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ORIGINAL RESEARCH

Complex dynamics in a nonlinear duopoly model with heuristic expectation formation and learning behavior

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Abstract

We develop a nonlinear duopoly model in which the heuristic expectation formation and learning behavior of two boundedly rational firms may engender complex dynamics. Most importantly, we assume that the firms employ different forecasting models to predict the behavior of their opponent. Moreover, the firms learn by leaning more strongly on forecasting models that yield more precise predictions. An eight-dimensional nonlinear map drives the dynamics of our approach. We analytically derive the conditions under which its unique steady state is locally stable and numerically study its out-of-equilibrium behavior. In doing so, we detect multiple scenarios with coexisting attractors at which the firms' behavior yields distinctively different market outcomes.

Keywords Duopoly model · Heuristic expectation formation · Learning behavior · Nonlinear dynamics · Stability and bifurcation analysis · Coexisting attractors

JEL Classification C73 · D43 · L12

1 Introduction

For firms that operate in oligopoly markets, it is of utmost importance to form expectations about the supply decisions of their competitors. In the classical duopoly model by Cournot (1838), for instance, firms have naïve expectations, that is, they expect their rival to continue to produce the same amount in the next period as it produces in the current period. Naïve expectations combined with a linear demand, constant marginal costs and a perfect knowledge of the market features (demand schedules, cost functions) lead to the global stability of the

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so-called Cournot-Nash equilibrium.[1] While the experimental evidence by Cox and Walker (1998) suggests that naïve expectations may be a reasonable description of firms' expectation formation behavior, the experimental evidence by Stahl (1996), Offerman et al. (2002), Bigoni (2010) and Assenza et al. (2015) paints a richer picture. In particular, these studies suggest that firms switch between a limited number of heuristic forecasting models to form their expectations.[2] Importantly, firms' boundedly rational expectation formation and learning behavior, rendering the dynamics of the underlying market nonlinear, may prevent them from reaching correct forecasts. As a result, firms may continuously revise their expectations, with the consequence that prices and quantities never reach their equilibrium values.

Against this backdrop, we propose a novel duopoly model in which two boundedly rational firms display a heuristic expectation formation and learning behavior. Within our nonlinear duopoly model, the firms use competing forecasting models to predict their opponent's behavior. To simplify matters, we assume that the firms rely on a growth forecasting model, which predicts that the opponent will increase its production, and a reduction forecasting model, which predicts that the opponent will decrease its production. Moreover, the firms learn in the sense that they adjust the weight they assign to a forecasting model according to an evolutionary fitness measure based on prediction accuracy. The dynamics of our nonlinear duopoly model is driven by an eight-dimensional nonlinear map. Our nonlinear duopoly model possesses a unique steady state at which prices and quantities correspond to the Cournot-Nash solution of the classical linear duopoly model with naïve expectations. While the steady state of that model is globally stable, we demonstrate that a Flip bifurcation may compromise the local stability of the steady state of our nonlinear duopoly model. In fact, our bifurcation analysis reveals that the stability loss of the steady state of our nonlinear duopoly model—which may occur, for instance, when firms predict that their opponent will strongly adjust its supply—is accompanied by the emergence of a period-two cycle.

We furthermore find that our nonlinear duopoly model is able to produce complex dynamics, involving, amongst others, multiple scenarios with coexisting attractors. Interestingly, these coexisting attractors may be associated with distinctively different market outcomes. For certain parameter constellations, for instance, the following two attractors coexist. One attractor implies that the supply of both firms evolves asynchronously, resulting in a relatively stable behavior of the firms' total supply and, ultimately, in rather calm price dynamics. The other attractor implies that the supply of both firms evolves synchronously, resulting in a relatively unstable behavior of the firms' total supply and, ultimately, in rather turbulent price dynamics. Observing the behavior of our nonlinear duopoly model from the outside, one may arrive at wrong policy conclusions. When the firms' supply evolves asynchronously, they alternately dominate the market, a situation that may appear as fierce competition. In contrast, the synchronous evolution of the firms' production decisions may be confused with collusion. However, both market outcomes are emergent phenomena of our nonlinear duopoly model; they result from different sets of initial conditions.

We continue as follows. In Sect. 2, we comment on related literature. In Sect. 3, we recall the classical linear duopoly model with naïve expectations. In Sect. 4, we present our nonlinear duopoly model. In Sect. 5, we conclude our paper. Appendices A and B contain proofs of our main analytical results.

[1] Puu and Sushko (2002), Kopel (2009) and Bischi et al. (2010) provide excellent surveys of dynamics duopoly models.

[2] Duffy (2006), Hommes (2011) and Bao et al. (2021) review related experimental evidence with respect to finance and macroeconomics. Indeed, such behavior is widespread.

2 Literature review

Our modeling approach is inspired by the seminal papers by Brock and Hommes (1997, 1998). In their cobweb (asset pricing) model, a farmer (speculator) switches between heterogeneous forecasting models according to an evolutionary fitness measure. In our nonlinear duopoly model, a firm aggregates the predictions of competing forecasting models using weights that reflect their forecasting performance. Such an approach was also taken in the famous exchange rate model by Frankel and Froot (1986, 1990) in which a portfolio manager pools the predictions of different forecasting models. One of the first models to explore the possibility of switching between different expectation formation rules in a Cournotian setting is that by Droste et al. (2002). They study a route to complex dynamics that may emerge when a simple expectation rule competes with a sophisticated but costly expectation rule, using a replicator dynamics approach with mutational noise. Anufriev et al. (2013) consider a Bertrand oligopoly model in which firms switch between least squares learning and gradient leaning for determining the price. Switching between such learning rules may induce endogenous dynamics. Kopel et al. (2014) adopt a similar mechanism for analyzing a scenario in which firms switch between a socially concerned and a profit-maximizing behavior. Cerboni Baiardi et al. (2015) build an evolutionary exponential replicator oligopoly model, focusing on the coexistence of strong and weak attractors. Similar switching principles are used in Bischi et al. (2015), Anufriev and Kopányi (2018), Lamantia and Radi (2018) and Bischi and Lamantia (2022).

Of course, endogenous duopoly dynamics may also emerge via other channels. Theocharis (1960), Puu (2008) and Hommes et al. (2018) show that the equilibrium of the classical linear duopoly model with naïve expectations becomes unstable in the presence of three or more firms. Moreover, Huang (2008) demonstrates that there are situations in which an improvement of the accuracy of information (e.g. a reduction in information lags) may be detrimental to the stability of an equilibrium. However, Huang (2003) proves that oligopoly firms which display simple supply strategies because of limited information may achieve higher relative profits than their competitors. By replacing the hypothesis of linearity of the demand function with a microfounded nonlinear (isoelastic) demand schedule, Puu (1991) shows that the steady state of the classical linear duopoly model with naïve expectations may become unstable, leading to persistent periodic and chaotic fluctuations. Bischi and Naimzada (2000) endow firms with a gradient learning rule that does not require the full knowledge of the demand and profit functions, but only a local one. According to this rule, firms are able to estimate, by market experiments, the slope of the profit function at the point corresponding to the current values of the strategic variables, and move towards the direction of the profit maximum at a speed regulated by a behavioral parameter. Another learning rule is the so-called Local Monopolistic Approximation (LMA), first proposed by Tuinstra (2004), to model a dynamic oligopoly where demand is not known by firms but estimated through market experiments. Bischi et al. (2007) show that the dynamics generated through LMA behavior may converge to a Nash equilibrium, also in cases where the classic best-response dynamics does not converge to it.

3 The classical linear duopoly model with naïve expectations

The goal of our paper is to show that the heuristic expectation formation and learning behavior of boundedly rational firms may give rise to complex duopoly dynamics. As a workhorse, we use the classical linear duopoly model in which firms have naïve expectations. In the remainder of this section, we first recall this model and then discuss some of its main properties. Our exposition follows Gandolfo (2009) and Kopel (2009).

Let us turn to the details of this model. Market equilibrium implies that consumers' demand equals the firms' total supply in each time step, i.e.

$$D_t = S_t. \tag{1}$$

Moreover, consumers' demand depends negatively on the current market price

$$D_t = \frac{a - P_t}{b}, \tag{2}$$

where a and b are positive parameters. The firms' total supply consists of the supply of firms A and B

$$S_t = q_t^A + q_t^B. \tag{3}$$

Obviously, the market price adheres to

$$P_t = a - b(q_t^A + q_t^B). \tag{4}$$

We assume that parameters a and b are such that prices and quantities are positive, both at the steady state and out of equilibrium.[3]

Let us next derive the supply of firm A. Firm A maximizes its expected profits. Since firm A has a linear cost function with constant marginal costs $c > 0$, its expected profits result in

$$E_t^A\left[\pi_t^A\right] = E_t^A\left[P_t q_t^A - c q_t^A\right] = \left(a - b\left(q_t^A + E_t^A\left[q_t^B\right]\right)\right)q_t^A - c q_t^A. \tag{5}$$

From the corresponding first-order condition, we can compute that the optimal supply of firm A is equal to

$$q_t^A = \frac{3}{2}\alpha - \frac{1}{2}E_t^A[q_t^B], \tag{6}$$

where, for ease of exposition, $\alpha = \frac{a-c}{3b} > 0$ reflects an aggregate parameter. Importantly, the supply of firm A depends negatively on the expected supply of firm B. For simplicity, we assume that firms A and B have identical cost functions. The supply of firm B, resulting from analogous considerations, follows as

$$q_t^B = \frac{3}{2}\alpha - \frac{1}{2}E_t^B[q_t^A] \tag{7}$$

and depends negatively on the expected supply of firm A.

Apparently, a crucial question in this strand of literature is how a duopoly firm predicts the behavior of its opponent. In the classical linear duopoly model with naïve expectations, we simply have that

$$E_t^A\left[q_t^B\right] = q_{t-1}^B \tag{8}$$

and

$$E_t^B\left[q_t^A\right] = q_{t-1}^A, \tag{9}$$

[3] For models that explicitly take such constraints into account, see Tramontana et al., (2010, 2011) and Gori and Sodini (2017). We also assume that firms are aware of the true demand function. For models that deviate from this assumption, see Bischi and Naimzada (2000), Tuinstra (2004) and Bischi et al. (2007).

respectively. Accordingly, firms A and B then predict that their opponent will not adjust its supply decision.

Combining (6) to (9) then reveals that

$$q_t^A = \frac{3}{2}\alpha - \frac{1}{2}q_{t-1}^B \tag{10}$$

and

$$q_t^B = \frac{3}{2}\alpha - \frac{1}{2}q_{t-1}^A, \tag{11}$$

reflecting the firms' best-response functions.

In Appendix A, we prove the following proposition (an overbar denotes steady-state quantities).

Proposition 1 The dynamics of the classical linear duopoly model with naïve expectations is driven by a two-dimensional linear map. Its unique steady state $SSL = (\overline{q}^A, \overline{q}^B) = (\alpha, \alpha)$ is globally stable.

With respect to the steady state of the classical linear duopoly model with naïve expectations, Proposition 1 reveals that the production quantities of firms A and B are given by $\overline{q}^A = \overline{q}^B = \alpha = \frac{a-c}{3b}$, commonly referred to as Cournot-Nash equilibrium. Consequently, the total steady-state supply of firms A and B results in the steady-state price $\overline{P} = \frac{a+2c}{3}$ such that steady-state profits realized by firms A and B amount to $\overline{\pi}^A = \overline{\pi}^B = b\alpha^2 = \frac{(a-c)^2}{9b}$. At the steady state, neither firm A nor firm B makes a prediction error, i.e. their naïve expectations are correct.

Figure 1 provides an example of the best-response dynamics of the classical linear duopoly model with naïve expectations. The blue and pink lines mark the best-response functions of firms A and B, respectively. Starting in the out-of-equilibrium point (q_0^A,q_0^B), the figure illustrates for a few best-response iterations how the duopoly market converges towards the Cournot-Nash equilibrium point (α,α). Note that the production quantities of both firms display a zigzag path for this set of initial conditions until they have reached the equilibrium point (α,α). Since firms' total production is either high or low along the adjustment route, the price path reflects a zigzag pattern, too.

There are two main reasons why we use the classical linear duopoly model with naïve expectations as a benchmark model. First, its setup is well known, relatively simple, and easy to extend. Second, its dynamics is globally stable. Hence, all deviations from its well-known behavior that we observe in the following are due to our model extensions, i.e. the firms' heuristic expectation formation and learning behavior, as introduced in the next section.

4 A nonlinear duopoly model

As in the previous section, we assume that the supply of firms A and B is given by (6) and (7), respectively. In this section, however, we modify their expectation formation behavior. Moreover, we present our main analytical results and simulate the out-of-equilibrium behavior of our nonlinear duopoly model.

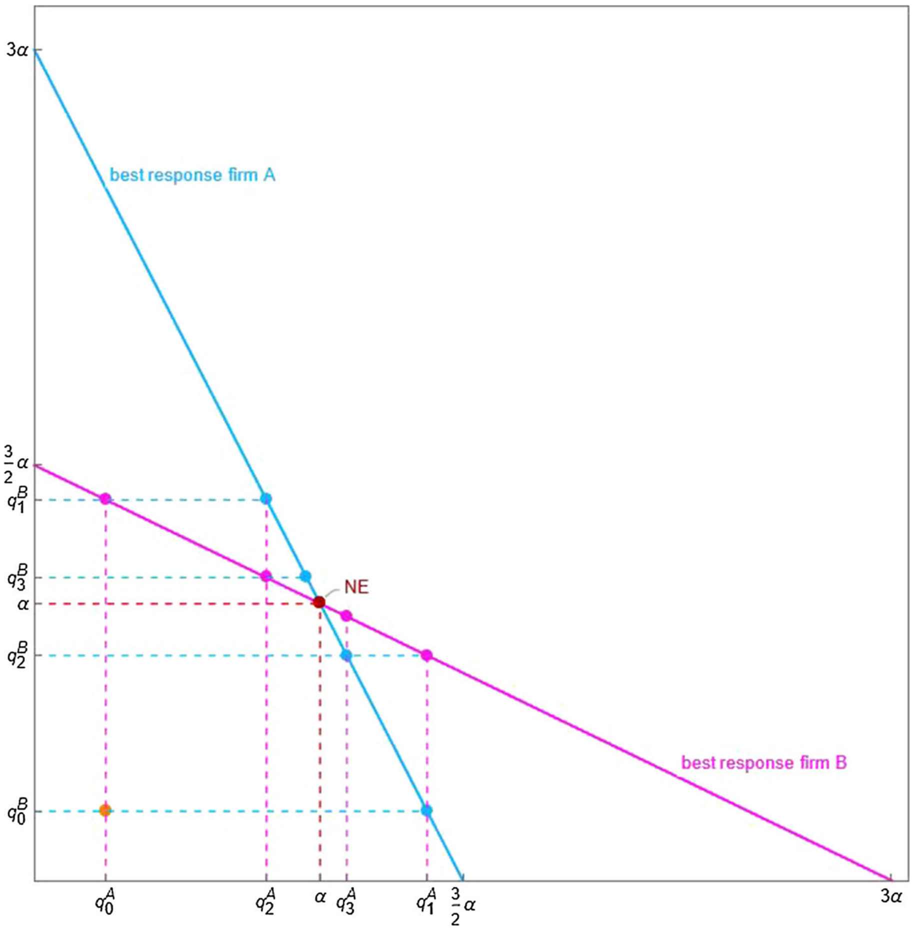

Fig. 1 Best-response dynamics within the classical linear duopoly model with naïve expectations. The blue and pink lines mark the best-response functions of firms A and B, respectively. Starting in the out-of-equilibrium point (q_0^A, q_0^B), the figure illustrates for a few time steps how the duopoly market converges towards the Cournot-Nash equilibrium point (α, α)

4.1 Firm's expectation formation and learning behavior

Let us turn to the key part of our nonlinear duopoly model, namely the question of how firms predict the behavior of their opponent and how they learn from their mistakes. Let us start with firm A. In general, firm A may employ several competing forecasting models to predict firm B's supply decision. Given our objective and to simplify matters, it is sufficient for us to consider two different forecasting models. In the first step, firm A predicts that firm B will either increase or decrease its supply. In the second step, firm A aggregates these predictions by assigning weights to them. The weights firm A uses to pool the predictions of its two forecasting models evolve over time, subject to an evolutionary performance measure based on prediction accuracies. In this sense, firm A displays a boundedly rational learning behavior. Firm B's expectation formation and learning behavior evolves along similar lines.

Let us formalize the firms' expectation formation and learning behavior. We express firm A's expectation about the supply of firm B as

$$E_t^A\left[q_t^B\right] = W_t^{A,G} M_t^{A,G} + W_t^{A,R} M_t^{A,R}, \tag{12}$$

where $M_t^{A,G}$ and $M_t^{A,R}$ are firm A's predictions using the growth and reduction forecasting models, and $W_t^{A,G}$ and $W_t^{A,R} = 1 - W_t^{A,G}$ are the weights it assigns to these two techniques.

Firm A's forecasting models predict that firm B will either increase or decrease its supply by a constant factor, represented by expectation parameter $0 < d < 1$. Accordingly, firm A's growth forecasting model is formalized as

$$M_t^{A,G} = (1 + d) q_{t-1}^B, \tag{13}$$

while its reduction forecasting model is captured by

$$M_t^{A,R} = (1 - d) q_{t-1}^B. \tag{14}$$

Experimental evidence for the growth and reduction forecasting models can be found in Assenza et al. (2015). Note that firm A's final prediction is given by (12). In general, there are three different scenarios. For $W_t^{A,G} = W_t^{A,R} = 0.5$, firm A predicts that firm B will not alter its supply decision. As we will see, this holds at the unique steady state of our nonlinear duopoly model. Out of equilibrium, we have that firm A predicts that firm B will either increase or decrease its supply decision. This depends on whether $W_t^{A,G}$ is larger or smaller than $W_t^{A,R}$.[4]

The fitness of the two forecasting models depends on current and past squared prediction errors. The fitness of firm A's growth and reduction forecasting models is defined as

$$F_t^{A,G} = e F_{t-1}^{A,G} - (1 - e)(q_{t-1}^B - M_{t-1}^{A,G})^2 \tag{15}$$

and

$$F_t^{A,R} = e F_{t-1}^{A,R} - (1 - e)(q_{t-1}^B - M_{t-1}^{A,R})^2, \tag{16}$$

respectively. Firm A's memory parameter is restricted to $0 < e < 1$.[5]

The weight firm A assigns to the growth and reduction forecasting models is due to

$$W_t^{A,G} = \frac{\exp[f F_t^{A,G}]}{\exp\left[f F_t^{A,G}\right] + \exp[f F_t^{A,R}]} \tag{17}$$

and

$$W_t^{A,R} = \frac{\exp[f F_t^{A,R}]}{\exp\left[f F_t^{A,G}\right] + \exp[f F_t^{A,R}]}, \tag{18}$$

[4] We do not argue that our setup is superior to setups that rely on more traditional heuristics such as extrapolative or regressive rules. We regard our setup as a reasonable alternative setup that may improve our understanding of the functioning of duopoly markets. However, we would like to point out that (13) and (14) do not rely on knowledge about firm B's long-run strategy, as would be typical for regressive rules. Moreover, (13) and (14) only rest on the previous supply decision of firm B. In contrast, extrapolative rules rely at least on the last two supply decisions of firm B, which blows up the dimension of the underlying dynamical system.

[5] Evolutionary models that also use squared prediction errors as a fitness indicator include Lines and Westerhoff (2010) and Schmitt and Westerhoff (2019). Alternatively, one may use past realized profits as a fitness criterion, such as in Brock and Hommes (1997, 1998).

respectively. The intensity of choice parameter $f > 0$ indicates how sensitively firm A reacts to differences in the fitness of its forecasting models. Two aspects of (17) and (18) are relevant to understanding firm A's learning behavior. First, the weight firm A assigns to a forecasting model increases in line with the forecasting model's fitness. Second, the weight firm A assigns to the forecasting model that yields the higher fitness increases in line with parameter f. In this respect, there are two extreme parameter constellations. For $f \to 0$, firm A weights both predictions of its forecasting model with 50 percent. For $f \to \infty$, firm A only uses the forecasting model with the higher fitness.[6]

Symmetrically, we obtain the following set of equations for firm B. We express firm B's expectation about the supply of firm A as

$$E_t^B\left[q_t^A\right] = W_t^{B,G} M_t^{B,G} + W_t^{B,R} M_t^{B,R}, \tag{19}$$

firm B's growth forecasting model as

$$M_t^{B,G} = (1+d) q_{t-1}^A, \tag{20}$$

firm B's reduction forecasting model as

$$M_t^{B,R} = (1-d) q_{t-1}^A, \tag{21}$$

the fitness of firm B's growth forecasting model as

$$F_t^{B,G} = eF_{t-1}^{B,G} - (1-e)(q_{t-1}^A - M_{t-1}^{B,G})^2, \tag{22}$$

the fitness of firm B's reduction forecasting model as

$$F_t^{B,R} = eF_{t-1}^{B,R} - (1-e)(q_{t-1}^A - M_{t-1}^{B,R})^2, \tag{23}$$

and the weights firm B assigns to its growth and reduction forecasting models as

$$W_t^{B,G} = \frac{\exp[f F_t^{B,G}]}{\exp\left[f F_t^{B,G}\right] + \exp[f F_t^{B,R}]}, \tag{24}$$

and

$$W_t^{B,R} = \frac{\exp[f F_t^{B,R}]}{\exp\left[f F_t^{B,G}\right] + \exp[f F_t^{B,R}]}, \tag{25}$$

respectively. Since the expectations of firms A and B become naïve when $f \to 0$, our nonlinear duopoly model nests the classical linear duopoly model with naïve expectations as a special case. The same is true when $d \to 0$.

We are particularly interested in how the firms' heuristic expectation formation and learning behavior affects the dynamics of our nonlinear duopoly model. The classical linear duopoly model with naïve expectations yields a globally stable steady state, i.e. permanent duopoly dynamics emerge in this setup only in the presence of exogenous shocks. In the following, we demonstrate that our nonlinear duopoly model may endogenously produce complex dynamics.

[6] See Hommes (2013), Franke and Westerhoff (2017) and Hommes (2021) for an in-depth discussion of this boundedly rational learning approach.

4.2 Analytical results

The following proposition, proven in Appendix B, summarizes our main analytical results.

Proposition 2 The dynamics of our nonlinear duopoly model is driven by an eight-dimensional nonlinear map. This map possesses a unique steady state, given by $SSN = \left(\overline{q}^A, \overline{F}^{A,G}, \overline{F}^{A,R}, \overline{x}^B, \overline{q}^B, \overline{F}^{B,G}, \overline{F}^{B,R}, \overline{x}^A\right) = \left(\alpha, -d^2\alpha^2, -d^2\alpha^2, \alpha, \alpha, -d^2\alpha^2, -d^2\alpha^2, \alpha\right)$. Steady state SSN is locally stable if and only if stability condition $d < d_{crit}^{Flip} = \frac{1}{2\alpha}\sqrt{\frac{(1+e)}{(1-e)f}}$ holds. A violation of this stability condition is associated with the emergence of a Flip bifurcation and the birth of a period-two cycle.

Proposition 2 deserves a few comments. Firm A's and B's steady-state supply, given by $\overline{q}^A = \overline{q}^B = \alpha = \frac{a-c}{3b}$, their steady-state profits, given by $\overline{\pi}^A = \overline{\pi}^B = b\alpha^2 = \frac{(a-c)^2}{9b}$, and the steady-state price, given by $\overline{P} = \frac{a+2c}{3}$, only depend on fundamental parameters a, b and c. Moreover, they are equal to those we encountered at the Cournot-Nash equilibrium of the classical linear duopoly model with naïve expectations. Why is this the case? Note that our nonlinear duopoly model ensures that neither firm A nor firm B makes any prediction errors at the steady state, i.e. both firms correctly predict the behavior of their opponent when the dynamics is at rest.[7] The same is true when firms A and B form naïve expectations. At the steady state, naïve expectations entail no prediction errors.

Solving stability condition $d < d_{crit}^{Flip} = \frac{1}{\alpha}\sqrt{\frac{(1+e)}{(1-e)f}}$ for memory parameter e reveals that the local stability of steady state SSN necessitates that $e > e_{crit}^{Flip} = 1 - \frac{2}{1+4\alpha^2 d^2 f}$. With respect to intensity of choice parameter f, we obtain the stability condition $f < f_{crit}^{Flip} = \frac{(1+e)}{4\alpha^2 d^2 (1-e)}$. Accordingly, the local stability of steady state SSN depends on all six model parameters. With respect to the fundamental parameters, we can conclude that an increase in parameter a is detrimental to market stability, while an increase in parameters b and c is beneficial for market stability. With respect to the behavioral parameters, we can conclude that an increase in expectation parameter d and intensity of choice parameter f may compromise market stability, while an increase in memory parameter e fosters market stability.[8]

4.3 Numerical results

Let us now study the out-of-equilibrium behavior of our nonlinear duopoly model. Our simulations rely on the following base parameter setting. For the fundamental parameters, capturing key demand and supply characteristics of the duopoly market, we assume that $a = 25$, $b = 2$ and $c = 1$, implying that the aggregate parameter α is equal to 4. For the behavioral parameters, capturing the firms' expectation formation and learning behavior, we assume that $d = 0.02$, $e = 0.9$ and $f = 475$. While we keep the fundamental parameters fixed, the behavioral parameters will also serve as bifurcation parameters. Using our analytical insights, we can compute that the steady-state supplies of firms A and B amount to $\overline{q}^A =$

[7] This is an important observation. While the growth and the reduction forecasting models deliver incorrect predictions, their pooled predictions are correct at the steady state. Since the firms' expectations should be correct at the steady state, we regard this as a desirable outcome.

[8] Given the dimension of map N, it is remarkable that we can precisely clarify the role played by each model parameter for the coordinates and local stability properties of steady state SSN. Such clear-cut analytical insights may justify one or the other simplifying assumption we made in setting up our nonlinear duopoly model, in particular the symmetric firm specification on which we focus mainly.

$\overline{q}^B = 4$. Moreover, the firms' total steady-state supply results in a steady-state price given by $\overline{P} = 9$. The steady-state profits realized by firms A and B are equal to $\overline{\pi}^A = \overline{\pi}^B = 32$. Once again, we remark that neither firm makes any prediction errors at the steady state, i.e. they correctly predict their opponent's supply decisions. Furthermore, Proposition 2 reveals that the steady state of our nonlinear duopoly model becomes unstable due to a Flip bifurcation when behavioral parameters d, e and f are about to violate the stability conditions $d < d_{crit}^{Flip} = 0.025$, $e > e_{crit}^{Flip} = 0.848$ and $f < f_{crit}^{Flip} = 742.2$, respectively. As we will see in the sequel, our nonlinear duopoly model then gives rise to a period-two cycle and, as these behavioral parameters change further, to endogenous dynamics that may involve coexisting attractors. In the following, we use bifurcation diagrams and time series plots to illustrate the functioning of our nonlinear duopoly model.

Figures 2, 3, 4 present bifurcation diagrams in which we depict the price, the supply of firm A, the supply of firm B and the firms' total supply versus expectation parameter d, using different sets of initial conditions. We chose the initial conditions with the goal of best visualizing the appearance and disappearance of coexisting attractors. Given the dimension of map N, this is a nontrivial task. Note that all bifurcation diagrams have in common that the dynamics of our nonlinear duopoly model converges towards its steady state SSN for $d < d_{crit}^{Flip} = 0.025$. As to be expected, we furthermore observe a Flip bifurcation and the birth of a locally stable period-two cycle at $d_{crit}^{Flip} = 0.025$. Note that $d = 0.025$ means that the firms' growth and reduction forecasting models predict that their opponent will increase or decrease its supply by 2.5 percent in the next period. Since the firms' eventual predictions are averages of the predictions of their two forecasting models, such a value for expectation parameter d seems to us not to be extreme. Figures 2, 3, 4 indicate that we may observe coexisting attractors for higher values of expectation parameter d. Initial conditions then determine the fate of the duopoly market. In general, we can also infer from Figs. 2, 3, 4 that an increase in expectation parameter d amplifies the amplitude of the dynamics, even if the coexistence of several attractors may create exceptions to this rule.

Comparing Figs. 2, 3, 4 at around $d = 0.04$ indicates that a period-two cycle, visible in Fig. 2, coexists with a cyclical attractor, visible in Figs. 3 and 4.[9] Let us explore this scenario in more detail. The panels in Fig. 5 report the price, the supply of firm A, the supply of firm B and the firms' total supply in the time domain for our base parameter setting, except that $d = 0.04$. The solid (dashed) horizontal lines mark the steady-state (average) values of the quantities depicted. Clearly, the left and right panels of Fig. 5 only differ with respect to their initial conditions. Interestingly, the period-two cycle presented in the left panels of Fig. 5 goes hand in hand with relatively strong price volatility. The main reason for this is that the supplies of firms A and B evolve synchronously, i.e. the supply of both firms is either high or low. In contrast, the cyclical dynamics depicted in the right panels of Fig. 5 is associated with relatively low price volatility. While the supplies of firms A and B again display a relatively high amplitude, comparable to the one observed for the case of the period-two cycle, their supplies behave asynchronously. When the supply of firm A is high (low), the supply of firm B is low (high). In fact, the bottom right panel of Fig. 5 reveals that the firms' total supply is

[9] Two technical comments are in order. First, all the cyclical attractors that we discuss in our paper were detected numerically, with the exception of the steady state. While it is clear that our nonlinear duopoly model may give rise to scenarios with at least four coexisting attractors, we cannot rule out the existence of further attractors. Second, it is difficult to judge the true nature of some of the cyclical attractors we will encounter in the following. They may stand for high-period cycles, quasiperiodic motion or chaotic dynamics. For a single time series, one may seek to identify their true nature by estimating the corresponding largest Lyapunov exponent or by exploring their attractors in phase space. However, since small changes in parameter d may render these results, we abstain from discussing this technical aspect in more detail.

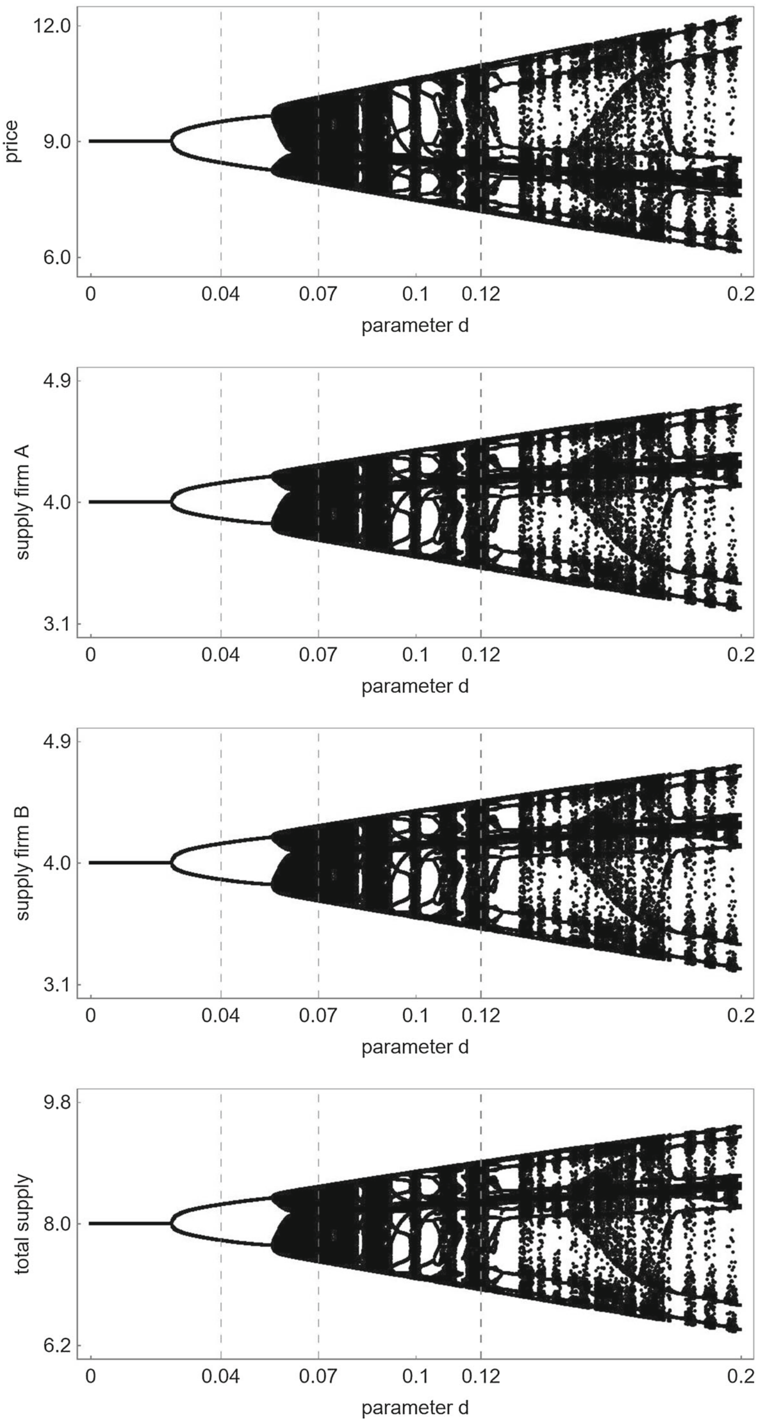

Fig. 2 First set of bifurcation diagrams for parameter d. The panels show the price, the supply of firm A, the supply of firm B and the firms' total supply for increasing values of parameter d. The other parameters correspond to our base parameter setting. Figures 2, 3, 4 only differ with respect to their initial conditions

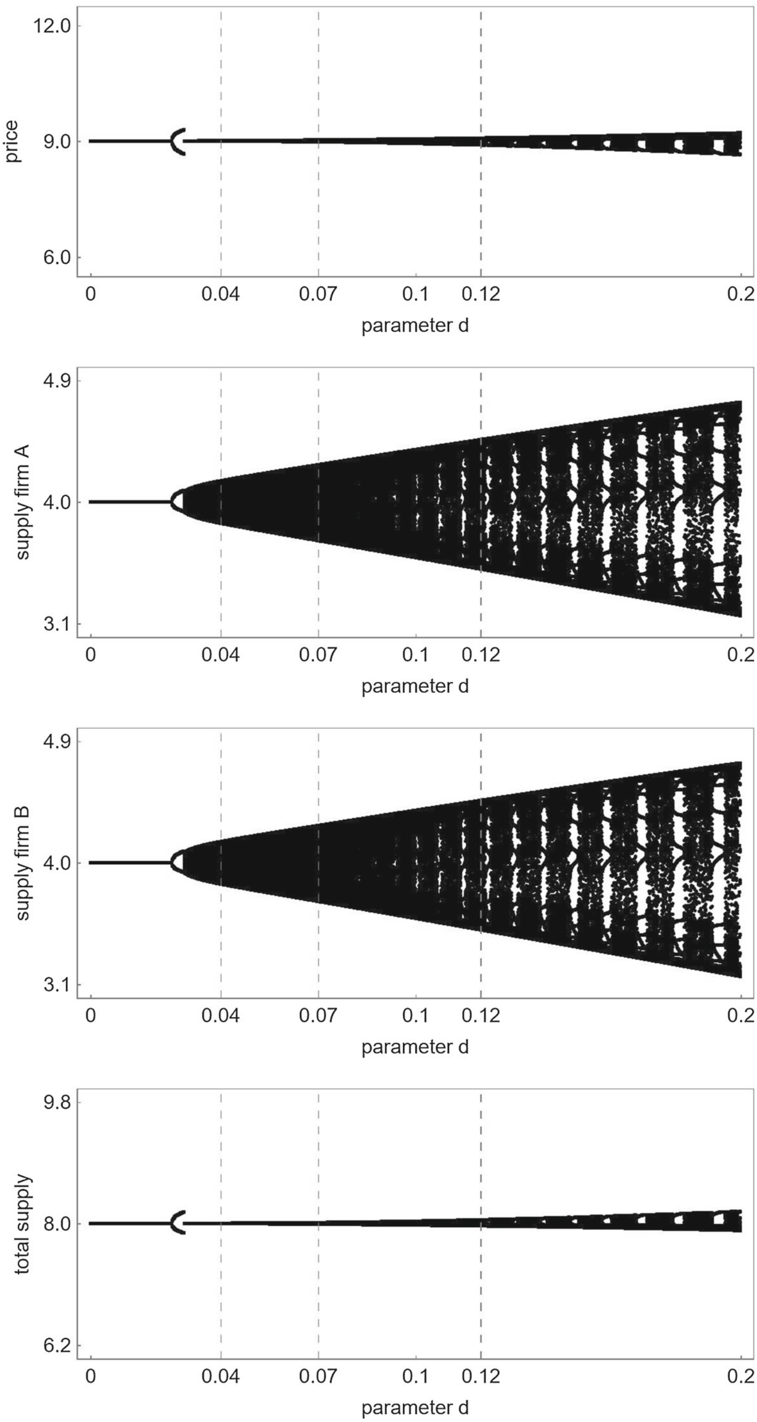

Fig. 3 Second set of bifurcation diagrams for parameter d. The panels show the price, the supply of firm A, the supply of firm B and the firms' total supply for increasing values of parameter d. The other parameters correspond to our base parameter setting. Figures 2, 3, 4 only differ with respect to their initial conditions

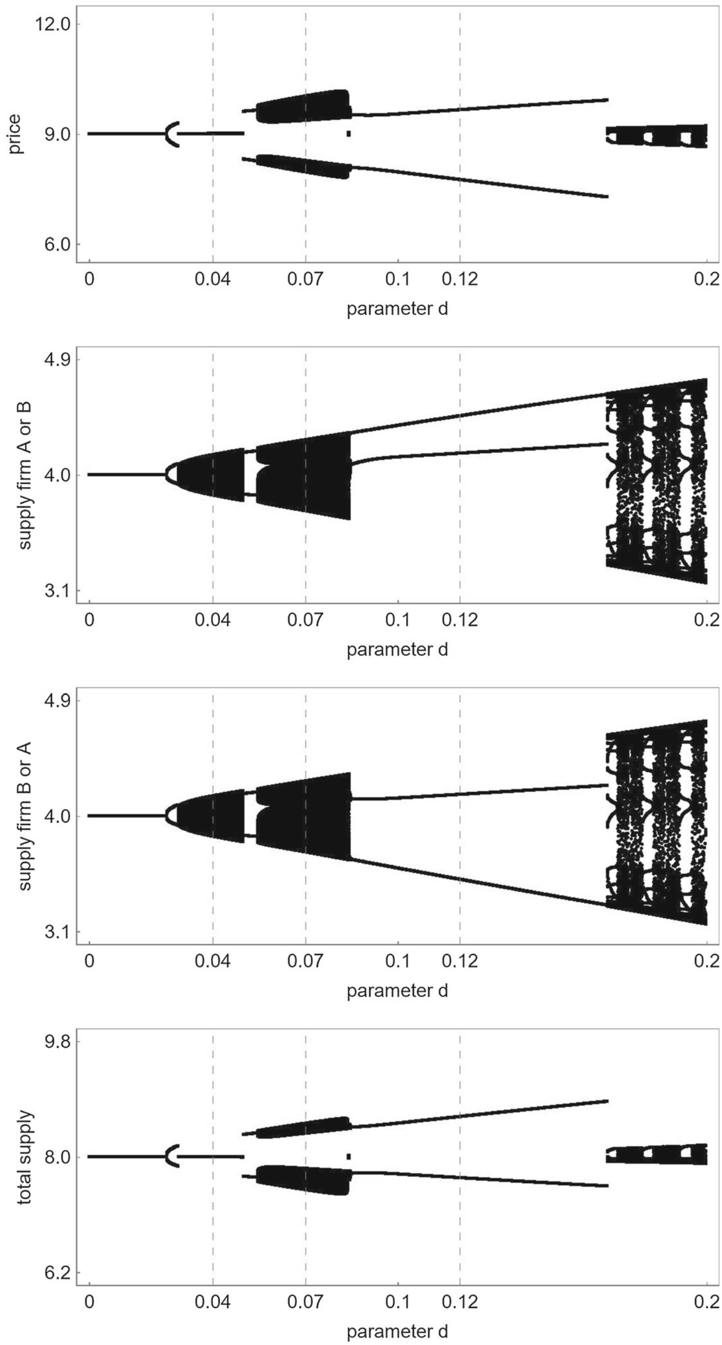

Fig. 4 Third set of bifurcation diagrams for parameter d. The panels show the price, the supply of firm A or firm B, the supply of firm B or firm A and the firms' total supply for increasing values of parameter d. The other parameters correspond to our base parameter setting. Figures 2, 3, 4 only differ with respect to their initial conditions

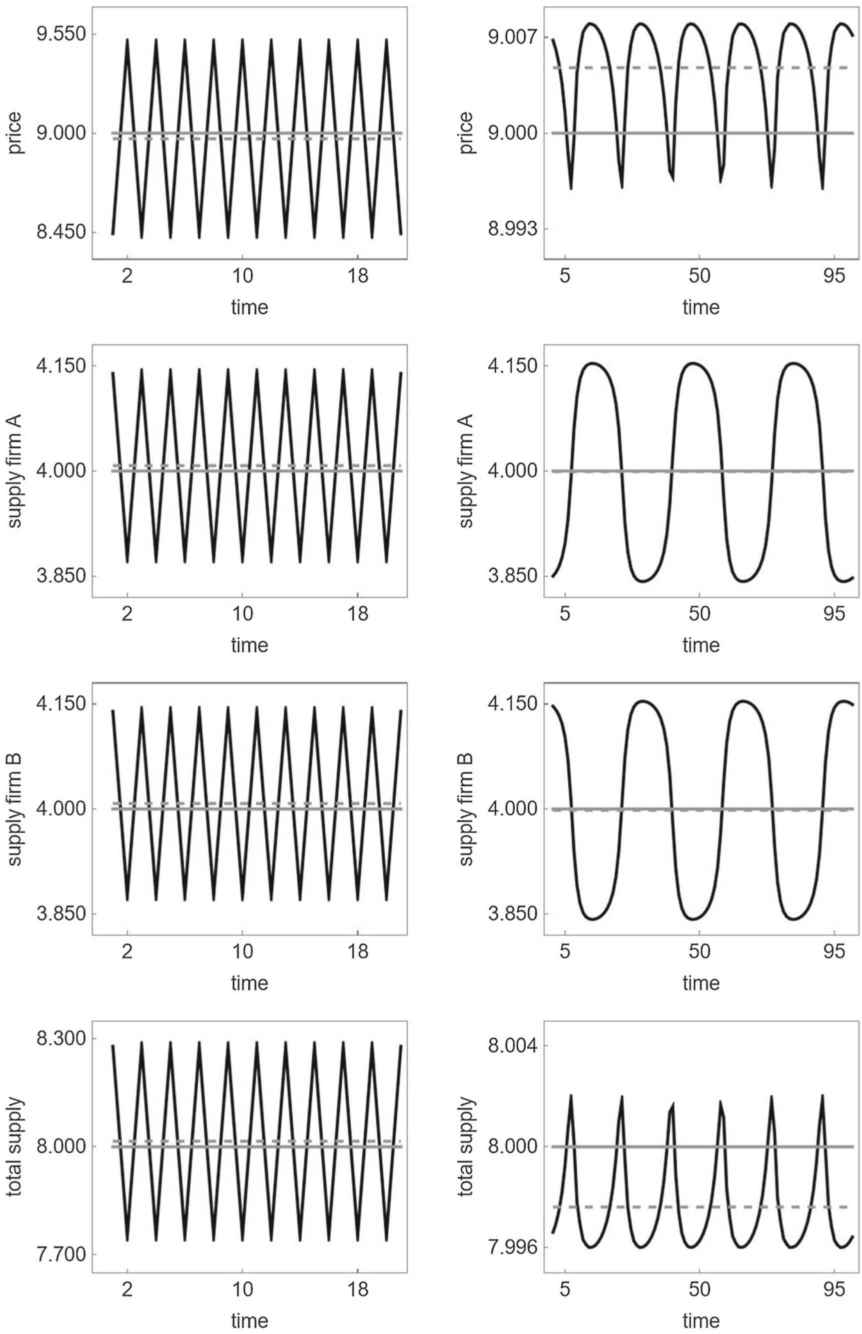

Fig. 5 First set of time series diagrams for parameter d. The panels show the price, the supply of firm A, the supply of firm B and the firms' total supply in the time domain. The simulations are based on our base parameter setting, except that $d = 0.04$. The left and right panels only differ with respect to their initial conditions. The solid (dashed) horizontal lines mark the steady-state (average) values of the quantities depicted

relatively stable, which explains why price volatility is low. A comparison of the solid and dashed lines in Fig. 5 reveals that, in the case of the period-two cycle, the average price is lower than the steady-state price, while it is the other way around for the cyclical attractor. At the cyclical attractor, firms' average profits amount to 32.01, comparable to their steady-state profits $\overline{\pi}^A = \overline{\pi}^B = 32$. At the period-two cycle, however, firms' average profits drop to 31.87. Overall, we have here the first example where our nonlinear duopoly model yields distinctively different market outcomes for the same parameter setting.

Continuing by increasing the value of expectation parameter d further, we can see from Fig. 2 that the period-two cycle undergoes a Neimark-Sacker bifurcation at a value of about $d \cong 0.057$, creating two closed invariant curves around the two points of the cycle, which in turn becomes unstable. As can further be observed, the amplitude of these cycles increases in line with parameter d. Scrutinizing Figs. 2, 3, 4 at $d = 0.07$ reveals that our nonlinear duopoly model now gives rise to three coexisting attractors. We study these scenarios in the time domain in Fig. 6, generated using our base parameter setting, except that $d = 0.07$. In the left panels of Fig. 6, we see relatively stable price dynamics due to asynchronous supply decisions of firms A and B. This scenario corresponds to the right-hand scenario depicted in Fig. 5. In the middle panels of Fig. 6, price volatility is much higher. This is a result of the synchronous supply behavior of firms A and B. In contrast to the left-hand scenario depicted in Fig. 5, however, the dynamics is not due to a period-two cycle, but a cyclical attractor, located around the two points of the aforementioned period-two cycle. In the right-hand panels of Fig. 6, we observe price dynamics that alternates between calm and turbulent periods. Price volatility is high when the firms' supply decisions evolve synchronously and low when they evolve asynchronously. In the latter case, the supplies of firms A and B, and consequently their total supply, are clearly higher than their steady-state quantities. As a result, prices are then lower than their steady-state value. This also holds on average. With respect to the firms' profitability, we can conclude that the dynamics depicted in the left panels of Fig. 6 yield average profits that, at 32.04, are quite comparable to those they achieve at the steady state. However, this does not hold for the dynamics depicted in the middle and right panels of Fig. 6, which are associated with average profits of 31.61 and 31.46, respectively.

The bifurcation diagrams reported in Fig. 4 further reveal that an additional period-two cycle suddenly appears at around $d \cong 0.085$, probably via a saddle-node bifurcation. Since the depicted simulations hold for firm A or firm B, there are in fact two new period-two cycles. Figure 7 presents the dynamics of our nonlinear duopoly model for our base parameter setting, except that $d = 0.12$. Now we are confronted with four coexisting attractors. The left panels of Fig. 7 show one of the two new-born period-two cycles. Note that the depicted period-two cycle is associated with relatively high price volatility. The firms' supplies now evolve not only synchronously—we can also observe the supply of one firm fluctuating modestly around a high level, while that of the other firm fluctuates more strongly around a low level. Total supply fluctuates significantly, as do prices. Moreover, the average quantities produced by the firms differ considerably from their steady-state values, as can be seen from the solid and dashed lines in Fig. 7. As a result, the average profits of firm A, at 33.08, are considerable higher than the average profits of firm B, namely 29.24 (of course, for other initial conditions this may be the other way around).

The middle and right panels of Fig. 7 depict the remaining two attractors. In the middle panels of Fig. 7, we again see an attractor at which prices are relatively stable. We already encountered this attractor in the left panels of Fig. 6 and in the right panels of Fig. 5. For this attractor, firms' average profits, at 32.13, are slightly higher than their steady-state profits. In the top-right panel of Fig. 7, we see price dynamics that alternate between calm and turbulent periods, as in the top-right panel of Fig. 6. In contrast to the previous figures, however, the

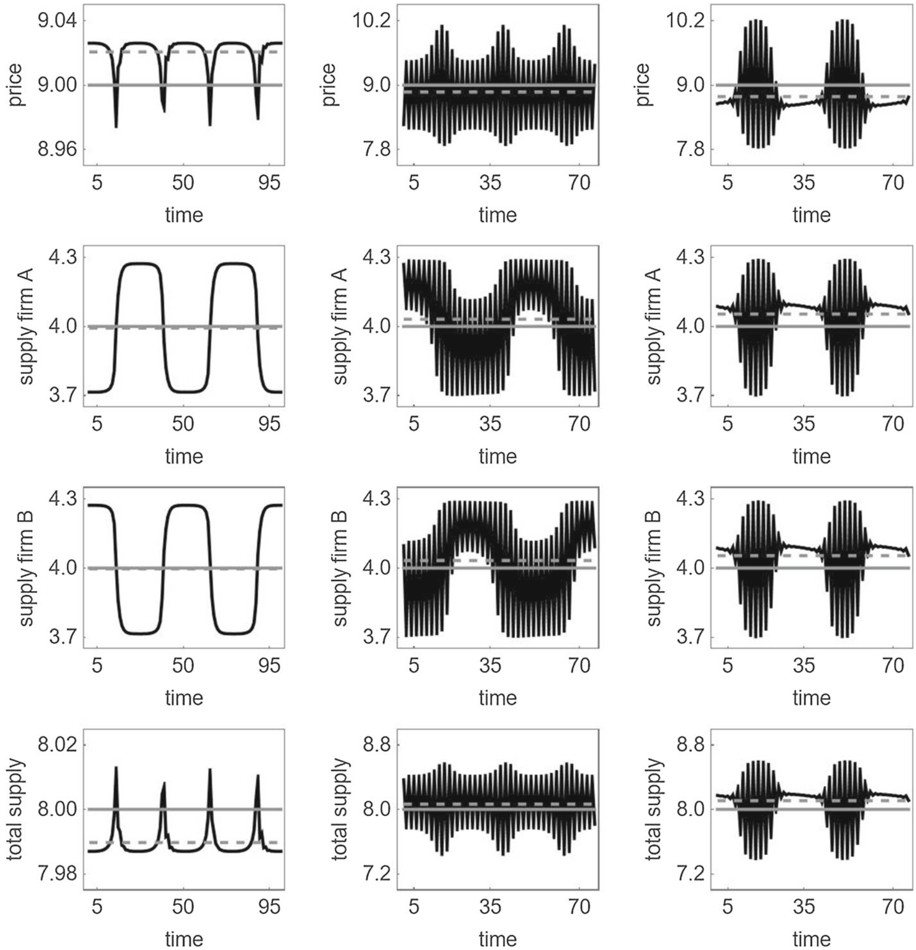

Fig. 6 Second set of time series diagrams for parameter d. The panels show the price, the supply of firm A, the supply of firm B and the firms' total supply in the time domain. The simulations are based on our base parameter setting, except that $d = 0.07$. The left, middle and right panels only differ with respect to their initial conditions. The solid (dashed) horizontal lines mark the steady-state (average) values of the quantities depicted

amplitude of the dynamics has increased. Of course, this has an impact on the average values of prices and the firms' total supply, which may deviate even more strongly from their steady-state value than before. For instance, the average price in the top-middle panel of Fig. 7 is equal to 9.07, while it equals 8.47 in the top-right panel of Fig. 7. The firms' average profits for the dynamics depicted in the right panels of Fig. 7 are equal to 30.75. Once again, our nonlinear duopoly model may give rise to distinctively different coexisting attractors.[10]

[10] Dieci et al., (2022, 2023) develop behavioral cobweb models in which firms switch between different production technologies. In their models, the volatility of the price dynamics increases when firms involuntarily coordinate on the same production technology. In our model, the volatility of prices increases when firms involuntarily coordinate on the same forecasting model.

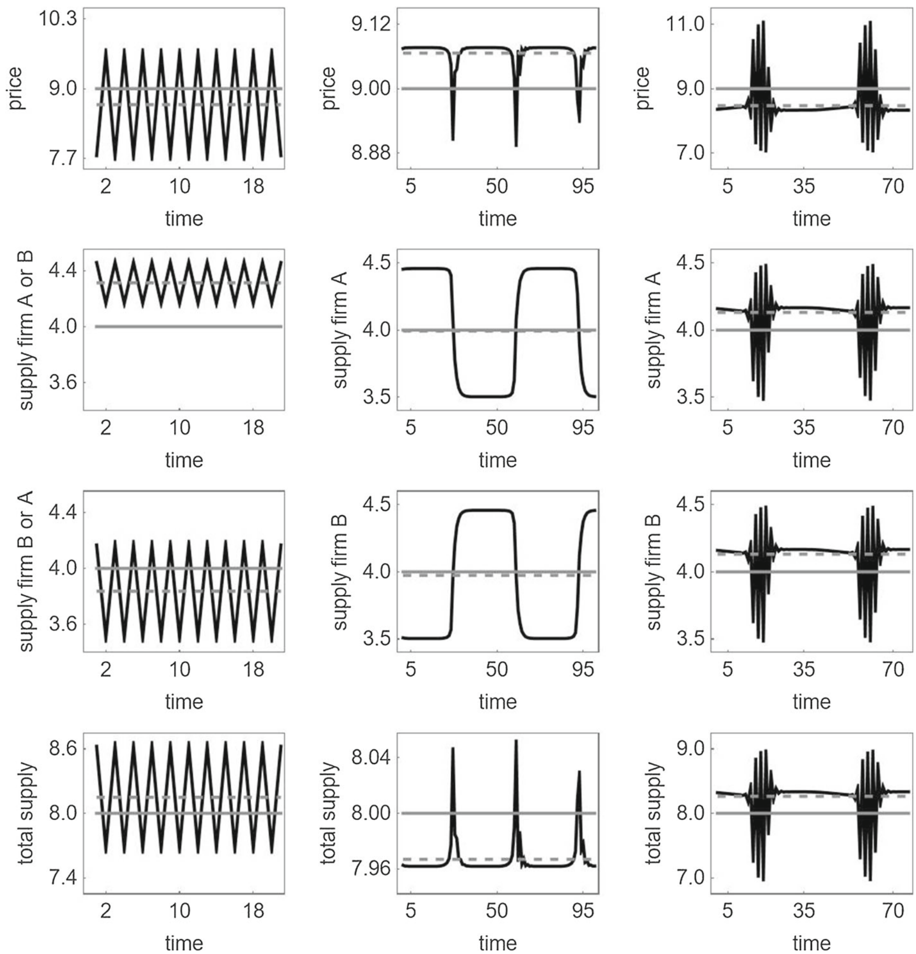

Fig. 7 Third set of time series diagrams for parameter d. The panels show the price, the supply of firm A, the supply of firm B and the firms' total supply in the time domain. The simulations are based on our base parameter setting, except that $d = 0.12$. The left, middle and right panels only differ with respect to their initial conditions. Since the period-two cycle depicted in the left panels exists for firm A and for firm B, there are four coexisting attractors in total. The solid (dashed) horizontal lines mark the steady-state (average) values of the quantities depicted

At around $d \cong 0.168$, the two period-two cycles, depicted in the left panels of Fig. 7, suddenly disappear, leaving space only for the two cyclical attractors, depicted in the middle and right panels of Fig. 7. One final remark is in order. The bifurcation diagrams in Figs. 2, 3, 4 highlight how attractors of different kinds suddenly appear, coexist for some time, and eventually disappear. In this respect, it is important to note that even a small change in expectation parameter d may have drastic consequences for the dynamics of our nonlinear duopoly model.

For completeness, let us briefly explore how the other two behavioral parameters may affect the dynamics of our nonlinear duopoly model. The top two panels of Fig. 8 present bifurcation diagrams in which the price and the supply of firm A is depicted for increasing

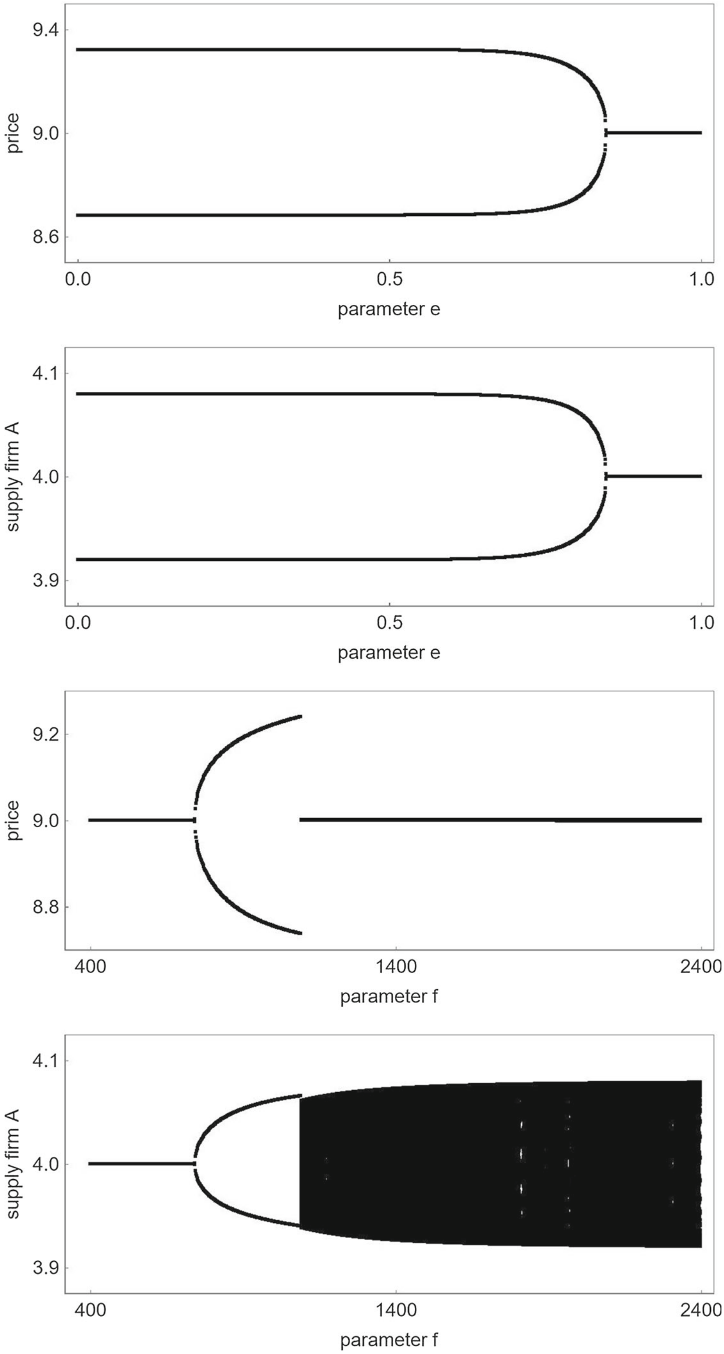

Fig. 8 Bifurcation diagrams for parameters e and f. The top two panels show the price and the supply of firm A for increasing values of parameter e. The bottom two panels show the price and the supply of firm A for increasing values of parameter f. The other parameters correspond to our base parameter setting

values of memory parameter e. In line with Proposition 2, we observe a Flip bifurcation at $e_{crit}^{Flip} = 0.848$. When memory parameter e is larger than $e_{crit}^{Flip} = 0.848$, the dynamics converges towards steady state SSN. When memory parameter e is smaller than $e_{crit}^{Flip} = 0.848$, our nonlinear duopoly model gives rise to a period-two cycle. Since the supply of firms A and B is either high or low, the variations in the firms' total production entail pronounced price fluctuations.

The bottom two panels of Fig. 8 display bifurcation diagrams depicting the price and the supply of firm A for increasing values of intensity of choice parameter f. As long as $f < f_{crit}^{Flip} = 742.2$, the dynamics of our nonlinear duopoly model converges to its steady state SSN. At $f = f_{crit}^{Flip} = 742.2$, we observe a Flip bifurcation and the birth of a period-two cycle. As the intensity of choice parameter f increases further, we observe the onset of cyclical dynamics. In contrast to the period-two cycle, the supplies of firms A and B now evolve asynchronously. When the supply of firm A is high, then the supply of firm B is low, and vice versa. Hence, the volatility of the firms' total production and price is rather low.

As can be seen in the top two panels of Fig. 8, a period-two cycle may exist for $e = 0$. Moreover, note that the dimension of the map of our nonlinear duopoly model decreases from eight to four for $e = 0$. While it is still difficult to explain in general how our nonlinear duopoly model functions for this parameter constellation, we are at least able to understand the emergence of a period-two cycle. For instance, the parameter setting $a = 25$, $b = 2$, $c = 1$, $d = 0.06$, $e = 0$ and $f = 10$ yields a locally stable period-two cycle at which the supplies of firms A and B are alternatingly given by ($q_1^A = 3.76$, $q_1^B = 3.76$) and ($q_2^A = 4.23$, $q_2^B = 4.23$). In Fig. 9, we portray the best-response dynamics of our nonlinear duopoly model for this cycle. To be precise, the light and dark blue lines mark the best-response functions of firm A, while the light and dark pink lines mark the same for firm B. Clearly, the different lines represent the best-response behaviors of a firm for period t as a function of the supply of its rival in period $t - 1$, assuming that the supply of its rival in period $t - 2$ is equal to the solution of the period-two cycle at that moment. Figure 9 shows that, given (q_1^A, q_1^B), the best-response of firms A and B are (q_2^A, q_2^B), and, given (q_2^A, q_2^B), the best-response of firms A and B are (q_1^A, q_1^B). In the presence of higher-order cycles, the behavior of each firm is characterized by more than two best-response functions, and a graphical representation becomes tedious. Nevertheless, it is easy to imagine that, given the past behaviors of firms' A and B, their actual best-response functions are constantly adjusting, giving rise to complex duopoly dynamics, as illustrated above.

5 Conclusions

In this paper, we develop a nonlinear duopoly model in which boundedly rational firms employ a set of different forecasting models to predict the supply decisions of their opponent. In particular, we focus on a setup in which the firms use a growth and a reduction forecasting model. Moreover, we consider that the firms display a learning behavior in the sense that they put more weight on a forecasting model that yields a higher prediction accuracy. As it turns out, the dynamics of our nonlinear duopoly model is due to an eight-dimensional nonlinear map. We analytically prove that our nonlinear duopoly model possesses a unique steady state. At the steady state, the production quantities of the two firms correspond to those that we observe at the Cournot-Nash equilibrium of the classical linear duopoly model with naïve expectations. Our analysis further reveals that the local stability of the steady state may only be compromised by a Flip bifurcation, an outcome that triggers endogenous dynamics

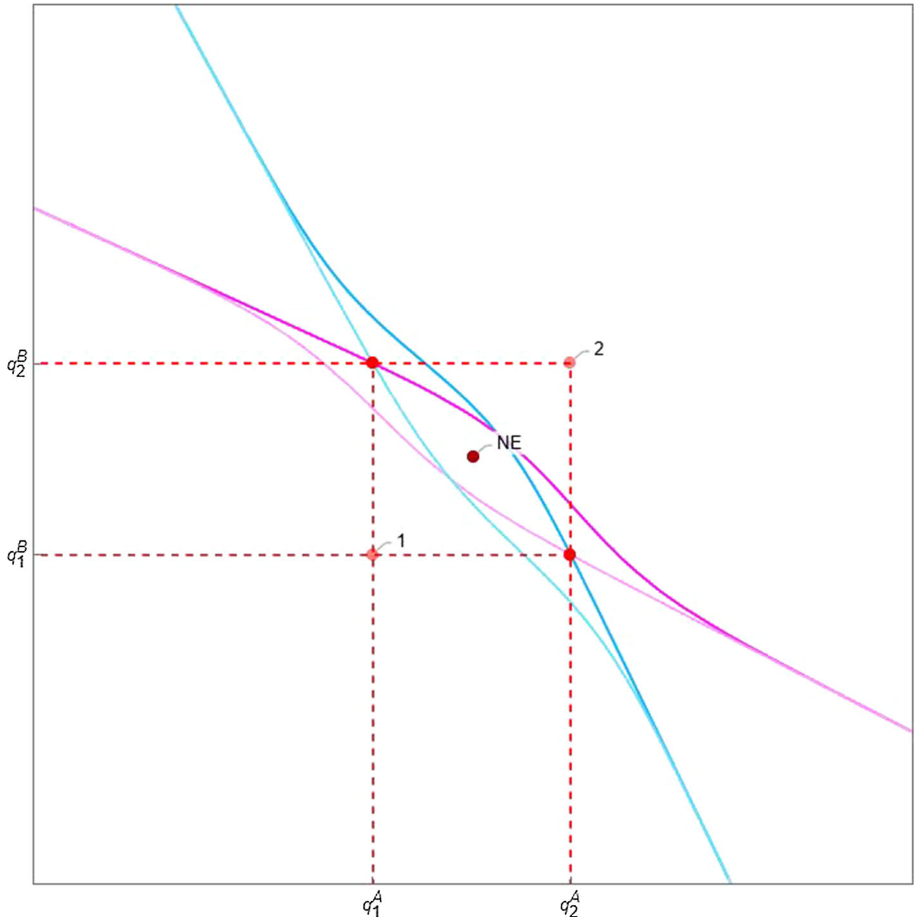

Fig. 9 Best-response dynamics within our nonlinear duopoly model for a period-two cycle. Parameter setting $a = 25$, $b = 2$ and $c = 1$, $d = 0.06$, $e = 0$ and $f = 10$ yields the period-two cycle ($q_1^A = 3.76$, $q_1^B = 3.76$) and ($q_2^A = 4.23$, $q_2^B = 4.23$). The light and dark blue lines mark the best-response functions of firm A for this period-two cycle; the light and dark pink lines mark the same for firm B. The figure shows that, given (q_1^A, q_1^B), the best-response of firms A and B are (q_2^A, q_2^B), and, given (q_2^A, q_2^B), the best-response of firms A and B are (q_1^A, q_1^B)

in the form of a period-two cycle. Such dynamics may, for instance, be set in motion when the firms' forecasting models predict that their opponent will adjust its supply sufficiently strongly.

Out of equilibrium, we observe that the firms' heuristic expectation formation and learning behavior may lead to complex dynamics. Amongst others, our nonlinear duopoly model gives rise to coexisting cyclical attractors, which in turn display distinctively different price and quantity properties. For instance, we may observe rather stable or unstable price and quantity dynamics for the same parameter setting. Price and quantity dynamics are more volatile when the supply of both firms evolves synchronously and less volatile when the supply of both firms evolves asynchronously. In the former case, the firms' total supply is relatively stable, while it is subject to larger fluctuations in the latter case. For an outside observer who only observes the price and quantity dynamics of our nonlinear duopoly model, the case in which the firms' supply evolves synchronously may appear as an instant of collusive behavior. Such a judgement would be wrong, since no collusive behavior is possible in our nonlinear duopoly model. Synchronous behavior is an emergent phenomenon in our model that occurs due to the firms' heuristic expectation formation and learning behavior.

Overall, we are impressed by the large number of distinctively different types of dynamic behavior that our nonlinear duopoly model can simultaneously generate. We conclude our paper by pointing out a number of possible model extensions. First, preliminary investigations reveal that our nonlinear duopoly model may produce intriguing dynamics when the firms differ, e.g. with respect to their fundamental or behavioral parameters. Studying an asymmetric version of our nonlinear cobweb model may be worthwhile. Second, one may regard our nonlinear duopoly model as a static, but repeated game. Against this backdrop, it may be interesting to assume that firms seek to condition their action on the discounted stream of future profits, taking into account their rival's future behavior. Third, we assume that firms maximize their absolute profits and that they weight the growth and reduction forecasting models based on absolute prediction accuracies. Schaffer (1989) and Vega-Redondo (1997) make a case for the use of relative instead of absolute performance indicators. Relatedly, Huck et al. (1999) report experimental evidence according to which firms also display imitative behavior. Incorporating these ideas may lead to even richer duopoly models. Fourth, we opted for a rather simple functional specification of firms' growth and reduction forecasting models—future work may consider alternative setups. We hope that our paper stimulates more research in this direction.

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Declarations

Conflict of interest All authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Appendix A: Proof of Proposition 1

In the following, we derive the map that drives the dynamics of the classical linear duopoly model with naïve expectations and show that its unique steady state is globally stable. From

(10) and (11) it immediately follows that the dynamics of this model is due to the two-dimensional linear map

$$L := \begin{array}{l} q_t^A = \frac{3}{2}\alpha - \frac{1}{2}q_{t-1}^B \\ q_t^B = \frac{3}{2}\alpha - \frac{1}{2}q_{t-1}^A \end{array} \tag{A1}$$

Straightforward computations indicate that map L has the unique steady state.

$$SSL = (\overline{q}^A, \overline{q}^B) = (\alpha, \alpha). \tag{A2}$$

Note that the total steady-state supply of firms A and B implies that the steady-state price results in $\overline{P} = \frac{a+2c}{3}$. At the steady state, the profits realized by firms A and B amount to $\overline{\pi}^A = \overline{\pi}^B = b\alpha^2$. The Jacobian matrix of map L reads as

$$J = \begin{matrix} 0 & -0.5 \\ -0.5 & 0 \end{matrix}, \tag{A3}$$

giving rise to the characteristic polynomial

$$P(\lambda) = \lambda^2 - 0.25. \tag{A4}$$

Since both eigenvalues of (A6) are equal to $\lambda_{1,2} = \pm 0.5$, we can conclude that the steady state SSL of map L is globally stable.

Appendix B: Proof of Proposition 2

In the following, we derive the map that drives the dynamics of our nonlinear duopoly model, compute its unique steady state and conduct a local stability analysis. Combining (6)-(7) with (12)-(25) enables us to express our nonlinear duopoly model in the form of an eight-dimensional nonlinear map, given by

$$N := \begin{cases} q_t^A = \frac{3}{2}\alpha - \frac{1}{2}\left(\frac{2dq_{t-1}^B}{1+\exp\left[f\left(e\left(F_{t-1}^{A,R} - F_{t-1}^{A,G}\right) - (1-e)4dx_{t-1}^B\left(q_{t-1}^B - x_{t-1}^B\right)\right)\right]} + (1-d)q_{t-1}^B\right) \\ x_t^B = q_{t-1}^B \\ F_t^{A,G} = eF_{t-1}^{A,G} - (1-e)\left(q_{t-1}^B - (1+d)x_{t-1}^B\right)^2 \\ F_t^{A,R} = eF_{t-1}^{A,R} - (1-e)\left(q_{t-1}^B - (1-d)x_{t-1}^B\right)^2 \\ q_t^B = \frac{3}{2}\alpha - \frac{1}{2}\left(\frac{2dq_{t-1}^A}{1+\exp\left[f\left(e\left(F_{t-1}^{B,R} - F_{t-1}^{B,G}\right) - (1-e)4dx_{t-1}^A\left(q_{t-1}^A - x_{t-1}^A\right)\right)\right]} + (1-d)q_{t-1}^A\right) \\ x_t^A = q_{t-1}^A \\ F_t^{B,G} = eF_{t-1}^{B,G} - (1-e)\left(q_{t-1}^A - (1+d)x_{t-1}^A\right)^2 \\ F_t^{B,R} = eF_{t-1}^{B,R} - (1-e)\left(q_{t-1}^A - (1-d)x_{t-1}^A\right)^2 \end{cases}, \tag{B1}$$

where $x_t^B = q_{t-1}^B$ and $x_t^A = q_{t-1}^A$ are auxiliary variables.

Setting $\overline{q}^A = q_t^A = q_{t-1}^A = \overline{x}^A = x_t^A = x_{t-1}^A$, $\overline{q}^B = q_t^B = q_{t-1}^B = \overline{x}^B = x_t^B = x_{t-1}^B$, $\overline{F}^{A,G} = F_t^{A,G} = F_{t-1}^{A,G}$, $\overline{F}^{A,R} = F_t^{A,R} = F_{t-1}^{A,R}$, $\overline{F}^{B,G} = F_t^{B,G} = F_{t-1}^{B,G}$ and $\overline{F}^{B,R} = F_t^{B,R} = F_{t-1}^{B,R}$, we find that map N possesses the unique steady state

$$\begin{aligned} SSN &= \left(\overline{q}^A, \overline{F}^{A,G}, \overline{F}^{A,R}, \overline{x}^B, \overline{q}^B, \overline{F}^{B,G}, \overline{F}^{B,R}, \overline{x}^A\right) \\ &= \left(\alpha, -d^2\alpha^2, -d^2\alpha^2, \alpha, \alpha, -d^2\alpha^2, -d^2\alpha^2, \alpha\right). \end{aligned} \tag{B2}$$

At the steady state, we furthermore have that $\overline{P} = \frac{a+2c}{3}$ and $\overline{\pi}^A = \overline{\pi}^B = b\alpha^2$.

Let us next study the local stability properties of steady state SSN. The Jacobian matrix of map N, evaluated at the steady state SSN, reads as

$$J(SSN) = \begin{matrix} \Phi & \Psi \\ \Psi & \Phi \end{matrix}, \tag{B3}$$

where

$$\Phi = \begin{pmatrix} 0 & \alpha^2 d^2 (1-e) f & -\frac{\alpha d e f}{4} & \frac{\alpha d e f}{4} \\ 0 & 0 & 0 & 0 \\ 0 & -2\alpha d(1+d)(1-e) & e & 0 \\ 0 & 2\alpha d(1-d)(1-e) & 0 & e \end{pmatrix} \tag{B4}$$

and

$$\Psi = \begin{pmatrix} -\frac{1}{2} - \alpha^2 d^2 (1-e) f & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2\alpha d(1-e) & 0 & 0 & 0 \\ -2\alpha d(1-e) & 0 & 0 & 0 \end{pmatrix}, \tag{B5}$$

respectively. Using the Laplacian expansion on the $(J - \lambda I)$ matrix, we may express the characteristic polynomial of $J(SSN)$ as

$$P(\lambda) = \lambda^2 (e-\lambda)^2 \left(\lambda^2 + \xi_1 \lambda + \xi_2\right)\left(\lambda^2 + \eta_1 \lambda + \eta_2\right), \tag{B6}$$

where $\xi_1 = -\left(\frac{1}{2} + e + \alpha^2 d^2 (1-e) f\right)$, $\xi_2 = \frac{e}{2} + \alpha^2 d^2 (1-e) f$, $\eta_1 = \left(\frac{1}{2} - e + \alpha^2 d^2 (1-e) f\right)$ and $\eta_2 = -\left(\frac{e}{2} + \alpha^2 d^2 (1-e) f\right)$.[11] Now, steady state SSN is locally stable when all eight eigenvalues of Jacobian matrix $J(SSN)$ are inside the unit circle.[12] In our local stability analysis, we are confronted with a situation in which two eigenvalues are equal to zero, i.e. $\lambda_1 = \lambda_2 = 0$. Moreover, two eigenvalues are equal to e, i.e. $\lambda_3 = \lambda_4 = e$. Since $0 < e < 1$, these two eigenvalues do not harm the local stability of the steady state. Put differently, the local stability of steady state SSN hinges on the remaining four eigenvalues, determined by two separate quadratic polynomials, namely $\left(\lambda_{5,6}^2 + \xi_1 \lambda_{5,6} + \xi_2\right)$ and $\left(\lambda_{7,8}^2 + \eta_1 \lambda_{7,8} + \eta_2\right)$. Let us first determine the necessary and sufficient conditions assuring that λ_5 and λ_6 are less than one in modulus, for which we have to check whether the stability conditions (i) $1 + \xi_1 + \xi_2 > 0$, (ii) $1 - \xi_1 + \xi_2 > 0$ and (iii)

[11] Instead of using this tedious procedure, it may be more comfortable to compute the characteristic polynomial using modern software tools such as Mathematica.

[12] Puu (1997), Medio and Lines (2001) and Gandolfo (2009) review standard tools to study the behavior of nonlinear dynamical systems.

$1 - \xi_2 > 0$ jointly hold. We find that stability conditions (i) and (ii) are always satisfied. In contrast, stability condition (iii) necessitates that

$$\frac{4-2e}{4} - \alpha^2 d^2 (1-e) f > 0. \tag{B7}$$

Let us next determine the necessary and sufficient conditions assuring that λ_7 and λ_8 are less than one in modulus. From the stability conditions (i) $1+\eta_1+\eta_2 > 0$, (ii) $1-\eta_1+\eta_2 > 0$ and (iii) $1-\eta_2 > 0$ we find that stability conditions (i) and (iii) are always satisfied, while stability condition (ii) requires that

$$\frac{1+e}{4} - \alpha^2 d^2 (1-e) f > 0. \tag{B8}$$

Since $4 - 2e > 1 + e$, stability condition (B8) is more binding than stability condition (B7). Solving stability condition (B8) for parameter d, we arrive at

$$d < \frac{1}{\alpha}\sqrt{\frac{(1+e)}{(1-e)f}}. \tag{B9}$$

As long as stability condition (B9) holds, steady state SSL is locally stable. If stability condition (B9) is violated, a Flip bifurcation renders steady state SSL unstable and a period-two cycle emerges. Finally, solving (B9) for parameters e and f reveals that

$$e > 1 - \frac{2}{1+4\alpha^2 d^2 f} \tag{B10}$$

and

$$f < \frac{(1+e)}{4\alpha^2 d^2 (1-e)}, \tag{B11}$$

respectively.

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