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Dynamics of 1D discontinuous maps with multiple partitions and linear functions having the same fixed point. An application to financial market modeling

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ABSTRACT

Piecewise smooth systems are intensively studied today in many application areas, such as economics, finance, engineering, biology, and ecology. In this work, we consider a class of one-dimensional piecewise linear discontinuous maps with a finite number of partitions and functions sharing the same real fixed point. We show that the dynamics of this class of maps can be analyzed using the well-known piecewise linear circle map. We prove that their bounded behavior, when unrelated to the fixed point, may consist of either nonhyperbolic cycles or quasiperiodic orbits densely filling certain segments, with possible coexistence. A corresponding model describing the price dynamics of a financial market serves as an illustrative example. While simulated model dynamics may be mistaken for chaotic behavior, our results demonstrate that they are quasiperiodic.

1. Introduction

Since the seminal work by Huang and Day [1] on bull and bear market dynamics, numerous models have been published that use one-dimensional (1D) piecewise linear (PWL) or piecewise smooth (PWS) maps to explain the behavior of financial markets (see, e.g., [2–6]). In these models, different types of market participants interact. The key actors shaping market dynamics, besides the market maker — who adjusts prices in response to excess demand — are fundamentalists, who buy or sell assets when the market is undervalued or overvalued, and chartists, who buy or sell assets when they perceive a bull or bear market. As has been shown, even simple low-dimensional deterministic models can capture the intricate price behavior of financial markets, exhibiting transitions from fixed-point dynamics to chaotic behavior under parameter changes.

The goal of our paper is twofold. First, we contribute to the line of research that studies the behavior of financial markets via nonlinear dynamical systems. Second, we derive mathematical results concerning the behavior of 1D PWL discontinuous maps that are of broader relevance, e.g., for applications in economics, finance, engineering, biology, and ecology. The main characteristic of our financial market model is that it corresponds to a 1D discontinuous PWL map, defined over several partitions, with the real fixed point being the same for the functions in all partitions. While such maps produce dynamics that may be mistaken for chaotic behavior, our insights allow us to conclude that they are, in fact, quasiperiodic.

To be precise, the financial market model we propose in the next section belongs to a general class of maps, defined as follows:

Definition 1 (*The Class of 1D Maps*)**.** We consider the class of 1D PWL maps with a finite number of discontinuity points and functions sharing the same real fixed point.

Before presenting our main results, formally stated in Theorem 1, a few comments are in order.

We always translate the unique real fixed point of the linear functions to the origin, denoted as O, thereby making the system homogeneous. Note that if the real fixed point belongs to a partition with slope equal to 1, so that it is nonhyperbolic, we first consider the system with the hyperbolic fixed point, translating it to O, and after it can be considered also the case of nonhyperbolic fixed point. From now on, we assume that the fixed point is at the origin.

An essential element in the class of maps stated in Definition 1 is discontinuity. The real fixed point O may either be internal to one

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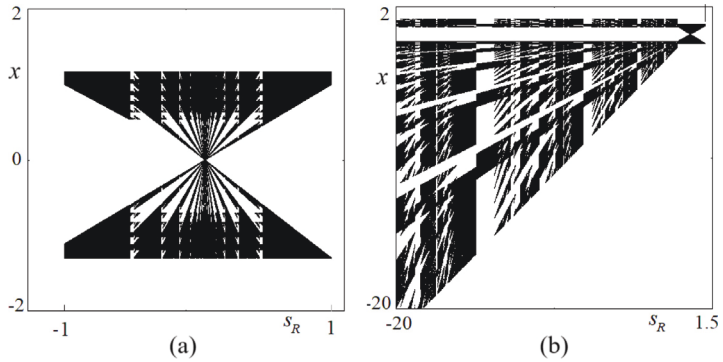

Fig. 1. One-dimensional bifurcation diagram of the map in (8) as a function of slope s_R. In (a), $Z_L = 0.6$, $Z_R = 1$, $s_L = -0.9$, $s_M = -1.3$. In (b), $Z_L = 1$, $Z_R = 0.6$, $s_L = -0.5$, $s_M = -1.1$.

partition or located at the boundary of two partitions. In the first case, the map is continuous and differentiable at the fixed point O (making it a virtual fixed point for the functions in the other partitions). In the second case, the map is continuous but piecewise smooth in the two partitions, with a kink at the fixed point O. In this case, there must be at least one additional discontinuity point to ensure a discontinuous map.

The map of the financial market model proposed in the next section satisfies Definition 1, both when the real fixed point is internal to one partition and when it is on the boundary of two partitions.

Despite being motivated by a financial market model, systems in the class of maps defined above also appear in other application areas, such as electronic engineering [7,8], biology, and ecology [9]. Moreover, a map belonging to our class may appear as a restriction or first return map on some straight-line segment of a two-dimensional PWL discontinuous map, as proposed in [10–12].

The one-dimensional bifurcation diagrams depicted in Fig. 1 illustrate the dynamics of the map related to the financial market model proposed in the next section. At first sight, the dynamics may appear chaotic. However, we will show that this conclusion is not justified. Recall that for a discontinuous 1D map $x' = G(x)$, the most widely used definition of chaos is as follows:

Definition 2. A map G is chaotic in a closed invariant set X if periodic points are dense in X and there exists an aperiodic trajectory dense in X (implying topological transitivity).

See, e.g., [13]. In this paper, "invariant"means mapped exactly onto itself,[1] $G(X) = X$.

We show that chaos cannot emerge in this class of maps, since nonhyperbolic cycles different from O cannot exist. We prove that the dynamics of the maps satisfying Definition 1 can be reduced to those of a PWL circle map [14,15]. As a result, the only bounded dynamics different from those related to the fixed point O are either associated with nonhyperbolic cycles or quasi-periodic trajectories densely filling certain intervals. We prove the following theorem:

Theorem 1. *Let $x' = G(x)$ be a 1D discontinuous PWL homogeneous map as in Definition 1. Then the following statements hold:*

(1) Map G cannot have a hyperbolic cycle different from the fixed point O (and thus, it cannot have a chaotic set).
(2) The only possible bounded invariant sets of map G, different from those related to the fixed point O (hyperbolic or nonhyperbolic), are those occurring in a PWL circle map. These consist of intervals dense with nonhyperbolic periodic points or quasiperiodic orbits dense in some intervals, with possible coexistence.
(3) Quasiperiodic orbits lead to (weak) sensitivity to initial conditions.
(4) The Lyapunov exponent is zero.

In point (3) the sensitivity to initial conditions is "weak"in the sense introduced by Kollar et al. in [7]. That is, the orbits of two nearby initial conditions are in general close to each other, except for the cases in which they are on opposite sides of a discontinuity point, but when this happens, after a few iterations the trajectories are again close to each other, as it occurs in a PWL circle map with one discontinuity point.

We continue as follows. In Section 2, we introduce the financial market model that leads to a discontinuous map in three or four partitions. In Section 3, we recall the properties of the PWL Lorenz map with one discontinuity to emphasize that the PWL circle map separates regular regimes from chaotic ones. The properties related to maps in our definition are considered in Section 4. First, we prove the result in the simplest case with a unique discontinuity (Lemma 1). Second, we consider two discontinuity points, showing several scenarios with the financial market model. Third, we consider the generic case, proving Theorem 1. Fourth, we discuss the role of maps satisfying Definition 1 as a separator of regular regimes from chaotic ones. Some conclusions are drawn in Section 5.

2. A simple nonlinear financial market model

The seminal paper by Day and Huang [16] has motivated numerous follow-up studies aimed at explaining the intricate behavior of financial markets (see [17–20] for surveys). In [1], the authors developed a piecewise linear version of their original financial market model. The simplicity of the piecewise linear nature of their model provides new insights into the properties of financial markets. Until recently, only a few piecewise linear models had been proposed, mainly because the theory of PWL maps was not well developed or widely known, at least in applied contexts. Fortunately, the properties of 1D and 2D PWL maps have received greater attention in recent years, leading to the emergence of many new models that enhance our understanding of the dynamics of financial markets (see, e.g., [2,3,21,22]).

Following the approach in [23], we consider a financial market model in which there are two types of chartists and two types of fundamentalists. Type 1 chartists buy assets when they perceive a bull market and sell assets when they perceive a bear market. Type 2 chartists follow the same trading strategy; however, they become active only if the price deviates by at least a certain distance from its fundamental value. Type 1 fundamentalists buy assets when the market is undervalued and sell assets when it is overvalued. Type 2 fundamentalists do the same but – similar to type 2 chartists – enter the market only when the price deviates by at least a certain distance from its fundamental value. Clearly, type 2 chartists and type 2 fundamentalists do not trust – and therefore do not act upon – their respective trading signals when these signals are too weak.

[1] In other papers, its meaning may be also $G(X) \subseteq X$, so that X may be strictly mapped into itself.

Let P_t be the log of the price in period t and F its log fundamental value. The market maker mediates speculators' transactions out of equilibrium and adjusts prices via excess demand using a log-linear price adjustment rule. To be precise, the market maker quotes the log price for period $t+1$ as

$$P_{t+1} = P_t + a\left(DC_t^1 + DF_t^1 + DC_t^2 + DF_t^2\right) \tag{1}$$

The orders placed by type 1 chartists are formalized as

$$DC_t^1 = \begin{cases} a^{1,b} + c^{1,b}\left(P_t - F\right) & \text{for} \quad P_t - F \geq 0 \\ -b^{1,d} + c^{1,d}\left(P_t - F\right) & \text{for} \quad P_t - F < 0 \end{cases} \tag{2}$$

Type 2 chartists are inactive when $-Z_L < P_t - F < Z_R$, where $Z_L > 0$, $Z_R > 0$ are threshold parameters. The orders of type 2 chartists are expressed as

$$DC_t^2 = \begin{cases} a^{2,b} + c^{2,b}\left(P_t - F\right) & \text{for} \quad P_t - F \geq Z_R \\ 0 & \text{for} \quad -Z_L < P_t - F < Z_R \\ -b^{2,d} + c^{2,d}\left(P_t - F\right) & \text{for} \quad P_t - F \leq -Z_L \end{cases} \tag{3}$$

Put differently, type 2 chartists trust and act upon their trading signals if they are sufficiently strong.

The orders placed by type 1 fundamentalists are formalized as

$$DF_t^1 = \begin{cases} -a^{1,b} + f^{1,b}\left(F - P_t\right) & \text{for} \quad P_t - F \geq 0 \\ b^{1,d} + f^{1,d}\left(F - P_t\right) & \text{for} \quad P_t - F < 0 \end{cases} \tag{4}$$

Type 2 fundamentalists are inactive when $-Z_L < P_t - F < Z_R$. Their orders are given by

$$DF_t^2 = \begin{cases} -a^{2,b} + f^{2,b}(F - P_t) & \text{for} \quad P_t - F \geq Z_R \\ 0 & \text{for} \quad -Z_L < P_t - F < Z_R \\ b^{2,d} + f^{2,d}(F - P_t) & \text{for} \quad P_t - F \leq -Z_L \end{cases} \tag{5}$$

Similar to type 2 chartists, type 2 fundamentalists enter the market when their trading signals are sufficiently strong.

We assume that all reaction parameters in equations from (2) to (5) are non-negative. Inserting (2)–(5) into (1), defining

$$\begin{aligned} s_1 &= a\left(c^{1,b} - f^{1,b}\right), \quad s_2 = a\left(c^{1,d} - f^{1,d}\right), \\ s_3 &= a\left(c^{2,b} - f^{2,b}\right), \quad s_4 = a\left(c^{2,d} - f^{2,d}\right) \end{aligned} \tag{6}$$

and expressing the log of the price in deviations from its log fundamental value, i.e., $x_t = P_t - F$, we obtain the following system

$$G_4 : \begin{cases} G_L : x' = s_L x = \left(1 + s_2 + s_4\right) x & \text{for} \quad x \leq -Z_L \\ G_{M-} : x' = s_{M-} x = \left(1 + s_2\right) x & \text{for} \quad -Z_L < x \leq 0 \\ G_{M+} : x' = s_{M+} x = \left(1 + s_1\right) x & \text{for} \quad 0 \leq x < Z_R \\ G_R : x' = s_R x = \left(1 + s_1 + s_3\right) x & \text{for} \quad x \geq Z_R \end{cases} \tag{7}$$

The prime symbol stands for the unit time advancement operator. Moreover, the parameters $s_L = \left(1 + s_2 + s_4\right)$, $s_{M-} = \left(1 + s_2\right)$, $s_{M+} = \left(1 + s_1\right)$ and $s_R = \left(1 + s_1 + s_3\right)$ represent the slopes of the map's four linear functions. The four slope parameters may take any sign, positive or negative. This is a map that satisfies Definition 1, with the fixed point located on the boundary of two partitions and two additional discontinuity points.

The particular case where $Z_L = Z_R$, $s_2 = s_1$ and $s_4 = s_3$ has been considered in [4,24], where it has been shown that the system can have cycles of any period that densely fill some segments and quasiperiodic orbits that are dense in certain intervals, leading to dynamics that may appear chaotic but are not.

Now let us relax some assumptions, assuming only that $s_2 = s_1$. We then have the following system:

$$G_3 : \begin{cases} G_L : x' = s_L x = \left(1 + s_1 + s_4\right) x & \text{for} \quad x \leq -Z_L \\ G_M : x' = s_M x = \left(1 + s_1\right) x & \text{for} \quad -Z_L < x < Z_R \\ G_R : x' = s_R x = \left(1 + s_1 + s_3\right) x & \text{for} \quad x \geq Z_R \end{cases} \tag{8}$$

with $s_M = \left(1 + s_1\right)$, $s_L = \left(s_M + s_4\right)$ and $s_R = \left(s_M + s_3\right)$. This map satisfies Definition 1 with the fixed point internal to the middle partition.

One of the threshold parameters Z_L and Z_R in (7) and (8) can be considered a scaling factor, and thus, be set at any value. However, we keep them because they are economically relevant.

We also notice that when parameter $a^{1,b}$ in the trading rule of type 1 chartists differs to that in the trading rule of type 1 fundamentalists, then the maps in (7) and (8) have some offsets (i.e., the function in some partition becomes affine instead of linear). The same applies for parameters $a^{2,b}$, $b^{1,d}$ and $b^{2,d}$. In such cases, the systems do not belong to the class of maps considered in this work. The PWL systems with some affine functions can clearly have chaotic dynamics (see, e.g., the systems in [21–23]).

Differently, the dynamics that can occur in the system in (7), as well as in the system in (8), cannot be chaotic, as is the case with any map satisfying Definition 1. However, from the applied point of view, the systems may nevertheless mimic important properties of financial market dynamics.

3. Preliminary properties of a PWL Lorenz map

In this section, we recall some properties of the dynamics of 1D PWL discontinuous maps with a single discontinuity point, known as 1D PWL Lorenz maps, and circle maps.

Any map $f : [a,b] \to [a,b]$ such that $f(a) = f(b) \in (a,b)$ with a unique discontinuity point $c \in (a,b)$, where $f = f_L$ is monotonically increasing in $[a,c)$ with $f_L(c) = b$ and $f = f_R$ is monotonically increasing in $(c,b]$ with $f_R(c) = a$, defines a homeomorphism of the circle ([14], p. 20). Note that then $f_L\left(f_R(c)\right) = f_R\left(f_L(c)\right)$.

A **PWL circle map** is a circle homeomorphism in which the two components are linear or affine functions.

A 1D **PWL Lorenz map** may be defined as follows:

$$f = \begin{cases} f_L : x' = a_L x + \mu_L & \text{for} \quad x < 0 \\ f_R : x' = a_R x + \mu_R & \text{for} \quad x > 0 \end{cases} \tag{9}$$

with $a_L > 0$, $a_R > 0$, and $\mu_L \neq \mu_R$ (so that f is discontinuous). For this class of maps, assuming $\mu_R < \mu_L$, the interval $I = \left[f_R(0), f_L(0)\right] = \left[\mu_R, \mu_L\right]$ is an absorbing interval (that is, a point in a neighborhood of I enters I in a finite number of iterations, and $f(I) \subseteq I$). All the properties of map f are also recalled in [15] (see pp. 95–100).

Considering the images of the discontinuity point $x = 0$, the dynamics of a PWL Lorenz map, as depicted in Fig. 2, can be classified into three different types (see [25], p. 184, or [26], p. 591):

(L1) In the case $f_R \circ f_L(0) < f_L \circ f_R(0)$, the map is called a **gap map**, where the gap is the interval $J = \left]f_R \circ f_L(0), f_L \circ f_R(0)\right[$ and $f(I) \subset I$, (in [26] it is called nonoverlapping). Each point $x \in I$ has either a unique preimage or no preimages inside I. The rotation number is well defined (see [14,26]); it is the same for any point in I, and its dynamics can only be as follows:

- If the rotation number of map f is rational, then the ω-limit set for all $x \in I$ is an attracting hyperbolic cycle.
- If the rotation number is irrational, then the ω-limit set for all $x \in I$ is a unique Cantor set attractor C,

$$C = \bigcap_{n \geq 0} f^n(I) = I \setminus \bigcup_{n \geq 0} f^n(J)$$

The Cantor set attractor C is the closure of quasiperiodic orbits. A Cantor set attractor has a self-similar structure, as is the case with any Cantor set.

In this class of maps, the irrational rotation is the nongeneric case,[2] which is not persistent under parameter perturbation, while the rational rotations occur in open regions of the parameter plane and are thus

[2] A case is called "generic"when it occurs for an open set of the values of the parameters in the considered space, "non generic"when it occurs for a set of zero measure of the values of the parameters.

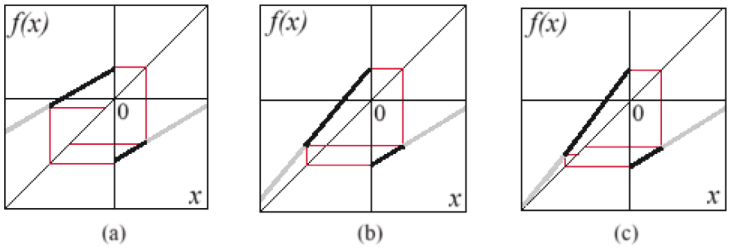

Fig. 2. PWL Lorenz map f is a gap map in (a), a circle map in (b), an overlapping map in (c).

persistent under parameter perturbations. The existence regions related to cycles are bounded by border collision bifurcation curves following a period adding rule and Farey summation rule [15].

(L2) Map f is a **circle map** if and only if the following equality holds: $f_R \circ f_L(0) = f_L \circ f_R(0)$. Now I is invariant, $f(I) = I$. In this case, we have a circle homeomorphism. It is known [14,26] that the rotation number is well defined; it is the same for any point of interval I, and the dynamics depend on the rotation number as follows:

- If the rotation number of map f is rational, then the absorbing interval $I = [f_R(0), f_L(0)]$ is filled with periodic points (of the same period).
- If the rotation number of map f is irrational, then the absorbing interval I is filled with quasiperiodic orbits dense in that interval (i.e., any orbit of f is quasiperiodic and dense in I, so that the ω-limit set of each orbit is the whole interval). Note that the case such that the minimal set is a Cantor set is excluded since the map is linear (hence its derivative has bounded variation automatically), according to Denjoy's theorem.

The irrational rotation is now the generic case, while the case of a rational rotation is not persistent under parameter perturbations (that is, in the parameter space satisfying $f_R \circ f_L(0) = f_L \circ f_R(0)$, the rotation number is irrational with a probability of 1, since the rational rotation occurs in a parameter set of zero Lebesgue measure).

(L3) In the case $f_R \circ f_L(0) > f_L \circ f_R(0)$, the map is called an **overlapping map**, since the interval $[f_L \circ f_R(0), f_R \circ f_L(0)]$ consists of points having two different rank-one preimages in the invariant interval $I = [f_R(0), f_L(0)]$, $f(I) = I$. This system can have different kinds of dynamics (see [25–27]), especially hyperbolic cycles and chaotic attracting or repelling sets (which necessarily imply the existence of infinitely many unstable cycles). In the PWL case of interest here, this regime is characterized by attracting chaotic intervals, bounded by images of the discontinuity point [27].

Thus, the case of a circle map denotes the transition from regular dynamics (without chaos, in a gap map) to chaotic dynamics (existence of chaotic sets, in the overlapping map), see [15,28].

4. 1D discontinuous PWL homogeneous maps

In this section, we study the properties of maps satisfying Definition 1. In Section 4.1, we consider maps with a unique discontinuity point. In Section 4.2, we examine maps with two discontinuity points. In Section 4.3, we prove Theorem 1. In Section 4.4, we discuss the role of maps satisfying Definition 1 as separators between regular regimes and chaotic ones. The last two subsections also include comments on the implications of our financial market model.

4.1. One discontinuity point

In the simplest case, the map consists of only two partitions and is defined as

$$F = \begin{cases} F_L : x' = s_L x & \text{for} \quad x < h \\ F_R : x' = s_R x & \text{for} \quad x > h \end{cases} \qquad h \neq 0 \tag{10}$$

In order to have a unique fixed point at the origin and a unique discontinuity point, it must be $h \neq 0$.

In the following lemma, we prove that besides the possible occurrence of divergent trajectories and trajectories related to the fixed point O, the only possible bounded dynamics are those occurring in a PWL circle map.

Lemma 1. *Let F be the map defined in (10). Then the only possible bounded ω-limit sets different from the fixed point O, and from nonhyperbolic cycles associated with the nonhyperbolic fixed point O, are those of a PWL circle map.*

Proof of Lemma 1. To fix the reasoning, let us assume $h > 0$ (which can also serve as a scaling factor, and be set to 1). The case $h < 0$ is topologically conjugate to $h > 0$.

When one of the two slopes is zero, then the only possible bounded ω-limit set is the fixed point O. So, let us consider $s_{L,R} \neq 0$. Moreover, if $|s_L| = 1$ and/or $|s_R| = 1$, the bounded ω-limit sets different from the fixed point O can only be nonhyperbolic fixed points or nonhyperbolic 2-cycles (filling densely some segments).

Let us now consider the generic case, proving the statement for the different signs of the slopes of the linear functions.

(i1) In the case of **positive slopes**, asymptotic dynamics different from the fixed point O or divergence can occur only for $F_L(h) > h > F_R(h)$, leading to $s_L > 1 > s_R$, and in this case it is $F_R \circ F_L(h) = s_R s_L h = F_L \circ F_R(h)$. Thus, we obtain a PWL circle map in the absorbing interval $I = [F_R(h), F_L(h)] = [s_R h, s_L h]$.

(i2) In the case of **one negative and one positive slope**, asymptotic dynamics different from the fixed point O or divergence can occur only for $s_L < 0$ and $s_R > 0$, with $F_L \circ F_L(h) > h > F_R(h)$, leading to $s_L^2 > 1 > s_R$. In this case, the dynamics can be obtained via the first return map in $x > 0$, that is given by $F_L \circ F_L(h)$ for $x < h$ and $F_R(h)$ for $x > h$, so that $F_R \circ F_L^2(h) = s_R s_L^2 h = F_L^2 \circ F_R(h)$. Thus, we obtain a PWL circle map in the absorbing interval $I = [F_R(h), F_L^2(h)] = [s_R h, s_L^2 h]$.

(i3) In the case of **negative slopes**, the dynamics can be studied via the second iterate, and asymptotic dynamics different from the fixed point O or divergence can occur only for $F_L \circ F_L(h) > h > F_L \circ F_R(h)$, leading to $s_L^2 > 1 > s_L s_R$. In this case, its dynamics are the same as those of the first return map, in partition $x > 0$, or in $x < 0$, in a neighborhood of the discontinuity point. In the partition $x > 0$, it is given by $F_L \circ F_L(h)$ for $x < h$ and $F_L \circ F_R(h)$ for $x > h$, so that $F_L \circ F_R \circ F_L^2(h) = s_R s_L^3 h = F_L^3 \circ F_R(h)$, and it is a PWL circle map in the absorbing interval $I = [F_L \circ F_R(h), F_L \circ F_L(h)] = [s_L s_R h, s_L^2 h]$. In the partition $x < 0$, it is given by $F_R \circ F_L(h)$ for $x < F_L^{-1}(h) = h/s_L$ and $F_L \circ F_L(h)$ for $x > F_L^{-1}(h) = h/s_L$, so that $F_L \circ F_L \circ F_R \circ F_L(h/s_L) = s_L^2 s_R h = F_R \circ F_L^3(h/s_L)$, and it is a PWL circle map in the absorbing interval $J = [F_L \circ F_L(h/s_L), F_R \circ F_L(h/s_L)] = [s_L h, z_L s_R h]$. Both lead to the same dynamics of map F. □

The three possible generic cases described in the proof of Lemma 1 are shown in Fig. 3.

Remarks. It is worth noting that, unlike the dynamics occurring in a gap map, and similarly to the comment in the case (L2) of a Lorenz map in the previous section (PWL circle map), the generic case is an

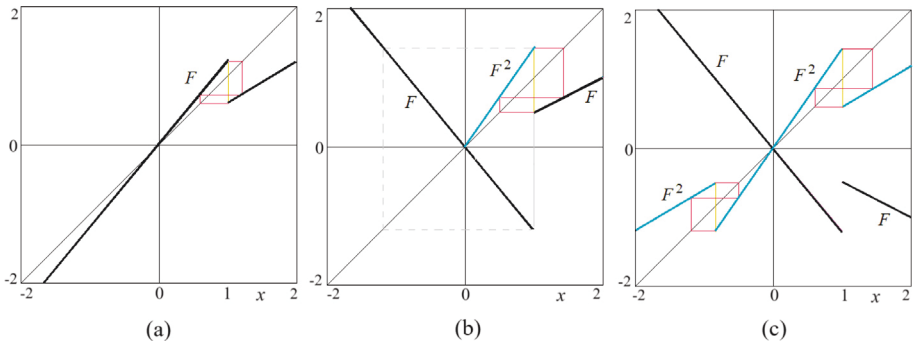

Fig. 3. Map F in (10) with $h = 1$. Function F is shown in black and its second iterate in light blue. In (a), case (i1); in (b), case (i2); in (c), case (i3).

irrational rotation number, and thus, the trajectories are quasiperiodic and dense in intervals.

For map F, the existence of dense cycles is associated with a set of zero Lebesgue measure in the (s_R, s_L) parameter plane of the possible slopes. In the case of positive slopes, cycles occur if and only if integers p and q exist such that $s_L^p s_R^q = 1$. This leads to nonhyperbolic cycles (filling densely interval I) of period $n = p+q$, assuming p and q are the smallest integers, and rotation number $\rho = p/n$ (or $(1-\rho) = q/n$).

It is also worth noting that the map known as "rigid rotation", that is, $R : x \to x+\rho \pmod 1$, for $\rho \in (0,1)$, is not included in our definition, since considering the map $x' = x+\rho$ for $0 \le x < (1-\rho)$, and $x' = x+\rho-1$ for $(1-\rho) < x \le 1$, the fixed point is at infinity. However, this circle homeomorphism with rotation number ρ is topologically conjugate to map F with positive slopes and the same rotation number.

4.2. Two discontinuity points

Let us comment on the case of two discontinuity points. For a 1D map satisfying Definition 1 with two discontinuities and the fixed point internal to the middle partition, the map is given, for $h_1 < h_2$, $h_1 \neq 0$, $h_2 \neq 0$, by:

$$G_1 : \begin{cases} G_L : x' = s_L x & \text{for} \quad x < h_1 \\ G_M : x' = s_M x & \text{for} \quad h_1 < x < h_2 \\ G_R : x' = s_R x & \text{for} \quad x > h_2 \end{cases} \tag{11}$$

A particular case of map (11) with $h_1 = -h_2$ and $s_R = s_L$ has been considered in [4,24]. In these works, it has been shown that the only bounded trajectories, different from the fixed point O, and the cycles related to the fixed point O when it is nonhyperbolic, are those occurring in a PWL circle map.

In the generic case, the same result can be obtained. A few examples with map G_1 in (11), which can be considered as our applied model in (8), with a different denomination of the discontinuity points, are shown in Figs. 4 and 5, where it is evidenced that it is sufficient to look for the first return map in a suitable interval (sometimes in a neighborhood of a discontinuity point, but not necessarily), to see that the first return map is a PWL circle map. The fact that coexistence of attractors is possible can be seen also from the function in light blue in Fig. 3(c) (although in that figure it is the second iterate of map F).

To get the first return map in some interval, each point x is iterated and when the trajectory returns for the first time to the interval, we assign to x that value, say $F_{ra}(x)$. Considering the symbolic sequences of the trajectory in the usual way, using the partitions involved in the trajectory, the first return has a fixed symbolic sequence. Due to the linearity, the definition will be the same for an interval of points (i.e., the same symbolic sequence), up to a point, say $x = c$, whose trajectory merges with a discontinuity point of the map. For $x > c$, the first return leads to a different function $F_{rb}(x)$, i.e., a different symbolic sequence. Since the composition of linear homogeneous functions is always a linear homogeneous function, the branches are functions with

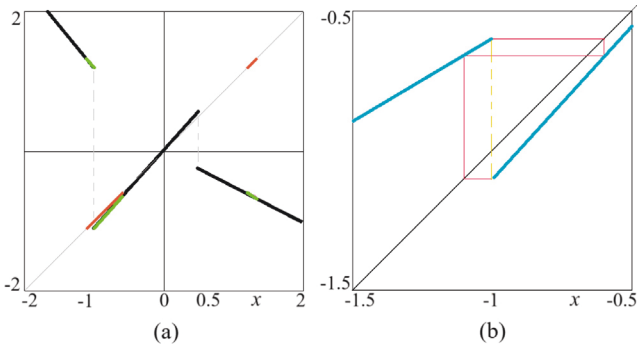

Fig. 4. Map G_3 in (8) with $Z_L = 1$, $Z_R = 0.5$, $s_L = -1.2$, $s_M = 1.2$, $s_R = -1.1$. Along the diagonal, the segments corresponding to the attracting set are shown in red. The values of the function along these red segments are indicated in green. In (b), first return map in a neighborhood of $Z_L = -1$.

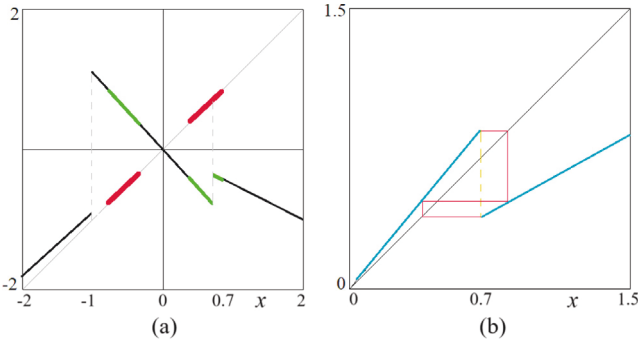

Fig. 5. Map G_3 in (8) with $Z_L = 1$, $Z_R = 0.7$, $s_L = 0.9$, $s_M = -1.1$, $s_R = -0.5$. Meaning of the red and green points as in Fig. 4(a). In (b), first return map in a neighborhood of $Z_R = 0.7$.

the same fixed point at the origin. Then we have a map such as map F in (10), and the dynamics can be ultimately described using $x' = F_{ra}(x)$ for $x < c$ and $x' = F_{rb}(x)$ for $x > c$ in the considered interval.

In the examples in Figs. 4 and 5, we have shown the first return map in an interval including a discontinuity point. The map shown in Fig. 6(a) has many intervals filled with quasiperiodic trajectories. The first return map in one segment is shown in Fig. 6(b). Considering its discontinuity point c, the rank-1 preimage c_{-1}, and the image c_1, we have two segments on the right side of the discontinuity point c. Each point in the segment (c, c_{-1}) has the trajectory $R(LM)^3 RM^2 (ML)^3$, and then it returns to the right side of c, while each point in segment (c_{-1}, c_1) has the trajectory $R(LM)^3 R(LM)^3 RM^2 (ML)^3$, and then it returns to the right side of c.

In the case that the product of the slopes of the symbolic sequence $R(LM)^3 RM^2 (ML)^3$ is 1, then the product of the two slopes in the first

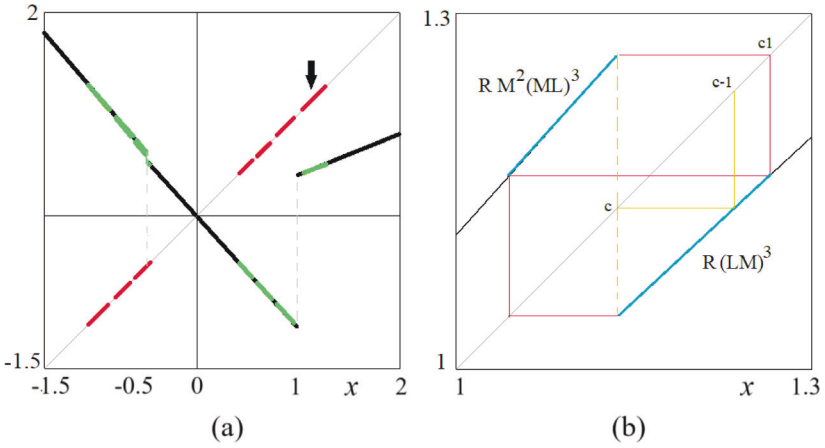

Fig. 6. Map G_3 in (8) with $Z_L = 0.5$, $Z_R = 1$, $s_L = -1.2$, $s_M = -1.1$, $s_R = -0.4$. Meaning of the red and green points as in Fig. 4(a). In (b), first return map in the rightmost interval, evidenced by an arrow in (a).

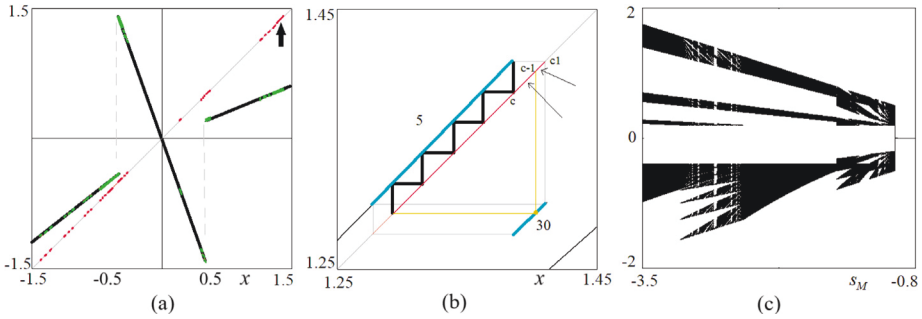

Fig. 7. Map G_3 in (8) with $Z_L = 0.5$, $Z_R = 0.5$, $s_L = 0.8$, $s_M = -2.82$, $s_R = 0.4$. Meaning of the red and green points as in Fig. 4(a). In (b), first return map in the rightmost interval, evidenced by an arrow in (a). In (c), 1D bifurcation diagram as a function of slope s_M.

return map is also 1 (so that it is filled with 2-cycles) and map G_3 has segments filled with periodic points of cycles of period 16.

In the example shown in Fig. 7, we consider the first return map in a small segment evidenced by an arrow in Fig. 7(a), which is shown in Fig. 7(b). This first return map takes 5 iterations for $x < c$ with a symbolic sequence, say S_{ra}, and 30 iterations for $x > c$, with a symbolic sequence, say S_{rb}. Looking at the attractor in Fig. 7(a), it seems chaotic in several intervals, but it is not, and its dynamics are predictable. Taking the preimages of the discontinuity point c of the first return map, the segment on the right side is separated into two parts. Each point in the first segment, close to c, has the trajectory $S_{rb}\left(S_{ra}\right)^5$ and returns in the right side of c, while each point in the second segment at the right side of c, has the trajectory $S_{rb}\left(S_{ra}\right)^4$, and is back again in the right side of c. Clearly, S_{rb} and S_{ra} are to be expanded with the 30 and 5 symbols, respectively (related to the partitions L, M, R), $S_{rb} = R^2ML^2\left(MRML^5\right)^3M$, and $S_{ra} = R^2MLM$. Fig. 7(c) shows a 1D bifurcation diagram as a function of slope s_M in the middle partition.

A similar behavior, as described in Theorem 1, always occurs in maps satisfying Definition 1, regardless of the number of partitions. Indeed, hyperbolic cycles different from the fixed point O cannot exist, so that homoclinic orbits (and hence chaotic invariant sets) are ruled out. The absence of hyperbolic cycles leads to the dynamics of a map of Lorenz type, but it cannot be a gap map because a gap map can have only hyperbolic cycles or Cantor set attractors, which are not possible here. Neither it is possible to get an overlapping map (which in the linear case has hyperbolic cycles different from the fixed point O, while they cannot exist here). It follows that the only possible bounded dynamics not converging to the fixed point O are segments filled with nonhyperbolic cycles (all with the same period) or quasiperiodic trajectories dense in some intervals, as occurring in a PWL circle map.

4.3. *Proof of Theorem 1*

We are now prepared to prove Theorem 1, which characterizes the attracting sets that may arise in 1D PWL maps, as defined in Definition 1, assuming, for simplicity (without loss of generality), that the unique fixed point of the functions defined in the different partitions has been set in the origin.

Proof of Theorem 1. (1) Assuming, by contradiction, that a map in our class can have a hyperbolic cycle of some period, with a given symbolic sequence, the periodic points must be fixed points of the related composite map (obtained following the symbolic sequence of the cycle). However, any composition of the linear maps in the definition is also a linear map that has only the fixed point O as its possible hyperbolic fixed point. Thus, a hyperbolic cycle different from the fixed point O cannot exist, and obviously nor can a chaotic set, since one of the properties of a chaotic set is that homoclinic hyperbolic cycles must be dense in it.

(2) Let us first recall the dynamics associated with the fixed point O. When the fixed point is internal to a partition it may be hyperbolic or not. If not (the eigenvalue is +1 or −1), then a segment exists filled with nonhyperbolic cycles (fixed points or 2-cycles), all with eigenvalue +1. When the fixed point O is a kink point, so that the map is continuous at the fixed point O, but not differentiable, the behavior of the points close to the fixed point O on its right and left sides may be different, depending on the related slopes. It may be attracting on one side and repelling on the other side, or attracting (repelling) on both sides, or nonhyperbolic on one or both sides. In any case, the related dynamics are known.

Now we turn to the possible bounded invariant sets of map G, different from those related to the fixed point O (hyperbolic or nonhyperbolic).

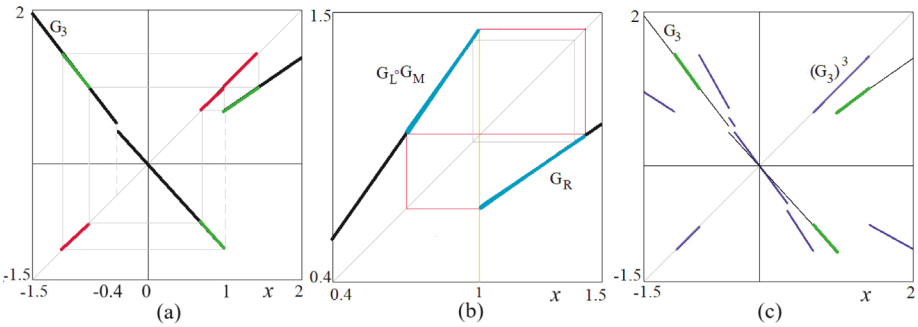

Fig. 8. Map G_3 in (8) with $Z_L = 0.4$, $Z_R = 1$, $s_L = -1.3$, $s_M = -1.1$, $s_R = 0.6993006993$. In this example, the map has segments filled with cycles of period 3 since $s_L s_M s_R = 1$. Meaning of the red and green points as in Fig. 4(a). In (b), first return map in a neighborhood of $Z_R = 1$. In (c), the map and its third iterate, having segments on the diagonal filled with fixed points.

If bounded ω-limit sets different from those related to the fixed point O exist, these must involve different partitions of the map (at least two), that is, at least one discontinuity point. Let A ($\neq O$) be a bounded ω-limit set of a trajectory of map G, then $G(A) = A$. If it is a k-cycle, it cannot be hyperbolic. Thus, it is nonhyperbolic, and nonhyperbolic cycles in PWL maps in our definition must be dense in some intervals. In fact, any iterate with map G^k of an interval I in a partition is such that $G^k(I)$ consists of segments belonging to straight lines issuing from the fixed point O, and thus cannot have a kink point with a contact on the main diagonal (nor can transversely intersect it). It follows that it can only be the case that the kth iterate of the map, G^k, includes segments on the main diagonal.

Note that any composition of the linear functions defining map G consists of linear branches, so that the first return map in any segment inside the absorbing interval is always a function with the same characteristics as map G, the branches have the same fixed point O. Hence the first return map in a suitable segment leading to a function with only one discontinuity point is necessarily as map F in (10), and by Lemma 1 it can be only a PWL circle map.

(3) Recall that a map $F : I \to I$ is said to have sensitive dependence on initial conditions on I if there exists a constant $\delta > 0$ such that, for any $x \in I$ and any neighborhood $U(x)$ of x, there exists an $x \in U(x)$ and a positive integer n such that $|F^n(x) - F^n(x)| > \delta$ (see, e.g., [13,29]). In our case, when there are quasiperiodic trajectories, these can be ultimately reduced to those of a PWL circle map, since the first return map in some suitable interval leads to a map as F in (10). Considering any two nearby points in the first return PWL circle map, in a finite number of iterations, the trajectories will have points separated by the discontinuity point. Thus, the definition cannot be satisfied. However, after one iteration, the trajectories will again be close to each other, leading to the property of weak sensitivity (a proof for a map as map F in (10) is also presented in [7]).

(4) Since the dynamics can ultimately be reduced to those of a PWL circle map as defined in (10), it is well known that it has Lyapunov exponent zero (a proof is also given in [7]). □

Some comments are in order. The example with map G_3 in (8) shown in Fig. 8 evidences that the first return map F has an interval filled with nonhyperbolic 2-cycles. Such cycles for map G_3 correspond to nonhyperbolic 3-cycles with symbolic sequence RLM, since it is $s_L s_M s_R = 1$. Fig. 8(c) shows, besides G_3, the third iterate $(G_3)^3$, which has segments on the main diagonal filled with nonhyperbolic fixed points.

The example with map G_4 in (7) shown in Fig. 9 evidences that the two partitions with the kink point in the fixed point O may not be involved in the asymptotic dynamics. Here, a right neighborhood of the fixed point O is mapped to the left side, and all the points in the left partition of the fixed point O are converging to an attractor belonging to the other partitions, as shown via the symbolic sequence occurring in the first return map (in the interval in the square of Fig. 9(a)). In this example, the slopes of the function in the left/right partitions of the fixed point O are $s_{M-} = 1.2$ and $s_{M+} = -1$, respectively.

However, changing the slope in the left partition M_- of the fixed point O, as in the example in Fig. 9(c), we have coexistence: the same intervals filled with quasiperiodic trajectories shown in Fig. 9(a) coexist with the attracting fixed point O, whose basin of attraction is the segment $(-1, 1)$. The remaining points converge to the other attractor. Now the slopes of the function in the kink point O are $s_{M-} = 0.9$ and $s_{M+} = -1$.

It is worth noting that the invariant set A that can occur in the class of 1D PWL maps considered here, different from the fixed point O, can be an attracting set or not (depending on the behavior of the map in points outside the invariant set). The fixed point O or the invariant segment related to the nonhyperbolic fixed point O may be attractors in Milnor sense (attracting a set of positive Lebesgue measure, [30]) and may coexist with intervals filled with quasiperiodic trajectories or periodic points. Note that when the map has segments filled with nonhyperbolic cycles, the segments lead to an attracting set, but not to an attractor, since such an invariant set does not have a dense orbit. It follows that the only attractor A (an invariant set with a dense orbit and a suitable neighborhood $U(A)$ such that $A = \cap_{n>0} G^n(U(A))$) different from the fixed point O may be the set of intervals filled with quasiperiodic trajectories.

Moreover, we remark that in the parameter space of the slopes of a system in our definition, the occurrence of segments filled with nonhyperbolic cycles is not structurally stable. This can occur only when the product of the related slopes (of the linear functions in the partitions with periodic points) leads exactly to the value 1 (as in the example shown in Fig. 8). Clearly, the generic case is the occurrence of quasiperiodic trajectories. This is also the reason why the numerical computations, as one parameter is varied, lead to figures that seem related to chaos (as in Fig. 1(a,b) and Fig. 7(c)).

The quasiperiodic character of a trajectory may be better observed when an applied system is investigated by using the versus-time trajectory, as the prices in the financial market model. A couple of examples with the case shown in Fig. 7 at $s_M = -1.5$ and $s_M = -2.82$ are reported in Fig. 10.

Recall that x represents the log price of the financial market in deviation from its log fundamental value. Hence, when $x > 0$, the financial market is overvalued, and when $x < 0$, the financial market is undervalued. Clearly, our financial market model is capable of producing endogenous price dynamics. Depending on the parameter setting, prices fluctuate around their fundamental value within a narrower (left) or broader band (right), qualitatively mimicking the excessively volatile boom-bust behavior of actual financial markets. The dynamics depicted in the right panel, Fig. 10(b), are particularly interesting in this respect. The financial market model more frequently produces a bull market than a bear market, though when a bear market emerges, it may be quite significant.

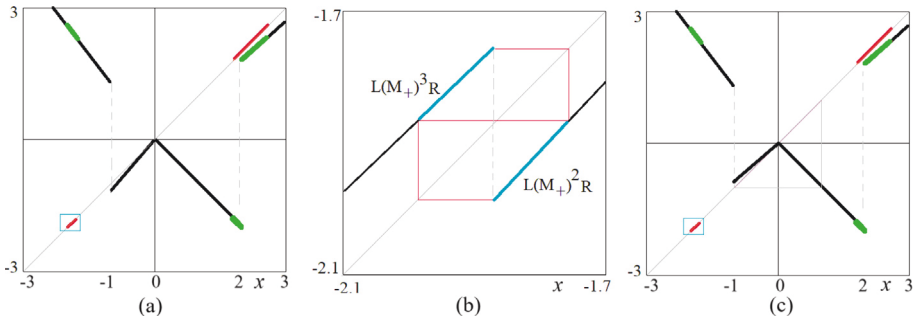

Fig. 9. Map G_4 (7) with $Z_L = 1$, $Z_R = 2$, $s_L = -1.3$, $s_{M-} = 1.2$, $s_{M+} = -1$, $s_R = 0.9$. In (b), first return map in the leftmost interval, evidenced by a rectangle in (a). In (c), the slopes of the function in the middle left partition is $s_{M-} = 0.9$. In (a) and (c) meaning of the red and green points as in Fig. 4(a).

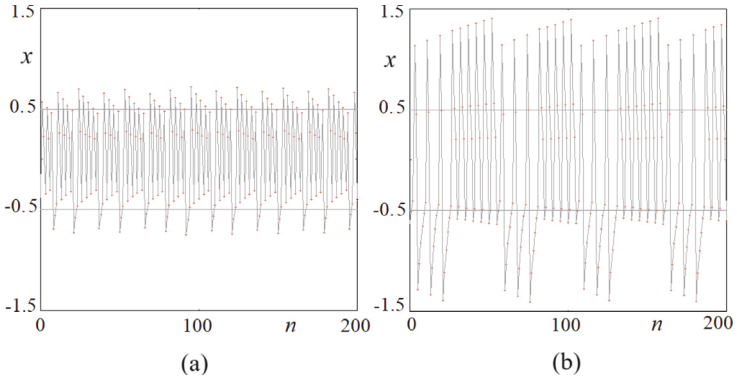

Fig. 10. Map G_3 in (8) with $Z_L = 0.5$, $Z_R = 0.5$, $s_L = 0.8$, $s_R = 0.4$. Versus time trajectory, for 200 iterations. In (a), $s_M = -1.5$. In (b), $s_M = -2.82$.

4.4. Class of maps separating regular/chaotic behaviors

The properties of the maps in our definition, described in Theorem 1, are due to the existence of a unique fixed point for all the functions in the partition. We may expect that when this assumption is not satisfied, then hyperbolic cycles different from the fixed point O may appear as well as chaos.

As for the PWL circle map in the Lorenz family recalled in Section 3, which separates regular regimes (with hyperbolic attracting cycles) from chaotic regimes, we conjecture that also the maps in our family have a similar property. As soon as a second fixed point, also virtual, exists for one of the functions defining the map, we may expect either chaos or some hyperbolic attracting cycle.

Let us discuss a numerical example. We add an offset μ_R to the function in the right partition of map G_3 in (8), obtaining the map

$$\widetilde{G}_3 : \begin{cases} G_L : x' = s_L x & \text{for} \quad x \le -Z_L \\ G_M : x' = s_M x & \text{for} \quad -Z_L < x < Z_R \\ G_R : x' = s_R x + \mu_R & \text{for} \quad x \ge Z_R \end{cases} \tag{12}$$

In our financial market model defined in Section 2, this map in (12) corresponds to a scenario in which the parameter $a^{1,b}$ and/or $a^{2,b}$ of the trading rule of chartists differs not only with respect to the sign from that of fundamentalists.[3] Let us reconsider the parameters as in the example shown in Fig. 7. A 2D bifurcation diagram of map $\widetilde{G}_3$ in (12) varying μ_R in the range $(-0.2, 0.4)$ and slope s_M in the middle partition in the range $(-0,3, -0.1)$ is shown in Fig. 11(a). The white region there shown, for $\mu_R < 0$, corresponds to the existence of chaotic dynamics. The colored regions for $\mu_R > 0$ denote the existence of attracting cycles. Different colors correspond to cycles of different periods. The yellow region denotes the existence of an attracting fixed point.

An example at $s_M = -2.82$ (as in Fig. 7(a,b)) and with $\mu_R = -0.01$ is shown in Fig. 11(b), with dynamics that is chaotic in some intervals, as proven via the first return map in one of the intervals (evidenced by an arrow in Fig. 11(b)) and reported in Fig. 11(c), which shows an overlapping Lorenz map, chaotic in that interval.

Differently, an example at $s_M = -2.82$ (as in Fig. 7(a,b)) and $\mu_R = 0.01$ is shown in Fig. 12(a), where the dynamics are converging to an attracting 19-cycle. The 19th iterate of the map is shown in Fig. 12(b), evidencing 19 segments crossing the main diagonal; an enlargement is also shown in Fig. 12(c) with the fixed point in $x^* = 1.2161$. As our simulations suggest, this cycle is the only stable one, attracting all the points except for the fixed point O (i.e., there are no other attracting or repelling cycles).

It may be interesting to compare the dynamics that can be observed in the time domain, starting from the same initial condition and after the same number of transient iterations. The three cases with $\mu_R = 0$, $\mu_R = -0.01$ and $\mu_R = 0.01$ are shown in Fig. 13.

All three parameter constellations can produce endogenous price fluctuations – this is a robust feature of our financial market model. At least at first glance, one may furthermore conclude that the differences in the dynamics are not so great. However, this is clearly not the case. The dynamics are quasiperiodic (left), chaotic (middle), or periodic (right). This may have relevant policy implications, e.g., with respect to stabilization policies. For instance, certain chaos control methods may help stabilize the dynamics in the middle panel, but it is not clear whether this will also be the case in the other parameter constellations. Put differently, it is important to be able to distinguish between different types of dynamic behaviors. Adding noise to the dynamics may have further significant implications. In the presence of a positive Lyapunov exponent, even small shocks may have a marked impact on the evolution of financial prices, while this is not the case when the Lyapunov exponent is zero or negative.

[3] In the specific case considered, the value of μ_R can be interpreted as the excess demand resulting from the order sizes placed by chartists and fundamentalists when the asset price equals its fundamental value.

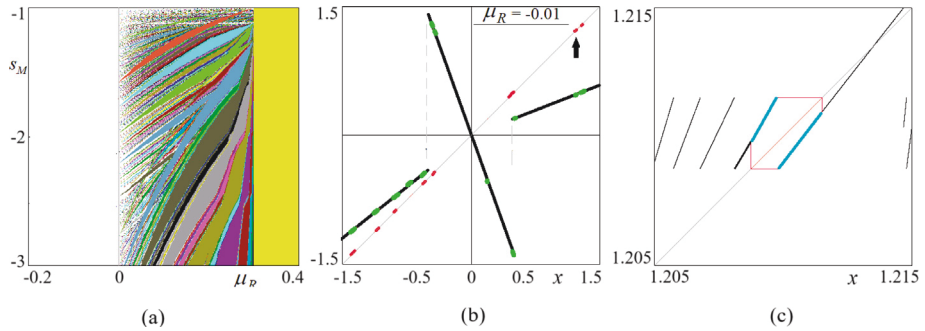

Fig. 11. In (a), 2D bifurcation diagram in the parameter plane (μ_R, s_M) of map $\widetilde{G}_3$ in (12) at $Z_L = 0.5$, $Z_R = 0.5$, $s_L = 0.8$, $s_R = 0.4$. In (b), an example at $\mu_R = -0.01$ and $s_M = -2.82$. Meaning of the red and green points as in Fig. 4(a). In (c), the first return map in the interval evidenced by an arrow in (b), which is an overlapping PWL Lorenz map, with chaos in the interval.

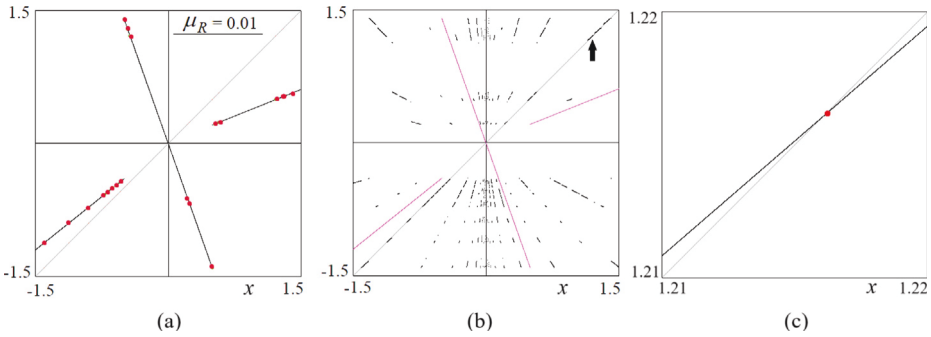

Fig. 12. In (a), map G_3 in (8) with $Z_L = 0.5$, $Z_R = 0.5$, $s_L = 0.8$, $s_M = -2.82$, $s_R = 0.4$, perturbed with $\mu_R = 0.01$, and the trajectories converge to a cycle of period 19. In (b), map G_3 is shown in pink and the 19th iterate of the map is shown in black. It has 19 segments crossing the main diagonal. An enlargement of the segment evidenced by an arrow is shown in (c), where the fixed point $x^* = 1.2161$ is marked in red.

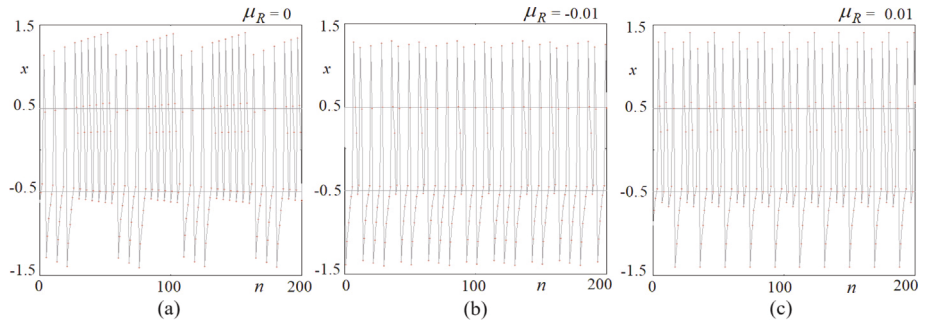

Fig. 13. Map G_3 in (8) $Z_L = 0.5$, $Z_R = 0.5$ and $s_L = 0.8$, $s_M = -2.82$, $s_R = 0.4$ perturbed with μ_R. Versus time trajectory, for 200 iterations. In (a) $\mu_R = 0$. In (b) $\mu_R = -0.01$. In (c) $\mu_R = 0.01$.

5. Conclusions and future works

In this work, we have characterized the dynamics that may occur in a wide class of 1D PWL discontinuous maps with a finite number of discontinuity points. We have shown that, although numerical simulations may suggest the existence of chaotic behavior, there is no chaos. The bounded dynamics, not related to the fixed point, can ultimately be described via a PWL circle map, so that either non-hyperbolic cycles or quasiperiodic orbits densely fill some segments. The only attractors different from the fixed point are segments filled with quasiperiodic trajectories. However, due to the existence of many partitions, quasiperiodic orbits mimic chaotic behavior, and this, in systems with practical applications, may still be interpreted as complex behavior. Moreover, our model, which represents the dynamics of a financial market, may be used to understand the mechanisms leading to oscillations in prices.

One interesting question left for future work is whether any map in the class defined here truly acts as a separator between regular and chaotic regimes. Another interesting question is whether a map satisfying Definition 1 can appear as a restriction, or first return map, on some segment of a straight line in a two-dimensional PWL discontinuous map, such as the one proposed in [11]. Such systems can be generalized, leading to families of 2D discontinuous maps whose first return map on some segments of a straight line, when they exist, lead to a map in our definition. That is, the 1D maps in our definition occur as particular cases of a larger class of 2D PWL discontinuous maps. Furthermore, in such cases, it is important to examine whether the role of these maps as separators between regular and chaotic regimes persists, so that under perturbation, chaos can emerge, as we observed in the financial market model when assuming that one parameter among $a^{1,b}$, $a^{2,b}$, $b^{1,d}$, $b^{2,d}$, in the trading rule of chartists differs from the one in the trading rule of fundamentalists.

CRediT authorship contribution statement

Laura Gardini: Methodology. **Davide Radi:** Writing – original draft, Conceptualization. **Noemi Schmitt:** Investigation. **Iryna Sushko:** Methodology. **Frank Westerhoff:** Writing – review & editing, Supervision, Software.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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