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Can you keep a secret? The dissemination of false rumors and the evolution of bubbles in perceived predatory trading games[☆]

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ABSTRACT

This research addresses the impact of fake rumors on the evolution of bubbles in stock prices in a perceived predatory trading game. In line with intuition, we find that the development of these bubbles is significantly different from the formation and progression of bubbles based on true rumors. Particularly, the first increase in stock prices may be followed by a crash when the rumor is not true. More volatility is observed when there are many players or when there is a lot of pressure to act on the victim. In contrast, the dissemination of the rumor has a dampening effect on the inflation of the bubble when the rumor is true. Moreover, it is more difficult to detect false rumors from stock prices when there are many predators. This observation makes herding among predators a particular challenging problem for regulators and policymakers.

1. Introduction

Stock price bubbles are deviations from the fundamental values of firms. They introduce additional volatility to stock markets and they can even become a threat to financial stability. Therefore, for practitioners and regulators it is important to understand their causes and their likely development. One possible reason for bubbles is a rumor among market participants that there are distressed traders who have to buy or sell a certain number of stocks. Such a rumor can trigger trading activities by other traders who try to exploit the pressure of the distressed traders to act. This behavior is known as predatory trading in the literature and generally describes the practice of exploiting a favorable trading environment that disadvantages investors who are compelled to trade.[1]

In this paper, we study the development of bubbles on the stock market due to false rumors (fake news). A particular focus is on the dissemination of the rumor. It is a characteristic of rumors that they emerge fast, uncontrollably and can spread quickly, especially via new media.[2] Therefore, we consider an arbitrary number of predators who believe that there is a distressed trader who has to trade a particular number of stocks. In contrast to the existing literature, though, we assume that the rumor is "fake news". In fact, it is an inherent characteristic of rumors that they can also be untrue. We are interested in the question how bubbles evolve and the impact of the dissemination of the false rumor as opposed to a true rumor.

We derive optimal strategies for strategic players and contribute to the literature in several aspects. Most importantly, we consider games with untrue rumors. Modeling under the assumption of fake news is a challenging task. In a first step, we address this problem by assuming that all players act as if the underlying information was correct. This solution is justified when every single player is fully convinced about the correctness of the perceived information. In a second step, the assumption is relaxed by introducing a correction mechanism to verify results. According to this correction mechanism, predators change their belief when observed prices deviate too far from expected ones.

To increase the realism, we assume a predatory trading game with "flexible ending conditions" for an arbitrarily large number of players. Intuitively, these ending conditions imply that market participants are willing to accept deviations from the target portfolio at the end of the game. However, deviations cause a penalty. As it will turn out, this additional flexibility is one of the reasons for significantly different bubble trajectories between those based on true rumors and those based

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[1] It should be noted that the definition used here merely assumes the existence of trading pressures. How these pressures arise is not the focus. While this definition aligns with the literature, such as Brunnermeier and Pedersen (2005) and Carlin, Lobo, and Viswanathan (2007), it should be pointed out that there are other approaches in which the creation of trading pressures is already considered part of predatory trading.

[2] The spread through social ties itself is studied in Arieli et al. (2019), Chen, Zenou, and Zhou (2018), Jackson and Zenou (2015) and Kobayashi, Ogisu, and Onaga (2023) to name just a few.

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on false rumors. Since games under flexible end conditions have a complex solution structure, we back up results by also considering games with "fixed ending conditions". Informally, these games can be seen as a limit case of flexible ending conditions with high penalties. One advantage of fixed ending conditions is that the solution structure is significantly simpler, making it explicitly analyzable.

In a nutshell, our paper extends the existing literature that usually considers (i) games with true rumors with (ii) mostly fixed ending conditions, and (iii) mainly concentrates on a small number of players. In line with intuition, we find that the truthfulness of the information has a decisive influence on the formation of a bubble. When the rumor is true the bubble inflates until the end of the game. In contrast, the bubble bursts during the game when the rumor is untrue. The price can even fall below the initial price, which means that price fluctuations in both directions must be taken into account. Moreover, expected trading profits of the predators turn into expected losses when rumors are untrue. The higher the penalty, that is, the higher the pressure on the predators not to deviate from their target portfolios, the less pronounced the bubble becomes in terms of the difference between the highest and the lowest expected price during the game.

More importantly, we establish a relationship between the price drop towards the end of a game and the dissemination of the rumor, that is, the number of players involved. As a rough rule of thumb, the more players believe the untrue rumor, the more rapidly the price falls. Thus, dissemination and extra volatility are positively related for fake news. We also find that bubbles burst earlier when the number of players is large and the rumor is untrue. This burst of the bubble never happens in the first half of the game, though. If the number of players is large then the complete life cycle of a bubble is always described and the bubble bursts at half time of a game with fixed ending condition. This is in contrast to the case of true information, where the expected price is monotonically increasing and the number of players has a dampening effect on the evolution of bubbles: The more informed players participate in the game, the smaller is the expected price increase during the game. There is less extra volatility.

To explore the implications of predators becoming aware of their mistake we introduce a correction mechanism. We assume that this is the case when market prices deviate from expected prices assuming that the rumor is true. We find that the effects found carry over under the condition that traders become aware of the incorrectness of their information during the game. It is noteworthy that the number of players characterizing the dissemination of fake news has an important impact on the detection of an untrue rumor. If the number of predators is high then it becomes unlikely that false beliefs are realized in an early stage of the game. Instead, predators exhibiting a stronger herding behavior are likely to correct their incorrect beliefs at a later point of time. This observation makes herding among predators a particular challenging problem for regulators and policymakers.

Our work is related to several strands in the literature. First, our approach extends the existing research on predatory trading, which focuses on true information trading and the resulting effects. Brunnermeier and Pedersen (2005) developed a deterministic model and consider spillover effects. Carlin et al. (2007) develop a stochastic framework and extend the research by the possibility of cooperative trading. Carmona and Yang (2011) generalize this model and search for a closed-loop equilibrium by using a game version of the Hamilton–Jacobi-Bellman approach. Our expansion within the information structure on possible misinformation adds a further perspective to the research. Second, we model the relatively new phenomenon of many tiny investors actively participating on the capital market in large scale. An empirical analysis with focus on the case of Robinhood traders is presented by Welch (2022). In conjunction, Umar et al. (2021) study the impact of social media driven sentiments and find that regulators and policymakers should continuously monitor the investing groups on social media platforms since they can create inefficiency in the market. This view is supported by our models. Third, the derivation of optimal trading recommendations and exploitation of market participants in distress is also linked to issues of behavioral finance and ethics. We extend the work of Sobolev and Clunie (2022) and show that also the predators can suffer losses. Fourth, we expand the literature of logical bubbles. The existence of such bubbles, even over longer periods of time, is discussed in Abreu and Brunnermeier (2003) and others.[3] We contribute to this line of literature by explaining new bubble progressions even over their whole life cycles.

The paper is organized as follows. Section 2 describes the model set-up and derives optimal strategies of the strategic players. Section 3 presents the effects of fake news on the development of bubbles and shows how these deviate from the case of true information. Section 4 studies the impact of model parameters on results in a comparative statics analysis. In particular, it addresses the number of players. In Section 5, we allow predators to identify their misinformation and correct it. Section 6 concludes. In order not to impair the flow of reading, all proofs can be found in the appendix. Furthermore, there is a multi-player justification in the appendix as well as some results about the predatory trading in the fixed case.

2. Model

In this paper, we assume that there are some market participants who believe that one of them is a perceived "victim" which has to adjust portfolio positions in a risky asset ("stock"). However, the rumor no longer needs to be the truth. In fact, our focus is on the implications when the underlying information is fake news. The question then is, what happens when investors start a predatory chase not because the conditions of such a hunt are actually met, but rather just because of their believe that the market environment is suitable. In doing so, we follow Brunnermeier and Pedersen (2005) and Carlin et al. (2007), who define predatory trading as trading that includes and/or exploits another investor's need to change their position.[4] This very broad definition particularly leaves open the question of why a trader has found themselves in such dire trading circumstances, thus differs from alternative definitions where predatory trading is the practice of creating a favorable trading environment (with low uncertainty) that disadvantages investors who have no choice but to trade, and making large profits from this manipulated trading setting.

The rumor initiates trading strategies that would be followed in predatory trading games. In our investigation, it is assumed here that the market participants' faith in the information presented to them is indeed so strong that they initiate their pursuit based on this information. As a framework that allows to work out the effects of this error we use a dynamic stochastic predatory trading game set-up. More precisely, we consider the general microstructure approach[5] as used in Carmona and Yang (2011) and Carlin et al. (2007).[6]

2.1. The general model

Trading and Price impact: In addition to the stock there is a risk-free asset.[7] Furthermore, the stock has imperfect liquidity and it is

[3] Logical bubbles are also tested and explained by Asako et al. (2020), Brunnermeier (2001), Brunnermeier and Morgan (2010) and Moinas and Pouget (2013) in a different model set-up.

[4] The same definition is also used in Carmona and Delarue (2013), Carmona and Yang (2011) and Schoeneborn and Schied (2009).

[5] The microstructure approach studies how trading mechanisms affect the price formation process and is one of the main sources of illiquidity investigated in the literature, see e.g. Eesley et al. (1996) and O'Hara (1995).

[6] Others that use this approach in a slightly different design are Brunnermeier and Pedersen (2005), Schied and Schoeneborn (2008) and Schoeneborn and Schied (2009) to name just a few.

[7] Without loss of generality, we set the interest rate to zero (see for instance Brunnermeier & Pedersen, 2005; Carlin et al., 2007 and Carmona & Yang, 2011).

actively traded by rational and risk-neutral market participants. Index the players, that is, the traders who believe to have the knowledge necessary to hunt a (perceived) distressed trader (and the victim itself), with $n = 1, \ldots, N$. Without loss of generality the first player is assumed to be the distressed player. Further $N \in \mathbb{N} \geq 2$ applies, such that a real game is established. By α_t^n we denote the associated trade rates[8] such that the traders position X_t^n in the asset at time $t \in \mathbb{R}^+$ is given by

$$X^n(t) = X^n(0) + \int_0^t \alpha^n(s)\mathrm{d}s. \tag{1}$$

Let X_t^0 denote the market (mid-)quote price of the traded asset at time t. Due to imperfect liquidity, the temporary trading price P_t, that is, the price at which the transaction takes place may differ from this price. The difference $P_t - X_t^0$ between these prices is known as the *temporary price impact*. Figuratively speaking, this describes how quickly market participants eat up the bid–ask limit-order book at given trading intensities. Thus, the temporary price impact is a trade rate dependent function. To be consistent with the existing literature, we choose a linear relationship in the trading rates,[8] which might be slightly disturbed by a random impact,[9] that is,

$$P_t - X_t^0 = \lambda \sum_{i=1}^{N} \alpha_t^i + \mathcal{N}(0, \xi). \tag{2}$$

The parameter λ is known as the *elasticity* factor and describes the depth of the limit-order book. It should be noted that this relationship is reversed, which means raising λ describes thinning out the limit-order book.

In addition to the temporary impact, there is also a *permanent price impact*. This describes the ongoing influence of trade rates on the mid-price, that is,

$$\mathrm{d}X_t^0 = \gamma \sum_{i=1}^{N} \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t. \tag{3}$$

The deterministic part of the above dynamics consists only of the permanent price impact, which is assumed to be linear[8]. This is because of the short time periods in which predatory trading takes place, the normal drift term $\mu(t)$ can be neglected.[10] The coefficient γ describes the *plasticity* factor of the market and is directly related to the information asymmetry that exists in the market regarding the risky asset.[11] Like the temporary price impact, the permanent price impact is also determined by the trade rate, which is backed up by numerous empirical studies.[12]

Noise trader: In addition to the predators and the victim, who are often referred to as *strategic players*, there are also the uninformed participants. These are those who have no knowledge of the purchase or liquidation needs of other players. For this part of the market, which represents the rest, the term *noise traders* has come into use. These also submit limit or market orders and thus influence the price movement over time. The influence of the noise traders is represented by the martingale diffusion process with volatility parameter σ in the price dynamics (3). The resulting influence of trading rates on asset prices can be found in a similar way in both empirical and theoretical studies like, for example, Almgren and Chriss (1999, 2001), Huberman and Stanzl (2004) and Kaul, Mehrotra, and Morck (2000).

Ending conditions: We identified three different groups that interact in the predatory trading game. The first group comprises market participants who are subject to trading constraints for various reasons. The second group consists of players who want to generate the greatest possible profit

$$-\int_0^T P_t \mathrm{d}X_t^n. \tag{4}$$

The third group are the uninformed players. While the latter have no direct influence on the game, the first two groups require to make sure that the profit or loss they generate results only from the game.[13]

Typically so a hands clean scheme is considered.[14] This scheme is a special case of *fixed ending conditions*,[15] which imply that strategic players have a target portfolio at the end of the game and that they do not accept any deviations from this target portfolio at all.

The set-up is generalized by assuming that predators do accept deviations from the initial portfolios. However, these deviations come at a cost. We refer to such ending conditions as *flexible ending conditions*. We introduce a penalty function[16]

$$g^n(X) := \frac{c}{2}\left(X^n - h\right)^2 \tag{5}$$

for each player, where X_t^n denotes the position of predator n at time $t \in \mathbb{R}^+$. This measures the financial losses incurred by a deviation from a given *target value* h. The parameter c describes the predators *penalty intensity*.

Remark on the victim's ending condition: In contrast to the predators, we do not allow the victim to deviate from its prescribed trading target. However, there are scenarios where such deviation might be conceivable. Hence, we aim to briefly address the implications of flexible ending conditions for the victim in Section 3. Since these invariably lead to the same expected effects, we choose to assign fixed ending conditions to the victim to keep things simple.

Differences and similarities of the terminal conditions: Flexible ending conditions account for the possibility that the player's trading constraints

[8] Even if the trading rates are actually piecewise constant, Carmona and Yang (2011) show that they can be reasonably approximated by continuous functions. It is further proven that the permanent price influence must be odd and linear if deterministic hands clean schemes (i.e., trade rates with the property $X_T - X_0 = \int_0^T \alpha_t \mathrm{d}t = 0$) with strictly positive expected return are excluded. Under these conditions, only more general restrictions can be derived for the temporary price impact, which are, however, fulfilled for linear functions.

[9] This simplified approach essentially yields the same effects as if the influence rate λ were set to be uncertain.

[10] This is in line with previous studies on predatory trading: Carlin et al. (2007), Carmona and Yang (2011) and Schoeneborn and Schied (2009). Furthermore, Huberman and Stanzl (2004) have shown that in a model without discounting no arbitrage assumptions are inconsistent with the existence of a drift term.

[11] It is important to distinguish between the prevailing information asymmetry regarding the asset, which is based on the market-microstructure approach and the misinformation regarding the purchase/liquidation needs as they are introduced at a later stage.

[12] See, for example, Chan and Lakonishok (1995), Holthausen, Leftwich, and Mayers (1990), Kraus and Stoll (1972), Madhavan and Cheng (1997), Meng et al. (2020) and Sadka (2006).

[13] Thus, the profit of a predator can be considered as the value of information and the loss of the distressed player can be interpreted as the price of information dissemination.

[14] A hands clean scheme specifies that the initial amount of risky assets held by the predators $\tilde{X}_0$ is equal to the amount held at the end of the game $\tilde{X}_T$.

[15] In this case, the final position $\tilde{X}_T$ which needs to be held at the end of the game is deterministic and given at the beginning of the game. Noting that the system described depends only on the stock changes and not on the absolute sizes one can choose the initial stocks by $X_0 := \tilde{X}_0 - \tilde{X}_T$, which gives the final conditions by $X_T^n = 0 \ \forall n \in \{1, \ldots, N\}$. Thus, under deterministic conditions, each player must have completely liquidated his stock at the end of the game, which is a typical way to express deterministic ending conditions.

[16] Penalty functions are a long-established concept of constraint optimization. Put simply, this is how we prevent a player from going outside a certain allowable range (too far) by penalizing them more severely the closer they get to the boundary. Even if technically the penalty functions used here play a somewhat different role, this motivation already offers a nice idea of their effects.

may not be ultimate. Given the existence of the penalty function, players can choose their strategies freely, implying

$$\alpha_t^n = (\alpha_t^n)_{0\le t\le T} \in \mathbb{A}^n = \mathcal{H}^2_{[0,T]} = \left\{ \alpha_t^n \middle| \text{ adapted and } \mathbb{E}\int_0^T \left|\alpha_t^n\right| \mathrm{d}t < \infty \right\}. \tag{6}$$

Letting the penalty intensity c soar makes players intuitively compelled to precisely meet their target value. Hence, fixed ending conditions can be considered as a limit case of flexible conditions.[17] In this scenario, the penalty function can be omitted from the outset, and only those strategy profiles

$$\alpha_t^n = (\alpha_t^n)_{0\le t\le T} \in \mathbb{A}^n = \left\{ \alpha_t^n \middle| \mathcal{H}^2_{[0,T]} \text{ and } X_T^n = 0 \right\} \tag{7}$$

that exactly hit the prescribed terminal value are permitted. To understand why a distinction is made between the two types of ending conditions, it should be noted that flexible ending conditions lead to interesting effects on the trajectory of the bubble, which also depend on the accuracy of the rumors involved. Fixed ending conditions, on the other hand, present a manageable solution structure that enables the explicit proof of numerous observations.

Stochastic differential game: Each player tries to maximize the personal profit (4), which is calculated from the transaction price and the associated trade rates. However, since the selected trading rates influence the transaction price through the temporary (2) and permanent (3) price impact, which is relevant for the optimization of each individual player, the players cannot optimize their trading profile independently of each other. Thus, the evolution of our economy is given by a *stochastic differential game.*[18] More precisely the $(N+1)$-dimensional system

$$\begin{cases} \mathrm{d}X_t^0 = \gamma \sum_{i=1}^N \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t, \\ \mathrm{d}X_t^n = \alpha_t^n \mathrm{d}t \quad \forall n \in \{1,\dots,N\}, \end{cases} \tag{8}$$

is considered, where X_t^0 is the mid-price of the risky asset at time t, X_t^n are the positions held by the players at time t, and α_t^n are their associated trading rates from admissible strategy sets $\mathbb{A}^n$ ((7) for the victim and (6) for the predators). For a given strategy profile $\boldsymbol{\alpha} = (\alpha^1,\dots,\alpha^N)$, each player tries to maximize[19] the personal revenue functional. For the predators it is given by

$$J^n(\boldsymbol{\alpha}) = \mathbb{E}\left(\int_0^T \alpha^n \left(X_t^0 + \lambda \sum_{i=1}^N \alpha_t^i + \mathcal{N} \right) \mathrm{d}t + g^n(X_T) \right) \tag{9}$$

with $g^n(\cdot)$ defined in (5) and for the victim the functional simplifies to

$$J^n(\boldsymbol{\alpha}) = \mathbb{E}\left(\int_0^T \alpha^n \left(X_t^0 + \lambda \sum_{i=1}^N \alpha_t^i + \mathcal{N} \right) \mathrm{d}t \right). \tag{10}$$

2.2. Modeling fake news

Definition of Fake News: Modeling games under false information can pose a challenging task even with basic models.[20] Therefore, we propose the following approach: Each predator n selects their trading strategy α_t^n in such a way that $\boldsymbol{\alpha} = (\boldsymbol{\alpha^1},\dots,\boldsymbol{\alpha^N})$ with

$$J^n(\boldsymbol{\alpha}) \le J^n(\boldsymbol{\alpha^{-n}}, \boldsymbol{\alpha^n}), \quad \forall n \in \{1,\dots,N\}, \quad \forall \boldsymbol{\alpha^n} \in \mathbb{A}^n \tag{11}$$

holds true under the assumption

$$x_{0,i}^1 = x_{0,n}^1 \qquad \forall\, i \in \{1,\dots,N\}, \tag{12}$$

where $x_{0,n}^1$ denotes the terminal value of the victim as assumed by player n. Further, as the rumor about the victim's trading requirements is just fake news, the victim is not really in the need to trade, and thus it will not do so. This is, the victims trading strategy is $\alpha^1 = 0$. This approach offers the crucial advantage of transferring the unique solvability of the problem without misinformation directly to the current case. Indeed, the corresponding trading strategies can be directly derived from the associated game under correct information.

Fake News vs. Uncertainty: Our understanding of fake news entails that each player is convinced of the correctness of their information and therefore must also assume that all other players possess this same information. Fake news thus lead to information asymmetry, as the individual players, specifically the predators and the victim, have different information, but do not mutually know that this is the case. This results in each player choosing the trading strategy that is perfectly suited under their given information. Due to the information asymmetry, the individual strategies are not optimally coordinated with each other. Fake news therefore induce errors into the system, leading to a deviation from equilibrium, in which we are interested in the process and consequences. It is important to delineate our understanding of false rumors from games under uncertainty. The latter assumes essentially complete availability of information, meaning all players are aware that their information might not be accurate and therefore assign a corresponding probability to it, and secondly, all players are aware of the uncertainty of other players and attempt to exploit this accordingly. Such cases typically lead to a shift, not a deviation, from equilibrium. We have decided against this approach, as it implicitly assumes that players who fall for fake news are aware from the beginning that it could be false, which does not seem accurate to us.

Updating: To investigate the question of how long false information can persist, it is necessary to allow for information updates during the game. These need to be adapted to the definition of fake news and are discussed separately in Section 5.

Open loop vs. Closed loop: For our investigation, we resort to *open-loop strategies*.[21] These are independent of the current state of the system but offer the advantage of being generally 'simpler' to handle mathematically. Especially explicit solutions in closed form are often feasible, which is of crucial importance for an increasing number of players and a explicit analysis of the results. It should be noted that predetermining strategies does not represent a limitation in terms of dealing with fake news, as these can still exist due to stochastic influences, even with (partial) observability. For further insights, refer to the subsequent discussion and Section 5.

Closed-loop strategies assume that the entire system can be observed, and trading decisions are made based on the states. Besides the fact that these are numerically hardly manageable for multiple players,[22] they also pose some difficulties regarding false information. Indeed, the predators could observe the victim's holdings at any time, particularly noticing if it does not trade at all. Without allowing for information updates, it is entirely unclear why this is the case. Apart from incorrect

[17] If $c \to \infty$ one expects $X^n = h^n\ \forall n \in \{1,\dots,N\}$. Thus, fixed ending conditions can be seen (in a motivational sense, not mathematically) as a limit case of the flexible counterpart with $\tilde{X}_T = h$.

[18] Among others, see Başar and Olsder (1999) for a general overview, An and Øksendal (2012) for stochastic differential games under partial information, Saporito (2019) for an extension to path-dependent dynamics and running costs and Carmona and Delarue (2013) for stochastic games with mean field interaction.

[19] To stay in line with the common notation within the optimization literature, we transform all the upcoming optimization problems into equivalent minimization problems.

[20] See e.g. Fudenberg and Levine (1993) for the case of self-confirming equilibria in a sequential two-step game.

[21] It should be noted that these terms are not used consistently in the literature. We speak of an open loop strategy if the controls are of the form $\alpha_t = \phi(X_0, t)$ with some deterministic function ϕ. In contrast, we speak of a closed loop strategy, if $\alpha_t = \phi(t, X_{[0,t]})$.

[22] As demonstrated in Carmona and Yang (2011), this is already the case for two players under correct information.

information, it could just as well be possible that the victim does not respond optimally. On the other hand, guaranteeing the rationality of the victim and allowing the predators to immediately correct their erroneous information upon deviations would lead to the instant detection of fake news under perfect observability.

Considering it highly unrealistic that all players can view the exact portfolio positions of others at all times, we deem the aforementioned problem not significant in practice. Finally, a middle ground between open and closed-loop strategies, the *partial information strategies*, is briefly discussed. These allow for the observability of parts of the system, such as price developments, but not portfolio positions. Since price developments are subject to stochastic disturbances, only the victim's position can be estimated from this. A deviation from the expected position could thus be due to incorrect information or the stochastic part of the price influence. Indeed, if the player is convinced of the correctness of their information, the latter will always be chosen. In this case, there is consequently no conflict with our understanding of fake news. It remains to clarify that even in this case, filtering out fake news must always occur via an updater and not through the strategy profile itself. Since partial strategies are even more challenging to determine than closed-loop strategies, they cannot be used for the desired investigation of larger numbers of players. However, we intend to revisit the idea of drawing conclusions about the victim's actions through partial observations in the discussion about updaters in Section 5.

2.3. Solution

Following the definition of fake news, the strategies of the players are given as those that would result if their information were true. Therefore, it is essential to solve the game under real conditions in order to subsequently use the solution found in case of false information. Thus we are looking for a Nash equilibrium, that means admissible strategies $\hat{\boldsymbol{\alpha}} = (\hat{\boldsymbol{\alpha}}^{\mathbf{1}}, \dots, \hat{\boldsymbol{\alpha}}^{\boldsymbol{N}})$ with

$$J^i(\hat{\boldsymbol{\alpha}}) \leq J^i(\hat{\boldsymbol{\alpha}}^{-\boldsymbol{i}}, \hat{\boldsymbol{\alpha}}^{\boldsymbol{i}}), \quad \forall i \in \{1, \dots, N\}, \quad \forall \hat{\boldsymbol{\alpha}}^{\boldsymbol{i}} \in \mathbb{A}^i. \tag{13}$$

The following theorem states the optimal trading strategies for a Nash-equilibrium:

Theorem 2.1. *Consider the game where all players try to maximize their personal profit with respect to their admissible trading strategies described above. Then the optimal open-loop trading strategies are given by*

$$\alpha_t^1 = ae^{-\frac{N-1}{N+1}\frac{\gamma}{\lambda}t} + b_1 e^{\frac{\gamma}{\lambda}t} \quad \text{for the victim and} \quad \alpha_t^n = ae^{-\frac{N-1}{N+1}\frac{\gamma}{\lambda}t} + b_2 e^{\frac{\gamma}{\lambda}t} \tag{14}$$

for the predators ($n \in \{2, \dots, N\}$). The coefficients $a \in \mathbb{R}$ and $b_n \in \mathbb{R}$, $n \in \{1, 2\}$ are determined by

$$a = -\frac{N-1}{N+1}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{N-1}{N+1}\frac{\gamma}{\lambda}T}\right)^{-1}\left(\frac{x_0^1}{N} + \frac{(N-1)x_0^p}{N}\right), \tag{15}$$

$$b_1 = \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1}\left(\frac{N-1}{N}\left(x_0^p - x_0^1\right)\right)$$

$$b_2 = \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1}\left(\frac{x_0^1 - x_0^p}{N}\right) \text{for } n \neq 1.$$

Hereby the quantity x_0^p is given as

$$\left(2ch + K_3 K_2^{-1}\right)\left(-2c + \gamma K_4 K_2^{-1}\right)^{-1} \tag{16}$$

with

$$K_1 = \frac{T\gamma}{\lambda}, \qquad K_2 = \left(e^{K_1} - 1\right)\left(e^{(N+1)^{-1}K_1} - e^{N(N+1)^{-1}K_1}\right)N,$$

$$K_3 = -e^{\frac{N+2}{N+1}K_1} N X_0^0 - e^{\frac{N}{N+1}K_1} N(X_0^0 - \gamma x_0^1) + e^{\frac{1}{N+1}K_1}(N X_0^0 - \gamma x_0^1) + e^{\frac{2N+1}{N+1}K_1}(N X_0^0 - \gamma N x_0^1 + \gamma x_0^1),$$

$$K_4 = -e^{\frac{N+2}{N+1}K_1} N - e^{\frac{N}{N+1}K_1} N(N-1) + e^{\frac{1}{N+1}K_1}(N-1) + e^{\frac{2N+1}{N+1}K_1}((N-1)N + 1).$$

Theorem 2.1 states that the case under consideration, in which the predators follow flexible ending conditions, can be reduced to the case of fixed ending conditions. This transformation is made by assuming that the predators are forced to trade the quantity x_0^p. Furthermore, it holds that $\lim_{c\to\infty} x_0^p = -h$, indicating that with (infinitely) high penalty intensity, the predators are compelled to exactly achieve their target value. In this case, the game with flexible ending conditions exactly matches the game under fixed ending conditions (without the need for a transformation over the target set x_0^p).

3. General effect of fake news

In this section we demonstrate the fundamental influence of fake news and address the most important differences between bubbles that result from true and untrue rumors. Furthermore, we address the role of the penalty parameter which drives the differences between flexible and fixed ending conditions. In some cases, due to the complex structure of the solution in the case of flexible ending conditions, influences of parameters and relationships can only be demonstrated numerically. In these instances, we resort to the limit case $c \to \infty$ to substantiate the observations with proofs.

In this chapter, we proceed under the assumption that trading strategies, once selected, are maintained until the end of the game. This scenario might be representative of algorithmic trading or instances where the predators remain convinced of the correctness of their information until the very end. This simplifying assumption allows us to demonstrate and describe the fundamental mechanisms through which fake news can influence outcomes. The possibility that the predators recognize their mistake and adjust their strategies accordingly is discussed in Section 5.

3.1. Bubble dynamics under fake news

We start by examining the impact of false rumors on the expected price trajectory. Utilizing Theorem 2.1, the explicit formulation of this impact can be specified as follows:

Theorem 3.1. *Consider the stochastic differential game , where N players try to maximize their personal profit within their given strategy sets. If all predators trade according to their open loop strategies (14) while the alleged distressed player 1 does not trade at all then the expected price development is given by*

$$\mathbb{E}\left(X_t^0\right) = X^0(0) + c_1\left(1 - e^{-\frac{N-1}{N+1}\frac{t\gamma}{\lambda}}\right) + c_2\left(e^{\frac{\gamma t}{\lambda}} - 1\right). \tag{17}$$

The coefficients $c_1, c_2 \in \mathbb{R}$ are determined by

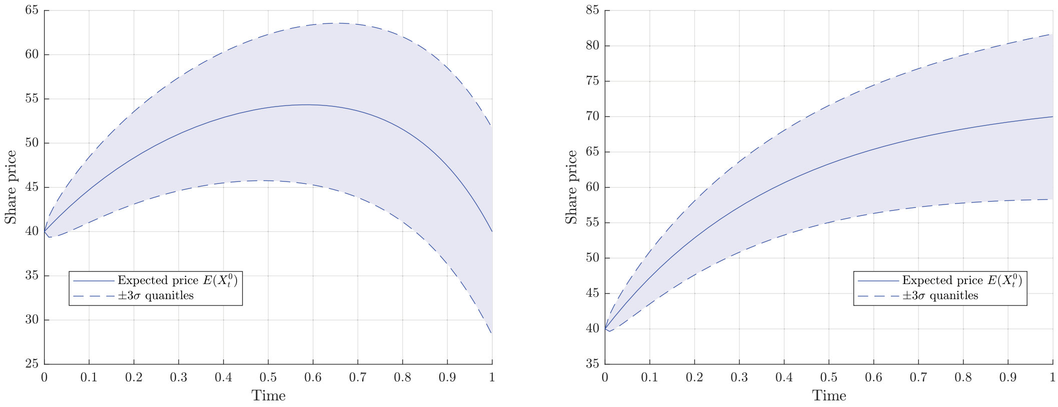
$$c_1 = -\frac{N-1}{N}\gamma\left(1 - e^{-\frac{N-1}{N+1}\frac{T\gamma}{\lambda}}\right)^{-1}\sum_{i=1}^{N} x_0^i, \tag{18}$$

$$c_2 = \gamma\left(e^{\frac{\gamma T}{\lambda}} - 1\right)^{-1}\left(\frac{(N-1)x_0^1}{N} - \sum_{i=2}^{N}\frac{x_0^i}{N}\right). \tag{19}$$

Further x_0^1 denotes the quantity wrongly assumed to be traded by the victim and $x_0^i = x_0^p$ with x_0^p as in Theorem 2.1 for $i = 2, \dots, N$.

The implications of Theorem 3.1 are demonstrated and elucidated through a representative simulation in the subsequent discussion.

Parameter: We follow the related literature and consider a low number of strategic players first ($N = 3$): There is one (perceived) victim (player 1) and two predators (player 2 and 3). In Section 4 below, we study the effect of a larger number of players when a rumor becomes more widespread through, for example, social media. In our



(a) Case: Untrue Rumor (two predators and one perceived victim)

(b) Case: True Rumor (two predators and one true victim)

Fig. 1. Expected Prices: Untrue vs. True Rumor with Flexible Ending Condition. The figure shows the expected price trajectories and the corresponding $\pm 3\sigma$ quantile the same game, once under untrue 1(a) and once under true information 1(b). The simulation is performed for a plastic market with a high penalty intensity for the predators. It is assumed that the victim wants to purchase another 300 units of the risky investment and the total number of players is set to $N = 3$.

base case,[23] the predators believe that the (perceived) victim has to buy 300 shares ($X_0^1 = -300$). In contrast, the predators do not face a trading compulsion, hence $h = 0$. Furthermore, the current price of the underlying stock is $X_0^0 = 40$. In line with the related literature,[24] we assume a plastic market, that is, the plasticity factor exceeds the elasticity factor and $\frac{\lambda}{\gamma} < 1$. We set $\gamma = 0.1$ and $\lambda = 0.2\gamma$. The duration of the game is $T = 1$.

Influence on bubble trajectory: Panel 1(a) shows the expected price developments for untrue rumors. In this situation, expected prices increase until they reach a maximum during the duration of the game. After that, the bubble bursts and expected prices decrease again. The burst of the bubble is endogenously explained within the model. The expected terminal price may deviate from the initial price. In the benchmark case, the initial price is 40 and increases to about 55. After that it decreases again to a terminal price of about 40.

In the first stage of the game, each predator buys the underlying asset. This is done to create an excess demand and thus drive the corresponding price upwards.

In the second stage, the predators plan to sell the acquired shares back to the victim at the elevated price. The optimal strategies of the predators can, therefore, be summarized as follows: Purchase shares at the beginning of the game to push the price up and then sell them at a higher price to the victim, who, due to the trading obligation, has no choice but to buy them at the given price. Since the rumor about the victim's buying obligation was false, the predators are forced to sell without the expected demand from the victim. The resulting surplus supply drives the price back down.

The severity of this price drop towards the end of the game significantly depends on the magnitude of penalties and the number of players involved. The corresponding effects are discussed in Sections 4 and 3.2.

The Limit Case: In the case of an 'infinite' penalty, the predators are compelled to engage in an exact zero-sum game. Thus, they are not allowed to sell more or fewer shares at the end of the game than they initially acquired. It is intuitively clear that in such a case, the intensity of the price drop cannot depend on the magnitude of the penalty (Indeed such a quantity does not exist in the limit case). Moreover, it turns out that, in the present case, it is also independent of the number of players.

Theorem 3.2. *Assume the situation as in the limit case of Theorem 2.1 (Theorem A.1) with the predators trading according to a hands clean scheme, then*

$$\mathbb{E}\left(X_T^0\right) = X_0^0. \tag{20}$$

This theorem states that the expected terminal price is equal to the initial price. This means that in the case of false rumors, the complete life cycle of a bubble is described by the model. When considering bubble formation from the perspective of underlying trading volumes, this result aligns with our intuition. Since no additional demand or supply is generated in mean throughout the course of the game, it is reasonable that false rumors do not have a long-term impact on the price. Thus the price needs to drop back to the initial fair value.

In fact, it is even possible to specify more precisely when this price drop begins. The following theorem summarizes the most important findings:

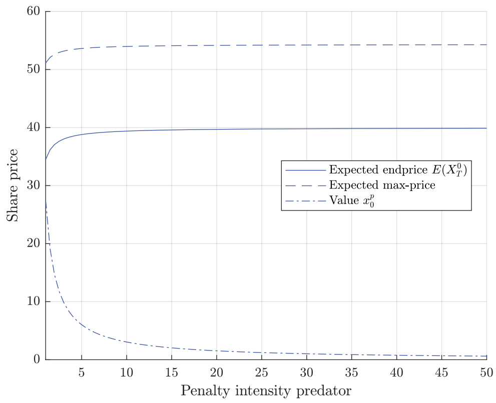
Theorem 3.3. *Assume the same situation as in Theorem 3.2, then the expected price changes direction during the second half of the game. More precisely, the expected price attains its maximum/minimum within the time interval $[\frac{T}{2}, T)$ if the rumor is untrue.*

Theorem 3.3 yields three results, which do not depend on the parameters of the game. First, untrue rumor creates bubbles. Second, the bubble is expected to burst during the game. Third, the burst of the bubble never happens in the first half of the game. This means the bubble inflates over a period of at least $\frac{T}{2}$ before it bursts.

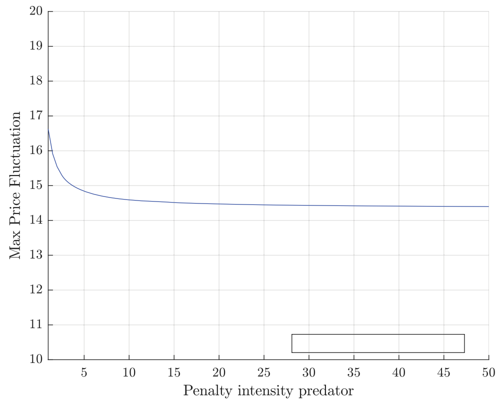
Fake News vs. True Information: To emphasize that the described effects indeed stem from false rumors, it is necessary to delineate the bubble trajectories from those that would be expected under true rumors. Panel 1(b) shows the expected price development (for the same game as shown in Panel 1(a)) when the rumor is true. In line with intuition, it highlights that implications are clearly different. This can be attributed to the fact that the victim now indeed has to purchase the shares. This demand offsets the supply from the predators in the second phase of the game, leading to a situation where, contrary to false rumors, the expected prices increase monotonically until the end of the game. Therefore, the expected terminal price reaches the maximum, and

[23] If not mentioned otherwise, these are the parameters used to generate the figures shown in this article. It should be noted that the described effects also occur in the numerous simulations carried out under different parameter selection. Furthermore, the most important statements are explicitly proved under fixed ending conditions. Thus, a sufficient robustness can be assumed and the concrete parameter choice serves only for demonstration purposes.

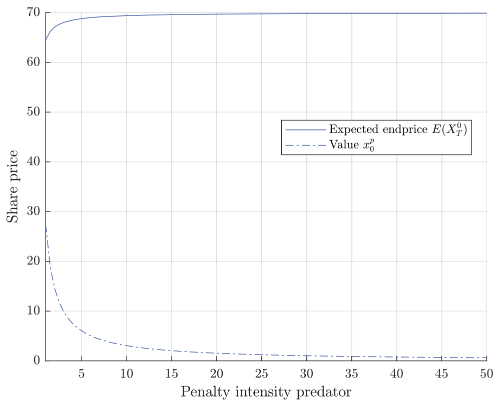
[24] See e.g. Carlin et al. (2007), Carmona (2016) and Carmona and Yang (2011).



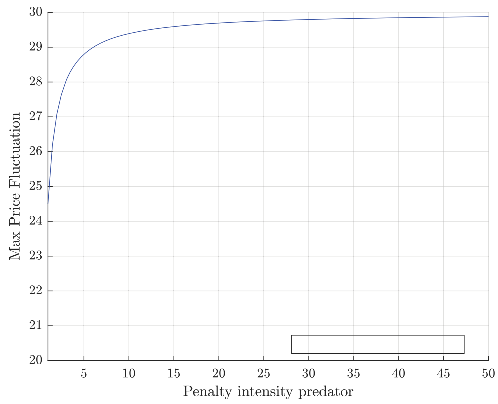
(a) Additional sales x_0^p in dependence of the predators penalty in the case of untrue rumor.



(b) Expected max. price fluctuation in dependence of the predators penalty in the case of untrue rumor.



(c) Additional sales x_0^p in dependence of the predators penalty in the case of true rumor.



(d) Expected max. price fluctuation in dependence of the predators penalty in the case of true rumor.

Fig. 2. Influence penalty parameter predator. The figure illustrates the interplay between the penalty intensity of the predators and the variables x_0^p (additional selling volume), the expected maximum price increase, the expected price decline at the end of the game, and the maximum price fluctuation, for both cases of true and false rumors.

whether or how it subsequently returns to its original price is not addressed in these models. Specifically, this means that in the case of true rumors, the bubble does not burst during the game. This is in line with existing literature on true rumors.[25] Furthermore, it is noticeable that the price increase is potentially more significant. This is not surprising, since the victim, who is under the obligation to sell, creates additional excess demand.

3.2. Penalty parameter

As shown in Theorem 3.2, in the limit case, the expected final price coincides with the initial price. Generally, however, deviations between these two prices can occur, as according to Theorem 3.1

$$\mathbb{E}\left(X_T^0\right) = X_0^0 - (N-1)\gamma x_0^p. \tag{21}$$

This demonstrates that the deviation from the fair price at the beginning of the game is driven by the quantity x_0^p. Recall that this quantity indicates how far the predator can deviate from their target value and accept the resulting penalty. Thus, it is evident that the bubble trajectories under the influence of fake news are closely related to the penalty function.

Predator's Penalty: Panel 2(a) shows the additional trading volume x_0^p of the predators as a function of the penalty intensity c. It is clearly observable that at very low penalty intensities, the sale of many additional share portions is planned, while for increasing penalty intensities, this quickly converges towards zero. This is intuitive, as selling additional shares at inflated prices becomes increasingly profitable as the negative consequences of these sales diminish. The trajectory of the final price, also seen in the figure, now aligns with (21). When the penalty is low, the additional sales volume is high, leading to a larger supply surplus towards the end of the game, causing the price to fall even below the initial value. However, if the penalty is high, the additional planned sales approach zero, and thus, there is hardly any supply surplus to drive the price below the original price. Besides the final price, the sales volume x_0^p can also affect the expected maximum share price during the game. This is likewise illustrated in Panel 2(a) and increases with the severity of the penalty. Revisiting the quantity x_0^p, it is unsurprising that with an overall higher planned sales volume, the price increase is less significant than with a lower volume.

Overall, it can be established that a higher penalty intensity leads, on one hand, to a lesser price drop below the original value but, on the other hand, to a greater price increase during the game. To consider the overall effect on the resulting bubble, we define:

Definition 3.1. We define the maximum expected price fluctuation during the time period $[t_*, t^*]$ as

$$MPF(t_*, t^*) := \max_{t_1, t_2 \in [t_*, t^*]} \left| \mathbb{E}(X_{t_1}^0) - \mathbb{E}(X_{t_2}^0) \right|. \tag{22}$$

Intuitively, the MPF is a measure of the intensity of a bubble. If the maximum in Definition 3.1 is reached for $t_1 = t_*$, the quantity is

[25] See, for example, Carlin et al. (2007) and Carmona and Yang (2011).

a measure for the maximum deviation from the initial price. If this is not the case, it is an indication that during the game and the associated bubble formations, distortions below as well as above the fair value are to be expected.

The maximum price fluctuation is illustrated in Panel 2(b). It is evident how it decreases with increasing penalty intensity. Thus, the positive effect of a reduced price plunge towards the end of the game outweighs the negative effect of a larger maximum price increase during the game.

Fake News vs. True Information: Panel 2(c) presents the game under true rumors. As before, the trading volume x_0^p decreases with increasing penalty intensity, consequently leading to an increase in the expected maximum price. Due to the monotonicity of the expected price trajectory under true rumors, this (the maximum price) therefore coincides with the expected final price. As a result, the maximum price fluctuation must also increase with rising penalty severity. This is demonstrated in Panel 2(d).

The effects of penalty severity thus fundamentally differ depending on whether the rumor is true or false. While true rumors lead to larger, but unidirectional, price fluctuations with increasing penalty intensity, under false rumors, one can expect smaller, yet bidirectional (rise above the original price and crash below the initial price), fluctuations.

Victim' Penalty: Although in the main model proposed by us the victim itself is not allowed to deviate from the trading objective set for them, we would like to briefly discuss the consequences of allowing such a deviation at this point. Panel 3(a) initially shows the adjusted trading volume of the victim x_0^1 as a function of their penalty intensity. If this is low, it is worthwhile to accept a penalty instead of fulfilling the trading obligation at the inflated prices in full. With increasing penalty severity, this incentive disappears and the actual trading volume of the victim converges towards the target specification. This, in turn, is picked up by the predators. Thus, for them, it is no longer profitable to drive up the price too much at a low penalty intensity of the victim, as otherwise, the victim would prefer to accept the penalty rather than continue buying. Consequently, the maximum price fluctuation increases with the victim's penalty intensity. This argumentation also applies in the case of true rumors, as shown in Panel 3(b).

In summary, two observations can be made. First, the penalty function of the victim influences bubble formation in the same direction, regardless of whether the rumor is true or false. Thus, the effect is not exclusively attributable to the study of fake news presented here. Second, predatory trading arises precisely because of a prevailing trading obligation. It would therefore be assumed that in such a situation, the penalty intensity of the victim is very high. Considering the rapid convergences presented, the influence of the penalty function in these areas is limited anyway. To keep the model simple, we have therefore decided to make the trading obligations of the victim binding (which corresponds to the limit case).

The Penalty Function: The intention of the predators to sell additional shares at the inflated prices is crucial for the dynamics demonstrated. These effects, while attributable to the penalty function, do not necessarily depend on the exact structure of this function. Indeed, it is merely important that the chosen penalty function ensures that further sales at elevated prices are profitable.

3.3. Profit and loss analysis

While we have studied how false rumors affect price stability so far, we will now focus on the impact on the players themselves. This means that we are looking for a profit and loss analysis regarding the predators. The trading profit can be computed as given by (9), where the system behaves as in Theorem 3.1.

Fake News: Panel 4(a) displays the penalty magnitude, the expected trading loss, and the associated total loss of the predators depending on their penalty intensity in the case of false rumors. Starting with trading profits, it is initially noticeable that these are consistently negative. This means that costs incurred to drive up the price cannot be offset by selling the shares into the crash towards the end of the game. With low penalty intensity, the trading losses of the predator are tendentiously lower, which is due to the sale of a larger quantity of shares, somewhat cushioning the losses. With increasing penalty intensity, this possibility of additional sales progressively disappears, making the trading losses greater. Conversely, the predator's penalty behaves. The additional sale at low penalty intensity deviates further from the target value, which entails higher subsequent costs. At high intensity, the opposite is the case. In summary, we observe two opposing movements. On the one hand, the increase in trading losses with penalty intensity, and on the other hand, the decrease in penalty costs with the increase in penalty intensity. In total, the trading costs outweigh, so that the predators suffer more loss, the more binding their target specification is.

Fake News vs. True Information: The expected profits under true information are presented in Panel 4(b). Initially, we observe that trading profits are now consistently positive. Moreover, they are significantly higher at low penalty intensity, as the sale of additional shares can now actually occur at the elevated price. The penalty costs behave as in the case of false rumors. In summary, the trading gains once again predominate. Hence, the pursuers consistently expect a profit, which decreases with increasing penalty intensity.

Thus, the question of true or false rumors has a significant impact on the predators themselves, as fake news can turn an expected profit into an expected loss.

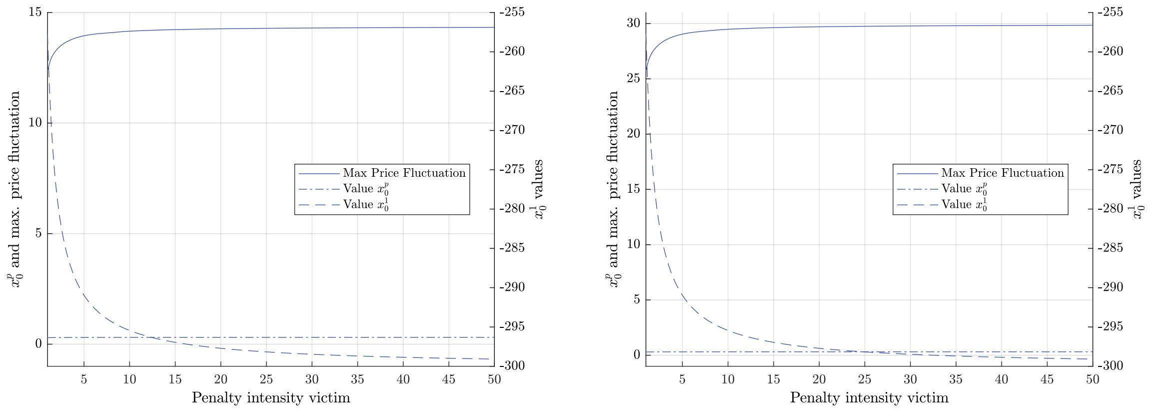
Limit Case: While the question of profits or losses generally always involves an interplay between trading profits and penalties, in the limit case, the total profit can be directly attributed to the trading profit. The following evidence reinforces the observation that all players incur an (expected) loss if the rumor turns out to be false.

Theorem 3.4. *Consider the situation as in the limit case of Theorem 2.1 (Theorem A.1) with the predators acting according to a hands clean scheme. Then all predators suffer an expected loss due to the game.*

The aforementioned theorem states that the predators must always expect losses when the underlying information is false. To explain this, we first note that the optimal trading strategy in the absence of a victim is constantly zero (and thus the trading profit is zero). This means that no player can move the market in their favor in such a way that they can expect a profit while not deviating from their optimal portfolio. Because the predators, due to the fake news, now believe that there is a victim, they are driven to act and thus deviate from their optimal strategy of inaction, which causes the expected loss.

4. Dissemination of fake news

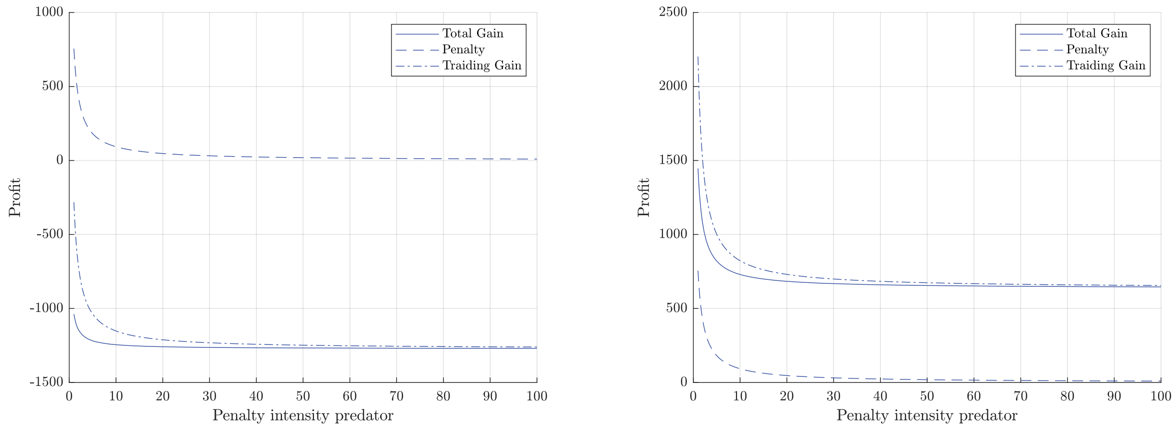
In this section, we examine at the impact of the quantity of strategic participants on the results. Nowadays, social media makes it easy to spread rumors, as seen with Gamestop. Thus, the spread of fake news and untrue rumors is a big concern in this context. Therefore, we address the relevance of dissemination of a rumor, which is measured by the number of strategic players. Note that, as shown in Appendix B, it is optimal, even for "big" players, to participate in the game with only a small amount of capital if the number of players grows. This justifies a unification of all predators making the number of players an appropriate measure of information dissemination.



(a) Adjustment of sales volume x_0^1 and max price fluctuation in dependence of the victims penalty in the case of false rumor.

(b) Adjustment of sales volume x_0^1 and max price fluctuation in dependence of the victims penalty in the case of true rumor.

Fig. 3. Influence penalty parameter victim. The figure illustrates the interplay between the penalty intensity of the victim and the variables x_0^1 (Adjusted sales volume), x_0^p (additional selling volume) and the maximum price fluctuation, for both cases of true and false rumors.



(a) Expected loss in dependence of the penalty intensity of the predators in the case of fake news.

(b) Expected loss in dependence of the penalty intensity of the predators in the case of true information.

Fig. 4. Profit and loss analysis. The figure shows a profit and loss analysis for a game under untrue rumor in a plastic market. The predators falsely assume that the alleged victim must acquire 300 units of a risky good. The results are stated in dependence of the penalty intensity of the predators.

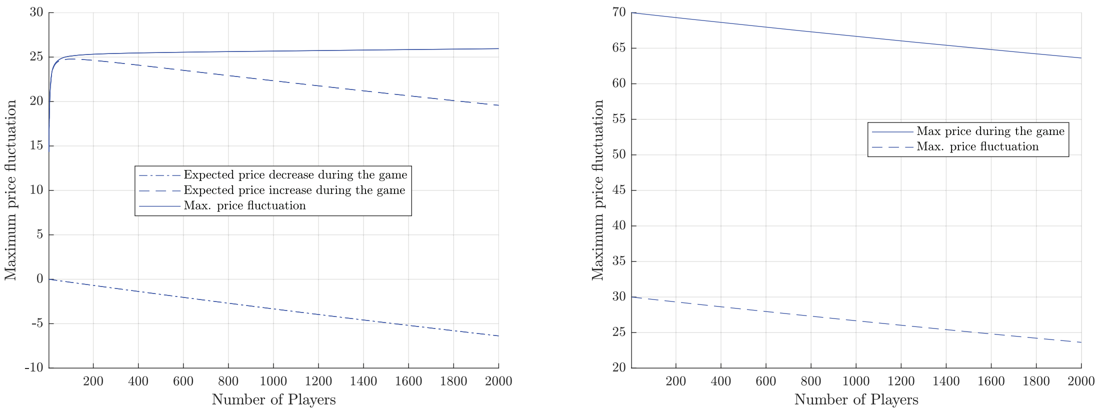
4.1. Effects on bubble trajectory

General Observations: We have already shown that stock prices may decrease sharply as the game approaches its end ("price crash"). As seen in (21), this crash intensity is not only in dependence of the penalty parameter, but also in dependence of the total number of players. Panel 5(a) concentrates on the effects associated with the spread of the rumor. More precisely, the figure highlights the maximum upwards and downwards deviation of the stock price in comparison to the initial stock price during the entire game. The maximum upwards deviation gives the price increase as the bubble inflates. The maximum downwards deviation follows from the price after the crash has occurred. The distance between the two indicates the total price drop after the burst of the bubble.

To begin with the maximum upwards deviation, we find, after a significant increase is observed for the first few players only, an almost linear decrease within the number of players. When the number of predators is limited, the demand they collectively generate to inflate the price predominates. However, as the number of players increases, the cumulative planned additional sales by all predators, and thus the increase in supply, gain dominance. Consequently, the impact of the dissemination of fake news is not monotonic. In essence, limited dissemination leads to stronger anticipated price increases, whereas widespread dissemination tends to result in more modest rises.

In contrast, the maximum downwards deviation exhibits a monotonic trend with respect to the number of players. The more predators are involved, the deeper the price drops expected towards the end of the game. This observation aligns with (21), noting that the additional sales quantities x_0^p also depend on the total number of players (indeed, converging to 0 as $N \to \infty$). However, the diminishing additional sales are overshadowed by the increasing number of players executing these sales, resulting in an overall rise in the planned additional sales. This overall larger supply then drives the price downwards at the end of the game. Thus, crashes become more severe when they are triggered by untrue rumors and the results indicate that a more widespread dissemination of untrue rumors (fake news) foster price drops and thus menace stability of financial markets.

With a reduced expected price increase during the game (at least beyond a certain number of players) and a more pronounced crash towards the end, the number of players has two potentially opposing effects on bubble magnitude. Fig. 5(a) illustrates the corresponding maximum price fluctuation, which exhibits a sharp increase transitioning from a lower to medium number of players (in this range, the increasing maximum price still amplifies the price drop), while only



(a) Maximum expected price fluctuation and expected price in/decrease in the stochastic case with true rumor.

(b) Upwards and downwards maximum expected price fluctuation in the stochastic case with untrue rumor.

Fig. 5. Maximum expected price fluctuation. Fig. 5(a) shows the maximum price fluctuation in terms of the lowest expected price and the highest expected price occurring during the game under wrong rumor (in dependence of the number of predators). Fig. 5(b) shows the expected maximum price fluctuation and the highest expected price in dependence of the number of predators in the case of true information. Both are samples of a plastic market.

marginal increases are observed for larger dissemination, as the opposing effects largely neutralize each other. Consequently, the overall bubble size tends to increase with the spread of the rumor, with the most significant growth observed at lower player counts.

Limit Case: In the general case, it is not possible to distinctly separate the direct effects of the number of players from the indirect effects mediated through the flexible sales quantity x_0^p. Since the transition to fixed end conditions in the limit case results in the latter being nonexistent ($x_0^p = 0$), the direct effects of the dissemination of fake news can be emphasized. This allows us to make explicit statements about the expected bursting time of the bubbles, presented by the following theorem:

Theorem 4.1. *Assume the situation as in Theorem A.2 with the predators following a hands clean strategy, then*

1. *If the number of players converges to ∞ the expected price breaks at time $\frac{T}{2}$ regardless of the other parameters of the game.*
2. *The more players participate in the game, the earlier the expected price development changes direction. Thus, the time of occurrence of the expected maximum/minimum is strictly monotonically decreasing in the number of players.*

The above Theorem 4.1 states that the more predators are involved, the sooner the bubble will burst. Consequently, the price decrease is observed over a longer period of time. These observations are independent of the particular model parameters like the elasticity and plasticity parameters. Moreover, as the number of predators is large enough, the bubble bursts at halftime of the game ($T/2$).

This result can be used to show that the severeness of the bubble indeed directly depends on the number of players. Therefore we extend Theorem 4.1 to show that the maximum expected price deviation grows with the number of players involved.

Theorem 4.2. *Assume the situation as in Theorem A.2 with the predators following a hands clean strategy, then the maximum expected price deviation is strictly monotonically increasing in the number of players and limited by*

$$\underset{N\to\infty}{MPF}(0,T) = \left|\gamma \frac{1-e^{\frac{T\gamma}{2\lambda}}}{1+e^{\frac{T\gamma}{2\lambda}}} x_0^1\right|. \tag{23}$$

The above theorem confirms the observation that a larger number of misinformed market participants lead to larger distortions. However, this influence is limited. It is evident that the larger the falsely assumed sales volume, the greater the maximum increase in the bubble.[26]

We have already demonstrated that, in the limit case (and approximately for very high penalty intensities), the expected final price aligns with the initial price. For widely disseminated rumors, this assertion can be further extended, as in such cases, a symmetrical bubble pattern is indeed anticipated:

Theorem 4.3. *Assume the situation as in Theorem A.2 with the predators trading according to a hands clean scheme, then the expected price trend for a large number of players is symmetric at time $\frac{T}{2}$. This is*

$$\mathbb{E}\left(X^0_{\frac{T}{2}-\epsilon}\right) \approx \mathbb{E}\left(X^0_{\frac{T}{2}+\epsilon}\right) \quad \forall \epsilon \in \left[0, \frac{T}{2}\right]. \tag{24}$$

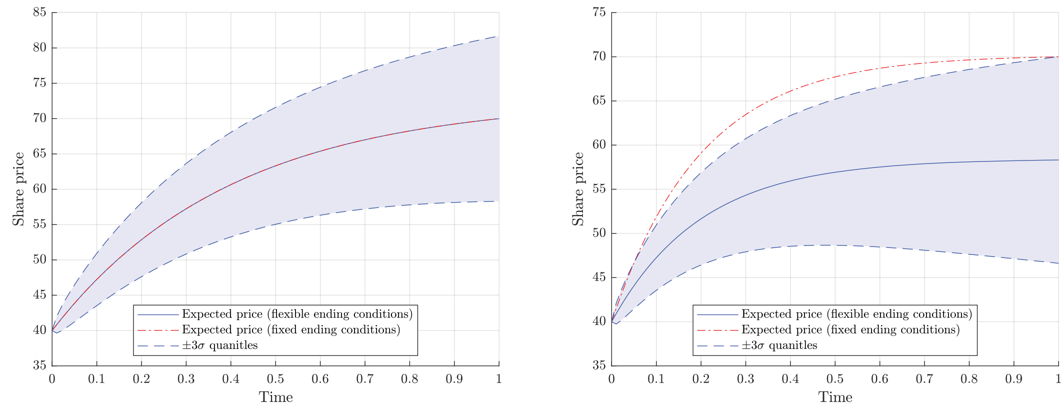
The theorem implies that the price falls in the same way as it rose before, if there is a large number of players. Hence a rapid (slow) price increase in the beginning of the game will lead to a sharp (moderate) decline towards the end. This result is independent of the model parameters like, for example, the plasticity and elasticity of the market and the number of assets to be sold or acquired by the distressed player. This is particularly important for market observers since it is enough to monitor the first half of the game to get an idea of how the price will behave in the second half.

Fake News vs. True Information: To ensure that the effects highlighted are indeed caused by fake news and not inherently part of the game, we compare the observed bubble patterns with those that would emerge under true information conditions. In the case of true rumors, the expected final price is given by (29) through the following alignment:

$$\mathbb{E}\left(X_T^0\right) = X_0^0 - \gamma\left(x_0^1 + (N-1)x_0^p\right). \tag{25}$$

Hence, it is anticipated that with an increasing number of players, the opposing effect to the increase induced by the forced trade volume (x_0^1) is mitigated. Due to the monotonicity of expected bubble patterns in this scenario, a dampening effect on the overall bubble height is presumed. It also becomes clear that in the limit case, the final price,

[26] Further, it can be seen that the maximum possible price fluctuation (23) depends on the running time of the game and converges to 0 for $T \to 0$ and to $|\gamma x_0^1|$ for $T \to \infty$. The longer the players have time to pursue their wrong strategy, the bigger the distortions they cause. This is in contrast to the case with true rumors, where the maximum price increase is independent of the duration (see (30)).



(a) Development of the expected price in the three-player case.

(b) Development of the expected price in the case of a game with 4000 players.

Fig. 6. Illustration of the dampening effect. The figure shows the expected price development as well as the $\pm 3\sigma$ limits for flexible ending conditions under the presence of true rumor. Further the expected price trend under fixed ending conditions is shown for comparison purposes.

and thus the maximum price fluctuation, must be independent of the number of players.

Panel 6 demonstrated these described effects. The figure contains the corresponding expected developments for both, flexible and fixed (limit case) ending conditions. We compare two illustrative set-ups including one with just three players and one with 4000 players. The maximum downwards deviation is not addressed since expected prices are monotonically increasing during the game if the rumor is true. Hence, the initial price is always the lowest price and the expected terminal price is always the highest one.

We find that for the case with only two predators the expected price and the $\pm 3\sigma$ quantile hardly differ between fixed and flexible ending conditions. However, when there are 4000 players expected price movements are clearly different. In particular, a large number of players has a dampening effect on the price increase which results from the true rumor that the victim has to buy stocks. This dampening effect can be explained again by the fact that strategic players accept a penalty for deviating from their target portfolios. After the price was driven up they plan to sell more stocks than they buy during the first stage. They can sell these stocks at a higher price because there really is demand by the victim. Thus, contrary to fake news, there is no oversupply to drive prices down, but enough to mitigate the victim's demand. The gains from trading the stock, therefore, outweighs the penalty that they have to accept. For this reason, the dampening effect is not observed with fixed ending conditions.[27]

Further, the more players are involved, the more will sell some additional stocks at the upwards driven price. More sales slow down the price increase to a greater extent and thus lead to lower price fluctuations. Thus, the greater the number of predators, the stronger the dampening effect. This result is demonstrated in Panel 5(b).

In a nutshell, fake news must be clearly distinguished from the case of true rumors. While the former leads to potentially more severe bidirectional fluctuations with greater dissemination and thus states a negative impact on financial market stability, the latter dampens the expected unidirectional fluctuations as dissemination increases. Although in both scenarios the importance of the number of players diminishes with increasing penalty intensity, in the case of false rumors, this results in symmetrical bubbles, whereas the bubble pattern under true rumors remains strictly monotonically increasing.

4.2. Profits and losses

Focusing on the individual player, it remains to be clarified what effects can be expected for an predator when not only she, but a multitude of market participants succumb to fake news.

Fake News: Panel 7(a) shows the expected loss of a player suffered by participation in a game believing an untrue rumor in dependence of the total number of participants. It is clear to see that the more players succumb to the false rumor, the smaller the loss suffered. This means that if a rumor turns out to be false, it is to the individual player's advantage if he is not the only one who has bought into the misconception. On the other hand, one can also fear that individual players are tempted not to check existing information in detail, especially if enough other players follow this information and the checking is associated with costs and effort. These considerations can be kept in mind specifically with regard to the question of new threats posed by social media.

Note, that even if the loss of each player is negligible, it does not mean that the total loss of all the players who were exposed to the false rumor is insignificant. As Panel 7(b) shows, the total loss suffered by the strategic players does not have to be monotonous in the number of those either, which makes it necessary to carry out specific investigations in individual cases in order to answer this question.

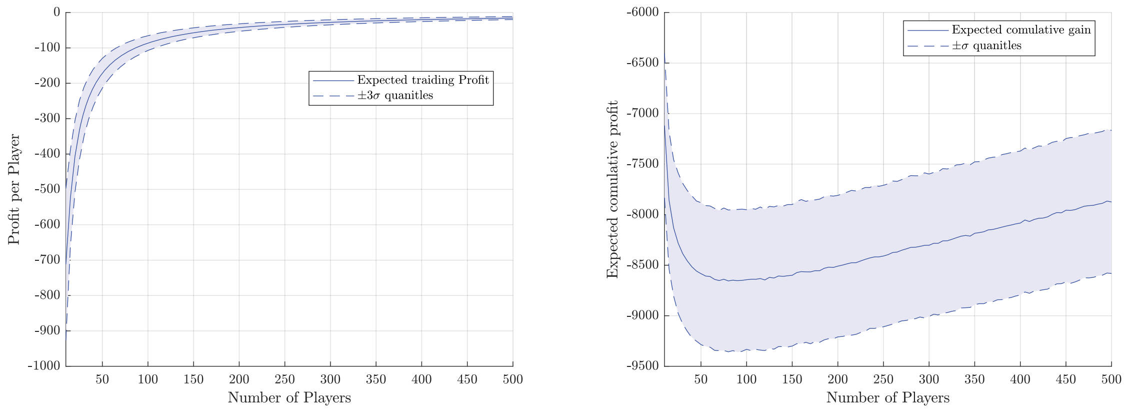
All in all, the information dissemination leads to the fact that the individual players have to take less risk thanks to the lower threat of loss, while the total damage caused can be significant due to the large number of participants.

Limit Case and True Information: To focus solely on pure trading profits, the limit case can be considered once again. Therefore, we can explicitly prove the above observation, which leads to the following theorem:

Theorem 4.4. *Consider the situation as in Theorem A.1 with the predators acting according to a hands clean scheme. Then, as the number of players increases, the expected loss of each player converges to zero.*

This theorem demonstrates that, (even) without penalties through the trading strategy of predators, a loss is expected, and the more players participate, the lesser the loss for each individual. Indeed, it may be noted that even in the case of true rumors, the profit for an individual player converges to zero as the number of predators increases.

[27] According to Eq. (29) the expected end price is given by $X_0^0 - \gamma x_0^1$ and thus independent of the number of players. See Appendix A for further details.



(a) Expected loss caused by the game in dependence of the number of players.

(b) Cumulative expected loss caused by the game in dependence of the number of players.

Fig. 7. Profit and loss vs. Number of players. The figure shows a profit and loss analysis for a game under false rumor in a plastic market. The predators falsely assume that the alleged victim must acquire 300 units of a risky good. The results are stated in dependence of the number of players.

The conclusions drawn from this, however, are fundamentally different. While under true rumors the actual expected profit is reduced, making it potentially bad for an individual player when others also learn of the victim and join the hunt, under fake news the exact opposite is true. This highlights once more the significant differences between true and false rumors.

5. Correction of misinformation

To articulate the individual effects of fake news, it has been consistently presumed that participants maintain belief in their false rumors until the end. As demonstrated, in this case, the impacts on the anticipated price trend are considerable. Particularly, these trends significantly diverge from those expected under true information. Therefore, participants are now afforded the opportunity to correct their misinformation should it appear implausible. It will become evident that this primarily leads to the same effects already known from the scenario without information updates, although the spread of the rumor assumes an even more pronounced key role.

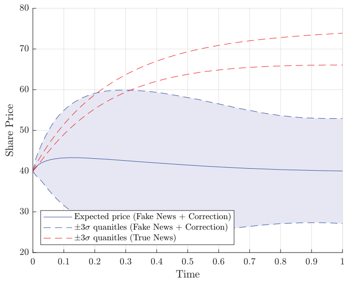
Information Update: Since continuously updating existing information scarcely aligns with our understanding of fake news (see Section 2), we shall, in a generalized manner, assume that an event occurs, leading followers to doubt the truth of their information. Given that such occurrence was not originally anticipated, this aligns with the existing modeling of false rumors. More precisely, we assume that the predators question their (false) information and receive the correct information as soon as the actual price moves far away from the expected price trend (if the information had been true). Let this be the case if the actual price trend leaves a certain confidence interval. As soon as this happens, the players consider the observed course sufficiently improbable that they question their information as a further attempt at explanation. Having done so, they finish the game knowing that the previous information was wrong. The question, therefore, is whether fake news can exist at all under such updating and, if so, for how long they can survive.

General effects: Generally, the game can now be divided into two parts. The first, which covers the timeline up until the recognition of the fake news, behaves exactly as previously described. The second part, describing the progression of the game after this event, differs from the situation known so far. Now, the predators realize that their information was incorrect and that no victim will provide the urgently needed demand. They are forced to change their strategy and find the best way to dispose their acquired shares under the new situation. Additionally, it is important to note that the sale of additional share portions can still be considered to exploit the possibly already very inflated price at this point. Ironically, each predator knows that the other predator faces the same dilemma, thus the predators become 'victims' themselves trying to unwind open positions. This competition leads to a price drop at the end of the game. It should be noted that although the expected price decline is also observable in the case without updating, the reasons for it are now significantly different.

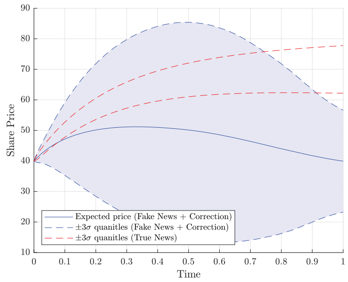
As shown in Panel 8 and Table 1, the expected trajectory of the bubble crucially depends on when the predators recognize their mistake. The earlier this recognition occurs, the shorter the period during which prices are erroneously driven up (part one). Consequently, fewer shares have been acquired by that point, which then need to be sold off in the second phase, leading to a milder decline in price. The opposite is true if the error is recognized late in the game. In summary, it can be concluded that the later the players become aware of their mistake, the more severe fluctuations are expected.

Dissemination of Fake News: The correction time at which predators realize that the rumor is false is a random number. Table 1 provides information on the distribution of this random number including expectation and standard deviation as well as the 25%, 50%, and 75% quantile. The distribution depends on the chosen tolerance level (measured in terms of standard deviations σ) and more importantly the number of players (10 players vs. 1000 players). Moreover, the Table also provides information on the expected maximum price increase, decrease and the total fluctuation. Focusing on the number of players first, we find that a large number of players and thus herding among predators makes the early discovery of a false believe less likely. In fact, in the case of many players (Panel 8(f)), the σ-quantiles under false and true rumors are almost identical at the beginning of the game. Thus, early detection of the error is almost impossible.

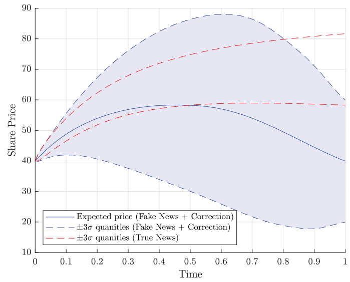
To explain the impact of the number of players on the correction time, let us consider the (erroneously assumed) trading strategy of the victim. These are indicated together with the total positions held in Panel 9. It is observed that the victim buys the fewer stocks at the beginning of the game the more predators participate in the hunt. For example, in the case of two players only, it is expected that the victim will have acquired over a third of the required stocks by the midpoint of the game, while at 20 players, an acquisition of about 10% is expected at this point. The lesser the trading intensity of the victim, the lesser is also their influence on the price. Since this entails a stochastic system, meaning the price also subjects to random components, the disappearance of this minor influence (because the victim does not exist) can just as plausibly be explained by random fluctuations. In fact, in the case of



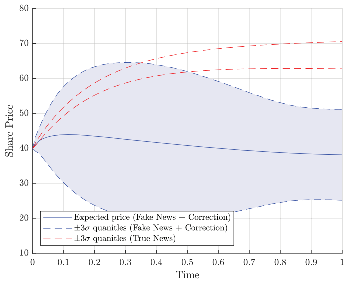
(a) Expected price development in the ten-player case and 1σ bound.



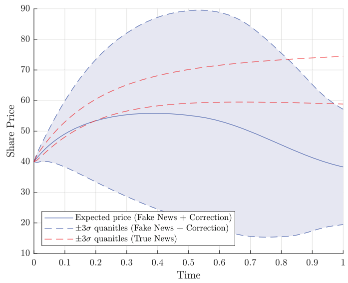
(b) Expected price development in the ten-player case and 2σ bound.



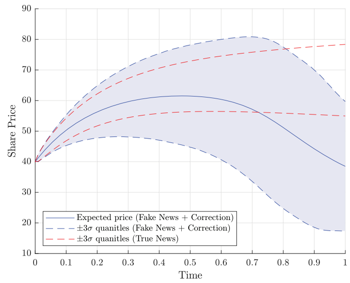
(c) Expected price development in the ten-player case and 3σ bound.



(d) Expected price development in the 1000-player case and 1σ bound.

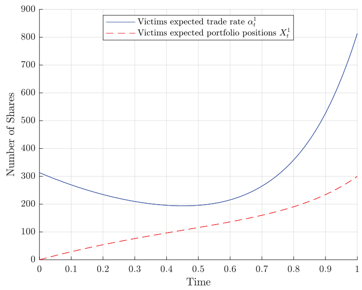


(e) Expected price development in the 1000-player case and 2σ bound.

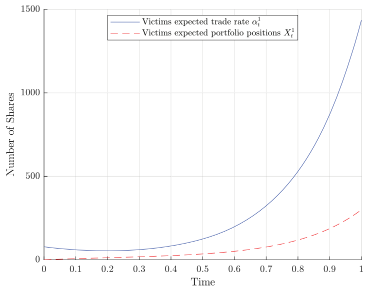


(f) Expected price development in the 1000-player case and 3σ bound.

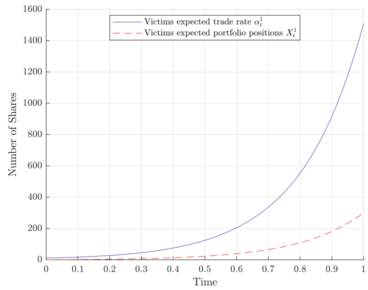
Fig. 8. Correction of Misinformation. The figure shows the expected price development as well as the ± 3σ limits for flexible ending conditions under the presence of untrue rumor. Further, the lower σ boundaries under true rumors are shown, which are used for noticing the misinformation. Comparing the graphs in the horizontal one gets an impression of the influence of the σ boundaries. Comparing the graphs in the vertical plane, the influence of dissemination is illustrated.



(a) 2 Player case



(b) 20 Player case



(c) 1000 Player case

Fig. 9. Control strategies. The figure shows the expected control strategies of the predators as well as the wrongly assumed control strategy of the victim for a different number of players. As several players can shoot up prices very fast by buying some units of shares at the beginning of the game. The victim is not willing to buy at these high prices first, but is forced to do so at the end (because of the threatening penalty). Since the predators are planning to take the opposite position, the strategy is also to start selling only towards the end of the game. In summary, an increase in the number of players leads to an increased focus of planned sales at the end of the game.

many players (see Panel 9(c)), the victim scarcely acts at the beginning of the game, leading the price to be driven exclusively by the actions of the predators and, consequently, inevitably develops just as they anticipate. Significant trading decisions by the victim are only expected late in the game, the absence of which results in such severe impacts on the price development that they can no longer be plausibly explained by random effects. In summary, it becomes increasingly difficult to identify fake news at an early stage as the rumor spreads further and the longer it persists, the more severe fluctuations are expected. Therefore, in the case of herding among predators it is very difficult to distinguish a true and an untrue rumor from observed market prices.

Tolerance Level: In Panel 8 and Table 1, the expected bubble trajectory and the expected correction time in dependence of the quantile limits is shown as well. These provide a measure of how long a player deems it plausible to attribute observed deviations to random impacts. Accordingly, the correction occurs sooner, the lower this boundary is set. Thus, the distortions are also less significant, the smaller this tolerance limit is.

However, it must be noted that this does not imply that choosing the smallest possible σ level is optimal. In fact, selecting too small a tolerance level would mean that players discard their information at the slightest deviations from the expected path. Due to stochastic influences, such deviation is quite probable and would inevitably lead to the discarding of information, even if it is correct. Since the premature discarding of true information reduces trading profits, or incurs costs, this must be weighted against the costs arising from the late detection of an untrue rumor. The determination of the optimal σ level is beyond the scope of this paper, though.

Different correction Mechanism: Instead of using the observed price for verification, (2) could also be utilized to directly estimate the trading volume of the victim. If it emerges that this volume is zero, indicating the victim is not trading at all, it becomes clear that it was a case

Table 1
The table summarizes the expected correction time and associated statistics for a game with few and many players regarding the correction via observed price.

	10 Player			1000 Player		
	1σ	2σ	3σ	1σ	2σ	3σ
Correction time						
Expectation	0.0674	0.3572	0.6463	0.0962	0.5184	0.7611
Standard deviation	0.1135	0.2815	0.2298	0.1572	0.2848	0.1407
25% Quantil	0.0100	0.0800	0.5800	0.0100	0.2400	0.7300
50% Quantil	0.0200	0.3000	0.7300	0.0300	0.6400	0.7900
75% Quantil	0.0600	0.6400	0.8000	0.0900	0.7500	0.8400
Expected max. price						
increase	6.2120	15.6548	21.2678	7.5102	19.5232	23.0113
decrease	2.6801	2.8438	2.9439	4.0319	3.9202	3.7422
fluctuation	8.8922	18.4986	24.2117	11.5422	23.4435	26.7535

Table 2
The table summarizes the expected correction time and associated statistics for a game with few and many players regarding the correction via the victim's estimated trading rate.

	10 Player			1000 Player		
	1σ	2σ	3σ	1σ	2σ	3σ
Correction time						
Expectation	0.0122	0.0225	0.1364	0.0306	0.1608	0.4348
Standard deviation	0.0053	0.0193	0.1571	0.0248	0.1174	0.1256
25% Quantil	0.0100	0.0100	0.0200	0.0100	0.0600	0.3800
50% Quantil	0.0100	0.0200	0.0600	0.0200	0.1400	0.4700
75% Quantil	0.0100	0.0300	0.1900	0.0400	0.2400	0.5200
Expected max. price						
increase	3.3675	3.9458	9.7921	4.1765	13.1036	22.4188
decrease	2.7516	2.6830	2.5522	4.1741	3.8177	3.6557
fluctuation	6.1191	6.6288	12.3443	8.3506	16.9213	26.0745

of fake news, necessitating a correction. It is crucial to note that this approach represents a 'genuine' estimation, meaning it provides only a stochastically biased impression of the actual trading rate. The corresponding results of this methodology are depicted in Table 2. Not surprisingly, the number of players and the σ quantiles affect the outcomes in the same direction as previously discussed. Indeed, the rationale behind this remains unchanged. However, it is notable that the correction time is generally expected to be somewhat earlier than when employing the price as an estimator. Therefore, correction via estimated trading volumes of the victim may serve as a 'better' estimator.

Final Notes: Firstly, it must be emphasized once again that the fact that fake news can survive in the system (at least for a certain period) is significantly based on the fact that players can only control on parts of the system that are stochastically disturbed. In this context, the specific estimator, whether it concerns the price, the trading strategy of the victim, or something else, plays only a secondary role. Indeed, fake news could hardly survive in deterministic systems or under perfect observability. See also the discussion in Section 2. Put differently, this fact is intuitive: under general uncertainty, fake news can also thrive.

Finally, it should be noted that in the present work, we have consistently described the course of 'classic' bubbles. Likewise, the model can explain 'inverse' bubbles by postulating that the victim is compelled to sell. The interpretations in this case remain essentially the same, except that the predators initially push down the price by selling shares and then plan to buy them back at a now cheaper price when the forced offer of the victim provides the necessary counterweight. If this counterweight does not occur, the prices rise again.

6. Conclusion

In this paper we study the impact of the number of players when there is a false rumor about a perceived distressed trader on the development of a stock market bubble. We derive optimal strategies for an arbitrary number of strategic players who optimize expected profits subject to a flexible or a fixed ending strategy. A explicit analysis in the limit case suggests that the bubble always bursts during the second half of the game and that the expected end price of the stock corresponds to the initial price. For the more general case of a flexible ending condition the burst of the bubble can lead to sharp crash and the stock price declines below the initial price. If there is a large number of predators more volatility is introduced to the market. We verify that this dynamic is observable even if the players realize that the rumor was wrong. More importantly, it turns out that it is more difficult to detect false rumors for market participants when there is herding among predators. This observation makes herding among predators a particular challenging problem.

The paper also highlights that the development of bubbles from false rumors is different from the one with true rumors depending on the ending condition. In particular, a large number of predators has a dampening effect on the evolution of a bubble. A summary of the most important differences can be found in Table 3. Concerning games and losses, we find that the predators suffer from unexpected losses in the case of an untrue rumor. These losses result from the fact that the predators trade on a future demand by the perceived victim, which does not materialize. The expected losses contrast the expected profits that predators can make by exploiting the pressure of the victim to act when the rumor is true. Note that expected losses suffered from an untrue rumor might even be worse when new predators enter the market and exploit the selling pressure of the original predators. The introduction of such predators would be an interesting follow-up question to our paper. It would also be conceivable to allow different market participants different trading periods. This is left for future research.

CRediT authorship contribution statement

Tobias J. Herzing: Conceptualization, Methodology, Formal analysis, Writing – original draft, Writing – review & editing, Visualization, Project administration. **Matthias Muck:** Conceptualization, Resources, Writing – original draft, Writing – review & editing, Supervision, Project administration.

Table 3
The table summarizes the impact of false and true rumors on the formation of bubbles. Here, an additional distinction is made between flexible and fixed ending conditions.

Rumor	Untrue		True	
Ending condition	Flexible	Fixed	Flexible	Fixed
Existence of a bubble	Yes	Yes	Yes	Yes
Bursting time explained by the model	Yes	Yes	No	No
Complete live circle of a bubble is explained	No	Yes	No	No
Price fluctuations on both sides, above and below the original level, are expected	Yes	No	No	No
Influence of information propagation on the bubble	Expected to grow	Expected to grow	Expected to decrease	No influence

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No.

Appendix A. Solution with fixed constraints (true rumor)

The following theorem describes the optimal open loop trading strategies under fixed constraints. According to Carlin et al. (2007),[28] the optimal trade rates of the N players are given by

Theorem A.1. *Consider the stochastic differential game , where N players try to maximize their personal profit (10) under the constraint (7). Looking for open loop strategies, the trade rate of player n is given by*

$$\alpha_t^n = ae^{-\frac{N-1}{N+1}\frac{\gamma}{\lambda}t} + b_n e^{\frac{\gamma}{\lambda}t} \tag{26}$$

in equilibrium. The coefficients $a \in \mathbb{R}$ and $b_n \in \mathbb{R}$, $n \in \{1, \dots, N\}$, are determined by

$$a = -\frac{N-1}{N+1}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{N-1}{N+1}\frac{\gamma}{\lambda}T}\right)^{-1}\sum_{i=1}^{N}\frac{x_0^i}{N}, \tag{27}$$

$$b_n = \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1}\left(\sum_{i=1}^{N}\frac{x_0^i}{N} - x_0^n\right). \tag{28}$$

It is important to note that the optimal trading rates depend only on the time t. Thus, the players' trading strategies are already fixed before the game starts.[29] We use this fact to underline the following statement about the expected price development:

Theorem A.2. *Consider the stochastic differential game , where N players try to maximize their personal profit (10) under the constraint (7). Then:*

1. *The expected price development of the risky asset is given by*

$$\mathbb{E}\left(X_t^0\right) = X^0(0) - \frac{1 - e^{-\frac{N-1}{N+1}\frac{t\gamma}{\lambda}}}{1 - e^{-\frac{N-1}{N+1}\frac{T\gamma}{\lambda}}}\gamma\sum_{i=1}^{N}x_0^i. \tag{29}$$

2. *According to the $Sign$ of $\sum_{i=1}^{N} x_0^i$, the expected price of the risky asset is strictly monotonically increasing/decreasing and reaches its maximum/minimum at time $t = T$.*[30]

According to Theorem A.2, the maximum expected price fluctuation is given by

$$MPF(0,T) = \max_{t\in[0,T]} |X_0^0 - \mathbb{E}(X_t^0)| = \left|\gamma\sum_{i=1}^{N}x_0^i\right|. \tag{30}$$

Thus, this depends less on the number of players than on the accumulated needs to deviate from the initial holdings. Especially in the hands clean case $\sum_{i=1}^{N} x_0^i = x_0^1$ applies, so that the maximum price fluctuation is completely independent of the number of players. Consequently, if a market participant worries that a market price of potential concern to him will be exceeded or undercut, the only determining factor is the plasticity factor of the market and the amount of assets to be acquired/liquidated by the distressed player.

Understanding the price inflow driven by the game more generally as a bubble formation, not only the expected maximum price deviation from the fundamental value, but also the speed of the inflation is of interest. This in turn is indeed dependent on the number of players, as can be read directly from Theorem A.2. In general, the larger the number of predators, the more rapidly the share price explodes or crashes right at the beginning of the game. However, this influence can be seen as limited to a certain extent, since the convergence of the term

$$\lim_{N\to\infty}\frac{1 - e^{-\frac{N-1}{N+1}\frac{t\gamma}{\lambda}}}{1 - e^{-\frac{N-1}{N+1}\frac{T\gamma}{\lambda}}} = \frac{1 - e^{-\frac{t\gamma}{\lambda}}}{1 - e^{-\frac{T\gamma}{\lambda}}} \tag{31}$$

holds. Thus, there is a maximum speed at which a bubble, driven by predatory trading, emerges.

Appendix B. Multi player justification

Classically, predatory trading models are used only with a few players. This is usually justified by the fact that information about the selling constraints of others is difficult to obtain and is only available to a few market participants. However, if we take into account the increasingly rapid dissemination of information via new media and the fact that in our research it may be a freely invented rumor that is being spread, this argument does not apply.

Furthermore, it is often argued that smaller market participants cannot muster the liquidity necessary to exert an influence on the price dynamics. For the case of fixed ending conditions, the following theorem exemplary refutes this claim:

[28] To be more precise, the authors restricted themselves to deterministic strategy profiles. However, by means of a subspace argument, it can be shown that the solutions found also represent the optimal open loop strategies for the larger space (7). See for instance (Carmona, 2016).

[29] For a detailed interpretation of the results, please refer to Carlin et al. (2007). Further, the authors have also discussed the more complex case of closed loop solutions. However, they found that no decisive differences are associated with this consideration.

[30] From an economic point of view, this result represents the well-known fact that the price is formed by supply and demand. If demand outweighs supply, prices are expected to rise, and if supply outweighs demand, prices are expected to fall. Remarkable, however, is the monotony over the entire period, which means that the 'inferior' side is not ahead at any time.

Theorem B.1. *Consider the stochastic differential game , where the distressed player is in the need to cover his positions and the $N-1$ predators trade according to a hands clean scheme. Then the expected required liquidity $\mathbb{E}\left(L_{total}\right)$ of a predator is given by*

$$\mathbb{E}\left(L_{total}\right) = \int_0^{\tilde{t}} \alpha_t^n \left(X_t^0 + \lambda \left(\sum_{i=1}^N \alpha_t^i \right) \right) dt \tag{32}$$

where we set

$$\tilde{t}(N,T,\lambda,\gamma) = \frac{(N+1)\lambda}{2N\gamma} \log \left(\frac{e^{\frac{N}{N+1}\frac{T\gamma}{\lambda}} \left(1 - e^{\frac{T\gamma}{\lambda}}\right)(N-1)}{\left(e^{\frac{1}{N+1}\frac{T\gamma}{\lambda}} - e^{\frac{N}{N+1}\frac{T\gamma}{\lambda}}\right)(N+1)} \right) \tag{33}$$

and decreases towards 0 for $n \to \infty$.

So one can imagine that even if each player has little liquidity available, the mass thrown together is enough to ensure a noticeable price impact.[31]

Appendix C. Proofs

C.1. *Theorem 2.1*

Proof of Theorem 2.1. It should first be noted that it suffices to find an optimal strategy for a predator, since all predators must follow the same strategy. Thus $x_0^2 = \cdots = x_0^N$. As we are searching for an open-loop equilibrium, it is necessary for the predators to decide from the outset how far they want to deviate from their target value. Once they have decided on a quantity x_0^p, it is reasonable to assume that this quantity is acted upon optimally. If this is the case, then the victim must also act optimally in response to the assumed quantity. Therefore, it is a reasonable guess to search for strategies of the form stated in Theorem A.1, where all players adopt the same chosen quantities for their actions, this is

$$\alpha_t^n = a e^{-\frac{N-1}{N+1}\frac{\gamma}{\lambda}t} + b_n e^{\frac{\gamma}{\lambda}t} \tag{34}$$

with the coefficients $a \in \mathbb{R}$ and $b_n \in \mathbb{R}$, $n \in \{1,\dots,N\}$, are determined by

$$a = -\frac{N-1}{N+1}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{N-1}{N+1}\frac{\gamma}{\lambda}T}\right)^{-1}\left(\frac{x_0^1}{N} + \frac{(N-1)x_0^p}{N}\right), \tag{35}$$

$$b_1 = \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1}\left(\frac{N-1}{N}\left(x_0^p - x_0^1\right)\right),$$

$$b_n = \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1}\left(\frac{x_0^1 - x_0^p}{N}\right) \text{ for } n \neq 1.$$

Since the solution found must also satisfy the conditions of a Nash equilibrium, we must further ensure that x_0^p is chosen in such a way that it is not advantageous for any predator to choose a different sales quantity, assuming all other players are adopting x_0^p. To do so, lets fix a arbitrary x_0^p and all players except one predator (without loss of generality we will assume the second player) are acting according to the strategies (34). Note that under this situation, should player two decide to choose a different trade quantity x_0, the optimal trading strategy that moves this quantity $\alpha_t^{x_0}$ is given by

$$\alpha_t^{x_0} = \alpha_t^2 - \frac{1}{T}\left(x_0 - x_0^p\right), \tag{36}$$

with α_t^2 as defined by (34). We now need to ensure that the optimal sales quantity x_p of the second player matches x_0^2, as otherwise, a switch from the assumed trade quantity x_0^2 to x_0 would be advantageous. This is equivalent to the requirement that the difference $D := x_0 - x_0^2$ is equal to zero. To begin with, the objective function of the second player is established and optimized with respect to D. This expression must be equal to zero to ensure that switching to a different trading quantity is not advantageous. Consequently, one obtains $x_0^2 = x_0^p$ as given in Theorem 2.1. The verification that the found candidate indeed represents a Nash equilibrium can be directly performed based on its definition. One selects an arbitrary predator n and fixes the strategies for all other players according to Theorem 2.1. It must be shown that $J^n(\boldsymbol{\alpha}) < J^n(\boldsymbol{\alpha^{-n}}, \boldsymbol{\alpha^n})$ $\forall \alpha^n \in \mathbb{A}^n$. Since for any sales quantity x_0, $J^n(\boldsymbol{\alpha^{-n}}, \boldsymbol{\alpha_t^{x_0}}) \leq J^n(\boldsymbol{\alpha^{-n}}, \boldsymbol{\alpha^n})$ holds, it suffices to restrict attention to the candidates of the structure (36). Among these, the objective function is minimized for $D = 0$ by construction, indicating that, should there be a better strategy under the given conditions, it would also have to trade the quantity x_0^p. Since our candidate precisely represents the equilibrium solution for this quantity, there can be no better response. As n was arbitrary, this argument applies to all predators, and the optimality of the victims strategy directly follows from the fact that it plays the equilibrium solution on x_0^p (and its own x_0^1 is fixed). Thus, the candidate found indeed represents the equilibrium solution.

C.2. *Theorem 3.1*

Proof of Theorem 3.1. Using Ito's Lemma, one easily verifies that the price process described by is given by

$$X_t^0 = X^0(0) + \gamma \int_0^t \sum_{i=1}^N \alpha_s^i ds + \sigma\left(W(t) - W(0)\right) \quad \forall t \in [0,T]. \tag{37}$$

Plugging in $\alpha_t^1 = 0$ and α_t^n according to (26) for the other players into the price process (37) and taking the expectation states (17) after some further straight forward algebraic simplifications. □

C.3. *Proof of Theorem 3.3*

C.3.1. *A technical lemma*

Lemma C.1 (*Technical Lemma*). *Let $T > 0$, $\lambda > 0$, $\gamma > 0$ be arbitrarily chosen and $N \in \{N \in \mathbb{R} \mid N \geq 2\}$, then the following properties apply to the function*

$$f(N) := \frac{(N+1)\lambda}{2N\gamma} \log \left(\frac{e^{\frac{N}{N+1}\frac{T\gamma}{\lambda}} \left(1 - e^{\frac{T\gamma}{\lambda}}\right)(N-1)}{\left(e^{\frac{1}{N+1}\frac{T\gamma}{\lambda}} - e^{\frac{N}{N+1}\frac{T\gamma}{\lambda}}\right)(N+1)} \right). \tag{38}$$

1. *The function $f(\cdot)$ is strictly monotonically decreasing.*
2. $\lim_{N\to\infty} f(N) \to \frac{T}{2}$.
3. *$f(\cdot)$ takes a value in the interval $[\frac{T}{2}, T)$.*

Proof of Lemma C.1. First, consider that obviously $f(N) \in \mathbb{R}$ holds. This is easy to see, since the argument in the logarithm is greater than zero because of $N \leq 2$, $e^{\frac{T\gamma}{\lambda}} > 1$ and $e^{\frac{N}{N+1}\frac{T\gamma}{\lambda}} > e^{\frac{1}{N+1}\frac{T\gamma}{\lambda}}$.

1. We will show that $f'(N) < 0$ holds true. After a few algebraic simplifications and substituting $\frac{T\gamma}{\lambda} > 0$ by $x > 0$ this requirement is equivalent to

$$2\frac{N+1}{N-1} < 2x\frac{e^{x\frac{1}{N+1}}}{e^{x\frac{N}{N+1}} - e^{x\frac{1}{N+1}}} + \frac{N+1}{N}\log\left(\frac{(N-1)(1-e^x)e^{x\frac{N}{N+1}}}{(N+1)\left(e^{x\frac{1}{N+1}} - e^{x\frac{N}{N+1}}\right)}\right). \tag{39}$$

Notice that the first summand of the right hand side of the inequality converges to $2\frac{N+1}{N-1}$ as $x \searrow 0$. Likewise the expression in the logarithm converges to 1 for $x \searrow 0$, so that the right hand side of the inequality coincides with the left hand side in the limit $x \searrow 0$. Thus because of the independence of the left hand side from x and the condition $x > 0$ the above inequality is satisfied if the right hand

[31] It is important that no illegal agreements on joint trading are made, but only the information is passed on that there is a market participant who is forced to buy or sell positions.

side is strictly monotonically increasing in x. This is the case, if the derivative of the right side, understood as a function in x, is greater than 0. This requirement can be reduced to the inequality

$$2(N-1)Ne^{-2x\frac{N}{N+1}}\left(-e^{2x\frac{N}{N+1}}+e^{x\frac{3N-1}{N+1}}+1\right)>0, \tag{40}$$

which holds true because of $N \geq 2$.

2. This can be shown directly with standard calculus.
3. We only have to show $f(N) < T$ as the rest of the statement is a direct consequence of the previous statements. After some algebraic simplifications and substituting $\frac{T\gamma}{\lambda} > 0$ by $x > 0$ this inequality is satisfied if

$$0 < e^{x}\left(e^{x\frac{N-1}{N+1}}(N+1)-2N\right)+N-1 \tag{41}$$

holds. To see that this is always satisfied, notice that the expression on the right hand side is equal to 0 for $x = 0$ and is further strictly monotonically increasing in x, which follows directly from the monotonicity of e^x. □

C.3.2. Theorem 3.3

Proof. In the hands clean case, (17) simplifies to

$$\mathbb{E}\left(X_t^0\right)=X^0(0)-\frac{N-1}{N}\gamma x_0^1\left(\frac{1-e^{-\frac{N-1}{N+1}\frac{t\gamma}{\lambda}}}{1-e^{-\frac{N-1}{N+1}\frac{T\gamma}{\lambda}}}-\frac{e^{\frac{\gamma t}{\lambda}}-1}{e^{\frac{\gamma T}{\lambda}}-1}\right), \tag{42}$$

so that from the first order condition $\frac{\partial \mathbb{E}(X^0(t))}{\partial t}=0$

$$t=\frac{(N+1)\lambda}{2N\gamma}\log\left(\frac{e^{\frac{NT\gamma}{\lambda(N+1)}}\left(1-e^{\frac{T\gamma}{\lambda}}\right)(N-1)}{\left(e^{\frac{T\gamma}{(N+1)\lambda}}-e^{\frac{NT\gamma}{(N+1)\lambda}}\right)(N+1)}\right) \tag{43}$$

follows. Since the second order condition

$$\frac{\partial \mathbb{E}(X^0(t))}{\partial t^2}(t)=\begin{cases}>0 \text{ if } x_0^1>0\\ <0 \text{ if } x_0^1<0\end{cases} \qquad \forall t\in\mathbb{R} \tag{44}$$

is always fulfilled because of $N \geq 2$, the assertion follows from Lemma C.1(3.). □

C.4. Theorem 3.2

Proof. The first assertion follows directly by substituting $t = T$ into (42). □

C.5. Theorem 3.4

Proof of Theorem 3.4. Plugging in $\alpha_t^1 = 0$ and α_t^n according to (26) for the other players into (10) and applying some further algebraic simplifications leads to

$$J^n(\boldsymbol{\alpha})=\lambda(N-1)\int_0^T\left(\alpha_t^n\right)^2\,\mathrm{d}t, \qquad \forall n\in\{2,\dots,N\}, \tag{45}$$

where we used the fact that $\alpha_t^2=\dots=\alpha_t^N$ are deterministic functions. Thus $J^n(\boldsymbol{\alpha}) \geq 0$ holds. However, by converting the original maximization problem into a minimization problem, this term must be corrected by -1 to obtain the actual profit. This proofs the assertion. □

C.6. Theorem 4.1

Proof. Applying Lemma C.1 on the turning point calculated in the proof of Theorem 3.3 shows all the assertions. □

C.7. Theorem 4.2

Proof. Note that

$$MPF(0,T)=\max_{t\in[0,T]}|X_0^0-\mathbb{E}(X_t^0)| = \left|\frac{N-1}{N}\gamma x_0^1\left(\frac{1-f(N)^{\frac{1-N}{2N}}}{1-e^{-\frac{N-1}{N+1}x}}-\frac{f(N)^{\frac{N+1}{2N}}-1}{e^x-1}\right)\right|,$$

where we used (42), (43) and set

$$f(N)=\frac{e^{\frac{N}{N+1}x}(1-e^x)(N-1)}{\left(e^{\frac{1}{N+1}x}-e^{\frac{N}{N+1}x}\right)(N+1)} \qquad x=\frac{T\gamma}{\lambda}. \tag{46}$$

Thus it is sufficient to show that

$$g(N):=\frac{1-f(N)^{\frac{1-N}{2N}}}{1-e^{-\frac{N-1}{N+1}x}}-\frac{f(N)^{\frac{N+1}{2N}}-1}{e^x-1} \tag{47}$$

is increasing in N, which is true because

$$f(N)^{\frac{1+N}{2N}} \quad \text{is decreasing and} \quad \frac{1-f(N)^{\frac{1-N}{2N}}}{1-e^{-\frac{N-1}{N+1}x}} \tag{48}$$

is increasing in N. This shows the first assertion. The second is obtained by taking $\lim_{N\to\infty} MPF(0,T)$. □

C.8. Theorem 4.3

Proof of Theorem 4.3. Taking the limit $N \to \infty$ in (42) and a few more algebraic modifications states

$$\mathbb{E}(X_t^0)=X_0^0-\gamma x_0^1\left(\frac{\left(e^{(T-t)\frac{\gamma}{\lambda}}-1\right)\left(e^{t\frac{\gamma}{\lambda}}-1\right)}{e^{T\frac{\gamma}{\lambda}}-1}\right). \tag{49}$$

Thus

$$\mathbb{E}\left(X_{\frac{T}{2}+\epsilon}^0\right)=X_0^0-\gamma x_0^1\left(\frac{\left(e^{\left(\frac{T}{2}-\epsilon\right)\frac{\gamma}{\lambda}}-1\right)\left(e^{\left(\frac{T}{2}+\epsilon\right)\frac{\gamma}{\lambda}}-1\right)}{e^{T\frac{\gamma}{\lambda}}-1}\right)=\mathbb{E}\left(X_{\frac{T}{2}-\epsilon}^0\right). \tag{50}$$

If we now denote by $\mathbb{E}_N(X_t^0)$ the expected price trend with N players, then due to the pointwise convergence of the function series against (49) the assertion already holds approximately for a large number of players. □

C.9. Theorem 4.4

Proof of Theorem 4.4. The assertion is a direct consequence of taking the limit
$\lim_{N\to\infty} N\int_0^T\left(\alpha_t^n\right)^2\mathrm{d}t=0$ in the proof of Theorem 3.4. □

C.10. Theorem A.2

Proof of Theorem A.2. By using the optimal strategies from (26) in (37) and taking the expected value, the first assertion is obtained directly. The strict monotonicity follows directly from the strict monotonicity of the exponential function and from this follows in turn the existence of the maximum or minimum at the endpoint T. □

C.11. Theorem B.1

Proof of Theorem B.1. The trading strategies of the players are given by α_t^n in (26). Since a positive α_t^n means that the player buys assets and a negative α_t^n stands for selling positions, the zeros of α_t^n stand for the turning points between buying and selling. Since the only zero of α_t^n is given at time $t = \tilde{t}$ with $\tilde{t}$ defined as in Theorem B.1, the player buys up to this time and then sells afterwards. This follows directly from the continuity of α_t^n combined with the fact that for a predator trading hands clean schemes

$$\alpha_0^n = X_0^1 \frac{\gamma}{N\lambda} \left(\left(e^{\frac{\gamma}{\lambda}T} - 1 \right)^{-1} - \frac{N-1}{N+1} \left(1 - e^{-\frac{N-1}{N+1}\frac{\gamma}{\lambda}T} \right)^{-1} \right) = \begin{cases} > 0 \mid X_0^1 < 0 \\ < 0 \mid X_0^1 > 0. \end{cases} \tag{51}$$

Thus, according to (4) the total capital need is given by (32). The convergence towards 0 for $n \to \infty$ follows for the fact $\lim_{N\to\infty} \tilde{t}(N) = \frac{T}{2}$ according to Lemma C.1 and

$$\lim_{N\to\infty} X_t^0 + \lambda \left(\sum_{i=1}^{N} \alpha_t^i \right) = X_0^0 - X_0^1 \frac{\left(\gamma + \lambda \left(e^{\frac{\gamma t}{\lambda}} - 1 \right) \right) e^{(T-t)\frac{\gamma}{\lambda}}}{e^{T\frac{\gamma}{\lambda}} - 1} \tag{52}$$

is bounded in the interval $[0, T]$. Thus the claim follows directly from $\lim_{N\to\infty} \alpha_t^n = 0$. □

References

Abreu, Dilip, & Brunnermeier, Markus K. (2003). Bubbles and crashes. *Econometrica*, *71*(1), 173–204.

Almgren, Robert, & Chriss, Neil (1999). Value under liquidation. *Journal of Risk*, *12*, 61–63.

Almgren, Robert, & Chriss, Neil (2001). Optimal execution of portfolio transactions. *Journal of Risk*, *3*, 5–40.

An, Ta, & Øksendal, Bernt (2012). A maximum principle for stochastic differential games with g-expectations and partial information. *Stochastics*, *84*, 137–155.

Arieli, Itai, et al. (2019). *The speed of innovation diffusion in social networks*. Economics Group, Nuffield College, University of Oxford, Economics papers 2019-W07.

Asako, Yasushi, et al. (2020). (A)symmetric information bubbles: Experimental evidence. *Journal of Economic Dynamics & Control*, *110*, Article 103744.

Başar, Tamer, & Olsder, Geret Jan (1999). *Dynamic noncooperative game theory*. SIAM Classics in Applied Mathematics.

Brunnermeier, Markus K. (2001). *Asset pricing under asymmetric information: Bubbles, crashes, technical analysis, and herding*. Oxford University Press.

Brunnermeier, Markus K., & Morgan, John (2010). Clock games: Theory and experiments. *Games and Economic Behavior*, *68*(2), 532–550.

Brunnermeier, Markus K., & Pedersen, Lasse Heje (2005). Predatory trading. *The Journal of Finance*, *60*(4), 1825–1863.

Carlin, Bruce Ian, Lobo, Miguel Sousa, & Viswanathan, S. (2007). Episodic liquidity crises: Cooperative and predatory trading. *The Journal of Finance*, *62*(5), 2235–2274.

Carmona, René (2016). *Lectures on BSDEs, stochastic control, and stochastic differential games with financial applications*. SIAM Society for Industrial and Applied Mathematics.

Carmona, René, & Delarue, François (2013). Probabilistic analysis of mean-field games. *SIAM Journal on Control and Optimization*, *51*(4), 2705–2734.

Carmona, René, & Yang, Joseph (2011). Predatory trading: a game on volatility and liquidity. *Quantitative Finance*, [under Revision?].

Chan, Louis K. C., & Lakonishok, Josef (1995). The behavior of stock prices around institutional trades. *The Journal of Finance*, *50*(4), 1147–1174.

Chen, Ying-Ju, Zenou, Yves, & Zhou, Junjie (2018). Multiple activities in networks. *American Economic Journal: Microeconomics*, *10*(3), 34–85.

Eesley, David, et al. (1996). Liquidity, information, and infrequently traded stocks. *The Journal of Finance*, *51*(4), 1405–1436.

Fudenberg, Drew, & Levine, David K. (1993). Self-confirming equilibrium. *Econometrica*, 523–545.

Holthausen, Robert W., Leftwich, Richard W., & Mayers, David (1990). Large-block transactions, the speed of response, and temporary and permanent stock-price effects. *Journal of Financial Economics*, *26*(1), 71–95.

Huberman, Gur, & Stanzl, Werner (2004). Price manipulation and quasi-arbitrage. *Econometrica*, *72*(4), 1247–1275.

Jackson, Matthew, & Zenou, Yves (2015). Games on networks. In *Handbook of game theory with economic applications*: *Vol. 4*, (pp. 95–163). Elsevier.

Kaul, Aditya, Mehrotra, Vikas, & Morck, Randall (2000). Demand curves for stocks do slope down: New evidence from an index weights adjustment. *The Journal of Finance*, *55*(2), 893–912.

Kobayashi, Teruyoshi, Ogisu, Yoshitaka, & Onaga, Tomokatsu (2023). Unstable diffusion in social networks. *Journal of Economic Dynamics & Control*, *146*, Article 104561.

Kraus, Alan, & Stoll, Hans R. (1972). Price impacts of block trading on the New York Stock Exchange. *The Journal of Finance*, *27*(3), 569–588.

Madhavan, Ananth, & Cheng, Minder (1997). In search of liquidity: Block trades in the upstairs and downstairs markets. *The Review of Financial Studies*, *10*(1), 175–203.

Meng, Qingbin, et al. (2020). The impact of block trades on stock price synchronicity: Evidence from China. *International Review of Economics & Finance*, *68*, 239–253.

Moinas, Sophie, & Pouget, Sebastien (2013). The bubble game: An experimental study of speculation. *Econometrica*, *81*(4), 1507–1539.

O'Hara, Maureen (1995). *Market microstructure theory*. Blackwell Publishers.

Sadka, Ronnie (2006). Momentum and post-earnings-announcement drift anomalies: The role of liquidity risk. *Journal of Financial Economics*, *80*, 309–349.

Saporito, Yuri F. (2019). Stochastic control and differential games with path-dependent influence of controls on dynamics and running cost. *SIAM Journal on Control and Optimization*, *57*(2), 1312–1327.

Schied, Alexander, & Schoeneborn, Torsten (2008). Risk aversion and the dynamics of optimal trading strategies in illiquid markets. *Finance and Stochastics*, *13*.

Schoeneborn, Torsten, & Schied, Alexander (2009). Liquidation in the face of adversity: Stealth vs. Sunshine trading. EFA 2008 Athens Meetings Paper.

Sobolev, Daphne, & Clunie, James (2022). Predatory trading: ethics judgments, legality judgments and investment intentions. *Review of Behavioral Finance*, *15*(3), 275–291.

Umar, Zaghum, et al. (2021). A tale of company fundamentals vs sentiment driven pricing: The case of GameStop. *Journal of Behavioral and Experimental Finance*, *30*, Article 100501.

Welch, Ivo (2022). The wisdom of the robinhood crowd. *The Journal of Finance*, *77*(3), 1489–1527.