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ANALYSIS OF A LOSS SYSTEM WITH MUTUAL OVERFLOW AND EXTERNAL TRAFFIC

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Abstract

We investigate the traffic behaviour in a network consisting of a local exchange and two exchanges of the long-distance network being connected to each other by two distinct both-way trunk groups. Modeling this network by means of a loss system with two Poissonian originating traffic streams following a mutual overflow routing scheme and two Poissonian external traffic streams, we derive a representation of the steady-state probabilities for the number of busy lines in both trunk groups. Our theorectical results include simple expressions of time and individual call-congestion rates of the arrival streams as well as a simple fixed-point approximation of these quantities. Finally, we point out that lost calls selected from the originating and external traffic streams offered to each trunk group form PH-renewal and MMPP processes.

Keywords
teletraffic theory, loss system, mutual overflow, time and call-congestion rates, fixed-point approximation, PH-renewal process, MMPP process

1. Introduction

Recently, considerable attention has been devoted to the analysis of advanced routing schemes in circuit-switched digital networks based on efficient modern signalling systems such as CCS CCITT No. 7.

In this paper we investigate the traffic behaviour in a subnetwork. It consists of a local exchange which is connected with two exchanges of the long-distance network by two distinct both-way trunk groups. The outgoing traffic of the local exchange is split into two portions and each portion is offered to an outgoing group. These partial traffic streams are routed according to a mutual overflow scheme (cf. [12], [16]). Additionally each trunk group carries external traffic (cf. Fig. 1).

This paper is organized as follows. In section 2 we give a detailed mathematical description of this subnetwork by means of a loss system. Assuming exponentially distributed interarrival and holding times, we derive a Markovian model for the number of busy lines in both trunk groups. In section 3 we represent its steady-state probabilities in terms of Brockmeyer polynomials (cf. [2], [20], [12]). In section 4 a simple fixed-point approximation of the average call-congestion rates of the total arrival streams offered to both trunk groups is presented. In section 5 we point out that the streams of lost calls corresponding to the outgoing and external traffic form PH-renewal and MMPP processes. Assuming that these streams are offered to a group with an infinite number of lines and exponentially distributed holding times, we calculate simple expressions for the factorial moments of busy lines. Based on these moment formulas the corresponding streams may be approximated by simple traffic streams such as IPP's employing standard moment-matching techniques (cf. [14], [8], [19], [18]).

2. A model of the overflow system

The subnetwork consisting of a local exchange and two exchanges of the long-distance network which are connected to each other by two both-way trunk groups may be described by a loss system being composed of two fully available trunk groups called systems 1 and 2 with N_1 and N_2 lines, respectively. Two originating traffic streams (streams 2 and 3) representing the portions of the outgoing traffic and two incoming external traffic streams (streams 1 and 4) are offered to the loss system (cf. Fig. 2). These arrival processes are modeled by mutually independent Poisson processes with positive rates λ_2, λ_3 and λ_1, λ_4, respectively.
The external traffic streams 1 and 4 offered to systems 1 and 2 follow a random hunting scheme for free lines. Their calls are lost without further impact on the system if the corresponding trunk group is busy upon arrival. The outgoing streams 2 and 3 follow a mutual overflow routing scheme. This means, that upon arrival at system 1 a call of flow 2, for instance, is searching for a free line. If possible, a free trunk is selected in a random manner and occupied. If system 1 is busy and there are free lines in system 2, the incoming call from flow 2 will immediately overflow to system 2 upon arrival and occupy a line selected at random. If both systems are busy, the call will be blocked and lost without further impact on the system (lost calls cleared).
Call holding times are considered to be mutually independent exponentially distributed random variables with a finite common mean $1/\mu$. They are also assumed to be independent of the arrival processes.
The occupation of groups can be modeled by an irreducible, homogeneous continuous-time Markov chain $\{X(t), t \geq 0\}$ with state variables $X_i(t)$ denoting the number of busy trunks in group i at time t, $i = 1, 2$, and a finite state space $S = \{(i,j) \mid 0 \leq i \leq N_1; 0 \leq j \leq N_2\}$. Its limiting distribution $P = (P_{ij})_{i=0,\ldots,N_1; j=0,\ldots,N_2}$, $P_{ij} = \lim_{t\to\infty} P(X(t) = (i,j))$, is the unique solution of the normalization condition $\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} P_{ij} = 1$ and the balance equations

$$\begin{aligned}
P_{ij} \quad & \cdot \quad [(\lambda_1(1-\delta_{i,N_1}) + \lambda_2)(1-\delta_{i,N_1}\delta_{j,N_2}) + i\mu \\
& + \quad (\lambda_3 + \lambda_4(1-\delta_{j,N_2}))(1-\delta_{i,N_1}\delta_{j,N_2}) + j\mu] \\
= \quad & \quad (\lambda_1 + \lambda_2 + \lambda_3\delta_{j,N_2})(1-\delta_{i,0}) \cdot P_{i-1j} + (i+1)\mu(1-\delta_{i,N_1}) \cdot P_{i+1j} \\
& + \quad (\lambda_3 + \lambda_4 + \lambda_2\delta_{i,N_1})(1-\delta_{j,0}) \cdot P_{ij-1} + (j+1)\mu(1-\delta_{j,N_2}) \cdot P_{ij+1} \, ,
\end{aligned} \tag{1}$$

$0 \leq i \leq N_1$, $0 \leq j \leq N_2$, where $\delta_{k,l} = 1$ for $k = l$ and 0 otherwise (cf. [6, Satz 12.1, p. 121]). Obviously, the steady-state probability vector $P = (P_{00}, P_{01}, \ldots, P_{0N_2}, \ldots, P_{N_1 0}, \ldots, P_{N_1 N_2})^t$ is the unique positive, normalized solution of the homogeneous linear system $A \cdot P = 0$ where $A = -Q^t$ is the negative transpose of the generator matrix Q associated with $\{X(t), t \geq 0\}$. A is an irreducible, singular M-matrix with block tridiagonal structure and tridiagonal regular M-matrices along its diagonal. Furthermore, $e^t A = 0$ holds where e denotes the vector with all ones. Thus, A is a weakly 2-cyclic consistently ordered Q-matrix (cf. [13], [25], [24], [1]). Taking advantage of the

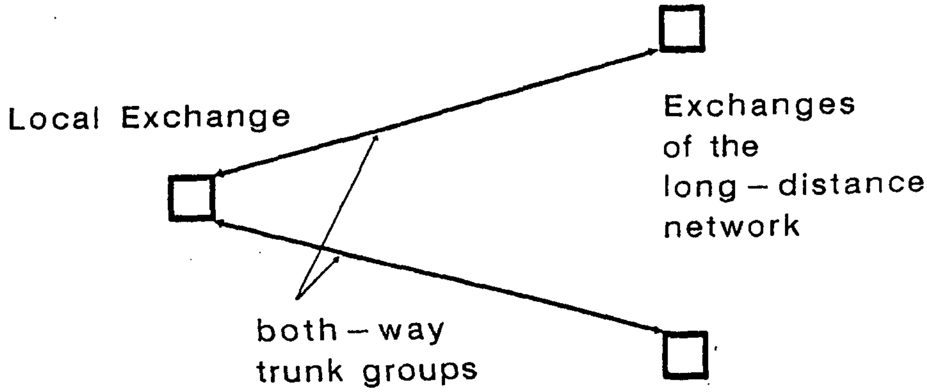

Figure 1: Model of the circuit-switched subnetwork

block structure of A, the steady-state vector P may be computed by an efficient convergent block iterative scheme derived from an R-regular splitting of A (cf. [13], [24], [19]).

3. Analysis of the steady-state distribution

Solving the difference equations (1) we apply a combined series representation and separation method which has been introduced by Morrison [20] for similar models. It is possible to prove that the steady-state probabilities may be represented by a linear combination of products of the well-known Brockmeyer polynomials $s_n(\lambda, A)$ (cf. [4, p. 226f], [20, Appendix A]). Due to the limitation of space the details will be omitted (cf. [13]). We only summarize the result in the following theorem.

Theorem 1
Let $A_1 = (\lambda_1 + \lambda_2)/\mu > 0$, $A_2 = (\lambda_3 + \lambda_4)/\mu > 0$ be the offered loads of the mutual overflow model. Then the unique limiting distribution $P = (P_{ij})$ of the ergodic Markov chain $\{X(t), t \geq 0\}$ modeling the number of busy lines in both trunk groups may be represented in the form of

$$\begin{aligned} P_{ij} = & \frac{A_1^i/i!}{\sum_{r=0}^{N_1} A_1^r/r!} \cdot \frac{A_2^j/j!}{\sum_{r=0}^{N_2} A_2^r/r!} \\ & + \sum_{k=1}^{N_1} C_k \cdot s_i(\rho_k, A_1) \cdot s_j(-\rho_k, A_2) + \sum_{l=1}^{N_2} D_l \cdot s_i(\beta_l, A_1) \cdot s_j(-\beta_l, A_2) \end{aligned} \tag{2}$$

for $0 \leq i \leq N_1$, $0 \leq j \leq N_2$.
$\{\rho_1, \ldots, \rho_{N_1}\} \subseteq (-\infty, -1)$ denote the N_1 simple zeros of $s_{N_1}(1 + \rho, A_1)$ whereas $\{\beta_1, \ldots, \beta_{N_2}\} \subseteq (1, \infty)$ are the N_2 simple roots of $s_{N_2}(1 - \beta, A_2)$. The unique real coefficients $C_1, \ldots, C_{N_1}, D_1, \ldots, D_{N_2}$ are the unique solution of the following inhomogeneous linear system of order $N_1 + N_2$:

$$\begin{aligned} & \sum_{k=1}^{N_1} C_k[(-\rho_k) \cdot s_i(\rho_k, A_1) \cdot s_{N_2}(1 - \rho_k, A_2) + \lambda_3/\mu \cdot s_i(-1 + \rho_k, A_1) \cdot s_{N_2}(-\rho_k, A_2)] \\ & \quad + \sum_{l=1}^{N_2} D_l[\lambda_3/\mu \cdot s_i(-1 + \beta_l, A_1) \cdot s_{N_2}(-\beta_l, A_2)] = -\frac{\lambda_3}{\mu} \cdot \frac{s_i(-1, A_1)}{s_{N_1}(1, A_1)} \cdot \frac{s_{N_2}(0, A_2)}{s_{N_2}(1, A_2)} \\ & \qquad 0 \leq i \leq N_1 - 1 \end{aligned} \tag{3}$$

$$\begin{aligned} & \sum_{l=1}^{N_2} D_l[\beta_l \cdot s_{N_1}(1 + \beta_l, A_1) \cdot s_j(-\beta_l, A_2) + \lambda_2/\mu \cdot s_{N_1}(\beta_l, A_1) \cdot s_j(-1 - \beta_l, A_2)] \\ & \quad + \sum_{k=1}^{N_1} C_k[\lambda_2/\mu \cdot s_{N_1}(\rho_k, A_1) \cdot s_j(-1 - \rho_k, A_2)] = -\frac{\lambda_2}{\mu} \cdot \frac{s_{N_1}(0, A_1)}{s_{N_1}(1, A_1)} \cdot \frac{s_j(-1, A_2)}{s_{N_2}(1, A_2)} \\ & \qquad 0 \leq j \leq N_2 - 1 \end{aligned} \tag{4}$$

□

4. Fixed-point approximation of congestion rates

4.1 Calculation of flow-dependent call-congestion rates

We assume the occupation process $X(t)$ to be in steady state and analyse an isolated trunk group, for instance, group 1. Three streams are offered to this subsystem: two Poisson processes with rates λ_1 and λ_2 and the stream formed by those calls of the Poisson process with rate λ_3 overflowing from group 2 to 1. If we mark the incoming calls of all Poisson streams by four different colours

(customer types), the isolated trunk group may be described by a $G_3/M/N_1/0$-model (cf. [5, p. 61]) . Let us denote the individual call-congestion rate of the customer stream of type j offered to group i by B_{ij}, $i \in \{1,2\}, j \in \{1,\ldots,4\}$.
Due to the PASTA-Theorem [9, 11-2, p. 391f] time- and call-stationary probabilities (cf. [5, p. 58ff], [7, §11, p. 517ff]) coincide in the case of arriving Poisson streams, thus $B_{11} = B_{12} = P_{N_1} = P(X_1 = N_1), B_{23} = B_{24} = P_{N_2} = P(X_2 = N_2)$. The time congestion of the entire system is denoted by $P_{N_1N_2} = P(X_1 = N_1, X_2 = N_2)$. Applying a well-known result [5, 5.3.5, p. 154] we conclude that the call-congestion rates of the overflow streams are given by $B_{13} = P_{N_1N_2}/P_{N_2}$ and $B_{22} = P_{N_1N_2}/P_{N_1}$. The average call-congestion rates corresponding to the total streams of calls offered to groups 1 and 2 read

$$B_1 = \frac{(\lambda_1+\lambda_2)\cdot P_{N_1} + \lambda_3 \cdot P_{N_1N_2}}{\lambda_1+\lambda_2+\lambda_3\cdot P_{N_2}}, \qquad B_2 = \frac{(\lambda_3+\lambda_4)\cdot P_{N_2} + \lambda_2 \cdot P_{N_1N_2}}{\lambda_3+\lambda_4+\lambda_2\cdot P_{N_1}} \tag{5}$$

(cf. [5, p. 154], [15, p.790]).
Obviously, it is sufficient to calculate the time-congestion rates of the entire system, $P_{N_1N_2}$, and of groups 1 and 2, P_{N_1}, P_{N_2}. On the other hand, the latter quantities are determined by the average blocking rates in a unique manner, too :

$$P_{N_1} = \frac{B_1(\lambda_3+\lambda_4)[\lambda_1+\lambda_2+\lambda_3B_2] - P_{N_1N_2}\lambda_3[\lambda_3+\lambda_4+\lambda_2B_1]}{(\lambda_1+\lambda_2)(\lambda_3+\lambda_4) - \lambda_2\lambda_3B_1B_2} \tag{6}$$

$$P_{N_2} = \frac{B_2(\lambda_1+\lambda_2)[\lambda_3+\lambda_4+\lambda_2B_1] - P_{N_1N_2}\lambda_2[\lambda_1+\lambda_2+\lambda_3B_2]}{(\lambda_1+\lambda_2)(\lambda_3+\lambda_4) - \lambda_2\lambda_3B_1B_2} \tag{7}$$

Numerical experiments have shown that the differences between customer specific call-congestion rates and average call-congestion rates decrease if the offered traffic increases.

4.2 Approximation of congestion rates

To approximate the call-congestion rates related to each trunk group we assume the overflow processes which are offered to each trunk group in addition to the originating and external traffic streams to be Poissonian. Moreover, these streams are supposed to be independent of the Poisson streams originally offered to the corresponding trunk groups.
Due to the PASTA-Theorem (cf. [9, §11-2, p. 391f]) originally offered and overflowing calls experience the same call-congestion rates under these assumptions. Thus, flow-dependent and average call-congestion rates coincide with time-congestion rates, i.e. $P_{N_1} = B_{11} = B_{12} = B_{13} = B_1$ and $P_{N_2} = B_{22} = B_{23} = B_{24} = B_2$.
Applying the upper bound $0 < \frac{\partial}{\partial A} E(m,A) < 1/A$ for the derivative of Erlang's loss formula $E(m,A)$, $m \in \mathbb{N}$, $A > 0$ (cf. [12], [11]) it is possible to derive the following fixed-point formula for estimating the average call-congestion rates of both trunk groups.

Lemma 1
Let $\lambda_2 > 0$, $\lambda_3 > 0$. Then there exists a unique fixed point $(B_1^*, B_2^*) \in (0,1)^2$ determined by the fixed-point equations:

$$B_1 = E(N_1, \frac{\lambda_1+\lambda_2}{\mu} + B_2\frac{\lambda_3}{\mu}), \qquad B_2 = E(N_2, \frac{\lambda_3+\lambda_4}{\mu} + B_1\frac{\lambda_2}{\mu}) \tag{8}$$

For each initial point $B_2^{(0)}$ it may be computed by the convergent iterative scheme $B_2^{(n+1)} = E(N_2, (\lambda_3+\lambda_4)/\mu + \lambda_2/\mu \cdot E(N_1, (\lambda_1+\lambda_2)/\mu + B_2^{(n)}\lambda_3/\mu))$, $n \geq 0$, setting $B_2^* = B_2^{(n)}$, $B_1^* = E(N_1, (\lambda_1+\lambda_2)/\mu + B_2^*\lambda_3/\mu)$ after convergence. □

All results derived so far, especially Theorem 1 and Lemma 1, also hold if the mean holding times in both trunk groups are different (cf. [13]). Only in the next conclusion do we have to assume explicitly that all holding times are equally distributed. Without loss of generality we suppose $\mu = 1$. Now the individual call-congestion rates B_{13}, B_{22} may be approximated by the average rates B_1, B_2. Based on the relation

$$P_{N_1 N_2}(\lambda_2 + \lambda_3) = \lambda_3 P_{N_2} \cdot B_{13} + \lambda_2 P_{N_1} \cdot B_{22} \approx \lambda_3 P_{N_2} \cdot B_1 + \lambda_2 P_{N_1} \cdot B_2 \ , \tag{9}$$

the substitution of the unknown time-congestion rates P_{N_1}, P_{N_2} by the formulas (6), (7) yields the following approximation of the overall time-congestion $P_{N_1 N_2}$ in terms of the average call-congestion rates B_1, B_2 which may be computed by means of (8):

$$\begin{aligned}
P_{N_1 N_2} &= \frac{Z(B_1, B_2)}{N(B_1, B_2)} && (10)\\
Z(B_1, B_2) &= E(N_1, \lambda_1 + \lambda_2 + B_2\lambda_3)(\lambda_1 + \lambda_2 + B_2\lambda_3)(\lambda_3 + \lambda_4)\lambda_3 B_1 \\
&\quad + E(N_2, \lambda_3 + \lambda_4 + B_1\lambda_2)(\lambda_3 + \lambda_4 + B_1\lambda_2)(\lambda_1 + \lambda_2)\lambda_2 B_2 && (11)\\
N(B_1, B_2) &= (\lambda_2 + \lambda_3)[(\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) - \lambda_2\lambda_3 B_1 B_2] \\
&\quad + \lambda_3^2(\lambda_3 + \lambda_4 + B_1\lambda_2)E(N_1, \lambda_1 + \lambda_2 + B_2\lambda_3) \\
&\quad + \lambda_2^2(\lambda_1 + \lambda_2 + B_2\lambda_3)E(N_2, \lambda_3 + \lambda_4 + B_1\lambda_2) && (12)
\end{aligned}$$

If the average call-congestion rates are known, it is possible to determine the time-congestion rates P_{N_1}, P_{N_2} by (6), (7), (10). As expected (cf. [10]), these estimates derived from the proposed fixed-point approach are very accurate for heavy traffic.

5. Analysis of the loss streams

The loss streams are formed by those calls of the originating and external traffic streams which cannot find a free line upon arrival. Subsequently, we assume that the ergodic Markov chain $\{X(t), t \geq 0\}$ is in steady state and $t = 0$ is an instant of a loss. Furthermore, we suppose $\lambda_2 > 0$, $\lambda_3 > 0$. Without loss of generality we shall only investigate the loss streams corresponding to the offered streams 1 and 2.

5.1 Stochastic structure of the loss streams

A call of stream 2 is lost if and only if it finds the overflow system in state (N_1, N_2) upon arrival. Thus, the corresponding loss stream can be described by a Markov-modulated Poisson process (MMPP) (cf. [7, §7.4.2, p. 361ff], [19], [21], [17]) with representation (Q, Λ_2). Here $Q \in \mathbb{R}^{N \times N}$, $N = (N_1 + 1) \cdot (N_2 + 1)$, is the generator matrix of $X(t)$ and $\Lambda_2 = \lambda_2 \cdot e_N \cdot e_N^t \in \mathbb{R}^{N \times N}$ is the rate matrix where $e_N \in \mathbb{R}^N$ denotes the Nth unit vector.
As lost calls can only occur during a sojourn of the controlling Markov chain $X(t)$ in state (N_1, N_2), the resulting loss stream is a stationary PH-renewal stream with an irreducible representation $(e_N, Q - \Lambda_2)$ (cf. [17, Theorem 2.5, p. 18], [21, Chap. 2]). Its corresponding interarrival time distribution is given by $H(x) = 1 - e_N^t \cdot e^{(Q-\Lambda_2)\cdot x} \cdot e \ , \quad x \geq 0$, and has the Laplace-Stieltjes transform (LST) $\Psi_2(s) = \int_0^\infty e^{-st} dH(t) = \lambda_2 \cdot e_N^t \cdot (sI - Q + \Lambda_2)^{-1} \cdot e_N \ , \quad \mathrm{Re}(s) \geq 0$ ([17, Theorem 2.5, p. 19]).
A call of the external traffic stream 1 is lost if and only if it finds the overflow system in some state (N_1, j), $j \in \{0, \ldots, N_2\}$, upon arrival. Hence, the stream of lost calls selected from flow 1 is a Markov-modulated Poisson process (MMPP) with the representation (Q, Λ_1) where $Q \in \mathbb{R}^{N \times N}$ is again the generator matrix of $X(t)$ and

$$\Lambda_1 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_1 I_{N_1} \end{pmatrix} \in \mathbb{R}^{N \times N} \tag{13}$$

is the rate matrix. This MMPP is a special Semi-Markov process (cf. [17, p. 9], [3, Chap. 10]) with the Semi-Markov matrix

$$F(x) = \int_0^x e^{(Q-\Lambda_1)\cdot t}\Lambda_1 \, dt = (I - e^{(Q-\Lambda_1)\cdot x}) \cdot (\Lambda_1 - Q)^{-1} \cdot \Lambda_1 \tag{14}$$

whose LST is $\Psi_1(s) = (sI - Q + \Lambda_1)^{-1}\Lambda_1$, $\mathrm{Re}(s) \geq 0$ ([17, p. 10]). The transition probability matrix $R = (-Q + \Lambda_1)^{-1}\Lambda_1$ of its embedded Markov chain has the stationary distribution

$$r = \frac{\Lambda_1 \cdot P}{P^t \Lambda_1 e} = \begin{pmatrix} 0 \\ P_{N_1}/(P_{N_1}^t \cdot e) \end{pmatrix}, \tag{15}$$

where P is the stationary distribution of Q and $P_{N_1}^t = (P_{N_1,0}, \ldots P_{N_1,N_2})$.
The distribution function G of the generic time distance between consecutive loss instants of the external traffic stream 1 is given by

$$G(x) = r^t \cdot F(x) \cdot e = 1 - r^t \cdot e^{(Q-\Lambda_1)\cdot x} \cdot e \quad , x \geq 0. \tag{16}$$

It is corresponding to a PH-renewal process with an irreducible representation $(r, Q - \Lambda_1)$ and the LST $\Phi(s) = r^t \cdot (sI - Q + \Lambda_1)^{-1} \cdot \Lambda_1 \cdot e$.

5.2 Approximation of the loss streams

If the overflow model with external traffic is a subsystem in a large network, it is necessary to approxmiate the streams of lost calls by simple point processes, for instance, IPP- or PH-renewal processes (cf. [14], [19], [18], [26]) in order to analyse the adjacent part of the network. A well-known approach is an approximation by IPP-processes based on Kuczura's three-moment-matching procedure (cf. [14], [19]).
In order to apply this technique to the loss stream resulting from stream 2, the first three factorial moments of the number of busy servers in a $GI/M/\infty$ system with mean holding time $1/\mu$ have to be computed for this PH-renewal process. Then this loss stream has to be approximated by an IPP with the same first three moments. Suppose the rate of this equivalent IPP is λ', its mean on-time $1/\gamma$ and its mean off-time $1/\omega$ ([14, p. 438]). Kuczura's three-moment-match yields the relations (cf. [19], [8, (20) - (22), p. 1220])

$$\lambda' = \mu\frac{\delta_0\delta_1 + \delta_1\delta_2 - 2\delta_0\delta_2}{2\delta_1 - \delta_0 - \delta_2}, \quad \omega = \mu\frac{\delta_0(\lambda'/\mu - \delta_1)}{\lambda'/\mu(\delta_1 - \delta_0)}, \quad \gamma = \mu\frac{\omega/\mu(\lambda'/\mu - \delta_0)}{\delta_0} \tag{17}$$

where $\delta_j = f_{j+1}/f_j$, $j = 0, 1, 2$, $f_0 = 1$, denote the ratios of factorial moments of the PH-renewal process.The latter are given by (cf. [23], [7, §2.2.2.2, p. 81])

$$f_1 = \lambda/\mu = \lambda_2 \cdot P_{N_1 N_2}/\mu \tag{18}$$

$$f_2 = \frac{\lambda}{\mu} \cdot e_N^t \cdot (\mu I - Q)^{-1} \cdot \Lambda_2 e \tag{19}$$

$$f_3 = 2 \cdot \frac{\lambda}{\mu} \cdot e_N^t \cdot (\mu I - Q)^{-1} \cdot \Lambda_2 e \cdot e_N^t \cdot (2\mu I - Q)^{-1} \cdot \Lambda_2 e \tag{20}$$

where $\lambda = \lambda_2 \cdot P_{N_1 N_2}$ denotes the mean arrival rate of the PH process and $\lambda^{-1} = e_N^t(\Lambda_2 - Q)^{-1}e$ is the mean interarrival time ([22, Corollary 1, p. 448]). $P_{N_1 N_2}$ is the steady-state probability of state (N_1, N_2).
These moments may be computed by means of the unique solutions of the linear systems

$$(\mu I - Q^t) \cdot y_2 = e_N \,, \qquad (2\mu I - Q^t) \cdot y_3 = e_N \tag{21}$$

applying a standard iterative scheme, for instance, a block Gauss-Seidel procedure (cf. [19, §4], [13]). Taking into account $f_2 = f_1 \cdot \lambda_2 \cdot (y_2)_N$, $f_3 = 2 \cdot \lambda_2 \cdot f_2 \cdot (y_3)_N$, the IPP parameters λ', γ, ω may be calculated on the basis of (17) by inserting

$$\delta_0 = \lambda_2 \cdot P_{N_1 N_2}/\mu \ , \quad \delta_1 = \lambda_2 \cdot (y_2)_N \ , \quad \delta_2 = 2 \cdot \lambda_2 \cdot (y_3)_N. \tag{22}$$

The MMPP stream formed by lost calls of the external traffic stream 1 can be approximated either directly by an IPP-renewal stream or by its associated PH-renewal stream corresponding to the generic interarrival time distribution $G(x) = 1 - r^t \cdot e^{(Q-\Lambda_1)\cdot x} \cdot e$ of consecutive calls. The corresponding factorial moments of these streams are given in Table 1.

Moment	$PH/M/\infty$	$MMPP/M/\infty$
f_1	$[\mu r^t(\Lambda_1 - Q)^{-1} e]^{-1}$	$\frac{1}{\mu} P^t \Lambda_1 e$
f_2	$f_1 \frac{r^t(\mu I - Q + \Lambda_1)^{-1}\Lambda_1 e}{1 - r^t(\mu I - Q + \Lambda_1)^{-1}\Lambda_1 e}$	$\frac{1}{\mu} P^t \Lambda_1 (\mu I - Q)^{-1} \Lambda_1 e$
f_3	$2 f_2 \frac{r^t(2\mu I - Q + \Lambda_1)^{-1}\Lambda_1 e}{1 - r^t(2\mu I - Q + \Lambda_1)^{-1}\Lambda_1 e}$	$\frac{2}{\mu} P^t \Lambda_1 (\mu I - Q)^{-1} \Lambda_1 (2\mu I - Q)^{-1} \Lambda_1 e$

Table 1: Factorial Moments

Obviously, the first moments are equal, $f_1^{PH} = f_1^{MMPP} = 1/\mu \cdot P^t \cdot \Lambda_1 \cdot e = 1/\mu \cdot \lambda_1 \cdot P_{N_1}^t \cdot e$, whereas the second and third moments are different. These moments may be computed by means of an efficient scheme which is similar to (21).

6. Conclusion

We have studied a telecommunication network consisting of a local exchange and two exchanges of the long-distance network being connected to each other by two distinct both-way trunk groups. Modeling this network by means of a loss system with two Poissonian originating traffic streams following a mutual overflow routing scheme and two Poissonian external traffic streams, we derived a representation of the steady-state probabilities for the number of busy trunks in both groups in terms of Brockmeyer polynomials.

Moreover, we have pointed out that lost calls selected from the originating and external traffic streams by each trunk group form PH-renewal and MMPP processes. The factorial moments of these streams resulting from their occupation on a $G/M/\infty$ system have been computed. In view of these results it is possible to construct appropriate approximations of these loss streams, for instance, IPP-renewal processes by a moment-matching approach (cf. [14]).

Furthermore, we have developed an efficient and accurate fixed-point approximation method for calculating time-congestion and call-congestion rates of the model.

Thus, our analysis is the basis for generating simple point processes which approximate the streams of blocked and carried calls of this telecommunication model.

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References

[1] A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, New York, 1979.

[2] E. Brockmeyer. Det simple overflowproblem i telefontrafikteorien. *Teleteknik*, 5(4),361–374, 1954.

[3] E. Çinlar. *Introduction to Stochastic Processes.* Prentice Hall, Englewood Cliffs, 1975.

[4] A. Erdélyi et al. *Higher Transcendental Functions.* Volume 2, McGraw-Hill, New York, 1953.

[5] P. Franken, D. König, U. Arndt and V. Schmidt. *Queues and Point Processes.* John Wiley, New York, 1982.

[6] K.W. Gaede. *Zuverlässigkeit, Mathematische Modelle.* Carl Hanser Verlag, München, 1977.

[7] B. W. Gnedenko and D. König. *Handbuch der Bedienungstheorie II.* Akademie-Verlag, Berlin, 1984.

[8] H. Heffes. Analysis of first-come first served queuing systems with peaked inputs. *Bell System Technical Journal*, 52(7),1215–1228, 1973.

[9] D. P. Heyman and M. J. Sobel. *Stochastic Models in Operations Research.* Volume I: Stochastic Processes and Operating Characteristics, McGraw-Hill, New York, 1982.

[10] F.P. Kelly. Blocking probabilities in large circuit-switched networks. *Adv. Appl. Prob.*, 18,473–505, 1986.

[11] U. Krieger. *Analysis of a Loss System with Mutual Overflow.* Technischer Bericht, 4302 TB 9 E, Forschungsinstitut der DBP, Darmstadt, 1988.

[12] U. Krieger. Analysis of a loss system with mutual overflow. In *International Teletraffic Seminar on Teletraffic and Network, Beijing, September 12–16, 1988, Proceedings*, pages 331–340, 1988.

[13] U. Krieger. *Untersuchung eines Nachrichtenverkehrsmodelles mit wechselseitigem Überlauf und Externverkehr.* Preprint, V 88-12-01, Forschungsinstitut der DBP, Darmstadt, 1988.

[14] A. Kuczura. The interrupted Poisson process as an overflow process. *Bell System Technical Journal*, 52(3),437–448, 1973.

[15] A. Kuczura. Loss systems with mixed renewal and Poisson inputs. *Operations Research*, 21,787–795, 1973.

[16] F. LeGall and J. Bernussou. An analytical formulation for grade of service determination in telephone networks. *IEEE Trans. on Communications*, 31(3),420–424, 1983.

[17] K. Meier. *A statistical procedure for fitting Markov-modulated Poisson processes.* PhD thesis, University of Delaware, 1984.

[18] K. Meier-Hellstern. *The analysis of a queue arising in overflow models.* Technical Report, AT&T Bell Laboratories, Holmdel, 1988.

[19] K. Meier-Hellstern. Parcel overflows in queues with multiple inputs. In *Proceedings,ITC 12, Turino, Italy*, pages 5.1B.3.1 – 5.1B.3.8, 1988.

[20] J. A. Morrison. Analysis of some overflow problems with queuing. *Bell System Technical Journal*, 59(8),1427–1462, 1980.

[21] M. F. Neuts. *Matrix-Geometric Solutions in Stochastic Models.* Johns Hopkins University Press, Baltimore, 1981.

[22] M.F. Neuts. Renewal processes of phase type. *Naval Research Log. Quart.*, 25,445–454, 1978.

[23] V. Ramaswami and M.F. Neuts. Some explicit formulas and computational methods for infinite-server queues with phase-type arrivals. *J. Appl. Prob.*, 17,498–514, 1980.

[24] D. Rose. Convergent regular splittings for singular M-matrices. *SIAM J. Alg. Disc. Meth.*, 7(3),390–398, 1986.

[25] R. Varga. *Matrix Iterative Analysis.* Prentice-Hall, Englewood Cliffs, 1962.

[26] W. Whitt. Approximation a point process by a renewal process, I: Two basic methods. *Operations Research*, 30(1),125–147, 1982.

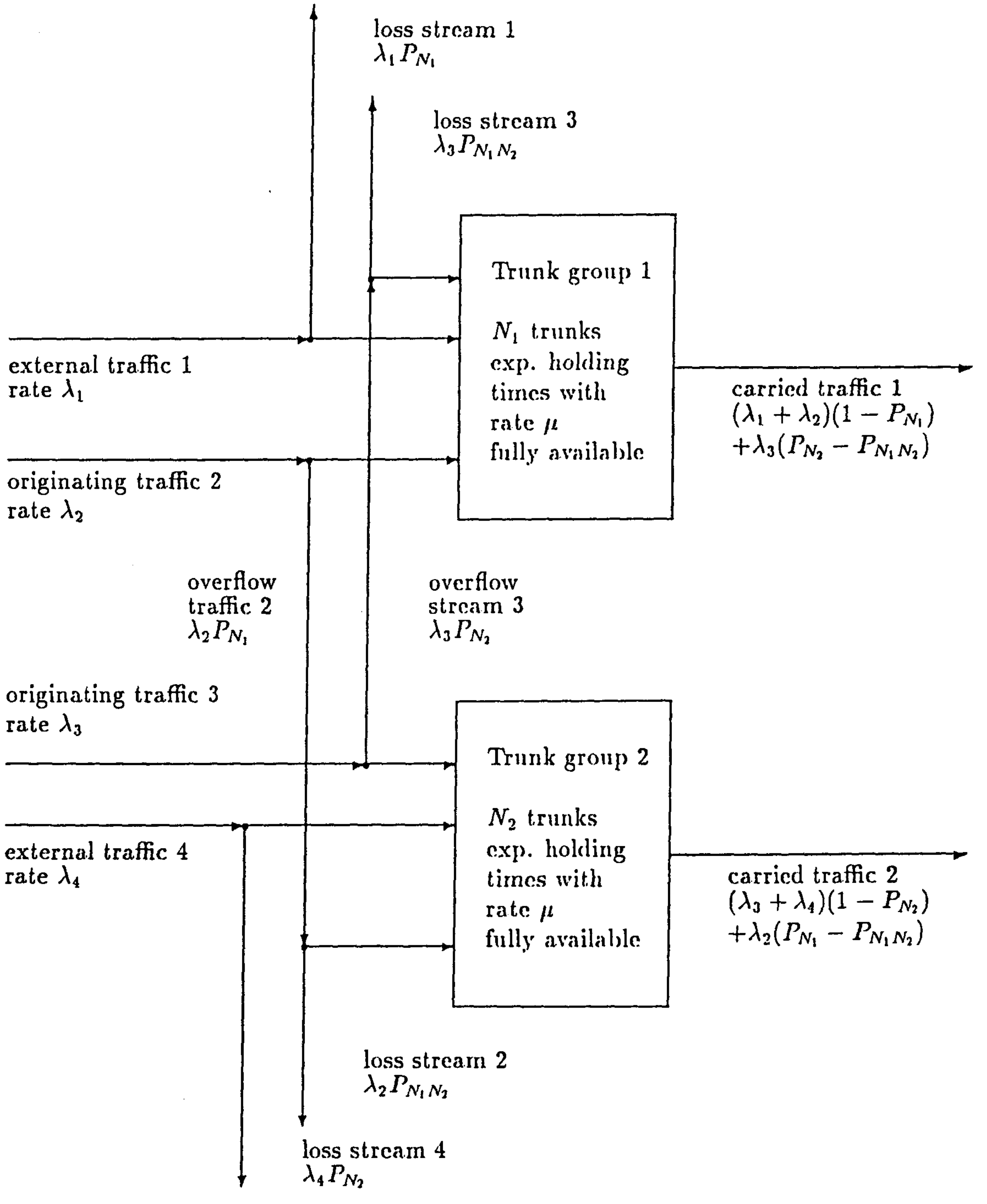

Figure 2: Loss system with mutual overflow and external traffic