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Measuring Power in Voting Bodies: Linear Constraints, Spatial Analysis, and a Computer Program

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1. Introduction

The purpose of this contribution is to present some extensions of an established method for measuring the distribution of power among the various groups within any voting body. If there are no formal alignments of individual voters – like parliamentary groups or fractions – within such a unit, and if voting occurs on a purely individual basis according to a "one man one vote" rule, all members obviously possess the same voting power in that they are all equally decisive for voting outcomes. If, on the other hand, voting patterns are mediated, so to speak, by a division of the voting body into several voting blocs with a fairly high probability of joint voting, as is the case for parliamentary groups within legislatures, then the voting power of individual members can vary between blocs depending upon the number of individuals in each bloc and the decision rule. In highly simplified economic terms this means that if one wants to "buy" the one decisive vote for a particular motion, then in the former case each member would be worth the same amount of money whereas in the latter case this amount would have to be weighted by the odds that an individual member will belong to the decisive voting bloc. In other words, the "price" paid for each voting bloc should covary with its probability of turning the outcome which is a straightforward measure of its power.

These basic notions about measuring power in voting bodies have received widespread acceptance and application following *Shapley's* [1953] and *Shapley's/Shubik's* [1954] seminal work. Their power index is presented as a standard tool for game-theoretical analysis of politics in textbooks [e.g., *Riker/Ordershook*, ch. 6], and it has been usefully applied to the analysis of the distribution of power in several legislatures [e.g., *Frey; Weiersmüller; Zerche; Holler/Kellermann*].

The intention of the present contribution is not to extend this series of applications of the "classical" power index but rather to attempt to overcome one of its serious deficiencies in empirical research. This shortcoming is the assumption implicit in calculations of the values of the index that all voting sequences within a given voting body are equally likely. After briefly discussing the conventional power index we will concern ourselves with various methods for avoiding this unrealistic assumption. We shall introduce either one-dimensional or spatial constraints on joint voting, or shall weight voting coalitions with the probability of joint voting of their participants. Finally, a computer program designed to perform this kind of modified computations of the power index will be presented together with an illustrative example.

2. Shapley's Power Index

If voting within a voting body is simply according to a "one man one vote" rule and if there are no voting blocs or alignments whatsoever, then there is no need for a power index. With N individual voters (and therefore N individual votes) each vote is equally likely to be decisive or "pivotal" for a given motion. Each voter clearly holds one N-th of the total voting power, i.e., in an infinite series of voting procedures his vote is expected to decide the outcome in one N-th of the total number of cases.

Empirically, voting bodies without any formal subdivisions are the exception rather than the rule. Whenever such subdivisions exist, then there is the problem of measuring their voting power. The simplest measure is the vote share commanded by each group or voting bloc, which is analogous to ascribing a power share of $1/N$ to each voter in the case of an unstructured voting body. The problem with this measure is that it is only in exceptional cases – e.g., if all voting blocs are of equal size – identical to the likelihood of each group to be decisive for voting outcomes. In most other circumstances vote shares and probabilities for being "pivotal" differ, the most extreme discrepancy occuring when one voting bloc commands just over half of the total number of votes.

This insight was the starting point for the development of Shapley's power index. It proceeds from the simple consideration that in a voting body with K units (fractions, parties, voting blocs), K blocs of votes are cast in any voting procedure if all units vote homogeneously. As blocs of votes are regarded as units, and not the individual votes, the weight of a unit is defined as the number of its individual votes.

If one assumes that in deciding on a particular motion the votes of all K units are homogeneously cast in a sequential fashion, unit after unit, there are $P_K = K!$ possible sequences of voting. If one further assumes that the votes of each bloc are homogeneously cast in identical direction (pro or con) to those of the previous blocs, one can for each of the P_K sequences determine which unit establishes a majority (pro or con) by adding its votes to those previously cast. For each unit i, let T_i denote the number of voting sequences in which i plays this pivotal role. Then obviously:

$$\sum_{i=1}^{K} T_i = P_K.$$

Shapley's power index S_i for each unit i is defined as $S_i = T_i/P_K$, so that naturally the values of S_i for all units sum to unity. This index can be substantially interpreted as the probability that unit i, by casting its votes, establishes a winning majority of minimum size in any voting sequence or, to put it differently, as the relative frequency with which unit i plays the pivotal role in an infinite number of random majority coalitions among the K units.

According to *Shapley* [1953] any attempt to measure the distribution of power among the units i in a voting body under his set of axioms either produces an index equivalent to S_i or leads to logical inconsistencies. It should be noted that Shapley's power index in its initial formulation is defined for the absolute majority rule, "majority" being the smallest integer greater than $N/2$. This restriction is not a logical necessity. In fact, it is possible to define any fraction of N as the required threshold of votes without affecting the logic of the power index itself or the logic of its computation. Therefore, the computer program to be presented below has been designed to compute the distribution of power, alternatively, for absolute and two-thirds-majority as the two most frequent decision rules for parliamentary voting.

The power index S_i represents an *a priori* distribution of power which does not take any restrictions on joint voting and on coalition formation into account – be they politically, sociologically, psychologically, or otherwise determined. Each numerically possible voting sequence which secures a majority is treated as equally likely. That is in many cases of course a gross distortion of reality. We will now turn to several strategies for avoiding this unrealistic assumption by introducing measures for the chances that various given units will jointly appear in a majority coalition into the computation of the power index. Let us start with linear constraints along a continuum.

3. One-Dimensional Constraints on Coalition Formation

In introducing linear constraints on coalition formation into the computation of the power index we proceed from the assumption of the existence of a one-dimensional ordinal policy space. This means that the K voting blocs can be arranged in an order conforming to their relative positions in this policy space. The obvious example naturally is an ideological left-right dimension where for any two voting blocs i and j the first unit i ideologically stands to the left of unit j if and only if $i < j$. With this kind of ordinal policy space it is assumed, of course, that the size of the intervals between positions has no relevance.

If such a transitive order of K voting blocs along any simple dimension of policy preferences exists, then obviously not all minimum winning coalitions are equally likely. The more dispersed a minimum winning coalition is along the relevant policy dimension the less likely is its formation, and vice versa, for the conflict of interest within a coalition depends upon its dispersion [*Axelrod*, p. 169]. The conflict of interest within a coalition is minimized – and therefore the coalition's utility for participating voting blocs is maximized – if dispersion along the policy dimension it kept to a minimum. Dispersion is lowest, however, if a coalition consists of adjacent voting blocs, or, to follow Axelrod's terminology, if it is connected.

More formally, the property of connectedness can be defined as follows: If K voting blocs have been arranged transitively along a one-dimensional policy space, then a coalition consisting of n voting blocs m is connected if and only if $\max(m) - \min(m) = n - 1$. If a minimum winning coalition is also connected, it is called a *minimum connected winning coalition.*

If we proceed from the basic logic of the Shapley-Shubik index and in addition assume that – given an ordinal policy space – *only* minimum connected winning coalitions will be formed, the voting power of any voting bloc i is clearly the share of all those coalitions in which it is pivotal. In order to find this share, all we have to do is find the number TA_i

of *connected* coalitions among the T_i minimum winning voting sequences in which i is pivotal. This can be easily performed by applying the numerical criterion given above [*Rattinger*, 1979]. We therefore obtain a modified power index for the units in a voting body which can be represented by an ordinal policy dimension:

$$SA_i = \frac{TA_i}{\sum_{i=1}^{K} TA_i}.$$

4. Spatial Constraints on Coalition Formation

The consideration that policy preferences along a policy dimension influence coalition formation, certainly represents an important step towards a more realistic measurement of voting power. However, one might argue that in many cases the assumption of a one-dimensional ordinal policy space still constitutes a gross oversimplification. It is conceivable that in a voting body dominated by ideological positions, coalitions among extremist voting blocs or factions from both left and right are more probable than coalitions comprising middle-of-the-road and extremist groups from either side. In such a situation certain minimum connected winning coalitions will *not* be formed whereas certain realistic coalitions are *not connected*.

Here generalizing the concept of connectedness for multi-dimensional ordinal policy spaces does not lead us any further. Instead, we have to assume that for each pair of two voting blocs i and j, out of the total of K voting blocs, we know whether or not i and j are willing to join forces in a coalition. Let $v_{ij} = 1$ if and only if i regards j as an acceptable partner in a coalition, and let $v_{ij} = 0$ otherwise. We then have a binary $K \times K$ adjacency matrix $V = [v_{ij}]$ with all entries v_{ii} on the main diagonal trivially being 1. Obviously matrix V does *not* have to be symmetrical as it is entirely conceivable that voting bloc i would accept j as a partner in a joint coalition, so that $v_{ij} = 1$, whereas j rejects i, so that $v_{ji} = 0$.

In this contribution we shall not concern ourselves with the problem of empirically arriving at the adjacency matrix V. We assume V to be given and demonstrate the opportunities for the measurement of voting power if information of the sort contained in V is available. Any minimum winning coalition can be formed if and only if for every pair of two voting blocs i and j in this coalition $v_{ij} = 1$, i.e., if each voting bloc is adjacent to all the other participants in this coalition. In graph-theoretical terms this means that if an empirical binary adjacency relation is represented by an adjacency digraph, then any empirically feasible minimum winning coalition has to correspond to a complete, reflexive, symmetric, and connected sub-digraph which, of course, is represented by a sub-matrix of V with entries of only 1. If at least one voting bloc i in a numerically possible minimum winning coalition is rejected by at least one other member j, i.e., $v_{ji} = 0$, then this coalition cannot be formed.

From these considerations the following concept of voting power constrained by spatial adjacency is derived: For each voting bloc i let TB_i be the number of those minimum winning coalitions for which i is pivotal *and* which are represented by a sub-matrix of V

containing only entries of 1, then a second modified Shapley-Shubik-type power index SB_i is given by:

$$SB_i = \frac{TB_i}{\sum_{i=1}^{K} TB_i}.$$

5. Policy Distance and Voting Power

The above concepts of adjacency or mutual acceptability between voting blocs within a voting body are obviously based upon a dichotomization of a more general concept of policy distance in multi-dimensional space. If policy distance between two units is *below* a certain threshold, they are said to be adjacent or mutually acceptable as partners in a joint coalition, and vice versa. All distances *above* the particular threshold are represented by an entry of zero in matrix V, entries of one correspond to distances *below* this threshold. Information on policy distance between i and j is reduced to information on whether or not they can appear together in one coalition. This logical relationship does not imply, however, that policy distances will have to be measured empirically before one can set up matrix V. In many cases it will be much easier to obtain the kind of categorical judgements contained in V, than to arrive at precise policy distance readings. Here SB_i is a useful improvement over Shapley's initial power index.

Let us now assume, however, that information on the policy distance between any two units i and j within a voting body is empirically available. Let $D = [d_{ij}]$ be a $K \times K$ distance matrix with $d_{ii} = 0$ and $0 \leqslant d_{ij} \leqslant Z$, where Z is any positive integer denoting the maximum value of the distance scale. d_{ij} is the policy distance perceived by unit i vis-à-vis voting bloc j. d_{ij} and d_{ji} can be equal empirically, but need not be identical for logical reasons.

If we assume coalition formation to be a process of policy distance minimization [*De Swaan*], then each numerically possible minimum winning coalition for which a given unit i is pivotal has to be weighted by its probability of occuring. This probability has to be an inverse function of the policy distances among the participating voting blocs. In order to arrive at appropriate probability weights, let us first define a $K \times K$ matrix $Q = [q_{ij}]$ with $q_{ij} = Z - d_{ij}$. $q_{ij} = 0$ if i sees j at maximum political distance from itself, and $q_{ij} = Z$ if i and j in i's judgement have identical policy positions. It should be noted that Q it *not* a probability matrix in the conventional sense. However, if we assume that the probability that i will join j in a coalition depends inversely and in a linear fashion upon the policy distance between i and j as perceived by i, and if we further assume that the maximum distance Z corresponds to a probability of zero then Q can be transformed into a probability matrix by an (unknown) similarity function $f(q_{ij}) = aq_{ij}$. But this kind of transformation is not required for the present purpose of weighting minimum winning coalitions, for which Q suffices.

Our problem now becomes how to assign weights to minimum winning coalitions which covary with their probabilities of formation. If such a coalition consists of only two voting blocs, the solution is fairly straightforward. But if there are more than two participants, there is no single compelling way of aggregating an overall probability weight. Instead, we have to rely on plausibility and face validity. One solution would be to compute a weighted mean of q_{ij} for all pairs of participants, weighting by n_i, the number of votes in each voting bloc i. Let us assume that a minimum winning coalition is established by voting blocs i, j, and k, with $n_i = n_j = n_k$, and $q_{ij} = q_{ji} = q_{jk} = q_{kj}$ with q_{ij} close to Z, and $q_{ik} = q_{ki}$ with q_{ik} close to zero. A weighted mean would yield a fairly high probability weight in the neighborhood of $2Z/3$, whereas obviously this coalition is extremely unlikely as i and k, for all practical purposes, will not join in a coalition. A second solution, therefore, would be to take the *smallest* probability score of joint membership in a coalition for all pairs of its participants as the probability weight of a minimum winning coalition.

For this second solution it proves inconvenient that Q need not be symmetric. Before formally describing the two weighting procedures we therefore define a symmetric $K \times K$ matrix $P = [p_{ij}]$ with $p_{ij} = p_{ji} = (n_i q_{ij} + n_j q_{ji})/(n_i + n_j)$. For any two units i and j, P contains a probability weight for their joint appearance in a coalition which is derived as a weighted average from both units' evaluations of their proximity.

Given matrix P it is now possible to formally define the two strategies (as above) for assigning probability weights to minimum winning coalitions. We shall start with the first approach. Let M_{im} be the set of voting blocs which form the m-th minimum winning coalition for which unit i is decisive, and let a_{im} be the number of units in this coalition. Then the weighted average probability score, W_{im}, for this minimum winning coalition is:

$$W_{im} = \frac{\sum_{j=1}^{K} \sum_{k=1}^{K} p_{jk}(n_j + n_k)}{\sum_{j=1}^{K} 2(a_{im} - 1)n_j} \qquad \text{where: } \begin{array}{l} j \in M_{im} \\ k \in M_{im} \\ j \neq k. \end{array}$$

If T_i is the total number of minimum winning coalitions for which i is pivotal, TC_i is the sum of the probability weights of all those coalitions:

$$TC_i = \sum_{m=1}^{T_i} W_{im}.$$

Then obviously a third modified power index taking policy distances into account can be defined as:

$$SC_i = \frac{TC_i}{\sum_{i=1}^{K} TC_i}.$$

Following the second strategy we obtain the probability weight W'_{im} for the m-th minimum winning coalition for which i is pivotal:

$$W'_{im} = \min(p_{jk}) \qquad \text{where: } \begin{array}{l} j \in M_{im} \\ k \in M_{im} \\ j \neq k. \end{array}$$

Therefore:

$$TD_i = \sum_{m=1}^{T_i} W'_{im}.$$

Accordingly, our fourth and final modified power index, in which minimum winning coalitions are discounted by minimum proximity among participants is:

$$SD_i = \frac{TD_i}{\sum_{i=1}^{K} TD_i}.$$

Whether one prefers SC or SD, when information on policy distances is available, will depend upon personal judgement and upon empirical patterns of distances. If policy distances are highly "intransitive", so to speak, i.e., if within many minimum winning coalitions policy distances between participating voting blocs vary widely, then both indices will yield divergent results on voting power. Comparing the findings from both methods might lead to useful insights.

6. The Program

The five indices of voting power here described, can be computed within one FORTRAN-program.[1]) The basic idea of the program is to start with the votes of the first voting bloc and to add the votes of the second, third, etc. voting blocs until a minimum winning coalition is reached. This process is repeated for all the $K!$ logically possible permutations. Each minimum winning coalition is ascribed to the pivotal voting bloc. For each unit i there are five "accounts". In the first account, T_i, we have all minimum winning coalitions which i can establish. In the second and third accounts, TA_i and TB_i, only those minimum winning coalitions (established by i) which meet the appropriate numerical criteria are included. The final two accounts, TC_i and TD_i, sum the probability weights of the minimum winning coalitions for which i is decisive. Standardization across all i, finally yields the five measures of voting power described in this contribution.

A significant reduction of computing time is derived from the following consideration: If i is pivotal for a minimum winning coalition consisting of m voting blocs, then there are $(K-m)!$ permutations which contain exactly the same coalition in the first m places. We cannot only write one, but $(K-m)!$ minimum winning coalitions into the accounts of i and, therefore, take only a fraction of $K!$ loops to go through all permutations.

The program currently exists for a maximum of $K = 12$ voting blocs. As input it requires the number of voting blocs, the total number of votes, a vector of the numbers of votes in each bloc, a parameter which switches the program to absolute or to two-thirds-majority, an adjacency matrix V, and, finally, a distance matrix D. One has the option to rearrange voting blocs, as normally in the computation of SA their position in the input vector will be interpreted as their position in a one-dimensional ordinal policy space. The program's output will now be illustrated by the following example.

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[1]) The program was written by Gertrud Steigmiller proceeding from a program written by myself for the purpose of analyzing voting power in the European Parliament [*Rattinger*].

7. An Example

In our example, K has the maximum value of 12. The program first prints back a symmetrized matrix V and matrix P. In the example these matrices look as follows:

Matrix V

1	1	1	1	1	1	1	1	0	1	1	0
1	1	1	1	1	1	1	1	0	1	1	0
1	1	1	1	1	1	1	0	0	1	1	0
1	1	1	1	1	1	1	0	0	1	1	0
1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	0	1	1	0
1	1	1	1	1	1	1	1	1	1	1	0
1	1	0	0	1	1	1	1	1	1	1	1
0	0	0	0	1	0	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	1	0	0	1	1	1	1	1

Matrix P

5,0	3,9	3,6	3,0	3,0	2,4	2,1	2,1	1,2	2,1	2,1	0,1
3,9	5,0	3,9	3,6	3,3	3,0	2,7	2,4	1,5	2,1	2,1	0,5
3,6	3,9	5,0	3,6	3,0	2,4	2,1	1,8	1,2	2,4	2,1	0,9
3,0	3,6	3,6	5,0	3,9	3,3	2,4	1,8	1,5	2,7	2,4	1,5
3,0	3,3	3,0	3,9	5,0	3,6	3,3	3,0	2,7	2,4	2,1	2,1
2,4	3,0	2,4	3,3	3,6	5,0	3,9	3,0	1,8	3,0	3,0	1,8
2,1	2,7	2,1	2,4	3,3	3,9	5,0	3,6	3,0	2,4	2,1	1,8
2,1	2,4	1,8	1,8	3,0	3,0	3,6	5,0	3,9	3,6	3,3	3,0
1,2	1,5	1,2	1,5	2,7	1,8	3,0	3,9	5,0	3,6	2,7	2,4
2,1	2,1	2,4	2,7	2,4	3,0	2,4	3,6	3,6	5,0	3,9	3,6
2,1	2,1	2,1	2,4	2,1	3,0	2,1	3,3	2,7	3,9	5,0	3,6
0,1	0,5	0,9	1,5	2,1	1,8	1,8	3,0	2,4	3,6	3,6	5,0

We then receive information on the total number of permutations, the number of minimum winning coalitions satisfying the criteria of connectedness and of spatial adjacency, the number of program loops, the total number of votes, the number of independent votes not belonging to any voting bloc, and, finally, the requested type of majority and its numerical value. These figures are all presented below. Note that, on the average, slightly above 63 permutations per loop were dealt with. In spite of this reduction, more then 90 minutes CPU-time were required.

ΣT	ΣTA	ΣTB	Program-loops
479001600	4440960	45688320	7585920

Total votes	Independent votes	Absolute Majority
380	10	191

	1	2	3	4	5	6	7	8	9	10	11	12
Votes	30	10	20	30	40	50	60	10	40	40	20	20
Votes Share	.079	.026	.053	.079	.105	.132	.158	.026	.105	.105	.053	.053
T	37929600	11923200	24503040	37929600	51788160	67063680	83358720	11923200	51788160	51788160	24503040	24503040
S	.079	.025	.051	.079	.108	.140	.174	.025	.108	.108	.051	.051
TA	0	0	29376	380160	587520	1054080	1261440	241920	414720	207360	0	0
SA	0	0	.066	.086	.132	.237	.284	.055	.093	.047	0	0
TB	4510080	1330560	2592000	3732480	6238080	7758720	9383040	466560	207360	6238080	3231360	0
SB	.099	.029	.057	.082	.137	.170	.205	.010	.005	.137	.071	0
SC	.077	.025	.051	.079	.110	.142	.175	.025	.106	.109	.051	.049
SD	.067	.025	.052	.084	.116	.150	.185	.027	.099	.116	.056	.025

The final and most important segment of the program's output is a table which for each unit contains a column with the following data: absolute number of votes, vote share, number of minimum winning coalitions (*T*), of minimum connected winning coalitions (*TA*), and of spatially adjacent minimum winning coalitions (*TB*) for which this unit is decisive, and the five indices of voting power described in this contribution.

As our example is purely fictitious, it would not be very useful to analyze it in great detail. Let us, therefore, conclude with some very obvious observations. First, with just 12 voting blocs we already see vote shares and Shapley's basic index converging as there are no clearly preponderant units. Second, the assumption that only minimum connected winning coalitions will be formed leads to a substantial concentration of voting power in the middle of the underlying policy dimension. Third, as the matrices *V* and *P* also exhibit a moderate one-dimensional pattern of coalition preferences along the main diagonal, *SB*, *SC*, and *SD* do not deviate drastically from *SA*. With different matrices *V* and *P*, these discrepancies between the various power indices could be much stronger. If this were the case it would be a matter of substantive reasoning to decide which version of the index should be preferred.

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