https://doi.org/10.20378/irbo-52626

16

Analysis of a Loss System with Mutual Overflow in a Markovian Environment

UDO R. KRIEGER Research Institute of the Deutsche Bundespost, Darmstadt, Germany

ABSTRACT

We investigate a loss system modeling mutual overflow between two fully available trunk groups. Assuming exponentially distributed call holding times and two independent Markov modulated Poisson processes to be offered as arrival streams, a Markovian model is derived and its steady-state distribution is computed by numerical methods. We extend Schweitzer's iterative A/D method to a larger class of regular splittings and improve Haviv's error analysis of this procedure. Moreover, formulas for the call-congestion rates of the arrival streams are derived.

1. INTRODUCTION

Recently, considerable attention has been devoted to the analysis of advanced routing schemes in nonhierarchical, circuit-switched digital networks based

on efficient modern signalling systems such as CCITT CCS No. 7. Especially adaptive routing procedures that balance the effects of random traffic fluctuations within the network have been studied intensively.

In this context, a new loss system modeling mutual overflow between two fully available trunk groups has been introduced and investigated (cf. [27], [22], [23]). It describes a telecommunication network that consists of an exchange A being connected to two exchanges B and C of the long-distance network by two distinct trunk groups (see Figure 1). Each route carries peaked traffic originating in A. If one route is blocked the corresponding stream is allowed to use the other route. Hence, the traffic follows a mutual overflow routing scheme, called symmetric grading in the case of one line per group (cf. [22], [36, p. 39]).

In this paper we study the traffic behavior in this telecommunication model assuming exponentially distributed call holding times and two independent Markov modulated Poisson processes to be offered as arrival streams. Thus we extend the assumptions of previous studies (cf. [27], [22]).

We describe simple algorithms for the calculation of the relevant steady-state performance characteristics of the model. Considering such procedures for the computation of the stationary distribution of a homogeneous continuous-time Markov chain (CTMC), our analysis offers new insights into the behavior of iterative methods accelerated by aggregation–disaggregation (A/D) steps. We extend Haviv's error analysis for iterative A/D methods (cf. [16], [5]). Our results are a natural extension of Chatelin and Miranker's work [5] to the case of semiconvergent matrices.

This paper is structured as follows. In Section 2 we give a detailed mathematical description of the system by means of a Markovian model. Section 3

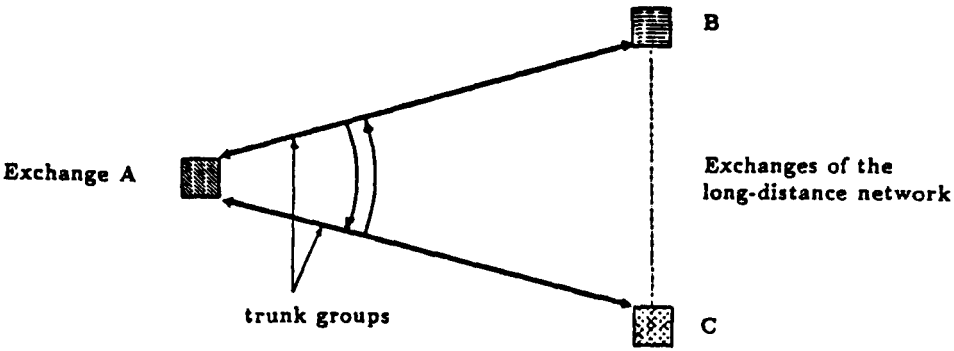

Figure 1 Network with mutual overflow routing.

provides the mathematical background for the calculation of the steady-state probability vector of a Markov chain. In Section 4 an extension of Schweitzer's A/D algorithm (cf. [40], [41]) to a larger class of iterative procedures is presented and Haviv's error analysis is improved. Furthermore, a stochastic interpretation of the block Gauss–Seidel method is provided. In Section 5 the proposed procedures are applied to the computation of the steady-state vector of the mutual overflow model. In Section 6 formulas are derived for the overall and individual call-congestion rates of the arrival streams.

2. A MATHEMATICAL MODEL OF THE OVERFLOW SYSTEM

The investigated network can be described by a loss system composed of two fully available trunk groups, called systems 1 and 2, with N_1 and N_2 lines. Following a standard approach in teletraffic theory (cf. [29], [30]), the offered traffic streams are modeled by two IPP-renewal processes resulting from a two-moment approximation of the peaked traffic streams 1 and 2. They will be represented by two mutually independent Markov modulated Poisson processes (MMPP) (see Appendix) with generator matrices

$$Q_1 = \begin{bmatrix} -\gamma_1 & \gamma_1 \\ \omega_1 & -\omega_1 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} -\gamma_2 & \gamma_2 \\ \omega_2 & -\omega_2 \end{bmatrix}$$

and rate vectors $\hat{\lambda}_1 = \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix}$ and $\hat{\lambda}_2 = \begin{bmatrix} \lambda_2 \\ 0 \end{bmatrix}$, respectively (cf. [29], [30], [28], [26]). Here, λ_i is the intensity of the Poisson process associated with the IPP stream i, $1/\gamma_i$ its mean on-time, and $1/\omega_i$ its mean off-time, $i \in \{1, 2\}$ (cf. [26, p. 438]).

The arrival streams 1 and 2 follow a mutual overflow routing scheme. This means that upon arrival at system 1 a call of flow 1, for instance, is searching for a free line. If possible, a free trunk is selected in a random manner and occupied. If system 1 is busy and there are free lines in system 2, the incoming call from flow 1 will immediately overflow to system 2 upon arrival and occupy a line selected at random. If both systems are busy, the call will be blocked and lost without further impact on the system (lost calls cleared) (see Figure 2).

Call holding times are supposed to be mutually independent, exponentially distributed random variables with a common finite mean $1/\mu$. They are also assumed to be independent of the arrival processes.

Let us denote the phase of the controlling CTMC of the MMPP $i \in \{1, 2\}$ at time $t \geq 0$ by $Y_i(t)$. Its associated irreducible generator is Q_i. The Markovian environment resulting from the composition of both arrival streams is given

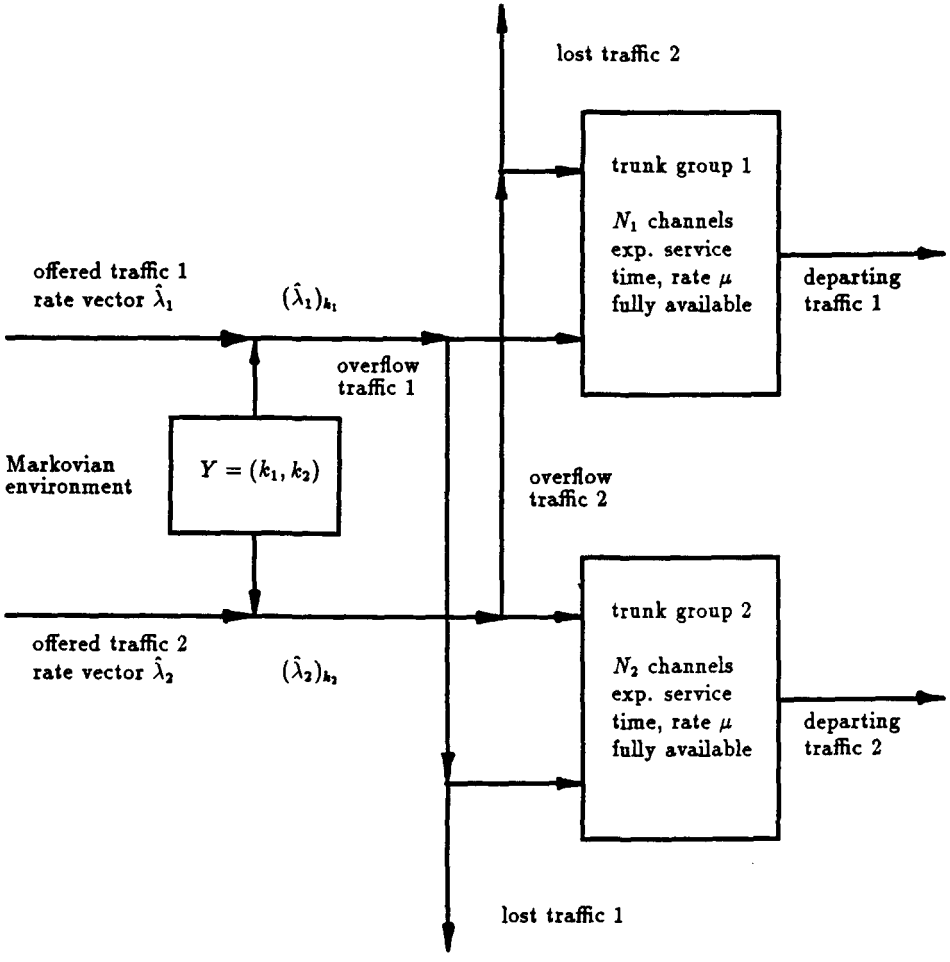

Figure 2 Queuing model of a loss system with mutual overflow in a Markovian environment.

by $Y(t) = (Y_1(t), Y_2(t))$. It possesses the irreducible generator

$$Q = Q_1 \oplus Q_2 = \begin{bmatrix} -\gamma_1 - \gamma_2 & \gamma_2 & \gamma_1 & 0 \\ \omega_2 & -\gamma_1 - \omega_2 & 0 & \gamma_1 \\ \omega_1 & 0 & -\omega_1 - \gamma_2 & \gamma_2 \\ 0 & \omega_1 & \omega_2 & -\omega_1 - \omega_2 \end{bmatrix} \in \mathbb{R}^{m \times m}$$

with $m = 4$. Its states $Y(t) = (k_1, k_2) \equiv k$ will be ordered lexicographically and enumerated by integers $k \in \{1,2,3,4\}$. The occupation of groups will be modeled by $X(t) = (X_1(t), X_2(t))$, $t \geq 0$, where the state variables $X_i(t)$ denote the number of busy trunks in the groups $i \in \{1,2\}$ at time t.

This overflow system in Markovian environment can be described by an irreducible CTMC $Z(t) = (X(t), Y(t))$, $t \geq 0$, with a finite state space $S = \{(i,j,k) \mid 1 \leq k \leq 4; 0 \leq i \leq N_1; 0 \leq j \leq N_2\}$. Its limiting distribution $P = (P_{ijk})_{i=0,\ldots,N_1; j=0,\ldots,N_2; k=1,\ldots,4}$, $P_{ijk} = \lim_{t\to\infty} P(Z(t) = (i,j,k))$, is the unique solution of the normalization condition $\sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \sum_{k=1}^{4} P_{ijk} = 1$ and the balance equations

$$\begin{aligned}
&[\Lambda(1 - \delta_{i,N_1}\delta_{j,N_2}) + i\mu I + j\mu I - Q^t] \cdot P_{ij} \\
&= (\Lambda_1 + \Lambda_2\delta_{j,N_2})(1 - \delta_{i,0}) \cdot P_{i-1j} + (i+1)\mu(1 - \delta_{i,N_1}) \cdot P_{i+1j} \\
&\quad + (\Lambda_2 + \Lambda_1\delta_{i,N_1})(1 - \delta_{j,0}) \cdot P_{ij-1} + (j+1)\mu(1 - \delta_{j,N_2}) \cdot P_{ij+1}
\end{aligned} \tag{1}$$

$0 \leq i \leq N_1$, $0 \leq j \leq N_2$, with $\delta_{k,l} = 1$ for $k = l$ and 0 otherwise. Here, $P_{ij}^t = (P_{ij1}, \ldots, P_{ij4})$ is the portion of the steady-state vector on the aggregate (i,j), $\Lambda_1 = \mathrm{Diag}(\hat{\lambda}_1) \otimes I_2 = \mathrm{Diag}(\lambda_1, \lambda_1, 0, 0)$, $\Lambda_2 = I_2 \otimes \mathrm{Diag}(\hat{\lambda}_2) = \mathrm{Diag}(\lambda_2, 0, \lambda_2, 0)$ and $\Lambda = \Lambda_1 + \Lambda_2$ are the arrival rate matrices. Assuming a lexicographical ordering of states, the steady-state probability vector $P^t = (P_{00}^t, P_{01}^t, \ldots, P_{0N_2}^t, \ldots, P_{N_10}^t, \ldots, P_{N_1N_2}^t)$ is the unique positive, normalized solution of the homogeneous linear system $A \cdot P = 0$. $A = -\tilde{Q}^t \in \mathbb{R}^{N \times N}$, $N = (N_1 + 1) \cdot (N_2 + 1) \cdot m$, is the negative transpose of the generator matrix $\tilde{Q}$ associated with $\{Z(t), t \geq 0\}$. It is an irreducible Q-matrix (cf. [37], [24]) having a block tridiagonal structure

$$A = \begin{bmatrix}
E_0 - B & D_0 & 0 & \ldots & \ldots & \ldots & 0 \\
B & E_1 - D_0 - B & D_1 & \ddots & \ldots & \ldots & \vdots \\
0 & B & \ddots & \ddots & \ddots & \ldots & \vdots \\
\vdots & \ddots & B & E_k - D_{k-1} - B & D_k & \ddots & \vdots \\
\vdots & \ldots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ldots & \ldots & \ddots & \ddots & \ddots & D_{N_1-1} \\
0 & \ldots & \ldots & \ldots & 0 & B & E_{N_1} - D_{N_1-1}
\end{bmatrix} \tag{2}$$

with

$$\begin{aligned}
B &= -(I_{N_2+1} \otimes \Lambda_1 + e_{N_2+1} \cdot e_{N_2+1}^t \otimes \Lambda_2) \\
D_i &= -(i+1) \cdot \mu \cdot I_L \qquad i = 0, \ldots N_1 - 1
\end{aligned}$$

where $L = (N_2 + 1) \cdot m$, and with irreducible block tridiagonal Q-matrices

$$E_i = \begin{bmatrix} -Q^t - G_i & T_0 & 0 & \dots & \dots & \dots & 0 \\ G_i & -Q^t - T_0 - G_i & T_1 & \ddots & \dots & \dots & \vdots \\ 0 & G_i & \ddots & \ddots & \ddots & \dots & \vdots \\ \vdots & \ddots & G_i & -Q^t - T_{k-1} - G_i & T_k & \ddots & \vdots \\ \vdots & \dots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & T_{N_2-1} \\ 0 & \dots & \dots & \dots & 0 & G_i & -Q^t - T_{N_2-1} \end{bmatrix}$$

for $i = 0, \dots N_1$ of order L along its diagonal, where

$$G_i = -(\Lambda_2 + \Lambda_1 \delta_{i,N_1})$$

$$T_j = -(j+1) \cdot \mu \cdot I_m \qquad j = 0, \dots N_2 - 1.$$

Here I_k is the identity matrix of order k and e_k the kth unit vector. Thus, A is a 2-cyclic consistently ordered Q-matrix with respect to this block partition (cf. [24], [45], [37], [3]). Taking advantage of the block structure of A, the steady-state vector P may be computed by a block iterative scheme derived from an R-regular splitting of A or an accelerated point iterative scheme such as JOR or SOR (cf. [24], [37], [30]).

3. MATHEMATICAL FOUNDATION OF COMPUTATIONAL METHODS FOR MARKOVIAN MODELS

In this section we provide the mathematical background for the calculation of the steady-state distribution p of a CTMC with finite state space $S = \{1, \dots, n\}$ and irreducible generator matrix $Q \in \mathbb{R}^{n \times n}$. We know that p is the unique positive solution of the linear system

$$A \cdot x = 0 \tag{3}$$

which satisfies additionally the normalization condition $e^t \cdot x = 1$. Here e denotes the vector with all ones and $A = -Q^t$ is the irreducible Q-matrix associated with Q (cf. [37], [24]).

In the following we adopt the notation of Berman, Plemmons [3, Chap. 2, p. 26] w.r.t. vector and matrix orderings: Let $x \in \mathbb{R}^n$, then $x \gg 0 \Leftrightarrow x_i > 0$ for each $i \in \{1, \dots, n\}$, $x > 0 \Leftrightarrow x_i \geq 0$ for each $i \in \{1, \dots, n\}$ and $x_j > 0$ for some $j \in \{1, \dots, n\}$, $x \geq 0 \Leftrightarrow x_i \geq 0$ for each $i \in \{1, \dots, n\}$.

Solving the system (3), we may employ either direct or iterative methods. The latter are usually based on a regular matrix splitting of the form $A = M - N$, $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$ (cf. [39, Def. 2.3, p. 410]). The associated nonnegative iteration matrix is $J = M^{-1} \cdot N$.

It is known that every regular splitting of an irreducible singular M-matrix is also graph compatible, weak regular (cf. [39, Def. 2.3, p. 410]). Thus we conclude from [39, Theorem 4.4, p. 420] and [3, Theorem 6.4.16, p. 146] that the spectral radius $\rho(J) = 1$ is a simple eigenvalue of J, especially $\text{index}_1(J) = 1$. Given a positive initial vector $x^{(0)}$, the convergence of the sequence

$$x^{(k+1)} = J \cdot x^{(k)} \qquad k = 0, 1, \ldots$$

to a solution of the eigenvalue problem

$$x = J \cdot x \tag{4}$$

is guaranteed if J is semiconvergent (cf. [3, p. 197]), that is, if J has no further eigenvalues on the unit circle apart from $\rho(J) = 1$. As this condition may be violated, it is necessary to enforce semiconvergence by proceeding to the extrapolated iteration matrix

$$J_\omega = (1-\omega)I + \omega J \qquad 0 < \omega < 1$$

It is well known that the sequence $\{x^{(k)} : k \in \mathbb{N}_0\}$ defined by the iterative scheme

$$x^{(k+1)} = J_\omega \cdot x^{(k)} \qquad k = 0, 1, \ldots, \tag{5}$$

called stationary first-order Richardson extrapolation, converges to a positive solution of equations (4) and (3), respectively, provided the initial vector $x^{(0)}$ is positive (cf. [2, p. 173]). The unique normalized solution coincides with p.

The convergence of the procedure (5) may be accelerated by inserting some aggregation–disaggregation (A/D) steps during the iteration (cf. [40], [41], [16], [4], [5], [34]). In order to apply Schweitzer's convergence result [41, Theorem 4, p. 328] in this context, we have to define a fallback procedure $x^{(k+1)} = T \cdot x^{(k)}$. It is based on a semiconvergent stochastic matrix T that converges to the normalized eigenvector x^* corresponding to the eigenvalue $\rho(T) = 1$ which is related to the stationary distribution p by some transformation (see equation (6); cf. [33, p. 126]).

Usually, neither J nor J_ω is stochastic. In order to construct a stochastic iteration matrix, we proceed to a nonnegative matrix T, called dual iteration matrix, by a similarity transformation:

$$T = M \cdot J \cdot M^{-1} = N \cdot M^{-1}$$

$$T_\omega = M \cdot J_\omega \cdot M^{-1} = (1-\omega)I + \omega T \qquad 0 < \omega < 1$$

Then T_ω, $0 < \omega \leq 1$, is column stochastic and $\|T_\omega\|_1 = \rho(T_\omega) = 1$ as well as $\text{index}_1(T_\omega) = 1$ hold. Furthermore, T_ω, $0 < \omega < 1$, is semiconvergent and $\rho(T_\omega) = 1$ is a simple eigenvalue. For all vectors $x > 0$ we conclude $T_\omega \cdot x > 0$.

PROPOSITION 1 Let $Q \in \mathbb{R}^{n\times n}$ be an irreducible generator matrix of a CTMC with steady-state distribution $p \gg 0$. Set $A = -Q^t$. Given a nontrivial regular splitting $A = M - N$, we denote the dual iteration matrix by $T = N \cdot M^{-1} \geq 0$ and set $T_\omega = (1-\omega)I + \omega T$ for some $0 < \omega < 1$. Then the sequence

$$x^{(k+1)} = T_\omega \cdot x^{(k)} \qquad k = 0, 1, \ldots$$

converges to a vector $x^* > 0$ with $e^t \cdot x^* = 1$, provided the initial vector $x^{(0)}$ satisfies $x^{(0)} > 0$, $e^t \cdot x^{(0)} = 1$. In this case the relation

$$p = \frac{M^{-1} \cdot x^*}{e^t \cdot M^{-1} \cdot x^*} \tag{6}$$

holds.

4. THE A/D ACCELERATED DUAL ITERATION

Let $T_\omega \geq 0$, $0 < \omega < 1$, be a stochastic, extrapolated dual iteration matrix corresponding to a regular splitting of the irreducible Q-matrix $A = M - N \in \mathbb{R}^{n\times n}$. Subsequently the subscript ω will be omitted. Let $x^* > 0$ denote the unique normalized eigenvector corresponding to the spectral radius $\rho(T) = 1$.

In this section we show that Schweitzer's iterative A/D procedure (IAD) and the corresponding convergence result [41, Theorem 4, p. 328] may be applied beyond point or block Jacobi and Gauss–Seidel splittings to a larger class of regular splittings. First we extend Haviv's error analysis of the IAD method (cf. [16]).

4.1 Error Analysis

We choose a partition $\Gamma = \{J_1, \ldots, J_m\}$ of the state space $S = \{1, \ldots, n\}$ into $m \geq 2$ disjoint sets J_i with $n_i \geq 1$ elements each. Without loss of generality we assume the elements of these sets to be enumerated in a consecutive order such that $i < j$ holds if $i \in J_l$, $j \in J_k$, $l < k$. Furthermore, w.l.g. let T and $x^{*t} = (x_1^{*t}, \ldots, x_m^{*t})$ be arranged according to this state space partition and ordering.

Following the approach of Chatelin and Miranker (cf. [5], [16], [4]), we define an aggregation matrix $R \in \mathbb{R}^{m\times n}$ by

$$R_{ij} = \begin{cases} 1 & \text{if } j \in J_i \\ 0 & \text{otherwise} \end{cases} \qquad 1 \le i \le m, 1 \le j \le n \tag{7}$$

For a fixed vector $x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} > 0$ with $e^t \cdot x = 1$ the prolongation matrix $P_{(x)} \in \mathbb{R}^{n\times m}$ is given by

$$P_{(x)_{ij}} = \begin{cases} (y_j)_i & \text{if } i \in J_j \\ 0 & \text{otherwise} \end{cases} \qquad 1 \le i \le n, 1 \le j \le m \tag{8}$$

where the vector $y = y_{(x)} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} > 0$ is defined for $j \in \{1,\dots,m\}$ as follows:

$$\mathbb{R}^{n_j} \ni y_{(x)_j} = \begin{cases} x_j/\alpha_{(x)_j} & \text{if } x_j > 0 \\ 1/n_j \cdot e & \text{if } x_j = 0 \end{cases}$$

$$\alpha_{(x)_j} = e^t \cdot x_j$$

According to the construction, the relations $e^t \cdot y(x)_j = 1$ and $x_j = \alpha_{(x)_j} \cdot y_{(x)_j}$ hold for each $j \in \{1,\dots,m\}$. Also, $x > 0$, $e^t \cdot x = 1$ implies $e^t \cdot \alpha = 1$ for $\alpha = \alpha_{(x)} = (\alpha_{(x)_1},\dots,\alpha_{(x)_m})^t > 0$. Hence, the matrices

$$R = \begin{bmatrix} e^t & 0 & \dots & 0 \\ 0 & e^t & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^t \end{bmatrix} > 0 \qquad P_{(x)} = \begin{bmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & y_m \end{bmatrix} > 0$$

satisfy $e^t \cdot P_{(x)} = e^t$, $e^t \cdot R = e^t$, and $R \cdot P_{(x)} = I$. For fixed $x > 0$ we define the nonnegative projection matrix by

$$\Pi = \Pi_{(x)} = P_{(x)} \cdot R = \begin{bmatrix} y_1 e^t & 0 & \dots & 0 \\ 0 & y_2 e^t & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & y_m e^t \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{9}$$

Then Π is column stochastic, that is, $e^t \cdot \Pi = e^t$. Furthermore, $R \cdot x = \alpha_{(x)}$ and $P_{(x)} \cdot \alpha_{(x)} = x$ induce $\Pi_{(x)} \cdot x = x$. $\Pi_{(x)} \cdot \Pi_{(x)} = \Pi_{(x)}$ implies that the matrix $\Pi_{(x)}$ is a projection onto the subspace $\Pi_{(x)}(\mathbb{R}^n)$ along $\mathrm{Kern}(\Pi_{(x)})$ (cf. [5, p. 20]).

Defining the projection of the dual iteration matrix T by

$$G = G_{(x)} = \Pi_{(x)} \cdot T$$

we see that $G \geq 0$ is a stochastic matrix satisfying $\rho(G) = 1$ and $\mathrm{index}_1(G) = 1$. Hence, the group inverse $(I - G)^{\#} = (I - \Pi_{(x)} \cdot T)^{\#}$ corresponding to $I - G$ exists (cf. [31, p. 445], [3, p. 119]). Obviously, the related projection matrix

$$F = F_{(x)} = T \cdot \Pi_{(x)}$$

is a stochastic matrix, too. Denoting the aggregated iteration matrix by

$$B = B_{(x)} = R \cdot T \cdot P_{(x)} \in \mathbb{R}^{m \times m} \tag{10}$$

it follows that $B \geq 0$ is stochastic. Hence, there exists a vector $\alpha_{(x)} > 0$ in $\mathbb{R}^m$ satisfying

$$B_{(x)} \cdot \alpha_{(x)} = \alpha_{(x)} \tag{11}$$

and $e^t \cdot \alpha_{(x)} = 1$. The disaggregated vector $\tilde{x}_{(x)} \in \mathbb{R}^n$ defined by

$$\tilde{x}_{(x)} = P_{(x)} \cdot \alpha_{(x)} \geq 0 \tag{12}$$

fulfills $e^t \cdot \tilde{x}_{(x)} = 1$ and $\Pi_{(x)} \cdot \tilde{x}_{(x)} = P_{(x)} \cdot \alpha_{(x)} = \tilde{x}_{(x)}$. Thus, there exists a nonnegative vector $\tilde{x}^* \in \Pi_{(x)}(\mathbb{R}^n)$ with $\Pi_{(x)} \cdot (I - T) \cdot \Pi_{(x)} \cdot \tilde{x}^* = 0$ and $e^t \cdot \tilde{x}^* = 1$. This solution approach may be interpreted as a Galerkin approximation in $\Pi_{(x)}(\mathbb{R}^n)$ (cf. [5, p. 20ff]). Obviously, the solution $\tilde{x}_{(x)}$ fulfills

$$(I - \Pi_{(x)} \cdot T) \cdot \tilde{x}_{(x)} = 0$$

As x^* is the unique nonnegative, normalized eigenvector of T corresponding to the spectral radius $\rho(T) = 1$, we conclude that

$$(I - \Pi_{(x)} \cdot T) \cdot (x^* - \tilde{x}_{(x)}) = (I - \Pi_{(x)}) \cdot (x^* - x) \tag{13}$$

holds (cf. [16, p. 954], [5, (4.3), p. 31]).

An A/D step consists of an aggregation step (11) followed by a disaggregation step (12). Equation (13) provides a simple relationship between the errors before an A/D step, $\epsilon = x^* - x$, and after an A/D step, $\tilde{\epsilon} = x^* - \tilde{x}_{(x)}$.

The following proposition reveals a relation between the steady-state vectors of the stochastic matrices G, F, and B. Here, Kern denotes the nullspace of a linear operator.

PROPOSITION 2 Let $T \in \mathbb{R}^{n \times n}$ be a stochastic matrix. Suppose $x \in \mathbb{R}^n$ satisfies $x > 0$ and $e^t \cdot x = 1$. Define the prolongation matrix $P_{(x)} \in \mathbb{R}^{n \times m}$, the

restriction matrix $R \in \mathbb{R}^{m \times n}$, and the projection matrix $\Pi_{(x)} \in \mathbb{R}^{n \times n}$ according to equations (8), (7), and (9).

Then the stochastic matrices $\Pi_{(x)} \cdot T$, $T \cdot \Pi_{(x)}$, and $R \cdot T \cdot P_{(x)}$ fulfill the following relations:

$$\begin{aligned}
\text{Kern}(I - \Pi_{(x)} \cdot T) &= P_{(x)}(\text{Kern}(I - R \cdot T \cdot P_{(x)})) \\
R(\text{Kern}(I - \Pi_{(x)} \cdot T)) &= \text{Kern}(I - R \cdot T \cdot P_{(x)}) \\
T(\text{Kern}(I - \Pi_{(x)} \cdot T)) &= \text{Kern}(I - T \cdot \Pi_{(x)}) \\
\Pi_{(x)}(\text{Kern}(I - T \cdot \Pi_{(x)})) &= \text{Kern}(I - \Pi_{(x)} \cdot T) \\
\text{Kern}(I - \Pi_{(x)} \cdot T) &\subseteq \text{Kern}(I - \Pi_{(x)})
\end{aligned}$$

Standard results from linear algebra yield the following relations.

REMARK 1

$$\begin{aligned}
\text{Dim}(T(\text{Kern}(I - \Pi_{(x)} \cdot T))) &= \text{Dim}(\text{Kern}(I - \Pi_{(x)} \cdot T)) \\
\text{Dim}(R(\text{Kern}(I - \Pi_{(x)} \cdot T))) &= \text{Dim}(\text{Kern}(I - \Pi_{(x)} \cdot T)) \\
\text{Dim}(P_{(x)}(\text{Kern}(I - R \cdot T \cdot P_{(x)}))) &= \text{Dim}(\text{Kern}(I - R \cdot T \cdot P_{(x)})) \\
\text{Dim}(\Pi_{(x)}(\text{Kern}(I - T \cdot \Pi_{(x)}))) &= \text{Dim}(\text{Kern}(I - T \cdot \Pi_{(x)}))
\end{aligned}$$

PROPOSITION 3 Suppose the assumptions of Proposition 2 hold. Then the following conditions are equivalent:

$$\text{Dim}(\text{Kern}(I - \Pi_{(x)} \cdot T)) = 1 \tag{14}$$

$$\text{Dim}(\text{Kern}(I - R \cdot T \cdot P_{(x)})) = 1 \tag{15}$$

$$\text{Dim}(\text{Kern}(I - T \cdot \Pi_{(x)})) = 1 \tag{16}$$

In this case, there exist vectors $\tilde{x}^* \in \mathbb{R}^n$, $z^* \in \mathbb{R}^n$, $\alpha^* \in \mathbb{R}^m$ satisfying

$$\begin{aligned}
&\tilde{x}^* > 0 \quad e^t \cdot \tilde{x}^* = 1 \quad z^* > 0 \quad e^t \cdot z^* = 1 \\
&\alpha^* > 0 \quad e^t \cdot \alpha^* = 1
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\text{Kern}(I - \Pi_{(x)} \cdot T) &= \text{span}(\tilde{x}^*) \\
\text{Kern}(I - R \cdot T \cdot P_{(x)}) &= \text{span}(\alpha^*) \\
\text{Kern}(I - T \cdot \Pi_{(x)}) &= \text{span}(z^*)
\end{aligned} \tag{18}$$

These vectors fulfill the relations:

$$\tilde{x}^* = P_{(x)} \cdot \alpha^* \quad \alpha^* = R \cdot \tilde{x}^* \quad \tilde{x}^* = \Pi_{(x)} \cdot z^* \quad z^* = T \cdot \tilde{x}^* \tag{19}$$

As the group inverse of the stochastic matrix $G = \Pi_{(x)} \cdot T$ exists (cf. [31, Theorem 2.1, p. 445]), the matrix

$$W = W_{(x)} = I - (I - G_{(x)}) \cdot (I - G_{(x)})^{\#}$$

is well defined and a projection onto $\text{Range}(W) = \text{Kern}(I - G_{(x)})$ along $\text{Kern}(W) = \text{Range}(I - G_{(x)})$. Furthermore, $e^t \cdot W = e^t$ and $W \cdot \bar{x}^* = \bar{x}^*$ hold for the steady-state vector $\bar{x}^*$ of $G_{(x)}$.

PROPOSITION 4 Suppose the assumptions of Proposition 2 and one of the conditions (14), (15), or (16) in Proposition 3 are fulfilled. Then

$$W = I - (I - \Pi_{(x)}T) \cdot (I - \Pi_{(x)}T)^{\#} = \bar{x}^* \cdot e^t = P_{(x)} \cdot \alpha^*_{(x)} \cdot e^t > 0 \qquad (20)$$

holds, where $\bar{x}^*$ and $\alpha^*_{(x)}$ satisfy the relations (17), (18), and (19) in Proposition 3. Furthermore, W is stochastic.

Proof. Cf. [32, Lemma 1, p. 142], [31, Theorem 2.3, p. 449], [15, Lemma 2.5, p. 37].

As the vectors x^* and $\bar{x}_{(x)}$ from (12) are normalized, the error $\bar{\epsilon}$ after an A/D step satisfies $e^t \cdot \bar{\epsilon} = 0$. Taking into account equation (13), we are now able to prove an extension of the error result [5, (4.3), p. 31] derived by Chatelin and Miranker.

THEOREM 1 Suppose the assumptions of Proposition 2 and one of the conditions (14), (15), or (16) in Proposition 3 hold. Let x^* be the stationary distribution of T. Denote the error before and after an A/D step by $\epsilon = x^* - x$ and $\bar{\epsilon} = x^* - \bar{x}_{(x)}$, respectively.

Then

$$\bar{\epsilon} = (I - \Pi_{(x)} \cdot T + W)^{-1} \cdot (I - \Pi_{(x)}) \cdot \epsilon \qquad (21)$$

$$= (I - \Pi_{(x)} \cdot T)^{\#} \cdot (I - \Pi_{(x)}) \cdot \epsilon \qquad (22)$$

hold with W as in equation (20). $Z_{(\bar{x}^*)} = (I - \Pi_{(x)} \cdot T + W)^{-1}$ is the fundamental matrix associated with the stochastic matrix $\Pi_{(x)} \cdot T$.

Proof. Equation (21) immediately follows from the assumptions, equation (13), and Proposition 4. In this case we conclude from [15, Corollary 3.3, p. 83] that the generalized fundamental matrix $(I - \Pi_{(x)} \cdot T + u \cdot e^t)^{-1}$ exists (cf. [19]). It has the representation

$$(I - \Pi_{(x)} \cdot T + u \cdot e^t)^{-1} = (I - \Pi_{(x)} \cdot T)^{\#} \cdot (I - u \cdot e^t) + \bar{x}^* \cdot e^t \qquad (23)$$

for any vector u satisfying $e^t \cdot u = 1$. Hence, equations (20) and (21) yield equation (22).

Suppose the assumptions of Theorem 1 hold. Then the existence of the generalized fundamental matrix $(I - \Pi_{(x)} \cdot T + u \cdot e^t)^{-1}$ is guaranteed for any vector u provided $e^t \cdot u \neq 0$ holds (cf. [19]). Hence, equation (23) yields an explicit representation of the error $\bar{\epsilon}$ in terms of the initial vector x.

COROLLARY 1 Suppose the assumptions of Theorem 1 hold. Let $x > 0$, $e^t \cdot x = 1$ be the initial vector and let u be an arbitrary vector satisfying $e^t \cdot u = 1$. Then

$$\bar{\epsilon} = (I - \Pi_{(x)} \cdot T + u \cdot e^t)^{-1} \cdot (I - \Pi_{(x)}) \cdot \epsilon$$

especially

$$\bar{\epsilon} = (I - \Pi_{(x)} \cdot T + x \cdot e^t)^{-1} \cdot (I - \Pi_{(x)}) \cdot \epsilon \tag{24}$$

hold.

The error formula (22) has been derived by Haviv (cf. [16, Theorem, p. 954]). The equivalent representations (21) and (24) have the advantage of being easily computable. They reveal the influence of the initial vector x, the prolongation matrix $P_{(x)}$, and the projection matrix $\Pi_{(x)}$ onto the error $\bar{\epsilon}$ after an A/D step. Haviv [16] has imposed the condition that $\Pi_{(x)}$ is irreducible in order to guarantee equation (22). The next result shows that this is the strongest condition, which is of little practical interest.

LEMMA 1 Suppose the assumptions of Proposition 2 hold. Furthermore, assume $\mathrm{Dim}(\mathrm{Kern}(I - \Pi_{(x)})) = 1$ is satisfied. A sufficient condition is the irreducibility of $\Pi_{(x)}$.

Then the conditions (14), (15), and (16) are satisfied. Moreover,

$$\mathrm{Kern}(I - \Pi_{(x)} \cdot T) = \mathrm{Kern}(I - \Pi_{(x)}) = \mathrm{span}(x)$$

and $\bar{x}^* = x$ hold where $\bar{x}^*$ is the unique stationary distribution of $\Pi_{(x)} \cdot T$.

Proof. The result immediately follows from Proposition 2 and $\Pi_{(x)} \cdot x = x$.

An important feature of the iterative A/D procedure is the error reduction that is achieved by inserting an A/D step (11) and (12) during the iteration $x^{(k+1)} = T \cdot x^{(k)}$. This improvement has been computed by Chatelin and Miranker for the case of a convergent iteration matrix T (cf. [5, section 4.2.4, p. 36f]). The next theorem extends their result to semiconvergent iteration matrices.

THEOREM 2 Suppose the assumptions of Proposition 2 and one of the conditions (14), (15), or (16) in Proposition 3 are satisfied.

Then $\mathrm{Kern}(I - T \cdot \Pi_{(x)}) = \mathrm{span}(z^*)$ holds where z^* is the unique stationary distribution of $T \cdot \Pi_{(x)}$.

Let $\epsilon = x^* - x$ be the error before an A/D step and $\hat{\epsilon} = x^* - \hat{x}_{(x)}$ the error after an A/D step (11) and (12) followed by an iteration step $\hat{x} = \hat{x}_{(x)} = T \cdot \tilde{x}_{(x)}$. Then

$$\hat{\epsilon} = (I - T \cdot \Pi_{(x)} + H)^{-1} \cdot T \cdot (I - \Pi_{(x)}) \cdot \epsilon \tag{25}$$

$$= (I - T \cdot \Pi_{(x)})^{\#} \cdot T \cdot (I - \Pi_{(x)}) \cdot \epsilon \tag{26}$$

with

$$H = I - (I - T \cdot \Pi_{(x)}) \cdot (I - T \cdot \Pi_{(x)})^{\#} = z^* \cdot e^t \tag{27}$$

follows. Moreover, $z^* = \hat{x}_{(x)}$ holds.

Proof. The assumptions imply $\mathrm{Dim}(\mathrm{Kern}(I - \Pi_{(x)} \cdot T)) = 1$. Proposition 2 yields $\Pi_{(x)}(\mathrm{Kern}(I - T \cdot \Pi_{(x)})) = \mathrm{Kern}(I - \Pi_{(x)} \cdot T) = \mathrm{span}(\tilde{x}^*)$ where $\tilde{x}^*$ is the unique stationary distribution of $\Pi_{(x)} \cdot T$. According to Proposition 3 $\mathrm{Kern}(I - T \cdot \Pi_{(x)})$ is generated by the unique stationary distribution z^* of $T \cdot \Pi_{(x)}$. Furthermore, the projection matrix H exists and satisfies equation (27).

From $\tilde{x}^* = \Pi_{(x)} \cdot \tilde{x}^*$ we conclude $\Pi_{(x)} \cdot \hat{x} = \Pi_{(x)} \cdot T \cdot \Pi_{(x)} \cdot \tilde{x}^* = \tilde{x}^*$. Hence, $\hat{x} = T \cdot \Pi_{(x)} \cdot \hat{x}$ holds, implying $z^* = \hat{x}_{(x)} > 0$, as $e^t \cdot \hat{x}_{(x)} = 1$, and

$$(I - T \cdot \Pi_{(x)}) \cdot (x^* - \hat{x}_{(x)}) = T \cdot (I - \Pi_{(x)}) \cdot (x^* - x)$$

This equation yields equation (25) if we take into account $H \cdot \hat{\epsilon} = z^* \cdot e^t \cdot (x^* - \hat{x}_{(x)}) = 0$. The proof of equation (26) follows along the lines of equation (23).

COROLLARY 2 Suppose the assumptions of Theorem 2 hold. Let $x > 0$, $e^t \cdot x = 1$ be the initial vector and let u be an arbitrary vector satisfying $e^t \cdot u = 1$. Then

$$\hat{\epsilon} = (I - T \cdot \Pi_{(x)} + u \cdot e^t)^{-1} \cdot T \cdot (I - \Pi_{(x)}) \cdot \epsilon$$

especially

$$\hat{\epsilon} = (I - T \cdot \Pi_{(x)} + x \cdot e^t)^{-1} \cdot T \cdot (I - \Pi_{(x)}) \cdot \epsilon \tag{28}$$

hold.

The gain of an iteration step following the A/D step may be computed by the ratio $\hat{\beta}$ of magnitudes of the errors $\hat{\epsilon} = x^* - \hat{x}_{(x)}$ and $\tilde{\epsilon} = x^* - \tilde{x}_{(x)}$. Taking into account $(I - \Pi_{(x)})^2 = I - \Pi_{(x)}$, we conclude from equations (28) and (13):

$$\hat{\beta} = \frac{\|x^* - \hat{x}_{(x)}\|}{\|x^* - \tilde{x}_{(x)}\|} = \frac{\|\hat{\epsilon}\|}{\|(I - \Pi_{(x)})\epsilon\|} \frac{\|(I - \Pi_{(x)})\epsilon\|}{\|\tilde{\epsilon}\|}$$

$$\leq \|(I - T \cdot \Pi_{(x)} + x \cdot e^t)^{-1}\| \cdot \|T \cdot (I - \Pi_{(x)})\| \cdot \|I - \Pi_{(x)} \cdot T\| \tag{29}$$

This upper bound (29) extends the corresponding result [5, (4.18), p. 37] of Chatelin and Miranker.

Analyzing the proofs, we recognize that only the assumptions $P_{(x)} \geq 0$, $e^t \cdot P_{(x)} = e^t, R \geq 0, e^t \cdot R = e^t, R \cdot P_{(x)} = I, \Pi_{(x)} = P_{(x)} \cdot R \geq 0, e^t \cdot \Pi_{(x)} = e^t$, and $\Pi_{(x)} \cdot x = x$ have to be imposed on the prolongation, aggregation, and projection matrix to guarantee our results. Thus, the error representations (21), (22), (24), (25), (26), and (28) also hold for partial aggregation methods such as the methods e and f in Haviv [16, p. 957f] (see also [44]).

4.2 Convergence Results

First we define a continuous semi-norm in $\mathbb{R}^n$ by

$$r(x) = \|(I - T) \cdot x\|_1 \qquad x \in \mathbb{R}^n$$

and a compact set $K = \{x \in \mathbb{R}^n : x \geq 0, e^t \cdot x = 1\}$. Let $L = \{x \in K : r(x) = 0\} = \{x \in K : x = T \cdot x\} = \{x^*\}$. We define a family of compact sets by $K_\varepsilon = \{x \in K : r(x) \leq \varepsilon\}$, $\varepsilon > 0$. Obviously, $\lim_{n\to\infty} K_{1/n} = \bigcap_{n>0} K_{1/n} = L$. According to Proposition 1 the scheme

$$x^{(k+1)} = T \cdot x^{(k)} \qquad k = 0, 1, \ldots$$

is a convergent fallback procedure for any $x^{(0)} > 0$, and $r(Tx) \leq r(x)$, $x \in \mathbb{R}^n$, holds due to $\|T\|_1 = \rho(T) = 1$. Now we may construct an iterative A/D algorithm according to the scheme of Schweitzer and Kindle for the generator Q of a CTMC (cf. [41, p. 326f]).

Dual IAD Algorithm

Assumption: Let $A = M - N$ be a regular splitting of the irreducible Q-matrix $A = -Q^t \in \mathbb{R}^{n\times n}$ with extrapolated dual iteration matrix $T = I - \omega A M^{-1}$ for some $\omega \in (0, 1)$. Select a partition $\Gamma = \{J_1, \ldots, J_m\}$ of $\{1, \ldots, n\}$ into $m \geq 2$ disjoint sets.

1. Initialization:
   Select an initial vector $x^{(0)} \gg 0$, $e^t x^{(0)} = 1$, and three real numbers $0 < \varepsilon, c_1, c_2 < 1$. Construct the matrices $P_{(x)} \in \mathbb{R}^{n\times m}$, $R \in \mathbb{R}^{m\times n}$, $T \in \mathbb{R}^{n\times n}$, and $B_{(x)} \in \mathbb{R}^{m\times m}$ according to equations (8), (7), and (10). Set $k = 0$.
2. A/D step:

Solve	$B_{(x^{(k)})} \cdot \alpha_{(x^{(k)})} = \alpha_{(x^{(k)})}$
subject to	$e^t \cdot \alpha_{(x^{(k)})} = 1, \alpha_{(x^{(k)})} > 0$
and compute	$\tilde{x} = P_{(x^{(k)})} \cdot \alpha_{(x^{(k)})}$

3. Iteration step:
   Compute $x^{(k+1)} = T \cdot \tilde{x}$

4. Convergence test:
   If $r(\tilde{x}) \leq c_1 \cdot r(x^{(k)})$
   then go to step 5
   else compute $x^{(k+1)} = T^m \cdot x^{(k)}$
   with $m = m(x^{(k)}) \in \mathbb{N}$ such that
   $r(x^{(k+1)}) \leq c_2 \cdot r(x^{(k)})$
   endif
5. Termination test:
   If $\|x^{(k+1)} - x^{(k)}\|_1 / \|x^{(k)}\|_1 = \|x^{(k+1)} - x^{(k)}\|_1 < \varepsilon$
   then go to step 6
   else k=k+1
   go to step 2
   endif
6. Normalization:

$$p = \frac{M^{-1} \cdot x^{(k+1)}}{e^t \cdot M^{-1} \cdot x^{(k+1)}}$$

From the mathematical point of view this algorithm is reasonable as seen by the following lemma.

LEMMA 2 Let Q be the irreducible generator matrix of a CTMC and choose a regular splitting $A = M - N$ of the Q-matrix $A = -Q^t$, such as an M-splitting like the block Gauss–Seidel splitting $A = D - L - U$. $T_1 = NM^{-1}$ is the corresponding dual stochastic iteration matrix and $T = (1-\omega)I + \omega T_1$, $0 < \omega < 1$, the extrapolated variant.

Then the dual IAD algorithm converges to the steady-state distribution p of Q for any initial vector $x^{(0)} \gg 0$ with $e^t x^{(0)} = 1$.

Moreover, $r(x^{(k+1)}) \leq \max(c_1, c_2) \cdot r(x^{(k)})$, $k \geq 0$ holds.

Proof. The proof follows the lines of Schweitzer and Kindle [41, Theorem 4].

4.3 Stochastic Interpretation of the Block Gauss–Seidel Method

The IAD procedure may be applied to any regular splitting of $A = -Q^t$, especially to a point or block Gauss–Seidel splitting $A = (D-L)-U$ that generates an M-splitting. Hence, the results are still valid if we apply the procedure to a variant of Rose's R-regular block splitting (cf. [37], [24, Def. 14, p. 67]).

Mitra and Tsoucas [33] have pointed out that, in analogy to the point Jacobi method, the point Gauss–Seidel procedure has a probabilistic interpretation (cf. [24, p. 56]). They have shown that the iterative scheme evolves identically

in law with a related DTMC, called high-stepping random walk. It is obtained by observing the embedded jump chain associated with the original CTMC at selected instants only. As the derivation of this interpretation is, to our point of view, somewhat cumbersome, we shall describe an equivalent but rather simple approach that is also applicable to the block Gauss–Seidel procedure.

Let $Q = (L + U) - D$ be the irreducible generator matrix of a CTMC that is partitioned into $m \geq 2$ blocks. Let $D = \mathrm{Diag}(D_{11}, \ldots, D_{mm})$ be the block diagonal, and $L \geq 0$ the strictly lower and $U \geq 0$ the strictly upper block triangular part of Q. As the diagonal blocks D_{ii}, $i = 1, \ldots, m$, are regular, D is a regular M-matrix. We construct a generalized jump chain with t.p.m. $P = D^{-1}(L+U)$. If a point partition is chosen, P is the t.p.m. of the associated jump chain embedded at successive transition epochs.

Now we construct a new DTMC by associating with each state i of the jump chain a new state i'. The original states, called up-states, are collected in a set S. The new states, called down-states, constitute a set S'. The block partition induces a natural aggregation of the states. The transition behavior of the new chain is specified as follows: Considering the natural ordering of numbers, only increasing transitions between up-state aggregates are possible. The transition probability for a transition from i to j is determined by the corresponding probability $(D^{-1}U)_{ij}$ of the jump chain. Each down-state i' can reach its corresponding up-state i with probability 1, but no other states. Hence, transitions are not allowed between down-states. Each down-state i' can be reached from an up-state j with probability $(D^{-1}L)_{ji}$. Thus the t.p.m. P^{udc} of the new chain has the following form:

$$P^{udc} = \begin{array}{c} \begin{matrix} S' & S \end{matrix} \\ \begin{bmatrix} 0 & I \\ D^{-1}L & D^{-1}U \end{bmatrix} \end{array} \begin{matrix} S' \\ S \end{matrix}$$

Now we construct the reduced Markov chain on the set S' of down-states according to Grassmann's state reduction approach (cf. [11], [24, p. 28ff]). This means that we observe the new chain only during its visits in the set of down-states. Hence, the t.p.m. of this reduced DTMC is given by (cf. [11], [38, p. 261])

$$P_{S'} = (I - D^{-1}U)^{-1} \cdot D^{-1}L$$

Regarding the original generalized jump chain, this procedure has a simple stochastic interpretation. First, we divide each sample path of the jump chain into consecutive sections of a (perhaps empty) monotone increasing sequence of aggregates followed by a single nonincreasing step with respect to the natural ordering of aggregates. The length of the increasing path may be zero if two nonincreasing transitions between aggregates follow each other. Let us

assume that i and j are two successive states reached after such an excursion. Then the probability for such a transition from i to j is given by $(P_{S'})_{ij}$.

If we associate the M-matrix $A = -Q^t = (D-(U+L))^t$ with the generator matrix Q, the iteration matrix of the forward Gauss–Seidel procedure, based on the splitting $A = M - N, M = D^t - U^t, N = L^t$, is defined by $J = (D^t - U^t)^{-1} \cdot L^t$. A similarity transformation with the matrix M yields

$$T_1 = M \cdot J \cdot M^{-1} = L^t \cdot (D^t - U^t)^{-1} = L^t \cdot D^{-t} \cdot (I - U^t \cdot D^{-t})^{-1}$$
$$= [(I - D^{-1}U)^{-1} \cdot D^{-1} \cdot L]^t = (P_{S'})^t$$

Hence, the dual iteration matrix T_1 corresponding to the Gauss–Seidel method has the same spectral properties as the stochastic matrix $P_{S'}$ associated with the constructed DTMC.

The extrapolated version $T_\omega = (1-\omega)I + \omega T_1, 0 < \omega < 1$, called stationary first-order Richardson extrapolation, is related to the t.p.m. of a modified generalized jump chain. This modification stems from damping the transition probabilities of the jump chain by a term ω and adding new transitions from each state into itself with probabilities $1-\omega$ (cf. [20, Chap. V: Ergodic Markov Chains, Theorem 5.1.1, p. 99], [24, section 6.4]).

5. CALCULATION OF THE STATIONARY DISTRIBUTION OF THE OVERFLOW MODEL

As no simple analytical expression for the stationary distribution P of the Markov chain $\{Z(t), t \geq 0\}$ is available, the steady-state vector has to be calculated as solution of the balance equations (1) by appropriate direct or iterative numerical methods (cf. [24], [18], [9], [11]). All procedures discussed subsequently are based on the Q-matrix representation (2) according to Section 3.

Considering the sparsity of the generator matrix $\tilde{Q}$, we restrict our attention to iterative solution techniques. Due to the ease and efficiency of implementation, especially the use of vectorizable and parallel executable algorithms, standard procedures based on point splittings $A = D-L-U$ such as the point Jacobi and Gauss–Seidel methods have again attracted considerable attention. Obviously, the latter are nontrivial M-splittings.

It can be shown that $A = -\tilde{Q}^t$ is an irreducible, 2-cyclic matrix with respect to the point partition and has property A. Thus, the point Jacobi matrix $J = D^{-1}(L + U)$ is cyclic of index 2, hence not semiconvergent (cf. [3, Theorem 2.2.30, p. 35], [24, Theorem 29, p. 56]), whereas the corresponding JOR and SOR procedures with the iteration matrices $J_\omega = (1 - \omega)I + \omega J$ and $L_\omega = (D - \omega L)^{-1}((1 - \omega)D + \omega U)$ are convergent for each relaxation parameter

$\omega \in (0,1)$ (cf. [24 p. 73], [2, p. 174], [1, Cor. 3, p. 395], [17, Theorem 3.4, p. 191]).

As $M_{21} = (D - L)_{21} = -\gamma_2 \neq 0$ and $N_{12} = U_{12} = \omega_2 \neq 0$ hold, we conclude from [39, Cor. 3.8, p. 417] that the point Gauss–Seidel procedure is convergent (see also [37, Cor. 2, p. 139], [24, Theorem 35, p. 66]). This point iteration may be accelerated by applying the standard relaxation technique (cf. [18]) or by employing a semi-iterative technique such as the stationary or nonstationary Chebyshev method or Eiermann's stationary fourth-order scheme (cf. [45, section 5], [6], [35], [7, Lemma 8.4, p. 28], [2], [12], [13]). The main difficulty with respect to these procedures is the determination of "optimal" relaxation parameters. As there is no a priori information about the location of the eigenvalues of the iteration matrices, heuristic procedures estimating approximately optimal parameters seem to be the only practicable approach (cf. [14, section 9.5, p. 223ff], [42], [12], [13]).

An alternative is given by block iterative schemes such as the block Gauss–Seidel procedure or its modified versions based on Rose's R-regular splitting (cf. [37], [30], [25]). Due to [37, Cor. 2, p. 139] the block Gauss–Seidel scheme derived from the given block tridiagonal structure (2) of $-\tilde{Q}^t$ is convergent as the diagonal blocks are irreducible regular M-matrices. All methods may be combined with A/D steps if Schweitzer's IAD procedure is used (see Section 4.2; cf. [34], [21], [44]).

6. COMPUTATION OF THE CONGESTION RATES

In teletraffic theory the most important steady-state performance characteristics of a loss system are the congestion rates. Recall that we distinguish between time-congestion and call-congestion rates with respect to the arrival streams of a model. In the subsequent section we assume the reader to be familiar with the notions of time-stationary and customer-stationary distributions and time congestion as well as customer-dependent and average call congestion of a $G_n/M/N/0$ model (cf. [8, p. 58ff], [10, section 11, p. 517ff]).

Furthermore, we assume the vector process $Z(t) = (X_1(t), X_2(t), Y(t))$ to be in steady state. Its steady-state distribution is denoted by P.

6.1 Overall Congestion Rates

The steady-state distribution Π of the occupation process $X(t) = (X_1(t), X_2(t))$ is given by the marginal distribution

$$\Pi_{ij} = \lim_{t \to \infty} P(X(t) = (i,j)) = e^t \cdot P_{ij}$$

Thus, the overall time-congestion of the overflow model is $\Pi_{N_1N_2} = e^t P_{N_1N_2}$. The time-congestion rates of groups 1 and 2 are $\Pi_{N_1} = \sum_{j=0}^{N_2} e^t P_{N_1 j}$ and $\Pi_{N_2} = \sum_{i=0}^{N_1} e^t P_{iN_2}$.

Let $R_i, i \in \{1,2\}$, denote the overall call-congestion rate of MMPP i, that is, the steady-state probability that a call of the arrival stream i selected at random finds no free trunk in the overflow system upon its arrival. Following the lines of Meier-Hellstern [29], we conclude that these overall call-congestion rates have the form

$$R_1 = \frac{P^t \cdot I_{(N_1+1)(N_2+1)} \otimes \Lambda_1 \cdot (e_{(N_1+1)(N_2+1)} \otimes e)}{e^t \cdot I_{(N_1+1)(N_2+1)} \otimes \Lambda_1 \cdot P}$$

$$= \frac{P_{N_1N_2 1} + P_{N_1N_2 2}}{\sum_{i=0}^{N_1} \sum_{j=0}^{N_2} (P_{ij1} + P_{ij2})}$$

$$R_2 = \frac{P^t \cdot I_{(N_1+1)(N_2+1)} \otimes \Lambda_2 \cdot (e_{(N_1+1)(N_2+1)} \otimes e)}{e^t \cdot I_{(N_1+1)(N_2+1)} \otimes \Lambda_2 \cdot P}$$

$$= \frac{P_{N_1N_2 1} + P_{N_1N_2 3}}{\sum_{i=0}^{N_1} \sum_{j=0}^{N_2} (P_{ij1} + P_{ij3})}$$

6.2 Flow-Dependent Call-Congestion Rates

In this section we analyze an isolated trunk group of the mutual overflow model and determine its corresponding call-congestion rates with respect to the offered traffic streams.

Considering group 1, for instance, there are two flows offered to this subsystem: the MMPP 1 and the stream of calls overflowing from group 2 to 1. The overflow stream is created by selecting the arrival instants $T_n^{(2)}$ corresponding to those customers of the second MMPP that find the system in one of the states (i,N_2,k), $i \in \{0,\ldots,N_1\}$, $k \in \{1,3\}$. If we mark incoming calls of the offered MMPPs by two different colors representing distinct customer types, the isolated primary trunk group may be described by a special $G_2/M/N_1/0$ model (cf. Franken [8, p. 61]).

Let us denote the individual call-congestion rates of the customer stream j offered to group i by B_{ij}, $i,j \in \{1,2\}$. Then these flow-dependent call-congestion rates with respect to groups 1 and 2 are given by

$$B_{11} = \frac{e^t \cdot (e_{N_1+1} e_{N_1+1}^t) \otimes I_{N_2+1} \otimes \Lambda_1 \cdot P}{e^t \cdot I_{(N_1+1)(N_2+1)} \otimes \Lambda_1 \cdot P} = \frac{\sum_{j=0}^{N_2} (P_{N_1 j1} + P_{N_1 j2})}{\sum_{i=0}^{N_1} \sum_{j=0}^{N_2} (P_{ij1} + P_{ij2})}$$

$$B_{12} = \frac{e^t \cdot (e_{(N_1+1)(N_2+1)} e_{(N_1+1)(N_2+1)}^t) \otimes \Lambda_2 \cdot P}{e^t \cdot I_{N_1+1} \otimes (e_{N_2+1} e_{N_2+1}^t) \otimes \Lambda_2 \cdot P} = \frac{P_{N_1N_2 1} + P_{N_1N_2 3}}{\sum_{i=0}^{N_1} (P_{iN_2 1} + P_{iN_2 3})}$$

$$B_{21} = \frac{e^t \cdot (e_{(N_1+1)(N_2+1)} e^t_{(N_1+1)(N_2+1)}) \otimes \Lambda_1 \cdot P}{e^t \cdot (e_{N_1+1} e^t_{N_1+1}) \otimes I_{N_2+1} \otimes \Lambda_1 \cdot P} = \frac{P_{N_1 N_2 1} + P_{N_1 N_2 2}}{\sum_{j=0}^{N_2} (P_{N_1 j 1} + P_{N_1 j 2})}$$

$$B_{22} = \frac{e^t \cdot I_{N_1+1} \otimes (e_{N_2+1} e^t_{N_2+1}) \otimes \Lambda_2 \cdot P}{e^t \cdot I_{(N_1+1)(N_2+1)} \otimes \Lambda_2 \cdot P} = \frac{\sum_{i=0}^{N_1} (P_{iN_2 1} + P_{iN_2 3})}{\sum_{i=0}^{N_1} \sum_{j=0}^{N_2} (P_{ij1} + P_{ij3})}$$

7. CONCLUSIONS AND PERSPECTIVES

We have investigated a loss system that describes the mutual overflow of two Markov modulated Poisson processes between two fully available trunk groups. In order to compute the steady-state distribution of the associated CTMC modeling the behavior of the system, we have presented convergent point and block iterative methods based on regular splittings of the corresponding generator matrix.

In this context, we extended Schweitzer's iterative A/D algorithm to a larger class of splittings and improved Haviv's error analysis. Furthermore, a stochastic interpretation of the block Gauss–Seidel method was discussed.

The proposed procedures may be employed to calculate the steady-state performance characteristics of the model, which have been specified in terms of the overall and individual call-congestion rates of the arrival streams.

At present we are developing a software package that provides the user with a Markovian model world describing circuit-switched networks with adaptive routing schemes and advanced flow-control mechanisms. The proposed accelerated point iteration methods and some direct methods for the computation of the steady-state vector of a CTMC (cf. [11], [9], [18], [2], [24]) constitute the numerical solver of the system. The package is endowed with a user-friendly graphical interface and will be running on a workstation.

Further investigations concerning the efficient implementation of direct and iterative methods on a vector computer (IBM 3090 Vf) and on a distributed-memory multiprocessor architecture (Transputer system), the comparison of these algorithms, and their application to the performance analysis of adaptive routing schemes in circuit-switched networks are a subject of current research and will be reported elsewhere.

APPENDIX: DEFINITION OF THE MARKOV MODULATED POISSON PROCESS

Informally speaking, a Markov modulated Poisson process (MMPP) $\{N(t), t \geq 0\}$ with $m \in \mathbb{N}$ phases is a variant of a nonhomogeneous Poisson process

whose intensity function $\lambda(t)$ is a stochastic process $\lambda(t) = \sum_{i=1}^{m} \lambda_i 1_{\{i\}}(Y(t))$ governed by an irreducible, homogeneous, continuous-time Markov chain $\{Y(t), t \geq 0\}$ with finite state space $\{1,\ldots,m\}$. This means that $\{\lambda(t), t \geq 0\}$ is a CTMC with state space $\{\lambda_1,\ldots,\lambda_m\}$. The resulting point process is a special case of a double stochastic Poisson process that is also known as conditional Poisson process ([43], [28]). Therefore, it is convenient to call an MMPP a Poisson process in a Markovian environment.

An MMPP is parametrized by the specification of the initial probability vector $p = (p_i) \in \mathbb{R}^m$ and the generator matrix $Q \in \mathbb{R}^{m \times m}$ of the controlling CTMC and the rate vector $0 < \lambda = [\lambda_1,\ldots,\lambda_m]^t \in \mathbb{R}^m$. Notice that some intensities may be zero. It is convenient to associate a diagonal matrix $\Lambda = \mathrm{Diag}(\lambda)$ with the rate vector. Henceforth, $Y(t)$ is called the phase of the controlling CTMC of the MMPP at time $t \geq 0$.

Serfozo [43] has proved that any Markov modulated Poisson process may be represented by an equivalent two-dimensional pure Markov jump process. Regarding this equivalent characterization, we introduce an MMPP by the following definition.

DEFINITION 1 A nonnegative integer-valued stochastic process $\{N(t), t \geq 0\}$ is called Markov modulated Poisson process with $m \in \mathbb{N}$ phases and representation (Q,Λ,p) if there exists a two-dimensional regular Markov chain $\{Z(t) = (N(t),Y(t)), t \geq 0\}$ with values in $\mathbb{N}_0 \times \{1,\ldots,m\}$ such that the following conditions are satisfied:

1. $0 < \lambda = [\lambda_1,\ldots,\lambda_m]^t \in \mathbb{R}^m$, $\Lambda = \mathrm{Diag}(\lambda)$, and $-Q \in \mathbb{R}^{m \times m}$ is an irreducible M-matrix with $Q \cdot e = 0$; $p = (p_i) \in \mathbb{R}^m$ is a probability vector.
2. $Z(t)$ can be represented by its embedded jump chain, that is, there exists a Markov process $\{(N_n, Y_n, T_n), n \in \mathbb{N}_0\}$ with values in $\mathbb{N}_0 \times \{1,\ldots,m\} \times [0,\infty)$ and $P(N_0 = 0, Y_n = i, T_0 = 0) = p_i$, $1 \leq i \leq m$, satisfying

$$P(N_{n+1} - N_n = k, Y_{n+1} = j, T_{n+1} - T_n > t \mid N_n, Y_n = i, T_n)$$

$$= \begin{cases} \dfrac{Q_{ij}}{\lambda_i - Q_{ii}} e^{-(\lambda_i - Q_{ii})t} & k = 0, i \neq j \\ \dfrac{\lambda_i}{\lambda_i - Q_{ii}} e^{-(\lambda_i - Q_{ii})t} & k = 1, i = j \\ 0 & \text{otherwise} \end{cases}$$

such that

$$(N(t), Y(t)) = (N_n, Y_n) \quad \text{if} \quad T_n \leq t < T_{n+1}$$

holds.

Obviously, the MMPP $\{N(t), t \geq 0\}$ may be characterized as a special semi-Markovian point process with embedded Markov chain $R = (\Lambda - Q)^{-1}\Lambda$ and semi-Markov matrix $F(x) = (P(\tilde{Y}_{n+1} = j, \tilde{T}_{n+1} - \tilde{Y}_n \leq x \mid \tilde{Y}_n = i))_{i,j=1,\ldots,m} = (I - e^{(Q-\Lambda)x}) \cdot R, x \geq 0$ (cf. [28, p. 9]). Here $\tilde{Y}_n$ denotes the phase of the controlling Markov chain $Y(t)$ immediately after the nth event of the MMPP $N(t)$, and $\tilde{T}_n$ is its corresponding arrival instant. Notice that we assume that $t = 0$ is an arrival instant. This means that we determine the synchronous version of the point process $N(t)$ whose distribution is given by the Palm distribution.

The MMPP has been studied by Serfozo [43] and Meier [28], among others. For details the reader is referred to their articles and the references therein.

REFERENCES

[1] G. P. Barker and R. J. Plemmons. Convergent iterations for computing stationary distributions of Markov chains. SIAM J. Alg. Disc. Methods, 7(3):390–398, 1986.

[2] G. P. Barker and S. J. Yang. Semi-iterative and iterative methods for singular M-matrices. SIAM J. Matrix Anal. Appl., 9(2):168–180, 1988.

[3] A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, New York, 1979.

[4] F. Chatelin. Iterative aggregation/disaggregation methods. in P. J. Courtois, G. Iazeolla and A. Hordijk, editors, *Mathematical Computer Performance and Reliability*, pp. 199–207, North-Holland, Amsterdam, 1984.

[5] F. Chatelin and W. Miranker. Acceleration by aggregation of successive approximations methods. Linear Algebra Appl., 43:17–47, 1982.

[6] M. Eiermann, I. Marek, and W. Niethammer. On the solution of singular linear systems of algebraic equations by semiiterative methods. Numer. Math., 53:265–283, 1988.

[7] M. Eiermann, R. Varga, and W. Niethammer. Iterationsverfahren für nichtsymmetrische Gleichungssysteme und Approximationsmethoden im Komplexen. Jahresbericht Math.-Verein., Teubner, 89:1–32, 1987.

[8] P. Franken, D. König, U. Arndt, and V. Schmidt. *Queues and Point Processes*. John Wiley, New York, 1982.

[9] R. Funderlic and R. J. Plemmons. A combined direct-iterative method for certain M-matrix linear systems. SIAM J. Alg. Disc. Methods, 5:33–42, 1984.

[10] B. W. Gnedenko and D. König. *Handbuch der Bedienungstheorie II.* Akademie-Verlag, Berlin, 1984.

[11] W. Grassmann, M. Taksar, and D. Heymann. Regenerative analysis and steady-state distributions for Markov chains. Operations Res., 33(5):1107–1116, 1985.

[12] A. Hadjidimos. On the optimization of the classical iterative schemes for the solution of complex singular linear systems. SIAM J. Alg. Disc. Methods, 6(4):555–566, 1985.

[13] A. Hadjidimos. Optimum stationary and nonstationary iterative methods for the solution of singular linear systems. Numer. Math., 51:517–530, 1987.

[14] L. A. Hageman and D. M. Young. *Applied Iterative Methods.* Academic Press, New York, 1981.

[15] W. J. Harrod. Rank Modification Method for Certain Singular Systems of Linear Equations. Ph.D. thesis, University of Tennessee, Knoxville, 1982.

[16] M. Haviv. Aggregation/disaggregation methods for computing the stationary distribution of a Markov chain. SIAM J. Numer. Anal., 42:183–198, 1982.

[17] M. Neumann, J. Buoni, and R. Varga. Theorems of Stein–Rosenberg type: III, The singular case. Linear Algebra Appl., 41:183–198, 1982.

[18] L. Kaufman, Matrix methods for queueing problems. SIAM J. Sci. Stat. Comput., 4:525–552, 1983.

[19] J. G. Kemeny, Generalization of a fundamental matrix. Linear Algebra Appl., 38:193–206, 1981.

[20] J. G. Kemeny and L. J. Snell. *Finite Markov Chains.* Springer, New York, 1976.

[21] J. R. Koury, D. F. McAllister, and W. J. Stewart. Iterative methods for computing stationary distributions of nearly completely decomposable Markov chains. SIAM J. Alg. Disc. Methods, 5(2):164–186, 1984.

[22] U. Krieger. Analysis of a loss system with mutual overflow. In *International Teletraffic Seminar on Teletraffic and Network, Beijing, September 12–16, 1988, Proceedings*, pp. 331–340, 1988.

[23] U. Krieger. Analysis of a loss system with mutual overflow and external traffic. In *Stochastiche Modelle und Methoden in der Informationstechnik, Nürnberg, April 12–14, 1989, Proceedings, ITG-Fachbericht 107*, pp. 145–153, VDE-Verlag, Berlin, 1989.

[24] U. Krieger. *Computational Methods for Markovian Queueing Models*. Technical Report 4302 TB 10E, Research Institute of the DBP, Darmstadt, 1989.

[25] U. Krieger. Untersuchung eines Nachrichtenverkehrsmodelles mit wechselseitigem Überlauf und Externverkehr. Preprint, V 88–12–01, Research Institute of the DBP, Darmstadt, 1988.

[26] A. Kuczura. The interrupted Poisson process as an overflow process. Bell System Tech. J. 52(3):437–448, 1973.

[27] F. LeGall and J. Bernussou. An analytical formulation for grade of service determination in telephone networks. IEEE Trans. Commun., 31(3):420–424, 1983.

[28] K. Meier. A Statistical Procedure for Fitting Markov-Modulated Poisson Processes. Ph.D. thesis, University of Delaware, 1984.

[29] K. Meier-Hellstern. The analysis of a queue arising in overflow models. IEEE Trans. Commun., 37(4):367–372, 1989.

[30] K. Meier-Hellstern. Parcel overflows in queues with multiple inputs. In *Proceedings, ITC 12, Turino, Italy*, pp. 5.1B.3.1–5.1B.3.8, 1988.

[31] C. D. Meyer. The role of the group generalised inverse in the theory of finite Markov chains. SIAM Rev., 17:443–464, 1975.

[32] C. D. Meyer and M. W. Stadelmaier. Singular M-matrices and inverse positivity. Linear Algebra Appl., 22:139–156, 1978.

[33] D. Mitra and P. Tsoucas. Convergence of relaxations for numerical solutions of stochastic problems. In O. J. Boxma, G. Iazeolla, and P. J. Courtois, editors, *Applied Mathematics and Performance/Reliability Models of Computer/Communication Systems*, pp. 119–133, North-Holland, Amsterdam, 1987.

[34] B. Müller. Zerlegungsorientierte, numerische Verfahren für Markovsche Rechensystemmodelle. Ph.D. thesis, Universität Dortmund, Abteilung Informatik, 1980.

[35] W. Niethammer and R. Varga. The analysis of k-step iterative methods for linear systems from summability theory. Numer. Math.,41:177–206, 1983.

[36] A. Ridder. Stochastic Inequalities for Queues. Ph.D. thesis, Rijksuniversiteit te Leiden, 1987.

[37] D. Rose. Convergent regular splittings for singular M-matrices. SIAM J. Alg. Disc. Methods, 5(1):133–144, 1984.

[38] R. Schassberger. An aggregation principle for computing invariant probability vectors in Semi-Markovian models. In P. J. Courtois, G. Iazeolla and A. Hordijk, editors, *Mathematical Computer Performance and Reliability*, pp. 259–272, North Holland, Amsterdam, 1984.

[39] H. Schneider. Theorems on M-splittings of a singular M-matrix which depend on graph structure. Linear Algebra Appl., 58:407–424, 1984.

[40] P. J. Schweitzer. Aggregation methods for large Markov chains. In P. J. Courtois, G. Iazeolla and A. Hordijk, editors, *Mathematical Computer Performance and Reliability*, pp. 275–285, North Holland, Amsterdam, 1984.

[41] P. J. Schweitzer and K. W. Kindle. An iterative aggregation-disaggregation algorithm for solving linear equations. Appl. Math. Comput.,18: 313–353, 1986.

[42] L. P. Seelen. An Algorithm for $PH/PH/c$ Queues. Research Report 131, December 1984, Vrije Universiteit, Dept. of Actuarial Sciences and Economics, Amsterdam, 1984.

[43] R. F. Serfozo. Conditional Poisson processes. J. Appl. Prob., 9:288–302, 1972.

[44] U. Sumita and M. Rieders. A New Algorithm for Computing the Probability Vector for Large Markov Chains: Replacement Process Approach. Working Paper Series No. QM88–08, William E. Simon Graduate School of Business Administration, University of Rochester, 1988.

[45] R. Varga. *Matrix Iterative Analysis*. Prentice-Hall, Englewood Cliffs, N. J., 1962.