs from $\mathfrak{S}$ by having the constraints $u{:}{+}C$ and $u{:}{+}D$. We claim that the model $(\mathcal{I}, \alpha)$ of $\mathfrak{S}$ also yields a model for $\mathfrak{S}'$. By assumption $u{:}{+}C \sqcap D$ is satisfied and from Def. 4.2.2 it follows that $\alpha(u)$ is in the intersection of $C^{\mathcal{I}}$ and $D^{\mathcal{I}}$. Therefore it holds that $\mathcal{I};\alpha \models u{:}{+C}$ and $\mathcal{I};\alpha \models u{:}{+D}$. Hence, $(\mathcal{I},\alpha)$ is also a model for $\mathfrak{S}' = (\mathfrak{C} \cup \{u{:}{+C}, u{:}{+D}\}, \mathfrak{A})$.

$(\rightarrow_{\sqcap-})$ Suppose the constraint system $\mathfrak{S}$ contains $u{:}{-C} \sqcap D$ with $u \in \mathfrak{A}$, and the rule $\rightarrow_{\sqcap-}$ has been applied to $\mathfrak{S}$. This yields the two constraint systems $\mathfrak{S}' = ((\mathfrak{C} \cup \{u{:}{-C}\}, \mathfrak{A})$ and $\mathfrak{S}'' = ((\mathfrak{C} \cup \{u{:}{-D}\}, \mathfrak{A})$. The goal is to show that the pair $(\mathcal{I},\alpha)$ yields either a model for $\mathfrak{S}'$ or $\mathfrak{S}''$. The constraint systems $\mathfrak{S}'$ and $\mathfrak{S}''$ differ from $\mathfrak{S}$ by $u{:}{-C}$ or $u{:}{-D}$ respectively. By assumption it holds that $\alpha(u) \notin (C \sqcap D)^{\mathcal{I}}$, *i.e.*, either $\alpha(u) \notin C^{\mathcal{I}}$ or $\alpha(u) \notin D^{\mathcal{I}}$. By case analysis, if $\alpha(u) \notin C^{\mathcal{I}}$ then it follows that $\mathcal{I};\alpha \models x{:}{-C}$, in the other case $\alpha(u) \notin D^{\mathcal{I}}$ yields $\mathcal{I};\alpha \models u{:}{-D}$. Together with the former assumption $\mathcal{I};\alpha \models \mathfrak{S}$ it follows that $(\mathcal{I},\alpha)$ also models either $\mathfrak{S}'$ or $\mathfrak{S}''$.

$(\rightarrow_{\sqcup+})$ Let us suppose that $\mathfrak{S} = (\mathfrak{C} \cup \{u{:}{+C} \sqcup D\}, \mathfrak{A})$ with $u \in \mathfrak{A}$. An application of the rule $\rightarrow_{(\sqcup+)}$ to $\mathfrak{S}$ yields the constraint systems $\mathfrak{S}' = (\mathfrak{C} \cup \{u{:}{+C}\}, \mathfrak{A})$ and $\mathfrak{S}'' = (\mathfrak{C} \cup \{u{:}{+D}\}, \mathfrak{A})$. By assumption $\mathcal{I};\alpha \models u{:}{+C} \sqcup D$, *i.e.*, $\alpha(u)$ is in the union of $C^{\mathcal{I}}$ and $D^{\mathcal{I}}$, which means $\alpha(u) \in C^{\mathcal{I}}$ or $\alpha(u) \in D^{\mathcal{I}}$. Then, $\mathcal{I};\alpha \models u{:}{+C}$ or $\mathcal{I};\alpha \models u{:}{+D}$. Thus, $(\mathcal{I},\alpha)$ is a model for $\mathfrak{S}'$ or $\mathfrak{S}''$.

$(\rightarrow_{\sqcup-})$ The application of the rule $(\rightarrow_{\sqcup-})$ is triggered by a constraint $u{:}{-C} \sqcup D$ in $\mathfrak{S}$ with $u \in \mathfrak{A}$, which yields the constraint system $\mathfrak{S}' = (\mathfrak{C} \cup \{u{:}{-C}, u{:}{-D}\}, \mathfrak{A})$. By assumption $\mathcal{I};\alpha \models u{:}{-C} \sqcup D$ holds, *i.e.*, $\alpha(u) \notin (C \sqcup D)^{\mathcal{I}}$ which means by Def. 4.2.2 that the valuation of u is neither contained in $C^{\mathcal{I}}$ nor in $D^{\mathcal{I}}$. From there it follows that $\mathcal{I};\alpha \models u{:}{-C}$ and $\mathcal{I};\alpha \models u{:}{-D}$ hold. Hence, $\mathcal{I};\alpha \models \mathfrak{S}' = (\mathfrak{C} \cup \{u{:}{-C}, u{:}{-D}\}, \mathfrak{A})$.

$(\rightarrow_{\supset+})$ Let us assume that $\mathfrak{S}$ contains a constraint $u{:}{+C} \supset D$ with $u \in \mathfrak{A}$. The application of the rule $(\rightarrow_{\supset+})$ yields the constraint systems $\mathfrak{S}' = (\mathfrak{C} \cup \{u{:}{-C}\}, \mathfrak{A})$ and $\mathfrak{S}'' = (\mathfrak{C} \cup \{u{:}{+D}\}, \mathfrak{A})$. By assumption $\mathcal{I};\alpha \models u{:}{+C} \supset D$, and from Def. 4.2.2 it follows that for all $\preceq$-successors x of $\alpha(u)$, if $x \in C^{\mathcal{I}}$ then $x \in D^{\mathcal{I}}$. In particular, $\alpha(u) \preceq^{\mathcal{I}} \alpha(u)$ by reflexivity. Hence, $\alpha(u) \notin C^{\mathcal{I}}$ or $\alpha(u) \in D^{\mathcal{I}}$, *i.e.*, $\mathcal{I};\alpha \models u{:}{-C}$ or $\mathcal{I};\alpha \models u{:}{+D}$. This implies that $\mathcal{I};\alpha$ is also a model for $\mathfrak{S}'$ or $\mathfrak{S}''$.

$(\rightarrow_{\supset-})$ Suppose that $\mathfrak{S}$ contains the constraint $u{:}{-C} \supset D$ with $u \in \mathfrak{A}$, which triggers the application of the rule $(\rightarrow_{\supset-})$ such that the constraint system $\mathfrak{S}' = (\mathfrak{C} \cup \{u \preceq u', u'{:}{+C}, u'{:}{-D}\}, \mathfrak{A} \cup \{u'\})$ is derived, where u' is a fresh active variable in *Supp*$(\mathfrak{S}')$. The assumption implies that $(\mathcal{I},\alpha)$ satisfies $u{:}{-C} \supset D$, *i.e.*, $\alpha(u) \notin (C \supset D)^{\mathcal{I}}$. Formally this means that the valuation of u is not contained in the interpretation of $C \supset D$. Taking into account Def. 4.2.2, there exists a refinement $x \in \Delta^{\mathcal{I}}$ of $\alpha(u)$ such that $x \in C^{\mathcal{I}}$ and $x \notin D^{\mathcal{I}}$, where the latter implies $x \notin \bot^{\mathcal{I}}$. We extend the valuation α by the $\mathcal{I}$-assignment $\alpha'(u') = x$. This extension yields a model for $\mathfrak{S}'$ such that $\mathcal{I}; \alpha' \models \mathfrak{S}'$.

$(\rightarrow_{\forall+})$ The application of the rule $(\rightarrow_{\forall+})$ is due to the constraints $u{:}{+}\forall R.C$ and $u\ R\ v$ in $\mathfrak{S}$, with $u \in \mathfrak{A}$. The result is the constraint system $\mathfrak{S}' = (\mathfrak{C} \cup \{v{:}{+}C\}, \mathfrak{A})$. By assumption the pair $(\mathcal{I}, \alpha)$ satisfies $u{:}{+}\forall R.C$, *i.e.*, all $\preceq^{\mathcal{I}}; R^{\mathcal{I}}$-successors of $\alpha(u)$ are included in the interpretation of C. Since by assumption $\alpha(u)\ R^{\mathcal{I}}\ \alpha(v)$ and $\alpha(u) \preceq^{\mathcal{I}} \alpha(u)$ by reflexivity of $\preceq^{\mathcal{I}}$, it follows that $\alpha(u) \preceq^{\mathcal{I}}; R^{\mathcal{I}}\ \alpha(v)$. Thus, $\alpha(v) \in C^{\mathcal{I}}$, *i.e.*, $\mathcal{I}; \alpha \models v{:}{+}C$. Hence $\mathfrak{S}'$ is satisfied by the model $(\mathcal{I}, \alpha)$.

$(\rightarrow_{\forall-})$ Suppose that the rule $(\rightarrow_{\forall-})$ has been applied to the constraint system $\mathfrak{S}$ with $u{:}{-}\forall R.C \in \mathfrak{C}$ and $u \in \mathfrak{A}$. This derives the constraint system $\mathfrak{S}' = (\mathfrak{C}', \mathfrak{A}')$ containing the additional constraints $u \preceq u'$ and $u'{:}{-}_{\exists R}C$, where u' is a fresh active variable and $u' \in \mathfrak{A}'$. We have to show that an extension of the model of $\mathfrak{S}$ also satisfies $\mathfrak{S}'$, *i.e.*, it has to satisfy the additional constraint of $\mathfrak{S}'$ and the valuation of u' has to be non-fallible. By assumption, it holds that $u{:}{-}\forall R.C$ is satisfied, that is $\alpha(u) \notin (\forall R.C)^{\mathcal{I}}$ which means that there exists a $\preceq^{\mathcal{I}}$-refinement x of $\alpha(u)$ and an $R^{\mathcal{I}}$-successor y of x such that y is not in the interpretation of C. Non-fallibility of x follows directly from the fact that by assumption $y \notin C^{\mathcal{I}}$ and thus y is non-fallible, and Prop. 4.2.1 establishes the non-fallibility of x. Taking the extended assignment $\alpha'(u') = x$ implies that $(\mathcal{I}, \alpha')$ satisfies $u{:}{-}_{\exists R}C$ by Def. 7.1.4. Hence, the extended model also satisfies $\mathfrak{S}'$.

$(\rightarrow_{\exists+})$ Assume that the constraint system $\mathfrak{S}$ including the constraint $u{:}{+}\exists R.C$ with $u \in \mathfrak{A}$ has a model. The application of the rule $(\rightarrow_{\exists+})$ yields the constraint system $\mathfrak{S}' = (\mathfrak{C}', \mathfrak{A})$ containing the two additional constraints $u\ R\ v$ and $v{:}{+}C$, where v is a fresh variable in $\mathit{Supp}(\mathfrak{S}')$. Note that v is not active, *i.e.*, its fallibility cannot be determined. We have to show that the interpretation $\mathcal{I}$ of $\mathfrak{S}$ with an appropriate α-assignment can be extended to also satisfy $\mathfrak{S}'$. By the assumption it holds that $\alpha(u) \in (\exists R.C)^{\mathcal{I}}$. Reflexivity of $\preceq^{\mathcal{I}}$ implies $\alpha(u) \preceq^{\mathcal{I}} \alpha(u)$ and according to Def. 4.2.2 there exists an $R^{\mathcal{I}}$-successor y of $\alpha(u)$ which lies in the interpretation of C. Then, the extension of the $\mathcal{I}$-assignment α by $\alpha'(v) =_{\mathit{df}} y$ implies that $\mathcal{I}; \alpha' \models \mathfrak{S}'$.

($\rightarrow_{\exists-}$) The application of rule ($\rightarrow_{\exists-}$) is caused due to the constraint $u{:}{-}\exists R.C \in \mathfrak{C}$ with $u \in \mathfrak{A}$. This results in the constraint system $\mathfrak{S}' = (\mathfrak{C} \cup \{u \preceq u', u'{:}{-}_{\forall R}C\}, \mathfrak{A} \cup \{u'\})$ where u' is a fresh variable. Since $(\mathcal{I}, \alpha)$ is a model for $\mathfrak{S}$, in particular $\mathcal{I}; \alpha \models u{:}{-}\exists R.C$, there exists a $\preceq^{\mathcal{I}}$-refinement x of $\alpha(u)$ such that all $R^{\mathcal{I}}$-successors of x are not in the interpretation of C. Formally this is expressed by $\alpha(u) \preceq^{\mathcal{I}} x$ and $\forall y \in \Delta^{\mathcal{I}}.\, x\ R^{\mathcal{I}}\ y \Rightarrow y \notin C^{\mathcal{I}}$. Non-fallibility of x follows from Prop. 4.2.1 and the fact that if x was fallible then there would exist a fallible $R^{\mathcal{I}}$-successor, but this is not possible. Therefore, the extension of $(\mathcal{I}, \alpha')$ by the $\mathcal{I}$-assignment $\alpha'(u') = x$ also satisfies $\mathfrak{S}'$.

($\rightarrow_{R_\forall-}$) Suppose that $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ contains the constraints $u{:}{-}_{\forall R}C$ and $u\ R\ v$ in $\mathfrak{C}$ with $u \in \mathfrak{A}$. An application of the rule ($\rightarrow_{R_\forall-}$) to $\mathfrak{S}$ derives the constraint system $\mathfrak{S}' = (\mathfrak{C} \cup \{v{:}{-}C\}, \mathfrak{A} \cup \{v\})$. Since $(\mathcal{I}, \alpha)$ is a model for $\mathfrak{S}$, it follows that $(\alpha(u), \alpha(v)) \in R^{\mathcal{I}}$ and in particular $\mathcal{I}; \alpha \models u{:}{-}_{\forall R}C$ which implies that all $R^{\mathcal{I}}$-successors of $\alpha(u) \in \Delta^{\mathcal{I}}$ are not contained in the interpretation of C. Therefore, $\alpha(v) \notin C^{\mathcal{I}}$, which also implies that $\alpha(v)$ is infallible. Hence, $\mathcal{I}; \alpha \models \mathfrak{S}'$.

($\rightarrow_{R_\exists-}$) Suppose that $u{:}{-}_{\exists R}C$ is in $\mathfrak{S}$ with $u \in \mathfrak{A}$. Moreover, assume that the rule ($\rightarrow_{R_\exists-}$) has been applied to the constraint system $\mathfrak{S}$, which derives the constraint system $\mathfrak{S}' = (\mathfrak{C}', \mathfrak{A}')$ containing the additional constraints $u\ R\ v$ and $v{:}{-}C$ in $\mathfrak{C}'$, where v is a fresh variable and $v \in \mathfrak{A}'$. By assumption $(\mathcal{I}, \alpha)$ is a model for $\mathfrak{S}$, and in particular $\mathcal{I}; \alpha \models u{:}{-}_{\exists R}C$, which implies that there exists an $R^{\mathcal{I}}$-successor $y \in \Delta^{\mathcal{I}}$ such that $y \notin C^{\mathcal{I}}$ and therefore $y \notin \bot^{\mathcal{I}}$. By extending α by the assignment $\alpha'(v) = y$ it follows that $(\mathcal{I}, \alpha')$ satisfies $\mathfrak{S}'$ as well.

($\rightarrow_{\preceq+}$) Suppose that the constraint system $\mathfrak{S}$ contains the constraint $u{:}{+}C$ with $u \in \mathfrak{A}$. Moreover, assume that there exists a $\preceq^{+}$-successor u' of u such that $\mathfrak{S}' = (\mathfrak{C} \cup \{u'{:}{+}C\}, \mathfrak{A})$ has been derived from $\mathfrak{S}$ by an application of rule ($\rightarrow_{\preceq+}$). By assumption $\alpha(u) \in C^{\mathcal{I}}$ and $\alpha(u) \preceq^{\mathcal{I}} \alpha(u')$ by transitivity of $\preceq^{\mathcal{I}}$. Then, it follows by Proposition 4.2.2 that $\alpha(u') \in C^{\mathcal{I}}$. Hence the model of $\mathfrak{S}$ is also a model for $\mathfrak{S}'$.

($\rightarrow_{\mathtt{ax}}$) Suppose that $\mathcal{I}; \alpha \models \mathfrak{S}$ w.r.t. the TBox Θ, and that rule ($\rightarrow_{\mathtt{ax}}$) has been applied to some $u \in \mathfrak{A}$. This yields a constraint system $\mathfrak{S}'$ with a TBox axiom from Θ additionally added to $\mathfrak{C}$ as a positive constraint bound to u. The goal $\mathcal{I}; \alpha \models \mathfrak{S}'$ holds trivially, since by assumption $\mathcal{I} \models \Theta$. □

We shall see that if a constraint system $\mathfrak{S}$ together with a TBox, *viz.* $(\mathfrak{S}, \Theta)$, is saturated and clash-free then this yields a constructive pre-model $(\mathcal{I}, \alpha)$ which can be utilized to build a constructive model.

Proposition 7.3.3. *If a constraint system is clashed then it is not satisfiable.* ∇

Proof. Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a constraint system and assume that $\mathfrak{S}$ contains a clash. We have the following cases for a clash:

(i) $\mathfrak{C}$ contains the constraints $u{:}{+}C$ and $u{:}{-}C$. Now assume to the contrary that $\mathfrak{S}$ is satisfiable, *i.e.*, by Def. 7.1.4 there is a pair $(\mathcal{I}, \alpha)$ such that $\mathcal{I}; \alpha \models \mathfrak{S}$. More precisely $\mathcal{I}; \alpha \models u{:}{+}C$ if $\alpha(u) \in C^{\mathcal{I}}$ and $\mathcal{I}; \alpha \models u{:}{-}C$ if $\alpha(u) \notin C^{\mathcal{I}}$. But it cannot be that $\alpha(u) \in C^{\mathcal{I}}$ and $\alpha(u) \notin C^{\mathcal{I}}$ at the same time which contradicts the former assumption. Hence, $\mathfrak{S}$ is not satisfiable.

(ii) $\mathfrak{C}$ contains a constraint $u{:}{+}\bot$ and $u \in \mathfrak{A}$. Assume again that $\mathfrak{S}$ is satisfiable, *i.e.*, there is a pair $(\mathcal{I}, \alpha)$ such that $\mathcal{I}; \alpha \models u{:}{+}\bot$ holds. This means $\alpha(u) \in \bot^{\mathcal{I}}$, *i.e.*, u is a fallible entity. By Def. 7.1.4 this contradicts the assumption that $(\mathcal{I}, \alpha)$ satisfies $\mathfrak{S}$, which requires that $\alpha(u)$ is infallible. Thus, $\mathfrak{S}$ is not satisfiable.

(iii) $\mathfrak{C}$ contains a constraint $u{:}{+}\bot$ and one of $u{:}{-}C$, $u{:}{-}_{\exists R}C$ or $u{:}{-}_{\forall R}C$. Assume that $\mathfrak{S}$ is satisfiable, *i.e.*, there is a pair $(\mathcal{I}, \alpha)$ such that $\mathcal{I}; \alpha \models u{:}{+}\bot$ holds. The assumption implies $\alpha(u) \in \bot^{\mathcal{I}}$, *i.e.*, $\alpha(u)$ is fallible. However, by Def. 7.1.4 from the other constraints it follows that at the same time u is not in the interpretation of concept C, there is one $R^{\mathcal{I}}$-successor of u that falsifies concept C, or in all $R^{\mathcal{I}}$ successors of u concept C is false. All cases are impossible for the fallible entity u, because by Lem. 4.2.2 $\bot^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for arbitrary concepts D, and fallibility is closed under $R^{\mathcal{I}}$. This contradicts the assumption that $\mathfrak{S}$ is satisfiable. Therefore, $\mathfrak{S}$ is not satisfiable. $\square$

Proposition 7.3.4 (Clashed GKB is unsatisfiable)**.** *If a generalised knowledge base $\mathcal{K}$ is clashed then it is not satisfiable.* ∇

Proof. Suppose that $\mathcal{K} = (\mathcal{M}, \Theta)$ is clashed. Then, it holds for all $\mathfrak{S} \in \mathcal{M}$ that $\mathfrak{S}$ contains a clash by Def. 7.2.9, and Proposition 7.3.3 implies that no $\mathfrak{S}$ is satisfiable. Hence, by Def. 7.2.2 it follows that $\mathcal{K}$ is not satisfiable. $\square$

Proposition 7.3.5 (Invariance of a constraint system)**.** *Let $\mathfrak{S}$ be a non-speculative constraint system, Θ an arbitrary TBox and ξ be a tableau rule of $c\mathcal{ALC}$. If $\mathfrak{S} \overset{\Theta}{\rightarrow}_{\xi} \mathcal{M}$ then $\mathfrak{S}$ is satisfiable w.r.t. Θ iff there exists $\mathfrak{S}'$ in $\mathcal{M}$ s.t. $\mathfrak{S}'$ is satisfiable w.r.t. Θ.* ∇

Proof. This follows directly from Prop. 7.3.2 and Def. 7.2.6. $\square$

The remaining section introduces the construction of a canonical interpretation that can be transformed into a model for a clash-free constraint system.

Definition 7.3.3 (Canonical interpretation)**.** For a constraint system $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$, the canonical interpretation $\mathcal{I}_{\mathfrak{S}} = (\Delta^{\mathcal{I}_{\mathfrak{S}}}, \preceq^{\mathcal{I}_{\mathfrak{S}}}, \perp^{\mathcal{I}_{\mathfrak{S}}}, \cdot^{\mathcal{I}_{\mathfrak{S}}})$ is defined by:

$$\Delta^{\mathcal{I}_{\mathfrak{S}}} =_{df} \mathit{Supp}(\mathfrak{C}); \tag{7.10}$$

$$\begin{aligned} A^{\mathcal{I}_{\mathfrak{S}}} =_{df} \{u \mid\ & u{:}{+}A \in \mathfrak{C} \text{ or} && (7.11)\\ & u{:}{+}\perp \in \mathfrak{C} \text{ or} && (7.12)\\ & u \notin \mathfrak{A}\} \text{ for every } A \in N_C; && (7.13) \end{aligned}$$

$$\begin{aligned} \preceq^{\mathcal{I}_{\mathfrak{S}}} =_{df} \{(u, u') \in \Delta^{\mathcal{I}_{\mathfrak{S}}} \times \Delta^{\mathcal{I}_{\mathfrak{S}}} \mid\ & \exists n > 0 . \exists u_1, u_2, \ldots, u_n . u = u_1 \wedge u' = u_n . \forall 1 \leq i < n . \\ & u_i \preceq u_{i+1} \in \mathfrak{C} \text{ or} && (7.14)\\ & u_i \text{ is blocked by some } w, \text{ such that } w \preceq u_{i+1} \in \mathfrak{C}\}; && (7.15) \end{aligned}$$

$$\begin{aligned} R^{\mathcal{I}_{\mathfrak{S}}} =_{df} \{(u, v) \in \Delta^{\mathcal{I}_{\mathfrak{S}}} \times \Delta^{\mathcal{I}_{\mathfrak{S}}} \mid\ & u\, R\, v \in \mathfrak{C} \text{ or} && (7.16)\\ & u{:}{+}\perp \in \mathfrak{C} \text{ and } u = v \text{ or} && (7.17)\\ & u \notin \mathfrak{A} \text{ and } u = v \text{ or} && (7.18)\\ & u \text{ is blocked by some } w, \text{ such that } w\, R\, v \in \mathfrak{C}\}; && (7.19) \end{aligned}$$

$$\perp^{\mathcal{I}_{\mathfrak{S}}} =_{df} \{u \mid u \notin \mathfrak{A}\}. \tag{7.20}$$

∇

Let $\mathcal{K}_0 = (\mathfrak{S}_0, \Theta)$ be a non-speculative GKB and assume that $\mathcal{K}^* = (\mathcal{M}, \Theta)$ is a Θ-saturated GKB obtained from $\mathcal{K}_0$ by applying the completion rules of Fig. 7.1. Let us choose $\mathfrak{S} \in \mathcal{M}$ such that $\mathfrak{S}$ is clash-free, and let $\mathcal{I}_{\mathfrak{S}}$ be the corresponding canonical interpretation. The following section demonstrates that $\mathcal{I}_{\mathfrak{S}}$ can be used to construct a model of $\mathfrak{S}$ w.r.t. Θ, similarly to the method of Baader, Buchheit and Hollander [23, p. 209]. Due to blocking, $\mathcal{I}_{\mathfrak{S}}$ is not necessarily a model for $\mathfrak{S}_0$, because for a blocked entity $u \in \mathit{Supp}(\mathfrak{S})$ there may exist relational constraints of the form $u\, R\, v \in \mathfrak{C}$ or $u \preceq v \in \mathfrak{C}$ for some $v \in \mathit{Supp}(\mathfrak{S})$, which have been introduced in the derivation sequence before u has been blocked. Such relational constraints must not be satisfied by the extended canonical interpretation and therefore we need to alter the constraint system by dropping such relational constraints w.r.t. a blocked entity.

Definition 7.3.4 (Reduced constraint system)**.** Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a Θ-saturated, clash-free and non-speculative constraint system. The *reduced* constraint system is given by $red(\mathfrak{S}) =_{df} (red(\mathfrak{C}), \mathfrak{A})$, where $red(\mathfrak{C})$ is defined by

$$\begin{aligned} red(\mathfrak{C}) =_{df} &\{\mathfrak{c} \mid \mathfrak{c} \text{ is a conceptual constraint in } \mathfrak{C}\} \\ &\cup \{u \preceq u' \mid u \preceq u' \in \mathfrak{C} \;\&\; u \text{ is not blocked}\} \\ &\cup \{u\, R\, v \mid u\, R\, v \in \mathfrak{C} \;\&\; u \text{ is not blocked}\}. \end{aligned}$$

We state that a constraint system is *reduced* if for all $u, u' \in \mathfrak{S}$ it holds that if $u \preceq u' \in \mathfrak{C}$ or $u\, R\, u' \in \mathfrak{C}$ then u is not blocked. Obviously, $red(\mathfrak{S})$ is a reduced constraint system. ∇

Lemma 7.3.2. *If a constraint system $\mathfrak{S}$ is Θ-saturated, clash-free and non-speculative then $red(\mathfrak{S})$ is Θ-saturated, clash-free and non-speculative.* ∇

Proof. Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a Θ-saturated, clash-free and non-speculative constraint system *i.e.*, no $c\mathcal{ALC}$ tableau rule is applicable to any constraint in $\mathfrak{C}$. The goal is to show that $red(\mathfrak{S}) = (\mathfrak{C}' = red(\mathfrak{C}), \mathfrak{A})$ is Θ-saturated, clash-free and non-speculative as well.

First observe that $red(\mathfrak{S})$ is clash-free and non-speculative as well. This follows from the fact that by Def. 7.3.4 the reduction does not add any conceptual constraint, but only removes relational constraints of blocked constraint variables.

Secondly observe that $Supp(\mathfrak{S}) = Supp(red(\mathfrak{S}))$. Let $u \in Supp(\mathfrak{S})$ be arbitrary, and

$$\begin{aligned} \kappa(u, \mathfrak{S}) =_{df} \{\mathfrak{c} \mid\ & \mathfrak{c} \text{ is a conceptual constraint for } u \text{ in } \mathfrak{C} \\ & \text{or for some } v \in Supp(\mathfrak{S}) \text{ either } \mathfrak{c} = u\, R\, v \in \mathfrak{C} \text{ or } \mathfrak{c} = u \preceq v \in \mathfrak{C}\}. \end{aligned}$$

We proceed by case analysis:

Case 1. If u is not blocked then $\kappa(u, \mathfrak{S}) = \kappa(u, red(\mathfrak{S}))$, *i.e.*, $\mathfrak{S}$ and $red(\mathfrak{S})$ coincide in the constraints of entities that are not blocked. Hence, no tableau rule can be applied to any conceptual constraint in which u occurs.

Case 2. If u is blocked by some $w \in Supp(\mathfrak{S})$ then by Def. 7.2.10 (blocking) and Def. 7.3.4 it holds that $CA(u, \mathfrak{S}) = CA(u, red(\mathfrak{S}))$, *i.e.*, $\mathfrak{S}$ and $red(\mathfrak{S})$ coincide in the conceptual constraint of each blocked entity, which means that u is saturated for all non-generating rules on the spot. Because u is blocked it follows that no generating rule can be applied to it.

Since u was arbitrarily chosen, it follows that no rule can be applied to $red(\mathfrak{S})$. Thus, by Def. 7.2.8 the reduced constraint system $red(\mathfrak{S})$ is Θ-saturated. $\square$

Proposition 7.3.6. *Let $\mathfrak{S}$ be a Θ-saturated, clash-free, non-speculative and reduced constraint system, then the canonical interpretation $\mathcal{I}_{\mathfrak{S}}$ is a constructive interpretation of $\mathfrak{S}$ in the sense of Def. 4.2.2.* ∇

Proof. Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a Θ-saturated, clash-free, non-speculative and reduced constraint system and let $\mathcal{I}_{\mathfrak{S}}$ be the canonical interpretation of $\mathfrak{S}$. We claim that $\mathcal{I}_{\mathfrak{S}}$ is a constructive interpretation of $\mathfrak{S}$ according to Def. 4.2.2.

- The set $\Delta^{\mathcal{I}_{\mathfrak{S}}} = \mathit{Supp}(\mathfrak{C})$ is non-empty, since by Definition 7.1.1 $\mathfrak{C}$ has to be non-empty for any constraint system.

- The interpretation of atomic concepts $A \in N_C$ is given by $A^{\mathcal{I}_{\mathfrak{S}}}$. Assume $u \in A^{\mathcal{I}_{\mathfrak{S}}}$ and $u \preceq^{\mathcal{I}_{\mathfrak{S}}} u'$. We claim $u' \in A^{\mathcal{I}_{\mathfrak{S}}}$.

  Case 1. If $u \notin \mathfrak{A}$ then $u = u'$, because by Def. 7.2.3 case (7.14) is not possible, *i.e.* inactive entities have no $\preceq$-successors, and by Def. 7.2.10 case (7.15) is not possible due to the fact that only active entities can be blocked. Therefore, $u \preceq u'$ because of $u = u'$, and the goal follows trivially.

  Case 2. Henceforth, assume that $u \in \mathfrak{A}$. By Def. 7.3.3 there is a non-empty chain $u_1, u_2, \ldots, u_n$ with $u = u_1$ and $u' = u_n$ and $\forall i. 1 \leq i < n. u_i \preceq u_{i+1} \in \mathfrak{C}$ or u_i is blocked by some w and $w \preceq u_{i+1} \in \mathfrak{C}$. By induction on i we show $u_i \in A^{\mathcal{I}_{\mathfrak{S}}}$ and $u_i \in \mathfrak{A}$.

  (i) For $i = 1$ this holds trivially by assumption.

  (ii) For $i > 1$ we have $u_i \in A^{\mathcal{I}_{\mathfrak{S}}}$ and $u_i \in \mathfrak{A}$ by induction hypothesis and the following two cases:

  Case 1. $u_i \preceq u_{i+1} \in \mathfrak{C}$, *i.e.*, u_i is not blocked:

    - If $u_i{:}{+}A \in \mathfrak{C}$ then by Def. 7.2.8 (saturation) $u_{i+1}{:}{+}A \in \mathfrak{C}$ which means $u_{i+1} \in A^{\mathcal{I}_{\mathfrak{S}}}$.
    - If $u_i{:}{+}\bot \in \mathfrak{C}$ then by Def. 7.2.8 $u_{i+1}{:}{+}\bot \in \mathfrak{C}$, hence $u_{i+1} \in A^{\mathcal{I}_{\mathfrak{S}}}$.

  Case 2. Otherwise, if some entity w blocks u_i and $w \preceq u_{i+1} \in \mathfrak{C}$ then by Def. 7.3.3 and blocking it holds that $w \in A^{\mathcal{I}_{\mathfrak{S}}}$ because of both, $u_i{:}{+}A \in \mathfrak{C}$ or $u_i{:}{+}\bot \in \mathfrak{C}$ imply $w{:}{+}A \in \mathfrak{C}$ or $w{:}{+}\bot \in \mathfrak{C}$ respectively. Since w is not blocked and $w \preceq u_{i+1} \in \mathfrak{C}$, we have $u_{i+1} \in A^{\mathcal{I}_{\mathfrak{S}}}$ by saturation.

  Thus, $A^{\mathcal{I}_{\mathfrak{S}}}$ is closed under refinement.

- By Def. 7.3.3 every entity $u \notin \mathfrak{A}$ that is inactive, *viz.*, every fallible entity, is included in the interpretation $A^{\mathcal{I}_{\mathfrak{S}}}$ for all $A \in N_C$. Hence, $\bot^{\mathcal{I}_{\mathfrak{S}}} \subseteq A^{\mathcal{I}_{\mathfrak{S}}}$ holds for all $A \in N_C$.

- The relation $\preceq^{\mathcal{I}_{\mathfrak{S}}}$ is reflexive and transitive by construction according to Def. 7.3.3.
- We have to show that $\perp^{\mathcal{I}_{\mathfrak{S}}}$ is closed under refinement and role-filling. First, we show that $\perp^{\mathcal{I}_{\mathfrak{S}}}$ is closed under refinement. Let us assume that $u \in \perp^{\mathcal{I}_{\mathfrak{S}}}$ and $u \preceq^{\mathcal{I}_{\mathfrak{S}}} u'$. The goal is to prove that $u' \in \perp^{\mathcal{I}_{\mathfrak{S}}}$. By the assumption $u \notin \mathfrak{A}$ and by Def. 7.2.10 (blocking) and 7.2.3 (non-speculative) follows $u = u'$, which is discussed as Case 1 for $A^{\mathcal{I}_{\mathfrak{S}}}$ above.

  Regarding closedness under role-filling, let us assume $u \in \perp^{\mathcal{I}_{\mathfrak{S}}}$:

  (i) We have to show that for all $u \notin \mathfrak{A}$ there exists a fallible filler, *i.e.*, for all $R \in N_R$ there exists an R-successor of u that is contained in $\perp^{\mathcal{I}_{\mathfrak{S}}}$. This comes by construction of $R^{\mathcal{I}_{\mathfrak{S}}}$, *viz.*, for every $u \notin \mathfrak{A}$ it holds that $(u, u) \in R^{\mathcal{I}_{\mathfrak{S}}}$ by construction of $R^{\mathcal{I}_{\mathfrak{S}}}$ (Def. 7.3.3 (7.18)). Hence, for every fallible entity there exists a fallible filler.

  (ii) Next, we have to show that all R-successors of a fallible entity are fallible. We argue this as follows: Let us assume that $u \in \perp^{\mathcal{I}_{\mathfrak{S}}}$ and $u\, R^{\mathcal{I}_{\mathfrak{S}}}\, v$. $u \in \perp^{\mathcal{I}_{\mathfrak{S}}}$ implies $u \notin \mathfrak{A}$. Since by Def. 7.2.3 inactive entities cannot have $\preceq$- or R-successors in $\mathfrak{S}$, $v = u$. Hence, $v \in \perp^{\mathcal{I}_{\mathfrak{S}}}$.

This shows that the canonical interpretation is indeed an interpretation for $\mathfrak{S}$. □

Definition 7.3.5 (Extended canonical interpretation (pre-model))**.** For a Θ-saturated, clash-free, reduced and non-speculative constraint system $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ the extended canonical interpretation (or *pre-model*) $(\mathcal{I}_{\mathfrak{S}}, \alpha_{\mathfrak{S}})$ is defined by:

- $\mathcal{I}_{\mathfrak{S}}$, the canonical interpretation of $\mathfrak{S}$ by Def. 7.3.3, and
- $\alpha_{\mathfrak{S}}(x) =_{df} x$, for all $x \in Supp(\mathfrak{C})$. ∇

Lemma 7.3.3. *Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a Θ-saturated and non-speculative constraint systemand $\mathcal{I}_{\mathfrak{S}}$ be the canonical interpretation of $\mathfrak{S}$. If $u \in \mathfrak{A}$, $u{:}{+}C \in \mathfrak{C}$ and $u \preceq^{\mathcal{I}_{\mathfrak{S}}} v$ then $v{:}{+}C \in \mathfrak{C}$ and $v \in \mathfrak{A}$.* ∇

Proof. Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a Θ-saturated, clash-free and non-speculative constraint system, $\mathfrak{S}' = red(\mathfrak{S}) = (\mathfrak{C}', \mathfrak{A})$ and $\mathcal{I}_{\mathfrak{S}}$ be the canonical interpretation of $\mathfrak{S}'$. By Prop. 7.3.2 it follows that $\mathfrak{S}'$ is Θ-saturated as well, and it is also non-speculative. Suppose that $u{:}{+}C \in \mathfrak{C}'$ and $u \preceq^{\mathcal{I}_{\mathfrak{S}}} v$. By Def. 7.3.3 there exists a non-empty chain $u_1, u_2, u_3, \ldots, u_n$ with $u = u_1$ and $v = u_n$ such that for all i, $1 \leq i < n$ either $u_i \preceq u_{i+1} \in \mathfrak{C}'$ or there exists $w \in Supp(\mathfrak{S}')$ such that u_i is blocked by w and $w \preceq u_{i+1} \in \mathfrak{C}'$. Let i be such that $1 \leq i < n$. We proceed by induction over i to prove $u_i{:}{+}C \in \mathfrak{C}$ and $u_i \in \mathfrak{A}$:

- In the base case $i = 1$, it trivially holds that $u_1 \in \mathfrak{A}$ and $u_1{:}{+}C \in \mathfrak{C}$ by assumption.

- In the inductive step let $1 \leq k < n$. We proceed by case analysis:
  Case 1. $u_k \preceq u_{k+1} \in \mathfrak{C}'$. The ind. hyp. implies that $u_k{:}{+}C \in \mathfrak{C}'$. Since $\mathfrak{S}'$ is non-speculative, it holds that $u_k \in \mathfrak{A}$ and from the fact that $\mathfrak{S}'$ is Θ-saturated we can conclude by saturation (rule $(\rightarrow_{\preceq+})$) that $u_{k+1}{:}{+}C \in \mathfrak{C}'$, Since $\mathfrak{S}$ is non-speculative, $u_{k+1} \in \mathfrak{A}$ follows from Prop. 7.2.1.
  Case 2. If u_k is blocked by some $w \in \mathit{Supp}(\mathfrak{S}')$ then $w \prec u_k$, $CA(u) \subseteq CA(w)$ and $w \preceq u_{k+1} \in \mathfrak{C}'$. The ind. hyp. implies that $u_k{:}{+}C \in \mathfrak{C}'$, and it follows from blocking that $w{:}{+}C \in \mathfrak{C}'$ and $w \in \mathfrak{A}$. Since $\mathfrak{S}'$ is Θ-saturated, it follows by saturation (rule $(\rightarrow_{\preceq+})$) that $u_{k+1}{:}{+}C \in \mathfrak{C}'$. Prop. 7.2.1 implies $u_{k+1} \in \mathfrak{A}$ from the fact that $\mathfrak{S}$ is non-speculative.

Hence, $u'{:}{+}C \in \mathfrak{C}'$ and $u' \in \mathfrak{A}$. □

Proposition 7.3.7 (Selfsatisfaction). *Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a Θ-saturated, non-speculative, clash-free and reduced constraint system, then the extended canonical interpretation of $\mathfrak{S}$ is a model for $\mathfrak{S}$.* ∇

Proof. Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a Θ-saturated, non-speculative, clash-free and reduced constraint system, $\mathcal{I}_{\mathfrak{S}}$ the canonical interpretation of $\mathfrak{S}$ and $(\mathcal{I}_{\mathfrak{S}}, \alpha_{\mathfrak{S}})$ the pre-model of $\mathfrak{S}$. We have to show that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models \mathfrak{S}$, *i.e.*, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models \mathfrak{c}$ for all $\mathfrak{c} \in \mathfrak{C}$ and $\alpha_{\mathfrak{S}}(x) \notin \bot^{\mathcal{I}_{\mathfrak{S}}}$ for all $x \in \mathfrak{A}$.

- By construction $\mathcal{I}_{\mathfrak{S}}, \alpha_{\mathfrak{S}}$ satisfies all constraints of the form $u \preceq u' \in \mathfrak{C}$. Note that u is not blocked because of the fact that $\mathfrak{S}$ is reduced.

- $\mathcal{I}_{\mathfrak{S}}, \alpha_{\mathfrak{S}}$ satisfies all relational constraints of the form $u\,R\,v \in \mathfrak{C}$ by construction.

- By Def. 7.3.3 all $u \in \mathfrak{A}.\alpha_{\mathfrak{S}}(u) \notin \bot^{\mathcal{I}}$, *i.e.* the assignment of all variables in $\mathfrak{A}$ is infallible.

- We show for an arbitrary concept C that for all $u \in \mathit{Supp}(\mathfrak{S})$,

$$u{:}{+}C \in \mathfrak{C} \Rightarrow \mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{+}C, \tag{7.21}$$

$$u{:}{-}C \in \mathfrak{C} \Rightarrow \mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{-}C, \tag{7.22}$$

$$u{:}{-}_{\exists R}C \in \mathfrak{C} \Rightarrow \mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{-}_{\exists R}C, \tag{7.23}$$

$$u{:}{-}_{\forall R}C \in \mathfrak{C} \Rightarrow \mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{-}_{\forall R}C, \tag{7.24}$$

simultaneously by induction on the structure of C following the order of the four cases above. In particular, we will justify the cases $u{:}-_{\exists R}C \in \mathfrak{C}$ and $u{:}-_{\forall R}C \in \mathfrak{C}$ for arbitrary C relying on the proof of the former two cases:

Case 1. If $u \notin \mathfrak{A}$ then (7.21) follows trivially from the fact that $u \in \bot^{\mathcal{I}_\mathfrak{S}}$ by Def. 7.3.3. The cases (7.21)–(7.24) are not possible, since $\mathfrak{S}$ is non-speculative and therefore u is optimistic by Def. 7.2.3.

Case 2. Otherwise, $u \in \mathfrak{A}$ and the cases (7.21)–(7.24) are as follows:

BC Base case: If $u{:}{+}A \in \mathfrak{C}$ with $A \in N_C$ then $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{+}A$ by construction of $A^{\mathcal{I}_\mathfrak{S}}$. Suppose that $u{:}{-}A \in \mathfrak{C}$. By assumption $\mathfrak{S}$ is non-speculative and clash-free. The former implies that $u \in \mathfrak{A}$ and the latter lets us conclude that $u{:}{+}A \notin \mathfrak{C}$ and $u{:}{+}\bot \notin \mathfrak{C}$. Hence, $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{-}A$.

If $u{:}{+}\bot \in \mathfrak{C}$ then $u \notin \mathfrak{A}$, since otherwise this would contradict the assumption that $\mathfrak{S}$ is clash-free, and therefore $u \in \bot^{\mathcal{I}_\mathfrak{S}}$ by definition of $\bot^{\mathcal{I}_\mathfrak{S}}$. Thus, $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{+}\bot$. Otherwise, if $u{:}{-}\bot \in \mathfrak{C}$ then the assumption (non-speculative) implies that $u \in \mathfrak{A}$. Therefore $u \notin \bot^{\mathcal{I}_\mathfrak{S}}$. Hence, $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{-}\bot$.

$(+C \sqcap D)$ If $u{:}{+}C \sqcap D \in \mathfrak{C}$ then by saturation (rule $(\rightarrow_{\sqcap+})$) we have $\{u{:}{+}C, u{:}{+}D\} \subseteq \mathfrak{C}$. The ind. hyp. yields $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{+}C$ and $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{+}D$. Then, $\alpha_\mathfrak{S}(u) \in C^{\mathcal{I}_\mathfrak{S}} \cap D^{\mathcal{I}_\mathfrak{S}}$, *i.e.*, $\alpha_\mathfrak{S}(u) \in (C \sqcap D)^{\mathcal{I}_\mathfrak{S}}$ and therefore $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{+}C \sqcap D$.

$(-C \sqcap D)$ If $u{:}{-}C \sqcap D \in \mathfrak{C}$ then by saturation of $\mathfrak{S}$ (rule $(\rightarrow_{\sqcap-})$) this implies that either $u{:}{-}C \in \mathfrak{C}$ or $u{:}{-}D \in \mathfrak{C}$. By ind. hyp. one can deduce $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{-}C$ or $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{-}D$, *i.e.*, $\alpha_\mathfrak{S}(u) \notin C^{\mathcal{I}_\mathfrak{S}}$ or $\alpha_\mathfrak{S}(u) \notin D^{\mathcal{I}_\mathfrak{S}}$ which means $\alpha_\mathfrak{S}(u) \notin C^{\mathcal{I}_\mathfrak{S}} \cup D^{\mathcal{I}_\mathfrak{S}}$. The latter implies that $\alpha_\mathfrak{S}(u) \notin (C \sqcap D)^{\mathcal{I}_\mathfrak{S}}$ and therefore $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{-}C \sqcap D$.

$(+C \sqcup D)$ If $u{:}{+}C \sqcup D \in \mathfrak{C}$ then by saturation (rule $(\rightarrow_{\sqcup+})$) of $\mathfrak{S}$ either $u{:}{+}C \in \mathfrak{C}$ or $u{:}{+}D \in \mathfrak{C}$. By ind. hyp. this yields $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{+}C$ or $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{+}D$, *i.e.*, $\alpha_\mathfrak{S}(u) \in C^{\mathcal{I}_\mathfrak{S}}$ or $\alpha_\mathfrak{S}(u) \in D^{\mathcal{I}_\mathfrak{S}}$, which means $\alpha_\mathfrak{S}(u) \in (C \sqcup D)^{\mathcal{I}_\mathfrak{S}}$. Hence, $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{+}C \sqcup D$.

$(-C \sqcup D)$ If $u{:}{-}C \sqcup D \in \mathfrak{C}$ then by saturation of $\mathfrak{S}$ (rule $(\rightarrow_{\sqcup-})$) this implies $u{:}{-}C \in \mathfrak{C}$ and $u{:}{-}D \in \mathfrak{C}$. By induction hypothesis we can deduce $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{-}C$ and $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{-}D$, *i.e.*, $\alpha_\mathfrak{S}(u) \notin C^{\mathcal{I}_\mathfrak{S}}$ and $\alpha_\mathfrak{S}(u) \notin D^{\mathcal{I}_\mathfrak{S}}$. This yields $\alpha_\mathfrak{S}(u) \notin (C \sqcup D)^{\mathcal{I}_\mathfrak{S}}$ and therefore $\mathcal{I}_\mathfrak{S}; \alpha_\mathfrak{S} \models u{:}{-}C \sqcup D$.

$(+C \supset D)$ Suppose that $u{:}{+}C \supset D \in \mathfrak{C}$. Let $u' \in \Delta^{\mathcal{I}_{\mathfrak{S}}}$ such that $u \preceq^{\mathcal{I}_{\mathfrak{S}}} u'$ and $u' \in C^{\mathcal{I}_{\mathfrak{S}}}$. From Lem. 7.3.3 it follows that $u'{:}{+}C \supset D \in \mathfrak{C}$ and $u' \in \mathfrak{A}$. Because of $u' \in \mathfrak{A}$, saturation (rule $(\rightarrow_{\supset +})$) implies that $u'{:}{-}C \in \mathfrak{C}$ or $u'{:}{+}D \in \mathfrak{C}$.

Case 1. If $u'{:}{-}C \in \mathfrak{C}$ then by ind. hyp. $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u'{:}{-}C$, *i.e.*, $u' \notin C^{\mathcal{I}_{\mathfrak{S}}}$.

Case 2. Otherwise, $u'{:}{+}D \in \mathfrak{C}$. It follows from the induction hypothesis that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u'{:}{+}D$, *i.e.*, $u' \in D^{\mathcal{I}_{\mathfrak{S}}}$.

Hence, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{+}C \supset D$.

$(-C \supset D)$ If $u{:}{-}C \supset D \in \mathfrak{C}$ then there are two cases:
Case 1. Suppose that u is not blocked. From saturation (rule $(\rightarrow_{\supset -})$) we can conclude that $\{u \preceq u', u'{:}{+}C, u'{:}{-}D\} \subseteq \mathfrak{C}$ and $u' \in \mathfrak{A}$, which implies $u' \notin \bot^{\mathcal{I}_{\mathfrak{S}}}$. From (7.14) it follows $u \preceq^{\mathcal{I}_{\mathfrak{S}}} u'$. The ind. hyp. lets us conclude that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u'{:}{+}C$ and $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u'{:}{-}D$, *i.e.*, $\alpha_{\mathfrak{S}}(u') \in C^{\mathcal{I}_{\mathfrak{S}}}$ and $\alpha_{\mathfrak{S}}(u') \notin D^{\mathcal{I}_{\mathfrak{S}}}$. Thus, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{-}C \supset D$.

Case 2. u is blocked by some $w \in \mathit{Supp}(\mathfrak{S})$, *i.e.*, $w \prec u$ and $CA(u) \subseteq CA(w)$, which implies $w{:}{-}C \supset D \in \mathfrak{C}$. Since $w \in \mathfrak{A}$ and w is not blocked, it follows from saturation that $\{w \preceq w', w'{:}{+}C, w'{:}{-}D\} \subseteq \mathfrak{C}$. By Def. 7.3.3 $w \preceq^{\mathcal{I}_{\mathfrak{S}}} w'$ and by ind. hyp. $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models w'{:}{+}C$ and $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models w'{:}{-}D$, *i.e.*, $\alpha_{\mathfrak{S}}(w') \in C^{\mathcal{I}_{\mathfrak{S}}}$ and $\alpha_{\mathfrak{S}}(w') \notin D^{\mathcal{I}_{\mathfrak{S}}}$. Since $w' \in \mathfrak{A}$ (non-speculative), it follows that $w' \notin \bot^{\mathcal{I}_{\mathfrak{S}}}$. Because u is blocked by w it follows by (7.15) that $u \preceq^{\mathcal{I}_{\mathfrak{S}}} w'$. Hence, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{-}C \supset D$.

$(+\exists R.C)$ If $u{:}{+}\exists R.C \in \mathfrak{C}$ then we have to show that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{+}\exists R.C$, which is the case if for all $\preceq^{\mathcal{I}_{\mathfrak{S}}}$-successors of u there exists an R-successor which is in the interpretation of C. Let $u' \in \Delta^{\mathcal{I}_{\mathfrak{S}}}$ such that $u \preceq^{\mathcal{I}_{\mathfrak{S}}} u'$. Lemma 7.3.3 implies $u'{:}{+}\exists R.C \in \mathfrak{C}$ and $u' \in \mathfrak{A}$. We proceed by case analysis:

Case 1. If u' is not blocked then it follows by saturation (rule $(\rightarrow_{\exists +})$) that $u' \, R \, v \in \mathfrak{C}$ and $v{:}{+}C \in \mathfrak{C}$. By (7.16) $u' \, R^{\mathcal{I}_{\mathfrak{S}}} \, v$, and the ind. hyp. implies that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models v{:}{+}C$, *i.e.*, $\alpha_{\mathfrak{S}}(v) \in C^{\mathcal{I}_{\mathfrak{S}}}$.

Case 2. Otherwise, u' is blocked by some w such that $w \prec u'$ and $CA(u') \subseteq CA(w)$, and in particular $w{:}{+}\exists R.C \in \mathfrak{C}$. It follows from saturation (rule $(\rightarrow_{\exists +})$) that $w \, R \, v \in \mathfrak{C}$ and $v{:}{+}C \in \mathfrak{C}$. Due to blocking we have $u' \, R^{\mathcal{I}_{\mathfrak{S}}} \, v$ and the ind. hyp. yields $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models v{:}{+}C$.

Hence, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{+}\exists R.C$.

$(-\exists R.C)$ If $u{:}-\exists R.C \in \mathfrak{C}$ then the goal is $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}-\exists RC$, which holds if there exists a refinement of u such that all its R-successors are not in $C^{\mathcal{I}_{\mathfrak{S}}}$.

Case 1. u is not blocked. Saturation (rule $(\rightarrow_{\exists -})$) implies that $\exists u' \in \mathfrak{A}$ such that $u \preceq u' \in \mathfrak{C}$ and $u'{:}-_{\forall R}C \in \mathfrak{C}$. Def. 7.3.3 (7.14) implies $u \preceq^{\mathcal{I}_{\mathfrak{S}}} u'$ and the induction hypothesis implies that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u'{:}-_{\forall R}C$, which means that all R-successors of u' are not in $C^{\mathcal{I}_{\mathfrak{S}}}$.

Case 2. u is blocked by some $w \in \mathfrak{A}$ such that $w \prec u$, $CA(u) \subseteq CA(w)$ and in particular $w{:}-\exists R.C \in \mathfrak{C}$. Since w is not blocked, saturation (rule $(\rightarrow_{\exists -})$) implies that $\exists w' \in \mathfrak{A}$ such that $w \preceq w'$ and $w'{:}-_{\forall R}C \in \mathfrak{C}$. The ind. hyp. yields $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models w'{:}-_{\forall R}C$, and due to blocking it follows from (7.15) that $u \preceq^{\mathcal{I}_{\mathfrak{S}}} w'$.

Therefore, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}-\exists RC$.

$(+\forall R.C)$ Suppose that $u{:}+\forall R.C \in \mathfrak{C}$. The goal is to prove $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}+\forall RC$, which holds if all $R^{\mathcal{I}_{\mathfrak{S}}}$-successors of all $\preceq^{\mathcal{I}_{\mathfrak{S}}}$-successors of u lie in the interpretation of C. Let $u', v \in \Delta^{\mathcal{I}_{\mathfrak{S}}}$ such that $u \preceq^{\mathcal{I}_{\mathfrak{S}}} u'$, $u' \, R^{\mathcal{I}_{\mathfrak{S}}} \, v$. Lemma 7.3.3 implies $u'{:}+\forall R.C \in \mathfrak{C}$ and $u' \in \mathfrak{A}$. We proceed by case analysis:

Case 1. If $u' \, R \, v \in \mathfrak{C}$ then u' cannot be blocked by any other entity due to the fact that $\mathfrak{S}$ is reduced (see Def. 7.3.4). From saturation (rule $(\rightarrow_{\forall +})$) it follows that $v{:}+C \in \mathfrak{C}$, which by the ind. hyp. yields $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models v{:}+C$.

Case 2. $u'{:}+\bot \in \mathfrak{C}$ and $u' = v$. It holds by Lem. 4.2.2 that $\bot^{\mathcal{I}_{\mathfrak{S}}} \subseteq C^{\mathcal{I}_{\mathfrak{S}}}$. Thus, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models v{:}+C$.

Case 3. Suppose that $u' \notin \mathfrak{A}$ and $u = v$. The argument is analogously to Case 2 above.

Case 4. u' is blocked by some $w \in \mathfrak{A}$ such that $w \prec u'$, $CA(u') \subseteq CA(w)$ and $w \, R \, v \in \mathfrak{C}$, in particular $w{:}+\forall R.C \in \mathfrak{C}$. Now, saturation (rule $(\rightarrow_{\forall +})$) implies $v{:}+C \in \mathfrak{C}$, and the ind. hyp. lets us conclude that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models v{:}+C$.

Hence, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}+\forall RC$.

$(-\forall R.C)$ If $u{:}-\forall R.C \in \mathfrak{C}$, then one has to prove that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}-\forall RC$, which holds if there exists a refinement of u with an R-successor that is not in the interpretation of C. We proceed by case analysis on u:

Case 1. u is not blocked, *i.e.*, by saturation (rule $(\rightarrow_{\forall -})$) it follows that $u \preceq u' \in \mathfrak{C}$, $u'{:}-_{\exists R}C \in \mathfrak{C}$, and $u' \in \mathfrak{A}$ (non-speculative). (7.14) implies $u \preceq^{\mathcal{I}_{\mathfrak{S}}} u'$ and the ind. hyp. lets us conclude that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u'{:}-_{\exists R}C$, which means that there exists an R-successor that is not in $C^{\mathcal{I}_{\mathfrak{S}}}$.

Case 2. u is blocked by some $w \in \mathfrak{A}$ such that $w \prec u$, $CA(u) \subseteq CA(w)$, *i.e.*, $w{:}{-}\exists R.C \in \mathfrak{C}$. Since w is not blocked, it follows from saturation (rule $(\rightarrow_{\forall -})$) that $w \preceq w' \in \mathfrak{C}$ and $w'{:}{-}_{\exists R}C \in \mathfrak{C}$. (7.15) lets us conclude that $u \preceq^{\mathcal{I}_{\mathfrak{S}}} w'$ and by ind. hyp. it follows that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models w'{:}{-}_{\exists R}C$.

Hence, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{-}\forall RC$.

$(-_{\exists R}C)$ Suppose $u{:}{-}_{\exists R}C \in \mathfrak{C}$ with $u \in \mathfrak{A}$. The goal is to demonstrate that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{-}_{\exists R}C$, *i.e.*, there exists an R-successor of u that is not in $C^{\mathcal{I}_{\mathfrak{S}}}$. By case analysis on u:

Case 1. u is not blocked. Saturation (rule $(\rightarrow_{R_{\exists}-})$) implies $\exists v \in \mathfrak{A}$ such that $u\,R\,v \in \mathfrak{C}$ and $v{:}{-}C \in \mathfrak{C}$. It follows from (7.16) that $u\,R^{\mathcal{I}_{\mathfrak{S}}}\,v$ and the induction hypothesis yields $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models v{:}{-}C$.

Case 2. u is blocked by some $w \in \mathfrak{A}$ such that $w \prec u$ and $CA(u) \subseteq CA(w)$, particularly $w{:}{-}_{\exists R}C \in \mathfrak{C}$. Since w is not blocked, saturation (rule $(\rightarrow_{R_{\exists}-})$) implies $\exists v \in \mathfrak{A}$ such that $w\,R\,v \in \mathfrak{C}$ and $v{:}{-}C \in \mathfrak{C}$. (7.19) lets us conclude that $u\,R^{\mathcal{I}_{\mathfrak{S}}}\,w$ and the ind. hyp. implies $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models v{:}{-}C$, which was to be shown.

Therefore, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{-}_{\exists R}C$.

$(-_{\forall R}C)$ Suppose $u{:}{-}_{\forall R}C \in \mathfrak{C}$ with $u \in \mathfrak{A}$. The task is to show that $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{-}_{\forall R}C$, which holds if all R-successors of u are not in $C^{\mathcal{I}_{\mathfrak{S}}}$. Let $v \in \Delta^{\mathcal{I}_{\mathfrak{S}}}$ be such that $u\,R^{\mathcal{I}_{\mathfrak{S}}}\,v$. We proceed by case analysis:

Case 1. If $u\,R\,v \in \mathfrak{C}$ then u cannot be blocked, because $\mathfrak{S}$ is reduced. Saturation (rule $(\rightarrow_{R_{\forall}-})$) implies $v{:}{-}C \in \mathfrak{C}$ and $v \in \mathfrak{A}$. The ind. hyp. yields $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models v{:}{-}C$.

Case 2. If $u{:}{+}\bot \in \mathfrak{C}$ then this contradicts our assumption that $\mathfrak{S}$ is clash-free, considering that $u \in \mathfrak{A}$.

Case 3. u is blocked by some $w \in \mathfrak{A}$, *i.e.*, $w \prec u$ and $CA(u) \subseteq CA(w)$. Then, $w{:}{-}_{\forall R}C \in \mathfrak{C}$ and $w\,R\,v \in \mathfrak{C}$. Saturation (rule $(\rightarrow_{R_{\forall}-})$) yields $v{:}{-}C \in \mathfrak{C}$. (7.19) lets us conclude that $u\,R^{\mathcal{I}_{\mathfrak{S}}}\,v$ and the ind. hyp. implies $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models v{:}{-}C$. Thus, $\mathcal{I}_{\mathfrak{S}}; \alpha_{\mathfrak{S}} \models u{:}{-}_{\forall R}C$. □

Proposition 7.3.8. *A Θ-saturated, non-speculative constraint system is satisfiable if and only if it contains no clash.* ∇

Proof. ($\Rightarrow$) Let $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ be a Θ-saturated, non-speculative and satisfiable constraint system, *i.e.*, no more rule is applicable to any element of $\mathfrak{C}$ and there is a pair $(\mathcal{I}, \alpha)$ which satisfies $\mathfrak{S}$. We have to show that $\mathfrak{S}$ contains no clash. Suppose that $\mathfrak{S}$ contains a clash. Then, this would imply by Proposition 7.3.3 that $\mathfrak{S}$ is not satisfiable, which contradicts our former assumption. Hence $\mathfrak{S}$ contains no clash.

($\Leftarrow$) For a saturated constraint system $\mathfrak{S} = (\mathfrak{C}, \mathfrak{A})$ that is non-speculative and clash-free, we can construct the reduced constraint system $\mathfrak{S}' = red(\mathfrak{S})$, which is saturated, non-speculative and clash-free by Lem. 7.3.2. By Proposition 7.3.7 $\mathfrak{S}'$ has a model, which implies that $\mathfrak{S}$ has a model as well. Therefore $\mathfrak{S}$ is satisfiable. $\square$

Theorem 7.3.1 (Soundness). *Let $\mathfrak{S}^*$ be a Θ-saturated constraint system, which is obtained by application of the derivation rules of Fig. 7.1 from a non-speculative constraint system $\mathfrak{S}$. $\mathfrak{S}$ is satisfiable if $\mathfrak{S}^*$ is clash-free.* ∇

Proof. Let $\mathfrak{S}^*$ be a clash-free saturation of $\mathfrak{S}$ obtained by application of the derivation rules in Figure 7.1. Proposition 7.3.8 implies that $\mathfrak{S}^*$ is satisfiable. Satisfiability of $\mathfrak{S}$ then follows by Proposition 7.3.5 and induction over the number of rule applications, *i.e.*, one exploits by induction over the number of rule applications that applying a tableau rule to an unsatisfiable constraint system does not yield a satisfiable constraint system. $\square$

Theorem 7.3.2 (Completeness). *Let $\mathfrak{S}$ be a finite, non-speculative constraint system and Θ a TBox. If $(\mathfrak{S}, \Theta)$ is satisfiable then there exists a finite, non-speculative, clash-free and Θ-saturated saturation $\mathfrak{S}^*$ of $\mathfrak{S}$ which is derived by the application of the completion rules from Fig. 7.1.* ∇

Proof. Let $\mathfrak{S}$ be a finite and non-speculative constraint system which is satisfiable. By Prop. 7.3.1 after a finite number of rule applications we can reach a finite, non-speculative Θ-saturation $\mathfrak{S}^*$ of $\mathfrak{S}$, which is still satisfiable by Prop. 7.3.5. By Prop. 7.3.8 $\mathfrak{S}^*$ is clash-free. $\square$

7.4 Towards Constructive ABox Reasoning – an Outlook

In classical $\mathcal{ALC}$, the knowledge about individuals is expressed in terms of a set of ABox assertions of the form $a : C$ and $a\,R\,b$, which are expressed w.r.t. individual names a, b from the alphabet N_I. The interpretation $\mathcal{I}$ is extended to ABoxes by mapping each individual name $a \in N_I$ to a possible world in the interpretation domain $\Delta^{\mathcal{I}}$. Then, an assertion $a : C$ is satisfied if the mapping $a^{\mathcal{I}}$ is part of $C^{\mathcal{I}}$, and $a\,R\,b$ holds if $a^{\mathcal{I}}\,R^{\mathcal{I}}\,b^{\mathcal{I}}$. Sometimes, a DL is required to satisfy the *unique name assumption* [16, p. 66], *i.e.*, distinct names a, b require that $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ holds. The most important inference tasks w.r.t. ABoxes (see Chap. 2.1) are the *consistency problem* and *instance checking*: Let us recall that an ABox is consistent if all individual names can be assigned to the elements of the interpretation domain in a way such that all ABox assertions are satisfied, and an assertion $a : C$ is entailed by an ABox $\mathcal{A}$ if every model of $\mathcal{A}$ satisfies $a : C$ as well.

The individual names of DLs correspond to nominals from hybrid logics [7]. According to Areces and de Rijke [7], nominals allow to refer explicitly to the single states (possible worlds) in a classical model. In DLs nominals can be seen as special atomic concepts of the form $\{a\}$ for all $a \in N_I$, which are interpreted as singleton sets, *i.e.*, $\{a\}^{\mathcal{I}} = \{a^{\mathcal{I}}\}$ [16, pp. 170 f.]. The interpretation of nominals as singleton sets carries over straightforward to DLs, which are based on an extension of standard intuitionistic Kripke semantics (see Chap. 3). This semantics separates the intuitionistic preorder from the modal accessibility relation by assigning a domain to each intuitionistic state of knowledge, which is the set of possible worlds relative to which the interpretation of roles takes place. Then, for each state of knowledge, the mapping of individual names is done by assigning each individual name to an element of the state's domain with the requirement that the assignment of names has to preserve monotonicity w.r.t. the intuitionistic preorder $\preceq$. Contrary, in the birelational semantics of c$\mathcal{ALC}$ the intuitionistic preorder and the modal accessibility relation are relations of the same domain, *i.e.*, the different states of a possible world are now elements of the same set. This leads to a further complication as Braüner points out, namely, under the birelational semantics the problem arises that '[...] if nominals are given their obvious interpretation, namely singleton sets, then the interpretation of nominals cannot be preserved by the partial order, thus, monotonicity is violated' [49, p. 178]. Therefore, the interpretation of nominals has to be adapted for the birelational semantics to incorporate that the domain of an interpretation (possibly) contains several states of an entity w.r.t. refinement $\preceq$. In the following, we will approach the problem of reasoning w.r.t. ABoxes relative to birelational semantics by means of two examples, which outline instance checking and countermodel construction.

Remark 7.4.1. Note that ABox assertions of the form $a : C$ and $a\ R\ b$ can only express *positive* knowledge about an individual name $a \in N_I$. Such assertions can be trivially satisfied in a fallible entity, *i.e.*, every ABox can be trivially satisfied by an interpretation $\mathcal{I}$ that consists of a single fallible entity x, and by mapping each name in N_I to x. However, we are more interested in non-trivial models of an ABox, in particular it seems to be appropriate to require that an ABox is satisfied w.r.t. a non-empty TBox if there exists a non-trivial (infallible) interpretation that satisfies both, the ABox and the TBox. ■

In the following two examples we will exemplify (i) a proof of instance checking w.r.t. an ABox by internalising the ABox into a TBox, and (ii) a disproof of instance checking w.r.t. an ABox and countermodel construction by extending the tableau calculus by constraints and rules to deal with individual names.

Example 7.4.1 (Instance checking)**.** In this example we extend the language of $c\mathcal{ALC}$ by nominals, that is, atomic concepts of the form $\{a\}$ for $a \in N_I$. They are used to internalise an ABox $\mathcal{A}$ into a TBox by taking the following translation [16, p. 171]:

$$\begin{aligned} t(a : C) &= \{a\} \supset C, \\ t(a\ R\ b) &= \{a\} \supset \exists R.\{b\}. \end{aligned}$$

Let us reconsider the *Food&Wine* knowledge base $\mathcal{K}_{F\&W} = (\mathcal{A}, \Theta)$ from Ex. 5.1.2, where the ABox $\mathcal{A}$ is described by

$$\begin{array}{rr} \mathtt{RED} : \mathsf{Colour}, & (\mathtt{RED}, \mathtt{BAROLO}) : \mathsf{isColourOf}, \\ \mathtt{WHITE} : \mathsf{Colour}, & (\mathtt{WHITE}, \mathtt{CHARDONNAY}) : \mathsf{isColourOf}, \\ \mathtt{FISH} : \mathsf{Food}, & (\mathtt{FISH}, \mathtt{WHITE}) : \mathsf{goesWith}, \\ \mathtt{MEAT} : \mathsf{Food}, & (\mathtt{MEAT}, \mathtt{RED}) : \mathsf{goesWith}, \\ \mathtt{BAROLO} : \mathsf{Wine}, & \mathtt{CHARDONNAY} : \mathsf{Wine}. \end{array}$$

and the TBox Θ is given by

$$\Theta = \{\mathsf{Food} \supset \exists \mathsf{goesWith}.\mathsf{Colour}, \mathsf{Colour} \supset \exists \mathsf{isColourOf}.\mathsf{Wine}\}.$$

Suppose we want to check whether `MEAT` is an instance of the concept

$$\mathsf{Food} \supset \exists \mathsf{goesWith}.(\mathsf{Colour} \sqcap \exists \mathsf{isColourOf}.\mathsf{Wine}).$$

The above instance checking problem can proved by showing that the constraint system $(\{u_0{:}+\{\mathtt{MEAT}\}, u_0{:}-\mathsf{Food} \supset \exists \mathsf{goesWith}.(\mathsf{Colour} \sqcap \exists \mathsf{isColourOf}.\mathsf{Wine})\}, \{u_0\})$ is not satisfiable w.r.t. TBox $\Theta \cup t(\mathcal{A})$, where $t(\mathcal{A})$ is the translation of the ABox $\mathcal{A}$ into a TBox, given by:

$$\begin{array}{rr}
\{\mathtt{RED}\} \supset \mathsf{Colour}, & \{\mathtt{RED}\} \supset \exists \mathsf{isColourOf}.\{\mathtt{BAROLO}\}, \\
\{\mathtt{WHITE}\} \supset \mathsf{Colour}, & \{\mathtt{WHITE}\} \supset \exists \mathsf{isColourOf}.\{\mathtt{CHARDONNAY}\}, \\
\{\mathtt{FISH}\} \supset \mathsf{Food}, & \{\mathtt{FISH}\} \supset \exists \mathsf{goesWith}.\{\mathtt{WHITE}\}, \\
\{\mathtt{MEAT}\} \supset \mathsf{Food}, & \{\mathtt{MEAT}\} \supset \exists \mathsf{goesWith}.\{\mathtt{RED}\}, \\
\{\mathtt{BAROLO}\} \supset \mathsf{Wine}, & \{\mathtt{CHARDONNAY}\} \supset \mathsf{Wine}.
\end{array}$$

We will use the abbreviations $\mathtt{r} = \{\mathtt{RED}\}$, $\mathtt{w} = \{\mathtt{WHITE}\}$, $\mathtt{f} = \{\mathtt{FISH}\}$, $\mathtt{m} = \{\mathtt{MEAT}\}$, $\mathtt{b} = \{\mathtt{BAROLO}\}$, $\mathtt{c} = \{\mathtt{CHARDONNAY}\}$, $R = \mathsf{goesWith}$, $S = \mathsf{isColourOf}$, $C = \mathsf{Colour}$, $F = \mathsf{Food}$ and $W = \mathsf{Wine}$ in the following derivation, in which we only expand the constraints that are relevant to obtain a clashing constraint system. Moreover, we use the admissible rule $(\rightarrow_{\mathtt{ax}+})$ stating that if $\mathfrak{C}, u{:}+A \,;\, \mathfrak{A}$ and $A \supset C \in \Theta$ then $\mathfrak{C}, u{:}+A, u{:}+C \,;\, \mathfrak{A}$, where A is an atomic concept.

$$\begin{array}{cl}
\mathfrak{C}_0 = u_0{:}+\mathtt{m}, u_0{:}-F \supset \exists R.(C \sqcap \exists S.W) \,;\, u_0 & \\ \hline
\mathfrak{C}_1 = \mathfrak{C}_0, u_0 \preceq u_1, u_1{:}+F, u_1{:}-\exists R.(C \sqcap \exists S.W) \,;\, u_0, u_1 & (\rightarrow_{\supset -}) \\ \hline
\mathfrak{C}_2 = \mathfrak{C}_1, u_1 \preceq u_2, u_2{:}-\forall_R C \sqcap \exists S.W \,;\, \mathfrak{A}_0 = u_0, u_1, u_2 & (\rightarrow_{\exists -}) \\ \hline
\mathfrak{C}_3 = \mathfrak{C}_2, u_2{:}+\mathtt{m}, u_2{:}+\exists R.\mathtt{r} \,;\, \mathfrak{A}_0 & (\rightarrow_{\preceq +}), (\rightarrow_{\mathtt{ax}+}) \\ \hline
\mathfrak{C}_4 = \mathfrak{C}_3, u_2 \, R \, v_0, v_0{:}+\mathtt{r} \,;\, \mathfrak{A}_0 & (\rightarrow_{\exists +}) \\ \hline
\mathfrak{C}_5 = \mathfrak{C}_4, v_0{:}-C \sqcap \exists S.W \,;\, \mathfrak{A}_1 = \mathfrak{A}_0, v_0 & (\rightarrow_{R_\forall -}) \\ \hline
\end{array}$$

$$\begin{array}{cc}
\begin{array}{cl}
\mathfrak{C}_5, v_0{:}-C \,;\, \mathfrak{A}_1 & \\ \hline
\mathfrak{C}_5, \underline{v_0{:}-C}, \underline{v_0{:}+C} \,;\, \mathfrak{A}_1 & (\rightarrow_{\mathtt{ax}+}) \\ \hline
\text{※} &
\end{array}
&
\begin{array}{cl}
\mathfrak{C}_6 = \mathfrak{C}_5, v_0{:}-\exists S.W \,;\, \mathfrak{A}_1 & (\rightarrow_{\sqcap -}) \\ \hline
\mathfrak{C}_7 = \mathfrak{C}_6, v_0{:}+\exists S.\mathtt{b} \,;\, \mathfrak{A}_1 & (\rightarrow_{\mathtt{ax}+}) \\ \hline
\mathfrak{C}_8 = \mathfrak{C}_7, v_0 \preceq v_1, v_1{:}-\forall_R W, v_1{:}+\exists S.\mathtt{b} \,;\, \mathfrak{A}_1, v_1 & (\rightarrow_{\exists -}), (\rightarrow_{\preceq +}) \\ \hline
\mathfrak{C}_9 = \mathfrak{C}_8, v_1 \, S \, w_0, w_0{:}+\mathtt{b}, \,;\, \mathfrak{A}_1, v_1 & (\rightarrow_{\exists +}) \\ \hline
\mathfrak{C}_{10} = \mathfrak{C}_9, w_0{:}-W \,;\, \mathfrak{A}_1, v_1, w_0 & (\rightarrow_{R_\forall -}) \\ \hline
\mathfrak{C}_{10}, \underline{w_0{:}-W}, \underline{w_0{:}+W} \,;\, \mathfrak{A}_1, v_1, w_0 & (\rightarrow_{\mathtt{ax}+}) \\ \hline
\text{※} &
\end{array}
\end{array}$$

■

Intuitively, the closed tableau derivation means that there exists no countermodel in the birelational semantics. Granted that there exists a standard Kripke semantics for c$\mathcal{ALC}$, with the property that there are more birelational models than Kripke models, and each Kripke model can be encoded into a birelational model while preserving the semantic consequence relation, it should follow that there exists no countermodel in the standard Kripke semantics for the above example as well.

However, in the case of a failed proof, *i.e.*, the tableau calculus stops with a saturated and clash-free constraint system, the approach of internalising an ABox into a TBox does not yet provide us with the construction of a countermodel that satisfies the ABox and refutes the instance assertion to be entailed by the ABox. This case will be discussed in the following example by introducing an interpretation of individual names for birelational interpretations and special rules that explicitly treat ABox assertions.

Example 7.4.2 (Countermodel construction)**.** Inspired by Chadha, Macedonio and Sassone [61], we interpret ABox assertions by extending the constructive interpretation $\mathcal{I}$ by a partial function $\cdot^{\mathcal{I}} : \Delta^{\mathcal{I}} \to N_I$ that maps an entity from $\Delta^{\mathcal{I}}$ to a single individual name from N_I such that for all $x, y \in \Delta^{\mathcal{I}}$ and $a \in N_I$ the following properties hold:

(i) *Coherence* [61, p. 11]: $x^{\mathcal{I}} = a$ and $x \preceq^{\mathcal{I}} y$ implies $y^{\mathcal{I}} = a$

(ii) *Refinement simulation*: $\preceq^{\mathcal{I}} \subseteq \underline{\rightarrow}$, or in other words $x \preceq^{\mathcal{I}} y \Rightarrow x \underline{\rightarrow} y$, where

  a) A relation $S \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ is called a *simulation* if it holds for all $(x, y) \in S$ that:

   - $\forall a \in N_I .\ x^{\mathcal{I}} = a \Rightarrow y^{\mathcal{I}} = a;$
   - $\forall x' \in \Delta^{\mathcal{I}}, \forall R \in N_R .\ x\, R^{\mathcal{I}}\, x' \Rightarrow \exists y' \in \Delta^{\mathcal{I}} .\ y\, R^{\mathcal{I}}\, y'$ and $(x', y') \in S$.

  b) Let $\underline{\rightarrow} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ be the maximal simulation relation given by

$$\underline{\rightarrow} =_{df} \bigcup \{S \mid S \text{ is a simulation}\}.$$

  Note that if S_1, S_2 are simulations then $S_1 \cup S_2$ is a simulation as well.

One can observe that (i) is a consequence of refinement simulation (ii), condition $\preceq^{\mathcal{I}} \subseteq \underline{\rightarrow}$.

Intuitively, an entity of $\Delta^{\mathcal{I}}$ in a birelational model corresponds to a single place at a specific state of a Kripke model. Let $\Delta^{\mathcal{I}}_a =_{df} \{x \in \Delta^{\mathcal{I}} \mid x^{\mathcal{I}} = a\}$ denote the set of entities in $\Delta^{\mathcal{I}}$ that are assigned to the name a. Then, an extended interpretation $\mathcal{I}$ models a concept assertion $a : C$ if $\forall x \in \Delta^{\mathcal{I}}_a . \mathcal{I}; x \models C$, and analogously $\mathcal{I}$ models a role assertion $a\, R\, b$ if $\forall x \in \Delta^{\mathcal{I}}_a .\ \exists y \in \Delta^{\mathcal{I}}_b .\ x\, R^{\mathcal{I}}\, y$. This is extended to a set (ABox) of assertional axioms in the usual style.

Moreover, we extend the tableau calculus by constraints and rules to handle nominals from the ABox explicitly. A *nominal constraint* is of the form $u \downarrow a$, where $a \in N_I$ and $N_I \cap V_E = \emptyset$, and denotes that variable u is assigned the name a (or name a is true at u). This constraint is a restricted form of the well-known satisfaction operator $@_u a$ from hybrid logics [49, pp. 5 ff.]. However, we will not extend the language $c\mathcal{ALC}$ by a hybrid logical satisfaction operator here.

Moreover, we extend a general knowledge base to include an ABox as well by taking $\mathcal{K} = (\mathcal{M}, \Theta, \mathcal{A})$, where $\mathcal{M}$ is a finite set of constraint systems, Θ a TBox and $\mathcal{A}$ an

ABox. The tableau rules of $c\mathcal{ALC}$ are extended accordingly to such extended GKBs, and we require additionally to Def. 7.2.5 (local tableau rule) that no rule alters the ABox, formally, $(\mathcal{M}, \Theta, \mathcal{A}) \rightarrow_\xi (\mathcal{M}', \Theta', \mathcal{A}')$ only if $\mathcal{A}' = \mathcal{A}$. The nominals are treated by the following rules w.r.t. an ABox $\mathcal{A}$:

$$\frac{\mathfrak{C}, u \downarrow a, u \preceq u' \,;\, \mathfrak{A}}{\mathfrak{C}, u' \downarrow a \,;\, \mathfrak{A}}\ (\rightarrow_{\preceq_a}),\ u \in \mathfrak{A}\ \&\ a \in \mathit{Supp}(\mathcal{A}) \qquad \frac{\mathfrak{C}, u \downarrow a \,;\, \mathfrak{A}}{\mathfrak{C}, u{:}{+}C \,;\, \mathfrak{A}}\ (\rightarrow_{C_a}),\ a{:}C \in \mathcal{A}$$

$$\frac{\mathfrak{C}, u \downarrow a \,;\, \mathfrak{A}}{\mathfrak{C}, v \downarrow b, u\, R\, v \,;\, \mathfrak{A}}\ (\rightarrow_{R_a}),\ a\, R\, b \in \mathcal{A},\ v \text{ is fresh}$$

The rules $(\rightarrow_{C_a})$ and $(\rightarrow_{R_a})$ are not restricted to active entities, but applicable to every $u \in \mathit{Supp}(\mathfrak{S})$ that satisfies their premise. Due to the fact that rule $(\rightarrow_{R_a})$ is a generating rule and also applicable to inactive entities, we need to reconsider termination and also allow blocking of inactive entities, since a cyclic ABox can lead to an infinite sequence of applications of rule $(\rightarrow_{R_a})$ for inactive entities. Therefore, we additionally require that if $u \notin \mathfrak{A}$ then rule $(\rightarrow_{R_a})$ is only applied if u is not blocked by any inactive entity in $\mathit{Supp}(\mathfrak{S})$, analogously to the definition of blocking (see Def. 7.2.10).

Let us reconsider the customer topology ABox from Ex. 4.2.5, using the abbreviations $\mathsf{hasCustomer} = R$ and $\mathsf{Insolvent} = I$:

$$\mathcal{A} =_{df} \{a\, R\, b, a\, R\, c, b\, R\, c, c\, R\, d, b{:}I,\ d{:}\neg I\}$$

A company is *credit-worthy* if it has an insolvent customer who in turn has at least one solvent customer, formalised by the concept:

$$CW = \exists R.(I \sqcap \exists R.\neg I).$$

Now, suppose we want to check if individual a is a credit worthy company. This holds iff the following constraint system is not satisfiable w.r.t. ABox $\mathcal{A}$:

$$\begin{aligned}\mathfrak{S}_0 =_{df} (\{&a_0 \downarrow a, b_0 \downarrow b, c_0 \downarrow c, d_0 \downarrow d,\\ &\qquad a_0\, R\, b_0, a_0\, R\, c_0, b_0\, R\, c_0, c_0\, R\, d_0,\\ &\qquad\qquad b_0{:}{+}I, d_0{:}{+}\neg I, a_0{:}{-}\mathsf{CW}\}, \{a_0\}),\end{aligned}$$

where the ABox is instantiated by fresh inactive variables, and variable a_0 is switched active. The derivation proceeds as follows:

$$
\dfrac{\begin{array}{c}\mathfrak{C}_0 = a_0 \downarrow a, b_0 \downarrow b, c_0 \downarrow c, d_0 \downarrow d, a_0\,R\,b_0, a_0\,R\,c_0, b_0\,R\,c_0, c_0\,R\,d_0,\\ b_0{:}{+}I, d_0{:}{+}\neg I, a_0{:}{-}\exists R.(I \sqcap \exists R.\neg I)\,;\,\mathfrak{A}_0 = \{a_0\}\end{array}}{\dfrac{\mathfrak{C}_1 = \mathfrak{C}_0, a_0 \preceq a_1, a_1{:}{-}_{\forall R} I \sqcap \exists R.\neg I, a_1 \downarrow a\,;\,\mathfrak{A}_0, a_1}{\dfrac{\mathfrak{C}_2 = \mathfrak{C}_1, a_1\,R\,b_1, b_1 \downarrow b, b_1{:}{+}I\,;\,\mathfrak{A}_0, a_1}{\dfrac{\mathfrak{C}_3 = \mathfrak{C}_2, b_1{:}{-}I \sqcap \exists R.\neg I\,;\,\mathfrak{A}_1 = \mathfrak{A}_0, a_1, b_1}{\dfrac{\mathfrak{C}_3, \underline{b_1{:}{-}I}, \underline{b_1{:}{+}I}\,;\,\mathfrak{A}_1}{\text{※}} \qquad \dfrac{\mathfrak{C}_4 = \mathfrak{C}_3, b_1{:}{-}\exists R.\neg I\,;\,\mathfrak{A}_1}{\dfrac{\mathfrak{C}_5 = \mathfrak{C}_4, a_1\,R\,c_1, c_1 \downarrow c\,;\,\mathfrak{A}_1}{\dfrac{\mathfrak{C}_6 = \mathfrak{C}_5, c_1{:}{-}I \sqcap \exists R.\neg I, c_1\,R\,d_1, d_1 \downarrow d\,;\,\mathfrak{A}_2 = \mathfrak{A}_1, c_1}{\begin{array}{c}\mathfrak{C}_7 = \mathfrak{C}_6, c_1{:}{-}I\,;\,\mathfrak{A}_2\\ \vdots\end{array} \qquad \dfrac{\mathfrak{C}_8 = \mathfrak{C}_6, c_1{:}{-}\exists R.\neg I\,;\,\mathfrak{A}_2}{\dfrac{\mathfrak{C}_{10} = \mathfrak{C}_9, c_1 \preceq c_2, c_2{:}{-}_{\forall R}\neg I\,;\,\mathfrak{A}_2, c_2}{\dfrac{\mathfrak{C}_{11} = \mathfrak{C}_{10}, c_2\,R\,d_2, d_2 \downarrow d\,;\,\mathfrak{A}_2, c_2}{\dfrac{\mathfrak{C}_{12} = \mathfrak{C}_{11}, \underline{d_2{:}{+}\neg I}, \underline{d_2{:}{-}\neg I}\,;\,\mathfrak{A}_2, d_1, d_2}{\text{※}}}\,(\to_{C_a}),(\to_{R_\forall -})}\,(\to_{R_a})}\,(\to_{\exists -})}\,(\to_{\sqcap -})}\,(\to_{R_\forall -}),(\to_{R_a})}\,(\to_{R_a})}\,(\to_{\sqcap -})}\,(\to_{R_\forall -})}\,(\to_{R_a}),(\to_{C_a})}\,(\to_{\exists -}),(\to_{\preceq_a})
$$

where the derivation of $\mathfrak{C}_7$ continues as follows:

$$
\dfrac{\mathfrak{C}_7 = \mathfrak{C}_6, c_1{:}{-}I\,;\,\mathfrak{A}_2}{\dfrac{\mathfrak{C}_{13} = \mathfrak{C}_7, b_1\,R\,c_3, c_3 \downarrow c, c_3\,R\,d_3, d_3 \downarrow d, d_3{:}{+}\neg I\,;\,\mathfrak{A}_2}{\dfrac{\mathfrak{C}_{14} = \mathfrak{C}_{13}, b_1 \preceq b_2, b_2 \downarrow b, b_2{:}{-}_{\forall R}\neg I\,;\,\mathfrak{A}_3 = \mathfrak{A}_2, b_2}{\dfrac{\mathfrak{C}_{14} = \mathfrak{C}_{14}, b_2\,R\,c_4, c_4 \downarrow c, c_4\,R\,d_4, d_4 \downarrow d, d_4{:}{+}\neg I\,;\,\mathfrak{A}_3}{\dfrac{\mathfrak{C}_{15} = \mathfrak{C}_{14}, c_4{:}{-}\neg I\,;\,\mathfrak{A}_4 = \mathfrak{A}_3, c_4}{\dfrac{\mathfrak{C}_{16} = \mathfrak{C}_{15}, c_4 \preceq c_5, c_5{:}{+}I\,;\,\mathfrak{A}_5 = \mathfrak{A}_4, c_5}{\mathfrak{C}_{17} = \mathfrak{C}_{16}, c_5\,R\,d_5, d_5 \downarrow d, d_5{:}{+}\neg I\,;\,\mathfrak{A}_5}\,(\to_{R_a}),(\to_{C_a})}\,(\to_{\supset -})}\,(\to_{R_\forall -})}\,(\to_{R_a})^2,(\to_{C_a})}\,(\to_{\exists -})}\,(\to_{R_a})^2,(\to_{C_a})
$$

The calculus finishes with the saturated and clash-free constraint system $(\mathfrak{C}_{17}, \mathfrak{A}_5)$, from which it follows that entity a is not an instance of the concept CW. The generated countermodel structure of the constraint system is depicted by Fig. 7.7, where solid lines represent role R, dotted lines refinement $\preceq$, filled dots ● active entities, and ○ stands for inactive entities.

Note that the countermodel structure contains redundant information w.r.t. its inactive entities, *i.e.*, one can simplify the structure and identify d_1–d_4 with d_0, and c_2 with c_0. We can generate an interpretation from the countermodel structure in Fig. 7.7 which is shown in Fig. 7.8. Observe, that this structure corresponds to the countermodel given in Fig. 4.4 of Ex. 4.2.5, and expresses the two choices for individual c under the EWOA, represented by c_1 where $\neg I$ holds (c_2 in Ex. 4.2.5), and c_4 that satisfies I (c_1 in Ex. 4.2.5).

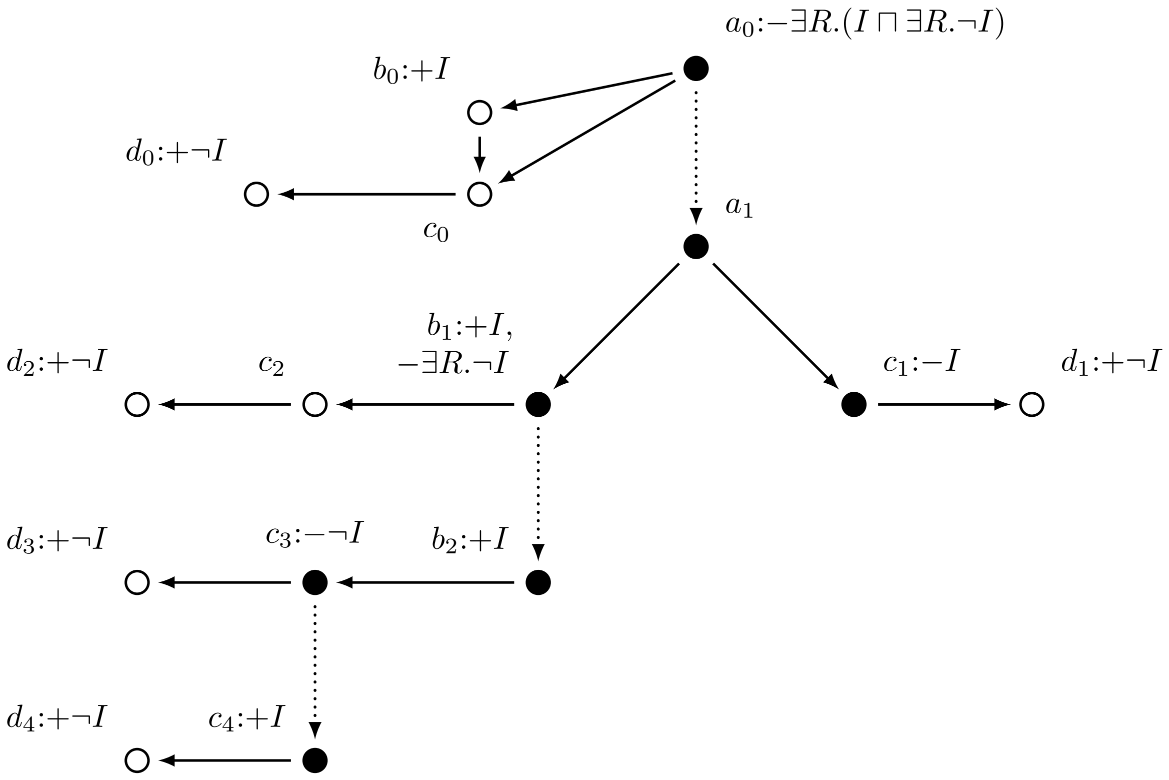

Figure 7.7: Countermodel structure for Ex. 7.4.2.

Formally, the interpretation is given by $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \perp^{\mathcal{I}}, \cdot^{\mathcal{I}})$, by taking

$$\begin{aligned}
\Delta^{\mathcal{I}} &=_{df} \{a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_3, c_4, d_0\};\\
\preceq^{\mathcal{I}} &=_{df} \{(a_0, a_0), (a_0, a_1), (a_1, a_1), (b_0, b_0), (b_1, b_1), (b_1, b_2), (b_2, b_2), (c_0, c_0), (c_1, c_1),\\
&\qquad (c_3, c_3), (c_3, c_4), (c_4, c_4), (d_0, d_0)\};\\
\perp^{\mathcal{I}} &=_{df} \{b_0, c_0, d_0\};\\
R^{\mathcal{I}} &=_{df} \{(a_0, b_0), (a_0, c_0), (b_0, c_0), (c_0, d_0), (a_1, b_1), (a_1, c_1), (b_1, c_0), (c_1, d_0), (b_2, c_3),\\
&\qquad (c_3, d_0), (c_4, d_0), (b_0, b_0), (c_0, c_0), (d_0, d_0)\};\\
I^{\mathcal{I}} &=_{df} \{b_0, c_0, d_0, b_1, b_2, c_4\},
\end{aligned}$$

and the mapping of entities to individual names is given by

$$\begin{aligned}
a_0^{\mathcal{I}} = a_1^{\mathcal{I}} &= a,\\
b_0^{\mathcal{I}} = b_1^{\mathcal{I}} = b_2^{\mathcal{I}} &= b,\\
c_0^{\mathcal{I}} = c_1^{\mathcal{I}} = c_3^{\mathcal{I}} = c_4^{\mathcal{I}} &= c,\\
d_0^{\mathcal{I}} &= d.
\end{aligned}$$

One can easily observe that the above interpretation $\mathcal{I}$ is a model of the initial ABox

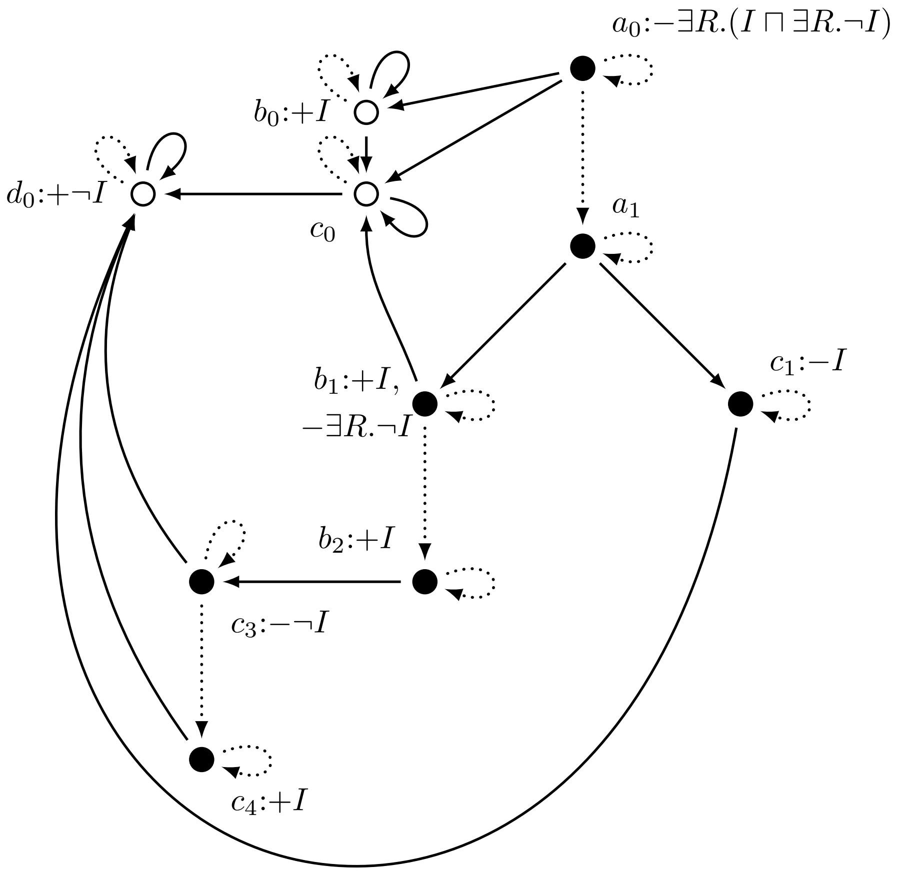

Figure 7.8: Countermodel for Ex. 7.4.2.

$\mathcal{A} =_{df} \{a\,R\,b, a\,R\,c, b\,R\,c, c\,R\,d, b{:}I,\ d{:}\neg I\}$ that satisfies the properties (i) coherence and (ii) simulation.

Let us proceed by inspecting the ABox assertions of $\mathcal{A}$. For instance, let us check the relational assertion $a\,R\,b$, which holds if $\forall x \in \Delta^{\mathcal{I}}_a = \{a_0, a_1\}. \exists y \in \Delta^{\mathcal{I}}_b = \{b_0, b_1, b_2\}$ such that $x\ R^{\mathcal{I}}\ y$. For a_0 there exists the entity b_0 with $b^{\mathcal{I}}_0 = b$, for a_1 there exists b_1 with $b^{\mathcal{I}}_1 = b$. Hence, $\mathcal{I} \models a\,R\,b$. The remaining relational assertions are argued analogously.

Secondly, let us discuss the conceptual assertions $b{:}I$ and $d{:}\neg I$. The first assertion holds if $\forall x \in \Delta^{\mathcal{I}}_b = \{b_0, b_1, b_2\}. \mathcal{I}; x \models I$, which follows immediately from the definition of $I^{\mathcal{I}}$. The second assertion $d{:}\neg I$ follows immediately from the fallibility of d_0.

We conjecture that the countermodel of Fig. 7.8 can be transformed into a finite standard Kripke model, which separates the intuitionistic preorder from the interpretation of the roles. Bozzato [39, p. 49, Ex. 4] has given such a countermodel for the same example in the logic $\mathcal{KALC}$. However, it seems we cannot obtain a finite standard Kripke model from birelational models involving cycles, like the one given by Fig. 5.3 in Ex. 5.2.3.

As a final remark, observe that the rules above, which treat ABox assertions explicitly, are just another representation of the existing rules of $\mathfrak{T}_{c\mathcal{ALC}}$, namely $(\rightarrow_{\preceq+})$ and $(\rightarrow_{\exists+})$, in the sense that we can achieve the same by internalising an ABox into a TBox and relaxing rule $(\rightarrow_{\exists+})$ such that it can be applied to inactive (unblocked) entities as well, in combination with an additional blocking condition on inactive entities. Then, the construction of the assignment of entities to individual names is by mapping each $x \in \{a\}^{\mathcal{I}}$ to name a, for an individual name $a \in N_I$. ■

7.5 Summary

This chapter presented a tableau-based calculus for $c\mathcal{ALC}$ in the spirit of DL-style tableau systems [16], showing soundness, completeness and termination. The tableau calculus is inspired by a contextual Gentzen calculus for multimodal CK [196], but in contrast, it relies on a constraint system as underlying data structure with labelled and signed formulæ. This calculus closely follows the $c\mathcal{ALC}$ Kripke semantics, with an explicit treatment of refinement $\preceq$ and roles R, and it uses an encoding of constraints similarly to the tableau calculus for $\mathcal{CALC}^{\mathsf{C}}$ by Odintsov and Wansing [219]. The notion of active/inactive entities is necessary to distinguish non-trivial realisations of entities in the form of active variables from fallible entities, which are represented by inactive variables. Moreover, this notions are used to control the applicability of the tableau rules in the construction of a tableau, *i.e.*, the generating rule for positive $\exists R.C$ is restricted to create an inactive successor only, in order to refute the axiom schema FS3/IK3 $= \neg\exists R.\bot$.

We discussed the difficulties of interpreting ABox assertions in birelational models on the basis of two example derivations in the tableau calculus, covering instance checking in both directions: validation and countermodel construction. In the latter example that covered countermodel construction, we sketched an alternative interpretation of nominals inspired by [61], by using a partial function that maps entities to individual names and respects the intuitionistic preorder of knowledge states, in contrast to the usual mapping of individual names to singleton sets of possible worlds as used in DLs.

Although, we presented a sound and complete tableau calculus that ensures termination by relying on anywhere blocking, the calculus is inefficient due to the treatment of duplication of intuitionistic implication. We leave it as future work to adapt the standard techniques from [43; 45; 88; 150; 200; 219] to avoid duplications, and the state of the art tableau optimisation techniques from DLs [16; 25] to the tableau calculus of $c\mathcal{ALC}$ in order to obtain an efficient and as-much-as-possible duplication-free decision procedure. We conjecture that an extension of the calculus with a proof strategy that avoids duplications and adapts the trace technique[19][261, p. 30] yields a decision procedure with an optimal worst case complexity, which should be not worse than PSpace.

Notes on Related Work

Tableaux for $\mathcal{KALC}$ Tableaux for $\mathcal{KALC}$ and $\mathcal{KALC}^{\infty}$ are introduced in [39; 45] and [43; 270]. Similarly to $\mathfrak{T}_{c\mathcal{ALC}}$, these calculi are based on sets of labelled and signed formulæ, but use the positive sign $\mathbf{T}$ to represent the notion of a *known* formula and the negative sign $\mathbf{F}$ to denote an *unknown* formula. In contrast to $\mathfrak{T}_{c\mathcal{ALC}}$, both calculi for $\mathcal{KALC}$ and $\mathcal{KALC}^{\infty}$ efficiently handle duplications in the treatment of implication.

The calculus for $\mathcal{KALC}$ [39, pp. 27 ff.] avoids duplications of implication by using a special sign $\mathbf{T_s}$, which refers to the future states of a world w.r.t. $\preceq$. An infinite expansion of TBox axioms is prevented by restricting the TBox to be acyclic and using an expansion strategy of TBox axioms similarly to lazy unfolding [16, p. 344 ff., rule U_3]. However, the calculus does not eliminate duplications completely, that is, unknown existential restriction ($\mathbf{F}(c{:}\exists R.C)$), known universal restriction ($\mathbf{T}(c{:}\forall R.C)$) and TBox formulæ ($\mathbf{T}(A \sqsubseteq C)$) must still be duplicated. Bozzato [39] introduces a decision procedure for $\mathcal{KALC}$ in the form of a graph-based expansion algorithm based on the tableau calculus for $\mathcal{KALC}$ and proves its termination, soundness and completeness w.r.t. the $\mathcal{KALC}$-realisability relation, but restricted to acyclic TBoxes. Since $\mathcal{KALC}$ uses a standard intuitionistic Kripke-semantics that separates the accessibility relation from the intuitionistic refinement relation, the interpretation of ABoxes works just like in classical DLs. Individual names are interpreted as singleton sets relative to one possible world, and this mapping of names to the possible worlds of the domain of a Kripke state is required to be static for all subsequent information states w.r.t. refinement $\preceq$.

Game-theoretic Decision Procedure for $c\mathcal{ALC}$ Sticht [254] proposes a tableau based game theoretic decision procedure for the logic $c\mathcal{ALC}$, which is a direct extension of the tableau calculus $\mathfrak{T}_{c\mathcal{ALC}}$ presented in this chapter. The decision procedure applies the ideas of dialogical logic introduced by Lorenzen and Lorenz [178] to $c\mathcal{ALC}$. This proof-theoretical approach interprets the proof of a logical statement as a game-based dialog between an opponent and a proponent. Both parties can apply rules and attack or defend a formula in a turn based fashion. The proof system differentiates two kinds of rules: (i) each move depends on a set of *particle rules*, which restrict the ways to reason about a formula and correspond to the rules that expand formulæ, and (ii) the

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[19] The idea of the *trace technique* is to restrict the required memory consumption to a single path of a tableau and to use a depth-first search strategy.

flow of the dialog is ruled out by so-called *structural rules*, which are differentiated into game-rules and frame-rules. Game-rules generically specify the laws of the dialog game, like for instance that the players apply game-moves alternately, and that a game is won by a player if his opponent cannot move anymore. The structural rules handle the organisation of a game and specify whether and how each player can access and use the information within a game. These rules correspond to frame-axioms of the underlying logic. The rules of the game theoretic decision procedure are derived from the tableau rules of $\mathfrak{T}_{c\mathcal{ALC}}$, but allow for a much finer control of the proof mechanics and seem to be promising as a base to investigate the structural constraints of the different IMLs within a single calculus. A prototype of the decision procedure has been implemented in the functional programming language Haskell[20]. It is an open problem whether this dialogical decision procedure for $c\mathcal{ALC}$ is sound and complete, and what its complexity is.

Tableau Calculus for $\mathcal{CALC}^{\mathsf{C}}$ The tableau calculus for $c\mathcal{ALC}$ is similar to the calculus for $\mathcal{CALC}^{\mathsf{C}}$ in [220] in that $\mathcal{CALC}^{\mathsf{C}}$ is based on birelational interpretations and its Fitting-style tableau rules use $\{+, -\}$-signed formulæ. The calculi differ in that $c\mathcal{ALC}$ relies on the active set to restrict the applicability of the tableau rules to infallible entities, while the calculus for $\mathcal{CALC}^{\mathsf{C}}$ treats constructive negation and its persistence with specific rules, and explicitly adds relational constraints to force reflexivity and transitivity of refinement $\preceq$, and the confluence of refinement w.r.t. role-filling. Similarly to our calculus, the system in [220] for $\mathcal{CALC}^{\mathsf{C}}$ is not duplication-free. This problem has been treated by the duplication-free tableau calculus for $\mathcal{CALC}^{\mathsf{C}}$ introduced in [219]. This calculus uses the notion of *active* and *passive* formulæ to distinguish the formulæ that have been expanded already or which need to be duplicated, that is, the rules are applied to active formulæ only, and such formulæ become passive after a rule application, except if they have to be duplicated. Like $c\mathcal{ALC}$, the logic $\mathcal{CALC}^{\mathsf{C}}$ uses a birelational semantics. However, the authors do not address the problem to constructively interpret ABox individuals in a birelational interpretation, *i.e.*, they use the classical interpretation of individual names by mapping them to single worlds of the domain of an interpretation. However, since $\mathcal{CALC}^{\mathsf{C}}$ is based on a birelational semantics, this interpretation of individual names does not preserve the partial order and therefore seems to break the monotonicity property.

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[20] See `http://www.haskell.org`.

CHAPTER 8

Conclusion

This thesis explored the model and proof theory of the constructive description logic $c\mathcal{ALC}$. The starting point for this research is motivated by the insight that classical DLs only support a static, closed interpretation of knowledge, and are insufficient to express and reason about partial and incomplete information (*cf.* Sec. 1.1). However, many application scenarios in the field of computer science, which are data- or process-driven, demand for a constructive notion of truth, which is robust under abstraction and refinement. Robustness of truth under abstraction and refinement are decisive to admit (i) the semantic abstractions that arise in data-driven applications, when factoring out non-relevant details or in terms of information compression to allow for the handling of very large data sets; or (ii) the partial nature of data generated by computational or business processes, which possibly never reach a final state, but are only defined up to contextual properties such as time or resources. Our key mission statement was to investigate a constructive variant of $\mathcal{ALC}$, that on the one hand allows for the model theoretical representation of partial and incomplete knowledge, and on the other hand is compatible with the Curry-Howard isomorphism in order to lay the grounds for a DL-based typing system.

We have presented a constructive semantics for $\mathcal{ALC}$ – denoted by *constructive* $\mathcal{ALC}$ ($c\mathcal{ALC}$) – that refines the classical semantics by a constructive notion of truth (*cf.* Chap. 4), and thereby generates a family of theories that admit the computational interpretation of proofs according to the Curry-Howard isomorphism (*cf.* Chap. 5). We have committed ourselves to base $c\mathcal{ALC}$ on the constructive modal logic CK [27; 188; 272], the main reason being, that (i) it is characterised by a minimal Hilbert-style axiomatisation; (ii) arguably, its axiomatisation reflects from our point of view the axiom schemata that appear to be computationally justified in computational and modal type theories [91; 159; 161; 162; 187; 198; 205; 209; 210; 226]; (iii) it does not require additional frame axioms in contrast to normal intuitionistic modal logics [96; 103; 228; 249]; and (iv) it has been demonstrated [27; 80; 194; 198] to admit a term-assignment relating proofs with λ-terms in Gentzen and natural deduction style, that is in line with the Curry-Howard isomorphism.

By means of studying several examples and application scenarios, we have demonstrated that the constructive semantics of $c\mathcal{ALC}$ supports the representation of partial and incomplete knowledge, and is crucial and to achieve consistency under abstraction as well as robustness under refinement (*cf.* Ex. 4.2.2, 4.2.5 and 4.2.6). Besides the model theoretical investigation we approached the proof theory of $c\mathcal{ALC}$ by characterising it in terms of a Hilbert-style axiomatisation, a Gentzen sequent calculus and a tableau-based decision procedure (*cf.* Chap. 5 and 7). The Hilbert calculus provides us with a clear proof-theoretical characterisation of $c\mathcal{ALC}$ in the form of a reference definition. The benefit of the Gentzen sequent calculus and the tableau calculus is, that the former is strongly related to the computational interpretation of $c\mathcal{ALC}$ according to the Curry-Howard isomorphism, while the latter establishes the base for an implementation and is also related to the game-theoretic perspective of dialogical logic [178]. The mentioned calculi support the standard reasoning services w.r.t. TBoxes, and admit decidable reasoning with proof extraction and countermodel construction. This thesis has also investigated the relation of $c\mathcal{ALC}$ to classical description logics (*cf.* Chap. 6) in terms of a faithful embedding, and by considering the sub-Boolean $\{\sqcup, \exists\}$-fragment $\mathcal{UL}$ [192], where the constructive semantics yields a complexity advantage over the classical descriptive semantics w.r.t. the subsumption problem.

Concluding, the system $c\mathcal{ALC}$ constitutes a well-behaved constructive description logic, which uses the same syntactical representation as classical $\mathcal{ALC}$, supports the standard DL inference services w.r.t. TBoxes in terms of decidable calculi, but is semantically more expressive and compatible with the Curry-Howard isomorphism.

8.1 Contributions

The central contributions of this thesis can be summarised as follows:

In Chapter 4, we have introduced $c\mathcal{ALC}$, that extends the classical semantics of $\mathcal{ALC}$ by an intuitionistic (epistemic) preorder and fallible entities. The former admits to semantically express states of knowledge that increase monotonically over time, and the latter correspond to specific states where any concept becomes true. The definition of the semantics stems from the study of several Kripke-style semantics of intuitionistic modal logics (*cf.* Sec. 4.1), that differ mainly in that they impose alternative truth conditions on the interpretation of the modalities $\Box$ and $\Diamond$, or require frame conditions for the intuitionistic and modal accessibility relations [160; 196; 249]. Looking for a minimal system as mentioned above, we decided to introduce $c\mathcal{ALC}$ as a multimodal generalisation of the modal logic CK, *i.e.*, $c\mathcal{ALC}$ is related to CK_m just like $\mathcal{ALC}$ can be viewed as a notational variant of the classical modal logic K_m (*cf.* Sec. 2.1.5). The semantics of $c\mathcal{ALC}$ is defined in terms of birelational Kripke frames, *i.e.*, the intuitionistic preorder and the modal accessibility relations are relations on the same domain. On the one hand this allows us to obtain a proof of the finite model property [188; 249, pp. 148 ff.; 118, pp. 24 ff.], but on the other hand it rules out the usual interpretation of ABox individual names and nominals as singleton sets [49, p. 178], like it is defined in classical DLs. For this reason, we focussed upon the investigation of $c\mathcal{ALC}$ w.r.t. TBoxes only. The constructive properties of $c\mathcal{ALC}$ were justified by showing that it is a conservative extension of intuitionistic propositional logic, and accepts only constructively accepted principles (*cf.* Examples 4.2.2–4.2.4 and 4.2.7). Moreover, $c\mathcal{ALC}$ is non-normal w.r.t. the interpretation of the possibility modality $\exists R$ ($\Diamond$) [103; 249] in the sense that it rejects the principle of disjunctive distribution in its binary $\mathsf{FS4}/\mathsf{IK4} =_{df} \exists R.(C \sqcup D) \supset \exists R.C \sqcup \exists R.D$ and nullary $\mathsf{FS3}/\mathsf{IK3} =_{df} \neg\exists R.\bot$ variant, where the latter is excluded by the addition of fallible entities [90]. Furthermore, $c\mathcal{ALC}$ also refutes the interaction schema $\mathsf{FS5}/\mathsf{IK5} =_{df} (\exists R.C \supset \forall R.D) \supset \forall R.(C \supset D)$. These schemata are usually accepted by normal intuitionistic modal and existing proposals for constructive description logics (*cf.* Sec. 2.2.2 and Chap. 3), but fail to have a uniform computational justification from our view. We have given proofs for the monotonicity property (*cf.* Proposition 4.2.2), the disjunction property (*cf.* Proposition 4.2.3), and the finite-model property (*cf.* Theorem 4.2.2) based on the filtration technique. Further, we illustrated a strengthening of the open world assumption (called *evolving open world assumption*) and suggested applications in the domain of auditing, to use $c\mathcal{ALC}$ as inference mechanism as well as a typing system for data streams. We believe that the definition of the semantics is general enough such that it should be applicable to other description logics as well.

Chapter 5 has demonstrated sound and complete Hilbert and Gentzen-style deduction systems that support the standard inference services of DLs w.r.t. TBoxes. The Hilbert calculus characterises $c\mathcal{ALC}$ in terms of axiom schemata and inference rules. Its deduction relation differentiates between local and global (TBox) hypotheses, and accordingly, we demonstrated a modal deduction theorem w.r.t. local and global premises. Soundness and completeness of the Hilbert calculus (*cf.* Theorem 5.1.2) follows from that of the Gentzen sequent calculus, by demonstrating that any deduction in either system can be translated into the other (*cf.* Proposition 5.2.1). A computational interpretation of DLs was illustrated by means of an information term semantics, inspired by [39; 41]. The Gentzen sequent calculus $\mathsf{G1}$ is an unlabelled multi-succedent sequent calculus with independent left- and right-introduction rules for the modalities $\exists R$ and $\forall R$, in contrast to the previous approaches [27, p. 2; 80, pp. 5 ff.] for CK, which use a single rule in which necessity and possibility are intertwined. We proved soundness and completeness of $\mathsf{G1}$ w.r.t. the birelational semantics (*cf.* Theorem 5.2.1), and argued its decidability (*cf.* Theorem 5.2.4). Moreover, the sequent calculus supports countermodel construction in the sense that each sequent specifies a single entity of the domain such that the countermodel is specified in terms of the graph that is generated by the sequent rules and equivalence classes of nodes (*cf.* Ex. 5.2.3). The final section discussed intermediate systems between $c\mathcal{ALC}$ and $\mathcal{ALC}$ by means of sound and complete extensions of the Hilbert and Gentzen calculus by the axiom schemata $\mathsf{FS3}/\mathsf{IK3}$ – $\mathsf{FS4}/\mathsf{IK4}$ and the Excluded Middle. We remark that the sequent calculus is not in proper first-order Gentzen format, because its rules involve sets rather than individual formulæ. Moreover, the proof of cut-elimination was obtained by semantic means (*cf.* Corollary 5.2.1). Thus, the $\mathsf{G1}$ calculus does not lend it self to a computational interpretation that admits to relate proofs with λ-terms. This problem has been approached by the contextual single-conclusion sequent calculus in [196] and its Curry-Howard term-assignment was explored in [194; 197; 198].

Chapter 6 explored in the first part the relation between $c\mathcal{ALC}$ and classical DLs. We introduced a faithful embedding (*cf.* Theorem 6.1.2) of $c\mathcal{ALC}$ into the fusion [103, Chap. 3] $\mathsf{S4}_n \otimes \mathsf{K}_m$, which corresponds to $\mathcal{ALC}$ extended by reflexive and transitive roles. The embedding allows us to transfer results from bimodal logics to $c\mathcal{ALC}$, namely, the finite model property (*cf.* Theorem 6.1.6), decidability (*cf.* Theorem 6.1.5), and an upper bound for the complexity of the subsumption (and satisfiability) problem in $c\mathcal{ALC}$. Moreover, the embedding of $c\mathcal{ALC}$ into classical DLs admits the use of existing highly optimised tableau reasoners from DL systems for the standard reasoning tasks of $c\mathcal{ALC}$ w.r.t. TBoxes. This has been evaluated by means of a case study relying on the reasoner *Racer*. We demonstrated that the problem of deciding satisfiability or subsumption of a $c\mathcal{ALC}$ concept without TBoxes is PSPACE-complete (*cf.* Theorem 6.1.1), while it becomes EXPTIME-complete in the presence of general TBoxes (*cf.* Theorem 6.1.4). The second part of Chapter 6 summarises the result from [192] that deciding the subsumption problem in the sub-Boolean $\{\sqcup, \exists\}$-fragment $\mathcal{UL}$ is tractable under the constructive semantics (*cf.* Theorem 6.2.2), but intractable under the classical descriptive semantics (*cf.* Theorem 6.2.1). It extends [192] by presenting the proof of the admissibility of the cut rule for the sequent calculus $\mathsf{G1}_{\mathcal{UL}}$ (*cf.* Proposition cut-admissible) of $\mathcal{UL}$. However, the complexity result for $\mathcal{UL}$ is mainly of theoretical interest, since the fragment $\mathcal{UL}$ is semantically very restricted, and at the moment it is unclear whether $\mathcal{UL}$ may find practical applications.

Chapter 7 introduced the decidable tableau-based calculus $\mathfrak{T}_{c\mathcal{ALC}}$, inspired by the contextual sequent calculus for CK_m [196], showing soundness, completeness and termination (*cf.* Theorem 7.3.1, 7.3.2 and Proposition 7.3.1). $\mathfrak{T}_{c\mathcal{ALC}}$ relies on a constraint system as underlying data structure, assigns labels and signs to formulæ to accommodate the constructive semantics of $c\mathcal{ALC}$, and uses blocking (loop-checking) to ensure termination. Fallible entities are emulated by the notion of active/inactive entities to allow for the distinction of non-trivial realisations of entities from fallible worlds. The novel aspect of the tableau is given by the extra constraints $u{:}-_{\forall R}C, u{:}-_{\exists R}C$ to specify that 'C is false in all R-successors' and 'C is false in some R-successor', their introduction by the independent right (negative) expansion rules $(\rightarrow_{\exists -}), (\rightarrow_{\forall -})$, and their treatment by the independent rules $(\rightarrow_{R_{\exists}-}), (\rightarrow_{R_{\forall}-})$. This independent treatment further generalises the focus shift rule Ax_f from the single conclusion sequent calculus for CK_n [196; 198]. The chapter closes by giving an outlook towards constructive ABox reasoning w.r.t. birelational semantics. We discuss the problems that arise when interpreting ABox assertions in birelational models, and sketch by means of two examples (*cf.* Example 7.4.1–7.4.2) a possible solution. Both examples cover instance checking. The first example (Ex. 7.4.1) presents a successful proof of instance checking, and uses nominal concepts and an internalisation of the ABox into the TBox. The second example (Ex. 7.4.2) discusses a failed proof of instance checking and illustrates countermodel construction. Hereby we use an alternative interpretation of individual names – inspired by [61] – in the form of a partial function that maps entities to individual names. This approach separates the ABox from the constraint system, and treats it more like a TBox. While this separation is not necessarily required, we believe that on the one hand it simplifies the treatment of ABox assertions for tableau calculi based on birelational semantics, and on the other hand it is in line with the idea of constructive reasoning, to have a clear distinction between assertional hypotheses from an ABox and their realisation by a concrete interpretation.

8.2 Future Perspectives

Future work on constructive DLs should be approached both from a theoretical and a practical point of view. On the one hand, this should also address the following open problems of this thesis: (i) A proof of the completeness of the R-infallible sequent calculus $\mathsf{G1}_{R^F}$ (see p. 176). (ii) A proof of the admissibility of the cut rule for the sequent calculus $\mathsf{G1}_D$ (see p. 187). (iii) A characterisation of the axiom schema $\mathsf{FS5}/\mathsf{IK5}$ in terms of a frame class/condition and an extension of the sequent rules of $\mathsf{G1}$ (see p. 169). (iv) A further investigation of the embedding of $c\mathcal{ALC}$ into $\mathcal{ALC}_{R^*}$ to see, whether reasoning of $c\mathcal{ALC}$ remains PSPACE-complete w.r.t. simple constructive TBoxes (see p. 215). (v) The complexity of the subsumption problem in the systems $\mathcal{UL}^-$ and $\mathcal{ELU}$ under the constructive semantics (see p. 226). (vi) A deeper investigation of the interpretation of ABox assertions w.r.t. the birelational semantics of $c\mathcal{ALC}$, possibly by extending our proposal for constructive ABox reasoning (see p. 268). Moreover, on the other hand, we suggest the following four possible directions for further investigations regarding theoretical work.

8.2.1 Theoretical Aspects

Semantics for constructive ABoxes

Firstly, a desirable further development is to investigate the interpretation of ABox assertions in $c\mathcal{ALC}$. In this thesis we have not developed a *general* theory for constructive reasoning w.r.t. ABoxes under the birelational semantics of $c\mathcal{ALC}$, but rather gave a sketch for a possible approach. The usual method in DLs and hybrid logics to give a semantics to individual names and nominals is to map individual names to single elements of the domain, and to interpret nominals as singleton sets. This semantics carries over without difficulty to intuitionistic DLs (see Sec. 3) and hybrid logics [49, pp. 171 ff.] that are based on standard intuitionistic Kripke semantics, where the intuitionistic preorder and the modal accessibility relations are separated. However, under the birelational semantics, this interpretation becomes infeasible, since it breaks with the monotonicity property [49, pp. 177 f.]. Based on an idea of Chadha, Macedonio and Sassone [61], we sketched an approach in terms of using a partial function that maps entities of the domain to individual names, and by separating ABox assertions from the constraint system to differentiate between ABox hypotheses and their concrete realisation. ABox assertions are realised by additional rules that manage the naming of elements of the domain and instantiate ABox hypotheses in the construction of birelational models. In order to satisfy an ABox, such models are required to possess a consistent naming of domain elements, expressed by the properties of *coherence* and *simulation*, *i.e.*, the naming of entities and the relational structure w.r.t. named entities must be robust under refinement. Our approach needs further investigation, for instance by demonstrating the correctness of the extended calculus. It also seems helpful to us to study this problem from the perspective of a standard intuitionistic Kripke semantics for $c\mathcal{ALC}$ and by investigating its relation to the birelational semantics. From our perspective the key questions still to be addressed are: 'What is the interpretation of individual names/nominals under intuitionistic birelational semantics?' and 'What is constructive ABox reasoning and how does it differ from its classical variant?'. We believe that an interpretation of individual names/nominals under a birelational Kripke-style semantics will not only be helpful for $c\mathcal{ALC}$ in particular, but also allow to further investigate birelational semantics for intuitionistic description [39; 43; 45; 64; 78; 125–127] and hybrid logics [52; 61].

More expressive constructive DLs

Secondly, it seems to be desirable to further enrich $c\mathcal{ALC}$ in terms of its expressivity with language extensions known from very expressive DLs as $\mathcal{SHROIQ}$ [173]. The most common extensions at the level of concept constructors are *qualified number restrictions* (a.k.a. *graded modalities* in modal logics) and *nominals*. Cardinality restrictions allow to restrict the number of role fillers in the form of *at-least* and *at-most* restrictions, *e.g.*,

$$\geq 1\mathsf{managedBy}.\mathsf{CEO}, \text{ and } \qquad \leq 4\mathsf{hasPart}.\mathsf{Tire},$$

where the former describes the set of individuals that are companies managed by at least one CEO, and the latter describes cars that possess at most four tires. Moreover, DLs permit the use of individual names not only in ABoxes but also at the concept level, denoted by *nominals*. Using the *set* constructor $\{i_1, i_2, \ldots, i_n\}$ that can be viewed as an enumeration of names i_k with $1 \leq k \leq n$, we can specify for instance the concept of the moons of planet Pluto by

$$\mathsf{MoonsOfPluto} \equiv \{\mathsf{Charon}, \mathsf{Nix}, \mathsf{Hydra}, \mathsf{Kerberos}, \mathsf{Styx}\}.$$

While the interpretation of nominals depends on the interpretation of ABox names as mentioned before, it is a general problem whether there exists a constructively acceptable interpretation of cardinality restrictions, since their semantics rely on the identification of individuals, which becomes delicate when considering partially determined objects under constructive semantics. We believe that extensions at the level of role restrictions are more feasible, for instance to extend $c\mathcal{ALC}$ by transitive roles, inverse roles, or role hierarchies and additional role constructors like *role inclusion* $R \sqsubseteq S$ and *role composition* $R \circ S$. Furthermore, besides the standard classical constructors, $c\mathcal{ALC}$ can consider non-standard language extensions like n-ary relations [16, pp. 221 ff.], temporal modalities for CK [162], McCarthy-style contexts [79; 156; 157; 188], or aggregating modalities to express statements up to resource bounds.

Computational Interpretation

One possible direction for future work is to explore $c\mathcal{ALC}$ from a type-theoretic perspective and in particular, to extract proof terms/natural deduction rules for existential and universal restriction ($\exists R$ and $\forall R$), to yield an extension of the simply-typed λ-calculus that expresses context-dependent computations in structured data.

Although not being part of this thesis, this program has been started in [194; 196; 198] (see Sec. 3.5.2). In [196], a contextual cut-free Gentzen sequent calculus in proper first-order representation has been introduced, which is derived from the multi-sequent calculus G1 for $c\mathcal{ALC}$. The work in [194; 196] pursues this direction, by exploring $c\mathcal{ALC}$ from a type-theoretical perspective, introducing its computational λ-calculus $\lambda\mathsf{CK}_n$ to give a computational interpretation for $c\mathcal{ALC}$ according to the Curry-Howard isomorphism. The modal type theory $\lambda\mathsf{CK}_n$ gives computational meaning for the modalities $\forall R$ and $\exists R$ as type operators with natural and independent constructors and destructors. Operationally, computations in $\lambda\mathsf{CK}_n$ are characterised by a form of a restricted top-down data flow within the contextual structure, *i.e.*, a computation that lives in some given context cannot resort to information from earlier or external contexts. Instead, the information flow is unidirectional from the actual context. This scheme of information flow resembles the axiomatisation of $c\mathcal{ALC}$, and operationally expresses the exclusion of the axiom schemata FS3/IK3 – FS5/IK5. The Gentzen-style typing system of $\lambda\mathsf{CK}_n$ is sound and complete, can be used for goal-directed proof search in CK_n, and the λ-calculus $\lambda\mathsf{CK}_n$ satisfies subject reduction, strong normalisation and confluence. Further issues that need to be investigated are decidability of typing and the existence of most general types. We believe that a system such as $\lambda\mathsf{CK}_n$ can establish the formal grounds for a contextually typed functional programming language, which is suitable as a programming paradigm in the domain of knowledge representation and semantic web applications.

Correspondence Theory

In Sec. 5.3, we have sketched the extension of $c\mathcal{ALC}$ by the axiom schemata FS3/IK3 – FS4/IK4 and the principle of the Excluded Middle in terms of sound and complete extensions of the corresponding Gentzen and Hilbert calculi. Note that the completeness of the sequent calculus $\mathsf{G1}_D$ that implements axiom FS4/IK4 relies on admissibility of the cut-rule, which we left as an open problem. For future work, it seems to be appropriate to systematically study extensions of $c\mathcal{ALC}$ in terms of developing a constructive correspondence theory [198]. For instance, other non-classical modal logics such as IK [96; 228; 249], CS4 [4; 159; 226], PLL [90], or Masini's deontic system[184] arise as specialised theories of CK, as sketched in [196, p. 5]. We believe that by taking $c\mathcal{ALC}$ (CK_m) as the base point of a constructive correspondence theory, it allows one to show that classical DLs (and modal logics) as well as intuitionistic DLs (and IMLs) arise as special theories of $c\mathcal{ALC}$, but also give rise to intermediate logics between $c\mathcal{ALC}$, intuitionistic and classical DLs. From our point of view, the constructive dimension of such a correspondence theory should relate theories and Kripke models from a model-theoretic perspective, and, from a proof-theoretic perspective, establish Curry-Howard correspondences between modal type theories and computational λ-calculi.

8.2.2 Towards Applications

Regarding practical future work, we propose to focus on the implementation of the calculi of $c\mathcal{ALC}$ and investigate its usage in potential applications.

Implementation

For the purpose of evaluating the practical importance and usability of $c\mathcal{ALC}$, it is indispensable to implement its calculi and review it in terms of large confirmatory case studies. Form our perspective, the critical factors for success are on the one hand to provide the necessary reasoning infrastructure for $c\mathcal{ALC}$ in terms of decidable and highly optimised decision procedures, and on the other hand to establish practical applications beyond classical DL-style knowledge representation, which extend the traditional inference services w.r.t. TBoxes and ABoxes to support dynamic and stream based knowledge. We believe that such an extension can constitute new reasoning tasks that account for several possible dimensions of dynamics, *e.g.* time, context, degree of information or up to resource bounds. Implementation-wise we see two key directions for further investigation.

Firstly, it is desirable to implement the tableau calculus $\mathfrak{T}_{c\mathcal{ALC}}$, aiming at the domain of knowledge representation. An early version of the tableau calculus for $c\mathcal{ALC}$ has been implemented by Sticht [254] in Haskell from a game-theoretic perspective. Moreover, $\mathfrak{T}_{c\mathcal{ALC}}$ was encoded in the generic tableau prover *LoTRec v2.0*[21] to evaluate several theorems of $\mathfrak{T}_{c\mathcal{ALC}}$ and obtain explicit countermodels for non-theorems. However, these proof-of-concept implementations only scratched the surface and did not employ optimisation methods from DL-style tableau reasoners. It is therefore our recommendation to implement $\mathfrak{T}_{c\mathcal{ALC}}$ systematically with a long-term perspective in mind, and to allow for a comparative evaluation with classical reasoning systems.

Secondly, we believe that an implementation of the sequent calculi and its associated λ-calculus seems promising. Regarding the computational interpretation of $c\mathcal{ALC}$, the calculus $\lambda\mathsf{CK}_n$ has been evaluated by means of an Haskell-implementation of its typing system and β-reduction in the context of the Bachelor's thesis of Gareis [107]. This work uses maps for an efficient nameless representation of variables in λ-terms [241], that is, a binary tree is used to indicate the position of bound variables, which gives rise to a λ-calculus without need for α-conversion when reducing a term to its normal form. However, proofs showing correctness of typing and β-reduction under the map-representation of $\lambda\mathsf{CK}_n$ are still open problems. Moreover, an untyped variant of $\lambda\mathsf{CK}_n$ was encoded in the Abella proof assistant [106], but solely used to give a formally verified Tait/Martin-Löf-style confluence proof [26] for β-reduction of $\lambda\mathsf{CK}_n$, borrowing the technique from Accattoli [2]. Though Abella is well-suited to formalise and reason about the meta-theory of logical systems, we believe it is not the adequate tool for a systematic implementation. Going one step further, a fully formal implementation in

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[21] LoTRec is available from http://www.irit.fr/Lotrec/.

a theorem prover like the *Coq Proof Assistant*[22] seems to be desirable, in spite of the tremendous effort required. A theorem prover like Coq provides the required facilities to experiment with logical system, to evaluate larger case studies, and it is tailor-made to suit the needs of a systematic implementation of the formal machinery of $c\mathcal{ALC}$. On the one hand it allows for a rigorous verification to ensure that the implementation behaves as expected, and on the other hand it allows for program extraction from proofs. In particular, the proof assistant Coq would permit to automate inference services and typing in $c\mathcal{ALC}$, by programming goal-oriented backward reasoning tactics. Moreover, the extraction of programs from proofs would yield verified code for key components implementing typing and β-reduction.

Applications

Besides the investigation of the classical inference services of DLs it seems desirable to put the emphasis of future work on non-standard reasoning problems w.r.t. dynamic and incomplete information. We believe that this approach may give rise to new application scenarios, which have been infeasible under the classical semantics so far.

In this thesis we motivated the utilisation of $c\mathcal{ALC}$ to represent and reason about evolving knowledge that is generated by ongoing processes in the form of data streams [189; 193]. One particular domain of interest is financial auditing [3; 59; 60; 231; 258; 259]. We believe that extensions of $c\mathcal{ALC}$ can be utilised as ontological specification languages to characterise the semantics of financial information flows in terms of strongly typed data streams. Constructive inference services like type-checking and model-checking can be used to ensure the trustworthiness of next-generation auditing tools, and to implement interactive audit scenarios based on stream-based executing models.

From a type-theoretic perspective, the computational interpretation of $c\mathcal{ALC}$ can form the cornerstone of a functional programming language, which employs DLs as programming language type systems, and is able to express context-dependent computations with restricted information flow in relational data structures such as DL-style knowledge bases, databases or data streams induced by ongoing processes. Ideally, this may give rise to a component-based programming environment, which uses as design language the combination of data flow and control flow programming. Then, $c\mathcal{ALC}$ types can specify the type of stream processing functions, and auditing procedures may be implemented in terms of audit agents [3]. Such software agents can play the role of intelligent audit procedures, which are acting as data flow processing nodes within information networks that connect enterprises. They process the information of business transactions in real-time and draw conclusions or fire events based on background

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[22] Further information about the Coq Proof Assistant is available from http://coq.inria.fr/.

knowledge specified in terms of DL-style ontologies, when detecting irregularities or in the presence of suspicious facts or fraud. Their implementation may arise from the proof of specific audit statements and their code may be extracted from proof terms based on a calculus. Concluding, we hope that $c\mathcal{ALC}$ may give rise to DL-typed functional programming languages that will find practical adoption in the domain of knowledge engineering and data processing languages.

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Description logics (DLs) represent a widely studied logical formalism with a significant impact in the field of knowledge representation and the Semantic Web. However, they are equipped with a classical descriptive semantics that is characterised by a platonic notion of truth, being insufficiently expressive to deal with evolving and incomplete information, as from data streams or ongoing processes. Such partially determined and incomplete knowledge can be expressed by relying on a constructive semantics. This thesis investigates the model and proof theory of a constructive variant of the basic description logic $\mathcal{ALC}$, called c$\mathcal{ALC}$. The semantic dimension of constructive DLs is investigated by replacing the classical binary truth interpretation of $\mathcal{ALC}$ with a constructive notion of truth. This semantic characterisation is crucial to represent applications with partial information adequately, and to achieve both consistency under abstraction as well as robustness under refinement, and on the other hand is compatible with the Curry-Howard isomorphism in order to form the cornerstone for a DL-based type theory. The proof theory of c$\mathcal{ALC}$ is investigated by giving a sound and complete Hilbert-style axiomatisation, a Gentzen-style sequent calculus and a labelled tableau calculus showing finite model property and decidability. Moreover, c$\mathcal{ALC}$ can be strengthened towards normal intuitionistic modal logics and classical $\mathcal{ALC}$ in terms of sound and complete extensions and hereby forms a starting point for the systematic investigation of a constructive correspondence theory.

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