Investigation of Automated and Augmented Covariate Imputation during the Estimation of two Confirmatory Multidimensional Bayesian Latent Variable Models

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Synopsis

The purpose of this thesis is to investigate parameter estimation and simultaneous imputation of covariates for two multidimensional Bayesian latent variable models. These models encompass a multivariate confirmatory factor analytic model and a multivariate normal-ogive item response theory (IRT) model. The basic idea for the research conducted stems from the positive findings for IRT models with one latent variable of Aßmann et al. (2015) and Gaasch (2017a) respectively, and the research of Preising (2018) concerning dynamic linear panel models, with presenting, aside from estimation, a relief for the burden of determining an imputation model for missing values in covariates by using an automated procedure, i.e. imputation via a classification and regression tree approach (CART).

The models considered within this thesis and later to be extended, originate from the class of confirmatory factor analytic and normal-ogive IRT settings, who find application in psychometric modelling, social and political sciences, medicine and many areas more. Their features get described by means of discussing concepts, first considerations with respect to estimation within a Bayesian framework, mathematical model formulations, different terminologies used in varying contexts, and an overview of considerations with respect to model and parameter identification.

By comparing different latent variable models (i.e. Aßmann et al., 2015, Gaasch, 2017a, and Conti et al., 2014) and their application, ways of including covariates and handling of missing values get discussed. Using Conti et al.'s (2014) model as a basis, our derivations transform their explorative factor model into a confirmatory matrix-variate factor and a normal-ogive IRT model with several latent variables and incorporating covariates in two different ways. In order to handle missing values within these covariates, CART imputation is employed. Next to binary and metric scaled measurements, estimation of ordered scaled items is facilitated with the help of an algorithm of Albert & Chib (1997),

which was already implemented and tested by Gaasch (2017a) in his research on IRT models with one latent variable. In order to allow the possibility of defining restrictions, the indicator matrix employed by Conti et al. (2014) gets extended to specify equality and multiplicative structures for loadings and item discrimination parameters respectively. In addition to the dedicated loading structure of Conti et al. (2014) this matrix can be used to specify cross-loading, such that a measurement can be related to several latent variables simultaneously.

In order to test the proposals made, nine factor analytic and twelve IRT model related simulation studies, and an empirical application to competency data stemming from the National Educational Panel study (see Blossfeld et al., 2011) are performed. The findings indicate the general ability to conduct the estimation and benefits for using the CART imputation within the IRT approach. However the factor analytic specification, although in general estimable without missing values, does not provide reasonable results compared to complete-case analysis when using the CART imputation approach. Thus, future directions of further research can be investigating reasons and alternatives for the shortcomings with respect to the factor analytic approach and general extensions to the IRT model.

Note, that this thesis uses data from the National Educational Panel Study (NEPS): Starting Cohort 4 – 9th Grade, doi:10.5157/NEPS:SC4:9.1.0. From 2008 to 2013, NEPS data were collected as part of the Framework Programme for the Promotion of Empirical Educational Research funded by the German Federal Ministry of Education and Research (BMBF). As of 2014, the NEPS survey is carried out by the Leibniz Institute for Educational Trajectories (LIfBi) at the University of Bamberg in cooperation with a nationwide network.

Contents

Chapter 1

Introduction

Latent variable models, like *item response theory* models (IRT models) or factor analytic models, are important statistical methods in psychometric and quantitative educational research–next to other scientific fields. Within the formers contexts, they specifically allow to investigate unobserved constructs like intelligence, personality traits or abilities. For example IRT models find their application in scaling of competency test items in the *National Educational Panel Study* (NEPS, see Pohl & Carstensen, 2013), factor models are applied to investigate health and psychological well-being using an explorative algorithm (see, e.g. Conti et al., 2014) or allow to relate aspects of personality to voting behaviour and formation of political attitudes (see Schoen & Schumann, 2007).

Basic models interconnect observed outcomes to latent variables via a dimensionality-reducing projection. However, it is also possible to include additional data in terms of covariates, such that available information can be increased (see amongst others Joreskog & Goldberger, 1975 or Mislevy, 1987). But this increase of data is also prone to increases in missing values, and in return might have a negative impact on estimates when simply being ignored. One prominent method of handling missing values, when aiming for example at classical frequentistic analysis, is multiple imputation (see Little & Rubin, 2002, or Rubin, 2004). For a Bayesian perspective, data augmentation seems to be a suited variant. It allows to model predictions for missing values and imputes them within Markov chain based estimation routines like Gibbs sampling (see Li, 1988, Tanner & Wong, 1987 and Enders, 2010, pp. 191ff.). As Aßmann et al. (2015) additionally argue, it allows to circumvent the application of combining rules[1] in order to access accuracy measures of

[1] See Rubin, 2004, p. 76ff., for details of the rules.

estimators. This approach opens up possibilities to analyse data during imputation or to impute while analyse. Aßmann et al. (2015), Gaasch (2017a) and Preising (2018) provide investigations for this combination of methods for different Bayesian models. The models they use, are IRT models with one latent variable and hierarchical and cluster structures in the case of Aßmann et al. (2015) and Gaasch (2017a), and linear dynamic panel models in the case of Preising (2018). In terms of missing values in covariates, they especially examined imputation using *classification and regression trees* (CART)[2], lowering the burden of specifying an additional imputation model next to the model of analysis. Comparing complete-case analysis with CART imputation, Gaasch (2017a, p. 56) speaks of reduced bias and reduced coverage rates, when only deleting list-wise and not imputing, and, although no bias and coverage declines seem to occur for complete-cases, Preising (2018, p. 5 and pp. 64ff.) points out gains in efficiency for parameter estimates (with respect to relative root mean squared errors), when using this latter automated imputation approach. Due to these positive results, within the research of the thesis presented, further Bayesian scenarios of possible applications for this imputation strategy will get investigated. These are multidimensional and multivariate normal-ogive IRT and confirmatory factor analytic models, encompassing several measurements (of different scale; also denoted as items, being dependent variables) and several latent variables as well. The benefit of using multiple measurements and latent variables within one model framework, is the ability to estimate several relationships at the same time and account for covariability between these items. For example psychometrics and educational research employ a great amount of survey items that relate to a competency, but competencies (which will be represented by latent variables), can be of different kind: For example science literacy, mathematical, reading or metacognitive competencies, to name a few (see Weinert et al., 2011, and Klieme et al., 2008, and Weinert, 2001, for a broader discussion about competencies) . However, this benefit can be accompanied by an increased computational burden and the aforementioned increase of missingness, which will be investigated within this thesis and whether and how this can be tackled.

In order to create an understanding of latent variable models in general and, specifically, factor analytic and IRT formulations, chapter 2, presents the fundamentals for them. Next to a brief overview of the historical development, applications and current research, first

[2]See Loh, 2014, for a historical overview of CART, and Burgette & Reiter (2010) and Doove et al. (2015) for imputation approaches applying this decision tree method.

mathematical notation and Bayesian estimation for both types of models under investigation, i.e. a confirmatory factor and a normal-ogive IRT specification, get introduced, interconnecting measurements and latent variables. In the case of IRT, terminology and different possible model formulations are depicted in more detail–not only with regards to further formulations that aren't "normal-ogive" but also to describe, what comparison of different research articles surfaced: The usage of same terminology for slightly different mathematical formulations. After both areas of models are layed out separately, the mathematical combination of normal-ogive IRT and factor analytic models are presented, and additionally issues and strategies of identification, when latent variables are involved, are highlighted.

The next chapter, chapter 3, investigates the inclusion of covariates into the multivariate latent variable models under consideration, and possibilities of handling of missing values. For this purpose, Bayesian approaches by Aßmann et al. (2015) and Gaasch (2017a) respectively, and by Conti et al., 2014 are examined. Work of the former researches concentrates on normal-ogive IRT models with one latent variable and measurements of binary or binary and ordered scaled measurements–including group- and cluster-specific effects. The latter present an explorative dedicated factor analytic approach, using multiple factors, measurements of binary and metric scale and an automated dimension reducing algorithm. When it comes to covariates and handling of missing values, the IRT and the factor analytic procedures differ. Within this thesis, this will be a basis for a model formulation that allows to combine several aspects and extend both perspectives and capabilities into a greater framework. Hence, chapter 4, presents a Bayesian sampling algorithm, which uses Conti et al.'s (2014) procedure as a starting point, transforms it into a matrix-variate formulation and extends it via including CART imputation, two different ways of incorporating covariates and the possibility of using measurements of ordered scale–next to binary and metric scale. Alongside the dedicated structure of Conti et al. (2014) it will also include the option of estimating cross-loading, i.e. one measurements loads onto several latent variables, and the potential to specify equality and linear restrictions for identification.

Using the statistical software R (R Core Team, 2017), chapter 5 presents the results of 21 simulation studies to test the proposed approach with respect to different model specifications, like loading structure, covariate influence and types of missingness. For this purpose, code implementation and the test environment will be described, next to the data generation, respective imposing of missingness and the calculations of measures of quality. In order to present the findings, the details of the scenarios are, at first, depicted individually and compared with one another afterwards. The results that we obtain, suggest that the implementation for a model where the covariates influence the measurements directly–coined factor analytic approach or factor analytic scenario–and where the latent variables are first of all influenced by the covariates (coined "IRT approach"), next to the aforementioned extensions, is in general possible. If however missingness gets imposed, complete-case analysis outperforms CART imputation within the factor analytic scenarios. But, within the IRT approach, applying CART imputation results similar to Preising (2018, p. 5 and pp. 64ff.) in gains of efficiency: Depending on the missingness mechanism, CART performs as good as complete-case analysis, or results in gains of eighteen percentage points on average (with respect to the relative root mean squared errors and parameter standard deviations) or up to seventy percentage points on average (for individual parameters of the most complex scenario, the gains are able outreach ninety percentage points).

With the previous results in mind and because imputation within the factor analytic approach seems to be problematic, a further empirical test is going to be carried out for the IRT approach only. Within this context, chapter 6 describes analysis using the IRT estimation applied to NEPS competency data. For this purpose, a similar scenario to Gaasch (2017a, pp. 58) using NEPS Starting Cohort 4 data (doi:10.5157/NEPS:SC4:9.1.0; see also Blossfeld et al., 2011) will be used, assigning mathematical literacy items to a mathematical competency. The similarity allows to have a reference to which the estimates can be compared to. In contrast to Gaasch (2017a), we include several more measurements that are assumed to be related to a second latent variable, i.e. reading competency. Thus we test, on the one hand, the applicability of the algorithm proposed on empirical data. But also, on the other hand, whether we are able to estimate a possible relationship between these two types of competencies. The overall results of the empirical application are throughout as intended.

In the last chapter we summarise the findings and point towards future directions.

Chapter 2

General and Specific Latent Variable Concepts in Statistics

Within this chapter, we are first of all concerned with the notion of latent variable models in general. But because the thesis is especially interested in two types of models, the descriptions subsequently focus on factor analysis and IRT modelling. Next to introducing terminology and first aspects of Bayesian estimation, this chapter also encompasses possibilities of mathematically combining both types of models and depicts strategies of identification.

2.1 The General Notion of Latent Variables

Perhaps the simplest description for a latent variable–and therefore covering a wide variety of analytical concepts and models–is, to define it as a variable that is not observed or not observable (see Koopmans & Reiersøl, 1950), or „whose realizations are hidden from us“ (Skrondal & Rabe-Hesketh, 2004, p. 1). This encompasses unobserved outcomes and predictors[3] and, as stated, not-observable variables[4] as well. Concerning this aspects of

[3]But these are thought as being potentially observable with some effort, like true measurements hidden via a measurement error as in basic item-response-theory and classical test-theory models, Hidden-Markov equations, counterfactual missing values or heterogeneity contained in random effects of multi-level / longitudinal models.

[4]„Not observable“ even with maximal effort due to the ideal or constructive nature. For example, hypothetical constructs like the “Big Five”-Personality scale, intelligence, but also assumptions of probit or logit models (see again, Skrondal & Rabe-Hesketh, 2004, e.g. chapter 1 for a discussion). Additionally, from our perspective, this latent concept might also refer to *knowledge atoms* (Smolensky, 1986, pp. 202ff.) and *hidden units* (Hinton & Sejnowski, 1986, p. 290) or *hidden layers* (see for example Mitchell, 1997, pp.

analysing different relationships, there is neither *the* latent variable model, nor *one* specific group of latent variable models. Instead, as many different approaches as they occur for solely observable / manifest variables, like in linear and other types of regression– e.g. logistic or (vector) autoregression–, (M)ANOVA or survival methods, seem to exist.

But despite the thoroughly abstract perception of a latent variable as an unobservable one–and nothing more, differing but nonetheless comparable statistical frameworks facilitating regressional structures were introduced. As an example, Kline (2016, p. 18) interweaves them with *structural equation models* (SEM). Rabe-Hesketh et al. (2004) and Skrondal & Rabe-Hesketh (2004) discuss several latent variable models facilitating the exponential family for responses and name it a scheme of *generalized-linear-latent-and-mixed-models* (GLLAMMs) encompassing factor, IRT, SEM and several other specifications (Ibid., 2004, Chapter 4). Similarly, Bartholomew et al. (2011, pp. 19ff.) promote a so called *general-linear-latent-variable-model* (GLLVM) framework. Further authors, e.g. Bartholomew & Knott (1999, p. 18), Bartholomew et al. (2011, p. 11) or Brachinger & Ost (1996, p. 710) use a classification connecting latent variables and their scales to manifest variables and the latter's respective levels of measurements (see table 2.1).

		y	
		discrete	*continuous*
θ	*discrete*	Latent Class Analysis	Latent Profile Analysis
	continuous	Dichotomous and Ordered Factor Analysis / Latent Trait Analysis	(Continuous) Factor Analysis

Table 2.1: Latent variable model classification based on Bartholomew & Knott (1999, p. 18), Bartholomew et al. (2011, p. 11), Brachinger & Ost (1996, p. 710), where θ denotes the latent variable and y the observed measurement.

From our point of view, and in line with several others (i.e. Bartholomew et al., 2011, p. 20, Borsboom et al., 2003, Mellenbergh, 1994, Rabe-Hesketh et al., 2004, Skrondal & Rabe-Hesketh, 2004, pp. 96ff., Thompson, 2004a, p. 109), all of these aforementioned approaches can be seen as facilitating and elaborating the idea of the *generalized linear*

106ff.) as they are discussed with regards to Boltzmann machines specifically and in the field of artificial neural networks in general.

model (GLM, see Nelder & Wedderburn, 1972) with respect to the observed variable y and a transforming function $f_y(\cdot)$, e.g. predictor function, link function or when including errors a probability density or probability mass function, corresponding parameters and probable further variables Ψ and error(s) ε, latent variables Θ and particularly accounting for their possible–because: also unobserved–level of measurement and a hence necessary second transformation [5] $f_\theta(\cdot)$, i.e.

$$y = f_y \circ f_\theta(\Theta, \Psi, \varepsilon). \tag{2.1.1}$$

Of course, when thinking of $f_\theta(\cdot)$ as a (latent) structural equation or latent variable model and of $f_y(\cdot)$ as a measurement model (see also Bollen, 1989, pp. 319ff.), the previously stated interconnection with structural equation models is likewise and easy to understand.

The interpretation of the associative or causal direction between the latent variable(s) and the manifest variable(s) is however perspective dependent. Whether the latent variables cause and explain–in a sense of conditioning regression–the variation of the measurements, or, by reversing the effective direction, whether the measurements create the hidden structure, e.g. in a principal-components' data-reduction way, isn't of course mathematically purported, instead it is imposed by theory and substantive thought.[6] Additional field specific interpretations might arise as well. For example, within different empirical analyses of different scientific fields the latent variables can be interpreted as portraying personality traits (see McCrae & Costa, 1987)[7] in general or as influence factors on partisan attitudes (see Schoen & Schumann, 2007), indicate ethnocentrism (see Lazarsfeld, 1973, p. 364f.) or are seen as determinants of water quality (see Kuppusamy & Giridhar, 2006).

A further common ground, but contingent on the suspected complexity (whether it is founded on theory or previous explorative research) for all of these approaches, is mostly a reduction of dimensionality with projecting high-dimensional space, $\mathbf{Y} \in \mathbb{S}_Y^M$, via a parameter set, onto lower-dimensional latent variables, i.e. $\boldsymbol{\Theta} \in \mathbb{S}_\Theta^K$ with $K \leq M$–where $\mathbb{S}$, in accordance with table 2.1, denotes a general set or space of discrete or continuous numbers.[8] Embodying these hidden variables can also result in necessities of restrictions

[5] Including the identity link.

[6] Depending on the direction, this is either called *reflective*, when the latent variable is assumed to cause the item values, or *formative*, when the items are assumed to cause the latent variables' values (see Skrondal & Rabe-Hesketh, 2004, p. 67 and Bollen, 1989, p. 65).

[7] Having five factors to consider, which are according to the authors: Neuroticism, extraversion, openness to experience, agreeableness-antagonism and conscientiousness-undirectedness.

[8] The equality $K = M$ in $K \leq M$ can be found for example in models of classical test theory (see Skrondal & Rabe-Hesketh, 2004, pp. 64f.).

for identifying purposes (see, e.g. Anderson & Rubin, 1956, Bollen, 1989, pp. 242ff. and pp. 326ff., Koopmans & Reiersøl, 1950, San Martín, 2015, or Skrondal & Rabe-Hesketh, 2004, pp. 135ff.), to which will be returned later.

Two prominent latent variable models portraying the main focus of this thesis, are the (confirmatory) factor model and the (normal-ogive) IRT model. Hence, the upcoming description is narrowed down to details solely about these model structures, applications, current developments and their interconnection.

2.2 Confirmatory Factor Analysis and IRT models

2.2.1 Specifics of Factor Analysis

The fundamentals of factor analysis, as one of the most well-known and earliest type of latent variable model, date almost back to the onset of academic psychology (see Fabrigar et al., 1999, tracing early principals of explorative factor analysis back to Spearman, 1904, and 1927). Ideas for factor analysis arose after examining psychological tests, where one of the aims, amongst others, was to discover "mental faculties" (see Thurstone, 1935, p. 53). Historically, two methodological perspectives were opened up: *explorative* first and *confirmatory* later (see Bollen, 1989, p. 226ff. and Thompson, 2004b, chapters 3 and 10). These terms describe, on the one hand, when the approach is explorative: Being concerned with estimating the number of latent factors and probable rotational aspects based on interpretative purposes with respect to the factor covariance, the dimensionality and relationships within the latent variables and between the latent and manifest variables. And, on the other hand, when defining the number of factors and impose (zero-)restrictions onto the loadings, e.g. setting some values of a loading matrix[9] $\boldsymbol{\alpha}_{(M\times K)}$ to $\alpha_{mk} := 0$, one aims at testing or *confirming* hypotheses with respect to the latent structure. As pointed out, the thesis focusses on the latter when it comes to factor analysis.

Although rooting in psychology–and nonetheless the main area of interest here lies in education and psychometrics as well–, factor analysis has become a prominent modelling tool across several disciplines. Mentionable examples from natural sciences are the usage in neuroimaging of EEG data via parallel factor analysis (see, e.g. Miwakeichi et al., 2004),

[9] Sometimes denoted with $\boldsymbol{\Lambda}$ (e.g. in Skrondal & Rabe-Hesketh, 2004, p. 67 or Bartholomew et al., 2011, p. 47) or $\mathbf{L}$ (for example in Brachinger & Ost, 1996).

investigations relating to radio-pharmaceuticals (see Oster et al., 1989), applications in pyrolysis-mass spectrometry (see Windig et al., 1982) or the aforementioned analysis of water quality (see Kuppusamy & Giridhar, 2006). Going back to social sciences, further research concentrates on methodological issues in explorative Bayesian analysis, for example, Aßmann et al. (2016) discuss an ex-post strategy to solve for rotational issues in a dynamic factor model, while Conti et al. (2014) establish a marginal-likelihood model-selection method that automatically adapts factor dimension and gets applied to a health related survey. In a confirmatory Bayesian scenario, there are also issues occurring by means of rotation or reflection (see e.g. Erosheva & Curtis, 2017).

As rich in variety the applications for factor analysis are, almost as many estimation procedures and estimation tasks seem to exist as well. They range from classical maximum likelihood methods (see Lawley, 1940), conceptualising expectation maximisation (see Rubin & Thayer, 1982) or a generalized version of this algorithm (see Cagnone & Viroli, 2014), alternating least squares in parallel factor analysis (see Harshman & Lundy, 1994), training of Hopfield neural networks (see Frolov et al., 2007)[10] or are conducted within a Bayesian MCMC framework (see Conti et al., 2014, Edwards, 2010, Geweke & Zhou, 1996 or Lopes & West, 2004). But factor models differ not only in their fields of application and inferential structure, they also diverge when it comes to processing of outcome variables with respect to their level of measurement (compare again table 2.1). As already indicated, similar to generalized regression models, one can find factor-analytic approaches for differing types of scales, i.e. classically one relies on metric scaled measurements, but there are approaches for ordered variables (for example in Nikele & Fahrmeir, 1999) as well. Furthermore, there are also combinations available, for example, binary and metric measurements (like Conti et al., 2014) and additionally ordered responses (see for example Raach, 2006[11]) or polytomous and continuous outcomes (see Shi & Lee, 1998).

Let us now introduce some first notation and estimation considerations, allowing to relate terminology with mathematical formulations and being one the statistical fundaments of the algorithm presented in chapter 4. Following Bartholomew et al. (2011, p. 47ff.) and Rubin & Thayer (1982), a basic linear expression without additional covariates for metric scaled measurements $\mathbf{y}_m = (y_{1m}, \ldots, y_{im}, \ldots, y_{Nm})'$, $m = 1, \ldots, M$, of an individual i, $i = 1, \ldots, N$ with $\mathbf{y}_i = (y_{i1}, \ldots, y_{im}, \ldots, y_{iM})'$, and a continuous latent variable $\boldsymbol{\theta}_i =$

[10] Although here, the factors, the measurements and even the loadings are binary.

[11] This is a general Bayesian approach to latent variable models, not only restricted to factor models.

$(\theta_{i1}, \ldots, \theta_{iK})'$, $k = 1, \ldots, K$, can be described via

$$\mathbf{y}_i \quad = \quad \boldsymbol{\mu}_y + \boldsymbol{\alpha}\boldsymbol{\theta}_i' + \boldsymbol{\varepsilon}_i \tag{2.2.1}$$

with complete-data likelihood, for now considering the latent variables as normal random variables with zero mean,

$$\begin{aligned} f_y(\mathbf{Y}|\Psi, \boldsymbol{\Theta}) = L(\Psi; \mathbf{Y}, \boldsymbol{\Theta}) \quad \propto \quad & \prod_{i=1}^{N} \exp\left\{-\frac{1}{2}\left(\mathbf{y}_i - \boldsymbol{\mu}_y - \boldsymbol{\alpha}\boldsymbol{\theta}_i'\right)' \boldsymbol{\Sigma}_{\varepsilon}^{-1}\left(\mathbf{y}_i - \boldsymbol{\mu}_y - \boldsymbol{\alpha}\boldsymbol{\theta}_i'\right)\right\} \cdot \\ & \prod_{i=1}^{N} \exp\left\{-\frac{1}{2}\boldsymbol{\theta}_i'\boldsymbol{\Sigma}_{\theta}^{-1}\boldsymbol{\theta}_i\right\} \end{aligned} \tag{2.2.2}$$

or

$$\begin{aligned} f_y(\mathbf{Y}|\Psi, \boldsymbol{\Theta}) \quad \propto \quad & \prod_{i=1}^{N}\prod_{m=1}^{M} \exp\left\{-\frac{1}{2\sigma_{\varepsilon,m}^2}\left(y_{im} - \mu_{y,m} - \boldsymbol{\theta}_i\boldsymbol{\alpha}_m'\right)^2\right\} \cdot \\ & \prod_{i=1}^{N} \exp\left\{-\frac{1}{2}\boldsymbol{\theta}_i'\boldsymbol{\Sigma}_{\theta}^{-1}\boldsymbol{\theta}_i\right\} \end{aligned} \tag{2.2.3}$$

assuming independence conditional on the factors $\boldsymbol{\theta}_i$. Herein, $\mathbf{Y}$ denotes the complete $N \times M$ matrix of measurements consisting of elements y_{im}, M columns $(\mathbf{y}_m)_{m=1}^{M}$ or N rows $\left[(\mathbf{y}_i)_{i=1}^{N}\right]'$. $\boldsymbol{\alpha}$ is a *factor loading matrix* of dimension $M \times K$, with M rows $\left[(\boldsymbol{\alpha}_m)_{m=1}^{M}\right]'$and K columns $(\boldsymbol{\alpha}_k)_{k=1}^{K}$, and $\boldsymbol{\Theta}$ the $N \times K$ matrix of *latent variables* of interest or *factors*, whose rows stem from a normal distribution with $MVN(\mathbf{0}, \boldsymbol{\Sigma}_{\theta})$. Depending on the covariance structure of the factors, i.e. $\boldsymbol{\Sigma}_{\theta,(K\times K)}$, one deals either with *orthogonal* factor analysis (in the case of diagonality) or an *oblique* factor model (when the factors covary), i.e. assuming a relationship amongst different latent variables or not. $\boldsymbol{\mu}_y = (\mu_{y,1}, \ldots, \mu_{y,m}, \ldots, \mu_{y,M})'$ is a $M \times 1$ individual independent mean vector and $\boldsymbol{\varepsilon}_i$ describes a $M \times 1$ multivariate Gaussian term with $MVN(\mathbf{0}, \boldsymbol{\Sigma}_{\varepsilon})$ described as *idiosyncratic error* (Conti et al., 2014) or *unique component* (Bollen, 1989, p. 220). The corresponding variance is $\boldsymbol{\Sigma}_{\varepsilon} = \mathrm{diag}(\sigma_{\varepsilon,1}^2, \ldots, \sigma_{\varepsilon,m}^2, \ldots \sigma_{\varepsilon,M}^2)$, which is called the *idiosyncratic variance* (Conti et al., 2014) or *uniqueness* (Rubin & Thayer, 1982). All information contained within the outcomes is then reduced onto the factors. This emphasizes that the overall covariation of the measurements $\mathbf{y}_i$ is separable into two parts,

$$\boldsymbol{\Sigma}_y = \boldsymbol{\alpha} \cdot \mathrm{E}\left[\boldsymbol{\Theta}'\boldsymbol{\Theta}\right] \cdot \boldsymbol{\alpha}' + \boldsymbol{\Sigma}_{\varepsilon} = \boldsymbol{\alpha}\boldsymbol{\Sigma}_{\theta}\boldsymbol{\alpha}' + \boldsymbol{\Sigma}_{\varepsilon} \tag{2.2.4}$$

i.e. the covariance of the loadings $\boldsymbol{\alpha}\boldsymbol{\Sigma}_{\theta}\boldsymbol{\alpha}'$ with respect to the factors and the covariance of the errors $\mathbf{\Sigma}_\varepsilon$.

As previously mentioned, estimation procedures differ. For factor analysis in a Bayesian setting[12] (see for example Quinn, 2004) one can rely on conjugate informative priors that allow the incorporation of previous information or (expert) assumptions. When connecting prior information for parameters, e.g. ψ with prior density $\pi(\psi)$, and a likelihood, e.g. $f_y(y|\psi)$, it results in posterior distributions for the parameters of interest, e.g. posterior density $f(\psi|y) = (f_y(y|\psi)\pi(\psi))/\int(f_y(y|\psi)\pi(\psi))d\psi$ or $f(\psi|y) \propto f_y(y|\psi)\pi(\psi)$ in an unnormalized depiction (see Gelman et al., 2014, p. 7). This differs from frequentist and maximum likelihood considerations, which assume one fixed true parameter as a standard instead.[13]

When not only using a multivariate normal prior distribution for a unrestricted loading vector $\boldsymbol{\alpha}_m^\Delta$, that contains only non-zero and unique elements (e.g. if a parameter is equal or a multiple of another parameter), with

$$\pi(\boldsymbol{\alpha}_m^\Delta) \propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{\alpha}_m^\Delta - \boldsymbol{\mu}_\alpha^0\right)'\left(\mathbf{\Sigma}_\alpha^0\right)^{-1}\left(\boldsymbol{\alpha}_m^\Delta - \boldsymbol{\mu}_\alpha^0\right)\right\}, \tag{2.2.5}$$

but also, first, an inverse-gamma distribution for $\sigma^2_{\varepsilon,m}$, where

$$\pi(\sigma^2_{\varepsilon,m}) \propto \left(\sigma^2_{\varepsilon,m}\right)^{-c_m^0-1}\exp\left(-\frac{C_m^0}{\sigma^2_{\varepsilon,m}}\right), \tag{2.2.6}$$

second, $\boldsymbol{\mu}_y = \mathbf{0}$, i.e. $\tilde{\mathbf{y}}_i = \mathbf{y}_i - \mathbf{0} = \mathbf{y}_i$, and, third, known factor covariance $\mathbf{\Sigma}_\theta$, e.g. $\mathbf{\Sigma}_\theta = \mathbf{I}$ for simplification, derivation of posterior distributions is possible in a closed form using these priors and the likelihood of (2.2.2). Computation is then conducted via the following Gibbs sampling scheme[14], which enables movements to traverse the parameter space of interest and produces samples establishing posterior information. Within an iterative scheme, several steps need to get carried out: First, draw the diagonal elements of $\mathbf{\Sigma}_\varepsilon$ via an inverse-gamma distribution with density

$$f(\sigma^2_{\varepsilon,m}|\mathbf{y}_i,\boldsymbol{\theta}_i,\boldsymbol{\alpha}_m) \propto \left(\sigma^2_{\varepsilon,m}\right)^{-c_m^\diamond-1}\exp\left(-\frac{C_m^\diamond}{\sigma^2_{\varepsilon,m}}\right), \tag{2.2.7}$$

where $c_m^\diamond = c_m^0 + \frac{N}{2}$ and $C_m^\diamond = C_m^0 + \frac{1}{2}\left\{(\mathbf{y}_i - \boldsymbol{\theta}_i\boldsymbol{\alpha}_m')'(\mathbf{y}_i - \boldsymbol{\theta}_i\boldsymbol{\alpha}_m')\right\}$. Second, draw elements

[12] See Box & Tiao (1973) or Gelman et al. (2014) for details on the Bayesian paradigm.
[13] Not neglecting the possibility of implementing random effects in hierarchical modelling.
[14] See for example Casella & George, 1992, Gelfand et al., 1990 and Geman & Geman, 1984 concerning general aspects of Gibbs sampling.

$\boldsymbol{\alpha}_m^{\Delta}$ of the loadings from a conditional multivariate normal distribution with

$$f(\boldsymbol{\alpha}_m^{\Delta}|\mathbf{y}_m, \boldsymbol{\theta}) \propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{\alpha}_m^{\Delta} - \boldsymbol{\mu}_{\alpha}^{\diamond}\right)' (\boldsymbol{\Sigma}_{\alpha}^{\diamond})^{-1} \left(\boldsymbol{\alpha}_m^{\Delta} - \boldsymbol{\mu}_{\alpha}^{\diamond}\right)\right\}, \tag{2.2.8}$$

using the posterior-expectation $\boldsymbol{\mu}_{\alpha}^{\diamond} = \boldsymbol{\Sigma}_{\alpha}^{\diamond}\left\{\boldsymbol{\theta}_i'\left(\boldsymbol{\Sigma}_{\varepsilon}^{-1}\right)\mathbf{y}_i + \left(\boldsymbol{\Sigma}_{\alpha}^{0}\right)^{-1}\boldsymbol{\mu}_{\alpha}^{0}\right\}$ and variance equal to $\boldsymbol{\Sigma}_{\alpha}^{\diamond} = \left\{\boldsymbol{\theta}_i'\left(\boldsymbol{\Sigma}_{\varepsilon}^{-1}\right)\boldsymbol{\theta}_i + \left(\boldsymbol{\Sigma}_{\alpha}^{0}\right)^{-1}\right\}^{-1}$. Finally, use a multivariate normal draw for $\boldsymbol{\theta}_i$, i.e.

$$f(\boldsymbol{\theta}_i|\mathbf{y}_i, \boldsymbol{\alpha}) \propto \exp\left\{-\frac{1}{2}(\boldsymbol{\theta}_i - \boldsymbol{\mu}_{\theta}^{\diamond})' (\boldsymbol{\Sigma}_{\theta}^{\diamond})^{-1} (\boldsymbol{\theta}_i - \boldsymbol{\mu}_{\theta}^{\diamond})\right\} \tag{2.2.9}$$

where $\boldsymbol{\mu}_{\theta}^{\diamond} = \boldsymbol{\Sigma}_{\theta}^{\diamond}\left(\boldsymbol{\alpha}'\boldsymbol{\Sigma}_{\varepsilon}^{-1}\mathbf{y}_i\right)$ and $\boldsymbol{\Sigma}_{\theta}^{\diamond} = \left\{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_{\varepsilon}^{-1}\boldsymbol{\alpha} + (\boldsymbol{\Sigma}_{\theta})^{-1}\right\}^{-1}$.

For item response theory based models, the latent variable structure can be similar and therefore estimated in a similar fashion, which will be depicted–next to introductory remarks–within the next section.

2.2.2 Specifics of IRT modelling

From a historical point of view, IRT originated from the same path factor analysis was built upon, i.e. the concept of latent variables. But perceived as an own class of methods, it is said to be first established in the 1960s (see Embretson & Reise, 2000, pp. 14ff. and Thissen & Steinberg, 1986). Although IRT models encompass a larger set of modelling tools, the central mathematical aspect is again the relationship between manifest variables and latent ones. Dependent on the formulation, the model contains parameters that get subtracted, multiplied and / or divided from the unobserved variables, and a linking function that maps the latent variables onto the observed measurements. Used in psychometrics, applications can be found, for example, for assessment studies, competency tests or attitude investigations, interpreting the latent variable as ability, skill, achievement level and others (see Wilson et al., 2008, p. 83)

Although further, nonetheless similar approaches on classification of IRT formulations exist (compare for example Thissen & Steinberg, 1986), these models can be differentiated based on the previously mentioned scale of the outcomes, the corresponding link function and the numbers of parameters on the item and person level included. Note, that some authors and therefore IRT models do not use all parameters and the same transformation $f_y(\cdot)$ mentioned in the next formulations. This is going to be addressed after a discussion of a more abstracted consideration referring to terminology and mathematical depiction.

A general representation for IRT uses

$$y_{im} = f_y\left(\alpha_{mk}\theta_{ik} - \beta_m\right)\left(1 - \Gamma_m\right) \tag{2.2.10}$$

as a core transformation of a measurement y_{im} for one individual–for example $y_{im} = 0$, if a person i gave a wrong answer (0) at item m, with a person-specific ability / latent variable θ_{ik}. While classical texts only refer to one latent variable, the index k is already included here to indicate possible multidimensionality of these variables (see Reckase, 1997, for a discussion on logistic multidimensional IRT (MIRT) models and Sheng & Wikle, 2007 for a normal-ogive oriented MIRT model). The parameter, Γ_m, is a *guessing*- or "*not knowing*"-*parameter* (see amongst others Béguin & Glas, 2001, Miyazaki & Hoshino, 2009 or Thissen & Steinberg, 1986), α_{mk} is called *item discrimination* and β_m is the *item difficulty* (see, e.g. Albert, 1992, Béguin & Glas, 2001, Christensen et al., 2013, Edwards, 2010, or Fox & Glas, 2001) or (negative) *item-intercept* (Bock & Aitkin, 1981). In the case of Bock & Aitkin (1981), the authors use *item threshold* for a β, where

$$y_{im} = f_y\left(\alpha_{mk}\left(\theta_{ik} - \beta_k\right)\right)\left(1 - \Gamma_m\right), \tag{2.2.11}$$

in the same way and position of the formulations by Edwards (2009), Lord et al. (1974, pp. 376 f.) or Samejima (1974), but they use *item difficulty.* The item-intercept in the expression of Bock & Aitkin (1981) and in terms of (2.2.11) is $-\alpha_{mk} \cdot \beta_k$ instead. Further formulations use

$$y_{im} = f_y\left(\alpha_{mk}\theta_{ik} + \beta_m\right)\left(1 - \Gamma_m\right) \tag{2.2.12}$$

and difficulty for $-\frac{\beta_m}{\alpha_{mk}}$, which is the negative fraction resulting from factoring out α_{mk}, (see Mellenbergh, 1994), or use (2.2.12), where β_m is once more–and without factoring out–referred to as item difficulty (see Molenaar et al., 2015).

From our point of view, these diverging naming conventions of item difficulty and item-intercept or item threshold respectively, lead to different model formulations and bear the risk of applying similar terminology to slightly different mathematical concepts. Therefore, we feel obliged to state that within the research of this thesis, the understanding presented in formula (2.2.10) with respect to β_m is used, and we refer to it from now on as item difficulty. First, because a change in β_m is still connected with the latent variable and is directly (not via recalculation with α_{mk}) related to the item. Second, when turning towards a multidimensional latent variable, dimensionality of this kind of intercept will be dependent of the dimensionality of the items and not the latent variables. From our point of view, the denotation as item difficulty and the mathematical formulation in use, seem to us therefore closest, when following the previously mentioned understanding of Albert (1992), Béguin & Glas (2001), Christensen et al. (2013), Edwards (2010) or Fox & Glas (2001).

When looking at all model aspects, for example, when outcomes (y_{im}) are dichotomous–like correct and incorrect–, this formulation can portray: there is a threshold that an answer for a specific question m is correct or not (α_{mk}), a person's general ability (θ_{ik}), a parameter incorporating the possibility of simply guessing the correct answer (Γ_m) and a general tendency towards a correct or incorrect answer (β_m) seen as, e.g. stemming from the difficulty of the item itself. Therefore, one major theoretical perspective of IRT is to be able to differentiate between amounts of the results, that stem from the item construction itself (the item parameters) and the actual ability of a person, i.e. the latent variable (see Embretson & Reise, 2000, p. 16).

Some derivations and scenarios do not include all parameters. Consequently, restricting parameters, e.g. using $\alpha_{mk} = 1$ or $\Gamma_m = 0$, determine the parameter number of the model and form, with additionally stating a link function, a naming convention like 1PL ("one parameter logistic model"): one parameter, where $\alpha_{mk} = 1$ and $\Gamma_m = 0$ (1P), two parameters, where only $\Gamma_m = 0$ (2P), or three parameters (3P), where all parameters are in use. As said, applying various link functions, $f_y(\cdot)$, allows the creation of relationships between the latent variable θ and measurements y of different scale. Hence, when using the stated understanding about item difficulty, the formulations and classifications are similar but not the same–due to the different specification–to Mellenbergh (1994), to whom it is referred for a more detailed depiction. But to portray the idea, table 2.2 gives a brief overview for some prominent IRT models.

Name	Parametrisation	Link	Outcome	Reference
Rasch Model, 1PL	$\theta_{ik} - \beta_m$; $\alpha_{mk} = 1$, $\Gamma_m = 0$	Logit	Binary	e.g. Christensen et al. (2013, p. 9)
Two-Parameter Logistic, 2PL	$\alpha_{mk}\theta_{ik} - \beta_m$; $\Gamma_m = 0$	Logit	Binary	e.g. Christensen et al. (2013, p. 220)
Partial Credit Model / Ordered Rasch Model	$\theta_{ik} - \beta_{mj}$; $\alpha_{mk} = 1$, $\Gamma_m = 0$	Multinomial for category j, $j = 1, \dots, J_m$	Ordered	e.g. Masters (1982)
Normal-Ogive, 2PNO (Probit)	$\alpha_{mk}\theta_{ik} - \beta_m$; $\Gamma_m = 0$	Gaussian	Binary	e.g. Albert (1992)
Normal-Ogive, 3PNO (Probit)	$\alpha_{mk}\theta_{ik} - \beta_m$; $\Gamma_m \neq 0$	Gaussian	Binary	e.g. Béguin & Glas (2001)
Logistic Model, 3PL	$\alpha_{mk}\theta_{ik} - \beta_m$; $\Gamma_m \neq 0$	Logit	Binary	e.g. Miyazaki & Hoshino (2009)

Table 2.2: Some prominent IRT models based on Mellenbergh (1994).

The "Outcome"-column of table 2.2 depicts that most measurements used in IRT modelling are categorical–originating from different outcome possibilities of (competency or ability) testing. Nonetheless, models for metric variables are also in use. Within these ability or competency scenarios, response time, i.e. how much time a test taker needs to give an answer, is also part of the modelling approach (see for example Molenaar et al., 2015, Thissen, 1983, and van der Linden, 2007). Leaving the basic formulations, current research on IRT models focusses for example on multilevel approaches for binary data (Fox & Glas, 2001), tree structures for response categories and latent variables (De Boeck & Partchev, 2012), the aforementioned response time inclusion (i.e. Molenaar et al., 2015, Thissen, 1983, and van der Linden, 2007) and handling of missing background variable values in a Bayesian (hierarchically structured) one-factor normal-ogive IRT model (see Aßmann et al., 2015, and Gaasch, 2017a).

Like before, there are differing approaches for estimation, i.e. using a maximum likelihood oriented routine (e.g. the marginal maximum likelihood EM-based approach of Bock & Aitkin, 1981) or techniques aiming at a Bayesian paradigm (see, e.g. Albert, 1992 for a normal-ogive IRT model and Levy, 2016, pp. 257ff., for a more general presentation concerning Bayesian approaches in IRT modelling). To give a brief idea of the later-to-be-extended normal-ogive IRT model, a Gibbs routine of a normal-ogive two-parameter model (2PNO) with binary measurements and one latent variable (i.e. $k = K = 1$) follows. Thus, let $\mathbf{y}_i = (y_{i1}, \dots, y_{im}, \dots, y_{iM})'$, $m = 1, \dots, M$, represent a vector of M binary outcomes for one individual i, $i = 1, \dots, N$, and $\mathbf{y}_i^* = (y_{i1}^*, \dots, y_{im}^*, \dots, y_{iM}^*)'$ denote a vector of M underlying continuous variables. $\mathbf{y}_i^*$ is connected with $\mathbf{y}_i$ based on a indicator function $\mathcal{I}_{\square}(\cdot)$, such that

$$y_{im} \quad = \quad \mathcal{I}_{[0,+\infty)}(y_{im}^*) = \begin{cases} 0 & \text{, where } y_{im}^* < 0 \\ 1 & \text{, where } y_{im}^* \geq 0. \end{cases} \tag{2.2.13}$$

$\mathbf{y}_i$ is then interconnected with the latent variable θ_i based on $\mathbf{y}_i^*$, i.e.

$$\mathbf{y}_i^* = \boldsymbol{\alpha}\theta_i - \boldsymbol{\beta} + \boldsymbol{\varepsilon}_i, \tag{2.2.14}$$

With establishing a probit-like (see Nelder & Wedderburn, 1972) probability structure using a Gaussian probability, i.e.

$$\mathcal{P}(Y_{im} = 1 | y_{im}^*) = F_y(y_{im}^*) = \Phi(y_{im}^*), \tag{2.2.15}$$

one is able to model the data-likelihood resting on (multivariate) normal densities, as will be shown. In a general parameter set $\Psi = \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \sigma_\theta^2\}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m, \dots, \alpha_M)'$ is a vector of M item discrimination parameters, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m, \dots, \beta_M)'$ represents a vector of item difficulties, where each element β_m corresponds to one measurement across all individuals and θ_i is a normally distributed (and scalar) latent variable with $\theta_i \sim N(0, \sigma_\theta^2)$. Similar to θ_i, the $M \times 1$-vector $\boldsymbol{\varepsilon}$ follows also a normal, but multivariate normal distribution, assuming an identity covariance structure, i.e. $\boldsymbol{\varepsilon} \sim MVN(\mathbf{0}, \mathbf{I})$. This results in the expectation and variance for $\mathbf{y}_i^*$ as

$$E\left[\mathbf{y}_i^* | \boldsymbol{\alpha}, \theta_i\right] = -\boldsymbol{\beta},\ \mathrm{Cov}\left[\mathbf{y}_i^* | \boldsymbol{\alpha}, \boldsymbol{\beta}, \theta_i\right] = \boldsymbol{\Sigma}_{y^*|\cdot} = \boldsymbol{\Sigma}_\varepsilon \text{ and } \boldsymbol{\Sigma}_{y^*} = \sigma_\theta^2 \boldsymbol{\alpha}\boldsymbol{\alpha}' + \mathbf{I}. \tag{2.2.16}$$

The likelihood for all individuals–based on all $N \times M$ observed measurements $\mathbf{Y}$, all underlying $N \times M$ values $\mathbf{Y}^*$, given all $N \times 1$ latent variable values $\boldsymbol{\theta} = (\theta_1, \dots, \theta_i, \dots, \theta_N)'$ and all parameters Ψ–can be written as

$$\begin{aligned} L(\Psi; \mathbf{Y}, \boldsymbol{\Theta}) =& f(\mathbf{Y}|\Psi) = \\ =& \left[\int \!\! \cdots \!\! \int_{\mathbb{R}} \left\{ \int \!\! \cdots \!\! \int_{\mathbb{R}} f_{y,y^*|\cdot}(\mathbf{Y}, \mathbf{Y}^* | \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}) f_\theta(\boldsymbol{\theta}|\sigma_\theta^2) d\left(\mathbf{Y}^*\right) \right\} d\left(\boldsymbol{\theta}\right) \right] \end{aligned} \tag{2.2.17}$$

with augmented (see specifically Albert & Chib, 1993, Tanner & Wong, 1987 and van Dyk

& Meng, 2001, in general on augmentation strategies) density

$$f_{y,y^*|\cdot}(\mathbf{Y},\mathbf{Y}^*|\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}) \propto \prod_{i=1}^{N}\left[\exp\left\{-\frac{1}{2}\left(\mathbf{y}_i^* - \boldsymbol{\alpha}\theta_i + \boldsymbol{\beta}\right)'\left(\mathbf{y}_i^* - \boldsymbol{\alpha}\theta_i + \boldsymbol{\beta}\right)\right\}\cdot \right.$$
$$\left.\prod_{m=1}^{M}\left\{\mathcal{I}_{(-\infty,0)}(y_{im}^*) + \mathcal{I}_{[0,\infty)}(y_{im}^*)\right\}\right], \tag{2.2.18}$$
$$f_\theta(\boldsymbol{\theta}|\sigma_\theta^2) \propto \prod_{i=1}^{N}\exp\left\{-\frac{\theta_i^2}{2\sigma_\theta^2}\right\}, \text{ with} \tag{2.2.19}$$
$$d\left(\boldsymbol{\theta}\right) = (d\theta_1\cdot\ldots\cdot d\theta_N),$$

and integrating for $d\left(\mathbf{Y}^*\right)$ with respect to every single binary variable, i.e. $d\left(\mathbf{Y}^*\right) = \prod_{i=1}^{N}\prod_{m=1}^{M} dy_{im}^* = dy_{11}^*\cdot\ldots\cdot dy_{NK}^*$.[15] With the help of multivariate normal priors for the item discrimination as well as the item difficulty parameters, i.e.

$$\pi(\boldsymbol{\alpha}) \quad\propto\quad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\alpha}-\boldsymbol{\mu}_\alpha^0\right)'\left(\boldsymbol{\Sigma}_\alpha^0\right)^{-1}\left(\boldsymbol{\alpha}-\boldsymbol{\mu}_\alpha^0\right)\right\} \text{ and} \tag{2.2.20}$$
$$\pi(\boldsymbol{\beta}) \quad\propto\quad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}-\boldsymbol{\mu}_\beta^0\right)'\left(\boldsymbol{\Sigma}_\beta^0\right)^{-1}\left(\boldsymbol{\beta}-\boldsymbol{\mu}_\beta^0\right)\right\}, \tag{2.2.21}$$

one ends up with full conditional posterior distributions from the same family. The item discrimination parameters are drawn from

$$f(\boldsymbol{\alpha}|\mathbf{y}_i^*,\boldsymbol{\theta}) \quad\propto\quad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\alpha}-\boldsymbol{\mu}_\alpha^\diamond\right)'\left(\boldsymbol{\Sigma}_\alpha^\diamond\right)^{-1}\left(\boldsymbol{\alpha}-\boldsymbol{\mu}_\alpha^\diamond\right)\right\} \tag{2.2.22}$$

with posterior moments $\boldsymbol{\mu}_\alpha^\diamond = \boldsymbol{\Sigma}_\alpha^\diamond\left\{\left(\mathbf{y}_i^*+\boldsymbol{\beta}\right)\theta_i + \left(\boldsymbol{\Sigma}_\alpha^0\right)^{-1}\boldsymbol{\mu}_\alpha^0\right\}$ and $\boldsymbol{\Sigma}_\alpha^\diamond = \left\{\theta_i^2\mathbf{I} + \left(\boldsymbol{\Sigma}_\alpha^0\right)^{-1}\right\}^{-1}$. The posterior specification for the item difficulties results in

$$f(\beta|\mathbf{y}_i^*,\boldsymbol{\theta}) \quad\propto\quad \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}-\boldsymbol{\mu}_\beta^\diamond\right)'\left(\boldsymbol{\Sigma}_\beta^\diamond\right)^{-1}\left(\boldsymbol{\beta}-\boldsymbol{\mu}_\beta^\diamond\right)\right\} \tag{2.2.23}$$

where the expectation is $\boldsymbol{\mu}_\beta^\diamond = \boldsymbol{\Sigma}_\beta^\diamond\left\{\left(\mathbf{y}_i^*-\boldsymbol{\alpha}\theta_i\right) + \left(\boldsymbol{\Sigma}_\beta^0\right)^{-1}\boldsymbol{\mu}_\beta^0\right\}$ and the posterior variance is given by $\boldsymbol{\Sigma}_\beta^\diamond = \left\{\mathbf{I} + \left(\boldsymbol{\Sigma}_\beta^0\right)^{-1}\right\}^{-1}$. The latent variable part bears two draws: First, a random generation for the latent variable's variance σ_θ^2 and, second, for the latent variable itself, i.e. using a conditional specification $f(\theta_i|\sigma_\theta^2)\pi(\sigma_\theta^2)$. When utilising an inverse-gamma prior density,

$$\pi(\sigma_\theta^2) \propto \left(\sigma_\theta^2\right)^{-c^0-1}\exp\left(-\frac{C_m^0}{\sigma_\theta^{-2}}\right), \tag{2.2.24}$$

[15]Instead of restricting the codomain of the densities via indicator functions $\mathcal{I}_\square(\cdot)$, a different notation rests upon restrictions on the domain for the integrals (see for a probit-approach Chib & Greenberg, 1998).

the respective posterior density results in

$$f(\sigma_\theta^2|\theta_i) \propto \left(\sigma_\theta^2\right)^{-\left(c^0+\frac{1}{2}\right)-1} \exp\left\{-\left(C^0 + \frac{1}{2}\theta_i^2\right)\sigma_\theta^{-2}\right\}. \tag{2.2.25}$$

Based on (2.2.19), the normal draws for the latent variable for an individual i rely on the following formulation

$$f_\theta(\theta_i|\mathbf{y}_i^*, \boldsymbol{\alpha}, \boldsymbol{\beta}) \propto \exp\left\{-\frac{1}{2\sigma_\theta^{2\diamond}}\left(\theta_i - \mu_\theta^\diamond\right)^2\right\} \tag{2.2.26}$$

where $\mu_\theta^\diamond = \sigma_\theta^{2\diamond}\left\{\boldsymbol{\alpha}'\left(\mathbf{y}_i^* + \boldsymbol{\beta}\right)\right\}$ and $\sigma_\theta^{2\diamond} = \left(\boldsymbol{\alpha}'\boldsymbol{\alpha} + \sigma_\theta^{-2}\right)^{-1}$. Completing the probit-like structure, one finally needs truncated draws for y_{im}^* and therefore facilitating

$$f_{y^*}(y_{im}^*|\beta_m, \theta_i, \alpha_m) \propto \begin{cases} \frac{1}{0.5-0}\exp\left\{-\frac{1}{2}\left(y_{im}^* - \alpha_m\theta_i\right)^2\right\}, & \text{if } y_{im} = 0 \\ \frac{1}{1-0.5}\exp\left\{-\frac{1}{2}\left(y_{im}^* - \alpha_m\theta_i\right)^2\right\}, & \text{if } y_{im} = 1. \end{cases} \tag{2.2.27}$$

When comparing the example estimation routines for the factor and the IRT depicted so far, similarities but also some dissimilarities can be seen. In order to resolve these differences, the next part sheds a light on combination possibilities of these two kinds of models.

2.2.3 Combining a Factor Analysis and a normal-ogive IRT model

Although bearing (general linear) regression-like structural relationships, a blunt statement that an IRT model and a factor model are technically the same, would not be true. Instead, transforming a factor model into an IRT model and vice versa, is not only dependent on terminology but also on the used scales of measurements for variables $\mathbf{Y}$, the respective link function $f(\cdot)$ and further general assumptions concerning model parameters. In the case of the previously described examples of Gibbs sampling built upon formulations (2.2.1) on page 10 and (2.2.14) on page 16, i.e.

$$\begin{aligned} (2.2.1,\text{FA})\text{: } \mathbf{y}_i &= \boldsymbol{\mu}_y + \boldsymbol{\alpha}\boldsymbol{\theta}_i' + \boldsymbol{\varepsilon}_i \\ (2.2.15,\text{IRT})\text{: } \mathbf{y}_i^* &= \boldsymbol{\alpha}\theta_i - \boldsymbol{\beta} + \boldsymbol{\varepsilon}_i \end{aligned}$$

a strong resemblance in these linear equations is given. And, it is therefore no surprise that comparisons and transformations of these kinds of models were already considered (see for example Edwards, 2010, McDonald, 1982, or Mellenbergh, 1994). Nonetheless, there are differences as well: Compared to standard factor analysis, the IRT model uses $\mathbf{y}_i^*$ instead of $\mathbf{y}_i$ and the item difficulty $\boldsymbol{\beta}$, i.e. a mean, is subtracted, while it is added in the factor analytic case of $\boldsymbol{\mu}_{y,i}$.

In order to extend one of the models with the capabilities of the other, first, the scales of the measurements and therefore the link function and an underlying variable $\mathbf{y}^*$ are taken into account. Specifically, the focus is shifted onto different scales of the measurements, basing further consideration upon single variables $\mathbf{y}_m$ and $\mathbf{y}_m^*$ for all individuals–instead of $\mathbf{y}_i$ and $\mathbf{y}_i^*$, which represent all variables for a single individual. For a transformation, one can say that the IRT model with the normal-ogive structure needs an identity link, i.e. $\mathbf{y}_m = \mathbf{y}_m^*$ or the factor model needs an additional underlying variable $\mathbf{y}_m^*$ and a probit link, if one wants to combine the estimation for continuous and binary data, such that

$$y_{im} = \begin{cases} \mathcal{I}_{[0,+\infty)}(y_{im}^*), & \text{if } \mathbf{y}_m \text{ is binary,} \\ y_{im}^*, & \text{if } \mathbf{y}_m \text{ is continuous.} \end{cases} \tag{2.2.28}$$

When it comes, second, to loadings and item discrimination parameters, and the factor models mean and item difficulty, there is more to tackle than defining one as the (negative) variant of the other, i.e. $\boldsymbol{\alpha}_{FA} \neq \boldsymbol{\alpha}_{IRT}$ and $\boldsymbol{\mu}_{FA} \neq -\boldsymbol{\beta}$ per se. This is caused by the different possible modelling perspectives and respective dimensionalities of the parameters. The dimensionality of the loading matrix and the dimensionality of the item discrimination parameters depend not only on the number of measurements but also on the number of latent variables assumed. When extending the normal-ogive model for multiple latent variables, $\boldsymbol{\theta}_i = (\theta_{i1}, \ldots, \theta_{iK})'$, $k = 1, \ldots, K$, the discrimination parameters $\boldsymbol{\alpha}$ need to get specified as a $M \times K$-matrix–similar to the factor analytic case; but they would still be parameters concerned with the items and not the individuals connecting the $N \times M$ matrix $\mathbf{Y}$ with the $N \times K$ matrix $\boldsymbol{\Theta}$. On the other hand, from a multivariate perspective, encompassing all individuals and all of the corresponding measurements: specifications for the mean to be a $N \times M$-matrix, as they are included in matrix-variate distributions and regression (see for example Gupta & Nagar, 2000, and Viroli, 2012), tend to undermine the item-oriented focus of the difficulty in the IRT model as stated above. So, instead of increasing the dimension of $\boldsymbol{\mu}_y$ and $\boldsymbol{\beta}$ to become a $N \times M$-matrix and $N \cdot M$ parameter values to estimate, one can facilitate a $N \times 1$ vector $\boldsymbol{\iota}_N$ of ones, such that $\boldsymbol{\iota}_N \boldsymbol{\beta}' = -\boldsymbol{\iota}_N \boldsymbol{\mu}_y'$ is still referring only to the items, but achieves the appropriate dimensionality needed.

Extending the latent variables employing $K > 1$ instead of one column in $\boldsymbol{\Theta}$, allows for different constructions of their relationships as well. Both of the previous examples used a normal distribution for each of the single variables, i.e.

$$\boldsymbol{\theta}_i \sim MVN(\mathbf{0}, \boldsymbol{\Sigma}_\theta). \tag{2.2.29}$$

The IRT variant was not able to include any interconnection between multiple θs, because of the obvious unidimensional perspective, and the factor model example employed a simplification: $\boldsymbol{\Sigma}_\theta = \mathbf{I}$, i.e. although there are several θs , these are assumed to be not interrelated. But of course, $\boldsymbol{\Sigma}_\theta$ can be utilized to model diverse understandings of latent variables: As already stated, specifically in factor analysis, when modelling off-diagonal elements that are zero, $\sigma_{\theta k,\bar{k}} = 0\,, \forall k \neq \bar{k}\,,\ k, \bar{k} = 1, \ldots, K$, one deals with *orthogonal* factors and if not, i.e. $\sigma_{\theta k,\bar{k}} \neq 0$, the factors are called *oblique* (see Rubin & Thayer, 1982) and are assumed to be interrelated. Technically, both ideas can be incorporated into the normal-ogive model as well–for example Sheng & Wikle (2007) use a correlation structure (with variances 1) for latent abilities in IRT, while for example Fox & Glas (2001) incorporate only variances ($\neq 1$) for the latent variables, but no covariation.

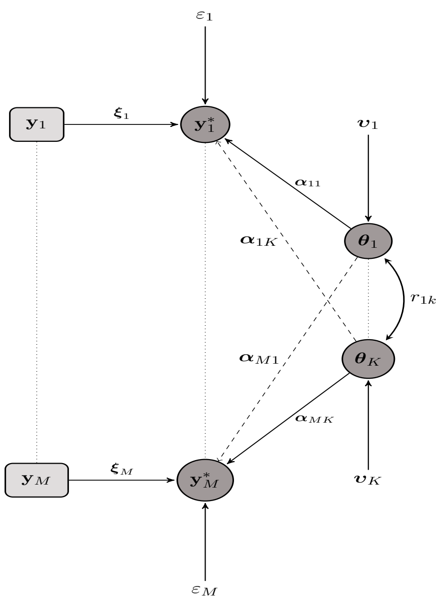

Figure 2.1: Path diagram of a basic multidimensional latent variable model.

The residual terms $\boldsymbol{\varepsilon}_i$, with

$$\boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon), \tag{2.2.30}$$

are multivariate normal in both model cases. But, particularly, the respective idiosyncratic variances of these errors, $\boldsymbol{\Sigma}_\varepsilon$, contain restrictions: first, the diagonality $\boldsymbol{\Sigma}_\varepsilon =$

$\text{diag}(\sigma^2_{\varepsilon 1}, \ldots, \sigma^2_{\varepsilon M})$ for factor analysis and IRT, and, second, in the IRT version with binary outcomes, the variances are restricted on a value of one. This relationship can be depicted with the help of a path diagram (see figure 2.1; where circles refer to unobserved variables, boxes refer to observed ones, arrows indicate a relationship amongst variables with given parameters next to them or with error terms (at the base points) – see also Skrondal & Rabe-Hesketh, 2004, pp. 2ff.) Parameters $\boldsymbol{\xi}_m$ refer to an implicit set of cut-off parameters, i.e. $\boldsymbol{\xi}_m = (-\infty, 0, +\infty)$ if the respective measurements $\mathbf{y}_m$ are binary, and aren't necessary otherwise–i.e. when measurements $\mathbf{y}_m$ are continuous, such that $\mathbf{y}_m = \mathbf{y}^*_m$.

All in all, one ends up with these two matrix-variate relationships,

$$\begin{aligned} \text{(FA): } \mathbf{Y}^* &= \boldsymbol{\Theta}\boldsymbol{\alpha}' + \boldsymbol{\iota}_N \boldsymbol{\mu}'_y + \boldsymbol{\varepsilon} \\ \text{(IRT): } \mathbf{Y}^* &= \boldsymbol{\Theta}\boldsymbol{\alpha}' - \boldsymbol{\iota}_N \boldsymbol{\beta}' + \boldsymbol{\varepsilon} \\ \text{with } & \text{vec}(\boldsymbol{\varepsilon}) \sim MVN(\text{vec}(\mathbf{0}), \boldsymbol{\Sigma}_\varepsilon \otimes \mathbf{I}_N), \end{aligned} \tag{2.2.31}$$

where $\text{vec}(\cdot)$ defines matrix vectorisation and $\otimes$ the Kronecker product (see Henderson & Searle, 1981 and Gupta & Nagar, 2000 for details, and Geweke, 1996 or Viroli, 2012 for some examples of applications on regression-type models). These formulations–only bearing one slight difference, i.e. the subtraction–and incorporating both continuous and binary outcomes via (2.2.28) are capable of being a basis for multiple-scale, multivariate and multidimensional IRT and factor model estimation, when imposing additional restrictions for model identification. A light will be shed on the application of these matrix-style formulations in section 4.1, while matters of identification and restrictions get discussed in the next section.

2.3 Identification and Parameter Restrictions

As shown, estimation algorithms are generally derivable, but without additional restrictions unique parameter and structural identification can not be guaranteed for the presented formulations given above. In this context one needs to realise that the parameter estimation based on the sample is not the gist of the matter. Instead, the initial thoughts rest solely upon the data generating distributions and respective model formulations (see San Martín, 2015, pp. 130ff., and Wechsler et al., 2013)–or in an earlier, to be clarified terminology: rests upon the structural relationships, structures and corresponding distributions (see Koopmans & Reiersøl, 1950, and Reiersøl, 1950). Thus, because of the models' structures containing unknown parameters and unknown variables: even with a vast amount of data, unique parameter estimates can not be guaranteed–and different parameter constellation lead to the same results. In order to resolve this problem, different analytical necessary or sufficient, and necessary and sufficient conditions–next to empirical algorithmic approaches–where proposed throughout the literature. The following section will therefore give a brief summary about classical, i.e. frequentist, and Bayesian considerations with regards to this topic and illustrates some varying perspectives. According to our knowledge, we presume that the considerations on identification to be presented, are applicable on both, IRT and factor models, for two reason: first, analysis on identification both model types seem to be based on the same origins–compare for example Koopmans & Reiersøl (1950), who present fundamental ideas on identification and use factor analysis as one example for demonstration, with San Martín, 2015, pp. 127ff., using amongst others the previous text, but transferring its ideas on IRT models. And, our second justification is of course the similar mathematical nature of IRT and factor models we have shown before.

General issues of identification were started being discussed up to a hundred years ago (see San Martín, 2015, pp. 127ff. and Fisher, 1976, p. 31, and compare Fisher, 1922, Haavelmo, 1944, Koopmans & Reiersøl, 1950 and Reiersøl, 1950). Following Koopmans & Reiersøl (1950) and Reiersøl (1950), identification in a classical understanding is seen as part of the model specification process that is concerned with the model formulations and their interrelation with a unique structure or unique distribution for an observed variable y with distribution $F_y(\cdot)$ and given further parameter aspects. This distribution is conditional on latent variables (including errors)–thus, representing the likelihood or data sampling distribution. Although identification is a necessity for inference, when considering whether a given structure and its parameters can be uniquely determined by a model and its respective specifications, it is not a part of estimation as such. This is because one focusses on one or more structures and observed variables and their distribution, not a finite sample (see Koopmans & Reiersøl, 1950). A model is then a set of structures, and a structure $\chi_{y|\theta,\varepsilon} = (F_{\theta\&\varepsilon}, f_{y.})$ is a combination of a fully specified distribution for the latent variables and errors, $F_{\theta\&\varepsilon}(\cdot)$, structural relationships $f_y(\cdot)$ and respective probability distributions $F_y(\cdot)$ between observed and unobserved variables, and known characteristics, like specific parameters. In terms of Reiersøl (1950) an alternative description is: "A structure thus is a particular realization of the model, and the model is the set of all structures compatible with the given specifications."

With using the model specifications from above, a simple example can be provided, i.e.

$$\mathbf{y}_i = \boldsymbol{\mu}_y + \boldsymbol{\alpha}\boldsymbol{\theta}_i' + \boldsymbol{\varepsilon}_i, \text{ with latent variables } \boldsymbol{\theta}_i \sim MVN(\mathbf{0}, \boldsymbol{\Sigma}_\theta) \text{ and } \boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon),$$

where

$$E\left[\mathbf{y}_i|\boldsymbol{\alpha}, \boldsymbol{\theta}_i\right] = \boldsymbol{\mu}_y \text{ and } \mathrm{Cov}\left[\mathbf{y}_i|\boldsymbol{\mu}_y\right] = \boldsymbol{\Sigma}_{y|\mu} = \boldsymbol{\alpha}\boldsymbol{\Sigma}_\theta\boldsymbol{\alpha}' + \boldsymbol{\Sigma}_\varepsilon \tag{2.3.1}$$

a unique structure (imposing several restrictions of fixed variances and covariances, and value restrictions for loadings) in a specific scenario could be

$$\mathbf{y}_i = \mathbf{0} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \boldsymbol{\theta}_i' + \boldsymbol{\varepsilon}_i, \tag{2.3.2}$$

with

$$\boldsymbol{\theta}_i \sim MVN(\mathbf{0}, \mathbf{I}) \text{ and } \boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \mathbf{I}) \tag{2.3.3}$$

and consequently,

$$E[\mathbf{y}_i|\boldsymbol{\alpha}, \boldsymbol{\theta}_i] = \boldsymbol{\mu}_y = \mathbf{0} \text{ and } \boldsymbol{\Sigma}_y = \mathbf{I}_2 \otimes \boldsymbol{\iota}_3\boldsymbol{\iota}_3' + \mathbf{I}_6. \tag{2.3.4}$$

When including binary outcomes, an additional set of latent variables $\mathbf{y}_i^*$ will be considered. Their distribution originates from $\boldsymbol{\mu}_y + \boldsymbol{\alpha}\boldsymbol{\theta}_i' + \boldsymbol{\varepsilon}_i$, and the relationship between these unobserved variables and the observed variables $\mathbf{y}_i$ is described by formula (2.2.28). Because identification is not as much concerned with "the" true specific parameter value, like $\boldsymbol{\mu}_y = \mathbf{0}$, but amongst others with the unique existence of a parameter, an illustration taken from Bollen (1989, p. 88f.) for a $\mathbf{y}_i|\theta_i \sim MVN(0, \boldsymbol{\Sigma}_y)$ might exemplify the idea: when $\boldsymbol{\Sigma}_y$ is known and can be related to further parameters $\boldsymbol{\Sigma}_\theta$ and $\boldsymbol{\Sigma}_\varepsilon$, like $\boldsymbol{\Sigma}_y = \boldsymbol{\Sigma}_\theta + \boldsymbol{\Sigma}_\varepsilon$, these are not identified without any further restriction such as $\boldsymbol{\Sigma}_\varepsilon = \boldsymbol{\Sigma}_\theta$ and therefore $\boldsymbol{\Sigma}_\varepsilon = \boldsymbol{\Sigma}_\theta = 0.5 \cdot \boldsymbol{\Sigma}_y$. As a counterexample, i.e. non-identification, we can examine loading parameters of an orthogonal factor model. In this scenario Anderson & Rubin (1956) point out that $\boldsymbol{\alpha\alpha}'$ can be re-formularised with the help of orthogonal matrices $\mathbf{K}$. Hence, $\boldsymbol{\alpha}\mathbf{I}\boldsymbol{\alpha}' = \boldsymbol{\alpha}\mathbf{K}'\mathbf{K}\boldsymbol{\alpha}' = \tilde{\boldsymbol{\alpha}}^{new}\tilde{\boldsymbol{\alpha}}^{new\prime}$, indicating missing parameter identification–and highlighting, in this case, rotational issues.

In the formal depiction of Koopmans & Reiersøl (1950)[16], two different structures $\chi_{y|\theta,\varepsilon}$ and $\tilde{\chi}_{y|\theta,\varepsilon}$ are said to be specified as *observationally equivalent*, if the same distributions[17] exist, i.e. $F_y(y|\chi_{y|\theta,\varepsilon}) = F_y(y|\tilde{\chi}_{y|\theta,\varepsilon})$, $\forall y$; and–given this observational equivalence–(structural) parameters Ψ are identified by a model, if $\Psi(\chi){=}\Psi(\tilde{\chi})$, i.e. the parameter has the same value or is the same for all structures; or, in other words: under identification unique results for $F_y(\cdot)$ will be received only with unique parameters and characteristics respectively. As Kadane (1975, pp. 176ff.) shows, the property of parameter identification can be extended to general functions of parameters as well.

Further texts (see, e.g. Bollen, 1989, p. 248, Rothenberg, 1971, or Skrondal & Rabe-Hesketh, 2004, p. 136) present additional considerations, i.e. global and local identification. *Global identification* of a parameter point (or parameter vector with all of its vector elements) is considered as having only one unique parameter for all structures everywhere in the parameter space. In contrast, *local identification*, which is a necessary condition for its global counterpart, requires only one unique parameter point or unique set of parameter vector elements in an open neighbourhood. If now all parameters are (globally or locally) identified, the model is said to be (globally or locally) identified as well.

So far, the focus rested mostly upon the models and hence the data generating process. When pursuing modelling attempts under a Bayesian perspective, several aspects, like the likelihood, parameter prior distributions, marginal distribution and resulting parameter posterior appear necessary. When reflecting on the various additional components, exclusively centring on the likelihood and data sampling distribution respectively, does not seem to be given at first. As the overview of Aldrich (2002) points out, thoughts on considering likelihood and prior distribution for identification correspondingly exist (see, e.g. Drèze, 1972). But other authors, like Kadane (1975, pp. 175ff.), Drèze (1975, pp. 159ff.), Poirier (1998), Gelfand & Sahu (1999)[18], Kociecki (2013) and Wechsler et al. (2013) mostly

[16] Compare also McCullagh (2002, p. 1226 and p. 1239)

[17] Or the same reduced form distributions are considered here instead (see Skrondal & Rabe-Hesketh, 2004, p. 136 for the definition on observational equivalence and Skrondal & Rabe-Hesketh, 2004, p. 127ff. for the reduced form specification).

[18] Were the authors include prior considerations with respect to properness and not with respect to identifiability itself.

concentrate their attention on the likelihood instead. Besides, in fact all of the mentioned subsequent authors rely explicitly or implicitly[19] on the probability- and measure-theoretic derivations of Kadane (1975). Without neglecting considerations on the properness of priors (see Poirier, 1998), we are also going to concentrate on the model formulation and the likelihood.

To illustrate the problem of identification for the factor analytic and IRT specifications here, a change in the model parameters is considered (compare also, e.g. Gelfand & Sahu, 1999, McCulloch et al., 2000, or Skrondal & Rabe-Hesketh, 2004, pp. 136f.). The mathematical variation of (2.2.1) is presented above, leads to a different parametrisation, but encompasses the same probabilistic relationship. I.e. formula (2.2.1) can be expanded with additional unknowns, where a and b are scalars, with $a, b \in \mathbb{R}$ and specifically $a \neq 0$. Therefore, the formula can be re-written as

$$\begin{aligned}
\mathbf{y}_i &= \boldsymbol{\alpha}\boldsymbol{\theta}_i' + \boldsymbol{\mu}_y + \boldsymbol{\varepsilon}_i \\
&= \boldsymbol{\alpha}\boldsymbol{\theta}_i' \cdot 1 + 0 + \boldsymbol{\mu}_y + \boldsymbol{\varepsilon}_i \\
&= \boldsymbol{\alpha}\boldsymbol{\theta}_i' \left(aa^{-1}\right) + \left(\boldsymbol{\alpha}a^{-1}b - \boldsymbol{\alpha}a^{-1}b\right) + \boldsymbol{\mu}_y + \boldsymbol{\varepsilon}_i \\
&= \boldsymbol{\alpha}a^{-1}\left(\boldsymbol{\theta}_i'a + \boldsymbol{\iota}_K'b\right) + \left(\boldsymbol{\mu}_y - \boldsymbol{\alpha}a^{-1}b\right) + \boldsymbol{\varepsilon}_i \\
&= \tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\theta}}_i' + \tilde{\boldsymbol{\mu}}_y + \boldsymbol{\varepsilon}_i.
\end{aligned} \tag{2.3.5}$$

That is, the models specification is altered via linear transformations $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}a^{-1}$, $\tilde{\boldsymbol{\mu}}_y = \boldsymbol{\mu}_y - \boldsymbol{\alpha}a^{-1}b$ and $\tilde{\boldsymbol{\theta}}_i' = \boldsymbol{\theta}_i'a + \boldsymbol{\iota}_K'b$. Moreover, $\boldsymbol{\alpha}\boldsymbol{\theta}_i'$ bears the additional possibility of sign switching, i.e. $(+\alpha_{mk})(+\theta_{ik}) = (-\alpha_{mk})(-\theta_{ik})$ (see also Erosheva & Curtis, 2017, and compare Stephens, 2000). Hence, one ends in the same solution where the distribution stays–with different parameters–the same, i.e. $F_y(y|\boldsymbol{\mu}_y, \boldsymbol{\theta}_i, \boldsymbol{\alpha}) = F_y(y|\tilde{\boldsymbol{\mu}}_y, \tilde{\boldsymbol{\theta}}_i, \tilde{\boldsymbol{\alpha}})$, overall rendering the parameters and the model unidentified. The previously shown relationship between a confirmatory factor model and a normal-ogive IRT model in (2.2.31) allows to transfer the considerations made from the factor analytic case to the IRT scenario easily, where

$$\begin{aligned}
\mathbf{y}_i^* &= \boldsymbol{\alpha}a^{-1}\left(\boldsymbol{\theta}_i'a - \boldsymbol{\iota}_K'b\right) + \left(\boldsymbol{\beta} + \boldsymbol{\alpha}a^{-1}b\right) + \boldsymbol{\varepsilon}_i \\
&= \tilde{\boldsymbol{\alpha}}\tilde{\boldsymbol{\theta}}_i' - \tilde{\boldsymbol{\beta}} + \boldsymbol{\varepsilon}_i.
\end{aligned} \tag{2.3.6}$$

[19] I.e. Gelfand & Sahu (1999) refer to Dawid (1979) and Poirier (1998), and Dawid (1979) and Poirier (1998) refer both to Kadane (1975).

Although the starting point, the initial model and the respective distributional aspects stay the same, there exist (further) approaches following different paths for identification, where the attention is drawn to seemingly different aspects. A first consideration whether the variables provide enough information for the estimable parameters, focuses on only one part, which is the models variance of (2.2.4) (see also Anderson & Rubin, 1956, Bekker, 1989, Bollen, 1989, pp. 242ff., Conti et al., 2014, Geweke & Zhou, 1996). A first approach here is referred to as *Ledermann bound* (by, e.g. Bekker, 1989 and Conti et al., 2014, see Ledermann, 1937) or *t-rule* (Bollen, 1989, p. 243). It compares known and available variances and covariances of the measurements (and covariates, if available, see Ibid., 1989, pp. 85ff. and p. 328) with the number of unknown parameters used in the covariance matrices of the loadings and factors / item discriminations and latent variables, and error term variances. More precise, the number of known and unknown parameters and the number of given linear equations of their covariance-based interrelationship is taken into account. With M measurements and K latent variables, this strategy results in the necessary (but insufficient) condition that the number of free parameters should not exceed a boundary of $0.5 \cdot (M^2 + M)$. This stems from the measurements variation allowing for information in (2.2.4) for $0.5 \cdot (M^2 + M)$ unique cases of $\mathbf{\Sigma}_y$, while there are (without restrictions in $\boldsymbol{\alpha}$, $\mathbf{\Sigma}_\theta$ and $\mathbf{\Sigma}_\varepsilon$) up to $K \cdot M + 0.5 \cdot K \cdot (K+1) + 0.5 \cdot (M^2 + M)$ unknown parameters on the right hand side of the equation $\boldsymbol{\alpha}\mathbf{\Sigma}_\theta\boldsymbol{\alpha}' + \mathbf{\Sigma}_\varepsilon$. And if there is at least as many information than unknowns, the parameters might be identified–although not sufficiently. Of course, if $M, K \in \mathbb{N}^+$, it follows that the inequality

$$0.5 \cdot (M^2 + M) \geq K \cdot M + 0.5 \cdot K \cdot (K+1) + 0.5 \cdot (M^2 + M) \tag{2.3.7}$$

will never indicate identification in the above given understanding, because $K \cdot M + 0.5 \cdot K \cdot (K+1) > 0$, $\forall M, K$. A solution lies in imposing restrictions, like the diagonality of $\mathbf{\Sigma}_\varepsilon$, with

$$0.5 \cdot (M^2 + M) \geq K \cdot M + 0.5 \cdot K \cdot (K+1) + M, \tag{2.3.8}$$

additional orthogonality of the factors, resulting in

$$0.5 \cdot (M^2 + M) \geq K \cdot M + K + M, \tag{2.3.9}$$

or fixating the factor covariance matrix to become a correlation matrix, i.e.

$$0.5 \cdot (M^2 + M) \geq K \cdot M + 0.5 \cdot K \cdot (K-1) + M. \tag{2.3.10}$$

When assuming, for example, that there are six measurements, utilising one or two latent variables will satisfy each of the latter three inequalities.

A more general method, encompassing the t-rule's and Ledermann bound's conception about the number of parameters in some situations, is also available (see Skrondal & Rabe-Hesketh, 2004, pp. 130ff. and pp. 135ff. and Bekker, 1989). It focusses on the rank of the matrix encompassing first derivatives, i.e. the Jacobian matrix $\mathcal{J}$, concerning the *reduced form* of a model–and as Bekker (1989) points out: the Ledermann bound turns out to be a special case of this Jacobian approach. The functional origin for this method is in fact a model, where on both sides of an equation endogenous variables can be found, e.g. possible initial formulations of structural models in structural equations and simultaneous equations modelling (see Skrondal & Rabe-Hesketh, 2004, pp. 78f.). However, in our scenario the endogenous and exogenous variables are already separated. Thus, our model equations of (2.2.31) are this necessary reduced form. In the case of a regular factor model–supposing normality of the factors and a mean of zero–, Bekker (1989) speaks of local identification that can be investigated via the non-redundant parameters in (2.2.4) and the unique parameters $\boldsymbol{\psi}_{unique}$ of (2.2.1). This model is said to be locally identified (or *locally isolated*) for a regular point $\boldsymbol{\psi}_{unique}$, if the Jacobian matrix,

$$\mathcal{J}(\boldsymbol{\psi}_{unique}) = \frac{\partial \mathrm{vec}\,(\boldsymbol{\Sigma}_{y^*})}{\partial \boldsymbol{\psi}'_{unique}}, \tag{2.3.11}$$

that encompasses all unique parameters as elements, is of full column rank. Examples for different models are given in Skrondal & Rabe-Hesketh (2004, pp. 138ff.). As Bollen (1989, pp. 90) points out, moments of a higher order should be considered in case of non-normal models, although Skrondal & Rabe-Hesketh (2004, pp. 137) warn about the fragility.

Further analytical aspects of identification stem from further algebraic and logical derivations. For example sufficient (but not necessary) conditions for model identification in factor analysis are the *three-indicator rule* and, alternatively, two versions of the *two-indicator rule* (see Bollen, 1989, pp. 244ff.). These two rules are concerned with the number of elements in $\boldsymbol{\alpha}$ being non-zero. Despite its name, the three-indicator rule describes a set of more than one requirement: When dealing with one or more latent variables, where the error-term variance matrix $\boldsymbol{\Sigma}_\varepsilon$ is demanded to be diagonal, at least three measurements should be influenced by one factor–which is also similarly depicted in Anderson & Rubin (1956) as *Theorem 5.6,* but in this earlier text it is said to be a necessary condition, and in Conti et al. (2014) it is presented as the (sufficient) *Theorem 1.* Moreover, in the case of multiple factors, it is required that one (and only one) element within a row of the loading matrix is non-zero; also denoted with *simple and dedicated factor structure* (by, amongst others, Conti et al., 2014). The two-indicator rule gets presented by Bollen (1989, pp. 244ff.), in two different versions, a basic and a more general one, and solely for multiple-factor models. In the first version, again one (and only one) element of a row of the loading matrix $\boldsymbol{\alpha}$ is non-zero, $\boldsymbol{\Sigma}_\varepsilon$is diagonal, but in this case $\boldsymbol{\Sigma}_\theta$ isn't and all latent variables are scaled via of a loading's value, i.e. $\exists \alpha_{mk} = 1$, $\forall k = 1, \ldots, K$. Furthermore, there is a dedicated structure where at least two measurements load onto a factor. Bollen's generalisation adds an additional restriction to the previous constraints, which is: each row of $\boldsymbol{\Sigma}_\theta$ should contain at least one non-zero off-diagonal element. In case of the orthogonal factor model, having uncorrelated latent variables, Anderson & Rubin (1956) present several more conditions, like the identification of the error-term variance matrix $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\alpha}\mathbf{K}$, where $\mathbf{K}$ is again an orthogonal matrix (sufficient, *Theorem 5.1*). Then a rule for identification of a one-factor model (necessary and sufficient, *Theorem 5.5*) is stated, demanding that three loadings are not zero. The previously mentioned *Theorem 5.6* (necessary), is in fact also concerned with the number of elements being non-zero (at least three elements) of the matrix $\boldsymbol{\alpha}\tilde{\mathbf{K}}$ resulting from multiplying the loading matrix with a general non-singular matrix $\tilde{\mathbf{K}}$ (which is not necessarily orthogonal, but includes the identity matrix). For a two-factor model, i.e. $K = 2$, the necessary and sufficient condition of *Theorem 5.7* focusses on ranks of submatrices of $\boldsymbol{\alpha}$. Necessary and sufficient is also *Theorem 5.8,* again being oriented on ranks of the submatrices of the loadings, but in combination with pairs of columns of $\boldsymbol{\alpha}\tilde{\mathbf{K}}$. The last theorem of Anderson & Rubin (1956) on identification, *Theorem 5.9* (sufficient), is interested in the local identification of $\boldsymbol{\alpha}$ and $\boldsymbol{\Sigma}_\varepsilon$ being given, if $\boldsymbol{\alpha}'\boldsymbol{\Sigma}_\varepsilon^{-1}\boldsymbol{\alpha}$ is diagonal and the off-diagonal elements of $\boldsymbol{\Sigma}_\varepsilon - \boldsymbol{\alpha}\left(\boldsymbol{\alpha}'\boldsymbol{\Sigma}_\varepsilon^{-1}\boldsymbol{\alpha}\right)^{-1}\boldsymbol{\alpha}'$ have a specifically size-ordered arrangement. Based on *Theorem 5.1* of Anderson & Rubin (1956), Conti et al. (2014) derive their sufficient condition for an explorative, dedicated oblique factor model with unknown loading structure, i.e. at least three measurements should load onto a factor or no loading should occur at all. In other words: each column of the matrix $\boldsymbol{\alpha}$ should contain at least three non-zero elements or all elements should be zero within this column. This condition is said to be able to identify the latter's model up to column- and sign-switching.

Depending on the number of parameters and the actual model structure, a proof of identification based on analytical approaches can become quite tedious. As an alternative, there exist ideas facilitating the estimated information matrix of the parameters (see Skrondal & Rabe-Hesketh, 2004, pp. 150f.). But aside from being data driven, one problematic for us seems to be, that this approach is bound to Maximum Likelihood estimation and has, to our best knowledge, no Bayesian counterpart.

Given the first considerations with regards to structures, the Ledermann bound, the Jacobian approach and the additional rank conditions, all of these methods point into one direction, if identification is in dispute: that is parameter restrictions in order to eliminate the possibility of observational equivalence under differing parameters, reduce the number of unknowns in general or re-establish possible full ranks. Additionally, depending on the model formulations: Indeterminacies need to get resolved as well–e.g. the aforementioned indeterminacy of the two unknowns $(+\alpha_{mk})(+\theta_{ik}) = (-\alpha_{mk})(-\theta_{ik})$ is avoidable by demanding that one of the unknowns should follow a positivity constraint, for example $\alpha_{mk} \geq 0$.

But choosing suited restrictions is not only a matter of mathematical necessity, because of the influence on interpretational aspects. Hence, theoretical or domain-specific concerns might also be taken into account, e.g. introducing positivity restrictions for loadings or explicitly omit these (see Reiersøl, 1950[20]). However, when contrasting theory based restrictions and solely mathematical-based restrictions for identifying purposes, one can easily realise the desire for both necessities to match, but one needs to be aware that this is not always and naturally given (see Manski, 1995, pp. 4ff. for a general discussion).

Either way, the parameter constraints that are then superimposed onto the models are first of all a mathematical intervention and can be of different type. Skrondal & Rabe-Hesketh (2004, pp. 117ff.) name several possibilities. These can be restricting a parameter's domain, or demanding inequalities, identities, linear or non-linear relationships between two or more parameters. Moreover, fixating parameters to specific values, is another way of restrictions in terms of removing the parameter's status of being unknown and therefore

[20]We specifically think of Reiersøl's (1950) reflection on propositions made by Thurstone in the context of factor analysis.

removing the necessity for estimating it. Examples are the previously mentioned “fixating of loading’s value”–also known as *anchoring* in order to provide the latent variables with a scale (Ibid., 2004, p. 66) or being a general way of identification, when an upper-triagonal part of loadings are set to zero (see Geweke & Zhou, 1996, where additionally the diagonal loading-elements’ domains are constrained to positivity). Or, setting variances to one–when there are binary measurements involved–, or, for example assigning zero to loadings, discrimination parameters or correlations, in order to denote no connection between latent variables and measurements or no interrelation amongst (latent) variables.

As we have seen, the historical origin in statistics concerned with identification is mostly similar. It aims at guaranteeing unique parameter-based influence and the relation of possible information contained in the data with these unknown parameters respectively. However, concrete implementations of identifying restrictions are model and parameter dependent–e.g. when a specific amount of parameters and a theoretically founded set of variables is of interest. And of course, this dependence on actual formulation also means that identifying restrictions for “this” logistic model do not necessarily allow identification for “that” factor model when transferred, restrictions imposed on factor loadings might not be suited for variances, etc. In order to resolve these issues several strategies are imaginable.

Not only because of identification and resulting options for restrictions, but also to retrieve an understanding of the necessary model structures, we need to get to know in a next step the domain where the models can be applied and how the data might look like. This specifically encompasses, how covariates might be included–where examples can be: whether possible covariates directly or at most via $\mathbf{y}*$ have an influence on measurements (e.g. Conti et al., 2014, use this for a Bayesian factor model applied to the analysis of abilities, or Carvalho et al., 2008, apply this strategy in gene-expression profiling) or indirectly via latent variables (see for example the **M**ultiple-**I**ndicators-**M**ult**I**ple-**C**auses-model, MIMIC-model for short of Joreskog & Goldberger, 1975, or Mislevy, 1987, in IRT).

Chapter 3

Applications of Latent Variable Concepts and Possible Ways of Including Background Information with Missing Values

So far, latent variables were mostly discussed as an abstract concept. Nonetheless possible applications in natural and social sciences were contoured within the previous chapters. In here some more explicit applications will be considered. This and reflecting on the usability of additional information in terms of background information, i.e. covariates, allow to display the fundament of the actual models under investigation. Specifically, two differing applications and types of data are considered. That is first, an application of a dedicated factor model on cognitive, psychological and physical traits for the *British Cohort Study* (BCS) in Conti et al. (2014) and, second, an IRT-approach to assessment of competencies (Aßmann et al., 2015, and Gaasch, 2017a), as they are–along with a multitude of additional educational topics–considered in the *National Educational Panel Study* (NEPS, Blossfeld et al., 2011). This chapter will allow not only to illustrate implementations but also reviewing different possibilities of including background information in latent variable models. Furthermore, because missing values in survey data are highly likely, techniques on imputation should be (and are nowadays often) borne in mind. Reviewing actual handling of missing values used, allows therefore to discuss some of the current approaches and further thoughts on implementational aspects.

3.1 Comparing Empirical Applications

Factor Model applied to British Cohort Study by Conti et al. (2014)

Conti et al. (2014) present a dedicated-structured explorative factor model algorithm and its application on data stemming from the British Cohort Study. According to Brown (2014), the British Cohort study started in 1970 as a medical survey with a cohort of more than 17000 individuals and, over the years, broadened the scope in order to investigate physical, educational and social development in the main study and further sub-studies. The Bayesian model that Conti et al. (2014) develop and apply to these cohort data, relies on Carvalho et al. (2008), where Conti et al. (2014) present the basic formulation for an individual i as

$$\mathbf{y}_i^* = \boldsymbol{\beta}\mathbf{x}_i' + \boldsymbol{\alpha}\boldsymbol{\theta}_i' + \boldsymbol{\varepsilon}, \text{ where } \boldsymbol{\varepsilon}_m \sim MVN(\mathbf{0}, \operatorname{diag}(\sigma_1^2, \ldots, \sigma_M^2)) \tag{3.1.1}$$

and

$$y_{im} = \begin{cases} \mathcal{I}_{[0,+\infty)}(y_{im}^*), & \text{if measurement } \mathbf{y}_m \text{ is binary,} \\ y_{im}^*, & \text{if } \mathbf{y}_m \text{ is continuous.} \end{cases} \tag{3.1.2}$$

Dimensions are, like before, M for the number of measurements (with index $m = 1, \ldots, M$), N the number of observations (with index $i = 1, \ldots, N$), K the number of latent variables $\boldsymbol{\Theta}$ (with index $k = 1, \ldots, K$) and Q the number of covariates $\mathbf{X}$ (with index $q = 1, \ldots, Q$). Thus, different from before, they use an additional set of covariates and respective parameters $\boldsymbol{\beta}$, having a similar structure like reduced-rank regression models (compare Baştürk et al., 2017, and Geweke, 1996, and for a least-squares weighted kernel reduced-rank perspective: De la Torre, 2012). The model of Conti et al. (2014) is of dedicated structure, such that loading of the measurements is restricted up to one factor at the same time, e.g.

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \\ \alpha_{31} & 0 \\ 0 & \alpha_{42} \\ 0 & \alpha_{52} \\ 0 & \alpha_{62} \end{pmatrix}, \tag{3.1.3}$$

the factors of an individual i can be correlated, i.e. dealing with an oblique structure.

$$\boldsymbol{\theta}_i \sim MVN(\mathbf{0}, \mathbf{R}), \tag{3.1.4}$$

where $\mathbf{R}$ is a correlation matrix with diagonal elements being one and off-diagonal elements that can be unequal to zero. Parameters' prior specifications use informative, conjugate distributions and all estimation is carried out using a specific Gibbs sampling approach.

Assuming, for matters of identification, that at least three measurements load onto a factor (or no loading should occur at all) and taking the Ledermann bound into account (see again chapter 2.3), the speciality of Conti et al.'s (2014) approach is, that the loading structure gets estimated based on a marginal likelihood strategy. This strategy samples the m-th row of the indicator matrix $\mathbf{\Delta}$, depicting the relationship between measurements and latent variables via combinations of zeros and ones. It considers whether a measurement is related to one latent variable or not, and whether it is related to one specific latent variable or an other. The respective probabilities rely on the posterior log odds, which in itself are calculated based on the marginal likelihood of the measurement under consideration (see for specific details Conti et al., 2014, Appendix A.2.1. and A.3.).[21] For a path diagram see figure 3.1.

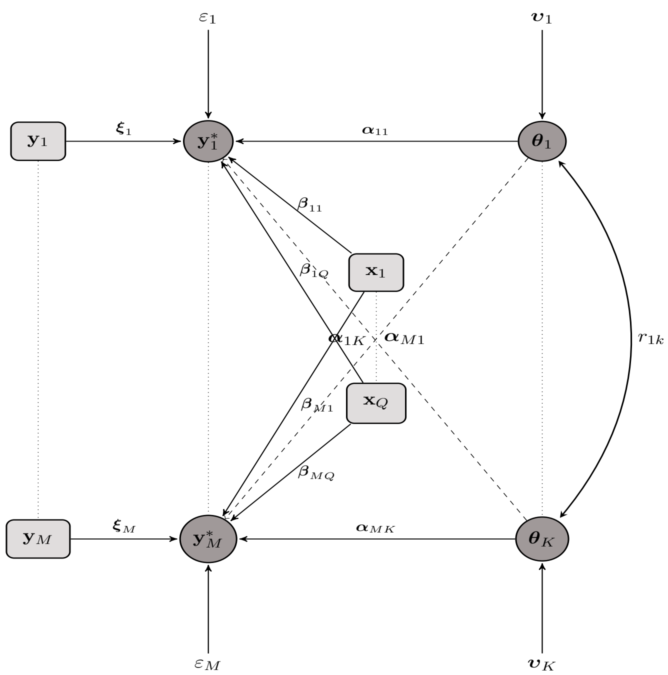

Figure 3.1: Relationship of observed and latent variables of the factor model of Conti et al. (2014)

[21]Of course the marginal likelihood differs with respect to the scale of this measurement.

Next to a simulation study–which, due to different scientific interest, doesn't encompass covariates and therefore the estimation of $\boldsymbol{\beta}$–they apply their algorithm on data stemming from the British Cohort Study. As the authors state: "We draw information on background characteristics from the birth survey, and on cognitive, mental and physical health measurements from the second sweep (age 10). We exclude children born with congenital abnormalities, non-whites, and respondents with missing information on the background characteristics. Individuals with missing observations on some of the cognitive, mental and physical health measurements are discarded from the sample, so we are left with a sample of 2,080 men." (Ibid., 2014, p. 54) The latent variables created, were based on outcome data including item batteries on cognitive ability, mental and physical health tests–partially recoding some of outcomes to a binary scale. Further covariates, i.e. $\mathbf{X}$, encompass seven control variables containing information on the participants' family backgrounds. Thirteen factors were uncovered, which can be attributed to each of the three test areas, e.g. all intelligence test items pertain to one, so called cognitive ability factor. Information for eleven of the latent variables can be attributed to the respective measurements of mental health tests, ranging from general behavioural problems, anxiety or depression to a "positive sense of self", and a final health related factor named "Body Build" was obtained as well (see Ibid., 2014, pp. 47ff. for a detailed overview).

Univariate IRT models applied to NEPS data by Aßmann et al. (2015) and Gaasch (2017a)

Being not a medical investigation, the National Educational Panel Study aims at investigating educational trajectories over the whole lifespan of individuals. As Blossfeld et al. (2009) argue, unlike cross-sectional approaches like the *Progress in International Reading Literacy Study* (PIRLS), the *Programme for International Student Assessment* (PISA) and *Trends in International Mathematics and Science Study* (TIMSS), the NEPS, being a panel-approach, allows to investigate time-dependent causal relationships over longer periods of time. Using several instruments, this German panel-study analyses educational decisions and pathways from pre-, in- and post-school periods of multiple cohorts (see Blossfeld et al., 2011). One set of instruments is concerned with different competencies (see, amongst others, Weinert et al., 2011, and Pohl & Carstensen, 2013), being Aßmann et al.'s (2015) and Gaasch's (2017a) area of interest and basis of their model-building process. Additionally, Gaasch (2017a) applies his IRT-approach also to SCOFF-based items incorporated in the NEPS. SCOFF–an acronym deriving from the questions content, that is Sick-Control-One-stone-Fat-Food–is a five-item-questionnaire, developed to investigate eating disorders (see Morgan et al., 1999, and Morgan et al., 2000, for its origin).

Relying on the normal-ogive specification of Edwards (2010) and Mislevy's (1987) idea of incorporating covariates into logistic IRT models[22], Aßmann et al. (2015) and Gaasch (2017a) use a model formulation (but presented in the notation we introduced in chapter 2), which is

$$y^*_{im} = \alpha_m \theta_i - \beta_m + \varepsilon_{im}, \text{ where } \varepsilon_{im} \sim N(0,1). \tag{3.1.5}$$

y^*_{im} is again a latent representation of an actual observed measurement y_{im} (of an individual i) being solely binary (in the case of Aßmann et al., 2015) or encompassing binary and ordered measurements (in the case of Gaasch, 2017a, pp. 12ff.)–hence, implementing a normal-ogive / probit-like structure. An additional equation with respect to the one-dimensional latent variable vector $\boldsymbol{\theta}$ considered, i.e.

$$\theta_i = \mathbf{x}_i' \boldsymbol{\gamma} + \upsilon_i, \text{ where } \upsilon_i \sim N(0, \sigma^2_\upsilon), \tag{3.1.6}$$

allows to incorporate $(Q \times 1)$ dimensioned background information $\mathbf{x}_i$ with covariate parameter vector $\boldsymbol{\gamma}$ of dimension $(Q \times 1)$ (see also figure 3.2).

By including a multigroup approach adapted from, amongst others, Mislevy (1985) and a cluster-specific effect (see Gaasch, 2017a, pp. 20ff. and pp. 35ff., which based on considerations of Aßmann & Boysen-Hogrefe, 2011, Frühwirth-Schnatter, 2011 and further authors), the equations change either to

$$\theta_i = \mathbf{x}_i' \boldsymbol{\gamma}_{g(1)} + \upsilon_{i,g(1)}, \text{ where } \upsilon_{i,g} \sim N(0, \sigma^2_{\upsilon,g(1)}) \tag{3.1.7}$$

with $g(1)$ denoting a specific group, $\boldsymbol{\gamma}_{g(1)}$ a group-specific effect and $\sigma^2_{\upsilon,g(1)}$ group-specific variance for the errors, or

$$y^*_{g(2),im} = \alpha_m \theta_{g(2),i} - \beta_m + \varepsilon_{im} \text{ and} \tag{3.1.8}$$

$$\theta_i = \mu_{\theta,g(2)} + \mathbf{x}'_{g(2),i} \boldsymbol{\gamma} + \upsilon_i, \text{ where } \upsilon_i \sim N(0, \sigma^2_{\upsilon,g(2)}) \tag{3.1.9}$$

$g(2)$ being now a cluster-indicator, and the newly introduced $\mu_{\theta,g(2)}$ represents a (1×1)-dimensional random intercept. Again, parameters' prior specifications use informative,

[22] Mislevy (1987) uses (weighted) logistic models with up to three parameters.

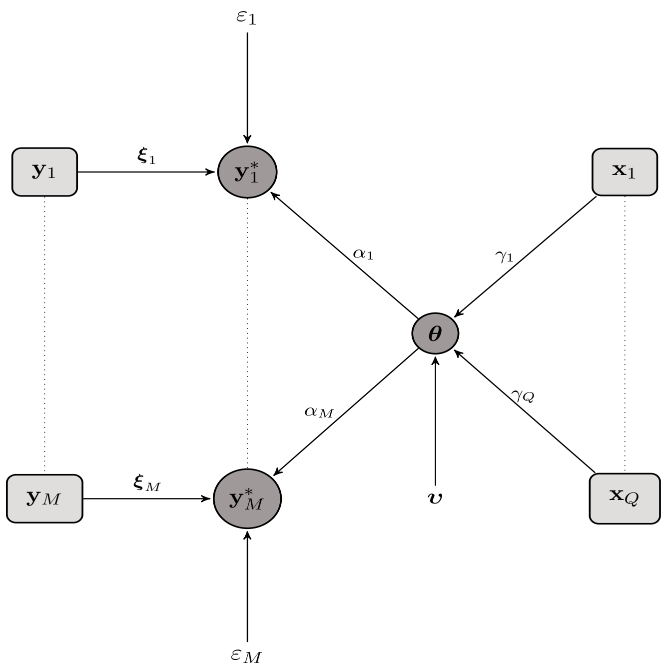

Figure 3.2: Relationship of observed and latent variables of the basic IRT model of Aßmann et al. (2015) and Gaasch (2017a)

conjugate distributions.

The applications of the researchers have a specific task in mind, i.e. imputation of missing values covariates in IRT, using a CART-type algorithm, while estimating their Bayesian $(N \times 1)$-latent variable model and respective parameters by means of a Metropolis-within-Gibbs approach.[23] Next to simulation studies, empirical data used, is competency data with background information stemming from the NEPS Starting Cohort 3 (version doi: 10.5157/NEPS: SC3:1.0.0), i.e. pupils in grade five, using information from wave 1 (see Aßmann et al., 2015), and NEPS Starting Cohort 4 (version doi: 10.5157/ NEPS: SC4:6.0.0), i.e. pupils in grade nine, using information from wave one and two (see Gaasch, 2017a, pp. 57ff.), respectively. In terms of identification the domains of the αs and βs are restricted, such that $\sum_{m=1}^{M} \beta_m = 0$ and $\prod_{m=1}^{M} \alpha_m = 1$, where additionally $\alpha_m \in \mathbb{R}^+$, $\forall m$. The latent variables estimated, then refer to mathematical competency (see Aßmann et al., 2015, and Gaasch, 2017a, pp. 58ff.) and a latent eating disorder score (see Gaasch, 2017a, pp. 63ff.) respectively. Depending on the original data, focus of the empirical application, the cases

[23]With regards to terminology, we refer to an algorithmic combination of genuine (full) conditional steps of Gibbs sampling and the usage of a specific Metropolis-Hastings algorithm, i.e. the approach from Albert & Chib (1997). This, of course, does not neglect the objections made by Chib & Greenberg (1995) in terms of understanding Gibbs sampling as a special case of the Metropolis(-Hastings) algorithm, and who therefore think of "Metropolis-within-Gibbs" being redundant naming.

and variables differ. I.e. Aßmann et al. (2015) use information from $N = 3615$ students and parents, $M = 23$ binary measurements and $Q = 6$ background variables. In terms of mathematical competency. Gaasch (2017a) uses $N = 13075$ units, giving answer to $M = 22$ items and having $Q = 5$ variables for background information. For investigating the concept of eating disorder with missing values in covariates, $N = 12460$ units with five items ($= M$) and two variables ($= Q$) with background information, i.e gender and body mass index (BMI), were utilized by Gaasch (2017a).

3.2 General Differences and Incorporation of Background Information

We have seen that the previous expressions use a similar but not the same structure. But before a more detailed depiction with regards to covariate incorporation is illustrated, some first discrepancies between the two approaches will be described. Although the factor analytic and the IRT formulations use both a latent representation $\mathbf{Y}^*$ for the measurements and are both interested in latent variables $\boldsymbol{\theta}$ as well, next to using solely categorical variables or including metric scaled measurements, they differ in several more aspects. While Conti et al. (2014) use a matrix of latent variables and thus a matrix of parameters $\boldsymbol{\alpha}$, Aßmann et al. (2015) and Gaasch (2017a) incorporate one unobserved variable $\boldsymbol{\theta}$ and the corresponding scalar α_m for a measurement m, and therefore a $(M \times 1)$-dimensional vector $\boldsymbol{\alpha}$ when all measurements $\mathbf{Y}$ are concerned. Another difference lies in the conditional mean structure of $\mathbf{Y}^*$: In Conti et al.'s (2014) formulation the intercept is missing, but there is one, that is the item difficulty β_m, in Aßmann et al.'s (2015) and Gaasch's (2017a) formulations. The variances of error terms and latent variables in the models of Aßmann et al. (2015) and Gaasch (2017a) are also different from Conti et al. (2014), not only with respect to the proposal for including group- or cluster-specific effects but also in its basic formulation. Because of the sole categorical nature of the approach and due to respective identifying restrictions, variances of error terms never will get estimated in the models of Aßmann et al. (2015) and Gaasch (2017a). Of course, estimation of these parameters will only get carried out in Conti et al.'s (2014) procedure, if they are related to metric scaled measurements. In the case of binary measurements, they are fixated to a value of one as well. In contrast to the errors, the latent variables variances get estimated in the case of Aßmann et al. (2015) and Gaasch (2017a) and are fixed to one in Conti et al. (2014).

After restructuring the formulations (3.1.1) and (3.1.5) the next step presents a closer look. Thus, we start with rewriting the first equations as

$$\begin{aligned}
\tilde{\mathbf{y}}_i^* = (\mathbf{y}_i^* - \boldsymbol{\beta}\mathbf{x}_i) = \mathbf{0} + \boldsymbol{\alpha}\boldsymbol{\theta}_i + \boldsymbol{\varepsilon}_i,& \\
\boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon),& \\
\boldsymbol{\theta}_i \sim MVN(\mathbf{0}, \mathbf{R}),& \qquad (3.2.1)
\end{aligned}$$

and the second as

$$\begin{aligned}
\mathbf{y}_i^* = -\boldsymbol{\beta} + \boldsymbol{\alpha}\theta_i + \boldsymbol{\varepsilon}_i,& \\
\boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \mathbf{I}_M),& \\
\tilde{\theta}_i^* = (\theta_i - \boldsymbol{\gamma}'\mathbf{x}_i) \sim N(0, \sigma_v^2).& \qquad (3.2.2)
\end{aligned}$$

It is now more clearly visible, how the explanation of variability in both formulations is different with respect to covariate influence. In (3.2.1) variability that is left when having measurements conditioned on background information and $\boldsymbol{\beta}$, $\tilde{\mathbf{y}}_i^*$, i.e. after removing covariate influence, is separated in a proportion pertaining to the latent variables and a proportion pertaining to the errors. By construction, conditional covariability will be part of the latent variables' covariance and not of the idiosyncratic errors. When taking a look at (3.2.2), we can see that the latent variables are an intermediator between the covariates and the measurements' latent representation $\mathbf{y}_i^*$ via $\boldsymbol{\alpha}\boldsymbol{\gamma}'$.

To give an illustrating example: Think of the latent variables representing a general ability for solving a test, the measurements indicate correct or incorrect answers to the test, and covariates might contain information on school curricula. We then can interpret the first formulation as: the measurements we observe are partially caused by–or at least correlated with–a persons genuine ability and additional school influence. While in the second case, we could interpret this as: the measurements we observe are caused by–or, again, at least correlated with–a persons ability. But this ability is in itself influenced by school curricula. A genuine ability could then be represented in $\tilde{\theta}_i^*$–of course, given the assumption that the school curricula information explains all variability except random noise.

3.3 Handling of Missing Values

In both empirical applications considered, missing values are present in the measurements as well as the covariates–so it is possible for some outcomes $\mathbf{y}_m$ and background variables $\mathbf{x}_q$ to be separable into missing and observed parts, i.e $(\mathbf{y}_m^{obs,\prime}, \mathbf{y}_m^{mis,\prime})'$ and $(\mathbf{x}_q^{obs,\prime}, \mathbf{x}_q^{mis,\prime})'$. As was cited above, the data stemming from the BCS and used by Conti et al. (2014), bears missingness in some of the cognitive, mental and physical health measurements, next to missing values in background information. Similarly, missingness is present in the outcomes of the NEPS-competency test data and data used to investigate eating disorders, and respective covariates analysed by Aßmann et al. (2015) and Gaasch (2017a). However, handling of missing values was conducted in completely different fashions.

Conti et al. (2014) deal with missing values in measurements and background information by simply excluding the respective cases, i.e. a restriction to $\mathbf{y}_m^{obs,\prime}$ and $\mathbf{x}_q^{obs,\prime}$, and cases where row-elements are observed but nevertheless deleted, because some other variables contain missings in the same position. Though not part of the missing-data problem, they also exclude individuals with "congenital abnormalities" and "non-whites" (see above). Aßmann et al. (2015) and Gaasch (2017a) combine two considerations: First, for the measurements' missing values. They are excluded for each column individually and not for all rows simultaneously, using different $\mathbf{y}_m^{obs,\prime}$, $\forall m$. The assumption underlying this approach, is a missing-by-design pattern of non-response, due to random administration of questions to participants (see Gaasch, 2017a, p. 3 and p. 34). Second, missing values in covariates get imputed by means of a CART-based imputation approach, relying on considerations of Burgette & Reiter (2010) and Doove et al. (2014) that get implemented similar to data augmentation.[24] This allows to carry out the Bayesian estimation and imputation at the same time, having information from $\mathbf{y}_m^{obs,\prime}$ and $(\mathbf{x}_q^{obs,\prime}, \mathbf{x}_q^{imputed,\prime})'$.

The problem Conti et al. (2014) solve, is not concerned with missing-data, but with their explorative algorithm. Thus, they do not investigate how their computations behave under different missing-data mechanisms. But as far as literature on missing-data problems goes, ignoring missing values by deleting them, as it is done by Conti et al. (2014), might, next to reduce the amount of values, lower the quality of estimates in terms of precision and bias, if missingness is not purely or completely random, or one deals with the case that

[24]Similar, because the imputation step is separated from the parameter estimation (compare also Enders, 2010, pp. 191ff.), such that CART creates parameters, being a basis for prediction / imputation, on its own and separate from the other conditional draws.

missingness of outcomes in a regression is a function of the covariates (see Enders, 2010, pp. 39f., Little & Rubin, 2002, pp. 41ff., and Little, 1992). Given the results of Aßmann et al. (2015) and Gaasch (2017a), using their combination of first of all ignoring missing values in $\mathbf{Y}$ and imputing missingness within background information seems to be a suited approach for their scenario.

In order to handle different covariate influence, multidimensional latent variables and handling of missing covariate-information, we combine both approaches and give an overview, how this can be achieved in the next chapter.

Chapter 4

Multidimensional Factor- and IRT-specification with Covariates and their Imputation

The first aim of combining both algorithms is, to include multidimensional latent variables and metric scaled measurements into the IRT framework. Multidimensionality allows for example to investigate simultaneously several competencies that can be found in psychometric modelling, like mathematical, reading or science literacy (see Weinert et al., 2011). Metric scaled measurements can be a basis of inclusion of process time, i.e. time someone spends on a specific task (see amongst others Thissen, 1983, pp. 179ff.). Second, it aims at incorporating ordered scaled measurements and background variable imputation into a factor-analytic approach. The aforementioned British Cohort Study also includes, for example, variables of ordered scale, like categorised periods of additional payments to the income a participant received or the number of jobs an interviewee has (see National Centre for Social Research, 2006, p. 44 and p. 55).

As mentioned above, in order to implement these and further steps, the approach of Conti et al. (2014) gets used as a starting point. We recreate and adapt the model specifications such that the respective likelihood can be formulated in terms of the IRT specification–that is, amongst others, including of covariates to have an influence on the latent variables and also a matrix-variate item difficulty. Additionally the Gibbs-based algorithm gets extended with ordered scaled measurements, imputation of covariates and the possibility of setting restrictions for the loading matrix. Moreover, to allow for further relationships between measurements and latent variables, the possibility of cross-loading will be included–i.e. measurements are able to load onto several latent variables at the same time.

4.1 Model Formulation and Prior Specifications

Model and Likelihood

We start with the model and move on to the prior specification afterwards. Consider again the matrix of measurements $\mathbf{Y} = (\mathbf{y}_1, \dots, \mathbf{y}_M)$ as it was already introduced in section 2.2 and respective subsection, with $\mathbf{y}_m = (y_{1m}, \dots, y_{Nm})'$, of dimension $N \times M$, containing variables of different scale in each column $m = 1, \dots, M$. Within this setup, these now comprise binary, newly included ordinal and already present metric scaled variables–representing for example correct / incorrect and cumulative correctness outcomes, and possible log-transformed process times or default metric scales, as they are, for example used by Conti et al. (2014). The model for $\mathbf{Y}$ can be formulated in terms of a latent continuous variable denoted as $\mathbf{Y}^* = (\mathbf{y}_1^*, \dots, \mathbf{y}_M^*)$, where $\mathbf{y}_m^* = (y_{1m}^*, \dots, y_{Nm}^*)'$ are the underlying variables, which are surjectively connected to the actual measurements $\mathbf{y}_{im}$ via a probit-based linking function if the latter are binary or of ordered scale, or bijective with the identity link in case of a metric scale. Let J_m denote the number of categories for each item with $J_m = 1$ indicating continuous, $J_m = 2$ binary, and $J_m > 2$ ordinal variables. The connection for every observation y_{im} with the corresponding latent variable y_{im}^* can then be stated as

$$y_{im} = \mathcal{I}_{\{1\}}(J_m) y_{im}^* + \mathcal{I}_{(1,+\infty)}(J_m) \sum_{j=1}^{J_m} j \mathcal{I}_{(\xi_{m,j}, \xi_{m,j+1})}(y_{im}^*), \tag{4.1.1}$$

where $\mathcal{I}_{\square}(\cdot)$ denotes again the indicator function and the distinction

$$\boldsymbol{\xi}_m = \begin{cases} (\xi_{m,1} = -\infty, \xi_{m,J_m+1} = +\infty), & \text{for } J_m = 1 \\ (\xi_{m,1} = -\infty, \xi_{m,2} = 0, \dots, \xi_{m,J_m+1} = +\infty), & \text{for } J_m > 1 \end{cases} \tag{4.1.2}$$

represents the cut-off parameters for the categorical variables, if $J_m > 1$, and portrays the full range of real numbers for continuous variables, when $J_m = 1$.

As before, the latent variables $\mathbf{Y}^*$ pertain to a regression model of the form

$$\mathbf{Y}^* = \boldsymbol{\Theta}\boldsymbol{\alpha}' + \mathbf{X}\boldsymbol{\beta}' + \boldsymbol{\varepsilon}, \text{ with vec}(\boldsymbol{\varepsilon}) \sim MVN(0, \boldsymbol{\Sigma}_\varepsilon \otimes \mathbf{I}_N), \tag{4.1.3}$$

where $\boldsymbol{\alpha}$ denotes the $M \times K$ matrix of loadings or discrimination parameters. This change from the multivariate equations of Conti et al. (2014) to a matrix-variate notation (again, see Geweke, 1996, or Viroli, 2012), allows an easier depiction, when restrictions on the loadings get included later on. For the latent factors $\boldsymbol{\Theta}$ of dimension $N \times K$ (with $k = 1 \ldots, K$) we assume

$$\boldsymbol{\Theta} = \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\upsilon}, \text{ with vec}(\boldsymbol{\upsilon}) \sim MVN(0, \mathbf{R} \otimes \mathbf{I}_N), \tag{4.1.4}$$

where $\mathbf{R}$ denotes a correlation matrix of dimension $K \times K$, thereby accounting for unidentifiable factor scale by fixating the amount of the factor variances to 1.

This setup results in two model perspectives: The first one belongs to classical factor analysis, assuming $\mathbf{Z} = \mathbf{0}$. Covariate information related to $\mathbf{Y}^*$ when $\mathbf{Z} = \mathbf{0}$, will be part of an $N \times Q$ matrix $\mathbf{X}$ with rows $\mathbf{x}_i = (x_{i1}, \ldots, x_{iQ})'$ and $q = 1, \ldots, Q$ (which can include an intercept or not). The latent variables $\boldsymbol{\Theta}$ are assumed to be correlated with zero mean and their relationship to with measurement is modelled via $\boldsymbol{\alpha}$. Parameters $\boldsymbol{\beta}$ that correspond to covariates are of dimension $M \times Q$–with $\boldsymbol{\beta}_m = (\beta_{m1}, \ldots, \beta_{mQ})'$ for every row m, and $\boldsymbol{\gamma}$ will not exist. As already indicated, the second perspective pertains to a model structure typically used in the context of item response theory models or MIMIC formulations, when $\mathbf{X} = -\boldsymbol{\iota}_N$ (where $\boldsymbol{\iota}_N$ is again a $N \times 1$ vector of ones). Given $\boldsymbol{\beta}$ is an $M \times 1$-vector, the resulting outer product $-\boldsymbol{\iota}_N\boldsymbol{\beta}'$ represents the matrix-variate version of item difficulty parameters (compare also formulas (2.2.31)). Unlike before, covariate information will be part of $N \times Q$ matrix of predictors $\mathbf{Z}$ with row vectors $\mathbf{z}_i = (z_{i1}, \ldots, z_{iQ})'$. These are part of the latent variables' mean, where respective parameters of covariate influence, denoted with $\boldsymbol{\gamma}$, are of dimension $K \times Q$.

For writing the model's likelihood, let $\boldsymbol{\Psi}$ denote the set of parameters, i.e.

$$\boldsymbol{\Psi} = \{\text{vec}(\boldsymbol{\beta}), \text{vec}(\boldsymbol{\alpha}), \text{vech}(\mathbf{R}), \text{diag}(\boldsymbol{\Sigma}_\varepsilon), \boldsymbol{\xi}\}\,. \tag{4.1.5}$$

This likelihood for the model setup is thus given as

$$
\begin{aligned}
f(\mathbf{Y}|\mathbf{X},\mathbf{Z},\boldsymbol{\Psi}) =& L(\boldsymbol{\Psi};\mathbf{X},\mathbf{Y},\mathbf{Z},\boldsymbol{\Theta}) \\
&= \left[\int\!\cdots\!\int_{\mathbb{R}} \left\{\int\!\cdots\!\int_{\mathbb{R}} f(\mathbf{Y},\mathbf{Y}^*|\boldsymbol{\Theta},\mathbf{X},\boldsymbol{\Psi})\right.\right. \\
&\left.\left. f(\boldsymbol{\Theta}|\mathbf{Z},\boldsymbol{\Psi}) d\left(\mathbf{Y}^*\right)\right\} d\left(\boldsymbol{\Theta}\right)\right], \quad \text{with} \qquad (4.1.6) \\
d\left(\boldsymbol{\Theta}\right) =& \left(d\theta_{11}\cdot\ldots\cdot d\theta_{NK}\right),
\end{aligned}
$$

and where integration is carried out with respect to those dy^*_{im}, whose m-th measurement is related to categorical variables only, i.e.

$$
d\left(\mathbf{Y}^*\right) = \prod_{i=1}^{N}\prod_{m=1}^{M}\left[\mathcal{I}_{\{1\}}(J_m) + \mathcal{I}_{(1,+\infty)}(J_m) dy^*_{im}\right]. \qquad (4.1.7)
$$

The functions within this integration correspond to the–with $\mathbf{Y}^*$ augmented–likelihood, using

$$
\begin{aligned}
f(\mathbf{Y},\mathbf{Y}^*|\boldsymbol{\Theta},\mathbf{X},\boldsymbol{\Psi}) \propto& |\boldsymbol{\Sigma}_\varepsilon|^{-\frac{N}{2}} \exp\left[-\frac{1}{2}\mathrm{tr}\left\{(\mathbf{Y}^* - \boldsymbol{\Theta}\boldsymbol{\alpha}' - \mathbf{X}\boldsymbol{\beta}')\boldsymbol{\Sigma}_\varepsilon^{-1}\right.\right. \\
&\left.\left.(\mathbf{Y}^* - \boldsymbol{\Theta}\boldsymbol{\alpha}' - \mathbf{X}\boldsymbol{\beta}')'\right\}\right] D, \text{ where} \qquad (4.1.8) \\
D =& \prod_{i=1}^{N}\prod_{m=1}^{M}\left[\mathcal{I}_{\{1\}}(J_m)\mathcal{I}_{(-\infty,+\infty)}(y^*_{im}) + \right. \\
&\left.\mathcal{I}_{(1,+\infty)}(J_m)\mathcal{I}_{(\xi_{m,j},\xi_{m,j+1})}(y^*_{im})\right] \\
=& \prod_{i=1}^{N}\prod_{m=1}^{M}\left[\mathcal{I}_{\{1\}}(J_m) + \mathcal{I}_{(1,+\infty)}(J_m)\mathcal{I}_{(\xi_{m,j},\xi_{m,j+1})}(y^*_{im})\right]. \qquad (4.1.9)
\end{aligned}
$$

The density with respect to latent variables of equation (4.1.4) is matrix-variate normal with

$$
f(\boldsymbol{\Theta}|\mathbf{Z},\boldsymbol{\Psi}) \quad \propto \quad |\mathbf{R}|^{-\frac{N}{2}} \exp\left[-\frac{1}{2}\mathrm{tr}\left\{(\boldsymbol{\Theta} - \mathbf{Z}\boldsymbol{\gamma})\mathbf{R}^{-1}(\boldsymbol{\Theta} - \mathbf{Z}\boldsymbol{\gamma})'\right\}\right]. \qquad (4.1.10)
$$

Aiming at missing values in covariates, still raises the question on how to handle missingness in $\mathbf{Y}$. Following Gaasch, 2017a, p. 34, one idea is, to remove the positions of missings in measurements. For this purpose, we can extend the considerations of Gaasch (2017a) using indexing and the previous formulations, with introducing vector and Kronecker product notation and particularly an elimination matrix $\mathbf{E}$. The latter is related to an $N \times M$ indicator matrix $\mathbf{O}^Y$ representing item responses, with elements

$$o_{im}^Y \quad = \quad \begin{cases} o_{im}^Y = 1, & \text{if } y_{im} \text{ is observed} \\ o_{im}^Y = 0, & \text{if } y_{im} \text{ is not observed,} \end{cases} \tag{4.1.11}$$

just like the response indicator of Rubin (2004, p. 30). With vectorising $\mathbf{O}^Y$ and write it as a diagonal matrix, one is able to state the obvious relationship $\text{diag}(\text{vec}(\mathbf{O}^Y)) = \mathbf{E}'\mathbf{E}$. Hence, $\mathbf{E}$ contains zeros and ones, and is of dimension $\sum_{i=1}^N \sum_{m=1}^M o_{im}^Y \times NM$. It condenses all elements of $\text{vec}(\mathbf{Y})$, with its respective dimension $NM \times 1$, and therefore $\text{vec}(\mathbf{Y}^*)$ to observed values, i.e. $\mathbf{E} \cdot \text{vec}(\mathbf{Y}) = \text{vec}(\mathbf{Y})^{obs}$ with dimension $\sum_{i=1}^N \sum_{m=1}^M o_{im}^Y \times 1$. Similarly, $\text{diag}(\text{vec}(\mathbf{o}_m^Y)) = (\mathbf{E}^{mm})'\mathbf{E}^{mm}$ and $\text{diag}(\text{vec}(\mathbf{o}_i^Y)) = (\mathbf{E}^{ii})'\mathbf{E}^{ii}$ describe the elimination (sub-)matrices for one column $\mathbf{o}_m^Y$ or one row $\mathbf{o}_i^Y$ of $\mathbf{O}^Y$, with $\mathbf{E}^{mm} \cdot \mathbf{y}_m = \mathbf{y}_m^{obs}$ and $\mathbf{E}^{ii} \cdot \mathbf{y}_i = \mathbf{y}_i^{obs}$.

Consequently, we are able to include denotation of ignoring missing values in $\mathbf{Y}$ and $\mathbf{Y}^*$ explicitly, where the observed data likelihood is

$$\begin{aligned} f(\mathbf{Y}^{obs}|\mathbf{X}, \mathbf{Z}, \mathbf{\Psi}) = \Big[\int \cdots \int_{\mathbb{R}} \Big\{ \int \cdots \int_{\mathbb{R}} & f(\mathbf{Y}^{obs}, \mathbf{Y}^{*,obs}|\mathbf{\Theta}, \mathbf{X}, \mathbf{\Psi}) \\ & f(\mathbf{\Theta}|\mathbf{Z}, \mathbf{\Psi}) d\left(\mathbf{Y}^{*,obs}\right) \Big\} d\left(\mathbf{\Theta}\right) \Big], \end{aligned} \tag{4.1.12}$$

and the augmentation to observed measurements only, is now given as

$$\begin{aligned} f(\mathbf{Y}^{obs}, \mathbf{Y}^{*,obs}|\mathbf{\Theta}, \mathbf{X}, \mathbf{\Psi}) \propto & |\tilde{\mathbf{\Sigma}}_\varepsilon|^{-\frac{1}{2}} \exp \Big\{ -\frac{1}{2} \text{vec}(\mathbf{Y}^* - \boldsymbol{\mu}_{y^*})' \mathbf{E}' \tilde{\mathbf{\Sigma}}_\varepsilon^{-1} \\ & \mathbf{E} \text{vec}(\mathbf{Y}^* - \boldsymbol{\mu}_{y^*}) \Big\} D^{obs}, \end{aligned} \tag{4.1.13}$$

where

$$\tilde{\mathbf{\Sigma}}_\varepsilon^{-1} = \mathbf{E}(\mathbf{\Sigma}_\varepsilon^{-1} \otimes \mathbf{I}_N)\mathbf{E}', \tag{4.1.14}$$

$$\boldsymbol{\mu}_{y^*} = \mathbf{\Theta}\boldsymbol{\alpha}' + \mathbf{X}\boldsymbol{\beta}', \tag{4.1.15}$$

$$d\left(\mathbf{Y}^{*,obs}\right) = \prod_{i=1}^N \prod_{m=1}^M \mathcal{I}_{\{1\}}(o_{im}^Y) \left[\mathcal{I}_{\{1\}}(J_m) + \mathcal{I}_{(1,+\infty)}(J_m) dy_{im}^* \right] \text{ and} \tag{4.1.16}$$

$$D^{obs} = \prod_{i=1}^N \prod_{m=1}^M \mathcal{I}_{\{1\}}(o_{im}^Y) \Big[\mathcal{I}_{\{1\}}(J_m) + \mathcal{I}_{(1,+\infty)}(J_m) \mathcal{I}_{(\xi_{m,j}, \xi_{m,j+1})}(y_{im}^*) \Big]. \tag{4.1.17}$$

To complete the model setup, the employed prior distributions are now discussed, which are mostly conjugate. Thereby, these priors are also accompanied by identifying restrictions guaranteeing unique parameter estimation in most cases. But, when using Conti et al. (2014) as a basis, a free estimation of the αs and θs in terms of unrestricted domains–as it is suggested by the authors–, sign-switching might occur, influencing the draws of the loadings / difficulties α_{mk}, latent variables θ_{ik} and latent variables' off-diagonal correlations $r_{k\bar{k}}$. A suggested approach relies on post-processing of the Gibbs-draws made, combining Conti et al. (2014) and Erosheva & Curtis (2017).

Priors for Loadings, Item Discrimination, Variances of the Idiosyncratic Errors next to Cut-Off Parameters

First parameters that are considered encompass the loadings and item difficulties respectively, and idiosyncratic variances. In order to estimate the matrix $\boldsymbol{\alpha}$, which might include with cross-loadings, multivariate normal draws are carried out. However, the draws for each (diagonally specified) variance are univariate. When deriving both respective full conditional posteriors in a next step (see section 4.2.6), the conditional prior structure that Conti et al. (2014) proposed is loosened, to allow independence in the compound probabilities of these parameters, i.e. $\pi(\boldsymbol{\alpha}|\boldsymbol{\Sigma}_\varepsilon)\pi(\boldsymbol{\Sigma}_\varepsilon) = \pi(\boldsymbol{\alpha})\pi(\boldsymbol{\Sigma}_\varepsilon)$.

Before the actual distribution for $\boldsymbol{\alpha}$ is portrayed, an additional matrix $\boldsymbol{\Delta}$ of zeros and ones is introduced, which allows to define whether there is a relationship between measurements and latent variables by 1, i.e. known as the measurements "load onto" the latent variables, or not (denoted with 0). Within the implementation of an algorithm, $\boldsymbol{\Delta}$ can then be utilized to index and retrieve desired elements of $\boldsymbol{\alpha}$ and, as will be shown, mark whether a specific form of identification is fulfilled or not. Based on this indicator matrix, that is also used in variable selection (see George & McCulloch, 1997, and Smith & Kohn, 1996), and employed by Conti et al. (2014) to denote this relationship between measurements and latent variables, one is able to include a more flexible structure than the latter authors seem to have intended. By specifying an $M \times K$ indicator matrix $\boldsymbol{\Delta}$, with elements

$$\delta_{mk} = 1 - \mathcal{I}_{\{0\}}(\alpha_{mk}) = \begin{cases} 1, & \text{if } \mathbf{y}_m \text{ loads onto } \boldsymbol{\theta}_k \\ 0, & \text{if } \mathbf{y}_m \text{ does not load onto } \boldsymbol{\theta}_k, \end{cases} \tag{4.1.18}$$

the aforementioned relationship between measurements and latent variables is depicted. Although, as pointed out in section 2.3, a loading of two outcomes onto one factor is sufficient for identification in multiple factor models, it might be problematic during the analysis (see Kline, 2016, p. 201). Therefore, we rely on Conti et al.'s (2014), said to be sufficient (see section 2.3 and compare again Anderson & Rubin, 1956, theorems 5.5 and

5.6 and the three-indicator rule of Bollen 1989, pp. 242ff.), Theorem 1, which requires that three or more measurements should load on one factor and one is able to described using the elements of $\boldsymbol{\Delta}$, i.e.

$$\sum_{m=1}^{M} \delta_{mk} \geq 3, \forall \mathbf{y}_m, k = 1, \ldots, K \text{ and } m = 1, \ldots, M, \tag{4.1.19}$$

where an exemplary structure of this requirement could look as follows,

$$\boldsymbol{\Delta} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \tag{4.1.20}$$

when six measurements load onto two latent variables, as it was already in terms of $\boldsymbol{\alpha}$ presented earlier (i.e. formulations in section 2.3 or subsequently to formula (3.1.1)).

With the help of this indicator matrix, the later characterised Gibbs draws for theαs full conditional can be simplified. Because without $\boldsymbol{\Delta}$, one would need to assume a mixture of a multivariate normal for the non-zero elements and a (degenerate) Dirac delta based distribution, if no loading, i.e. $\alpha_{mk} = 0$ occurs. By using this indicator matrix instead and focussing solely on a confirmatory setting, zero restrictions in $\boldsymbol{\alpha}$ are deterministic and, for example, removable via elimination with a matrix $\mathbf{E}_\Delta$ of dimension $(\sum_{m=1}^{M} \sum_{k=1}^{K} \delta_{mk} \times MK)$, where $\text{diag}(\text{vec}(\boldsymbol{\Delta})) = \mathbf{E}'_\Delta \mathbf{E}_\Delta$and $\mathbf{E}_\Delta \text{vec}(\boldsymbol{\alpha}) = \text{vec}(\boldsymbol{\alpha})^\Delta$. Thus, a multivariate normal specification for the non-zero unique elements of $\boldsymbol{\alpha}$, i.e. $\boldsymbol{\alpha}^\Delta$ is

$$\pi(\text{vec}(\boldsymbol{\alpha})^\Delta) \propto \exp\left\{-\frac{1}{2}\left(\text{vec}(\boldsymbol{\alpha})^\Delta - \boldsymbol{\mu}_\alpha^0\right)'\left(\boldsymbol{\Sigma}_\alpha^0\right)^{-1}\left(\text{vec}(\boldsymbol{\alpha})^\Delta - \boldsymbol{\mu}_\alpha^0\right)\right\}. \tag{4.1.21}$$

The indicator matrix thereby allows to incorporate further possibilities in describing the loading structure. As will be shown, it is, first, possible to load measurements onto several factors simultaneously. Therefore, their relationships are not only represented within the factor-correlation but also explicitly presented by the loadings, either to test for cross-loading (see Asparouhov et al., 2015) or explicitly demand it, in terms of, e.g., bifactor formulations (see Erosheva & Curtis, 2017). Second, in order to incorporate possible restrictions of identification, e.g. equality and linear restriction – as described in section 2.3–, one can extend the domain of the elements δ_{mk} to $\mathbb{N}_0$. $\bar{\boldsymbol{\Delta}}$ will be used to denote this extended variant, where one is able to specify zero restrictions ($\bar{\delta}_{mk} = 0$), free estimation of an α_{mk} ($\bar{\delta}_{mk} = 1$) and equality restrictions ($\bar{\delta}_{mk} > 1$), that is setting some elements of $\boldsymbol{\alpha}$ to be equal to one or more other elements at once. As will be depicted in detail during the parameter estimation within section 4.2.6, one can use $\boldsymbol{\Delta}$ and $\bar{\boldsymbol{\Delta}}$ respectively as basis of an elimination-and-duplication algorithm. This approach reduces the complete matrix $\boldsymbol{\alpha}$ to determine posterior draws of the non-zero and unique elements α_{mk} and facilities duplication for expanding these updates back to all elements, i.e. zero, non-zero and multiple occurring components–including linear transformations of $\boldsymbol{\alpha}$s elements.

When now taking the idiosyncratic variances $\boldsymbol{\Sigma}_{\varepsilon} = \operatorname{diag}(\sigma^2_{\varepsilon,1}, \ldots, \sigma^2_{\varepsilon,m}, \ldots, \sigma^2_{\varepsilon,M})$ into account, for elements belonging to continuous scaled observations $\mathbf{y}_m$ a univariate inverse-gamma distribution is utilized, i.e.

$$\pi(\sigma^2_{\varepsilon,m}) \propto (\sigma^2_{\varepsilon,m})^{-c^0_m-1} \exp\left(-\frac{C^0_m}{\sigma^2_{\varepsilon,m}}\right). \tag{4.1.22}$$

For categorical models in general and specifically for categorical measurements in the setup presented, two restrictions are introduced. First, defining the variances $\sigma^2_{\varepsilon,m}$ as 1 and, second, as shown in (4.1.2) fixing the first two and the last cut-off parameter of $\boldsymbol{\xi}_m$ to $\xi_{m,1} = -\infty$, $\xi_{m,2} = 0$ and $\xi_{m,J_m+1} = +\infty$. So we follow the general literature (that is Albert & Chib, 1993, Albert & Chib, 1997, Cowles, 1996, and Jackman, 2009, p. 399) in order to guarantee identification when sampling the cut-off values.

While it is understandable, that all binary outcomes use the same fixed set of cut-off parameters $\boldsymbol{\xi}$, i.e.$(-\infty, 0, +\infty)$, these kind of parameters need further considerations for ordinal measurements. A simple structure, like a normal prior, is not possible without violating the necessary ordering restriction, which demands by definition that the next element is greater than the previous one. Therefore, the algorithm proposed by Albert & Chib (1997), which is incorporated here, takes a detour and is based on a (strictly increasing) log-transform conditioning and the respective inverse map onto $\boldsymbol{\xi}_m$ with

$$\zeta_{m,1} = \log\left(\xi_{m,2}\right) = -\infty;\ \zeta_{m,j} = \log\left(\xi_{m,j+1} - \xi_{m,j}\right),\ \forall 3 \leq j < J_m + 1. \tag{4.1.23}$$

The prior structure of these newly introduced auxiliary parameters, where $\boldsymbol{\zeta}_m$ corresponds to one ordered variable $\mathbf{y}_m$, is

$$\pi(\boldsymbol{\zeta}_m) \quad \propto \quad \exp\left\{-\frac{1}{2}(\boldsymbol{\zeta}_m - \boldsymbol{\mu}^0_{\zeta,m})'(\boldsymbol{\Sigma}^0_{\zeta,m})^{-1}(\boldsymbol{\zeta}_m - \boldsymbol{\mu}^0_{\zeta,m})\right\}, \tag{4.1.24}$$

where $\boldsymbol{\mu}^0_{\zeta,m} = \mathbf{0}$ and $\boldsymbol{\Sigma}^0_{\zeta,m} = \mathbf{I}_{J_m}$, i.e. standard normal.

Both aspects induce for the upcoming Gibbs algorithm that variances won't get estimated for binary or ordered data, and, especially for ordered measurements, only the $J_m - 2$ remaining and non-fixed cut-off parameters get updated in the sampling procedure.

Distribution of Latent Variable Covariances / Correlations

When it comes to the latent variables, their expected values are either $\mathrm{E}[\boldsymbol{\Theta}] = \mathbf{0}$ for the factor model or $\mathrm{E}[\boldsymbol{\Theta}] = \mathbf{Z}\boldsymbol{\gamma}$ in the IRT version. As pointed out by equation (4.1.4), the covariance used to describe the (conditional) relationship amongst the latent variables is a correlation matrix, and thus variances are, by definition, restricted to 1. This matrix $\mathbf{R}$ is described as $\mathrm{vech}(\mathbf{R}) = (1, \ldots, r_{K1}, 1, \ldots, r_{K2}, \ldots, 1)'$ and $-1 \leq r_{k\bar{k}} < 1, \forall k \neq \bar{k}$. Albeit $\mathbf{R}$ can not be sampled directly, one can rely on a separation and augmentation scheme (see Barnard et al., 2000, Zhang et al., 2006, or Conti et al., 2014), exploiting the relationship between a covariance $\boldsymbol{\Omega}$, the respective variance $\boldsymbol{\Lambda}$ and correlation $\mathbf{R}$, which is

$$\begin{aligned} &\boldsymbol{\Omega} = \boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{R}\boldsymbol{\Lambda}^{\frac{1}{2}} \text{ and} \\ &\mathbf{R} = \boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-\frac{1}{2}}. \end{aligned} \tag{4.1.25}$$

Thus, a conditional inverse-Wishart distribution can be imposed on extended latent variables $\ddot{\boldsymbol{\theta}}_i = \boldsymbol{\theta}_i\boldsymbol{\Lambda}^{\frac{1}{2}}$ with covariance $\boldsymbol{\Omega}$, i.e.

$$\pi(\boldsymbol{\Omega}|\mathbf{S}^0_\Omega) \propto |\boldsymbol{\Omega}|^{-\frac{\nu^0_\Omega+K+1}{2}} |\mathbf{S}^0_\Omega|^{\frac{\nu^0_\Omega}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}^0_\Omega\boldsymbol{\Omega}^{-1}\right)\right\} \tag{4.1.26}$$

using a gamma prior for the scale matrix's diagonal elements, $\mathbf{S}^0_\Omega = \mathrm{diag}(s^0_1, \ldots, s^0_K)$, with

$$\pi(s^0_k) \propto (s^0_k)^{-0.5} \exp\left\{-\frac{s^0_k}{2(\nu^0_\Omega - K + 1)A^2_k}\right\}. \tag{4.1.27}$$

Prior for Covariate Parameters and Item Difficulty respectively

When incorporating previous knowledge for the regression parameters, likewise in the factor model or in the IRT approach, it is multivariate normal. Thus, the prior for $\boldsymbol{\beta}$ as a vectorised matrix refers to

$$\pi(\mathrm{vec}(\boldsymbol{\beta})) \propto \exp\left\{-\frac{1}{2}\left(\mathrm{vec}(\boldsymbol{\beta}) - \boldsymbol{\mu}^0_\beta\right)'\left(\boldsymbol{\Sigma}^0_\beta\right)^{-1}\left(\mathrm{vec}(\boldsymbol{\beta}) - \boldsymbol{\mu}^0_\beta\right)\right\}, \tag{4.1.28}$$

and for a vectorised version of the covariates influence, $\text{vec}(\boldsymbol{\gamma})$, it is

$$\pi(\text{vec}(\boldsymbol{\gamma})) \propto \exp\left\{-\frac{1}{2}\left(\text{vec}(\boldsymbol{\gamma}) - \boldsymbol{\mu}_{\gamma}^{0}\right)'\left(\boldsymbol{\Sigma}_{\gamma}^{0}\right)^{-1}\left(\text{vec}(\boldsymbol{\gamma}) - \boldsymbol{\mu}_{\gamma}^{0}\right)\right\}. \tag{4.1.29}$$

Note however that unlike covariate parameters in factor analysis, inclusion of covariates' parameter influence $\text{vec}(\boldsymbol{\gamma})$ and therefore estimation in IRT takes place in relationship to $\boldsymbol{\Theta}$. As will be shown in the next section, $\boldsymbol{\Theta}$ will be sampled in terms of an expansion, based on the latent variables' covariance $\boldsymbol{\Omega}$ and variance $\boldsymbol{\Lambda}$, not its correlation $\mathbf{R}$. But this expansion affects $\text{vec}(\boldsymbol{\gamma})$ as well, such that we need to define a prior for latent variance expanded covariate parameters in IRT, i.e. $\text{vec}(\ddot{\boldsymbol{\gamma}})$, with

$$\pi(\text{vec}(\ddot{\boldsymbol{\gamma}})) \propto \exp\left\{-\frac{1}{2}\left(\text{vec}(\ddot{\boldsymbol{\gamma}}) - \boldsymbol{\mu}_{\ddot{\gamma}}^{0}\right)'\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{0}\right)^{-1}\left(\text{vec}(\ddot{\boldsymbol{\gamma}}) - \boldsymbol{\mu}_{\ddot{\gamma}}^{0}\right)\right\}, \tag{4.1.30}$$

where $\ddot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\boldsymbol{\Lambda}^{\frac{1}{2}}$ (see also 4.2.7). Based on this specifications, a Gibbs-based algorithm can be presented in order to conduct the estimation.

4.2 Estimation Steps

Relying on a Bayesian perspective and thus using distributional assumptions for parameters, makes it necessary to carry out estimation based on a joint formulation for all unknown variables and parameters, conditioned on observed variables, i.e.

$$f(\mathbf{X}^{mis}, \mathbf{Y}^{*}, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\Sigma}, \boldsymbol{\alpha}^{\boldsymbol{\Delta}}, \boldsymbol{\Theta}, \boldsymbol{\Omega}, \boldsymbol{\gamma}|\mathbf{Y}, \mathbf{X}^{obs}) \text{ or} \tag{4.2.1}$$

$$f(\mathbf{Z}^{mis}, \mathbf{Y}^{*}, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\beta}, \boldsymbol{\Sigma}, \boldsymbol{\alpha}^{\boldsymbol{\Delta}}, \boldsymbol{\Theta}, \boldsymbol{\Omega}, \boldsymbol{\gamma}|\mathbf{Y}, \mathbf{Z}^{obs}). \tag{4.2.2}$$

Creating such a joint distribution is mostly cumbersome, if not impossible, when avoiding to give up flexibility in modelling variables and parameters individually. One strategy of evaluating this distribution nevertheless, is Gibbs sampling (see amongst others Casella & George, 1992, Robert & Casella, 2004, chapters 9 and 10, and Roberts, 1996) and full conditional specification (see Gilks, 1996) that rests the estimation upon conditional and posterior distributions instead, e.g. $f(\mathbf{X}^{mis}|\mathbf{X}^{obs}, \mathbf{Y})$ or $f(\mathbf{Z}^{mis}|\mathbf{Z}^{obs}, \boldsymbol{\Theta})$ in IRT, $f(\mathbf{Y}^{*}|\mathbf{X}, \boldsymbol{\beta}, \boldsymbol{\Theta}, \boldsymbol{\alpha}, \boldsymbol{\Sigma}, \boldsymbol{\xi})$, $f(\boldsymbol{\xi}|\boldsymbol{\zeta}, \mathbf{X}, \boldsymbol{\beta}, \boldsymbol{\Theta}, \boldsymbol{\alpha})$, $f(\boldsymbol{\beta}|\mathbf{Y}^{*}, \mathbf{X}, \boldsymbol{\Theta}, \boldsymbol{\Sigma}_{\varepsilon}, \boldsymbol{\alpha})$ and some more to be described. This approach uses temporary draws of unknown parameters and variables given previously conducted draws of necessary parameters and variables, producing Markov chains. Draws get iteratively repeated up until convergence is approximated (or a large, fixed amount of iterations was carried out). Thus, the joint space of (4.2.1) gets separated into individual parts, simplifying the mathematical modelling and therefore estimation task, and resulting in the desired imputation and posterior draws.

For sampling the necessary parameters, one can establish the following scheme:

(**A1S0**) *Initialisation*: To be able to conduct a first imputation, use hyperparameters, initial start values for parameters and unobserved variables, and create draws without replacement from the set of observed values for every column of covariates containing missing values. Use these draws as first hotdeck imputations for respective missing values.

(**A1S1**) Carry out (T) iterations, where in every iteration (t).

(*A1S1a*) $f(\mathbf{X}^{mis,(t)}|\mathbf{X}^{complete,(t-1)},\mathbf{Y})$ or $f(\mathbf{Z}^{mis,(t)}|\mathbf{Z}^{complete,(t-1)},\boldsymbol{\Theta})$: Impute missing covariates in $\mathbf{X}$ or $\mathbf{Z}$ based on a CART-type algorithm using the information of the observed parts of $\mathbf{X}$ or $\mathbf{Z}$, and the observed data $\mathbf{Y}$ (factor analysis) and $\boldsymbol{\Theta}$ (IRT) respectively, to create a set of completed covariates.

(*A1S1b*) $f(\mathbf{Y}^{*,(t)}|\mathbf{X}^{complete,(t)},\boldsymbol{\beta}^{(t-1)},\boldsymbol{\Theta}^{(t-1)},\boldsymbol{\alpha}^{(t-1)},\boldsymbol{\xi}^{(t-1)})$: Draw each latent y^*_{im} based on a truncated normal posterior, if the respective $\mathbf{y}_m$ is of ordered or binary scale.

(*A1S1c*) $f(\boldsymbol{\xi}^{(t)}_{nonfixed}|\boldsymbol{\zeta}^{(t-1)},\mathbf{X}^{complete,(t)},\boldsymbol{\beta}^{(t-1)},\boldsymbol{\Theta}^{(t-1)},\boldsymbol{\alpha}^{(t-1)})$: Draw the non-fixed cut-off parameters in $\boldsymbol{\xi}_m$ for ordered measurements $\mathbf{y}_m$.

(*A1S1d*) $f(\boldsymbol{\beta}^{(t)}|\mathbf{Y}^{*,(t)},\mathbf{X}^{complete,(t)},\boldsymbol{\Theta}^{(t-1)},\boldsymbol{\Sigma}_{\varepsilon}^{(t-1)},\boldsymbol{\alpha}^{(t-1)})$: Draw regression weights $\boldsymbol{\beta}$ in factor analysis or item difficulties in IRT from a multivariate normal full conditional distribution.

(*A1S1e*) $f(\sigma^{2,(t)}_{\varepsilon,m}|\mathbf{X}^{complete,(t)},\boldsymbol{\beta}^{(t)},\mathbf{Y}^{*,(t)},\boldsymbol{\Theta}^{(t-1)})$: Draw idiosyncratic variances $\sigma^2_{\varepsilon,m}$ for all continuous $\mathbf{y}_m$ using an inverse-gamma distribution. For purposes of identification $\sigma^2_{\varepsilon,m}$ is fixed to one, if the measurements are categorical.

(*A1S1f*) $f(\text{vec}(\boldsymbol{\alpha})^{\bar{\boldsymbol{\Delta}},(t)}|\boldsymbol{\Theta}^{(t-1)},\boldsymbol{\Sigma}_{\varepsilon}^{(t)},\mathbf{Y}^{*,(t)},\mathbf{X}^{complete,(t)},\boldsymbol{\beta}^{(t)})$: Create non-zero and unrestricted elements α_{mk} of the loading / item discrimination matrix $\boldsymbol{\alpha}$ using a element-restricted multivariate normal draw.

(*A1S1g1*) $f(\boldsymbol{\Theta}^{(t)}|\mathbf{R}^{(t-1)},\boldsymbol{\Sigma}_{\varepsilon}^{(t)},\boldsymbol{\alpha}^{(t)},\mathbf{Y}^{*,(t)},\boldsymbol{\beta}^{(t)},\mathbf{X}^{(t)})$, $f(\boldsymbol{\Omega}^{(t)}|\boldsymbol{\Theta}^{(t)})$, $f(\mathbf{R}^{(t)}|\boldsymbol{\Omega}^{(t)})$: Use a multivariate-normal draw for latent variables $\boldsymbol{\Theta}$. Their respective correlations $\mathbf{R}$ are created via facilitating a correlation-separation into latent variable variance $\mathbf{\Lambda}$ and variance-covariance matrix $\mathbf{\Omega}$, where the latter is created via an inverse-Wishart draw.

($A1S1g2$) *IRT only*: $f(\boldsymbol{\gamma}^{(t)}|\mathbf{\Theta}^{(t-1)}, \mathbf{R}^{(t-1)}, \mathbf{Z}^{complete,(t)})$: When the model suggests that background information has an influence on latent variables $\mathbf{\Theta}$, respective parameters $\boldsymbol{\gamma}$ get drawn using a multivariate-normal draw, and the conditioning for $\mathbf{\Theta}$ changes to

$$f(\mathbf{\Theta}^{(t)}|\mathbf{R}^{(t-1)}, \mathbf{\Sigma}_{\varepsilon}^{(t)}, \boldsymbol{\alpha}^{(t)}, \mathbf{Y}^{*,(t)}, \boldsymbol{\beta}^{(t)}, \mathbf{Z}^{complete,(t)}, \boldsymbol{\gamma}^{(t)})$$

and instead of $f(\mathbf{\Omega}^{(t)}|\mathbf{\Theta}^{(t)})$ we have

$$f(\mathbf{\Omega}^{(t)}|\mathbf{\Theta}^{(t)}, \mathbf{Z}^{complete,(t)}, \boldsymbol{\gamma}^{(t)}).$$

(**A1S2**) After (T) steps, address sign-changes in estimates for $\boldsymbol{\alpha}$ and $\mathbf{R}$, and $\boldsymbol{\gamma}$ if necessary for IRT.

Of course the hotdeck step of (**A1S0**), the imputation in step ($A1S1a$) and the estimation of the regression coefficients in step ($A1S1d$) for factor analysis and ($A1S1g2$) in IRT respectively, will not be necessary in the absence of covariates. Similarly, draws for idiosyncratic variances in ($A1S1e$) aren't needed, when all measurements are categorical; or creation of cut-off values in ($A1S1c$) and latent $\mathbf{Y}^*$s in ($A1S1b$) is skipped, if all measurements are solely continuous.

Details for each of these steps are presented subsequently, while for convenience the superscripts (t) and $(t-1)$ respectively are dropped within these steps. Additionally note: while very short and more complex mathematical parts are depicted directly, standard derivations of posteriors, that are necessary, but simple to understand and nonetheless lengthy in form, are depicted in Appendix B on page 178.

4.2.1 Imputing Covariates

Because the same considerations apply to covariates $\mathbf{X}$ (in factor analysis) and $\mathbf{Z}$ (in IRT), assume for the moment, that we only have $\mathbf{X}$ for background information, and note that all steps can be adopted for $\mathbf{Z}$ and elements z_{iq}, or vectors $\mathbf{z}_q$ and $\mathbf{z}_i$. Several strategies throughout the literature can be found for handling missing values within these covariates. This refers especially to interaction terms, e.g. $\mathbf{x}_q\mathbf{x}_{q+1}$, and transformed independent variables, like potential labour market experience raised by the power of two, $\mathbf{x}_q^2$ in the Mincer (1958) earnings function, or a different reformulation, e.g. $\log(\mathbf{x}_q)$. Approaches range from using passive imputation, i.e. imputing then transforming, as Von Hippel (2009) named it, using altered predictors as if they weren't transformed ("Just another variable approach" / "transform then impute", Von Hippel, 2009, Seaman et al., 2012), multilevel approaches (see Goldstein et al., 2009, and Goldstein et al., 2014) or CART-algorithms (Burgette & Reiter, 2010, and Doove et al., 2014).

In setting presented, the latter method is incorporated to guarantee a degree of flexibility via an automated model creation method for imputation. This means, if covariates, e.g. background variables in a competency test, contain missing values, they get imputed within every iteration by a classification and regression tree approach stemming from Burgette & Reiter (2010) and Doove et al. (2014) respectively. In doing so, both ideas are combined by using *rpart* (Therneau et al., 2015) for the tree creation as *mice.impute.cart* (Doove et al., 2015, and Doove et al., 2014) does, but using a Bayesian bootstrap (see Rubin, 1981, for details) to draw donors within the tree's leaves, as Burgette & Reiter (2010) state "In order to reflect uncertainty about the population distributions in the leaves[...]". This strategy was successfully tested for covariates in a linear dynamic panel model with continuous responses (see Preising, 2018, pp. 49f.), and normal-ogive IRT models with one latent variable and binary- and ordered measurements (see again Aßmann et al., 2015 and Gaasch, 2017a). Based on Burgette & Reiter (2010) and Doove et al., 2014, one is able to describe the imputation routine as follows:

(**A2S1**) Define $\mathbf{X}^{toImp}$ of dimension $N \times Q_1$ as the sub-matrix of predictors' columns containing missing values, i.e. for each column of $\mathbf{X}$ one has a separation in $\mathbf{x}_q^{toImp} = (\mathbf{x}_q^{\prime,mis}, \mathbf{x}_q^{\prime,obs})'$. Also define $\mathbf{X}^{noImp}$ with dimension $N \times (Q - Q_1)$ consisting of completely observed vectors, such that $\mathbf{X} = (\mathbf{X}^{toImp}, \mathbf{X}^{noImp})$.

(**A2S2**) Sort the columns of $\mathbf{X}^{toImp}$ according to the frequency of missing values, gaining a monotonicity pattern with respect to the information available.

(**A2S3**) Add auxiliary (observed) data as an additional source of information, i.e. $\mathbf{Y}$ when conducting factor analysis and Θ when carrying out the item response approach.

(**A2S4**) For every variable $\mathbf{x}_q^{toImp}$ with missing values:

(*A2S4a*) Depending on the scaling of the covariates to impute, either fit a classification- or a regression-tree constructed with

$$\mathbf{x}_q^{toImp}|\mathbf{X}_{-q}^{toImp}, \mathbf{X}^{noImp}, \mathbf{Y}, \text{ (for factor analysis)}$$

or

$$\mathbf{x}_q^{toImp}|\mathbf{X}_{-q}^{toImp}, \mathbf{X}^{noImp}, \mathbf{\Theta}, \text{ (for IRT)},$$

where the index $-q$ denotes all covariates except the q-th predictor currently imputed.

(*A2S5b*) Based on the emerging recursive partitioned tree from before, create an unique indicator for the leaf nodes.

(*A2S4c*) Relying on the indicators and respective leave nodes, predict new responses, with the parts of $\mathbf{X}_{-q}^{toImp}, \mathbf{X}^{noImp}$ that correspond to the missing values in input data.

(*A2S4d*) Create a Bayesian bootstrap sample based on Rubin (1981), resulting in the completed covariate $\mathbf{x}_q^{comp}$. This is done by, first, select donor values using $\mathbf{x}_q^{toImp}$ that pertain to the positions of the predicted values that correspond to the leave nodes. Second, create ascending uniform draws and finally sample with replacement from the donors, where the sorted uniform values' pairwise differences facilitate the probability weights. The latter sample are then used as imputed values.

(**A2S5**) Repeat steps of (**A2S4**) for all covariates with missing values until the complete set of predictors $\mathbf{X}^{comp}$ (that also includes the $\mathbf{X}^{noImp}$ part) is finished.

The implementation itself and the resulting $\mathbf{X}^{comp}$ is then used to impute like the imputation step of a data augmentation scheme (see Li, 1988, and Tanner & Wong, 1987, respectively). It now allows to conduct the rest of the estimation with unobserved, but recreated parts of the background information $\mathbf{X}$ or $\mathbf{Z}$ in every iteration of the Gibbs sampler.

4.2.2 Drawing $\mathbf{Y}^*$ for Categorical $\mathbf{y}_m$s

The elements of the latent measurements $\mathbf{y}_m^{obs,*} = \mathbf{E}^{mm}\mathbf{y}_m^*$ if the m-th measurement is categorical, are drawn from a truncated normal distribution-if, however, they are continuous, draws will not get carried out. Using a standard probit understanding (see, e.g. Albert & Chib, 1993, or Chib & Greenberg, 1998), this derives directly of the basic model formulation of (4.1.3) in terms of $\mathbf{Y}^*$. If the respective observed measurement is binary, one has for every i

$$f(y_{im}^{*,obs}|\mathbf{x}_i, \boldsymbol{\beta}_m, \boldsymbol{\theta}_i, \boldsymbol{\alpha}_m) \propto \begin{cases} \frac{1}{0.5-0} \exp\left\{-\frac{1}{2}\left(y_{im}^* - \mu_{y^*}\right)^2\right\}, & \text{if } y_{im} = 0 \\ \frac{1}{1-0.5} \exp\left\{-\frac{1}{2}\left(y_{im}^* - \mu_{y^*}\right)^2\right\}, & \text{if } y_{im} = 1 \end{cases} \tag{4.2.3}$$

and $\mu_{y^*} = \mathbf{x}_i'\boldsymbol{\beta}_m + \boldsymbol{\theta}_i'\boldsymbol{\alpha}_m$, given that $o_{im}^Y = 1$, i.e. y_{im} is not missing.

If on the other hand $\mathbf{y}_m$ is of ordered scale, for the corresponding elements of $\mathbf{y}_m^{obs,*}$ a truncated normal draw is also in use but now facilitates the cut-off parameters $\boldsymbol{\xi}_m$ to denote the categories needed. This is one part of Albert and Chib's (1997) algorithm for the Bayesian ordered-probit model–whose second part is shown below and used to create the related boundaries. Hence, the result is

$$f(y_{im}^{*,obs}|\mathbf{x}_i, \boldsymbol{\beta}_m, \boldsymbol{\theta}_i, \boldsymbol{\alpha}_m) \propto \frac{\exp\left\{-\frac{1}{2}\left(y_{im}^* - \mu_{y^*}\right)^2\right\}}{\Phi\left(\xi_{m,j+1} - \mu_{y^*}\right) - \Phi\left(\xi_{m,j} - \mu_{y^*}\right)}, \tag{4.2.4}$$

where $\Phi(\cdot)$ again describes the normal cumulative distribution function with mean $\mu_{y^*|\cdot}$ as before, and given that $o_{im}^Y = 1$.

4.2.3 Drawing Cut-Off Parameters

As already mentioned, for drawing each $\boldsymbol{\xi}_m$ for ordered measurements, an algorithm described by Albert & Chib (1997) is in use. This so-called cumulative approach is a Metropolis-Hastings procedure using a Newton-type minimisation, and creating the acceptance ratio by relying on a multivariate t- and normal distributions. Like in every probit model for categorical variables, the idea is to use a latent normal structure, which is transformed via truncation into categories. The main difference between a binary and an ordered model lies within the values of $\boldsymbol{\xi}_m = (\xi_{m,1}, \ldots, \xi_{m,J_m+1})'$.

While the binary formulation typically has its tipping point at the fixed latent value $\xi_{m,2} = 0$ in order to differentiate the outcome, the ordered probit uses in general $J_m + 1$

parameters to split the support of the latent variable into J_m disjunct intervals for every ordered y_{im}, with probability

$$\mathcal{P}(y_{im}^{obs} \leq j|\mathbf{x}_i, \boldsymbol{\beta}_m, \boldsymbol{\theta}_i, \boldsymbol{\alpha}_m, \sigma^2_{\varepsilon,m} = 1) \quad = \quad \Phi\left(\xi_{m,j}^{obs} - \mu_{y^*}^{obs}\right), \tag{4.2.5}$$

and $j = 1, \ldots, J_m + 1$.

Furthermore, it should be clear that the cut-off parameters exhibit an ordering restriction. Because they limit the regions where the latent variables y^*_{im} are assigned to the categorical measurements y_{im}, these borders need to be disjoint and are not to be switched, i.e. $\xi_{m,1} < \ldots < \xi_{m,j} < \ldots < \xi_{m,J_m+1}$.

Albert and Chib's approach from 1997 for cumulative models[25], which also has been successfully implemented in IRT-modelling with one latent variable (by Gaasch, 2017b,a), relies on an informative prior and an additional set of parameters denoted with ζ_m. These auxiliary parameters are then transformed, as the relationships in (4.1.23) describe, to keep up the monotonicity of the boundaries, i.e. $-\infty < 0 < \xi_{m,3} < \ldots < \infty$.

With the help of the likelihood with respect to $\boldsymbol{\xi}_m$ that derives from formula (4.2.5), i.e.

$$\begin{aligned} f_\xi(\boldsymbol{\xi}_m|\mathbf{x}_i^{comp}, \boldsymbol{\beta}_m, \boldsymbol{\theta}_i, \boldsymbol{\alpha}_m, \sigma^2_{\varepsilon,m} = 1) = \prod_{j=1}^{J_m} \prod_{i:y_{im}=j} & \left[\Phi\left(\xi_{m,j+1} - \mu_{y^*}\right)\right. \\ & \left. - \Phi\left(\xi_{mj} - \mu_{y^*}\right)\right], \end{aligned} \tag{4.2.6}$$

and the multivariate normal prior for $\boldsymbol{\zeta}_m$ stated in (4.1.24), the algorithm for one ordered vector of measurement, $\mathbf{y}_m$, during the (t)-th run of the complete sampler, can be described as follows: Minimise over the negative log-posterior $f_{\zeta\xi}\left(\boldsymbol{\zeta}_m|\boldsymbol{\xi}_m, \mu_{y^*}\right)$, i.e.

$$f_{\zeta\xi}\left(\boldsymbol{\zeta}_m|\boldsymbol{\xi}_m, \mu_{y^*}\right) \quad = \quad -\left[\log f_\xi(\xi_m|\mu_{y^*}) + \pi(\boldsymbol{\zeta}_m)\right] \tag{4.2.7}$$

with cumulative log-posterior

$$\log f_\xi(\xi_m|\mu_{y^*}) \quad = \quad \sum_{j=2}^{J_m} \log\left(\Phi\left(\xi_{m,j+1} - \mu_{y^*}\right) - \Phi\left(\xi_{m,j} - \mu_{y^*}\right)\right). \tag{4.2.8}$$

[25] An early version to estimate these bounding parameters, i.e. Albert & Chib (1993), was one of the first Metropolis-Hastings-based algorithms provided. It facilitates draws from an uniform interval, stemming from the full conditional of the parameters and a diffuse prior. As Cowles (1996) points out: this scheme results in bad or at least very slow mixing when the sample-size is large, because the bins within the intervals spanned by the ξs can be "overfull". Ways to overcome this problem are using informative priors (e.g., Cowles, 1996; Nikele & Fahrmeir, 1999) or additionally applying a Metropolis-based blocking strategy borrowing ideas of Liu (1994) but with some changes (again: Cowles, 1996).

The minimisation is carried only for $\zeta_{m,2}, \ldots, \zeta_{m,J_m-1}$, because $\zeta_{m,1}$ and ζ_{m,J_m} are fixed, obtaining an estimate $\hat{\boldsymbol{\zeta}}_m = (\hat{\zeta}_{m,2}, \ldots, \hat{\zeta}_{m,J_m-1})'$. Proposal values are drawn, i.e. $\boldsymbol{\zeta}^*_m = (\zeta^*_{m,2}, \ldots, \zeta^*_{m,J_m-1})'$, using a multivariate Student's t-density $f_{st}(\cdot|\cdot)$ conditional on respective parameters (see below), with moments $\hat{\boldsymbol{\zeta}}_m$ as the location, the scale matrix $\hat{\mathbf{S}}_\zeta$, i.e. the respective inverse Hessian estimated during optimisation from beforehand, and (externally specified) degrees of freedom ν_{st}. The proposals are now compared to retrieve an acceptance ratio for the Metropolis-step, with

$$acc_\zeta = \frac{\exp\left\{f_{\zeta\xi}\left(\boldsymbol{\zeta}_m|\boldsymbol{\xi}_m, \mu_{y^*}\right) - f_{\zeta\xi}\left(\boldsymbol{\zeta}^*_m|\boldsymbol{\xi}^*_m, \mu_{y^*}\right)\right\} \cdot f_{st}(\boldsymbol{\zeta}_m|\hat{\boldsymbol{\zeta}}_m, \hat{\mathbf{S}}_\zeta, \nu_{st})}{f_{st}(\boldsymbol{\zeta}^*_m|\hat{\boldsymbol{\zeta}}_m, \hat{\mathbf{S}}_\zeta, \nu_{st})}. \tag{4.2.9}$$

By comparing $u < acc_\zeta$, $u \sim Unif\,(0,1)$, $\boldsymbol{\zeta}^*_m$ and therefore $\xi^*_{m,j} = \sum_{j=1}^{J_m}\left\{\exp\left(\zeta^*_{m,j}\right)\right\}$, are accepted as new estimates; otherwise, the algorithm does not move and reuses values from the step before. Note that our incorporation of the algorithm, is able to reuse $\hat{\boldsymbol{\zeta}}_m$ and $\hat{\mathbf{S}}_\zeta$ from the most recent step, where the minimisation had gotten carried out, if the current optimisation can not find an appropriate minimum.

4.2.4 Drawing Regression Parameters or Item Difficulties

The posterior distribution to create regressions parameters, is similar to drawing them in a multivariate Bayesian linear model using a multivariate normal prior. While the factor model uses the multivariate normal draw for the vectorised matrix,

$$\begin{aligned} f(\mathrm{vec}(\boldsymbol{\beta})|\mathbf{Y}^*, \mathbf{X}, \boldsymbol{\Theta}, \boldsymbol{\Sigma}_\varepsilon, \boldsymbol{\alpha}) \propto \exp\Bigg\{-\frac{1}{2}\left(\mathrm{vec}(\boldsymbol{\beta}) - \boldsymbol{\mu}^\diamond_\beta\right)'\left(\boldsymbol{\Sigma}^\diamond_\beta\right)^{-1} \\ \left(\mathrm{vec}(\boldsymbol{\beta}) - \boldsymbol{\mu}^\diamond_\beta\right)\Bigg\}, \end{aligned} \tag{4.2.10}$$

the draws for the item difficulties are restricted to the vector $\boldsymbol{\beta}$, that is $\boldsymbol{\beta} = \mathrm{vec}(\boldsymbol{\beta})$. The parameters for the posterior moments are almost the same, i.e.

$$\begin{aligned} \boldsymbol{\Sigma}^\diamond_\beta =& \left\{\mathbf{V}'_\beta\left(\mathbf{I}_M \otimes \mathbf{X}\right)'\mathbf{E}'\left(\tilde{\boldsymbol{\Sigma}}_\varepsilon\right)^{-1}\mathbf{E}\left(\mathbf{I}_M \otimes \mathbf{X}\right)\mathbf{V}_\beta + \left(\boldsymbol{\Sigma}^0_\beta\right)^{-1}\right\}^{-1} \text{ and} \\ \boldsymbol{\mu}^\diamond_\beta =& \boldsymbol{\Sigma}^\diamond_\beta\left\{\mathbf{V}'_\beta\left(\mathbf{I}_M \otimes \mathbf{X}\right)'\mathbf{E}'\left(\tilde{\boldsymbol{\Sigma}}_\varepsilon\right)^{-1}\mathbf{E}\left(\mathrm{vec}\left(\mathbf{Y}^*\right) - \mathrm{vec}(\boldsymbol{\Theta}\boldsymbol{\alpha}')\right) + \left(\boldsymbol{\Sigma}^0_\beta\right)^{-1}\boldsymbol{\mu}^0_\beta\right\}, \end{aligned}$$

except that the matrix of covariates $\mathbf{X}$ in (4.2.10) needs to be replaced by the vector of negative ones, $-\boldsymbol{\iota}_N$. Additionally, in terms of depicting a consistent mathematical formulation,[26] $\mathbf{V}_\beta$ denotes a vector-permutation matrix (see Henderson & Searle, 1981,

[26] Note however, that in the early stages of the development, $\mathrm{vec}(\boldsymbol{\beta}')$, was used. Although the mathematical derivations now incorporate $\mathbf{V}_\beta$, current R-code implements $\mathrm{vec}(\boldsymbol{\beta}')$ and therefore, does not contain

Magnus & Neudecker, 1979, or Tracy & Dwyer, 1969 for details and characteristics) of dimension $MQ \times MQ$ with

$$\begin{aligned}
\operatorname{vec}(\boldsymbol{\beta}') &= \mathbf{V}_\beta \operatorname{vec}(\boldsymbol{\beta}) \\
\Leftrightarrow \mathbf{V}_\beta' \operatorname{vec}(\boldsymbol{\beta}') &= \operatorname{vec}(\boldsymbol{\beta}).
\end{aligned}$$

The necessity of using this vector-permutation stems from connecting the diverging element-order of the vectorised loadings from the model in (4.1.3), i.e. $\operatorname{vec}(\mathbf{X}\boldsymbol{\beta}') = (\mathbf{I}_M \otimes \mathbf{X})\operatorname{vec}(\boldsymbol{\beta}') = (\mathbf{I}_M \otimes \mathbf{X})\mathbf{V}_\beta \operatorname{vec}(\boldsymbol{\beta})$, with the prior's untransposed variant of (4.1.28).

Finally, because the influence of the covariates within the IRT representation is part of the latent factor specification, and the respective estimation is similarly carried out within the same augmentation scheme, posterior draws $\operatorname{vec}(\boldsymbol{\gamma})$, will be explained below.

4.2.5 Drawing Diagonal Elements of Σ_ε

With the help of the inverse-gamma distribution of (4.1.22), one is able to retrieve a posterior for every error term variance–corresponding to metric measurements m–, being

$$f(\sigma^2_{\varepsilon,m}|\mathbf{X}, \boldsymbol{\beta}_m, \mathbf{y}^*_m, \boldsymbol{\Theta}, \mathbf{E}^{mm}) \propto (\sigma^2_{\varepsilon,m})^{-c^N_m - 1} \exp\left(-\frac{C^N_m}{\sigma^2_{\varepsilon,m}}\right), \tag{4.2.11}$$

with

$$\begin{aligned}
c^N_m =& c^0_m + \frac{\sum_{i=1}^N o^Y_{im}}{2} \text{ and} \\
C^N_m =& C^0_m + \frac{1}{2}\left\{(\mathbf{y}^*_m - \mathbf{X}\boldsymbol{\beta}_m - \boldsymbol{\Theta}\boldsymbol{\alpha}_m)' (\mathbf{E}^{mm})'\mathbf{E}^{mm} (\mathbf{y}^*_m - \mathbf{X}\boldsymbol{\beta}_m - \boldsymbol{\Theta}\boldsymbol{\alpha}_m)\right\}.
\end{aligned}$$

As stated before, variances pertaining to measurements of binary and ordered scale are fixed to 1.

4.2.6 Posterior Draws for the Factor Loadings and Item Discrimination $\boldsymbol{\alpha}$ including Restrictions

Drawing the non-zero elements of the matrix $\boldsymbol{\alpha}$ that represent loadings within the (confirmatory) factor analysis and discrimination parameters within the IRT framework, is carried out with the help of a specific multivariate normal posterior. We integrate, as

vector-permutation.

already indicated when introducing the prior in (4.1.21), the ability to dedicate measurements to more than one factor simultaneously, as it is done for example in bifactor models (see Erosheva & Curtis, 2017), and sometimes coined as cross-loading (see amongst others Asparouhov et al., 2015[27]), via a vectorised notation of the matrices. Aside from specifying zero restrictions, i.e. the absence of loading, more easily, this allows to formalize the cross-loading as well. Additionally, identity restrictions and loading of several elements equally up to a linear transformed amount, e.g. $\alpha_{11} = 2 \cdot \alpha_{21} + 1$ as another possibility to specify identification, are employed. The latter parts use up to two algorithms: First, a duplication-and-elimination algorithm grounding on the information contained in the indicator matrix $\bar{\boldsymbol{\Delta}}$, and second, a conversion algorithm building on the posterior draws under equality and accomplishing the actual linear transformation.

The posterior itself for the restricted non-zero and unique values of the vectorised loading matrix, i.e. $\text{vec}(\boldsymbol{\alpha})^{\bar{\Delta}} = \mathbf{E}_{\bar{\Delta}}\text{vec}(\boldsymbol{\alpha})$ is

$$\begin{aligned} f(\text{vec}(\boldsymbol{\alpha})^{\bar{\Delta}}|\mathbf{Y}^*, \mathbf{X}, \boldsymbol{\beta}, \boldsymbol{\Theta}, \boldsymbol{\Sigma}_\varepsilon) \propto \exp &\left\{ -\frac{1}{2} \left(\text{vec}(\boldsymbol{\alpha})^{\bar{\Delta}} - \boldsymbol{\mu}_\alpha^\diamond \right)' (\boldsymbol{\Sigma}_\alpha^\diamond)^{-1} \right. \\ &\left. \left(\text{vec}(\boldsymbol{\alpha})^{\bar{\Delta}} - \boldsymbol{\mu}_\alpha^\diamond \right) \right\}, \end{aligned} \tag{4.2.12}$$

where the parameters are

$$\begin{aligned} \boldsymbol{\Sigma}_\alpha^\diamond =& \left\{ \mathbf{V}_\alpha' \mathbf{E}_{\bar{\Delta}}' \left(\mathbf{I}_M \otimes \boldsymbol{\Theta} \right)' \mathbf{E}' \left(\tilde{\boldsymbol{\Sigma}}_\varepsilon \right)^{-1} \mathbf{E} \left(\mathbf{I}_M \otimes \boldsymbol{\Theta} \right) \mathbf{E}_{\bar{\Delta}} \mathbf{V}_\alpha + (\boldsymbol{\Sigma}_\alpha^0)^{-1} \right\}^{-1} \quad \text{and} \\ \boldsymbol{\mu}_\alpha^\diamond =& \boldsymbol{\Sigma}_\alpha^\diamond \left\{ \mathbf{V}_\alpha' \mathbf{E}_{\bar{\Delta}}' \left(\mathbf{I}_M \otimes \boldsymbol{\Theta} \right)' \mathbf{E}' \left(\tilde{\boldsymbol{\Sigma}}_\varepsilon \right)^{-1} \mathbf{E} \left(\text{vec}(\mathbf{Y}^*) - \text{vec}(\mathbf{X}\boldsymbol{\beta}') \right) \right. \\ & \left. + (\boldsymbol{\Sigma}_\alpha^0)^{-1} \boldsymbol{\mu}_\alpha^0 \right\}. \end{aligned}$$

In this case, $\mathbf{E}_{\bar{\Delta}}$ is similar to the aforementioned matrix $\mathbf{E}_\Delta$ restricting $\text{vec}(\boldsymbol{\alpha})$ to non-zero elements, but also denoting the reduction to unique elements $\alpha_{mk} \neq 0$ with $\bar{\delta}_{mk} > 1$, $\forall m, k$. Like before, $\mathbf{V}_\alpha$ is a vector-permutation matrix of dimension $MK \times MK$ with $\text{vec}(\boldsymbol{\alpha}') = \mathbf{V}_\alpha \text{vec}(\boldsymbol{\alpha})$.

The elimination matrix $\mathbf{E}_{\bar{\Delta}}$ that allows to condense the estimation of the loadings onto unique elements (and if necessary re-expand these later on), can be constructed via the following algorithm:

(**A3S1**) Compose the vectorisation of $\bar{\Delta}$ and restrict this vectorisation onto non-zero elements, i.e. onto $\{\bar{\delta}_{mk} | \bar{\delta}_{mk} \neq 0\}, \forall m = 1, \ldots, M, k = 1, \ldots, K$,

[27]Although in their explorative setting something unwanted and the actually try to avoid.

obtaining a vector $\bar{\boldsymbol{\delta}}^{aux} = (\bar{\delta}_1^{aux}, \ldots, \bar{\delta}_g^{aux}, \ldots)$ and a copy $\bar{\boldsymbol{\delta}}^{aux}$.

(**A3S2**) Create an identity matrix of dimension $MK \times MK$, and remove the rows whose position correspond to non-zero elements in $\bar{\boldsymbol{\delta}}^{aux}$ to get the matrix $\mathbf{E}^{aux}$

(**A3S3**) Create an identity matrix $\mathbf{D}^{aux}$ with dimension $\mathfrak{D} \times \mathfrak{D}$. Here, $\mathfrak{D}$ describes as many rows and columns as there are non-zero elements in Δ and $\bar{\Delta}$ respectively, i.e. $\mathfrak{D} = \sum_{m=1}^{M} \sum_{k=1}^{K} \delta_{mk}$.

(**A3S4**) Create an empty matrix $\mathbf{E}^{aux2}$ with numbers of rows concurring to the number of elements of equality indicators and $\mathfrak{D}$ columns.

(**A3S5**) For every element u that is greater 1, where $u = 1, \ldots, \left[\max\left(\bar{\boldsymbol{\delta}}^{aux}\right) - 1\right]$:

(*A3S5a*) Take the row of $\mathbf{E}^{aux2}$ that corresponds to the current value of u. Set all row elements, whose position indicate the u-th equality restriction in $\bar{\boldsymbol{\delta}}^{aux}$, to the proportion of the amount of occurence of this restriction, i.e. to $\left[\sum_{g=1}^{\mathfrak{D}} \mathcal{I}_{\{u\}}\left(\bar{\boldsymbol{\delta}}_g^{aux}\right)\right]^{-1}$.

(*A3S5b*) If $u > 1$, remove all further same-valued elements in $\bar{\boldsymbol{\delta}}^{aux2} \neq 1$.

(*A3S5c*) Find the positions of the equality restriction within the previously reduced vector $\bar{\boldsymbol{\delta}}^{aux2}$ (or within the very first version of $\bar{\boldsymbol{\delta}}^{aux2}$, if $u = 1$).

(*A3S5d*) Copy the u-th row of $\mathbf{E}^{aux2}$ in the row of $\mathbf{D}^{aux}$ that matches the u-th equality restriction in $\bar{\boldsymbol{\delta}}^{aux2}$. Afterwards, keep the row that corresponds to the first appearance of the u-th element and remove all later appearances, getting a reduced matrix $\mathbf{D}^{aux}$.

(**A3S6**) Return the actual elimination matrix $\mathbf{E}_{\bar{\Delta}} = \mathbf{D}^{aux}\mathbf{E}^{aux}$ and $\mathbf{D}_{\bar{\Delta}} = \mathbf{E}'_{\bar{\Delta}} = (\mathbf{D}^{aux}\mathbf{E}^{aux})'$, i.e. the actual duplication matrix.

When this algorithm gets carried out, and $\text{vec}(\boldsymbol{\alpha})^{\bar{\Delta}}$ is drawn, the values for $\text{vec}(\boldsymbol{\alpha})$ are retrieved by applying $\mathbf{D}_{\bar{\Delta}}$, such that $\text{vec}(\boldsymbol{\alpha}) = \mathbf{D}_{\bar{\Delta}}\text{vec}(\boldsymbol{\alpha})^{\bar{\Delta}}$.

If instead or additionally, elements have linear restrictions, they get equally changed by the algorithm above, but are further altered with the next method: Let $\boldsymbol{\alpha}_e$ be a vector of N_α elements, being in a linear relationship with a second set of vector elements, i.e. the equations $\boldsymbol{\alpha}_{lin} = slopes \odot \boldsymbol{\alpha}_e + intercepts$ with vectors containing individual slopes (simply denoted with "*slopes*" and "*intercepts*", and including slopes of 1 and intercepts of 0, i.e. no transformation), and $\odot$ represents entry-wise multiplication / the Hadamard product. In order to obtain a suited set of loadings, $\boldsymbol{\alpha}_e$ won't get estimated directly and the linearly transformed. Instead, the aforementioned duplication and elimination algorithm for equality restriction is the basis, and the (equal) loading created there, e.g. an $\boldsymbol{\alpha}_d$, will get split according to the desired conversion. Hence, we set $\boldsymbol{\alpha}_d = \boldsymbol{\alpha}_e$ and gain $\boldsymbol{\alpha}_{lin}$ with the help of

$$\boldsymbol{\alpha}_{lin} = slopes \odot \left(\frac{N_\alpha \cdot \boldsymbol{\alpha}_e - \sum_{i=1}^{M \cdot K} intercepts_i}{\sum_{i=1}^{M \cdot K} slopes_i} \right) + intercepts. \tag{4.2.13}$$

4.2.7 Latent Variables, their Correlations and Covariate Parameters in IRT

Within this subsection, several draws are under consideration–these are draws for the latent variables, establishing the respective correlation matrix, and possible draws of covariate influence $\boldsymbol{\gamma}$, when dealing with IRT. For the factor analytic part, we follow Conti et al. (2014), but for the IRT variant–that employs background information $\mathbf{Z}$ and parameters $\boldsymbol{\gamma}$–, their equations get extended.

In order to create proposals for latent variables correlations, one option is the aforementioned separation strategy, i.e. $\mathbf{R} = \boldsymbol{\Lambda}^{-\frac{1}{2}} \boldsymbol{\Omega} \boldsymbol{\Lambda}^{-\frac{1}{2}}$, where $\boldsymbol{\Lambda} = \operatorname{diag}(\boldsymbol{\Omega})$. Thus, $\boldsymbol{\Omega}$ can be drawn based on a conditional inverse-Wishart distribution. However, the latent variables of $\boldsymbol{\Theta}$ and conditioning parameters $\boldsymbol{\alpha}$ (and $\boldsymbol{\gamma}$ if applicable) are related to $\mathbf{R}$ not $\boldsymbol{\Omega}$. A solution is to expand these parameters and variables with respective variances $\boldsymbol{\Lambda}$, i.e.

$$\ddot{\boldsymbol{\theta}}_i = \boldsymbol{\theta}_i \boldsymbol{\Lambda}^{\frac{1}{2}},\ \ddot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \boldsymbol{\Lambda}^{-\frac{1}{2}},\ \ddot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \boldsymbol{\Lambda}^{\frac{1}{2}} \tag{4.2.14}$$

Now these expanded latent variables can get updated for each row, that is

$$f(\ddot{\boldsymbol{\theta}}_i | \mathbf{y}_i^*, \mathbf{x}_i, \boldsymbol{\beta}, \ddot{\boldsymbol{\alpha}}, \boldsymbol{\Sigma}_\varepsilon, \boldsymbol{\Omega}, \mathbf{z}_i) \propto \exp\left\{ -\frac{1}{2} \left(\ddot{\boldsymbol{\theta}}_i - \boldsymbol{\mu}_\theta^\diamond \right)' (\boldsymbol{\Sigma}_\theta^\diamond)^{-1} \left(\ddot{\boldsymbol{\theta}}_i - \boldsymbol{\mu}_\theta^\diamond \right) \right\} \tag{4.2.15}$$

with

$$\boldsymbol{\Sigma}_\theta^\diamond = \left\{ \ddot{\boldsymbol{\alpha}}' \mathbf{E}^{ii,\prime} \mathbf{E}^{ii} \boldsymbol{\Sigma}_\varepsilon^{-1} \mathbf{E}^{ii,\prime} \mathbf{E}^{ii} \ddot{\boldsymbol{\alpha}} + \boldsymbol{\Omega}^{-1} \right\}^{-1}$$

$$\boldsymbol{\mu}_\theta^\diamond = \begin{cases} \boldsymbol{\Sigma}_\theta^\diamond \left\{ \ddot{\boldsymbol{\alpha}}' \mathbf{E}^{ii,\prime} \mathbf{E}^{ii} \boldsymbol{\Sigma}_\varepsilon^{-1} \mathbf{E}^{ii,\prime} \left(\mathbf{E}^{ii} \mathbf{y}_i^* - \mathbf{E}^{ii} \boldsymbol{\beta} \mathbf{x}_i \right) \right\}, & \text{in the factor setting} \\ \boldsymbol{\Sigma}_\theta^\diamond \left\{ \ddot{\boldsymbol{\alpha}}' \mathbf{E}^{ii,\prime} \mathbf{E}^{ii} \boldsymbol{\Sigma}_\varepsilon^{-1} \mathbf{E}^{ii,\prime} \left(\mathbf{E}^{ii} \mathbf{y}_i^* - \mathbf{E}^{ii} \boldsymbol{\beta} \mathbf{x}_i \right) + \boldsymbol{\Omega} \ddot{\boldsymbol{\gamma}} \mathbf{z}_i \right\}, & \text{within the IRT setting,} \end{cases}$$

and where $\mathbf{E}^{ii}$ is the elimination matrix with $\mathbf{E}^{ii} \mathbf{y}_i = \mathbf{y}_i^{obs}$ from above. Afterwards, the diagonal elements of the scale matrix $\mathbf{S}_\Omega^0 = \operatorname{diag}(s_1, \ldots, s_K)$, needed for the factor covariance receive an update from a gamma distribution's draw (with diagonal element $(\mathbf{\Omega})_{kk} = (\mathbf{\Lambda})_k$), i.e.

$$f(s_k|\mathbf{\Omega}) \quad \propto \quad (s_k)^{\frac{\nu_\Omega^0+1}{2}-1} \exp\left[-\frac{(\mathbf{\Lambda}_k)^{-1} + \{A_k^2(\nu_\Omega^0 - K + 1)\}^{-1}}{2} s_k\right]. \tag{4.2.16}$$

resulting in the same dimensioned matrix $\mathbf{S}_\Omega^\diamond$ with these draws as new diagonal elements. The covariance matrix gets updated next by

$$f(\mathbf{\Omega}|\mathbf{S}_\Omega^0, \ddot{\theta}) \quad \propto \quad |\mathbf{\Omega}|^{-\frac{\nu_\Omega^\diamond+K+1}{2}} |\mathbf{S}_\Omega^\diamond|^{\frac{\nu_\Omega^\diamond}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_\Omega^\diamond \mathbf{\Omega}^{-1}\right)\right\} \tag{4.2.17}$$

where

$$\begin{aligned} \nu_\Omega^\diamond &= \nu_\Omega^0 + N \\ \mathbf{S}_\Omega^\diamond &= \begin{cases} \mathbf{S}_\Omega^0 + \ddot{\mathbf{\Theta}}'\ddot{\mathbf{\Theta}}, & \text{in the factor setting} \\ \mathbf{S}_\Omega^0 + \left(\ddot{\mathbf{\Theta}} - \mathbf{Z}\ddot{\gamma}\right)'\left(\ddot{\mathbf{\Theta}} - \mathbf{Z}\ddot{\gamma}\right), & \text{within the IRT setting.} \end{cases} \end{aligned}$$

Here, the diagonal elements of this newly create covariance establish a new $\mathbf{\Lambda} = \mathrm{diag}(\mathbf{\Omega})$.

When it comes to background information on latent variables during item response estimation and parametrised with the help of $\ddot{\boldsymbol{\gamma}}$,

$$f(\mathrm{vec}(\ddot{\gamma})|\mathbf{\Theta}, \mathbf{\Omega}) \quad \propto \quad \exp\left\{-\frac{1}{2}\left(\mathrm{vec}(\ddot{\gamma}) - \boldsymbol{\mu}_{\ddot{\gamma}}^\diamond\right)'\left(\mathbf{\Sigma}_{\ddot{\gamma}}^\diamond\right)^{-1}\left(\mathrm{vec}(\ddot{\gamma}) - \boldsymbol{\mu}_{\ddot{\gamma}}^\diamond\right)\right\}, \tag{4.2.18}$$

where

$$\begin{aligned} \mathbf{\Sigma}_{\ddot{\gamma}}^\diamond &= \left\{(\mathbf{I}_K \otimes \mathbf{Z})'(\mathbf{\Omega} \otimes \mathbf{I}_N)^{-1}(\mathbf{I}_K \otimes \mathbf{Z}) + \left(\mathbf{\Sigma}_{\ddot{\gamma}}^0\right)^{-1}\right\}^{-1} \quad \text{and} \\ \boldsymbol{\mu}_{\ddot{\gamma}}^\diamond &= \mathbf{\Sigma}_{\ddot{\gamma}}^\diamond\left\{(\mathbf{I}_K \otimes \mathbf{Z})'(\mathbf{\Omega} \otimes \mathbf{I}_N)^{-1}\mathrm{vec}(\mathbf{\Theta}) + \left(\mathbf{\Sigma}_{\ddot{\gamma}}^0\right)^{-1}\boldsymbol{\mu}_{\ddot{\gamma}}^0\right\}, \end{aligned}$$

is incorporated.[28] In fact, these covariate parameters will get drawn before the creation of conditional random proposal for $\mathbf{\Omega}$. Finally, the back transformation takes place, as Conti et al. (2014) describe it, extended with respect to $\ddot{\boldsymbol{\gamma}}$ and thus $\boldsymbol{\gamma}$:

$$\boldsymbol{\theta}_i = \ddot{\boldsymbol{\theta}}_i\mathbf{\Lambda}^{-\frac{1}{2}},\ \mathbf{R} = \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{\Omega}\mathbf{\Lambda}^{-\frac{1}{2}},\ \boldsymbol{\gamma} = \ddot{\boldsymbol{\gamma}}\mathbf{\Lambda}^{-\frac{1}{2}}. \tag{4.2.19}$$

[28]Next to these posterior specifications, inclusion of two prior draws, i.e. the first one depicted in formula (4.1.27) and, the seocnd one, an inverse-Gamma prior for each variance element $(\mathbf{\Omega})_{kk}$ of $\mathbf{\Lambda}$, i.e. $f((\mathbf{\Lambda})_k) = f((\mathbf{\Omega})_{kk}) \propto ((\mathbf{\Omega})_{kk})^{-0.5\cdot\nu_\Omega^0-1} \exp\left[-2\cdot(s_{\Omega,k}^0)^{-1}\cdot(\mathbf{\Omega})_{kk}\right]$ within the Gibbs algorithm adds additional variability.

This the last step, of the core of the estimation algorithm. The next step is concerned with sign-changes, after all draws are made.

4.2.8 Addressing Sign-Changes in $\boldsymbol{\alpha}$, $\mathbf{R}$ and $\boldsymbol{\gamma}$

As was stated in section (2.3), without restrictions, the products of latent variables and respective parameters, i.e. $\boldsymbol{\Theta}\boldsymbol{\alpha}'$ result in sign-switching, in terms of having a the same results for the product, but the possibility of flipped signs for individual components in contrast to true values. With respect to variability and bias, the effects can be severe for MCMC-based sampling. Figure 4.1, being an a example of sign-changes during estimation, gives an illustration of the behaviour.

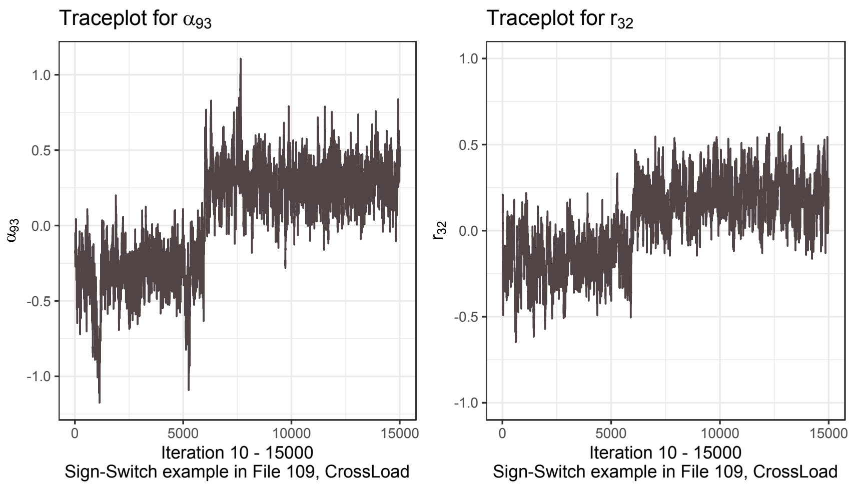

Figure 4.1: Example of Sign-Switching in loading α_{93} and factor correlation r_{32} of complete-case factor analysis with cross-loading (see Aßmann et al., 2016, and Erosheva & Curtis, 2017, for further depictions of sign problems).

As can be seen, the possibility of two solutions, sometimes creates reflected draws above (or below) the zero line. This can result in an increased variability and a posterior mean estimate of the randomly created values that is closer to zero, even if the true value isn't. This phenomenon is known as sign-switching (see Conti et al., 2014), or sign-changing and sign-invariance respectively (see Erosheva & Curtis, 2017). The latter authors further argue that inclusion of restrictions for the parameter space prior to estimation, results might still be problematic in terms of posterior mode changes. Both of these articles mentioned, use techniques processing the MCMC chains after their creation. The methods they apply, which address the reflection, consider derivations in terms of an approach by Stephens (2000) for label-switching in mixture modelling. But, Conti et al.'s (2014) model does not contain cross-loading, in contrast, Erosheva and Curtis' (2017) approach does, but only in an orthogonal setting without latent variables' correlation–thus, combining both ideas might me a suited compromise.

The approach we start with, is the procedure derived by Erosheva & Curtis (2017). Their algorithm is a simplification and adaption towards signs of loading matrices, relying on Stephens' (2000) method. Using post-processing after Gibbs sampling and the posterior draws for the non-zero loadings in $\boldsymbol{\alpha}$, Erosheva & Curtis (2017) use a Bayesian risk approach in order to estimate appropriate sign change indicators. The algorithm can be described as follows:

> Given (T), $(t) = 1, \ldots, (T)$, draws from the Gibbs sampler, ending in (T) $M \times K$ matrices of loadings or item discrimination parameters $\boldsymbol{\alpha}$ and $K \times K$, $k = 1, \ldots, K$, latent variables' correlations:

(**A4S1**) Initialise an indicator matrix $\boldsymbol{\Delta}^{aux3} = \left[\delta^{aux3}_{(t)k}\right]$, of dimension $(T) \times K$ with values dependent on user's choice, i.e.

- Either draw $(T) \cdot K$ times randomly from elements $\{\{-1\}, \{1\}\}$, based on a Bernoulli-trial with probability 0.5, or
- for every row (t) and column k in $\boldsymbol{\Delta}^{aux3}$ use the sign (positive: 1, negative -1) of the sum of non-zero elements corresponding to the k-th latent variable of a draw (t) of $\boldsymbol{\alpha}$.

Thus, the value for every element $\delta^{aux3}_{(t)k}$ is either -1 or $+1$, indicating negative or positive signs for the α_{mk}s.

(**A4S2**) Estimate, corresponding to every latent variable $\boldsymbol{\theta}_k$ and non-zero loadings / item discrimination the mean vector $\hat{\boldsymbol{\mu}}_{\alpha(T)k}$ and standard deviation vector $\hat{\boldsymbol{\sigma}}_{\alpha(T)k}$ for the (T) draws of the $\boldsymbol{\alpha}$s, with elements

$$\begin{aligned}
\hat{\mu}_{\alpha(T)mk} =& \frac{1}{(T)} \sum_{(t)=1}^{(T)} \alpha^{(t)}_{mk} \\
\widehat{\sigma^2}_{\alpha(T)mk} =& \frac{1}{(T)-1} \sum_{(t)=1}^{(T)} \left(\alpha^{(t)}_{mk} - \hat{\mu}_{\alpha(T)mk}\right)^2
\end{aligned}$$

(**A4S3**) For every (t), calculate the negative loss for every non-zero element in $\boldsymbol{\alpha}$ corresponding to one latent variable $\boldsymbol{\theta}_k$, α_{mk}, based on the log of a normal probability density, i.e.

$$\phi\left(\frac{\delta_{(t)k}^{aux3}\cdot\alpha_{mk}-\hat{\mu}_{\alpha(T)mk}}{\widehat{\sigma^2}_{\alpha(T)mk}}\right), \text{ and therefore}$$

$$\ell_k^{0(t)} = -\sum_{m=1}^{M}\mathcal{I}_{\{1\}}(\delta_{mk})\cdot\log\left\{\phi\left(\frac{\delta_{(t)k}^{aux3}\cdot\alpha_{mk}-\hat{\mu}_{\alpha(T)mk}}{\widehat{\sigma^2}_{\alpha(T)mk}}\right)\right\}.$$

(**A4S4**) Iterate–as long as a user-specified maximal number of iterations $maxIter$, $iter = 1,\ldots,maxIter$, is reached or sign-change estimates between iterations do not differ anymore, i.e. $\sum_{(t)=1}^{(T)}\sum_{k=1}^{K}\left|\delta_{(t)k}^{aux3,iter-1}-\delta_{(t)k}^{aux3,iter}\right| = 0$–over

($A4S4a$) Calculate a second loss, based on changed signs, i.e. $-\delta_{(t)k}^{aux3}$ instead of $\delta_{(t)k}^{aux3}$, where the loss is

$$\ell_k^{iter,(t)} = -\sum_{m=1}^{M}\mathcal{I}_{\{1\}}(\delta_{mk})\cdot\log\left\{\phi\left(\frac{-\delta_{(t)k}^{aux3}\cdot\alpha_{mk}-\hat{\mu}_{\alpha(T)mk}}{\widehat{\sigma^2}_{\alpha(T)mk}}\right)\right\}$$

($A4S4b$) Compare $\ell^{0(t)}$ and $\ell^{iter,(t)}$ and change $\delta_{(t)k}^{aux3}$ and $\ell^{0(t)}$ accordingly, i.e.

$$\delta_{(t)k} = \begin{cases} -\delta_{(t)k}^{aux3} & \text{, if } \ell^{0(t)} > \ell^{iter,(t)} \\ \delta_{(t)k}^{aux3} & \text{, otherwise,} \end{cases}$$

and

$$\ell^{0(t)} = \begin{cases} \ell^{iter(t)} & \text{, if } \ell^{0(t)} > \ell^{iter,(t)} \\ \ell^{0(t)} & \text{, otherwise.} \end{cases}$$

($A4S4c$) Based on probable changes in $\delta_{(t)k}^{aux3}$, re-estimate the mean vector $\hat{\boldsymbol{\mu}}_{\alpha(T)k}$ and standard deviation vector $\hat{\boldsymbol{\sigma}}_{\alpha(T)k}$ for the (T) draws of the $\boldsymbol{\alpha}$s.

This procedure allows handling of sign changes–by forcing unimodality–occurring during estimation, i.e. when the Gibbs draws change the sign after some iterations (resulting in bi- or even multimodality, when several estimators are observed at once). But one should be cautious, because multimodality is not always the case and the algorithm is not able to find the correct sign as such. The latter is still a decision the analyst needs to make.

The above algorithm from Erosheva & Curtis (2017), does however–due to design–not include considerations for the latent variables' correlation, and covariate coefficients for the IRT structure. The correlations are instead considered by Conti et al. (2014), i.e. changes in $\boldsymbol{\alpha}$ co-occur in $\boldsymbol{\Theta}$ and thus the respective correlations $\mathbf{R}$ (influencing directions of the latent variable vectors). In order to include a remedy for sign-switching with respect to $\mathbf{R}$, the updated final row of $\boldsymbol{\Delta}^{aux3}$ is in use, that is $\boldsymbol{\delta}_{(t)}^{aux3,maxIter} = (\delta_{(t)1}^{aux3,maxIter}, \ldots, \delta_{(t)K}^{aux3,maxIter})$ and each draw (t) of $\mathbf{R}$ is recalculated via

$$\mathbf{R}^{(t)} = \left(\boldsymbol{\delta}_{(t)}^{aux3,maxIter} \cdot \boldsymbol{\delta}_{(t)}^{aux3,maxIter,\prime}\right) \odot \mathbf{R}^{(t)}. \tag{4.2.20}$$

For coefficients of background information $\boldsymbol{\gamma}$ in IRT, $\boldsymbol{\delta}_{(t)}^{aux3,maxIter}$ can again be applied via

$$\boldsymbol{\gamma}^{(t)} = \left(\boldsymbol{\iota}_Q \cdot \boldsymbol{\delta}_{(t)}^{aux3,maxIter,\prime}\right) \odot \boldsymbol{\gamma}^{(t)}. \tag{4.2.21}$$

In order to test the complete model estimation presented, code for the statistical software R was created and simulation studies for IRT and factor analysis were conducted, where investigated scenarios and retrieved results are depicted in the next chapter.

Chapter 5

Simulation Study

The upcoming chapter is concerned with the implementation of the proposed algorithm in R and a test under given and known circumstances. This means, the statistical accuracy of the Bayesian estimation including handling missing values in covariates $\mathbf{X}$ and $\mathbf{Z}$ respectively is investigated, and the following simulation studies with replicated data sets are used for this purpose: Both model variants are considered, i.e. the confirmatory factor model (F) and the IRT formulation (I) and test their correct implementation under known information, i.e. we start with data without missing values incorporated (before deletion, BD).

In order to learn about the strengths and shortcomings of the data completion procedure when using CART, the next step is estimation when missing values are present (see for an overview of missing-data mechanisms Little & Rubin, 2002, p. 12). First, a missing completely-at-random mechanism (MCAR) is considered to govern the absence of values, such that missingness within a variable is purely random and not related to the variable itself or any other variable. A comparison of complete-case analysis' results with estimate properties of imputed values will give us first insights on the quality of the imputation as such. During the design-phase of the simulation study for the factor analytic and the IRT model, an additional missing-at-random mechanism (MAR) is considered next, i.e. missingness in one variable is governed by a or more parts of other variables. But, the first results employing MCAR already revealed problems for the estimates when conducting imputation in the factor analytic setting. Therefore, we instead focussed on MAR for the IRT model only, and concentrated on different MCAR scenarios and data situations in the factor analytic case, to investigate the difficulties in more detail. The, to be described results of the additional scenarios for factor analysis, suggested that having missing values created via MCAR and using the CART-imputation respectively, has a negative effect on the results. For changes in sample-size, number of measurements and their scales, improvements were found only in parts–however, performance never reached acceptable results. Thus, we ended pursuing the path applying CART-imputation for covariates in the factor model at the MCAR-stage and the more complex missing-at-random mechanism (MAR) was only applied to IRT.

5.1 R-Code and Test Environment

The steps of the algorithm and the diagnostics find their implementation in the statistical programming language R and partially in C++ (using *Rcpp,* Eddelbuettel et al., 2016, and *RcppArmadillo*, Eddelbuettel & Sanderson, 2014). Three central functions were created, conducting the estimation, post-processing with regards to sign-switching and calculation of measures of diagnostics.

The main estimation routine–with more than 1700 physical lines of code–contains over twenty-five sub-routines for each of the steps given above–e.g. the multivariate normal draws for the loadings or item discrimination parameters, additional posterior draws and further pre-processing purposes. Two of these sub-routines, the imputation step and Albert & Chib's (1997) approach for ordered scaled variables' cut-off values, contain code based on *mice.impute.cart* (Doove et al., 2015) in the first case and, in the second case, code based on the implementation of Jean-Christoph Gaasch (2017b) for the latters Metropolis-Hastings algorithm. Fifty-three arguments allow to specify not only the data, prior hyper-parameters and starting values, but also additional options controlling further aspects of the code. The output this main routine creates, is a list-type object containing all draws for the parameters, means and covariances of the covariates, estimated plausible values, i.e. $\hat{\boldsymbol{\Theta}}$ and additional informations. Further details on in- and outputs are presented in Appendix C on page 188.

Post-processing with regard to sign-switching is based on a wrapper function. It, first, prepares the outputs of the main function in order to work with the sign-switching routine *relabel.matrix* of the R-package *relabeLoadings* (Curtis & Erosheva, 2016, see additionally Erosheva & Curtis, 2017, for details and section 4.2.8 above), second, creates results based on this R routine, which, third, adjusts the draws of the loading and item discrimination parameters respectively, the latent variables' correlations and the regression coefficients in IRT, if applicable.

The diagnostics routine uses the outputs from the main routine or the relabelling wrapper in order to calculate several estimates, encompassing the posterior mean estimate, it's standard deviation, central posterior regions and so forth. Details of the calculations are described during the evaluation of the results on page 75.

Two different environments are used in order to conduct the entire analysis. First, data sets get created locally on a standard laptop running R-Version 3.4.1 on Windows 7. Second, these files are transferred to a SLURM-operated Linux cluster, running SuSE Linux Enterprise, hosted by the "Leibniz-Rechenzentrum (LRZ) der Bayerischen Akademie der Wissenschaften" (see Yoo et al., 2003, for SLURM and Weinberg, 2015, for information on the Linux distribution used). It allows to conduct 28 parallel estimations running R-3.3.1 (up until October 2018) and, due to changes made by the LRZ, R-3.3.2 afterwards. Because of these changes, parts of the calculation needed adaptions in the version of auxiliary R-packages–details are again specified in Appendix C on page 188. Finally, after the results are collected, they get re-transferred to the laptop and evaluation of the outputs are carried out locally.

5.2 Data Generation and Prior-specification for Estimation

While a general description of the parameter and data generating processes are given here, specific details of true values can be found in Appendix D on page 201 (these are the first columns of results of the simulation studies, denoted with ψ). In order to create several data sets for testing, a (for all data sets) fixed set of parameters is created via one respective draw for each parameter type. However, the indicator matrix of the loading structure and the cut-off parameters weren't drawn but set to specific values. The parameter values retrieved, are then the basis for the actual data creation, following the upcoming model structures.

We begin with creating a dedicated specification, i.e.$\mathbf{\Delta}_1$, and move on to cross-loading, i.e. $\mathbf{\Delta}_2$, and use also a cross-loading $\bar{\mathbf{\Delta}}$ (similar to $\mathbf{\Delta}_2$) but with an equality restriction, which is denoted with 2 in the upcoming depictions–however note, it is always the case that at least three measurements load onto a latent variable:

$$\boldsymbol{\Delta}_1 = \begin{pmatrix} 1 & 0 & 0 \\ \vdots & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \vdots & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \vdots \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\Delta}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \bar{\boldsymbol{\Delta}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Non-zero elements of $\boldsymbol{\alpha}$, following $\boldsymbol{\Delta}_1$, $\boldsymbol{\Delta}_2$ and $\bar{\boldsymbol{\Delta}}$, are drawn with the same scheme used by Conti et al. (2014), which we also applied to elements of $\boldsymbol{\beta}$ (including draws for an intercept, if needed) and $\boldsymbol{\gamma}$ respectively. Hence, one has

$$\begin{aligned} \alpha_{mk}^{\Delta} &= (-1)^{\phi_{mk}}\sqrt{a_{mk}^{*}}, \text{ where } a_{mk}^{*} \sim Unif(0.04, 0.64) \text{ and } \phi_{mk} \sim Bernoulli(0.5), \\ \beta_{mq} &= (-1)^{\phi_{mk}}\sqrt{b_{mq}^{*}}, \text{ where } b_{mq}^{*} \sim Unif(0.04, 0.64) \text{ and } \phi_{mq} \sim Bernoulli(0.5), \\ \gamma_{kq} &= (-1)^{\phi_{kq}}\sqrt{g_{kq}^{*}}, \text{ where } g_{kq}^{*} \sim Unif(0.04, 0.64) \text{ and } \phi_{kq} \sim Bernoulli(0.5). \end{aligned}$$

Similarly based on Conti et al. (2014), the latent variables correlations $\mathbf{R}$ are created with the help of an inverse-Wishart draw via retransformed covariances $\boldsymbol{\Omega}$ for K latent variables, i.e.

$$\begin{aligned} &\boldsymbol{\Omega} \sim InvWish(K + 5, \mathbf{I}_K), \\ &\boldsymbol{\Lambda} = \text{diag}(\boldsymbol{\Omega}), \\ &\mathbf{R} = \boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-\frac{1}{2}} \end{aligned}$$

For metric measurements corresponding to diagonal elements of $\boldsymbol{\Sigma}_{\varepsilon}$, idiosyncratic variances are created via

$$\sigma_{\varepsilon,m}^2 \sim Unif(0.2, 0.8),$$

while otherwise, they are fixed to 1. Fixed to specific values are, as stated above, also the cut-off or threshold parameters $\boldsymbol{\xi}_m$ for binary and ordered scaled measurements. All of them are set to either $(-\infty, 0, +\infty)$ in the dichotomous case or $(-\infty, 0, 0.75, 1.25, +\infty)$ for ordered scaled measurements.

In a next step, covariates $\mathbf{X}$ or $\mathbf{Z}$ (with or without a column of ones for an intercept), latent variables $\boldsymbol{\Theta}$ and error terms $\boldsymbol{\varepsilon}$ get created using multivariate normal draws, i.e.

$$\begin{aligned}
\text{vec}(\mathbf{X}) &\sim MVN\left(\text{vec}(\mathbf{0}), \boldsymbol{\Sigma}_X \otimes \mathbf{I}_N\right) \text{ or}\\
\text{vec}(\mathbf{Z}) &\sim MVN\left(\text{vec}(\mathbf{0}), \boldsymbol{\Sigma}_Z \otimes \mathbf{I}_N\right),\\
\text{vec}(\boldsymbol{\Theta}) &\sim MVN\left(\text{vec}(\boldsymbol{\mu}_\theta), \mathbf{R} \otimes \mathbf{I}_N\right),\\
\text{vec}(\boldsymbol{\varepsilon}) &\sim MVN(\text{vec}(\mathbf{0}), \boldsymbol{\Sigma}_\varepsilon \otimes \mathbf{I}_N).
\end{aligned}$$

For scenarios with three covariates, the covariances in creating them, are

$$\boldsymbol{\Sigma}_X = \boldsymbol{\Sigma}_Z = \begin{pmatrix} 1.0 & 0.5 & 0.3 \\ 0.5 & 1.0 & 0.5 \\ 0.3 & 0.5 & 1.0 \end{pmatrix}.$$

For additional factor analytic models with ten covariates, we employ

$$\boldsymbol{\Sigma}_X = \begin{pmatrix} 1.00 & 0.50 & 0.30 & 0.10 & 0.21 & 0.01 & -0.10 \\ 0.50 & 1.00 & 0.50 & -0.53 & 0.20 & 0.31 & 0.05 \\ 0.30 & 0.50 & 1.00 & 0.20 & 0.07 & 0.23 & -0.11 \\ 0.10 & -0.53 & 0.20 & 1.00 & 0.30 & 0.21 & 0.01 \\ 0.21 & 0.20 & 0.07 & 0.30 & 1.00 & 0.30 & 0.13 \\ 0.01 & 0.31 & 0.23 & 0.21 & 0.30 & 1.00 & -0.01 \\ -0.10 & 0.05 & -0.11 & 0.01 & 0.13 & -0.01 & 1.00 \end{pmatrix}$$

and the covariance matrix for only two independent variables is

$$\boldsymbol{\Sigma}_X = \begin{pmatrix} 1.0 & 0.3 \\ 0.3 & 1.0 \end{pmatrix}.$$

Because of the different possible structures for covariates and how they might have an influence on $\mathbf{Y}^*$ (i.e. directly in the factor analytic specification or indirectly via $\boldsymbol{\Theta}$ in the the IRT case), the expectation $\boldsymbol{\mu}_\theta$ of $\boldsymbol{\Theta}$ varies. For the factor analytic model, we have $\boldsymbol{\mu}_\theta = \mathbf{0}$, and in the IRT formulation it is $\boldsymbol{\mu}_\theta = \mathbf{Z}\boldsymbol{\gamma}$. With the help of these parameters and variables, $\mathbf{Y}^*$ gets created, using the already known relationships

$$
\begin{aligned}
\text{(FA): } \mathbf{Y}^* =& \mathbf{X}\boldsymbol{\beta}' + \boldsymbol{\Theta}\boldsymbol{\alpha}' + \boldsymbol{\varepsilon} \text{ or} \\
\text{(IRT): } \mathbf{Y}^* =& -\boldsymbol{\beta} + \boldsymbol{\Theta}\boldsymbol{\alpha}' + \boldsymbol{\varepsilon} \text{ and} \\
\boldsymbol{\Theta} =& \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\upsilon}.
\end{aligned}
$$

Binary and ordered scaled measurements in $\mathbf{Y}$ are now created by splitting some of the columns of $\mathbf{Y}^*$–those who correspond to measurements that should not be of metric scale–using the aforementioned cut-off parameters.

In order to investigate the accuracy of the results, the data generation gets replicated to retrieve 125 before deletion data sets. With 100 replications it would be questionable, whether to interpret the third position after the decimal point, e.g. for coverage rates. With some more replications, at least a tendency with respect of rounding of the second decimal place–given the third–seems to be somewhat given. If now missingness is under investigation, it gets imposed onto each of these before deletion data sets via a MCAR or an MAR mechanism on covariates $\mathbf{x}_1$ or $\mathbf{x}_2$. MCAR is created with the help of a Bernoulli-trial based indicator, with a fixed success probability (i.e. probability of missingness) of 15%. For the missing-at-random mechanism, again a Bernoulli-trial is used, but the success probability is, first, dependent from the values of $\mathbf{x}_3$. Except for the final scenario, the MAR mechanism is based on $\Phi(\mathbf{x}_3 - q_{0.95}(\mathbf{x}_3))$–that is: the success probability is created with using a 95%-quantile-shifted version of $\mathbf{x}_3$ as an input to the normal distribution. This results in a probability of missingness for a variable under consideration with an (linear) increase in values of $\mathbf{x}_3$. The next scenario of IRT uses, second, a cubic influence of $\mathbf{x}_3$, with $\Phi(\mathbf{x}_3^3 - q_{0.95}(\mathbf{x}_3))$, such that the higher values result in even more probability of missingness. Finally, an additional influence for missingness in $\mathbf{x}_2$ is included, which is $\Phi(\boldsymbol{\theta}_3^2 - \mathbf{x}_3^2 - q_{0.95}(\mathbf{x}_3) - q_{0.95}(\boldsymbol{\theta}_2))$, and furthermore: Variables $\mathbf{y}_2$ to $\mathbf{y}_9$ have 10% of MCAR missingness (altogether).

The priors are conjugate to the posterior distributions and presented in section 4.1. For hyperparameters and further specifications that were included for estimation, the setup described in table 5.1 is used, startvalues were drawn from respective prior distributions and variable processes similar to those from above, and all scenarios investigated are presented in table 5.2.

Prior specifications:		
$\sigma^2_{\varepsilon,m} \sim InvGamma(2,1)$	$\text{vec}(\boldsymbol{\alpha}^{\Delta}) \sim MVN(\mathbf{0}, 0.25 \cdot \mathbf{I})$	$\text{vec}(\boldsymbol{\beta}) \sim MVN(\mathbf{0}, \mathbf{I})$
For each $\sigma^2_{\varepsilon,m}$ pertaining to a metric scaled measurement	For non-zero elements $\boldsymbol{\alpha}^{\Delta}$	Dimension of $\mathbf{I}$ dependent on FA with $(M \cdot (Q+1))$ or without $(M \cdot Q)$ intercept, or IRT (M)
$\text{vec}(\boldsymbol{\gamma}) \sim MVN(\mathbf{0}, \mathbf{I}_{K \cdot M})$	$\nu^0_{\Omega} = K + 5$	$A^2_k = sqrt(1/6)$
Only in IRT	For prior degrees of freedom of latent variable covariance	For scaling matrix element of latent variable covariance, with $k = 1, \ldots, K$
$\nu_{st} = 10$	$\sigma^0_{\zeta,m,j} = 1$	$\sigma^0_{\xi,m,j} = 1$
Prior degrees of freedom for multivariate Student's t-density	Prior standard deviation for transformed cut-off $\zeta_{m,j}$ of measurement m and category j	Prior standard deviation referring to cut-off parameter $\boldsymbol{\xi}_m$ and category j
Further options set:		
Metropolis-Hastings maximal number of iterations: 1000	Minimal number of buckets of CART: 5	Splitting information decrease bound: 0.0001

Table 5.1: Simulation Study hyperparameters and further options used.

No.	Model	Scales	N	M	K	Q	Intercept	dedicated	Missingness	Imputation	Miss. Amount (1)	Miss. Amount (2)
(F1a)	FA	BOM	1000	9	3	4	yes	yes	BD	no	0%	0%
(F1b)	FA	BOM	1000	9	3	4	yes	yes	MCAR	no (CC)	15% (MCAR, $\mathbf{x_1}$)	15% (MCAR, $\mathbf{x_2}$)
(F1c)	FA	BOM	1000	9	3	4	yes	yes	MCAR	yes (CART)	15% (MCAR, $\mathbf{x_1}$)	15% (MCAR, $\mathbf{x_2}$)
(F2a)	FA	BOM	1000	9	3	7	yes	yes	MCAR	no (CC)	15% (MCAR, $\mathbf{x_1}$)	15% (MCAR, $\mathbf{x_2}$)
(F2b)	FA	BOM	1000	9	3	7	yes	yes	MCAR	yes (CART)	15% (MCAR, $\mathbf{x_1}$)	15% (MCAR, $\mathbf{x_2}$)
(F3a)	FA	BOM	2500	15	3	4	yes	yes	MCAR	no (CC)	15% (MCAR, $\mathbf{x_1}$)	15% (MCAR, $\mathbf{x_2}$)
(F3b)	FA	BOM	2500	15	3	4	yes	yes	MCAR	yes (CART)	15% (MCAR, $\mathbf{x_1}$)	15% (MCAR, $\mathbf{x_2}$)
(F4a)	FA	B	1000	10	1	2	no	yes	MCAR	yes (CART)	15% (MCAR, $\mathbf{x_1}$)	15% (MCAR, $\mathbf{x_2}$)
(F4b)	FA	B	1000	10	1	2	no	yes	MCAR	yes (CART with $\mathbf{Y}^*$)	15% (MCAR, $\mathbf{x_1}$)	15% (MCAR, $\mathbf{x_2}$)
(I1a)	IRT	BOM	1000	9	3	3	no	yes	BD	no	0%	0%
(I1b)	IRT	BOM	1000	9	3	3	no	yes	MCAR	no (CC)	15% (MCAR, $\mathbf{x_1}$)	15% (MCAR, $\mathbf{x_2}$)
(I1c)	IRT	BOM	1000	9	3	3	no	yes	MCAR	yes (CART)	15% (MCAR, $\mathbf{x_1}$)	15% (MCAR, $\mathbf{x_2}$)
(I1d)	IRT	BOM	1000	9	3	3	no	yes	MCAR/MAR	no (CC)	15% (MCAR, $\mathbf{x_1}$)	≈12.3% (MAR, $\mathbf{x_2}$)
(I1e)	IRT	BOM	1000	9	3	3	no	yes	MCAR/MAR	yes (CART)	15% (MCAR, $\mathbf{x_1}$)	≈12.3% (MAR, $\mathbf{x_2}$)
(I2a)	IRT	BOM	1000	9	3	3	no	no	BD	no	0%	0%
(I2b)	IRT	BOM	1000	9	3	3	no	no	MCAR/MAR	no (CC)	15% (MCAR, $\mathbf{x_1}$)	≈12.3% (MAR, $\mathbf{x_2}$)
(I2c)	IRT	BOM	1000	9	3	3	no	no	MCAR/MAR	yes (CART)	15% (MCAR, $\mathbf{x_1}$)	≈12.3% (MAR, $\mathbf{x_2}$)
(I3a)	IRT	BOM	1000	9	3	3	no	no	MAR	no (CC)	≈12.3% (MAR, $\mathbf{x_1}$)	≈16% (MAR, $\mathbf{x_2}$)
(I3b)	IRT	BOM	1000	9	3	3	no	no	MAR	yes (CART)	≈12.3% (MAR, $\mathbf{x_1}$)	≈16% (MAR, $\mathbf{x_2}$)
(I4a)	IRT	BOM	1000	9	3	3	no	no	MAR	no (CC)	≈12.3% (MAR, $\mathbf{x_1}$)	≈12.3% (MAR, $\mathbf{x_2}$)
(I4b)	IRT	BOM	1000	9	3	3	no	no	MAR	yes (CART)	≈12.3% (MAR, $\mathbf{x_1}$)	≈12.3% (MAR, $\mathbf{x_2}$)

Table 5.2: Nine factor analytic and twelve IRT scenarios under investigation, with Models: Models under investigation, Scales (of measurement): **B**inary, **O**rdered, **M**etric (continuous), N: Number of observations, M: Number of measurements, K: Number of latent variables, Q: Number of covariates (incl. intercept, if available), Intercept: Presence of intercept, dedicated: Presence of dedicated structure (no cross-loading) or not (cross-loading), Missingness: Type of missingness mechanism, Imputation: Handling of missing values (no imputation, no imputation but complete-case analysis (CC), imputation with CART (CART)), Miss. Amount: Percentage of missingness imposed on variable $\mathbf{x_1}$ or $\mathbf{x_2}$.

5.3 Detailed Evaluation of Results

5.3.1 Measures of Diagnostics and their Preparation

For every data set of the simulation studies, the basic estimates and estimated measures that are considered, encompass the estimated posterior mean, the bias and absolute bias (with target value of zero), the root mean squared error (RMSE; with again a target value zero) and the relative RMSE (with a target value of one), and the standard deviation of the parameter draws. Furthermore, upper and lower bound central posterior region estimates based on 2.5%- and 97.5%-quantile estimates, and, for comparison, upper and lower bound central posterior region estimates based on parameter posterior mean estimates, the estimated standard deviations and respective 2.5%- and 97.5%-normal quantiles (i.e. ± 1.96) get calculated as well.[29] In order to create a measure of reference, using 125 files for replication, one would expect for this proportions (under normality approximation, see Rohatgi & Saleh, 2015, pp. 513 ff.) to lie within an interval ranging from $0.95 - 1.96\sqrt{\frac{0.95 \cdot 0.05}{125}} = 0.912$ to $0.95 + 1.96\sqrt{\frac{0.95 \cdot 0.05}{125}} = 0.988$–and assuming an error of 5%: Up to five rates for parameters that exceed these bounds (deriving from $\lfloor 0.05 \cdot (1 - 0.05) \cdot 125 \rfloor = \lfloor 5.938 \rfloor$), are be tolerable.

Because of the correlated draws, estimates based on the effective sample-size, denoted with $(\hat{T})_{eff}$, are additionally considered. For these, the effective sample-size, that adjusts for autocorrelation, gets estimated (see Robert & Casella, 2004, pp. 499f.) using the R package *coda* (see Plummer et al., 2006) and a thinned subset of the draws–with burn-in removed and length $(T)_{eff}$–gets created, using each $\left(\left\lfloor \frac{(T) - burnIn}{(\hat{T})_{eff}} \right\rfloor + 1\right)$-th element. The idea of the latter is, first, gaining knowledge about actual independent information of the draws. Second, we additionally investigate, whether a reduction of draws has an effect–in this case: The higher the autocorrelation and therefore the lower the effective sample-size, draws more apart from one another are chosen. Because thinning can be used to reduce the amount of data stored and therefore reduce the amount of memory needed in computation, it is something considered, when dealing with larger amounts of data (see Raftery & Lewis, 1996, p. 115) and we would like to additionally investigate it.

To give an overview, table 5.3 presents the calculations made.

[29] Although mostly known, see for definitions, amongst others: Rohatgi & Saleh (2015, pp. 79, 339, 366 and 518f.), Chambers (1986) and Patil et al. (1979).

Basic estimators:		
$\bar{\check{\psi}} = \frac{1}{(T)-burnIn} \sum_{(t)=burnIn}^{(T)} \check{\psi}_{(t)}$	$bias\left(\bar{\check{\psi}}\right) = \left(\psi - \bar{\check{\psi}}\right)$	$\lvert bias \rvert = absBias\left(\bar{\check{\psi}}\right) = \left\lvert bias\left(\bar{\check{\psi}}\right)\right\rvert$
$\hat{\sigma}\left(\check{\boldsymbol{\psi}}\right) = \sqrt{\frac{1}{(T)-burnIn-1} \sum_{(t)=burnIn}^{(T)} \left(\check{\psi}_{(t)} - \bar{\check{\psi}}\right)^2}$	$RMSE\left(\check{\boldsymbol{\psi}}\right) = \sqrt{\left(\hat{\sigma}^2\left(\check{\boldsymbol{\psi}}\right) + \left\{bias\left(\bar{\check{\psi}}\right)\right\}^2\right)}$	$RRMSE\left(\check{\boldsymbol{\psi}}\right) = \frac{RMSE(\check{\boldsymbol{\psi}})}{\hat{\sigma}(\check{\boldsymbol{\psi}})}$
$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right) = \begin{cases} \frac{(\check{\boldsymbol{\psi}}_{(((T)-burnIn)\cdot 2.5\%)} + \check{\boldsymbol{\psi}}_{(((T)-burnIn)\cdot 2.5\%+1)})}{2}, & \text{if } ((T)-burnIn)\cdot 2.5\% \text{ is integer} \\ \check{\boldsymbol{\psi}}_{(\lfloor((T)-burnIn)\cdot 2.5\%\rfloor+1)} & \text{otherwise} \end{cases}$		
$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right) = \begin{cases} \frac{(\check{\boldsymbol{\psi}}_{(((T)-burnIn)\cdot 97.5\%)} + \check{\boldsymbol{\psi}}_{(((T)-burnIn)\cdot 97.5\%+1)})}{2}, & \text{if } ((T)-burnIn)\cdot 97.5\% \text{ is integer} \\ \check{\boldsymbol{\psi}}_{(\lfloor((T)-burnIn)\cdot 97.5\%\rfloor+1)} & \text{otherwise} \end{cases}$		
$CPR_q\left(\check{\boldsymbol{\psi}}\right) = \left[q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right), q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)\right]$	$CPR_{CI}\left(\check{\boldsymbol{\psi}}\right) = \left[\bar{\check{\psi}} - 1.96 \cdot \hat{\sigma}\left(\check{\boldsymbol{\psi}}\right), \bar{\check{\psi}} + 1.96 \cdot \hat{\sigma}\left(\check{\boldsymbol{\psi}}\right)\right]$	
Estimators using effective sample-size:		
$\bar{\check{\psi}}_{eff} = \frac{1}{(T)_{eff}} \sum_{(t)_{eff}} \check{\psi}_{(t)_{eff}}$	$bias_{eff}\left(\bar{\check{\psi}}_{eff}\right) = \left(\psi - \bar{\check{\psi}}_{eff}\right)$	$\lvert bias_{eff} \rvert = absBias_{eff}\left(\bar{\check{\psi}}_{eff}\right) = \left\lvert bias_{eff}\left(\bar{\check{\psi}}_{eff}\right)\right\rvert$
$\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right) = \sqrt{\frac{1}{(T)_{eff}} \sum_{(t)_{eff}} \left(\check{\psi}_{(t)_{eff}} - \bar{\check{\psi}}_{eff}\right)^2}$	$RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right) = \sqrt{\left(\hat{\sigma}_{eff}^2\left(\check{\boldsymbol{\psi}}_{eff}\right) + \left\{bias_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)\right\}^2\right)}$	
$CPR_{CI,eff}\left(\check{\boldsymbol{\psi}}_{eff}\right) = \left[\bar{\check{\psi}}_{eff} - 1.96 \cdot \hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right), \bar{\check{\psi}}_{eff} + 1.96 \cdot \hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)\right]$		

Table 5.3: Basic and effective sample-size based estimators for every data set, where ψ denotes the true value and $\check{\psi}_{(t)}$ the (t)-th draw from the sampler.

Except for the central posterior regions, these estimates get combined in terms of means for the complete simulation study, for the basic calculations as well as the ones relying on effective sample-size. Thus, an average absolute bias, average parameter standard deviations, an average RMSE and relative RMSE (RRMSE) for comparability, and average effective sample-size, are retrieved at first. And afterwards, the central posterior regions are used to create coverage rates for posterior means and rates of how many zeros are covered.

For presenting results for 125 replications with each having 50000 iterations and between 30 and 93 parameters to estimate, three levels of aggregation are introduced: First, each set of parameter draws of the 50000 iterations (with 20% removed for burn-in) gets combined in terms of posterior mean estimates, standard deviations, bias, absolute bias and so forth–resulting in 125 sets of measures for diagnostics of individual parameter values (i.e. about 33750 to 101250 values). Second, the 125 replications are combined in terms of means and additionally standard deviation estimates for the measures considered, and are now accompanied by coverage rates–with 18 measures per parameter, or about 540 or 1640 values, which are presented in Appendix D (starting on page 201). In order to remove the amount of tables needed, third, another level of aggregation gets introduced: the estimated measures of the second combination get averaged for each set of parameters (that is $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and so on), and are accompanied with ranges, i.e. minimum and maximum values. These results are depicted and described in the next sections, sometimes referring to individual parameter estimates or sets of parameter estimates, if they seem of specific interest.

We start with the basic scenarios for factor analysis, and additionally present further MCAR based simulation studies with increased and decreased number of parameters and variables. For the IRT model, the basic scenarios of dedicated structure get presented, MAR based simulations and simulation studies that include cross-loading. Next to tables containing all information retrieved for the third aggregation level, descriptions focus–for individual results–mostly on the absolute bias, relative RMSE (being scale invariant and therefore allow comparisons between parameters more easily) and coverage rates. Afterwards, section 5.4 summarises the individual findings and compares the different scenarios with respect to RMSE and standard deviation estimates, enabling insights into efficiency (as, for example, presented by Gaasch, 2017a, p. 56 and pp. 105ff., and Preising, 2018, pp. 64ff., as well).

5.3.2 Individual Results for Factor Analysis on Basic Scenarios

(F1a) FA, Before Deletion, dedicated structure, N=1000, M=9, Q=3+1, K=3

The very first scenarios, (F1a) to (F1c), use a dedicated structure with 1000 observations, nine measurements, three covariates (plus intercept) and three latent variables.

In the before deletion case of factor analysis (F1a), all results seem as intended. With three dichotomous, three ordered and three metric scaled measurements, actual and absolute biases are close to zero, and so are the root mean squared estimates. The relative root mean squared errors are, as desired, close to one, and the coverage rates are also close to 0.95. Nevertheless, three from 57 parameters ($\beta_{1,2}$, $\beta_{2,5}$ and $\beta_{4,5}$) are close to the lower bound, but still smaller (with 0.904, 0.896 and 0.904) and one parameter ($\sigma^2_{\varepsilon,3}$) from 57 is with respect to a quantile estimated central posterior region close to the upper bound, but still higher (with 0.992). Although, when concentrating on other characteristics, only the coverage seems to be somewhat problematic (compare table 5.4 and Appendix D, pages 201ff.), but nonetheless there are less than five outliers.

When taking a look at effective sample-sizes, it tends to be low for some parameters, when compared to 40000 draws (50000 minus 20% burn-in) for each of them. For example one of the σ^2_ε-parameter estimates has an effective sample-size of 469.867 and for one of the $\boldsymbol{\alpha}^\Delta$-chains a value of 531.298 is obtained,which roughly 1.3% ($\approx 531/40000$) of the draws , that were carried out, can be viewed as independent posterior draws. Nonetheless and as already mentioned, all results seem as intended. Using the effective sample-size as a basis for thinning, tends to have a slight effect on coverage rates–sometimes in opposite directions, like a slight increase for one of the ξ-cut-off parameters and a β-parameter or decrease of a tiny amount for one of the α-parameter draws. Root mean squared errors and standard deviations do not seem to be affected–when investigating individual files, discrepancies start to get visible at the fourth position after the decimal point.

	$\overline{\overline{bias}}$	$\overline{\overline{\vert bias\vert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ_ε^2	-0.018	0.067	0.093	0.121	1.027	0.965	0.981
(range)	(-0.026, -0.013)	(0.060, 0.079)	(0.083, 0.112)	(0.108, 0.144)	(1.011, 1.057)	(0.936, 0.984)	(0.968, 0.992)
α^Δ	0.003	0.063	0.084	0.110	1.013	0.960	0.965
(range)	(-0.015, 0.026)	(0.050, 0.079)	(0.062, 0.108)	(0.084, 0.139)	(0.988, 1.052)	(0.936, 0.976)	(0.936, 0.984)
β	0.000	0.042	0.053	0.072	1.009	0.950	0.950
(range)	(-0.026, 0.021)	(0.023, 0.061)	(0.030, 0.074)	(0.042, 0.102)	(0.974, 1.061)	(0.896, 0.984)	(0.896, 0.984)
r	0.002	0.044	0.058	0.076	0.994	0.976	0.971
(range)	(-0.002, 0.004)	(0.043, 0.044)	(0.056, 0.060)	(0.074, 0.078)	(0.982, 1.000)	(0.968, 0.984)	(0.960, 0.976)
ξ	0.001	0.050	0.061	0.083	0.995	0.948	0.951
(range)	(-0.001, 0.005)	(0.036, 0.069)	(0.044, 0.083)	(0.060, 0.114)	(0.984, 1.000)	(0.936, 0.960)	(0.936, 0.960)

	$\overline{(\hat{T})}_{eff}$	$\bar{\hat{\sigma}}_{eff}\left(\check{\psi}_{eff}\right)$	$\overline{RMSE}_{eff}\left(\check{\psi}_{eff}\right)$	$\overline{Cover}_{CI,eff}$
σ_ε^2	589.475	0.093	0.121	0.965
(range)	(469.867, 689.198)	(0.083, 0.112)	(0.108, 0.144)	(0.936, 0.984)
α^Δ	1088.464	0.084	0.110	0.959
(range)	(531.298, 1926.719)	(0.063, 0.108)	(0.084, 0.139)	(0.936, 0.976)
β	7651.658	0.053	0.072	0.951
(range)	(2009.597, 16031.135)	(0.030, 0.074)	(0.042, 0.102)	(0.896, 0.992)
r	4589.160	0.058	0.076	0.976
(range)	(3820.038, 5526.988)	(0.056, 0.060)	(0.074, 0.078)	(0.968, 0.984)
ξ	3920.819	0.061	0.083	0.948
(range)	(1270.829, 7862.985)	(0.044, 0.083)	(0.060, 0.114)	(0.944, 0.960)

Table 5.4: Scenario (F1a): Aggregated estimates and effective sample-size estimates. Before Deletion, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2,5,8, with 3 covariates plus intercept, 3 latent variables.

(F1b) FA, MCAR with CC, dedicated structure, N=1000, M=9, Q=3+1, K=3

For complete-case analysis of the factor model, again, almost all results seem as intended. Using a MCAR mechanism on the data from (F1a) with about 15% missingness on covariates $\mathbf{x}_1$ and $\mathbf{x}_2$ (while $\mathbf{x}_3$ is completely observed), actual and absolute biases are–like before–close to zero, and so are the root mean squared error estimates. The relative root mean squared errors are close to one and the coverage rates are also close to 0.95. But, again, some parameters are slightly outside of the bounds desired. This means that two from all 57 parameters (β_{12} and β_{45}) are close to the lower bound, but still smaller (with 0.904–for the quantile based estimate only–and 0.896) and one parameter ($\sigma^2_{\varepsilon,9}$) from 57 is with respect to a quantile estimated central posterior region close to the upper bound, but still higher (again 0.992). As before, when concentrating on other characteristics, only the coverage seems to be somewhat problematic, and results based effective sample-size seem to be similar to the previous outcomes (compare table 5.5 and pages 205ff. within Appendix D).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias\rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	-0.012	0.070	0.103	0.131	1.014	0.973	0.984
(range)	(-0.023, 0.001)	(0.065, 0.080)	(0.094, 0.122)	(0.120, 0.154)	(0.990, 1.039)	(0.960, 0.984)	(0.976, 0.992)
α^Δ	0.003	0.070	0.097	0.126	1.008	0.959	0.967
(range)	(-0.015, 0.025)	(0.053, 0.090)	(0.071, 0.123)	(0.094, 0.160)	(0.991, 1.032)	(0.920, 0.984)	(0.928, 0.984)
β	-0.000	0.050	0.062	0.084	1.016	0.947	0.946
(range)	(-0.035, 0.026)	(0.029, 0.076)	(0.035, 0.086)	(0.048, 0.119)	(1.000, 1.084)	(0.896, 0.984)	(0.896, 0.984)
r	0.003	0.050	0.068	0.089	1.000	0.971	0.971
(range)	(-0.002, 0.007)	(0.046, 0.055)	(0.067, 0.071)	(0.087, 0.091)	(1.000, 1.000)	(0.952, 0.984)	(0.952, 0.984)
ξ	0.000	0.058	0.071	0.097	0.998	0.959	0.945
(range)	(-0.006, 0.012)	(0.042, 0.071)	(0.052, 0.096)	(0.071, 0.126)	(0.989, 1.000)	(0.944, 0.968)	(0.936, 0.960)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	643.020	0.103	0.131	0.979
(range)	(541.935, 722.393)	(0.094, 0.122)	(0.120, 0.154)	(0.976, 0.984)
α^Δ	1220.805	0.097	0.126	0.959
(range)	(617.510, 2029.579)	(0.071, 0.123)	(0.094, 0.160)	(0.920, 0.984)
β	7834.791	0.062	0.084	0.947
(range)	(2306.761, 15813.547)	(0.035, 0.087)	(0.048, 0.119)	(0.896, 0.984)
r	4605.329	0.068	0.089	0.971
(range)	(4001.456, 5355.163)	(0.067, 0.071)	(0.087, 0.091)	(0.952, 0.984)
ξ	4296.105	0.071	0.097	0.960
(range)	(1456.661, 8208.731)	(0.052, 0.096)	(0.071, 0.126)	(0.944, 0.968)

Table 5.5: Scenario (F1b): Aggregated estimates and effective sample-size estimates. MCAR using complete-case analysis, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2,5,8, with 3 covariates plus intercept, 3 latent variables.

(F1c) FA, MCAR with CART, dedicated structure, N=1000, M=9, Q=3+1, K=3

When using the CART-based covariate imputation (with the original measurements $\mathbf{Y}$, and observed and imputed covariates as additional source of information), with the MCAR mechanism imposed on the data from (F1a) and amounts of missingness as described in (F1b), results start to differ. This encompasses estimates referring to almost all parameters except the cut-off parameters $\boldsymbol{\xi}$. With regards to suited coverage rates 24 of all 57 parameters are below the lower bound, up to nine are below a coverage of 80% and four are below 50%. Further characteristics seem to be especially problematic for estimates pertaining to $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\alpha}^\Delta$. Bias, absolute bias as well as (R)RMSE are most shifted from intended values for these parameters. But although 17 of 36 β-parameters have also lower coverage than 91%, all of these have an absolute bias below 0.1. However, examining the RMSE and relative RMSE, several parameters with increased values for these measures of quality can be found, indicating problems in variation for the sampled parameter proposals (compare tables 5.6 and 5.7 and Appendix D, starting on page 209).

When examining effective sample-sizes, results exhibit slightly increased information compared to previous scenarios, ranging from 1149.161 to 19930.948, i.e. about 2.5% to 50% of the draws made, can be viewed as independent. Nonetheless, the effect of thinning does not have a general direction. While the RMSE estimates are similar to the outcomes without thinning, coverages are slightly lower, for example for one of the cut-off parameters ξ, but slightly increased for one of the non-zero α-parameter chains.

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol{\psi}}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.179	0.179	0.057	0.188	4.312	0.043	0.037
(range)	(0.141, 0.208)	(0.141, 0.208)	(0.048, 0.067)	(0.149, 0.219)	(4.056, 4.683)	(0.016, 0.056)	(0.008, 0.056)
α^Δ	-0.035	0.095	0.082	0.134	1.826	0.749	0.731
(range)	(-0.115, 0.141)	(0.069, 0.141)	(0.048, 0.120)	(0.117, 0.172)	(1.011, 3.361)	(0.264, 0.992)	(0.256, 0.976)
β	-0.002	0.053	0.054	0.081	1.223	0.873	0.874
(range)	(-0.085, 0.066)	(0.026, 0.092)	(0.031, 0.078)	(0.042, 0.120)	(1.000, 2.514)	(0.352, 0.984)	(0.344, 0.984)
r	0.003	0.050	0.068	0.089	1.000	0.971	0.971
(range)	(-0.002, 0.019)	(0.053, 0.057)	(0.068, 0.075)	(0.090, 0.099)	(1.000, 1.051)	(0.976, 0.992)	(0.976, 0.992)
ξ	0.010	0.050	0.062	0.084	1.024	0.955	0.951
(range)	(0.001, 0.028)	(0.040, 0.059)	(0.046, 0.081)	(0.065, 0.105)	(1.000, 1.102)	(0.936, 0.968)	(0.928, 0.968)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	3756.072	0.057	0.188	0.043
(range)	(2636.365, 5039.009)	(0.048, 0.067)	(0.149, 0.219)	(0.016, 0.056)
α^Δ	2423.487	0.082	0.134	0.752
(range)	(1149.161, 4496.426)	(0.048, 0.120)	(0.117, 0.172)	(0.272, 0.992)
β	9905.912	0.054	0.081	0.874
(range)	(2670.985, 19930.948)	(0.031, 0.077)	(0.042, 0.120)	(0.344, 0.976)
r	3549.601	0.071	0.094	0.984
(range)	(3222.324, 3909.205)	(0.068, 0.075)	(0.090, 0.099)	(0.976, 0.992)
ξ	4259.236	0.062	0.084	0.955
(range)	(1965.214, 7110.082)	(0.046, 0.081)	(0.065, 0.105)	(0.936, 0.968)

Table 5.6: Scenario (F1c): Aggregated estimates and effective sample-size estimates. MCAR using CART imputation, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2, 5, 8, with 3 covariates plus intercept, 3 latent variables.

	ψ	$\bar{\bar{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.672	0.141	0.141	0.036	0.048	0.003	0.149	0.035	4.056	0.056	0.056
$\sigma^2_{\varepsilon,6}$	0.490	0.678	0.188	0.188	0.041	0.056	0.004	0.196	0.039	4.683	0.016	0.008
$\sigma^2_{\varepsilon,9}$	0.528	0.736	0.208	0.208	0.051	0.067	0.005	0.219	0.048	4.196	0.056	0.048
$\alpha_{2,1}$	0.674	0.712	0.038	0.069	0.056	0.102	0.009	0.130	0.039	1.099	0.992	0.976
$\alpha_{3,1}$	0.601	0.486	-0.115	0.115	0.036	0.048	0.002	0.126	0.033	3.361	0.264	0.256
$\alpha_{6,2}$	0.700	0.595	-0.104	0.104	0.041	0.049	0.002	0.117	0.036	2.732	0.432	0.424
$\alpha_{8,3}$	-0.378	-0.458	-0.080	0.086	0.059	0.075	0.008	0.119	0.047	1.552	0.896	0.832
$\alpha_{9,3}$	-0.709	-0.567	0.141	0.141	0.047	0.060	0.004	0.155	0.043	3.170	0.336	0.328
$\beta_{5,2}$	-0.744	-0.684	0.060	0.072	0.048	0.059	0.004	0.099	0.036	1.381	0.760	0.776
$\beta_{6,2}$	0.711	0.625	-0.085	0.086	0.035	0.038	0.001	0.096	0.031	2.514	0.352	0.344
$\beta_{7,2}$	0.735	0.650	-0.085	0.092	0.057	0.068	0.004	0.120	0.043	1.612	0.704	0.720
$\beta_{8,2}$	0.308	0.276	-0.032	0.047	0.034	0.046	0.002	0.070	0.025	1.204	0.856	0.856
$\beta_{9,2}$	0.562	0.509	-0.053	0.055	0.033	0.038	0.001	0.070	0.026	1.778	0.744	0.744
$\beta_{1,3}$	-0.632	-0.587	0.045	0.070	0.048	0.073	0.005	0.107	0.033	1.181	0.904	0.912
$\beta_{2,3}$	0.769	0.718	-0.051	0.065	0.051	0.069	0.004	0.102	0.035	1.262	0.888	0.888
$\beta_{3,3}$	0.794	0.774	-0.020	0.036	0.028	0.039	0.001	0.056	0.021	1.098	0.888	0.888
$\beta_{6,3}$	-0.693	-0.627	0.066	0.068	0.037	0.042	0.001	0.082	0.030	1.878	0.664	0.664
$\beta_{7,3}$	-0.435	-0.379	0.056	0.073	0.050	0.067	0.004	0.106	0.036	1.290	0.824	0.824
$\beta_{8,3}$	-0.470	-0.432	0.038	0.055	0.037	0.052	0.002	0.080	0.027	1.200	0.880	0.880
$\beta_{9,3}$	-0.693	-0.629	0.064	0.066	0.038	0.042	0.001	0.081	0.032	1.767	0.696	0.688
$\beta_{1,4}$	0.764	0.712	-0.053	0.073	0.046	0.071	0.005	0.107	0.032	1.246	0.904	0.904
$\beta_{2,4}$	-0.407	-0.353	0.055	0.065	0.042	0.057	0.003	0.090	0.032	1.400	0.824	0.832
$\beta_{3,4}$	-0.505	-0.476	0.029	0.039	0.026	0.036	0.001	0.056	0.019	1.270	0.872	0.872
$\beta_{5,4}$	0.764	0.723	-0.041	0.063	0.047	0.060	0.004	0.092	0.035	1.164	0.832	0.840
$\beta_{9,4}$	0.241	0.227	-0.014	0.035	0.025	0.038	0.001	0.055	0.018	1.049	0.904	0.904
$r_{1,2}$	-0.111	-0.091	0.019	0.053	0.033	0.068	0.003	0.090	0.021	1.051	0.992	0.992

Table 5.7: Scenario (F1c): Interesting parameters with respect to coverage rates. MCAR using CART imputation, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2, 5, 8, with 3 covariates plus intercept, 3 latent variables.

5.3.3 Individual Results for Factor Analysis with additional MCAR Scenarios

Within the next few subsections additional scenarios get presented and are concerned with further different data structures. This is, first, an increased amount of data in two dimensions, i.e. the amount of covariates (being seven instead of three–plus an additional intercept) in (F2a) and (F2b), and an increased amount of measurements and observations in (F3a) and (F3b), which is 2500 rows and fifteen measurements (instead of 1000 rows) and ten columns for $\mathbf{Y}$. Second, scenarios (F4a) and (F4b) aim at the opposite direction, which is a decrease of the number of latent variables and covariates resulting in an overall reduced amount of parameters. (F4b) additionally uses different information for imputation, that is, next to covariates, $\mathbf{Y}^*$ instead of $\mathbf{Y}$ gets utilized. With these extra simulation studies, we try to cover several possible aspects, i.e. too little information used in imputation (i.e. more covariates), or too little information as such (i.e. more measurements and 2500 instead of 1000 observation), too many parameters that are considered, or too little variability in imputation, when incorporating fixed observations $\mathbf{Y}$.

(F2a) FA, MCAR with CC, dedicated structure, N=1000, M=9, Q=7+1, K=3

Using complete-case analysis, there are slightly increased coverage rates in $\sigma^2_{\varepsilon,9}$ and in 20 of 72 β-parameters (except β-parameters pertaining to the intercept) or in other words: in 21 of a total of 93 parameters. Nonetheless, there are parameters having decreased coverage rates–but only within $\boldsymbol{\beta}$ (14 of 72, or 14 of 93). Both types are accompanied by increased RRMSEs and bias estimates (compare table 5.8, and tables 7.20, 7.22 and 7.23 of Appendix D from page 214 onwards). Specifically, a pattern for decreased coverages emerges: with the exception for parameters of $\boldsymbol{\beta}$ referring to the intercept of $\mathbf{X}$, all parameters of $\boldsymbol{\beta}$ that are related to the metric scaled measurements five and nine (but not the continuous outcome three) have lower coverage rates than the lower bound of 0.912, but no parameter has a coverage below 0.5 (compare table 5.9).

Compared to the 40000 iterations used, effective sample-size seems to be rather low for some of the parameter draws, i.e. it ranges from 395.836 (for a σ^2_{ε}-parameter) to 34776.232 (for one of the β parameters), which is information of about 1% to 87%. Thinning tends to have almost no effect, except for one of the β-parameters, it gets diminished by a tiny amount–from a coverage rate of 0.984 to 0.976, when using a confidence interval estimation.

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ_ε^2	-0.013	0.073	0.111	0.140	1.024	0.971	0.971
(range)	(-0.032, 0.007)	(0.070, 0.079)	(0.095, 0.139)	(0.124, 0.167)	(1.000, 1.060)	(0.944, 1.000)	(0.944, 1.000)
α^Δ	-0.007	0.074	0.104	0.134	1.022	0.959	0.961
(range)	(-0.034, 0.027)	(0.051, 0.090)	(0.073, 0.124)	(0.094, 0.162)	(0.988, 1.075)	(0.944, 0.976)	(0.944, 0.984)
β	0.010	0.141	0.154	0.221	1.960	0.931	0.931
(range)	(-0.276, 0.573)	(0.028, 0.573)	(0.035, 0.335)	(0.048, 0.661)	(0.986, 5.583)	(0.632, 1.000)	(0.632, 1.000)
r	0.002	0.052	0.070	0.092	1.000	0.971	0.963
(range)	(-0.008, 0.009)	(0.046, 0.057)	(0.067, 0.073)	(0.086, 0.098)	(1.000, 1.000)	(0.960, 0.984)	(0.952, 0.976)
ξ	-0.001	0.062	0.078	0.105	1.017	0.943	0.945
(range)	(-0.020, 0.027)	(0.049, 0.073)	(0.063, 0.098)	(0.085, 0.129)	(1.000, 1.047)	(0.920, 0.952)	(0.920, 0.960)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ_ε^2	554.809	0.111	0.140	0.973
(range)	(395.836, 693.114)	(0.095, 0.139)	(0.124, 0.167)	(0.944, 1.000)
α^Δ	1020.738	0.104	0.134	0.959
(range)	(453.603, 1557.031)	(0.073, 0.124)	(0.094, 0.162)	(0.944, 0.976)
β	17661.918	0.154	0.221	0.931
(range)	(2520.532, 34776.232)	(0.035, 0.335)	(0.048, 0.661)	(0.632, 1.000)
r	4180.465	0.070	0.092	0.968
(range)	(3510.380, 5084.858)	(0.067, 0.073)	(0.086, 0.098)	(0.960, 0.976)
ξ	3134.686	0.078	0.105	0.943
(range)	(1520.613, 5582.982)	(0.063, 0.098)	(0.086, 0.129)	(0.920, 0.952)

Table 5.8: Scenario (F2a): Aggregated estimates and effective sample-size estimates. MCAR using complete-case analysis, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2, 5, 8, with 7 covariates plus intercept, 3 latent variables.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{5,2}$	-0.298	-0.524	-0.227	0.227	0.069	0.145	0.003	0.272	0.058	3.386	0.840	0.840
$\beta_{9,2}$	0.768	0.541	-0.227	0.227	0.065	0.133	0.001	0.265	0.054	3.631	0.656	0.648
$\beta_{5,3}$	-0.525	0.048	0.573	0.573	0.108	0.325	0.002	0.661	0.093	5.398	0.720	0.728
$\beta_{9,3}$	-0.239	0.319	0.558	0.558	0.138	0.313	0.002	0.643	0.119	4.167	0.688	0.688
$\beta_{5,4}$	0.427	0.151	-0.276	0.276	0.066	0.160	0.002	0.321	0.056	4.303	0.688	0.688
$\beta_{9,4}$	0.623	0.357	-0.266	0.266	0.074	0.151	0.001	0.308	0.063	3.730	0.632	0.632
$\beta_{5,5}$	-0.647	-0.189	0.457	0.457	0.090	0.259	0.002	0.528	0.079	5.178	0.760	0.752
$\beta_{9,5}$	-0.765	-0.322	0.443	0.443	0.110	0.249	0.002	0.511	0.095	4.145	0.680	0.672
$\beta_{5,6}$	0.228	0.099	-0.129	0.132	0.062	0.089	0.002	0.164	0.048	2.164	0.768	0.768
$\beta_{9,6}$	-0.351	-0.457	-0.106	0.106	0.049	0.079	0.001	0.136	0.038	2.388	0.848	0.840
$\beta_{5,7}$	0.716	0.558	-0.158	0.161	0.072	0.116	0.004	0.203	0.056	2.256	0.792	0.792
$\beta_{9,7}$	0.617	0.445	-0.172	0.172	0.055	0.103	0.001	0.202	0.046	3.273	0.704	0.704
$\beta_{5,8}$	0.237	0.167	-0.069	0.073	0.046	0.066	0.003	0.103	0.033	1.686	0.864	0.872
$\beta_{9,8}$	-0.550	-0.615	-0.065	0.067	0.036	0.055	0.001	0.090	0.027	1.900	0.840	0.848

Table 5.9: Scenario (F2a): Interesting parameters with respect to coverage rates. MCAR using complete-case analysis, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2,5,8, with 7 covariates plus intercept, 3 latent variables.

(F2b) FA, MCAR with CART, dedicated structure, N=1000, M=9, Q=7+1, K=3

When using CART imputation instead of complete-case analysis for data of scenario (F2a), the previously seen pattern for metric scaled measurements $\mathbf{y}_5$ and $\mathbf{y}_9$ disappears, but not in a positive way. Instead, almost all parameters under consideration are affected by low coverage rates. In total 53 of 92 have a coverage rate lower than 0.912 (while 41 are below 0.8 and 30 are below 0.5), and a coverage rate higher than 0.988 is obtained for two parameters. When taking a look at absolute biases, 45 parameter estimates exceed 0.1 (and eighteen are above 0.25). Similarly, with respect to relative root mean squared errors high estimates are visible, i.e. 50 are above 1.5 and 35 exceed a value of 2 (compare table 5.10 for and overview and, because of the amount of parameter affected, compare also tables for (F2b) in Appendix D from page 219 onwards). Similar to complete-case analysis, effective sample-size tends to be somewhat low for some parameters, ranging from 860.441 (for an α-parameter) to 27300.584 (for one of the β-parameters). Thinning increases the coverage for one of the cut-off parameters, and increased and decreased rates for some of the σ_ε^2 parameters can be found.

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.121	0.131	0.068	0.155	2.807	0.456	0.461
(range)	(0.022, 0.191)	(0.055, 0.191)	(0.051, 0.078)	(0.099, 0.207)	(1.062, 4.026)	(0.120, 0.960)	(0.112, 0.968)
α^Δ	-0.003	0.087	0.089	0.132	1.505	0.854	0.841
(range)	(-0.101, 0.132)	(0.042, 0.132)	(0.050, 0.117)	(0.077, 0.166)	(1.000, 3.057)	(0.424, 0.992)	(0.416, 0.984)
β	0.015	0.175	0.088	0.206	2.565	0.603	0.603
(range)	(-0.311, 0.687)	(0.024, 0.687)	(0.031, 0.195)	(0.042, 0.715)	(1.000, 8.405)	(0.000, 0.992)	(0.000, 0.992)
r	-0.015	0.060	0.069	0.097	1.102	0.933	0.933
(range)	(-0.061, 0.017)	(0.046, 0.081)	(0.063, 0.077)	(0.082, 0.118)	(1.000, 1.250)	(0.864, 0.984)	(0.864, 0.984)
ξ	0.016	0.056	0.069	0.094	1.027	0.955	0.947
(range)	(0.006, 0.033)	(0.042, 0.074)	(0.052, 0.092)	(0.070, 0.125)	(1.000, 1.070)	(0.912, 0.984)	(0.896, 0.984)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	2198.374	0.069	0.155	0.453
(range)	(871.824, 4060.014)	(0.051, 0.078)	(0.099, 0.208)	(0.120, 0.960)
α^Δ	1583.400	0.089	0.132	0.853
(range)	(860.441, 3774.893)	(0.050, 0.117)	(0.077, 0.166)	(0.416, 0.984)
β	11407.999	0.088	0.206	0.603
(range)	(2215.996, 27300.584)	(0.031, 0.195)	(0.042, 0.715)	(0.000, 0.992)
r	3587.994	0.069	0.097	0.933
(range)	(2709.078, 4451.964)	(0.063, 0.077)	(0.082, 0.118)	(0.864, 0.984)
ξ	2859.179	0.069	0.094	0.956
(range)	(1275.205, 4526.385)	(0.052, 0.092)	(0.070, 0.125)	(0.912, 0.984)

Table 5.10: Scenario (F2b): Aggregated estimates and effective sample-size estimates. MCAR using CART imputation, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2, 5, 8, with 7 covariates plus intercept, 3 latent variables.

(F3a) FA, MCAR with CC, dedicated structure, N=2500, M=15, Q=3+1, K=3

With an increased amount of observations from N=1000 to N=2500, additional measurements, i.e. fifteen instead of nine, but the same amount of latent variables and covariates, for complete-case analysis almost all results seem again as intended. Bias and absolute bias never exceed absolute values of 0.1 and the relative root mean squared errors are always below 1.5. Without any apparent pattern visible, 4 out of all 87 parameters are slightly below the lower bound of coverages (i.e. $\beta_{5,1}$, $\beta_{6,3}$, $\beta_{4,4}$ and $\beta_{9,4}$), and 5 are slightly above (i.e. $\beta_{4,1}$, $\beta_{1,2}$, $\beta_{12,2}$, $\beta_{1,3}$ and $\beta_{8,4}$, and compare tables 5.11 and 5.12, with details starting at page 232).

Effective sample-size ranges for the scenario with 2500 observations from 1239.924 up to 14529.037 (i.e. between 3% and 36% can be viewed as stemming from an i.i.d. sample). With reference to thinning, like before, only slight changes in coverage rates are visible, resulting in increases for the smallest coverage rates of α^{Δ}-, β- and ξ-parameters and a decrease of of 0.008 for one of the σ^2_{ε}-chain based estimates.

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias\rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol{\psi}}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	-0.003	0.023	0.029	0.039	1.033	0.963	0.963
(range)	(-0.010, 0.004)	(0.012, 0.029)	(0.015, 0.037)	(0.020, 0.050)	(1.000, 1.071)	(0.944, 0.976)	(0.936, 0.984)
α^Δ	0.000	0.046	0.056	0.077	1.009	0.946	0.946
(range)	(-0.018, 0.017)	(0.018, 0.063)	(0.022, 0.081)	(0.030, 0.106)	(1.000, 1.027)	(0.920, 0.968)	(0.928, 0.976)
β	0.000	0.034	0.044	0.059	1.009	0.952	0.952
(range)	(-0.022, 0.018)	(0.012, 0.059)	(0.015, 0.064)	(0.020, 0.093)	(0.955, 1.069)	(0.904, 0.992)	(0.896, 0.992)
r	-0.005	0.030	0.039	0.052	1.000	0.965	0.965
(range)	(-0.007, -0.001)	(0.030, 0.031)	(0.037, 0.040)	(0.050, 0.053)	(1.000, 1.000)	(0.944, 0.984)	(0.944, 0.984)
ξ	0.002	0.039	0.048	0.066	1.003	0.945	0.945
(range)	(-0.007, 0.012)	(0.031, 0.049)	(0.037, 0.063)	(0.051, 0.084)	(1.000, 1.017)	(0.920, 0.976)	(0.912, 0.984)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	3567.393	0.029	0.039	0.963
(range)	(2029.029, 4711.920)	(0.015, 0.037)	(0.020, 0.050)	(0.944, 0.976)
α^Δ	2855.265	0.056	0.077	0.943
(range)	(1239.924, 4682.337)	(0.022, 0.082)	(0.030, 0.106)	(0.912, 0.976)
β	6903.551	0.044	0.059	0.953
(range)	(2147.958, 14529.037)	(0.015, 0.064)	(0.020, 0.093)	(0.904, 0.992)
r	6504.964	0.039	0.052	0.965
(range)	(5644.037, 7553.948)	(0.037, 0.040)	(0.050, 0.053)	(0.944, 0.984)
ξ	4915.659	0.048	0.066	0.947
(range)	(3129.231, 6563.428)	(0.037, 0.063)	(0.051, 0.084)	(0.920, 0.984)

Table 5.11: Scenario (F3a): Aggregated estimates and effective sample-size estimates. MCAR using complete-case analysis, 2500 observations, dedicated, 15 measurements, dichotomous measurements: 1, 2, 3, 6, 7, 8, 11, 12, 13, ordered measurements: 4, 9, 14, with 3 covariates plus intercept, 3 latent variables.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\|bias\|}$	$\hat{\sigma}(\|bias\|)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
Below lower bound of coverage:												
$\beta_{5,1}$	-0.601	-0.602	-0.001	0.019	0.015	0.023	0.000	0.032	0.010	1.000	0.904	0.912
$\beta_{6,3}$	-0.666	-0.661	0.005	0.059	0.047	0.064	0.004	0.093	0.034	1.000	0.904	0.896
$\beta_{4,4}$	-0.376	-0.361	0.016	0.040	0.028	0.043	0.002	0.062	0.020	1.043	0.904	0.896
$\beta_{9,4}$	-0.227	-0.226	0.001	0.040	0.030	0.044	0.002	0.063	0.021	1.000	0.904	0.896
Above higher bound of coverage:												
$\beta_{4,1}$	-0.217	-0.213	0.004	0.030	0.020	0.040	0.001	0.052	0.012	1.000	0.992	0.992
$\beta_{1,2}$	-0.527	-0.524	0.003	0.034	0.026	0.048	0.002	0.062	0.017	1.000	0.992	0.992
$\beta_{12,2}$	-0.746	-0.735	0.010	0.040	0.028	0.056	0.003	0.072	0.018	1.021	0.992	0.992
$\beta_{1,3}$	0.223	0.222	-0.001	0.037	0.025	0.049	0.002	0.064	0.016	1.000	0.992	0.992
$\beta_{8,4}$	-0.716	-0.715	0.001	0.034	0.025	0.054	0.003	0.067	0.016	1.000	0.992	0.992

Table 5.12: Scenario (F3a): Interesting parameters with respect to coverage rates. MCAR using complete-case analysis, 2500 observations, dedicated, 15 measurements, dichotomous measurements: 1, 2, 3, 6, 7, 8, 11, 12, 13, ordered measurements: 4, 9, 14, with 3 covariates plus intercept, 3 latent variables.

(F3b) FA, MCAR with CART, dedicated structure, N=2500, M=15, Q=3+1, K=3

As we have already seen for the first simulations, applying CART imputation in factor analysis, creates inferior estimates compared to complete-case analysis. And the scenario that employs 2500 observations is of no exception. With 87 parameters in total, 43 are below the lower bound of acceptable coverage rates (with 35 β-parameters and the remaining ones stemming from $\mathbf{\Sigma}_\varepsilon$, $\boldsymbol{\alpha}$ and $\mathbf{R}$). More severe: nine parameters are below 50% (where seven stem from $\boldsymbol{\beta}$ and one from $\boldsymbol{\alpha}$ and from $\mathbf{\Sigma}_\varepsilon$). Additionally, one parameter, i.e. $\xi_{9,2}$, lies above 0.988. When taking a look at RRMSE and (absolute) bias, the reason seems obvious. For 21 parameters, the relative RMSE is above 1.5, and eleven are above 2.0–but only three parameters have an absolute bias above 0.1. Therefore, the draws created traverse the parameter space very narrowly, resulting in very small central posterior region estimates. Hence, although the posterior mean estimates are close to the true value, the coverage is problematic (compare also 5.13 and Appendix D, pages 237ff.).

When it comes to effective sample-size and thinning, results are obtained that tend to be comparable to the complete-case analysis: between, again, 3% (i.e. 1237.186/40000) and now up to 40% (i.e. 16021.212/40000) can be viewed as stemming from an i.i.d. sample. When considering coverages, σ^2_ε-parameter rates increase from 0.096 to 0.112 for the smallest rate and from 0.848 to 0.872 for the greatest coverage rate. The smallest coverage for one of the ξ-parameters decreases (from 0.992 to 0.968) and all other ranges remain the same.

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias\rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ_ε^2	0.039	0.042	0.023	0.050	2.254	0.581	0.563
(range)	(0.016, 0.076)	(0.017, 0.076)	(0.013, 0.031)	(0.023, 0.081)	(1.333, 3.762)	(0.120, 0.864)	(0.096, 0.848)
α^Δ	-0.008	0.044	0.048	0.069	1.176	0.898	0.897
(range)	(-0.070, 0.018)	(0.015, 0.075)	(0.018, 0.070)	(0.025, 0.098)	(1.000, 2.571)	(0.472, 0.976)	(0.464, 0.976)
β	0.002	0.045	0.037	0.062	1.438	0.793	0.796
(range)	(-0.127, 0.109)	(0.010, 0.127)	(0.013, 0.055)	(0.018, 0.137)	(1.000, 2.900)	(0.216, 0.984)	(0.216, 0.984)
r	-0.013	0.030	0.034	0.048	1.092	0.936	0.933
(range)	(-0.021, -0.004)	(0.028, 0.032)	(0.033, 0.035)	(0.046, 0.050)	(1.000, 1.182)	(0.920, 0.960)	(0.904, 0.968)
ξ	-0.004	0.033	0.040	0.055	1.008	0.953	0.955
(range)	(-0.010, 0.002)	(0.028, 0.041)	(0.031, 0.051)	(0.044, 0.069)	(1.000, 1.029)	(0.920, 0.984)	(0.920, 0.992)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ_ε^2	5421.733	0.023	0.050	0.581
(range)	(4935.552, 5987.954)	(0.013, 0.031)	(0.023, 0.081)	(0.112, 0.872)
α^Δ	3238.054	0.048	0.069	0.897
(range)	(1237.186, 6021.211)	(0.018, 0.070)	(0.025, 0.098)	(0.464, 0.976)
β	7661.869	0.037	0.062	0.794
(range)	(2130.518, 16021.212)	(0.013, 0.055)	(0.017, 0.137)	(0.216, 0.984)
r	6029.508	0.034	0.048	0.933
(range)	(5507.828, 6800.382)	(0.033, 0.035)	(0.046, 0.050)	(0.904, 0.968)
ξ	5255.212	0.040	0.055	0.949
(range)	(3802.571, 6628.184)	(0.031, 0.051)	(0.044, 0.069)	(0.920, 0.968)

Table 5.13: Scenario (F3b): Aggregated estimates and effective sample-size estimates. MCAR using CART imputation, 2500 observations, dedicated, 15 measurements, dichotomous measurements: 1, 2, 3, 6, 7, 8, 11, 12, 13, ordered measurements: 4, 9, 14, with 3 covariates plus intercept, 3 latent variables.

(F4a) FA, MCAR with CART, dedicated structure, N=1000, M=10, Q=2+0, K=1

As already pointed out, the last two additional scenarios for factor analysis have a reduced amount of parameters, which are thirty in total and encompass only non-zero loading parameters $\boldsymbol{\alpha}^{\Delta}$ and covariate coefficients $\boldsymbol{\beta}$. When looking at the results after estimation and post-processing of sign-switches: sixteen are below the lower bound of the coverage rate of 0.912 (while none is below 0.8) and none is above 0.988. Similarly, no parameter exceeds a relative root mean squared error of 1.5, and the absolute bias is below 0.1 throughout (compare 5.14, with details in Appendix D, pages 242ff.). Percentages with respect to i.i.d. samples, compared to the total amount of Markov chain draws without burn-in, range from 7% to 28%. When compared coverage rate estimates using quantiles, the confidence-interval based thinned coverage rates stay almost the same, or are slightly decreased when compared to confidence-interval based coverage proportions.

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol{\psi}}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
α^{Δ}	0.006	0.062	0.072	0.101	1.056	0.929	0.938
(range)	(-0.030, 0.042)	(0.053, 0.072)	(0.059, 0.081)	(0.084, 0.113)	(1.000, 1.141)	(0.888, 0.968)	(0.912, 0.968)
β	-0.004	0.056	0.057	0.084	1.154	0.882	0.883
(range)	(-0.066, 0.054)	(0.046, 0.069)	(0.049, 0.063)	(0.074, 0.097)	(1.000, 1.673)	(0.808, 0.968)	(0.808, 0.952)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
α^{Δ}	3654.645	0.072	0.101	0.928
(range)	(2789.510, 5566.525)	(0.059, 0.081)	(0.084, 0.113)	(0.888, 0.968)
β	7413.950	0.057	0.084	0.882
(range)	(4962.619, 11324.461)	(0.049, 0.063)	(0.074, 0.097)	(0.808, 0.968)

Table 5.14: Scenario (F4a): Aggregated estimates and effective sample-size estimates. MCAR using CART imputation, 1000 observations, dedicated, 10 measurements, all dichotomous, with 2 covariates and no intercept, 1 latent variable.

(F4b) FA, MCAR with CART and $\mathbf{Y}^*$, dedicated structure, N=1000, M=10, Q=2+0, K=1

With employing $\mathbf{Y}^*$ as additional information in imputation, next to observed parts of the covariates, the results are similar to (F4a), but slightly worsen: seventeen parameter estimates lie below a coverage rate of 0.912 and five lie beneath 0.8 (while none is lower than 0.5). For the upper bound, no parameter estimate has a larger coverage rate than 0.988. Different from before, three parameter estimates can be located that have a RRMSE of 1.5 and above, but at least none exceeds an absolute bias of 0.1 (compare also 5.15 and pages 244ff. in Appendix D). Finally, whether we use $\mathbf{Y}^*$ or $\mathbf{Y}$ as additional information, does not seem to have any effect on effective sample-size or thinning compared to complete-cases.

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias\rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol{\psi}}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
α^{Δ}	0.007	0.062	0.072	0.101	1.070	0.924	0.927
(range)	(-0.031, 0.043)	(0.052, 0.072)	(0.059, 0.082)	(0.084, 0.113)	(1.000, 1.141)	(0.888, 0.960)	(0.888, 0.960)
β	-0.006	0.057	0.057	0.086	1.227	0.875	0.879
(range)	(-0.076, 0.063)	(0.044, 0.078)	(0.050, 0.064)	(0.074, 0.103)	(1.000, 1.875)	(0.776, 0.976)	(0.784, 0.984)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
α^{Δ}	3661.676	0.072	0.101	0.924
(range)	(2777.616, 5546.948)	(0.059, 0.082)	(0.084, 0.113)	(0.888, 0.960)
β	7422.508	0.057	0.086	0.874
(range)	(4954.093, 10981.074)	(0.050, 0.064)	(0.074, 0.103)	(0.776, 0.976)

Table 5.15: Scenario (F4b): Aggregated estimates and effective sample-size estimates. MCAR using CART imputation with $\mathbf{Y}^*$ instead of $\mathbf{Y}$, 1000 observations, dedicated, 10 measurements, all dichotomous, with 2 covariates and no intercept, 1 latent variable.

5.3.4 Individual Results for the IRT model

(I1a) IRT, BD, dedicated structure, N=1000, M=9, Q=3, K=3

The first scenarios for IRT, (I1a) to (I1e), encompass 39 parameters in total. Given before deletion data, (I1a), two parameters are below a coverage rate of 0.912 and one is above 0.988. With respect to absolute bias and relative root mean squared error, all values are small, i.e. no absolute bias exceeds 0.1 and all RRMSEs are below 1.5 (in fact, none is above 1.1). Compared to the basic scenarios of factor analysis, the effective sample-sizes are increased, ranging from 1770.864 to 22056.174 (around 4.4% and 55.1%); but once again, thinning does not tend to have any severe negative effects with regards to RMSE and coverage (compare 5.16, with detailed depictions starting on page 246 in Appendix D).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol{\psi}}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ_ε^2	0.006	0.045	0.054	0.074	1.005	0.933	0.928
(range)	(0.002, 0.008)	(0.041, 0.050)	(0.048, 0.063)	(0.067, 0.085)	(1.000, 1.016)	(0.912, 0.952)	(0.912, 0.936)
α^Δ	-0.000	0.042	0.049	0.068	1.008	0.934	0.938
(range)	(-0.015, 0.012)	(0.030, 0.053)	(0.035, 0.069)	(0.050, 0.093)	(1.000, 1.019)	(0.904, 0.968)	(0.904, 0.976)
γ	-0.007	0.051	0.061	0.085	1.010	0.945	0.940
(range)	(-0.019, 0.009)	(0.042, 0.063)	(0.052, 0.073)	(0.072, 0.103)	(0.986, 1.027)	(0.928, 0.960)	(0.904, 0.960)
β	-0.002	0.037	0.045	0.062	1.004	0.948	0.947
(range)	(-0.007, 0.006)	(0.022, 0.048)	(0.030, 0.057)	(0.039, 0.078)	(1.000, 1.034)	(0.912, 0.976)	(0.912, 0.968)
r	0.000	0.046	0.059	0.079	1.000	0.968	0.968
(range)	(-0.005, 0.003)	(0.041, 0.050)	(0.057, 0.061)	(0.074, 0.084)	(1.000, 1.000)	(0.952, 0.992)	(0.952, 0.992)
ξ	0.006	0.051	0.065	0.088	1.000	0.944	0.943
(range)	(0.001, 0.010)	(0.042, 0.062)	(0.052, 0.080)	(0.073, 0.108)	(1.000, 1.000)	(0.928, 0.960)	(0.912, 0.960)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ_ε^2	2616.684	0.054	0.074	0.928
(range)	(1770.864, 3136.689)	(0.048, 0.063)	(0.067, 0.085)	(0.912, 0.944)
α^Δ	3531.798	0.049	0.068	0.935
(range)	(1811.704, 4660.419)	(0.035, 0.069)	(0.050, 0.093)	(0.904, 0.968)
γ	7236.640	0.061	0.085	0.945
(range)	(3179.582, 10478.072)	(0.052, 0.073)	(0.072, 0.102)	(0.928, 0.960)
β	8934.318	0.045	0.062	0.948
(range)	(5277.909, 13680.625)	(0.030, 0.057)	(0.039, 0.078)	(0.912, 0.976)
r	5467.063	0.059	0.079	0.965
(range)	(5120.515, 5914.957)	(0.057, 0.061)	(0.074, 0.084)	(0.952, 0.992)
ξ	14760.374	0.065	0.088	0.943
(range)	(7993.104, 22056.174)	(0.052, 0.080)	(0.073, 0.108)	(0.920, 0.960)

Table 5.16: Scenario (I1a): Aggregated estimates and effective sample-size estimates. Before deletion, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2, 5, 8, with 3 covariates without intercept, 3 latent variables.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
Below lower bound of coverage:												
$\alpha_{9,3}$	-0.709	-0.697	0.012	0.040	0.035	0.045	0.002	0.064	0.027	1.019	0.904	0.904
$\gamma_{1,3}$	-0.564	-0.580	-0.016	0.056	0.047	0.063	0.007	0.091	0.035	1.014	0.928	0.904
Above higher bound of coverage:												
$r_{2,1}$	-0.111	-0.108	0.003	0.041	0.029	0.058	0.003	0.074	0.019	1.000	0.992	0.992

Table 5.17: Scenario (I1a): Interesting parameters with respect to coverage rates. Before deletion, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2, 5, 8, with 3 covariates without intercept, 3 latent variables.

(I1b) IRT, MCAR with CC, dedicated structure, N=1000, M=9, Q=3, K=3

When imposing missingness on two covariates, using a missing-completely-at-random mechanism, and estimate the total amount of 39 parameters after a list-wise deletion, six estimates lie slightly below the coverage rate's lower bound of 0.912 (none is below 0.8 or even 0.85, having the least favourable value at 0.888). Focussing on the upper bound, one parameter lies above 0.988. All (absolute) biases are below 0.1 and all RRMSEs are below 1.5–in fact none exceeds a value of 1.1.

When considering effective sample-size, it ranges from 4.2 % to 54.6% of information compared to independent draws. For thinning, some slight changes with respect to coverages are visible, most times towards a better result, but for one of the α-parameters, it gets decreased by one percentage point. RMSE estimates are however unaffected (see 5.18, and pages 250ff.).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias\rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ_ε^2	0.006	0.051	0.064	0.087	1.006	0.947	0.939
(range)	(0.002, 0.010)	(0.045, 0.061)	(0.057, 0.074)	(0.077, 0.102)	(1.000, 1.018)	(0.944, 0.952)	(0.920, 0.952)
α^Δ	-0.000	0.048	0.058	0.080	1.013	0.929	0.932
(range)	(-0.018, 0.015)	(0.036, 0.059)	(0.042, 0.081)	(0.059, 0.106)	(1.000, 1.028)	(0.888, 0.960)	(0.896, 0.960)
γ	-0.007	0.061	0.073	0.101	1.008	0.945	0.940
(range)	(-0.021, 0.012)	(0.049, 0.080)	(0.062, 0.087)	(0.084, 0.125)	(1.000, 1.022)	(0.904, 0.984)	(0.896, 0.976)
β	-0.004	0.043	0.053	0.073	1.006	0.936	0.934
(range)	(-0.010, 0.005)	(0.027, 0.054)	(0.035, 0.067)	(0.047, 0.090)	(0.976, 1.065)	(0.904, 0.984)	(0.896, 0.984)
r	-0.001	0.053	0.069	0.091	1.000	0.981	0.981
(range)	(-0.011, 0.005)	(0.050, 0.057)	(0.068, 0.072)	(0.088, 0.096)	(1.000, 1.000)	(0.960, 1.000)	(0.960, 1.000)
ξ	0.008	0.061	0.077	0.103	1.004	0.956	0.951
(range)	(-0.001, 0.020)	(0.052, 0.073)	(0.061, 0.095)	(0.086, 0.126)	(0.988, 1.023)	(0.944, 0.960)	(0.928, 0.960)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ_ε^2	2540.296	0.064	0.087	0.944
(range)	(1687.053, 3086.611)	(0.057, 0.074)	(0.077, 0.102)	(0.944, 0.944)
α^Δ	3463.674	0.058	0.080	0.931
(range)	(1739.888, 4611.805)	(0.042, 0.081)	(0.059, 0.106)	(0.880, 0.968)
γ	7062.963	0.073	0.101	0.944
(range)	(3104.920, 10107.735)	(0.062, 0.087)	(0.084, 0.125)	(0.904, 0.984)
β	8934.192	0.053	0.073	0.937
(range)	(5302.103, 13649.428)	(0.035, 0.067)	(0.047, 0.090)	(0.896, 0.984)
r	5403.360	0.069	0.091	0.984
(range)	(5099.227, 5808.332)	(0.068, 0.072)	(0.088, 0.096)	(0.960, 1.000)
ξ	14597.580	0.077	0.103	0.956
(range)	(7972.868, 21849.549)	(0.061, 0.095)	(0.086, 0.126)	(0.944, 0.960)

Table 5.18: Scenario (I1b): Aggregated estimates and effective sample-size estimates. MCAR using complete-case analysis, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2, 5, 8, with 3 covariates without intercept, 3 latent variables.

(I1c) IRT, MCAR with CART, dedicated structure, N=1000, M=9, Q=3, K=3

When conducting imputation by means of CART, four parameters lie below 0.912 for the coverage (with none below 0.8, and with the least favourable value at 0.856 with respect to a confidence-interval based coverage rate). The upper bound of 0.988 gets exceeded by one parameter. However, absolute biases are close to zero and the relative root mean squared errors are close to one. Similar to the complete-case scenario, effective sample-size ranges from roughly 4.2 % to 54.8% with respect to information compared to independent draws. Concerning thinning, almost no changes can be seen, except for an improvement for the γ-parameters of about 3 percentage points for the minimal coverage (compare 5.19, where details are again in Appendix D, beginning with page 254).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol{\psi}}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ_ε^2	0.004	0.046	0.055	0.076	1.000	0.931	0.928
(range)	(-0.001, 0.007)	(0.043, 0.052)	(0.049, 0.064)	(0.069, 0.087)	(1.000, 1.000)	(0.928, 0.936)	(0.928, 0.928)
α^Δ	0.000	0.042	0.050	0.069	1.002	0.941	0.945
(range)	(-0.010, 0.010)	(0.030, 0.055)	(0.036, 0.071)	(0.050, 0.095)	(1.000, 1.015)	(0.904, 0.976)	(0.904, 0.976)
γ	-0.005	0.054	0.063	0.088	1.022	0.929	0.927
(range)	(-0.025, 0.025)	(0.042, 0.063)	(0.053, 0.074)	(0.071, 0.102)	(1.000, 1.061)	(0.880, 0.952)	(0.856, 0.952)
β	-0.002	0.037	0.045	0.062	1.004	0.950	0.950
(range)	(-0.007, 0.006)	(0.021, 0.049)	(0.030, 0.057)	(0.039, 0.079)	(1.000, 1.034)	(0.904, 0.976)	(0.904, 0.976)
r	0.003	0.046	0.059	0.079	0.994	0.968	0.968
(range)	(0.000, 0.006)	(0.043, 0.050)	(0.058, 0.061)	(0.076, 0.084)	(0.982, 1.000)	(0.952, 0.992)	(0.952, 0.992)
ξ	0.006	0.051	0.065	0.088	0.995	0.948	0.945
(range)	(0.001, 0.009)	(0.042, 0.062)	(0.052, 0.081)	(0.073, 0.108)	(0.982, 1.000)	(0.920, 0.968)	(0.928, 0.960)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ_ε^2	2405.607	0.055	0.076	0.931
(range)	(1696.084, 2980.008)	(0.049, 0.064)	(0.069, 0.087)	(0.928, 0.936)
α^Δ	3408.974	0.050	0.069	0.944
(range)	(1765.269, 4579.275)	(0.036, 0.071)	(0.050, 0.095)	(0.904, 0.976)
γ	7065.748	0.063	0.088	0.929
(range)	(3202.542, 10595.365)	(0.053, 0.074)	(0.071, 0.102)	(0.880, 0.952)
β	8786.850	0.045	0.062	0.950
(range)	(5038.853, 13377.637)	(0.030, 0.057)	(0.039, 0.079)	(0.904, 0.976)
r	5434.522	0.059	0.079	0.965
(range)	(5109.214, 5914.336)	(0.058, 0.061)	(0.076, 0.084)	(0.944, 0.992)
ξ	14324.510	0.065	0.088	0.949
(range)	(7160.784, 21930.193)	(0.052, 0.081)	(0.073, 0.108)	(0.920, 0.968)

Table 5.19: Scenario (I1c): Aggregated estimates and effective sample-size estimates. MCAR using CART imputation, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2, 5, 8, with 3 covariates without intercept, 3 latent variables.

(I1d) IRT, MCAR & MAR with CC, dedicated structure, N=1000, M=9, Q=3, K=3

This scenario imposes missingness based on MCAR like before, but also facilitates a missing-at-random mechanism, such that the values in covariate $\mathbf{x}_2$ get removed, based on a Bernoulli-trial with probability $\Phi(\mathbf{x}_3 - q_{0.95}(\mathbf{x}_3))$, i.e. dependent on the third covariate. Complete-case analysis results in no parameter, which lies below the lower bound of the coverage rate we are aiming at, and one parameter ($r_{1,2}$) that lies above the upper bound. No bias and absolute bias exceeds 0.1 and the relative RMSE stays beneath 1.5 (and even 1.1).

Effective sample-size remains in the same regions as scenarios (I1a), (I1b) and (I1c) of roughly 4% to 55%. When using the effective sample-size in order to create a subset of the draws made, and calculate standard-error, RMSE and confidence-interval based coverage rates, again, severe negative effects are not visible (compare 5.20, and details can be found in Appendix D at page 254 and onwards).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias\rvert}}$	$\overline{\overline{\hat\sigma(\check{\boldsymbol\psi})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.006	0.052	0.063	0.087	1.000	0.933	0.931
(range)	(0.001, 0.011)	(0.047, 0.059)	(0.056, 0.073)	(0.078, 0.100)	(1.000, 1.000)	(0.928, 0.944)	(0.928, 0.936)
α^Δ	-0.002	0.048	0.057	0.080	1.018	0.931	0.929
(range)	(-0.024, 0.014)	(0.036, 0.062)	(0.042, 0.081)	(0.059, 0.108)	(1.000, 1.056)	(0.912, 0.944)	(0.912, 0.952)
γ	-0.009	0.062	0.073	0.101	1.012	0.939	0.931
(range)	(-0.025, 0.012)	(0.052, 0.077)	(0.061, 0.090)	(0.086, 0.125)	(1.000, 1.033)	(0.912, 0.976)	(0.912, 0.968)
β	-0.003	0.042	0.053	0.071	1.005	0.950	0.950
(range)	(-0.010, 0.004)	(0.024, 0.056)	(0.035, 0.066)	(0.045, 0.087)	(1.000, 1.028)	(0.920, 0.984)	(0.920, 0.984)
r	-0.003	0.051	0.069	0.091	0.994	0.971	0.971
(range)	(-0.007, 0.000)	(0.046, 0.056)	(0.067, 0.071)	(0.086, 0.096)	(0.982, 1.000)	(0.944, 1.000)	(0.944, 1.000)
ξ	0.007	0.060	0.077	0.104	1.002	0.953	0.953
(range)	(0.000, 0.015)	(0.048, 0.073)	(0.062, 0.094)	(0.084, 0.126)	(1.000, 1.012)	(0.936, 0.968)	(0.936, 0.968)

	$\overline{\overline{(\hat T)_{eff}}}$	$\overline{\overline{\hat\sigma_{eff}\left(\check{\boldsymbol\psi}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol\psi}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	2511.681	0.063	0.087	0.936
(range)	(1718.228, 3062.757)	(0.056, 0.073)	(0.078, 0.100)	(0.928, 0.944)
α^Δ	3461.110	0.057	0.079	0.931
(range)	(1755.924, 4490.600)	(0.042, 0.081)	(0.059, 0.108)	(0.912, 0.944)
γ	6940.827	0.073	0.101	0.939
(range)	(3172.456, 10017.565)	(0.061, 0.090)	(0.086, 0.125)	(0.912, 0.976)
β	8602.196	0.053	0.071	0.949
(range)	(5303.379, 12582.184)	(0.035, 0.067)	(0.045, 0.087)	(0.912, 0.984)
r	5344.354	0.069	0.091	0.971
(range)	(4993.658, 5753.691)	(0.067, 0.071)	(0.086, 0.096)	(0.944, 1.000)
ξ	14659.014	0.077	0.104	0.953
(range)	(8101.580, 22035.851)	(0.062, 0.094)	(0.084, 0.126)	(0.936, 0.968)

Table 5.20: Scenario (I1d): Aggregated estimates and effective sample-size estimates. Before deletion, 1000 observations, cross-loading, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2, 5, 8, with 3 covariates without intercept, 3 latent variables.

(I1e) IRT, MCAR & MAR with CART, dedicated structure, N=1000, M=9, Q=3, K=3

When looking at the aggregated results of 5.21 and detailed tables beginning at page 262, we can detect 1 parameter of 39 that is below ($\gamma_{1,1}$) and none above the outer bounds of suited coverage rates. RRMSE and (absolute) bias seem to be once again of no concern, i.e. no values exceed 1.5 (or even 1.1) for the former and no estimated results for the latter are 0.1 or above.

Just like the previous settings of (I1), outcomes for the measures of interest are not affected by thinning–specifically no parameter gets pushed outside of the desired coverage rate limits, that wasn't already (i.e. $\gamma_{1,1}$). Furthermore, the effective sample-size's minimal value, which is 1721.328 (4.3%), and the maximum value of 21949.784 (54.8%) are very close to the results of the before deletion scenario (I1a).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias\rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol{\psi}}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.005	0.046	0.055	0.076	1.007	0.939	0.931
(range)	(-0.001, 0.008)	(0.041, 0.051)	(0.049, 0.064)	(0.068, 0.086)	(1.000, 1.020)	(0.920, 0.952)	(0.912, 0.944)
α^Δ	0.000	0.042	0.049	0.069	0.996	0.940	0.939
(range)	(-0.009, 0.009)	(0.030, 0.055)	(0.035, 0.071)	(0.050, 0.095)	(0.974, 1.016)	(0.912, 0.976)	(0.912, 0.976)
γ	-0.004	0.053	0.063	0.088	1.018	0.935	0.935
(range)	(-0.025, 0.026)	(0.043, 0.062)	(0.053, 0.073)	(0.072, 0.102)	(1.000, 1.065)	(0.888, 0.968)	(0.896, 0.960)
β	-0.001	0.037	0.045	0.062	1.004	0.945	0.947
(range)	(-0.006, 0.007)	(0.022, 0.048)	(0.030, 0.057)	(0.039, 0.078)	(1.000, 1.037)	(0.912, 0.968)	(0.912, 0.968)
r	0.005	0.047	0.059	0.079	0.994	0.955	0.960
(range)	(0.002, 0.007)	(0.041, 0.052)	(0.058, 0.061)	(0.075, 0.085)	(0.983, 1.000)	(0.944, 0.976)	(0.944, 0.984)
ξ	0.006	0.051	0.065	0.088	1.000	0.945	0.943
(range)	(0.001, 0.010)	(0.042, 0.062)	(0.052, 0.081)	(0.073, 0.108)	(0.986, 1.014)	(0.928, 0.952)	(0.912, 0.952)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	2419.963	0.055	0.076	0.939
(range)	(1721.328, 2980.984)	(0.049, 0.064)	(0.068, 0.086)	(0.920, 0.952)
α^Δ	3428.841	0.050	0.069	0.939
(range)	(1790.578, 4585.050)	(0.036, 0.071)	(0.050, 0.095)	(0.912, 0.976)
γ	7149.664	0.063	0.088	0.934
(range)	(3291.985, 10731.156)	(0.053, 0.073)	(0.072, 0.102)	(0.888, 0.960)
β	8778.536	0.045	0.062	0.943
(range)	(5039.458, 13398.150)	(0.030, 0.057)	(0.039, 0.078)	(0.912, 0.968)
r	5447.920	0.059	0.079	0.957
(range)	(5164.060, 5940.726)	(0.058, 0.061)	(0.075, 0.085)	(0.944, 0.976)
ξ	14289.482	0.065	0.088	0.944
(range)	(7218.064, 21949.784)	(0.052, 0.081)	(0.073, 0.108)	(0.920, 0.952)

Table 5.21: Scenario (I1e): Aggregated estimates and effective sample-size estimates. Before deletion, 1000 observations, cross-loading, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2, 5, 8, with 3 covariates without intercept, 3 latent variables.

(I2a) IRT, BD, cross-loading, N=1000, M=9, Q=3, K=3

The remaining scenarios all deal with cross-loading as depicted with matrices $\mathbf{\Delta}_2$ and $\bar{\mathbf{\Delta}}$ in section 5.2, i.e. several measurements are related to more than one latent variable at the same time. Note, that these settings also use a changed position of binary, ordered and metric scaled measurements, providing a test and safeguard for correct scale detection and estimation within the algorithm.

Starting with a before deletion setting with 42 parameters. After estimation and aggregation one parameter ($\alpha_{4,2}$, and not related to cross-loading) has a coverage of 0.896–being the only one below 0.912. An additional parameter ($\alpha_{5,1}$, related to the cross-loading) lies at 0.992 for confidence interval calculated coverage rates and is thus above 0.988. An absolute bias being slightly above 0.1 can be found for $\alpha_{8,3}$ and $\alpha_{9,3}$, but there is no relative RMSE above 1.5–although there is one with 1.239 ($\sigma^2_{\varepsilon,4}$) and with 1.184 (again $\alpha_{4,2}$).

Effective sample-sizes tends to be low for some parameters when compared to 40000 draws. These range from 850.033 (2.1%) for one of the σ^2_{ε}-parameter estimates up to 22238.019 (55.6%) for one parameter of $\boldsymbol{\alpha}^{\Delta}$. If the effective sample-size is utilized for thinning, no severe changes for measures of interest are discovered. Tables that contain aggregates are the following table 5.22 and further details are presented in tables 7.91 to 7.96 starting on page 266.

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol{\psi}}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.014	0.036	0.046	0.062	1.080	0.947	0.949
(range)	(0.001, 0.034)	(0.024, 0.043)	(0.033, 0.053)	(0.043, 0.072)	(1.000, 1.239)	(0.928, 0.960)	(0.928, 0.960)
α^Δ	-0.003	0.041	0.052	0.070	1.019	0.950	0.946
(range)	(-0.024, 0.009)	(0.018, 0.071)	(0.027, 0.088)	(0.034, 0.120)	(1.000, 1.184)	(0.896, 0.984)	(0.896, 0.992)
γ	-0.005	0.047	0.057	0.078	1.031	0.948	0.944
(range)	(-0.023, 0.024)	(0.039, 0.053)	(0.051, 0.067)	(0.069, 0.090)	(1.000, 1.088)	(0.912, 0.984)	(0.912, 0.968)
β	-0.003	0.040	0.050	0.068	1.004	0.950	0.949
(range)	(-0.012, 0.008)	(0.024, 0.059)	(0.028, 0.067)	(0.039, 0.094)	(1.000, 1.017)	(0.928, 0.984)	(0.928, 0.976)
r	0.000	0.039	0.054	0.071	1.000	0.944	0.944
(range)	(-0.005, 0.003)	(0.034, 0.046)	(0.047, 0.060)	(0.061, 0.080)	(1.000, 1.000)	(0.936, 0.952)	(0.936, 0.952)
ξ	0.001	0.055	0.067	0.091	1.000	0.948	0.941
(range)	(-0.004, 0.009)	(0.042, 0.065)	(0.055, 0.080)	(0.073, 0.108)	(1.000, 1.000)	(0.920, 0.960)	(0.920, 0.960)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	2952.443	0.046	0.062	0.944
(range)	(850.033, 4141.984)	(0.033, 0.053)	(0.043, 0.072)	(0.928, 0.952)
α^Δ	3134.309	0.052	0.070	0.951
(range)	(1171.735, 6718.598)	(0.027, 0.088)	(0.034, 0.120)	(0.896, 0.984)
γ	7240.295	0.058	0.078	0.948
(range)	(3178.322, 11897.433)	(0.051, 0.067)	(0.069, 0.090)	(0.912, 0.984)
β	6556.455	0.050	0.067	0.951
(range)	(3264.276, 15834.633)	(0.028, 0.067)	(0.039, 0.094)	(0.928, 0.976)
r	6737.927	0.054	0.071	0.944
(range)	(4731.329, 9607.063)	(0.047, 0.060)	(0.061, 0.080)	(0.936, 0.952)
ξ	11897.726	0.067	0.091	0.947
(range)	(2269.954, 22238.019)	(0.055, 0.080)	(0.073, 0.108)	(0.920, 0.960)

Table 5.22: Scenario (I2a): Aggregated estimates and effective sample-size estimates. Before deletion, 1000 observations, cross-loading, 9 measurements, dichotomous measurements: 2, 5, 9, ordered measurements: 3, 6, 8, with 3 covariates without intercept, 3 latent variables.

(I2b) IRT, MCAR & MAR with CC, cross-loading, N=1000, M=9, Q=3, K=3

Scenario (I2b) already uses the $\Phi(\mathbf{x}_3 - q_{0.95}(\mathbf{x}_3))$-MAR mechanism next to MCAR missingness. When creating the complete-cases based on list-wise deletion, 2 of 42 parameters are slightly beneath 0.912 (with the smallest coverage rate of 0.896 for $\alpha_{4,2}$ and $\gamma_{2,2}$ with a rate of 0.904 with respect to confidence intervals). For 125 replications the coverage rate calculated via a confidence-interval for $\alpha_{5,1}$ results in a value of 0.992 and thus exceeds 0.988. An absolute bias above 0.1 is present for $\alpha_{5,2}$, $\alpha_{8,3}$ and $\alpha_{9,3}$, although no relative root mean squared error above 1.5 is visible. However, for $\alpha_{4,2}$ an RRMSE of 1.341 and for $\sigma^2_{\varepsilon,4}$ an RRMSE value of 1.412 is obtained.

Finally, behaviour of thinning and effective sample-size are comparable to the before deletion scenario (I2a) (compare table 5.23 and Appendix D, pages 270ff.).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias\rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.020	0.044	0.053	0.073	1.137	0.949	0.947
(range)	(0.003, 0.051)	(0.027, 0.057)	(0.038, 0.062)	(0.049, 0.087)	(1.000, 1.412)	(0.928, 0.984)	(0.928, 0.976)
α^Δ	-0.004	0.048	0.061	0.082	1.028	0.950	0.953
(range)	(-0.036, 0.011)	(0.023, 0.082)	(0.031, 0.103)	(0.041, 0.140)	(0.980, 1.341)	(0.896, 0.976)	(0.896, 0.992)
γ	-0.006	0.055	0.068	0.093	1.039	0.956	0.948
(range)	(-0.031, 0.034)	(0.047, 0.062)	(0.058, 0.077)	(0.079, 0.104)	(1.000, 1.125)	(0.912, 0.984)	(0.904, 0.984)
β	-0.003	0.046	0.057	0.078	1.008	0.945	0.947
(range)	(-0.016, 0.014)	(0.028, 0.065)	(0.033, 0.075)	(0.047, 0.105)	(1.000, 1.026)	(0.912, 0.968)	(0.912, 0.968)
r	0.001	0.047	0.063	0.083	1.000	0.968	0.968
(range)	(-0.001, 0.003)	(0.043, 0.055)	(0.055, 0.069)	(0.073, 0.092)	(1.000, 1.000)	(0.968, 0.968)	(0.968, 0.968)
ξ	0.005	0.063	0.077	0.105	1.001	0.953	0.952
(range)	(-0.001, 0.019)	(0.050, 0.080)	(0.063, 0.096)	(0.085, 0.129)	(0.986, 1.022)	(0.920, 0.976)	(0.920, 0.976)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	3027.270	0.053	0.073	0.949
(range)	(953.507, 4282.876)	(0.038, 0.062)	(0.049, 0.087)	(0.928, 0.984)
α^Δ	3191.047	0.061	0.082	0.952
(range)	(1308.667, 6320.116)	(0.031, 0.103)	(0.041, 0.140)	(0.904, 0.976)
γ	7357.397	0.068	0.093	0.956
(range)	(3471.344, 12090.957)	(0.058, 0.077)	(0.079, 0.104)	(0.912, 0.984)
β	6431.067	0.057	0.078	0.945
(range)	(3567.048, 14183.534)	(0.033, 0.075)	(0.047, 0.106)	(0.912, 0.968)
r	6500.088	0.063	0.083	0.971
(range)	(4670.824, 9235.532)	(0.055, 0.069)	(0.073, 0.092)	(0.968, 0.976)
ξ	11855.615	0.077	0.105	0.953
(range)	(2312.282, 21847.756)	(0.063, 0.096)	(0.085, 0.129)	(0.920, 0.976)

Table 5.23: Scenario (I2b): Aggregated estimates and effective sample-size estimates. MCAR & MAR using complete-case analysis, 1000 observations, cross-loading, 9 measurements, dichotomous measurements: 2, 5, 9, ordered measurements: 3, 6, 8, with 3 covariates without intercept, 3 latent variables.

(I2c) IRT, MCAR & MAR with CART, cross-loading, N=1000, M=9, Q=3, K=3

Applying CART-based imputation on the data sets of (I2a) with MCAR & MAR missingness imposed: 2 to 3 of 42 parameters lie below 0.912 (with the smallest coverage 0.872 for $\gamma_{3,2}$, $\gamma_{2,1}$ with 0.904–only with respect to confidence interval calculations–, and $\gamma_{3,1}$ and r_{23} with 0.904 each, but only for the quantile estimated rates). As with before deletion and complete-case considerations, the coverage of $\alpha_{5,1}$ is 0.992 and therefore above the upper bound. Absolute bias and RRMSE lie below 0.1 and 1.5 respectively–maximum values for the relative root mean squared error above 1.1 range from 1.266 and 1.314 and thus are lower than some of the instances in complete-case analysis. When it comes to thinning and effective sample-size, we once again end up with almost unaffected results that are comparable to scenarios (I2a) and (I2b). Again, compare table 5.24 for aggregates and Appendix D (page 274 and onwards).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias\rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.015	0.037	0.048	0.064	1.096	0.949	0.947
(range)	(0.001, 0.038)	(0.024, 0.046)	(0.034, 0.056)	(0.044, 0.075)	(1.000, 1.289)	(0.944, 0.960)	(0.936, 0.960)
α^Δ	-0.002	0.042	0.054	0.072	1.010	0.959	0.958
(range)	(-0.015, 0.014)	(0.019, 0.074)	(0.027, 0.096)	(0.035, 0.129)	(1.000, 1.079)	(0.928, 0.992)	(0.936, 0.992)
γ	-0.003	0.051	0.058	0.082	1.079	0.923	0.922
(range)	(-0.049, 0.038)	(0.042, 0.067)	(0.053, 0.066)	(0.072, 0.100)	(1.000, 1.266)	(0.872, 0.968)	(0.872, 0.944)
β	-0.003	0.041	0.050	0.068	1.009	0.942	0.945
(range)	(-0.012, 0.008)	(0.024, 0.058)	(0.029, 0.067)	(0.040, 0.094)	(1.000, 1.033)	(0.912, 0.968)	(0.912, 0.968)
r	-0.010	0.045	0.053	0.075	1.111	0.936	0.939
(range)	(-0.045, 0.010)	(0.036, 0.055)	(0.047, 0.059)	(0.063, 0.082)	(1.000, 1.314)	(0.904, 0.960)	(0.912, 0.960)
ξ	0.002	0.056	0.068	0.093	0.999	0.945	0.939
(range)	(-0.002, 0.010)	(0.043, 0.070)	(0.055, 0.083)	(0.074, 0.114)	(0.982, 1.013)	(0.928, 0.960)	(0.920, 0.960)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	2540.392	0.048	0.064	0.944
(range)	(810.466, 3994.589)	(0.034, 0.056)	(0.044, 0.075)	(0.928, 0.960)
α^Δ	2963.004	0.054	0.072	0.959
(range)	(1133.608, 6530.739)	(0.027, 0.096)	(0.035, 0.129)	(0.928, 0.992)
γ	7430.745	0.058	0.082	0.922
(range)	(3704.023, 11314.488)	(0.053, 0.066)	(0.072, 0.100)	(0.872, 0.968)
β	6366.934	0.050	0.068	0.942
(range)	(3177.273, 15337.074)	(0.029, 0.067)	(0.040, 0.094)	(0.912, 0.968)
r	6957.417	0.053	0.075	0.933
(range)	(4847.945, 9573.687)	(0.047, 0.059)	(0.063, 0.082)	(0.904, 0.952)
ξ	11349.316	0.068	0.093	0.945
(range)	(1781.345, 21313.943)	(0.055, 0.083)	(0.074, 0.115)	(0.928, 0.960)

Table 5.24: Scenario (I2c): Aggregated estimates and effective sample-size estimates. MCAR & MAR using CART imputation, 1000 observations, cross-loading, 9 measurements, dichotomous measurements: 2, 5, 9, ordered measurements: 3, 6, 8, with 3 covariates without intercept, 3 latent variables.

(I3a) IRT, MAR & MAR (cubic) with CC, cross-loading, N=1000, M=9, Q=3, K=3

Scenarios (I3a) and (I3b) are build upon the before deletion data of (I2a) that includes cross-loading. They additionally introduce an equality restriction onto parameters $\alpha_{8,1}$ and $\alpha_{9,1}$, such that the actual amount of parameters to estimate reduces to 41. Furthermore and as stated, missingness imposed on covariate $\mathbf{x}_2$ is dependent on a cubic relationship facilitating the normal CDF $\Phi(\mathbf{x}_3^3 - q_{0.95}(\mathbf{x}_3))$ as the inherent probability of missingness.

Applying complete-case analysis, i.e. (I3a), no parameter is outside the coverage rate bounds 0.912 and 0.988, and an absolute of 0.1 does not get exceeded (see table 5.25, and details within the appendix start at page 278) . With regards to relative root mean squared errors no values lie beyond 1.5 (but some are higher than 1.1, ranging from 1.119 to 1.420). When it comes to effective sample-size and thinning, the results here (and for (I3b) as well) are similar to the settings pertaining to (I2).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.022	0.047	0.055	0.077	1.145	0.936	0.939
(range)	(0.004, 0.051)	(0.029, 0.056)	(0.037, 0.068)	(0.050, 0.094)	(1.000, 1.420)	(0.912, 0.952)	(0.920, 0.952)
α^Δ	-0.007	0.047	0.058	0.079	1.032	0.952	0.950
(range)	(-0.034, 0.016)	(0.021, 0.076)	(0.031, 0.092)	(0.039, 0.127)	(0.982, 1.286)	(0.920, 0.984)	(0.920, 0.984)
γ	-0.009	0.059	0.071	0.097	1.042	0.948	0.941
(range)	(-0.032, 0.035)	(0.048, 0.068)	(0.058, 0.085)	(0.081, 0.115)	(1.000, 1.119)	(0.920, 0.984)	(0.904, 0.984)
β	-0.005	0.047	0.057	0.078	1.006	0.951	0.950
(range)	(-0.016, 0.011)	(0.029, 0.064)	(0.034, 0.072)	(0.048, 0.102)	(0.982, 1.031)	(0.928, 0.968)	(0.928, 0.968)
r	0.003	0.044	0.063	0.081	1.006	0.973	0.979
(range)	(-0.003, 0.010)	(0.036, 0.048)	(0.054, 0.070)	(0.068, 0.090)	(1.000, 1.017)	(0.968, 0.984)	(0.968, 0.992)
ξ	0.006	0.059	0.075	0.101	1.005	0.947	0.949
(range)	(-0.006, 0.028)	(0.050, 0.075)	(0.060, 0.097)	(0.085, 0.130)	(0.985, 1.043)	(0.920, 0.976)	(0.920, 0.976)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	2676.453	0.055	0.077	0.939
(range)	(896.556, 4227.751)	(0.038, 0.068)	(0.050, 0.094)	(0.912, 0.952)
α^Δ	3216.372	0.058	0.079	0.951
(range)	(1209.230, 6101.002)	(0.031, 0.092)	(0.039, 0.127)	(0.920, 0.984)
γ	6429.036	0.071	0.097	0.948
(range)	(3204.125, 11995.563)	(0.058, 0.085)	(0.081, 0.115)	(0.920, 0.984)
β	5920.104	0.057	0.078	0.951
(range)	(3777.181, 10909.110)	(0.034, 0.073)	(0.048, 0.102)	(0.928, 0.968)
r	6213.785	0.063	0.081	0.976
(range)	(4327.272, 9250.753)	(0.054, 0.070)	(0.068, 0.090)	(0.968, 0.984)
ξ	12111.913	0.075	0.101	0.948
(range)	(2982.159, 21335.018)	(0.060, 0.097)	(0.085, 0.130)	(0.920, 0.984)

Table 5.25: Scenario (I3a): Aggregated estimates and effective sample-size estimates. MAR & MAR (Cubic) using complete-case analysis, 1000 observations, cross-loading, 9 measurements, dichotomous measurements: 2, 5, 9, ordered measurements: 3, 6, 8, with 3 covariates without intercept, 3 latent variables, equality restriction on loadings 8 and 9 (of factor 1).

(I3b) IRT, MAR & MAR (cubic) with CART, cross-loading, N=1000, M=9, Q=3, K=3

Imputation by utilising the classification and regression tree approach results for (I3b) in two parameters that lie beneath 0.912 ($\gamma_{2,2}$ and $\gamma_{3,2}$), and one that is above 0.988 (either $\alpha_{5,1}$, when using the quantile estimate, or $\alpha_{1,1}$, when focussing on coverage boundaries adapted from the confidence interval). The maximum values for the relative root mean squared error is 1.373 and for the absolute bias is 0.071. Again, effective sample-size and effective sample-size based thinning is similar to before, although slightly diminished coverage rates for $\boldsymbol{\gamma}$-parameters are discernible (compare table 5.26, and Appendix D, pages 282ff.) .

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol{\psi}}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.016	0.040	0.051	0.068	1.081	0.960	0.957
(range)	(0.005, 0.033)	(0.026, 0.050)	(0.034, 0.065)	(0.045, 0.087)	(1.000, 1.244)	(0.944, 0.976)	(0.944, 0.976)
α^Δ	-0.003	0.040	0.053	0.070	1.008	0.961	0.959
(range)	(-0.012, 0.004)	(0.019, 0.068)	(0.028, 0.087)	(0.036, 0.117)	(0.986, 1.027)	(0.936, 0.992)	(0.928, 0.992)
γ	-0.003	0.053	0.061	0.085	1.082	0.926	0.930
(range)	(-0.052, 0.033)	(0.044, 0.071)	(0.053, 0.072)	(0.074, 0.107)	(1.000, 1.265)	(0.888, 0.952)	(0.896, 0.960)
β	-0.003	0.040	0.050	0.068	0.998	0.946	0.945
(range)	(-0.014, 0.006)	(0.024, 0.058)	(0.029, 0.067)	(0.040, 0.094)	(0.968, 1.017)	(0.928, 0.968)	(0.928, 0.968)
r	-0.010	0.046	0.056	0.076	1.132	0.957	0.957
(range)	(-0.048, 0.010)	(0.034, 0.058)	(0.048, 0.062)	(0.061, 0.085)	(1.000, 1.373)	(0.920, 0.976)	(0.920, 0.976)
ξ	0.004	0.053	0.068	0.091	1.004	0.948	0.945
(range)	(-0.002, 0.011)	(0.042, 0.063)	(0.055, 0.080)	(0.074, 0.108)	(1.000, 1.013)	(0.936, 0.960)	(0.928, 0.960)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	2300.739	0.051	0.068	0.960
(range)	(797.108, 3799.549)	(0.034, 0.065)	(0.045, 0.087)	(0.944, 0.976)
α^Δ	3012.597	0.053	0.070	0.961
(range)	(1119.200, 6537.632)	(0.028, 0.087)	(0.036, 0.117)	(0.936, 0.992)
γ	6702.794	0.061	0.085	0.926
(range)	(3715.143, 10927.022)	(0.053, 0.072)	(0.074, 0.108)	(0.888, 0.952)
β	6653.181	0.050	0.068	0.945
(range)	(3203.478, 16443.524)	(0.029, 0.067)	(0.040, 0.094)	(0.920, 0.968)
r	6667.216	0.056	0.076	0.957
(range)	(4490.448, 9658.884)	(0.048, 0.062)	(0.061, 0.085)	(0.920, 0.976)
ξ	11803.071	0.068	0.091	0.948
(range)	(2643.632, 21522.825)	(0.055, 0.080)	(0.074, 0.108)	(0.936, 0.968)

Table 5.26: Scenario (I3b): Aggregated estimates and effective sample-size estimates. MAR & MAR (Cubic) using CART imputation, 1000 observations, cross-loading, 9 measurements, dichotomous measurements: 2, 5, 9, ordered measurements: 3, 6, 8, with 3 covariates without intercept, 3 latent variables, equality restriction on loadings 8 and 9 (of factor 1).

(I4a) IRT, MAR & MAR (quadratic with $\mathbf{\Theta}$) MCAR in $\mathbf{Y}$ with CC, cross-loading, N=1000, M=9, Q=3, K=3

With including $\boldsymbol{\theta}_3^2$, next to $\mathbf{x}_3^2$, as an additional influence on missingness in $\mathbf{x}_2$ (i.e. $\Phi(\boldsymbol{\theta}_3^2 - \mathbf{x}_3^2 - q_{0.95}(\mathbf{x}_3) - q_{0.95}(\boldsymbol{\theta}_2)))$, seven parameters lie outside the coverage rate bounds 0.912 and 0.988, six below and one above, when employing complete-case analysis. When investigating further measures, eight absolute biases are above 0.1 (with 0.131 as the maximum value for $\alpha_{8,3}$) and one parameter has a relative root mean squared error above 1.5 (with a maximum value of 1.729, $\sigma_{\varepsilon,4}^2$). Effective sample-size and thinning behave however as before. Compare for this scenario table 5.27 and pages 286ff. of Appendix D.

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias\rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.037	0.073	0.081	0.115	1.246	0.923	0.917
(range)	(0.007, 0.083)	(0.046, 0.089)	(0.057, 0.106)	(0.078, 0.146)	(1.000, 1.729)	(0.896, 0.944)	(0.888, 0.944)
α^Δ	-0.014	0.074	0.089	0.123	1.055	0.933	0.938
(range)	(-0.055, 0.041)	(0.041, 0.131)	(0.047, 0.144)	(0.067, 0.207)	(0.988, 1.345)	(0.872, 0.976)	(0.896, 0.976)
γ	-0.013	0.096	0.120	0.162	1.048	0.958	0.940
(range)	(-0.064, 0.068)	(0.070, 0.119)	(0.088, 0.157)	(0.120, 0.207)	(0.993, 1.189)	(0.928, 0.976)	(0.880, 0.968)
β	-0.005	0.069	0.083	0.114	1.045	0.931	0.932
(range)	(-0.034, 0.057)	(0.040, 0.086)	(0.049, 0.108)	(0.067, 0.145)	(1.000, 1.158)	(0.896, 0.960)	(0.904, 0.968)
r	0.001	0.071	0.101	0.129	1.004	0.981	0.984
(range)	(-0.009, 0.016)	(0.064, 0.075)	(0.083, 0.113)	(0.109, 0.139)	(1.000, 1.012)	(0.968, 0.992)	(0.976, 0.992)
ξ	0.008	0.093	0.113	0.154	1.009	0.944	0.941
(range)	(-0.010, 0.043)	(0.078, 0.110)	(0.093, 0.141)	(0.128, 0.188)	(1.000, 1.045)	(0.928, 0.968)	(0.928, 0.968)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\psi}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	2589.302	0.081	0.115	0.920
(range)	(1310.928, 4052.987)	(0.057, 0.106)	(0.077, 0.146)	(0.896, 0.944)
α^Δ	3194.251	0.089	0.123	0.934
(range)	(1782.842, 5977.070)	(0.047, 0.143)	(0.067, 0.207)	(0.880, 0.976)
γ	6794.001	0.117	0.159	0.959
(range)	(3347.301, 10978.707)	(0.088, 0.149)	(0.120, 0.198)	(0.928, 0.976)
β	7229.288	0.083	0.114	0.932
(range)	(3979.735, 16684.923)	(0.049, 0.108)	(0.067, 0.145)	(0.904, 0.968)
r	5241.783	0.101	0.128	0.981
(range)	(3577.212, 8049.449)	(0.083, 0.113)	(0.109, 0.139)	(0.968, 0.992)
ξ	11965.092	0.113	0.154	0.943
(range)	(3195.827, 20608.135)	(0.093, 0.141)	(0.128, 0.188)	(0.928, 0.968)

Table 5.27: Scenario (I4a): Aggregated estimates and effective sample-size estimates. MAR & MAR (quadratic, with $\boldsymbol{\theta}_3$) using complete-case analysis, 1000 observations with 10% missingness overall in $\mathbf{y}_2$ to $\mathbf{y}_9$, cross-loading, 9 measurements, dichotomous measurements: 2, 5, 9, ordered measurements: 3, 6, 8, with 3 covariates without intercept, 3 latent variables, equality restriction on loadings 8 and 9 (of factor 1).

(I4b) IRT, MAR & MAR (quadratic with $\mathbf{\Theta}$) MCAR in $\mathbf{Y}$ with CART, cross-loading, N=1000, M=9, Q=3, K=3

Compared to complete-case analysis, using CART imputation here, results in slightly better outcomes, i.e. 4 measures for 41 parameters lie below the lower bound for coverage rates and one above. No (absolute) bias exceeds a value of 0.1, and the highest RRMSE that can be found, is 1.365 for the latent variable correlation r_{23}. Once again, effective sample-size and thinning behave just like before (see table 5.28 and tables 7.127 to 7.132 starting at page 290).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\psi})}}$	$\overline{\overline{RMSE\left(\check{\psi}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	0.020	0.044	0.055	0.074	1.132	0.944	0.944
(range)	(0.004, 0.045)	(0.027, 0.053)	(0.036, 0.070)	(0.047, 0.094)	(1.000, 1.383)	(0.920, 0.968)	(0.928, 0.968)
α^Δ	-0.002	0.043	0.056	0.075	1.008	0.951	0.955
(range)	(-0.021, 0.005)	(0.021, 0.074)	(0.029, 0.095)	(0.037, 0.128)	(0.977, 1.128)	(0.928, 0.984)	(0.928, 0.984)
γ	-0.004	0.053	0.063	0.087	1.062	0.933	0.934
(range)	(-0.061, 0.033)	(0.044, 0.079)	(0.054, 0.075)	(0.074, 0.115)	(1.000, 1.310)	(0.864, 0.968)	(0.880, 0.960)
β	-0.003	0.043	0.052	0.072	1.016	0.941	0.940
(range)	(-0.015, 0.008)	(0.026, 0.061)	(0.029, 0.070)	(0.042, 0.098)	(1.000, 1.065)	(0.904, 0.968)	(0.904, 0.968)
r	-0.013	0.047	0.059	0.080	1.122	0.979	0.979
(range)	(-0.049, 0.006)	(0.037, 0.059)	(0.050, 0.065)	(0.065, 0.091)	(1.000, 1.365)	(0.952, 1.000)	(0.952, 1.000)
ξ	0.005	0.057	0.072	0.097	1.004	0.939	0.941
(range)	(-0.001, 0.014)	(0.044, 0.068)	(0.058, 0.085)	(0.078, 0.114)	(1.000, 1.012)	(0.920, 0.960)	(0.912, 0.968)

	$\overline{\overline{(\hat{T})_{eff}}}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	2174.804	0.055	0.074	0.944
(range)	(818.059, 3524.200)	(0.036, 0.070)	(0.047, 0.094)	(0.920, 0.968)
α^Δ	2822.613	0.056	0.075	0.950
(range)	(1099.718, 5985.415)	(0.029, 0.095)	(0.038, 0.128)	(0.928, 0.984)
γ	6137.631	0.063	0.087	0.934
(range)	(3472.285, 10592.928)	(0.054, 0.075)	(0.074, 0.115)	(0.864, 0.976)
β	6821.184	0.052	0.072	0.941
(range)	(3201.846, 16966.896)	(0.029, 0.070)	(0.042, 0.098)	(0.904, 0.976)
r	5725.118	0.059	0.080	0.979
(range)	(4011.115, 8368.640)	(0.050, 0.065)	(0.065, 0.091)	(0.952, 1.000)
ξ	11447.201	0.072	0.097	0.939
(range)	(2433.886, 20791.158)	(0.058, 0.085)	(0.078, 0.114)	(0.920, 0.968)

Table 5.28: Scenario (I4b): Aggregated estimates and effective sample-size estimates. MAR & MAR (quadratic, with $\boldsymbol{\theta}_3$) using CART imputation, 1000 observations with 10% missingness overall in $\mathbf{y}_2$ to $\mathbf{y}_9$, cross-loading, 9 measurements, dichotomous measurements: 2, 5, 9, ordered measurements: 3, 6, 8, with 3 covariates without intercept, 3 latent variables, equality restriction on loadings 8 and 9 (of factor 1).

5.4 Summary of Results

The overall outcomes of the simulation studies show differences between factor analysis and item response theory based modelling. Although comparable results with respect to effective sample-size (with small percentages for idiosyncratic variances and high information for $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$) and thinning (that has almost no effect) are given, discrepancies of estimates' standard deviations, (absolute) biases, (relative) root mean squared errors and coverage rates are non-negligible. To summarise the results for the simulation studies conducted, we concentrate on the factor analysis scenarios first, because distinct and obvious problems emerged and can therefore be concisely depicted. Afterwards, a review of the results for IRT follows. But because of the similarities of the latters' outcomes, more detailed investigations seem to be necessary–especially examining possible gains that can be achieved by CART imputation compared to complete-case analysis.

Scenarios of Confirmatory Factor Analysis

When it comes to Bayesian confirmatory factor analysis, the first three scenarios, (F1a), (F1b) and (F1c), employ a dedicated structure, where (F1a) and (F1b) demonstrate the correct implementation of the estimation algorithm. When contrasting complete-case analysis with CART imputation, the latter's results are poor in comparison to the former's. For example, complete-case analysis ends up with three coverage rates outside of the targeted bounds, but for CART imputation there are more than 25 visible. Additionally, increased (absolute) biases and (relative) root mean squared errors can be found, pointing towards problems when CART imputation is conducted. An increase in the number of covariates–(F2a) and (F2b)–is not able to enhance the results. On the contrary, the quality of complete-case gets diminished, with respect to the measures considered, as well. Although complete-case nevertheless performs better than CART imputation, this increase of the numbers of covariates seems to be a problem for the sampler as such. Further investigations with these seven independent variables suggest that this problem is already present given before deletion (depicted in Appendix D, pp. 225). With increased numbers of observation and measurements, i.e. N = 2500 and M = 15, complete-case analysis (F3a) creates almost favourable results, although four too many rates exceed the coverage limits of 0.912 and 0.988. Yet, applying CART imputation, (F3b), shows that an increase of measurements is not a remedy: More than 40% of the estimated coverages lie outside of the bounds we are aiming for. For this setting, the biases tend to be low, but so are the draws' standard deviations–suggesting that the posterior draws traverse the parameter space in a region being too small, resulting in narrowed bounds and true values are therefore infrequently covered.

Scenarios of IRT models

In order to investigate gains (or losses) of efficiency and accuracy for complete-case analysis and CART imputation in IRT, we compare estimated standard deviations and root mean squared errors for every parameter under consideration. Specifically, we compare: first, complete-case results with before deletion, second, CART imputation results with before deletion and, third, complete-case analysis with CART imputation–for scenarios of dedicated structure, i.e. (I1a), (I1b), (I1c), (I1d) and (I1e), and cross-loading structure, i.e. (I2a), (I2b), (I2c), (I3a), (I3b), (I4a) and (I4b). Just like before, the results presented here, are highly aggregated (i.e. final means and ranges), while tables for single parameters are described in Appendix E, pp. 294ff.). Depicted are differences and fractions for estimators of interest of associated scenarios, i.e.

$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences:

$$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{CC} - \overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{BD}, \qquad \overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{CART} - \overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{BD}, \qquad \overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{CC} - \overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{CART}$$

$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions:

$$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{CC} \;/\; \overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{BD}, \qquad \overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{CART} \;/\; \overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{BD}, \qquad \overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{CC} \;/\; \overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}_{CART}$$

$\overline{RMSE}$ differences:

$$\overline{RMSE}_{CC} - \overline{RMSE}_{BD}, \quad \overline{RMSE}_{CART} - \overline{RMSE}_{BD}, \quad \overline{RMSE}_{CC} - \overline{RMSE}_{CART}$$

$\overline{RMSE}$ fractions:

$$\overline{RMSE}_{CC} \;/\; \overline{RMSE}_{BD}, \quad \overline{RMSE}_{CART} \;/\; \overline{RMSE}_{BD}, \quad \overline{RMSE}_{CC} \;/\; \overline{RMSE}_{CART}.$$

Of course, if the respective settings' measures of interest are equal, we either end with zero for differences or one for the fractions.

When closely investigating tables 5.29, 5.30 and 5.31 for the dedicated structure in use and tables 5.32, 5.33, 5.34 and 5.35 if cross-loading occurs, similar results emerge throughout: complete-case analysis always gets outperformed on average by CART imputation with respect to standard deviations and relative root mean squared errors.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
σ^2_ε:				
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences	0.010	0.001	0.009	0.001
(range)	(0.009, 0.011)	(0.001, 0.002)	(0.008, 0.010)	(0.001, 0.002)
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions	1.180	1.025	1.167	1.025
(range)	(1.175, 1.188)	(1.016, 1.039)	(1.159, 1.176)	(1.016, 1.039)
$\overline{RMSE}$ differences	0.013	0.002	0.013	0.002
(range)	(0.010, 0.017)	(0.002, 0.003)	(0.011, 0.015)	(0.001, 0.004)
$\overline{RMSE}$ fractions	1.169	1.032	1.171	1.028
(range)	(1.149, 1.200)	(1.024, 1.043)	(1.164, 1.176)	(1.012, 1.057)
α^Δ:				
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences	0.009	0.001	0.009	0.001
(range)	(0.006, 0.012)	(0.000, 0.002)	(0.006, 0.012)	(0.000, 0.002)
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions	1.182	1.018	1.179	1.012
(range)	(1.163, 1.200)	(1.000, 1.029)	(1.143, 1.229)	(1.000, 1.029)
$\overline{RMSE}$ differences	0.012	0.001	0.012	0.001
(range)	(0.008, 0.016)	(0.000, 0.002)	(0.008, 0.015)	(0.000, 0.002)
$\overline{RMSE}$ fractions	1.171	1.012	1.171	1.014
(range)	(1.140, 1.232)	(1.000, 1.025)	(1.116, 1.203)	(1.000, 1.036)
γ:				
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences	0.012	0.002	0.012	0.001
(range)	(0.009, 0.014)	(0.000, 0.003)	(0.008, 0.017)	(-0.001, 0.003)
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions	1.188	1.025	1.192	1.022
(range)	(1.170, 1.206)	(1.000, 1.049)	(1.151, 1.236)	(0.986, 1.049)
$\overline{RMSE}$ differences	0.016	0.003	0.016	0.003
(range)	(0.011, 0.022)	(-0.002, 0.007)	(0.014, 0.022)	(-0.001, 0.007)
$\overline{RMSE}$ fractions	1.184	1.036	1.195	1.032
(range)	(1.125, 1.214)	(0.981, 1.089)	(1.159, 1.260)	(0.990, 1.089)

Table 5.29: Efficiency and accuracy comparisons for dedicated IRT, CC and CART compared with Before Deletion (part 1).

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
β:				
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences	0.008	0.000	0.007	0.000
(range)	(0.005, 0.010)	(0.000, 0.000)	(0.005, 0.009)	(0.000, 0.000)
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions	1.175	1.000	1.164	1.000
(range)	(1.157, 1.194)	(1.000, 1.000)	(1.137, 1.194)	(1.000, 1.000)
$\overline{RMSE}$ differences	0.011	0.000	0.009	0.001
(range)	(0.007, 0.017)	(0.000, 0.001)	(0.006, 0.013)	(0.000, 0.001)
$\overline{RMSE}$ fractions	1.181	1.007	1.156	1.009
(range)	(1.108, 1.243)	(1.000, 1.023)	(1.115, 1.214)	(1.000, 1.024)
r:				
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences	0.011	0.000	0.010	0.000
(range)	(0.010, 0.011)	(0.000, 0.001)	(0.010, 0.010)	(0.000, 0.001)
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions	1.182	1.006	1.171	1.006
(range)	(1.172, 1.193)	(1.000, 1.018)	(1.164, 1.175)	(1.000, 1.018)
$\overline{RMSE}$ differences	0.013	0.000	0.012	0.001
(range)	(0.012, 0.014)	(-0.001, 0.002)	(0.012, 0.013)	(0.000, 0.001)
$\overline{RMSE}$ fractions	1.162	1.005	1.157	1.008
(range)	(1.143, 1.189)	(0.987, 1.027)	(1.143, 1.167)	(1.000, 1.014)
ξ:				
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences	0.012	0.000	0.012	0.000
(range)	(0.009, 0.015)	(0.000, 0.001)	(0.009, 0.016)	(0.000, 0.001)
$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions	1.178	1.002	1.189	1.002
(range)	(1.158, 1.188)	(1.000, 1.012)	(1.158, 1.208)	(1.000, 1.012)
$\overline{RMSE}$ differences	0.016	0.000	0.016	0.000
(range)	(0.013, 0.019)	(0.000, 0.001)	(0.011, 0.024)	(0.000, 0.001)
$\overline{RMSE}$ fractions	1.180	1.004	1.180	1.002
(range)	(1.167, 1.203)	(1.000, 1.014)	(1.151, 1.238)	(1.000, 1.014)

Table 5.30: Efficiency and accuracy comparisons for dedicated IRT, CC and CART compared with Before Deletion (part 2).

	(I1b)/(I1c) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I1d)/(I1e) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I1b)/(I1c) $(\overline{RMSE})$	(I1d)/(I1e) $(\overline{RMSE})$
σ_ε^2	1.151	1.139	1.133	1.139
(range)	(1.132, 1.163)	(1.132, 1.143)	(1.110, 1.172)	(1.108, 1.163)
α^Δ	1.162	1.166	1.157	1.154
(range)	(1.141, 1.174)	(1.141, 1.229)	(1.116, 1.232)	(1.116, 1.185)
γ	1.159	1.167	1.145	1.159
(range)	(1.127, 1.192)	(1.109, 1.250)	(1.065, 1.238)	(1.105, 1.260)
β	1.175	1.164	1.172	1.146
(range)	(1.157, 1.194)	(1.137, 1.194)	(1.108, 1.225)	(1.114, 1.186)
r	1.175	1.164	1.157	1.148
(range)	(1.172, 1.180)	(1.155, 1.172)	(1.143, 1.169)	(1.129, 1.167)
ξ	1.176	1.187	1.175	1.177
(range)	(1.158, 1.185)	(1.158, 1.208)	(1.155, 1.188)	(1.147, 1.238)

Table 5.31: Efficiency and accuracy comparisons for dedicated IRT, CC compared with CART.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
σ^2_ε:				
$\overline{\hat{\sigma}(\check{\psi})}$ differences	0.007	0.002	0.009	0.005
(range)	(0.005, 0.009)	(0.001, 0.003)	(0.004, 0.015)	(0.001, 0.012)
$\overline{\hat{\sigma}(\check{\psi})}$ fractions	1.152	1.042	1.180	1.098
(range)	(1.135, 1.170)	(1.030, 1.057)	(1.121, 1.283)	(1.030, 1.226)
$\overline{RMSE}$ differences	0.011	0.002	0.015	0.006
(range)	(0.006, 0.015)	(0.001, 0.003)	(0.007, 0.023)	(0.001, 0.016)
$\overline{RMSE}$ fractions	1.172	1.036	1.227	1.095
(range)	(1.140, 1.208)	(1.023, 1.042)	(1.163, 1.324)	(1.014, 1.225)
α^Δ:				
$\overline{\hat{\sigma}(\check{\psi})}$ differences	0.009	0.002	0.008	0.002
(range)	(0.004, 0.015)	(0.000, 0.008)	(0.004, 0.013)	(-0.001, 0.010)
$\overline{\hat{\sigma}(\check{\psi})}$ fractions	1.168	1.030	1.163	1.043
(range)	(1.145, 1.208)	(1.000, 1.091)	(1.045, 1.310)	(0.989, 1.238)
$\overline{RMSE}$ differences	0.012	0.002	0.011	0.001
(range)	(0.007, 0.020)	(-0.001, 0.009)	(0.002, 0.019)	(-0.003, 0.012)
$\overline{RMSE}$ fractions	1.173	1.021	1.176	1.025
(range)	(1.126, 1.220)	(0.982, 1.075)	(1.019, 1.305)	(0.946, 1.203)
γ:				
$\overline{\hat{\sigma}(\check{\psi})}$ differences	0.011	0.001	0.013	0.003
(range)	(0.007, 0.014)	(-0.001, 0.002)	(0.007, 0.025)	(0.000, 0.007)
$\overline{\hat{\sigma}(\check{\psi})}$ fractions	1.189	1.019	1.233	1.059
(range)	(1.137, 1.250)	(0.985, 1.039)	(1.125, 1.417)	(1.000, 1.113)
$\overline{RMSE}$ differences	0.014	0.004	0.019	0.007
(range)	(0.010, 0.017)	(0.000, 0.010)	(0.009, 0.032)	(0.000, 0.017)
$\overline{RMSE}$ fractions	1.182	1.044	1.241	1.084
(range)	(1.145, 1.233)	(1.000, 1.111)	(1.120, 1.386)	(1.000, 1.189)

Table 5.32: Efficiency and accuracy comparisons for cross-loading IRT, CC and CART compared with Before Deletion (part 1).

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
β:				
$\overline{\hat{\sigma}(\check{\psi})}$ differences	0.008	0.000	0.007	0.000
(range)	(0.005, 0.010)	(0.000, 0.001)	(0.005, 0.013)	(-0.003, 0.005)
$\overline{\hat{\sigma}(\check{\psi})}$ fractions	1.158	1.008	1.157	1.011
(range)	(1.119, 1.179)	(1.000, 1.036)	(1.075, 1.228)	(0.953, 1.088)
$\overline{RMSE}$ differences	0.010	0.001	0.010	0.001
(range)	(0.007, 0.014)	(0.000, 0.002)	(0.007, 0.016)	(-0.003, 0.005)
$\overline{RMSE}$ fractions	1.161	1.010	1.168	1.012
(range)	(1.117, 1.205)	(1.000, 1.026)	(1.082, 1.233)	(0.965, 1.063)
r:				
$\overline{\hat{\sigma}(\check{\psi})}$ differences	0.009	-0.001	0.009	0.002
(range)	(0.008, 0.009)	(-0.001, 0.000)	(0.007, 0.011)	(0.001, 0.002)
$\overline{\hat{\sigma}(\check{\psi})}$ fractions	1.161	0.988	1.172	1.030
(range)	(1.150, 1.170)	(0.982, 1.000)	(1.149, 1.200)	(1.021, 1.036)
$\overline{RMSE}$ differences	0.012	0.004	0.011	0.005
(range)	(0.012, 0.012)	(-0.001, 0.011)	(0.007, 0.015)	(0.000, 0.014)
$\overline{RMSE}$ fractions	1.172	1.058	1.150	1.070
(range)	(1.150, 1.197)	(0.988, 1.155)	(1.115, 1.211)	(1.000, 1.197)
ξ:				
$\overline{\hat{\sigma}(\check{\psi})}$ differences	0.011	0.001	0.008	0.001
(range)	(0.006, 0.016)	(0.000, 0.005)	(0.003, 0.017)	(0.000, 0.004)
$\overline{\hat{\sigma}(\check{\psi})}$ fractions	1.159	1.020	1.125	1.012
(range)	(1.105, 1.200)	(1.000, 1.064)	(1.053, 1.212)	(1.000, 1.071)
$\overline{RMSE}$ differences	0.014	0.002	0.010	-0.000
(range)	(0.005, 0.021)	(-0.001, 0.006)	(0.001, 0.022)	(-0.006, 0.003)
$\overline{RMSE}$ fractions	1.153	1.017	1.111	0.998
(range)	(1.058, 1.219)	(0.990, 1.056)	(1.009, 1.205)	(0.944, 1.039)

Table 5.33: Efficiency and accuracy comparisons for cross-loading IRT, CC and CART compared with Before Deletion (part 2).

	(I4a)&(I2a)	(I4b)&(I2a)	(I4a)&(I2a)	(I4b)&(I2a)
	σ^2_ε:		α^Δ:	
$\overline{\hat{\sigma}(\check{\psi})}$ differences	0.035	0.009	0.037	0.004
(range)	(0.024, 0.053)	(0.003, 0.017)	(0.017, 0.056)	(-0.013, 0.014)
$\overline{\hat{\sigma}(\check{\psi})}$ fractions	1.755	1.176	1.724	1.084
(range)	(1.538, 2.000)	(1.091, 1.321)	(1.279, 2.119)	(0.787, 1.333)
$\overline{RMSE}$ differences	0.053	0.012	0.053	0.005
(range)	(0.035, 0.075)	(0.004, 0.023)	(0.026, 0.087)	(-0.015, 0.017)
$\overline{RMSE}$ fractions	1.855	1.185	1.781	1.078
(range)	(1.694, 2.056)	(1.093, 1.324)	(1.321, 2.119)	(0.815, 1.288)
	γ:		β:	
$\overline{\hat{\sigma}(\check{\psi})}$ differences	0.063	0.005	0.033	0.003
(range)	(0.037, 0.097)	(0.002, 0.010)	(0.021, 0.044)	(0.000, 0.009)
$\overline{\hat{\sigma}(\check{\psi})}$ fractions	2.075	1.091	1.678	1.055
(range)	(1.725, 2.617)	(1.037, 1.161)	(1.562, 1.772)	(1.000, 1.158)
$\overline{RMSE}$ differences	0.084	0.009	0.047	0.004
(range)	(0.051, 0.122)	(0.001, 0.025)	(0.028, 0.064)	(0.001, 0.012)
$\overline{RMSE}$ fractions	2.051	1.109	1.708	1.064
(range)	(1.739, 2.470)	(1.013, 1.278)	(1.543, 1.884)	(1.012, 1.152)
	r:		ξ:	
$\overline{\hat{\sigma}(\check{\psi})}$ differences	0.047	0.005	0.046	0.005
(range)	(0.036, 0.053)	(0.003, 0.008)	(0.036, 0.061)	(0.003, 0.008)
$\overline{\hat{\sigma}(\check{\psi})}$ fractions	1.859	1.098	1.687	1.076
(range)	(1.766, 1.927)	(1.064, 1.145)	(1.615, 1.762)	(1.053, 1.143)
$\overline{RMSE}$ differences	0.058	0.010	0.063	0.006
(range)	(0.048, 0.067)	(0.004, 0.020)	(0.048, 0.080)	(0.004, 0.010)
$\overline{RMSE}$ fractions	1.823	1.137	1.694	1.068
(range)	(1.738, 1.944)	(1.062, 1.282)	(1.600, 1.808)	(1.037, 1.132)

Table 5.34: Efficiency and accuracy comparisons for cross-loading IRT, CC and CART compared with Before Deletion (part 3).

	(I2b)/(I2c) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I3a)/(I3b) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I2b)/(I2c) $(\overline{RMSE})$	(I3a)/(I3b) $(\overline{RMSE})$
σ_ε^2	1.106	1.076	1.132	1.123
(range)	(1.093, 1.118)	(1.046, 1.093)	(1.114, 1.160)	(1.080, 1.178)
α^Δ	1.134	1.115	1.150	1.149
(range)	(1.073, 1.208)	(1.053, 1.204)	(1.085, 1.218)	(1.050, 1.245)
γ	1.168	1.165	1.134	1.148
(range)	(1.094, 1.233)	(1.033, 1.308)	(1.040, 1.213)	(1.024, 1.292)
β	1.149	1.145	1.150	1.154
(range)	(1.119, 1.172)	(1.075, 1.207)	(1.111, 1.175)	(1.085, 1.233)
r	1.175	1.137	1.112	1.079
(range)	(1.169, 1.185)	(1.125, 1.158)	(1.012, 1.165)	(1.012, 1.115)
ξ	1.137	1.112	1.134	1.113
(range)	(1.096, 1.200)	(1.053, 1.212)	(1.062, 1.203)	(1.039, 1.204)

	(I4a)/(I4b) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I4a)/(I4b) $(\overline{RMSE})$
σ_ε^2	1.492	1.567
(range)	(1.379, 1.583)	(1.488, 1.660)
α^Δ	1.592	1.652
(range)	(1.500, 1.686)	(1.587, 1.811)
γ	1.896	1.848
(range)	(1.630, 2.275)	(1.622, 2.204)
β	1.591	1.606
(range)	(1.530, 1.724)	(1.480, 1.761)
r	1.694	1.610
(range)	(1.660, 1.738)	(1.516, 1.677)
ξ	1.569	1.585
(range)	(1.500, 1.659)	(1.524, 1.692)

Table 5.35: Efficiency and accuracy comparisons for cross-loading IRT, CC compared with CART.

Given a dedicated structure, the highest difference in fraction averages stems from the ordered-scale cut-off parameters with 0.187 (=1.189-1.002, for $\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$) and 0.178 (=1.180-1.002, $\overline{\overline{RMSE}}$) and therefore complete-case analysis standard-deviations and root mean squared errors are on average roughly 18 percentage points higher than those of CART imputation (compared to before deletion, when considering the cut-off parameters). If individual values are examined, the highest difference of fractions stems from $\gamma_{3,1}$ with 0.236 (1.236-1.000, for $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$) and 0.260 (1.260-1.000, for $\overline{RMSE}$). Hence, complete-case analysis standard-deviations and root mean squared errors are roughly 24 and 26 percentage points higher than those of CART imputation (compared to before deletion, when considering $\gamma_{1,3}$) . This increase of variation for complete-case analysis, and therefore the increase in efficiency of CART, is visible for most of the parameters. But, there is one parameter ($\gamma_{3,3}$) for CART, whose $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ results, and three parameters ($\gamma_{2,3}$,$\gamma_{3,3}$, r_{23}) , whose $\overline{RMSE}$ results, are diminished, i.e. below fractions of 1, when compared to before deletion. However, we are able to include this diverging reduction, i.e. if a reduction $a < 1$, we compare $(1 - a) + 1 = (2 - a)$ instead of solely a. After this recalculation CART still performs better than complete-case analysis (see table 5.36 on page 134).

With a changed structure that has multiple measurements (without missingness considered first) that load onto several latent variables, the overall outcomes are similar. The highest difference in fraction averages originates in the estimation of covariate regression parameters $\hat{\boldsymbol{\gamma}}$ with 0.174 (=1.232-1.059, for $\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$) and 0.157 (=1.241-1.084, $\overline{\overline{RMSE}}$) and therefore complete-case analysis standard-deviations are about 17 and root mean squared errors are about 16 percentage points on average higher than those of CART imputation (compared to before deletion, when considering the regression parameters $\boldsymbol{\gamma}$). If, once more, individual values are examined, the highest difference of fractions stems from $\gamma_{3,3}$ with 0.333 (1.416-1.083, for $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$) and 0.313 (1.386-1.072, for $\overline{RMSE}$). Hence, complete-case analysis standard-deviations and root mean squared errors are roughly 33 and 31 percentage points higher than those of CART imputation (compared to BD, when considering $\gamma_{1,3}$). Investigating parameters, whose performance for CART imputation is diminished with respect to before deletion (i.e. five for standard deviations and eight for RMSE), is, when compared with complete-case analysis, again better in performance–with $\alpha_{9,3}$ and $\xi_{3,8}$ as exceptions (the RMSEs here, are roughly 1% and 5% higher than those of complete-case analysis).

Increase	Parameter	Measure	comparison
Dedicated:			
0.219 (=1.233-(2-0.986))	$\gamma_{3,3}$	$\overline{\hat{\sigma}(\check{\psi})}$ fraction	(I1d)/(I1a) with (I1e)/(I1a)
0.204 (=1.214-(2-0.990))	$\gamma_{3,3}$	$\overline{RMSE}$ fraction	(I1d)/(I1a) with (I1e)/(I1a)
0.153 (=1.167-(2-0.986))	$\gamma_{3,2}$	$\overline{RMSE}$ fraction	(I1b)/(I1a) with (I1c)/(I1a)
0.194 (=1.214-(2-0.981))	$\gamma_{3,2}$	$\overline{RMSE}$ fraction	(I1b)/(I1a) with (I1c)/(I1a)
0.141 (=1.154-(2-0.987))	$r_{1,2}$	$\overline{RMSE}$ fraction	(I1b)/(I1a) with (I1c)/(I1a)
Cross-loading:			
0.034 (=1.045-(2-0.989))	$\alpha_{8,3}$	$\overline{\hat{\sigma}(\check{\psi})}$ fraction	(I3a)/(I2a) with (I3b)/(I2a)
0.134 (=1.149-(2-0.985))	$\gamma_{2,3}$	$\overline{\hat{\sigma}(\check{\psi})}$ fraction	(I2b)/(I2a) with (I2c)/(I2a)
0.031 (=1.078-(2-0.953))	$\beta_{9,1}$	$\overline{\hat{\sigma}(\check{\psi})}$ fraction	(I3a)/(I2a) with (I3b)/(I2a)
0.133 (=1.150-(2-0.983))	$r_{1,3}$	$\overline{\hat{\sigma}(\check{\psi})}$ fraction	(I2b)/(I2a) with (I2c)/(I2a)
0.145 (=1.163-(2-0.982))	$r_{2,3}$	$\overline{\hat{\sigma}(\check{\psi})}$ fraction	(I2b)/(I2a) with (I2c)/(I2a)
0.065 (=1.278-(2-0.787))	$\alpha_{9,1}$	$\overline{\hat{\sigma}(\check{\psi})}$ fraction	(I4a)/(I2a) with (I4b)/(I2a)
0.179 (= 1.196-(2-0.982))	$\alpha_{4,2}$	$\overline{RMSE}$ fraction	(I2b)/(I2a) with (I2c)/(I2a)
0.125 (= 1.179-(2-0.947))	$\alpha_{4,2}$	$\overline{RMSE}$ fraction	(I3a)/(I2a) with (I3b)/(I2a)
0.033 (= 1.058-(2-0.975))	$\alpha_{8,3}$	$\overline{RMSE}$ fraction	(I3a)/(I2a) with (I3b)/(I2a)
-0.010 (= 1.019-(2-0.971))	$\alpha_{9,3}$	$\overline{RMSE}$ fraction	(I3a)/(I2a) with (I3b)/(I2a)
0.138 (= 1.150-(2-0.988))	$r_{1,3}$	$\overline{RMSE}$ fraction	(I2b)/(I2a) with (I2c)/(I2a)
0.050 (= 1.063-(2-0.988))	$\xi_{3,2}$	$\overline{RMSE}$ fraction	(I3a)/(I2a) with (I3b)/(I2a)
0.049 (= 1.058-(2-0.990))	$\xi_{3,3}$	$\overline{RMSE}$ fraction	(I2b)/(I2a) with (I2c)/(I2a)
-0.046 (= 1.009-(2-0.944))	$\xi_{8,3}$	$\overline{RMSE}$ fraction	(I3a)/(I2a) with (I3b)/(I2a)
0.136 (= 1.321-(2-0.815))	$\alpha_{9,1}$	$\overline{RMSE}$ fraction	(I4a)/(I2a) with (I4b)/(I2a)

Table 5.36: Efficiency and accuracy comparisons for individual parameters in IRT, CC/BD compared with CART/BD.

When additionally investigating the outcomes when missingness is present within the measurements and missingness within the covariates is also dependent on $\boldsymbol{\theta}_3^2$, the differences between CART and complete-case analysis grow. Again, draws for covariate regression parameters $\boldsymbol{\gamma}$ have the highest difference in fraction averages being now 0.984 (=2.075-1.091, for $\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$) and 0.942 (=2.051 - 1.109, $\overline{\overline{RMSE}}$), i.e. more than 90 percentage points–on average it is 70 percentage points for all parameters. $\gamma_{3,3}$ is once more the parameter with the highest individual difference in percentage points (i.e. 147 and 135 points respectively), which is 1.466 (2.617-1.150, for $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$) and 1.349 (2.470-1.120, for $\overline{RMSE}$). $\alpha_{9,1}$ is the only parameter whose $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ and is $\overline{RMSE}$ lower than one, when compared to before deletion, but still outperforms complete-case analysis, even if we account for this reduced value, again see table 5.36 on page 134 for details.

Overall, imputation using CART for background information that influences outcomes directly, as it is modelled in the factor analytic scenarios, does not seem to create sufficient results. While we are able to model correlated latent variables, include ordered scaled measurements next to binary and metric scaled ones, the automated and augmented algorithm tends to be not as robust as complete-case analysis. Because the same procedure gets applied to IRT with far better performance, at least two reasons for the problems in factor analysis might be possible: a) The CART procedure is not suited, while further procedures, like predictive mean matching (see Little, 1988), might work. b) Imputation is problematic as such–because when missing values occur, not only the latent variables are unknown but also parts of the covariates, and information that needs to get condensed onto the latent variables and the missing parts in the background information is not separable between these, when both, $\boldsymbol{\Theta}$ and $\mathbf{X}$, are structured to have a direct linear influence, i.e. $\mathbf{X}\boldsymbol{\beta}' + \boldsymbol{\Theta}\boldsymbol{\alpha}'$.

Within the IRT settings, imputation using CART creates more efficient results than complete-case analysis in terms of RMSE and parameter standard deviations. The latter approach of handling of missing values tends to be prone to create higher variation and thus inaccuracies for estimates than before deletion and CART imputation respectively. Dependent on the missingness mechanism considered, we do reach 40% in gains like Preising (2018, p. 64). Scenarios (I4a) and (I4b) exhibit up to 146 percentage points in differences for single parameters and roughly 70 percentage points on average. The next simpler scenarios ((I3a) and (I3b)) do not exceed 40% in gains, but 33 percentage points in differences for single parameters and roughly 10 to 18 percentage points on average are still present. Hence, the application of the automated and augmented imputation strategy has positive effects. In order to finally test the algorithm under empirical conditions, we apply the IRT procedure to competency data.

Chapter 6

Empirical Application using NEPS-Data

The data, which will be used for the empirical application, stems from the National Educational Panel Study. As was briefly outlined in section 3.1, when discussing the approaches of Aßmann et al. (2015) and Gaasch (2017a), this panel-study analyses educational decisions and pathways from pre-, in- and post-school periods in Germany of multiple cohorts (again, see Blossfeld et al., 2011). Parts of this study are concerned with investigations of different kinds of competencies, where an overview of all competencies and abilities included are described in Weinert et al. (2011). According to the latter authors, the NEPS interest with regards to competency assessment lies especially in “acquisition of education-dependent, domain-specific competencies“ (Weinert et al., 2011), but there are three further domains that are considered, encompassing domain-general cognitive abilities/capacities, metacompetencies and social competencies, and stage-specific attainments. In order to allow a basic comparability to an IRT scenario, i.e. the scenario Gaasch (2017a, pp. 57ff.), we also use data from pupils of grade nine (i.e. Starting Cohort 4) from the NEPS (Version: doi:10.5157/NEPS:SC4:9.1.0) for the empirical application. Of course, using multidimensional latent variables and no general parameter restriction, like Gaasch (2017a) does, comparison is not completely given, but provides at least some reference points. Given a to be created data set, two main scenarios will get investigated: the first one without covariates and the second one using the full data available. For completeness, another set of scenarios is created using subsetted data and gets depicted in Appendix F (pp. 324) allowing, alongside parameter estimation, for approximate insights into process times, given different amounts of data.

6.1 Data Preparation and Scenario Description

Within the version of the scientific-use-file present, there are nine waves, with the number of persons participating, ranging from 15577 (wave 1) to 403 (wave 9); and wave 4 and 6 are without any participants. Having the most competencies present, we restrict the application to wave 1. Competencies that get considered are concerned with reading and mathematical literacy–having 22 and 31 measurements respectively, such that $M = 53$. Although one major interest lies in the general applicability of the IRT algorithm from above, the intuitions using these domains are the following: First, the mathematical items are present in the investigations of Gaasch (2017a), thus we have the aforementioned reference for comparison. Second, when investigating the tasks within these mathematical competencies, it is easy to recognise, that the question that get administered use language as the given medium and aren't purely mathematical (compare Schnittjer & Duchhardt, 2015, and Gehrer et al., 2012). Thus, reading seems as a competency that might be necessary for understanding the mathematical oriented test items as well, and therefore a relationship between mathematical competency and reading competency could be found. Hence, two latent variables ($K = 2$) are assumed for a model of dedicated structure, where these latent variables are additionally correlated (i.e. $|r_{12}| > 0$).

When it comes to covariates, background information gets included similar to Gaasch (2017a, p. 110), using gender, a transformed ISEI-08 prestigiousness score[30] for the occupation of the mother and the father (denoted with *HighISEI*), whether the school year was repeated (denoted with *Repeat*) and the type of school (denoted with *School*, restricted to Hauptschule, Realschule and Gymnasium). The transformed ISEI-08 score HighISEI is not original to the data itself, but results from data preparation: It stems from comparing the father's and the mother's ISEI-08 score of an individual and using the maximum value only. Thus, it is the maximal ISEI-08 score of the parents of a person under consideration.

Further data preparation is described as follows and is solely carried out with the help of the statistical software R. In order to obtain one data set that contains all information desired, three data sets from the scientific-use-files need to get merged: *pTarget* (containing background information), *CohortProfile*, which includes the school-type variable, and *xTargetCompetencies* (for competencies) are merged via target-IDs, i.e. the interviewees

[30] ISEI is short for International Socio-Economic Index of occupational status and represents one of the scores for occupation surveyed within the NEPS. See Ganzeboom (2010) for its origin.

ID-variable; and get restricted to wave 1. Additionally, negative values that indicate missingness within the original Stata files get retransformed to R's NA format. From the dependent variables $\mathbf{Y}$ 47 binary measurements are coded 0-1 and six items, having three to five categories, are coded such that the first category gets denoted with 1 and the variables are interpreted as being of ordered scale. The background information encompasses one continuous variable (*HighISEI*) and two already dichotomous variables (*Gender*: 0=male, 1=female, *Repeat*: 0=repeated, 1=not repeated). The school-type variable gets dummy coded using two variables (*SchoolRS* and *SchoolGY*), with 0 0, i.e. the base category, denotes Hauptschule, 1 0 represents Realschule and 0 1 describes Gymnasium–including respective tracks of comprehensive schools. With therefore restricting to only observed types of schools and, furthermore, in order to have sufficient information, restricting only to those individuals where at least three observations within the measurements are present (see Gaasch, 2017a, p. 58), one ends up with the final amount of available cases. Thus, there are 12340 individuals giving answer to 53 items, which all contain missings ranging from 0.10% to 25.07% and a total amount of missingness of 5.59%. With the dummy coded variables included, 3 of 5 covariates contain missings ranging from 0.21% to 20.80% and a total amount of missingness of 4.72%. Details of single variables are presented in Appendix F (pp. 324ff.), wherein table 7.223 contains information on the measurements and table 7.224 presents information for covariates.[31]

Given this prepared data, two main scenarios are under investigation, where the first 22 mathematical measurements are specified to load onto a latent variable $\boldsymbol{\theta}_{1,math}$ and the next 31 reading competency measurements load onto a second latent variable $\boldsymbol{\theta}_{2,read}$, i.e. $\boldsymbol{\Theta} = (\boldsymbol{\theta}_{1,math}, \boldsymbol{\theta}_{2,read})$–an overview is given in table 6.1. The first scenario, (M1), encompasses a general test without background information included, i.e. using a model with latent representation

$$\begin{aligned}
\text{(M1):}\ \ (\mathbf{y}_1^*, \ldots, \mathbf{y}_{53}^*) =& \boldsymbol{\Theta}\boldsymbol{\alpha}' - \boldsymbol{\iota}_N\boldsymbol{\beta}' + \boldsymbol{\varepsilon}, \\
\text{vec}(\boldsymbol{\varepsilon}) \sim& MVN(\mathbf{0}, \mathbf{I}_M \otimes \mathbf{I}_N) \text{ and} \\
\text{vec}(\boldsymbol{\Theta}) \sim& MVN(\mathbf{0}, \mathbf{R} \otimes \mathbf{I}_N).
\end{aligned} \tag{6.1.1}$$

The next scenario encompasses all variables described, i.e. the 53 measurements and, next to Gender, Repeat and HighISEI, the two above described dummy coded variables for the

[31] Note, that no information of individual persons is contained–only abstracted aggregates, like percentages of missingness or measurements of location (i.e. the mode or the mean) of a variable.

type of school, which are SchoolRS and SchoolGY.

$$
\begin{aligned}
\text{(M2):}\quad (\mathbf{y}_1^*, \ldots, \mathbf{y}_{53}^*) =& \boldsymbol{\Theta}\boldsymbol{\alpha}' - \boldsymbol{\iota}_N\boldsymbol{\beta}' + \boldsymbol{\varepsilon}, \qquad (6.1.2)\\
\operatorname{vec}(\boldsymbol{\varepsilon}) \sim& MVN(\mathbf{0}, \mathbf{I}_M \otimes \mathbf{I}_N),\\
\boldsymbol{\Theta} =& (\mathbf{x}_{1,Gender}, \mathbf{x}_{2,Repeat}, \mathbf{x}_{3,HighISEI}, \mathbf{x}_{4,SchoolRS}, \mathbf{x}_{5,SchoolGY})\boldsymbol{\gamma} + \boldsymbol{\upsilon} \qquad (6.1.3)\\
\text{and } \operatorname{vec}(\boldsymbol{\upsilon}) \sim& MVN(\mathbf{0}, \mathbf{R} \otimes \mathbf{I}_N).
\end{aligned}
$$

The auxiliary scenarios include covariates Gender, Repeat and HighISEI only and encompass three subsets of the data–each one pertaining to one of the school-types: Hauptschule (a), Realschule (b) and Gymnasium (c). The results are depicted in Appendix F, p. 336 and the following pages. The latent representations are therefore similar to before, the differences are present in the formulations of the latent variables and in a general subsetting of measurements and covariates, i.e.

$$
\begin{aligned}
\text{(M3a):}\quad (\mathbf{y}_1^*, \ldots, \mathbf{y}_{53}^*)_{|\text{School=Hauptschule}} =& \boldsymbol{\Theta}\boldsymbol{\alpha}' - \boldsymbol{\iota}_N\boldsymbol{\beta}' + \boldsymbol{\varepsilon},\\
\operatorname{vec}(\boldsymbol{\varepsilon}) \sim& MVN(\mathbf{0}, \mathbf{I}_M \otimes \mathbf{I}_N),\\
\boldsymbol{\Theta} =& (\mathbf{x}_{1,Gender}, \mathbf{x}_{2,Repeat}, \mathbf{x}_{3,HighISEI})\boldsymbol{\gamma} + \boldsymbol{\upsilon} \text{ and}\\
\operatorname{vec}(\boldsymbol{\upsilon}) \sim& MVN(\mathbf{0}, \mathbf{R} \otimes \mathbf{I}_N),\\
\text{(M3b):}\quad (\mathbf{y}_1^*, \ldots, \mathbf{y}_{53}^*)_{|\text{School=Realschule}} =& \boldsymbol{\Theta}\boldsymbol{\alpha}' - \boldsymbol{\iota}_N\boldsymbol{\beta}' + \boldsymbol{\varepsilon},\\
\operatorname{vec}(\boldsymbol{\varepsilon}) \sim& MVN(\mathbf{0}, \mathbf{I}_M \otimes \mathbf{I}_N),\\
\boldsymbol{\Theta} =& (\mathbf{x}_{1,Gender}, \mathbf{x}_{2,Repeat}, \mathbf{x}_{3,HighISEI})\boldsymbol{\gamma} + \boldsymbol{\upsilon} \text{ and}\\
\operatorname{vec}(\boldsymbol{\upsilon}) \sim& MVN(\mathbf{0}, \mathbf{R} \otimes \mathbf{I}_N),\\
\text{(M3c):}\quad (\mathbf{y}_1^*, \ldots, \mathbf{y}_{53}^*)_{|\text{School=Gymnasium}} =& \boldsymbol{\Theta}\boldsymbol{\alpha}' - \boldsymbol{\iota}_N\boldsymbol{\beta}' + \boldsymbol{\varepsilon},\\
\operatorname{vec}(\boldsymbol{\varepsilon}) \sim& MVN(\mathbf{0}, \mathbf{I}_M \otimes \mathbf{I}_N),\\
\boldsymbol{\Theta} =& (\mathbf{x}_{1,Gender}, \mathbf{x}_{2,Repeat}, \mathbf{x}_{3,HighISEI})\boldsymbol{\gamma} + \boldsymbol{\upsilon} \text{ and}\\
\operatorname{vec}(\boldsymbol{\upsilon}) \sim& MVN(\mathbf{0}, \mathbf{R} \otimes \mathbf{I}_N). \qquad (6.1.4)
\end{aligned}
$$

No.	Subset	N	M	K	Q	Intercept	Miss in Y	Miss in X	Covars in model
(M1)	–	12340	53	2	–	no	5.59%	–	–
(M2)	–	12340	53	2	5	no	5.59%	4.72%	Gender, Repeat, HighISEI, SchoolRS, SchoolGY
(M3a)	(Hauptschule)	3412	53	2	3	no	9.22%	10.84%	Gender, Repeat, HighISEI
(M3b)	(Realschule)	3978	53	2	3	no	5.88%	8.49%	Gender, Repeat, HighISEI
(M3c)	(Gymnasium)	4950	53	2	3	no	2.85%	5.31%	Gender, Repeat, HighISEI

Table 6.1: Five dedicated IRT scenarios under investigation, with Subset: Whether data was subsetted to which category of school-type, N: Number of observations, M: Number of measurements, K: Number of latent variables, Q: Number of covariates, Intercept: Presence of intercept, Miss in Y: Percentage of missingness in all measurements, Miss in X: Percentage of missingness in all covariates, Covars in model: Covariates that are included.

In order to estimate the parameters of interest, the above algorithm for an IRT model is used and post-processing of sign-changes gets conducted, applying similar hyperparameter specifications as stated in table 5.1 on page 73, with three exceptions: First, to not exceed memory restrictions 25000 instead of 50000 iterations are carried out.[32] Second, the number of maximal possible iterations for the minimisation with respect to cut-off parameters is set to 10000. Third, after investigation of first traceplots (see figure 6.1 for examples) the amount of burn-in is increased from 20% to 50% (i.e. a minimum of 6000 iterations and additionally 6500 as a definitive safeguard[33]).

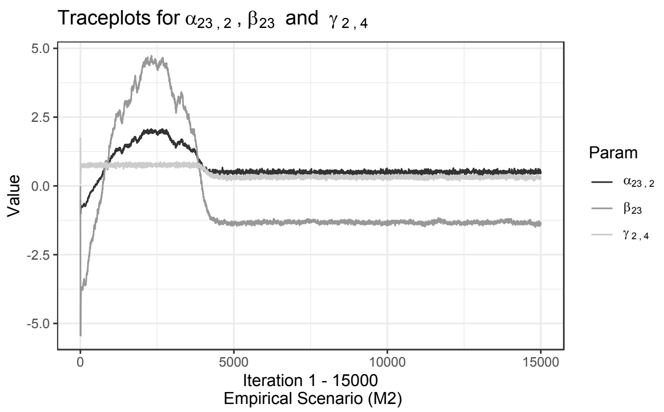

Figure 6.1: Example traceplot of empirical scenario (M2) for draws of $\alpha_{23,2}$, $\beta_{23,1}$, $\gamma_{2,4}$.

6.2 Results of the Estimation

With up to 128 parameters to estimate and if the loss of information using aggregation is bearable, aggregates are presented for the two main scenarios in tables 6.2 and 6.3 below–most times separated with respect to the competency domains if reasonable. The measures that are calculated are the mean estimate from the draws ($\bar{\check{\psi}}$), the effective sample-size ($(\hat{T})_{eff}$), the draws' standard deviations ($\hat{\sigma}(\check{\psi})$), the lower ($q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$) and the upper bound ($q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$) of an estimated 95% central posterior region. For aggregates the average of the parameters' measures are contained. However, detailed results for all parameters, and the auxiliary scenarios as well, are depicted in Appendix F, pp. 326ff.

[32] However, it turned out that thinning, which was used first, was not necessary and thus the estimation was carried out for a second time without thinning

[33] A Rafterty and Lewis based diagnostic (see Raftery & Lewis, 1992) using coda (see Plummer et al., 2006) for $\alpha_{23,2}$ of (M2) suggested an optimal iteration length of 864764, after discarding the first 10000 draws, it shrunk to 16254 with a lower bound of 3746. Thus, the additional safeguard.

Scenario (M1) *Aggregates*:					
	$\bar{\bar{\check{\psi}}}$	$\overline{(\hat{T})_{eff}}$	$\overline{\hat{\sigma}(\check{\psi})}$	$\overline{q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)}$	$\overline{q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)}$
α_{math}	0.66	2053.20	0.02	0.62	0.69
(range)	(0.409, 0.981)	(776.741, 3435.672)	(0.013, 0.024)	(0.381, 0.936)	(0.437, 1.028)
α_{read}	0.70	1366.74	0.02	0.66	0.74
(range)	(0.332, 1.17)	(117.076, 3296.379)	(0.014, 0.048)	(0.304, 1.103)	(0.358, 1.239)
β_{math}	-0.20	1441.47	0.02	-0.23	-0.17
(range)	(-1.551, 1.539)	(674.263, 2483.479)	(0.013, 0.024)	(-1.586, 1.492)	(-1.517, 1.586)
β_{read}	-1.12	1071.11	0.02	-1.17	-1.08
(range)	(-2.585, 0.28)	(94.505, 4050.805)	(0.012, 0.064)	(-2.716, 0.257)	(-2.461, 0.305)
Scenario (M1) *Some Single Parameters*:					
	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
r_{12}	0.700	1722.609	0.007	0.686	0.713
$\xi_{3,2}$	0.620	2699.334	0.014	0.594	0.648
$\xi_{3,3}$	1.797	1769.937	0.019	1.761	1.834
$\xi_{16,2}$	1.246	1796.668	0.018	1.211	1.281
$\xi_{16,3}$	1.848	1535.298	0.020	1.811	1.887
$\xi_{26,2}$	0.537	1807.744	0.017	0.504	0.572
$\xi_{26,3}$	1.058	1016.134	0.020	1.019	1.098
$\xi_{26,4}$	2.068	763.100	0.023	2.023	2.114
$\xi_{27,2}$	0.489	1498.299	0.018	0.453	0.525
$\xi_{35,2}$	0.674	710.182	0.026	0.626	0.726
$\xi_{35,3}$	1.225	587.068	0.028	1.171	1.281
$\xi_{46,2}$	0.390	3714.422	0.012	0.367	0.413

Table 6.2: Aggregates and some single parameters for empirical example (M1).

Scenario (M2) *Aggregates*:					
	$\bar{\bar{\check{\psi}}}$	$\overline{(\hat{T})_{eff}}$	$\overline{\hat{\sigma}(\check{\psi})}$	$\overline{q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)}$	$\overline{q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)}$
α_{math}	0.46	1258.04	0.01	0.44	0.49
(range)	(0.295, 0.677)	(416.693, 2267.738)	(0.009, 0.017)	(0.275, 0.644)	(0.315, 0.71)
α_{read}	0.54	713.43	0.02	0.50	0.57
(range)	(0.256, 0.911)	(121.752, 1485.864)	(0.01, 0.033)	(0.234, 0.86)	(0.276, 0.965)
β_{math}	0.41	190.72	0.03	0.36	0.47
(range)	(-1.105, 2.207)	(109.401, 325.123)	(0.021, 0.043)	(-1.148, 2.126)	(-1.06, 2.294)
β_{read}	-0.26	243.41	0.03	-0.32	-0.20
(range)	(-1.712, 1.208)	(68.416, 621.651)	(0.022, 0.04)	(-1.786, 1.132)	(-1.637, 1.287)
Scenario (M2) *Some Single Parameters*:					
	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\gamma_{1,1}$	-0.633	1336.341	0.023	-0.677	-0.588
$\gamma_{1,2}$	0.293	306.094	0.029	0.236	0.349
$\gamma_{1,3}$	0.007	213.330	0.001	0.006	0.008
$\gamma_{1,4}$	0.715	3161.184	0.029	0.658	0.772
$\gamma_{1,5}$	2.025	3527.053	0.036	1.956	2.095
$\gamma_{2,1}$	0.254	1877.325	0.021	0.213	0.296
$\gamma_{2,2}$	0.192	358.323	0.027	0.138	0.246
$\gamma_{2,3}$	0.007	121.726	0.001	0.005	0.008
$\gamma_{2,4}$	0.840	3036.306	0.029	0.783	0.896
$\gamma_{2,5}$	1.764	3619.418	0.033	1.699	1.829
r_{12}	0.552	1658.101	0.010	0.532	0.572
$\xi_{3,2}$	0.623	2913.910	0.014	0.595	0.650
$\xi_{3,3}$	1.803	1973.044	0.019	1.766	1.840
$\xi_{16,2}$	1.243	1869.965	0.017	1.209	1.277
$\xi_{16,3}$	1.844	1738.007	0.019	1.806	1.882
$\xi_{26,2}$	0.540	1888.893	0.017	0.507	0.573
$\xi_{26,3}$	1.064	1115.362	0.020	1.026	1.103
$\xi_{26,4}$	2.078	884.976	0.022	2.034	2.121
$\xi_{27,2}$	0.487	1904.984	0.018	0.452	0.522
$\xi_{35,2}$	0.678	681.573	0.025	0.629	0.728
$\xi_{35,3}$	1.231	509.915	0.028	1.177	1.285
$\xi_{46,2}$	0.388	4413.358	0.012	0.366	0.412

Table 6.3: Aggregates and some single parameters for empirical example (M2).

The first aspect to notice, is that the assumed relationship between latent variables can be found within all scenarios (i.e., $\hat{r}_{12(M1)} = 0.7$, $\hat{r}_{12(M2)} = 0.552$ and compare also figure 6.2[34]), and with the help of the estimates for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ additional relationships between measurements and the competencies are visible. Except for one–that is 0.4% of all α-estimates of the empirical scenarios–of the items' discrimination parameters, within the auxiliary scenario (M3a) for Hauptschule (i.e. $\hat{\alpha}_{49,2}$ depicted in table 7.240 on page 338), all non-zero α-parameter central posterior region estimates do not cover a zero.

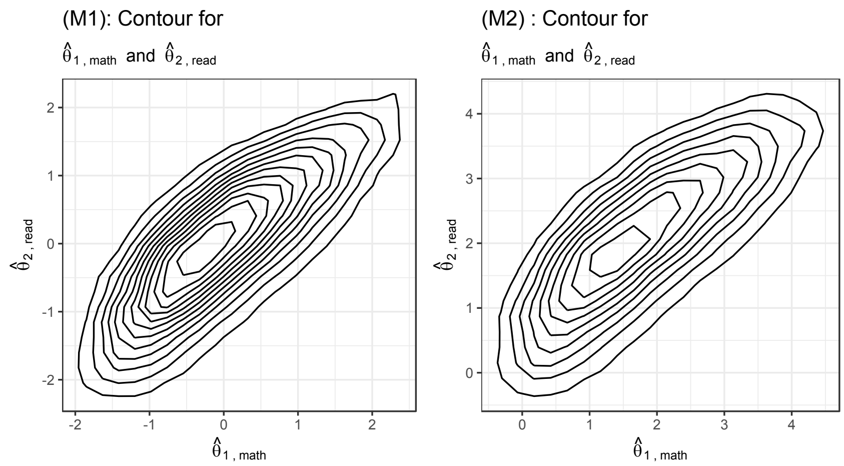

Figure 6.2: Contour plots for mean estimates of the latent variables for all individuals, i.e. $\hat{\boldsymbol{\theta}}_{1,math}$ and $\hat{\boldsymbol{\theta}}_{2,read}$ for empirical scenarios (M1) without covariates and (M2) with covariates indicating correlation between these variables.

When considering background information in (M2), the same directions of relationships–compared to Gaasch (2017a, p. 113)–between the covariates and the *math*-items related latent variable $\boldsymbol{\theta}_{1,math}$ are visible: That is, first, a negative relationship between female gender and the (mathematical) competency ($\hat{\gamma}_{1,1} = \hat{\gamma}_{math,genderFemale} = -0.633$). Second, persons who indicated to not have repeated a class have a positive relationship with this competency ($\hat{\gamma}_{1,2} = \hat{\gamma}_{math,Repeat} = 0.293$)–or: repeaters have a negative relationship. Third, pupils of Realschule and Gymnasium have an increased positive relationship ($\hat{\gamma}_{1,4} = \hat{\gamma}_{math,SchoolRS} = 0.715$, $\hat{\gamma}_{1,4} = \hat{\gamma}_{math,SchoolGY} = 2.025$) compared to Hauptschule, and, finally, the relationship with HighISEI is rather low ($\hat{\gamma}_{1,3} = \hat{\gamma}_{math,HighISEI} = 0.007$), in comparison with the other estimates. However, there is a difference to the scenario depicted compared to Gaasch (2017a, p. 113), which is the amount of the values. But a possible reason can be: Gaasch (2017a) includes an intercept, which isn't included here.

[34]Because of the positive relationship in (M1) and the positive estimate $\hat{r}_{12(M2)}$,$\hat{\boldsymbol{\theta}}_{1,math}$ for (M2) was adapted with a switched sign, otherwise the relationship cor $\left(\hat{\boldsymbol{\theta}}_{1,math}, \hat{\boldsymbol{\theta}}_{2,read}\right)$ would have been negative.

If one now focusses on the second latent variable, i.e. reading competency ($\boldsymbol{\theta}_{2,read}$), the directions of the relationships are similar (compare table 6.3) with two exceptions: First, the amount of the values are slightly decreased, and, second, the relationship between female gender and the (reading) competency is positive with $\hat{\gamma}_{2,1} = \hat{\gamma}_{read,genderFemale} = 0.254$. The directions of these relationships can be similarly found in scenarios (M3a), (M3b) and (M3c)–see tables 7.238 (p. 336), 7.246 (p. 342) and 7.254 (p. 348).

After investigating the estimations as such and comparing these to findings of Gaasch (2017a), a more technical aspect is considered next, i.e. computation time. When it comes to duration, the highest amount of time needed was 55 hours and 24 minutes in (M2). The quickest estimation was possible for (M3a), which used 14 hours and 19 minutes. If the computing times get sorted (see table 6.4), the number of cases, N, seems to have the highest impact, followed by the number of covariates.

Duration	No.	Subset	N	Q	Miss in Y	Miss in X	No. of Covars
14h 19m	(M3a)	(Hauptschule)	3412	3	9.22%	10.84%	3
16h 58m	(M3b)	(Realschule)	3978	3	5.88%	8.49%	3
21h 43m	(M3c)	(Gymnasium)	4950	3	2.85%	5.31%	3
45h 22m	(M1)	–	12340	–	5.59%	–	0
55h 24m	(M2)	–	12340	5	5.59%	4.72%	5

Table 6.4: Computation times needed for estimation of the empirical scenarios in ascending order.

However, considering that a few effective sample-size values are low for some of the estimates and therefore might indicate a higher amount of iterations needed, computation times as such seem to be rather high, leading to the general idea that several more parts of the code should be transformed into a compiled language like C++.

After the simulations studies and the empirical application got conducted, we are going to provide a brief summary, concluding remarks and point towards future prospects of research.

Chapter 7

Concluding Remarks and Future Prospects

This thesis investigated two statistical latent variable models and the possibility of imputation of covariates, while estimating the model parameters. The idea goes back to the positive findings for IRT models with one latent variable of Aßmann et al. (2015) and Gaasch (2017a) respectively, and the research of Preising (2018) concerning dynamic linear panel models. The models that were investigated within this thesis, stem from the class of confirmatory factor analytic and normal-ogive IRT models, whose underlying features were described by discussing concepts, first considerations with respect to estimation within a Bayesian framework, mathematical model formulations and varying terminologies used in different contexts. The introduction of the models was completed by presenting alternative possibilities of determining identification and options for reliefs of this problem via parameter restrictions.

By comparing different latent variable models (i.e. Aßmann et al., 2015, Gaasch, 2017a, and Conti et al., 2014) and their application, alternative ways of including covariates and handling of missingness were discussed next. Using Conti et al.'s (2014) model as a basis, chapter 4, presented the derivation of a Bayesian Gibbs sampling algorithm, where the explorative model of Conti et al. (2014) was transformed into a confirmatory matrix-variate factor and a normal-ogive IRT model that incorporates covariates in two different ways: a) as an influence to the measurements and b) as an influence to latent variables. In order to handle missing values within these covariates CART imputation was included. Next to binary and metric scaled items, estimation of ordered scaled measurements was incorporated as well–applying an algorithm of Albert & Chib (1997), which was already implemented and tested by Gaasch (2017a) in his approach. In order to allow the possibility of defining restrictions, the indicator matrix, used by Conti et al. (2014), was extended to specify equality and multiplicative relationships for loadings and item discrimination parameters respectively. In addition to the dedicated loading structure of Conti et al. (2014) this matrix was also used to specify cross-loading, in a way that a measurement can be related to several latent variables simultaneously.

In order to test the algorithm proposed, nine factor analytic and twelve IRT model related simulation studies were conducted. Using relative root mean squared errors, absolute biases and coverage rates as first measures of quality, the findings indicate that the algorithm as such works as intended–both, for the factor analytic and the IRT specification. However, when including missing values in covariates using a simple missing-completely-at-random mechanism, CART imputation performed poorer compared to complete-case analysis especially with respect to coverage rate. These problems continued to exist, even when the amount of available information and parameters to estimate was changed, i.e. de- or increased. This indicates that either the CART imputation or imputation of covariates as such seems to be not beneficial for the factor analytic formulation. Within the IRT approach, the results are however different: dependent on the missingness mechanism, CART imputation is able to outperform complete-case analysis. When using the root mean squared error and the estimated parameter standard deviations as measures of quality, CART imputation achieves better outcomes than complete-case analysis–with gains up to 90 percentage points on average.

Finally the IRT approach was used for an empirical application–it utilized NEPS data in a way to have, at first, comparability to already conducted research of Gaasch (2017a). The results indicated that the general direction of the latter's resulting relationships can be verified. With additionally incorporating measurements for reading competency, we were able, to investigate this latent variable concept at the same time. The relationship between the covariates (gender, whether a class was not or was repeated, the maximum ISEI-08 score of the parents of a participant, and the type of school) and the latent variable were–although having different strengths–positive throughout. Furthermore, a positive correlation between the competencies could be uncovered. Added scenarios of different data size gave, next to the empirical results, insights into computation time–ranging from roughly 14 to 55 hours, and suggesting that tests for suitedness of compiled programming languages as a replacement or as an addition to R should be considered for further enhancements .

For future research several aspects can be pursued, like imputation of the factor analytic approach and CART imputation as the respective method or imputation of measurements. As was pointed out, imputation within the factor analytic setting did not seem to work. This naturally raises the question, why this is the case. Given the applicability of the CART imputation for the IRT scenarios, this might indicate that either the CART approach for the factor model is not suited or imputation as such is problematic here–and could one aspect to investigate. When it comes to automated classification procedures further algorithms next to CART might be of interest with respect to benefits or drawbacks as well, like random forests (see e.g. Shah et al., 2014, for an application)–not only with respect to the factor model, but also to the IRT scenarios. And although not focus of this thesis, of course, how might the behaviour of the algorithms change, if missings in measurements get imputed as well? The investigations here, and those of Aßmann et al. (2015) and Gaasch (2017a), removed missingness within the items–although not list-wise but element-wise. Nonetheless, what if the underlying assumption of MCAR does not hold?[35] What will happen, with respect to performance, when the measurements get imputed as well? Furthermore and with respect to the the factor analytic scenario that used a high amount of parameters, i.e. 93, and even complete-case analysis was problematic: The possible boundariers concerning the number of parameters for the IRT model might also of interest. This and the previous considerations seem to be interesting areas of research with regards to applications of imputation techniques and aspects of model complexity.

When speaking of model formulations, additional aspects for future examination arise. First, the test for metric scaled measurements was restricted to the simulation study only, because the data set of the NEPS used did not contain process time, thus the idea of using log-transformed time still needs an empirical test. Second, next to multidimensionality within the latent domain, group or cluster effects, as employed by Gaasch (2017a) might be considered for inclusion using a multivariate latent domain as well. Third, we have seen the ability to impute within the multidimensional normal-ogive IRT model that facilitates normally distributed latent variables. In addition to logistic IRT models or categorical latent variables, there exist, as San Martín (2015, p. 135) points out, applications where the normality assumption is doubted and more general distributional settings are con-

[35]For example implied by Pohl et al. (2014) and Rose et al. (2017), if items were not answered within a given amount of time.

sidered. With respect to these additional scenarios, investigating extensions and possible capabilities of the presented impute-while-analyse approach seems to be a fruitful venture of further research, allowing for new insights in limitations and benefits of this method.

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Appendices

Appendix A – Abbreviations & Notation

Abbreviations

#PL: #-Parameter Logistic Model, where # refers to the number of parameters, e.g., 2PL denotes the two parameter logistic model.

#PNO: #-Parameter Normal Ogive Model, where # refers to the number of parameters, e.g., 2PN denotes the two parameter normal ogive model.

BD: Before Deletion

CFA: Confirmatory Factor Analysis

EEG: Electroencephalography

EFA: Explorative Factor Analysis

GRM: Graded Response Model

GLIRT: Generalized Item Response Model

GLM: Generalized Linear Model

GLLAMM: Generalized Latent Linear and Mixed Model

GLLVM: Generalized Linear Latent Variable Model

IRT: Item Response Theory

ISEI: International Socio-Economic Index of occupational status

MAR: Missing At Random

MANOVA: Multivariate Analysis Of Variance

MCAR: Missing Completely At Random

MCMC: Markov Chain Monte Carlo

NEPS: National Educational Panel Study

PARAFAC: Parallel Factor Analysis

PCA: Principal Component Analysis

PCM: Partial Credit Model

PIRLS: Progress in International Reading Literacy Study

PISA: Programme for International Student Assessment

RMSE: Root mean squared error

RRMSE: Relative root mean squared error

SCOFF: Sick-Control-One-Stone-Fat Food (scale)

SEM: Structural Equation Model

SLES: SuSE Linux Enterprise Server

SLURM: Simple Linux Utility for Resource Management

TIMSS: Trends in International Mathematics and Science Study

Remarks on notation

Random variables, random parameters and realisations

We do not explicitly differentiate between random variables and realised data, which sometimes can be found as y vs. Y.

Vectors

The standard notation for vectors here is that they are always column vectors if not stated otherwise–even, when they we refer to rows of a matrix. If a row vector is needed, we use transposition denoted with $'$. For example, the i-th row of the matrix of latent variables

$$\mathbf{\Theta} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \theta_{i1} & \cdots & \theta_{iK} \\ \vdots & \vdots & \vdots \end{pmatrix}$$

when written as a vector $\boldsymbol{\theta}_i = (\theta_{i1}, \ldots, \theta_{iK})'$ is a column vector with dimension $K \times 1$ and $\boldsymbol{\theta}_i' = (\theta_{i1}, \ldots, \theta_{iK})$ is the row vector of dimension $1 \times K$.

Notation

A

a: General variable.

$absBias(\cdot)$ & $|bias|$: Absolute bias with respect to a true parameter ψ and estimates $\cdot$, dimension (1×1).

A_k^2: Hyper parameter for scaling matrix element of latent variable covariance, dimension (1×1).

$\boldsymbol{\alpha}$: Matrix of loading parameters (in Factor Analysis) or item discrimination parameters (in IRT), dimension $(M \times K)$.

$\boldsymbol{\alpha}_k$: Vector of a column of the loading matrix (in Factor Analysis) or item discrimination matrix (in IRT) $\boldsymbol{\alpha}$referring to the k-th measurement, dimension $(M \times 1)$.

$\boldsymbol{\alpha}_m$: Vector of a row of the loading matrix (in Factor Analysis) or item discrimination matrix (in IRT) $\boldsymbol{\alpha}$referring to the m-th measurement, dimension $(K \times 1)$.

α_{mk}: One element of the above matrix $\boldsymbol{\alpha}$, dimension (1×1).

$\boldsymbol{\alpha}^{\Delta}$: Matrix of loading parameters (in Factor Analysis) or item discrimination parameters (in IRT) reduced to non-zero unique elements only, variable dimension

$(M \times \underline{K})$.[36]

$\boldsymbol{\alpha}_m^{\Delta}$: m-th row vector of $\boldsymbol{\alpha}$, but reduced to non-zero unique elements only, variable dimension $(\underline{K} \times 1)$.

$\text{vec}(\boldsymbol{\alpha})^{\Delta}$: Vector of non-zero elements of loading / discrimination matrix $\boldsymbol{\alpha}$, dimension $(M\underline{K} \times 1)$.

acc_{ζ}: Acceptance ratio within the Metropolis-Hastings algorithm stemming from Albert & Chib (1997), dimension (1×1).

B

b: General variable.

$bias(\cdot)$: Bias with respect to a true parameter ψ and estimates $\cdot$, dimension (1×1).

$burnIn$: Amount of burn-in used, dimension (1×1).

$\boldsymbol{\beta}$: Matrix of influence of covariates $\mathbf{X}$ (factor model) or vector of item difficulties (IRT), dimension $(M \times Q)$ (without intercept), $(M \times Q + 1)$ (with intercept) or $(M \times 1)$.

β_k: Item difficulty (IRT) referring to the k-th latent variable, dimension (1×1).

β_m: Item difficulty (IRT) referring to the m-th measurement, dimension (1×1).

$\boldsymbol{\beta}_m$: Vector of a row of the influence of covariates $\mathbf{X}$ (factor model) referring to the m-th measurement, dimension $(Q \times 1)$ or $(Q + 1 \times 1)$.

$\boldsymbol{\beta}_q$: Vector of a column of the influence of covariates $\mathbf{X}$ (factor model) referring to the m-th measurement, dimension $(M \times 1)$.

C

c_m^0: Shape hyperparameter for prior of σ_m^2 referring to the m-th (continuous) measurement, dimension (1×1).

[36] Because of the possibility that the number of zeros or unique elements of row or column can differ within the matrix, the common understanding of a matrix needs to be relaxed–hence, *variable* dimension.

C_m^0: Scale hyperparameter for prior of σ_m^2 referring to the m-th (continuous) measurement, dimension (1×1).

$c_m^\diamond$: Shape hyperparameter for posterior of σ_m^2 referring to the m-th (continuous) measurement, dimension (1×1).

$C_m^\diamond$: Scale hyperparameter for posterior of σ_m^2 referring to the m-th (continuous) measurement, dimension (1×1).

$\chi_{y|\theta,\varepsilon}$: Structure with variable y , given θ and ε.

$\tilde{\chi}_{y|\theta,\varepsilon}$: Additional structure with variable y , given θ and ε.

D

d & $d\,(\cdot)$: Differential(s) used in integration.

D : Domain restriction of an integral.

$\mathbf{D}^{aux}$: Auxiliary identity matrix, used in elimination and duplication algorithm A3, dimension $(\mathfrak{D} \times \mathfrak{D})$. Used in algorithm A3.

diag$(\cdot)$: Diagonal matrix with diagonal elements $\cdot$.

$\boldsymbol{\Delta}$: Indicator matrix with $0-1$-elements for loadings and item discrimination values respectively, dimension $(M \times K)$.

δ_{mk}: One element of the above matrix $\boldsymbol{\Delta}$, dimension (1×1).

$\bar{\boldsymbol{\Delta}}$: Indicator matrix with $\mathbb{N}_0$ elements for loadings and item discrimination values respectively, dimension $(M \times K)$.

$\bar{\delta}_{mk}$: One element of the above matrix $\bar{\boldsymbol{\Delta}}$, dimension (1×1).

$\bar{\boldsymbol{\delta}}^{aux}$: Vector of non-zero elements of vec$(\bar{\boldsymbol{\Delta}})$, dimension $(\sum_{m=1}^{M}\sum_{k=1}^{K}\mathcal{I}_{(1,+\infty)}(\bar{\delta}_{mk}) \cdot \bar{\delta}_{mk} \times 1)$ or simply $(\sum_{m=1}^{M}\sum_{k=1}^{K}\delta_{mk} \times 1)$. Used in algorithm A3.

$\bar{\boldsymbol{\delta}}^{aux2}$: Copy of $\bar{\boldsymbol{\delta}}^{aux}$ or by equality-indicating elements reduced version of $\bar{\boldsymbol{\delta}}^{aux}$, dimension ranging from $(\sum_{m=1}^{M}\sum_{k=1}^{K}\mathcal{I}_{(1,+\infty)}(\bar{\delta}_{mk}) \cdot \bar{\delta}_{mk} \times)$ to (1×1). Used in algorithm A3.

$\boldsymbol{\Delta}^{aux3}$: Indicator matrix with (-1) & 1-elements for signs of loadings and item discrimination values respectively, dimension $((T) \times K)$. Used in algorithm A4.

$\mathfrak{D}$: Number of non-zero elements in $\boldsymbol{\Delta}$ and $\bar{\boldsymbol{\Delta}}$ respectively, dimension (1×1).

E

$E[\cdot]$ & $E[\cdot|\cdot]$: General and conditional expectation respectively.

$\mathbf{E}$: Elimination matrix pertaining to the missing values in $\mathbf{Y}$, dimension $(\sum_{i=1}^{N}\sum_{m=1}^{M} o_{im}^{Y} \times NM)$.

$\mathbf{E}^{mm}$: Elimination matrix pertaining to the missing values one measurement $\mathbf{y}_m$, dimension $(\sum_{i=1}^{N} o_{im}^{Y} \times N)$.

$\mathbf{E}^{ii}$: Elimination matrix pertaining to the missing values one individuals measurements $\mathbf{y}_i$, dimension $(\sum_{m=1}^{M} o_{im}^{Y} \times M)$.

$\mathbf{E}_{\Delta}$: Elimination matrix creating a non-zero vectorised version of the matrix $\boldsymbol{\alpha}$, i.e., $\mathbf{E}_{\Delta}\mathrm{vec}(\boldsymbol{\alpha}) = \mathrm{vec}(\boldsymbol{\alpha})^{\Delta}$, dimension $(\sum_{m=1}^{M}\sum_{k=1}^{K} \delta_{mk} \times MK)$.

$\mathbf{E}^{aux}$: $(MK \times MK)$-Identity matrix, with rows removed that pertain to non-zero elements in $\bar{\boldsymbol{\delta}}^{aux}$, dimension $(\sum_{m=1}^{M}\sum_{k=1}^{K} \delta_{mk} \times MK)$.

$\mathbf{E}^{aux2}$: Auxiliary matrix of algorithm A3, containing the proportions of equality restrictions, with numbers of rows concurring to the number of elements of equality indicators and $\mathfrak{D}$ columns.

$\exp(\cdot)$: Exponential function.

ε: Error term for a general model.

$\boldsymbol{\varepsilon}_i$: Vector of error terms pertaining to one individual, dimension $(M \times 1)$.

F

$f_{\zeta\xi}(\zeta_m|\cdot)$: Negative (conditional) log-posterior for cut-off parameter transforms ζ_m, used in ordered-scale variables' cut-off value estimation.

$f_{st}(\cdot|\cdot)$: Multivariate Student's t-density, used in ordered-scale variables' cut-off value estimation.

$f_\theta(\cdot)$: Sometimes (context dependent) probability density function or link function or general mapping referring to latent variable θ.

$f_y(\cdot)$: Sometimes (context dependent) probability density function or link function or general mapping referring to variable y.

$f_{y,y*|\cdot}(\cdot)$: Conditional probability density function with y and y^*, and $\mathbf{Y}$ and $\mathbf{Y}^*$ respectively as dependent variables.

G

g: Index for number of non-zero vectorised elements in $\mathbf{\Delta}$ and $\bar{\mathbf{\Delta}}$ respectively, dimension (1×1). Used in algorithm A3.

$g(1)$: Group-indicator of a specific group in the model of Gaasch (2017a, pp. 20ff. and pp. 35ff.).

$g(2)$: Cluster-indicator of a specific cluster in the model of Gaasch (2017a, pp. 20ff. and pp. 35ff.).

Γ_m: Guessing-parameter in three-parameter IRT models referring to one measurement m, dimension (1×1).

$\boldsymbol{\gamma}$: Matrix of influence of covariates $\mathbf{Z}$ in IRT, dimension $(K \times Q)$.

I

i: Index for individuals / of the rows of measurements, covariates or latent variables, $i = 1, \ldots, N$, dimension (1×1).

$\mathbf{I}_\square$: Identity matrix, dimension $(\square \times \square)$.

$\boldsymbol{\iota}_\square$: Vector of ones, dimension $(\square \times 1)$.

$\mathcal{I}_\square(\cdot)$: Indicator function over an interval or (sub)set (of) $\square$, where $\mathcal{I}_\square(\cdot) = 1$, if $\cdot \in \square$ and $\mathcal{I}_\square(\cdot) = 0$ otherwise.

J

J_m: Number of categories of a categorical variable $\mathbf{y}_m$ (Note, if $J_m = 1$ we assume we deal with a continuous variable instead), dimension (1×1).

j: Index for categories pertaining to the numbers of categories J_m from before, $j = 1, \ldots, J_m$, dimension (1×1).

$\mathcal{J}(\cdot)$: Jacobian matrix of a parameter vector, i.e., the matrix of first order partial derivatives.

K

k: Index of the columns of the latent variables $\boldsymbol{\Theta}$, rows of the loadings and rows of the regression parameters $\boldsymbol{\gamma}$ for covariates $\mathbf{Z}$ in IRT, $k = 1, \ldots, K$, dimension (1×1).

$\bar{k}$: Specific index of the columns of the latent variables $\boldsymbol{\Theta}$ that don't refer to the columns k of $\boldsymbol{\Theta}$, dimension (1×1).

K: Number of columns of latent variables, dimension (1×1).

$\underline{K}$: Number of columns of latent variables with number of zero loadings / discriminations removed, dimension (1×1).

$\mathbf{K}$: General orthogonal matrix.

$\tilde{\mathbf{K}}$: General non-singular matrix.

L

$\mathbf{L}$: Matrix of Loadings, different notation sometimes found in literature, Dimension $(M \times K)$.

$L(\cdot;\cdot)$: Likelihood function, also $f_y()$.

$\log f(\cdot)$ & $\log f(\cdot|\cdot)$: Logarithmic transform of a function $f(\cdot)$ or conditional function $f(\cdot|\cdot)$, e.g. log-likelihood function or log-posterior.

$\boldsymbol{\Lambda}$: Variance component, i.e. variance matrix, stemming from variance-covariance matrix$\boldsymbol{\Omega}$, dimension $(K \times K)$.

M

M: Number of columns of measurements, dimension (1×1).

$M_{i,obs}$: Number of columns of measurements with number of missing values in $\mathbf{y}_i$ removed / only number of observed values, dimension (1×1).

m: Index of the columns of measurements (and rows of loadings and rows of covariates), $m = 1, \ldots, M$, dimension (1×1).

$MVN(\cdot, \square)$: Indicating multivariate normal distribution of a variable with moments $\cdot$ (mean) and $\square$ (variance).

μ & $\boldsymbol{\mu}$: Expectations in general.

$\boldsymbol{\mu}_\alpha^0$: Prior mean / mean hyperparameter for $\boldsymbol{\alpha}_m$ or $\boldsymbol{\alpha}_m^\Delta$, dimension $(K \times 1)$ or variable dimension $(\underline{K} \times 1)$, or $(MK \times 1)$ or $(M\underline{K} \times 1)$, when used in a vec$(\cdot)$-setting.

$\boldsymbol{\mu}_\alpha^\diamond$: Posterior mean for $\boldsymbol{\alpha}_m$ or $\boldsymbol{\alpha}_m^\Delta$, dimension $(K \times 1)$ or variable dimension $(\underline{K} \times 1)$, or $(MK \times 1)$ or $(M\underline{K} \times 1)$, when used in a vec$(\cdot)$-setting.

$\boldsymbol{\mu}_\beta^0$: Prior mean / mean hyperparameter for $\boldsymbol{\beta}$, $\boldsymbol{\beta}_q$ or vec$(\boldsymbol{\beta})$, dimension $(M \times 1)$, or $(MQ \times 1)$ or $(M(Q+1) \times 1)$ (with intercept).

$\boldsymbol{\mu}_\beta^\diamond$: Posterior mean for $\boldsymbol{\beta}_m$ or vec$(\boldsymbol{\beta})$, dimension $(M \times 1)$ or $(MQ \times 1)$ or $(M(Q+1) \times 1)$ (with intercept).

$\boldsymbol{\mu}_\gamma^0$: Prior mean / mean hyperparameter for $\boldsymbol{\gamma}_q$ or vec$(\boldsymbol{\gamma})$, dimension $(K \times 1)$ or $(KQ \times 1)$.

$\boldsymbol{\mu}_{\ddot{\gamma}}^0$: Prior mean / mean hyperparameter for expanded $\boldsymbol{\gamma}_q$ or vec$(\boldsymbol{\gamma})$, i.e. $\ddot{\boldsymbol{\gamma}}_q$ or vec$(\ddot{\boldsymbol{\gamma}})$, dimension $(K \times 1)$ or $(KQ \times 1)$.

$\boldsymbol{\mu}_{\ddot{\gamma}}^\diamond$: Posterior mean for $\ddot{\boldsymbol{\gamma}}_q$ or vec$(\ddot{\boldsymbol{\gamma}})$, dimension $(K \times 1)$ or $(KQ \times 1)$.

$\boldsymbol{\mu}_\Theta$: General (conditional) mean of latent variable matrix $\boldsymbol{\Theta}$, dimension $(N \times K)$.

$\boldsymbol{\mu}_\theta^\diamond$: (Conditional) mean of latent variable, referring to $\boldsymbol{\theta}_i$, i.e. the i-th row when drawn during Gibbs sampling, dimension $(K \times 1)$ or $(NK \times 1)$, when used in a vec$(\cdot)$-setting.

$\boldsymbol{\mu}_y$: Marginal expectation for vector of all measurements $\mathbf{y}_i$, dimension $(M \times 1)$.

$\boldsymbol{\mu}_{y|\cdot}$: Conditional expectation for vector of all measurements $\mathbf{y}_i$, dimension $(M \times 1)$.

$\boldsymbol{\mu}_{y^*|\cdot}$: Conditional expectation for vector of all latent representations $\mathbf{y}_i^*$ of measurements $\mathbf{y}_i$ or for matrix of all latent representations $\mathbf{Y}^*$ of measurements $\mathbf{Y}$, dimension $(M \times 1)$ and dimension $(N \times M)$ respectively.

$\mu_{y^*|\cdot}$: Conditional expectation for a latent representation y_{im}^* of a measurement, dimension (1×1).

$\mu_{y,m}$ & $\mu_{y|\cdot,m}$: Marginal and conditional expectation element referring to one measurement m, dimension (1×1).

N

N: (Maximal) number of rows of the measurements, latent variables and covariates, dimensions (1×1).

$N_{m,obs}$: (Maximal) number of rows of the measurements $\mathbf{y}_m$ without missing values, dimensions (1×1).

$\mathbb{N}$: Set of positive integers (without zero).

$\mathbb{N}_0$: Set of positive integers including zero.

ν_Ω^0: Prior degrees of freedom, used for inverse-gamma and inverse-Wishart draws, pertaining to latent variables' covariances, i.e. $\mathbf{\Omega}$, or its scale matrix ($\mathbf{S}_\Omega^\diamond$) diagonal element draws, dimension (1×1).

$\nu_\Omega^\diamond$: Expanded (with $\nu_\Omega^\diamond = \nu_\Omega^0 + N$) degrees of freedom, used for inverse-Wishart draws, pertaining to latent variables' covariance, i.e. $\mathbf{\Omega}$, dimension (1×1).

ν_{st}: Prior degrees of freedom for multivariate Student's t-density $f_{st}(\cdot|\cdot)$, dimension (1×1), used in ordered-scale variables' cut-off value estimation, dimension (1×1).

O

$\mathbf{O}^Y$: Indicator matrix of observed values in $\mathbf{Y}$, dimension $(N \times M)$.

o^Y_{im}: One element of the indicator matrix $\mathbf{O}^Y$ referring to the i-th unit and m-th measurement y_{im}, dimension (1×1).

$\mathbf{\Omega}$: Expanded latent variable variance-covariance matrix, dimension $(K \times K)$.

$(\mathbf{\Omega})_{kk}$: k-th diagonal element of $\mathbf{\Omega}$, which also corresponds to the k-th nonzero element in $\mathbf{\Lambda}$, dimension (1×1).

P

P: Number of columns of (covariates) $\mathbf{Z}$ in IRT, dimension (1×1)

p : Index for columns of covariates $\mathbf{Z}$ in IRT,$p = 1, \ldots, P$, dimension (1×1)

$\mathcal{P}(\cdot)$: Cumulative distribution function.

$\mathcal{P}(\cdot|\cdot)$: Conditional cumulative distribution function.

$\prod_{i=1}^{N}$ & $\prod_{m=1}^{M}$: Product symbol referring to all individuals / units and measurements respectively.

$\Phi(\cdot)$: Cumulative normal distribution function of a univariate or multivariate random variable

$\phi(\cdot)$: Normal density function of a univariate or multivariate random variable

$\pi(\cdot)$: Function or functional denoting a prior density.

ψ: Placeholder for a (general) true parameter.

$\check{\boldsymbol{\psi}}$: Placeholder for the set of all $((T) - burnIn)$-draws of a (general parameter), dimension $(((T) - burnIn) \times 1)$

$\check{\psi}$: Placeholder for a draw of a (general) parameter.

$\check{\psi}_{(t)}$: Placeholder for the (t)-th draw of a (general) parameter.

$\boldsymbol{\psi}_{unique}$: Vector of unique parameters of a reduced form.

Ψ: Set combining several parameters.

$\Psi(\cdot)$: Structural parameters of a structure $\cdot$.

Q

q: Index for columns of covariates,$q = 1, \dots, Q$, dimension (1×1).

$q_{\square}(\cdot)$: $(\square\%)$-quantile estimate for $\cdot$, dimension (1×1).

Q: Number of columns of (covariates) $\mathbf{X}$ or $\mathbf{Z}$ respectively, dimension (1×1).

R

$\mathbf{R}$: Matrix of latent variable correlation, dimension $(K \times K)$.

$\mathbf{R}_0$: Skew-symmetric version of Matrix of latent variable correlation, i.e. $\mathbf{R}$ with diagonal elements set to 0, dimension $(K \times K)$.

r One element of von R

r_{kk} k-th diagonal element of $\mathbf{R}$, dimension (1×1).

$\mathbb{R}$: Set of real numbers (without zero).

$\mathbb{R}_0$: Set of real numbers (including zero).

$\mathbb{R}^+$: Set of positive real numbers (without zero).

S

$\mathbb{S}$, $\mathbb{S}_Y^M$, $\mathbb{S}_\Theta^K$: General sets of numbers (spaces), with, e.g. dimension M or K for variables Y or Θ.

$\mathbf{S}_\Omega^0$: Prior scale matrix used in draws for latent variables' covariances $\mathbf{\Omega}$, dimension $(K \times K)$.

$\mathbf{S}_\Omega^\diamond$: Posterior scale matrix used in draws for latent variables' covariances $\mathbf{\Omega}$, dimension $(K \times K)$.

$\hat{\mathbf{S}}_\zeta$ Estimated (i.e., using inverse-Hessian) scale-matrix for student-t-distribution within the Metropolis-Hastings algorithm stemming from Albert & Chib (1997), dimension $((J_m - 2) \times (J_m - 2))$

$\mathbf{\Sigma}$: Variance-covariance matrix in general.

$\mathbf{\Sigma}_\varepsilon$: (Diagonal) variance-covariance matrix of (idiosyncratic) error $\boldsymbol{\varepsilon}_i$, dimension $(M \times M)$

$\sigma^2_{\varepsilon,m}$: One diagonal element of $\mathbf{\Sigma}_\varepsilon$ referring to the m-th measurement, dimension (1×1)

$\mathbf{\Sigma}_\theta$: Variance-covariance matrix of the latent variables $\boldsymbol{\theta}_i$, dimension $(K \times K)$

$\sigma^2_{\theta k,\bar{k}}$: One off-diagonal element of $\mathbf{\Sigma}_\theta$ referring to the k-th and $\bar{k}$-th column of $\mathbf{\Theta}$, with $k \neq \bar{k}$, dimension (1×1)

$\mathbf{\Sigma}_y$: Unconditional variance-covariance matrix of the measurements $\mathbf{y}_i$, dimension $(M \times M)$

$\mathbf{\Sigma}_{y^*}$: Unconditional variance-covariance matrix of the latent representations $\mathbf{y}_i^*$ corresponding to measurements $\mathbf{y}_i$, dimension $(M \times M)$

$\mathbf{\Sigma}_{y^*|\cdot}$: Conditional variance-covariance matrix of the latent representations $\mathbf{y}_i^*$ corresponding to measurements $\mathbf{y}_i$, dimension $(M \times M)$

$\mathbf{\Sigma}^0_\alpha$: Prior variance-covariance matrix referring , dimension $(M \times M)$, or $(MK \times MK)$ or $(M\underline{K} \times M\underline{K})$, when used in a vec$(\cdot)$-setting.

$\mathbf{\Sigma}^\diamond_\alpha$ Posterior variance of the loadings / item discrimination matrix, dimension $(M \times M)$, or $(MK \times MK)$ or $(M\underline{K} \times M\underline{K})$, when used in a vec$(\cdot)$-setting.

$\mathbf{\Sigma}^0_\beta$: Prior variance-covariance matrix referring to $\boldsymbol{\beta}$, dimension $(M \times M)$, or $(MQ \times MQ)$ or $(M(Q+1) \times M(Q+1))$, when used in a vec$(\cdot)$-setting.

$\mathbf{\Sigma}^\diamond_\beta$: Posterior variance-covariance matrix referring to $\boldsymbol{\beta}$, dimension $(M \times M)$, or $(MQ \times MQ)$ or $(M(Q+1) \times M(Q+1))$, when used in a vec$(\cdot)$-setting.

$\mathbf{\Sigma}^0_\gamma$: Prior variance-covariance matrix referring to $\boldsymbol{\gamma}$ or vec$(\boldsymbol{\gamma})$, dimension $(K \times K)$ or $(KQ \times KQ)$, when used in a vec$(\cdot)$-setting.

$\mathbf{\Sigma}^0_{\ddot{\gamma}}$: Prior variance-covariance matrix referring to $\ddot{\boldsymbol{\gamma}}_q$ or vec$(\ddot{\boldsymbol{\gamma}})$, dimension $(K \times K)$ or $(KQ \times KQ)$, when used in a vec$(\cdot)$-setting.

$\mathbf{\Sigma}^\diamond_{\ddot{\gamma}}$: Posterior variance-covariance matrix referring to $\ddot{\boldsymbol{\gamma}}$, dimension $(K \times K)$ or $(KQ \times KQ)$, when used in a vec$(\cdot)$-setting.

$\boldsymbol{\Sigma}^0_{\zeta,m}$: Prior variance-covariance matrix referring to $\boldsymbol{\zeta}_m$, dimension $((J_m-2)\times(J_m-2))$. Normally an identity matrix.

$\sigma^0_{\zeta,m,j}$: Prior standard deviation from matrix $\boldsymbol{\Sigma}^0_{\zeta,m}$ referring to $\boldsymbol{\zeta}_m$ and category j, dimension (1×1). Normally 1.

$\sigma^0_{\xi,m,j}$: Prior standard deviation referring to cut-off parameter $\boldsymbol{\xi}_m$ and category j within, dimension (1×1). Normally 1.

T

(t): Index for an iteration of the sampling algorithm, $(t)=1,\ldots,(T)$, dimension (1×1).

(T): Maximum number of iteration of the sampling algorihtm, dimension (1×1).

$\Theta\&\theta$: Latent variables / factors in general.

$\boldsymbol{\Theta}$: Matrix of all factors / latent variables, dimension $(N\times K)$

$\boldsymbol{\theta}_i$: Vector of one individual / unit i with respect to all latent variables, dimension $(K\times 1)$

$\boldsymbol{\theta}_k$: Vector of one / the k-th latent variable with respect to all individuals / units, dimension $(N\times 1)$

$\mathrm{tr}(\cdot)$: Trace operator or trace function.

U

υ: Error term for the latent variable model

u: Indicator index of elements pertaining to equality restrictions in $\bar{\boldsymbol{\Delta}}$, where $u=1,\ldots,\left[\max\left(\bar{\boldsymbol{\delta}}^{aux}\right)-1\right]$, dimension (1×1). Used in algorithm A3.

$Unif(\cdot,\square)$: Uniform draw with range starting at $\cdot$ and ending in $\square$.

V

$\mathbf{V}$: Vector permutation matrix.

$\text{vec}(\cdot)$: Vectorisation function.

$\text{vech}(\cdot)$: Vectorisation function with respect to half of the elements of a (symmetric) matrix.

X

$\mathbf{X}$: Covariates in the factor case or a vector of ones in the IRT case, dimension for factor analysis $(N \times Q)$ or $(N \times Q + 1)$ with intercept column, or $(N \times 1)$ in IRT.

$\boldsymbol{\xi}_m$: Vector of cut-off parameter for m-th measurement, dimension $(J_m + 1 \times 1)$

$\xi_{m,j}$: j-th element of the above vector of cut-off parameter for m-th measurement, dimension (1×1)

Y

y: Dependent variable in general.

Y_{im}: Random variable of a measurement m for one unit / individual i, (1×1).

Y^*: Latent representation of measurements

$\mathbf{Y}$: Matrix of all measurements / items and all individuals / units, dimension $(N \times M)$

$(\mathbf{y}_m)_{m=1}^M$ & $\left[(\mathbf{y}_i)_{i=1}^N\right]'$: Matrix of measurements accentuating the columns m and rows i respectively, dimension $(N \times M)$

$\mathbf{y}_i$: Vector of one individual / unit i with respect to all measurements / items, dimension $(M \times 1)$

$\mathbf{y}_i^*$: Latent continuous representation of vector of one individual / unit i with respect to all measurements / items–interconnected with corresponding $\mathbf{y}_i$, dimension $(M \times 1)$

$\tilde{\mathbf{y}}_i$: Mean-centred (or in general transformed) of the vector of one individual / unit i with respect to all measurements / items, dimension $(M \times 1)$

$\mathbf{y}_m$: Vector of one measurement m with respect to all individuals / units, dimension $(N \times 1)$

$\mathbf{y}_m^*$: Latent continuous representation of vector of one measurement m with respect to all individuals / units–interconnected with corresponding $\mathbf{y}_m$, dimension $(M \times 1)$

y_{im}: Element of one measurement m with respect one individual / unit i, dimension (1×1)

y_{im}^*: Latent continuous representation of element of one measurement m with respect one individual / unit i–interconnected with corresponding y_{im}, dimension (1×1)

$\mathbf{y}_m^{obs,*}$: Part of observed values in measurement $\mathbf{y}_m$.

$y_{im}^{*,obs}$: Observed value in latent representation pertaining to measurement y_{im} of an individual i.

Z

$\mathbf{Z}$: Covariates in the IRT case, dimension $(N \times Q)$; or a 0 during factor analysis.

$\boldsymbol{\zeta}_m$: Vector of transformed cut-off parameter for m-th measurement, dimension $(J_m - 2 \times 1)$

$\boldsymbol{\zeta}_m^*$: Proposal draws (via minimisation) of vector of transformed cut-off parameter for m-th measurement, dimension $(J_m - 2 \times 1)$

$\hat{\boldsymbol{\zeta}}_m$: Location estimate (via minimisation) of vector of transformed cut-off parameter for m-th measurement, dimension $(J_m - 2 \times 1)$

Further Symbols

$\cdot$ or $\square$: General place holder, where $\square$ will be used, when $\cdot$ can be confused with the (dot) product, is additionally involved or visually problematic.

$\tilde{\Box}$: General transformation or change of a variable $\Box$.

$\Box^{-1}$: Reciprocal or inverse for scalar or matrix $\Box$.

$\ddot{\Box}$: Variance transformation of latent variable or parameter $\Box$ (inside separation and augmentation strategy) .

$\Box'$: Vector or matrix transpose for vector or matrix $\Box$.

$\Box^{comp}$: Vector or matrix of completed (imputed) variable(s) $\Box$.

$\Box^{mis}$: Vector or matrix of missing / unobsered parts of variable(s) $\Box$.

$\Box^{obs}$: Vector or matrix of observed parts of variable(s) $\Box$.

$\int \cdot d\Box$: Integral of $\cdot$ and differential $d\Box$.

$\int \cdots \int_{\mathbb{R}} \cdot d\,(\Box)$: Multiple integral of $\cdot$ with differentials $d\,(\Box) = d\Box_1 \cdot \ldots \cdot d\Box_j \cdot \ldots$.

$\otimes$: Kronecker product.

$\odot$: Entry-wise (Hadamard) product.

∂: (Partial) derivative.

$\propto$: “Proportional to”

$\circ$: General binary operation.

Appendix B – Derivations

Within Appendix B, we focus on some of the derivations needed for the sampling algorithm. As pointed out, Conti et al. (2014) will be the basis, but with matrix-variate adaptions. In terms of (full) conditional considerations, general Bayes textbooks, i.e. Gelman et al. (2014) and Jackman (2009), and considerations for MCMC methods (Gilks et al., 1996) , Gibbs-sampling (i.e. Casella & George, 1992, Gelfand et al., 1990, Robert & Casella, 2004, chapters 9 and 10, and Roberts, 1996) and full conditionals (i.e. Gilks, 1996) are used as our reference. For specific linear algebra and matrix calculation, we rely on Eaton (2007), Gupta & Nagar (2000), Henderson & Searle, 1981, Henderson & Searle, 1979, Magnus & Neudecker, 1979, Meyer (2000), Tracy & Dwyer, 1969 and for usage on matrix regression-like structures, we refer to Geweke (1996) and Viroli (2012). Finally, in terms of separation of covariances and correlations respectively, we base the considerations, next to Conti et al. (2014), on Barnard et al. (2000) and Zhang et al. (2006).

Let $\tilde{\mathbf{\Sigma}}_{\varepsilon}^{-1} = \mathbf{E}'(\mathbf{\Sigma}_{\varepsilon}^{-1} \otimes \mathbf{I}_N)\mathbf{E}$, $\tilde{\mathbf{Y}}_{\theta}^{*} = \mathbf{Y}^* - \mathbf{\Theta}\boldsymbol{\alpha}'$, $\tilde{\mathbf{Y}}_{X}^{*} = \mathbf{Y}^* - \mathbf{X}\boldsymbol{\beta}'$

Observed-data likelihood for underlying variable $\mathbf{Y}^*$

Matrix-variate likelihood

$$\begin{aligned} f_{y*}(\mathbf{Y}^*|\cdot) \propto |\mathbf{\Sigma}_{\varepsilon} \otimes \mathbf{I}_N|^{-\frac{MN}{2}} \exp\Big[-\frac{1}{2}\left\{\text{vec}\left(\mathbf{Y}^*\right) - \text{vec}\left(\mathbf{\Theta}\boldsymbol{\alpha}'\right) - \text{vec}\left(\mathbf{X}\boldsymbol{\beta}'\right)\right\}' \cdot \\ \tilde{\mathbf{\Sigma}}_{\varepsilon}^{-1}\left\{\text{vec}\left(\mathbf{Y}^*\right) - \text{vec}\left(\mathbf{\Theta}\boldsymbol{\alpha}'\right) - \text{vec}\left(\mathbf{X}\boldsymbol{\beta}'\right)\right\}\Big] \end{aligned}$$

Multivariate likelihood for measurement m

$$\begin{aligned} f_{y*}(\mathbf{y}_m^*|\cdot) \propto \left(2\sigma_m^2\right)^{-\frac{N}{2}} \exp\Big[-\frac{1}{2\sigma_{\varepsilon,m}^2}\left\{\mathbf{y}_m^* - \mathbf{\Theta}\boldsymbol{\alpha}_m' - \mathbf{X}\boldsymbol{\beta}_m'\right\}' \mathbf{E}^{mm\prime}\left\{\mathbf{I}_N\right\} \\ \cdot \mathbf{E}^{mm}\left\{\mathbf{y}_m^* - \mathbf{\Theta}\boldsymbol{\alpha}_m' - \mathbf{X}\boldsymbol{\beta}_m'\right\}\Big] \end{aligned}$$

Multivariate likelihood for individual i

$$\begin{aligned} f_{y*}(\mathbf{y}_i^*|\cdot) \propto |\Sigma_{\mathbf{Y}^*}|^{-\frac{M}{2}} \exp\Big[-\frac{1}{2}\left\{\mathbf{y}_i^{\prime *} - \boldsymbol{\alpha}\boldsymbol{\theta}_i' - \boldsymbol{\beta}\mathbf{x}_i'\right\}' \mathbf{E}^{ii\prime}\left\{\mathbf{\Sigma}_{\varepsilon}^{-1}\right\} \\ \cdot \mathbf{E}^{ii}\left\{\mathbf{y}_i^{\prime *} - \boldsymbol{\alpha}\boldsymbol{\theta}_i' - \boldsymbol{\beta}\mathbf{x}_i'\right\}\Big] \end{aligned}$$

Posterior for β

Prior for β

$$\pi(\boldsymbol{\beta}) \propto \exp\left\{-\frac{1}{2}\left(\text{vec}(\boldsymbol{\beta}) - \boldsymbol{\mu}_\beta^0\right)' \left(\boldsymbol{\Sigma}_\beta^0\right)^{-1} \left(\text{vec}(\boldsymbol{\beta}) - \boldsymbol{\mu}_\beta^0\right)\right\}$$

Joint distribution with respect to $\mathbf{Y}^*$ and Prior

$$\begin{aligned}
f(\mathbf{Y}^*|\cdot)\pi(\boldsymbol{\beta}) \propto & |\boldsymbol{\Sigma}_\varepsilon \otimes \mathbf{I}_N|^{-\frac{MN}{2}} \left|\boldsymbol{\Sigma}_\beta^0\right|^{-\frac{MQ}{2}} \\
& \exp\left[-\frac{1}{2}\left\{\text{vec}\left(\tilde{\mathbf{Y}}_\theta^*\right) - (\mathbf{I}_M \otimes \mathbf{X})\mathbf{V}_\beta \text{vec}(\boldsymbol{\beta})\right\}' \tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1} \left\{\text{vec}\left(\tilde{\mathbf{Y}}_\theta^*\right) - (\mathbf{I}_M \otimes \mathbf{X})\mathbf{V}_\beta \text{vec}(\boldsymbol{\beta})\right\}\right] \\
& \exp\left[-\frac{1}{2}\left\{\text{vec}(\boldsymbol{\beta}) - \boldsymbol{\mu}_\beta^0\right\}' \left(\boldsymbol{\Sigma}_\beta^0\right)^{-1} \left\{\text{vec}(\boldsymbol{\beta}) - \boldsymbol{\mu}_\beta^0\right\}\right] \\
\propto & \exp\left[-2\text{vec}(\boldsymbol{\beta})' \left\{\mathbf{V}_\beta' (\mathbf{I}_M \otimes \mathbf{X})' \tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1} \text{vec}\left(\tilde{\mathbf{Y}}_\theta^*\right) + \left(\boldsymbol{\Sigma}_\beta^0\right)^{-1} \boldsymbol{\mu}_\beta^0\right\} + \text{vec}(\boldsymbol{\beta})' \left\{\mathbf{V}_\beta' (\mathbf{I}_M \otimes \mathbf{X})' \tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1} (\mathbf{I}_M \otimes \mathbf{X}) \mathbf{V}_\beta + \left(\boldsymbol{\Sigma}_\beta^0\right)^{-1}\right\} \text{vec}(\boldsymbol{\beta})\right]
\end{aligned}$$

Posterior for β

$$f(\text{vec}(\boldsymbol{\beta})|\mathbf{Y}^*, \mathbf{X}, \boldsymbol{\Theta}, \boldsymbol{\Sigma}_\varepsilon, \boldsymbol{\alpha}) \propto \exp\left\{-\frac{1}{2}\left(\text{vec}(\boldsymbol{\beta}) - \boldsymbol{\mu}_\beta^\diamond\right)' \left(\boldsymbol{\Sigma}_\beta^\diamond\right)^{-1} \left(\text{vec}(\boldsymbol{\beta}) - \boldsymbol{\mu}_\beta^\diamond\right)\right\}$$

with

- $\boldsymbol{\Sigma}_\beta^\diamond = \left\{\mathbf{V}_\beta' (\mathbf{I}_M \otimes \mathbf{X})' \mathbf{E}' (\boldsymbol{\Sigma}_\varepsilon \otimes \mathbf{I}_N)^{-1} \mathbf{E} (\mathbf{I}_M \otimes \mathbf{X}) \mathbf{V}_\beta + \left(\boldsymbol{\Sigma}_\beta^0\right)^{-1}\right\}^{-1}$
- $\boldsymbol{\mu}_\beta^\diamond = \boldsymbol{\Sigma}_\beta^\diamond \left\{\mathbf{V}_\beta' (\mathbf{I}_M \otimes \mathbf{X})' \mathbf{E}' (\boldsymbol{\Sigma}_\varepsilon \otimes \mathbf{I}_N)^{-1} \mathbf{E}\text{vec}\left(\tilde{\mathbf{Y}}_\theta^*\right) + \left(\boldsymbol{\Sigma}_\beta^0\right)^{-1} \boldsymbol{\mu}_\beta^0\right\}$

Posterior for σ_m^2 for metric measurements

Prior for σ_m^2

$$\pi(\sigma_{\varepsilon,m}^2) \propto (\sigma_{\varepsilon,m}^2)^{-c_m^0 - 1} \exp\left(-\frac{C_m^0}{\sigma_{\varepsilon,m}^2}\right)$$

Joint distribution with respect to $\mathbf{Y}^*$ and Prior, given $\pi\left(\boldsymbol{\alpha}_m|\sigma^2_m\right) = \pi\left(\boldsymbol{\alpha}_m\right)\pi\left(\sigma^2_m\right)$

$$
\begin{aligned}
f_{y*}(\mathbf{y}^*_m|\cdot)\pi(\sigma^2_{\varepsilon,m}) \propto & \left(2\sigma^2_{\varepsilon,m}\right)^{-\frac{N}{2}}\left(\sigma^2_{\varepsilon,m}\right)^{-c^0_m-1}\exp\left[-\frac{1}{2\sigma^2_m}\left\{\mathbf{y}^*_m-\boldsymbol{\Theta}\boldsymbol{\alpha}'_m-\mathbf{X}\boldsymbol{\beta}'_m\right\}'\right. \\
& \left.\cdot\mathbf{E}^{mm\prime}\left\{\mathbf{I}_N\right\}\mathbf{E}^{mm}\left\{\mathbf{y}^*_m-\boldsymbol{\Theta}\boldsymbol{\alpha}'_m-\mathbf{X}\boldsymbol{\beta}'_m\right\}-\frac{C^0_m}{\sigma^2_{\varepsilon,m}}\right] \\
=& 2^{-\frac{N}{2}}\left(\sigma^2_{\varepsilon,m}\right)^{-\frac{N}{2}}\left(\sigma^2_{\varepsilon,m}\right)^{-c^0_m-1}\exp\left[-\frac{1}{\sigma^2_m}\left\{C^0_m+\right.\right. \\
& \left.\left.\frac{1}{2}\left(\mathbf{y}^*_m-\boldsymbol{\Theta}\boldsymbol{\alpha}'_m-\mathbf{X}\boldsymbol{\beta}'_m\right)'\mathbf{E}^{mm\prime}\mathbf{E}^{mm}\left(\mathbf{y}^*_m-\boldsymbol{\Theta}\boldsymbol{\alpha}'_m-\mathbf{X}\boldsymbol{\beta}'_m\right)\right\}\right]
\end{aligned}
$$

Posterior for σ^2_m, when considering missing values in measurements

$$
f(\sigma^2_{\varepsilon,m}|\mathbf{X},\boldsymbol{\beta}_m,\mathbf{y}^*_m,\boldsymbol{\Theta},\mathbf{E}^{mm}) \propto \left(\sigma^2_{\varepsilon,m}\right)^{-c^N_m-1}\exp\left(-\frac{1}{\sigma^2_{\varepsilon,m}}\cdot C^N_m\right)
$$

with

- $c^N_m = c^0_m + \frac{\sum_{i=1}^N o^Y_{im}}{2}$
- $C^N_m = C^0_m + \frac{1}{2}\left\{\left(\mathbf{y}^*_m-\mathbf{X}\boldsymbol{\beta}_m-\boldsymbol{\Theta}\boldsymbol{\alpha}_m\right)'\left(\mathbf{E}^{mm}\right)'\mathbf{E}^{mm}\left(\mathbf{y}^*_m-\mathbf{X}\boldsymbol{\beta}_m-\boldsymbol{\Theta}\boldsymbol{\alpha}_m\right)\right\}$

Posterior for $\boldsymbol{\alpha}$

Prior for $\boldsymbol{\alpha}$

$$
\begin{aligned}
\pi(\mathrm{vec}(\boldsymbol{\alpha})^\Delta) \propto \exp & \left\{-\frac{1}{2}\left(\mathrm{vec}(\boldsymbol{\alpha})^\Delta-\boldsymbol{\mu}^0_\alpha\right)'\left(\boldsymbol{\Sigma}^0_\alpha\right)^{-1}\right. \\
& \left.\left(\mathrm{vec}(\boldsymbol{\alpha})^\Delta-\boldsymbol{\mu}^0_\alpha\right)\right\}
\end{aligned}
$$

Joint distribution with respect to $\mathbf{Y}^*$ and Prior, given $\pi(\boldsymbol{\alpha}|\boldsymbol{\Sigma}) = \pi(\boldsymbol{\alpha})\,\pi(\boldsymbol{\Sigma})$

$$
\begin{aligned}
f_{y^*}(\mathbf{Y}^*|\cdot)\pi(\mathrm{vec}(\boldsymbol{\alpha})^{\Delta}) \propto & \exp\left[-\frac{1}{2}\left\{\mathrm{vec}\left(\tilde{\mathbf{Y}}_X^*\right) - (\mathbf{I}_M \otimes \boldsymbol{\Theta})\,\mathbf{V}_\alpha \mathrm{vec}\,(\boldsymbol{\alpha})\right\}'\right.\\
& \left.\tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1}\left\{\mathrm{vec}\left(\tilde{\mathbf{Y}}_X^*\right) - (\mathbf{I}_M \otimes \boldsymbol{\Theta})\,\mathbf{V}_\alpha \mathrm{vec}\,(\boldsymbol{\alpha})\right\}\right]\cdot\\
& \exp\left\{-\frac{1}{2}\left(\mathrm{vec}(\boldsymbol{\alpha})^{\Delta} - \boldsymbol{\mu}_\alpha^0\right)'\left(\boldsymbol{\Sigma}_\alpha^0\right)^{-1}\left(\mathrm{vec}(\boldsymbol{\alpha})^{\Delta} - \boldsymbol{\mu}_\alpha^0\right)\right\}\\
\propto & \exp\left[-2\mathrm{vec}(\boldsymbol{\alpha})^{\Delta,\prime}\left\{\mathbf{V}_\alpha'\,(\mathbf{I}_M \otimes \boldsymbol{\Theta})'\,\tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1}\right.\right.\\
& \left.\mathrm{vec}\left(\tilde{\mathbf{Y}}_X^*\right) + \left(\boldsymbol{\Sigma}_\alpha^0\right)^{-1}\boldsymbol{\mu}_\alpha^0\right\} + \mathrm{vec}(\boldsymbol{\alpha})^{\Delta,\prime}\left\{\mathbf{V}_\alpha'\,(\mathbf{I}_M \otimes \boldsymbol{\Theta})'\cdot\right.\\
& \left.\left.\tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1}\,(\mathbf{I}_M \otimes \boldsymbol{\Theta})\,\mathbf{V}_\alpha + \left(\boldsymbol{\Sigma}_\beta^0\right)^{-1}\right\}\mathrm{vec}(\boldsymbol{\alpha})^{\Delta}\right]
\end{aligned}
$$

Posterior for $\boldsymbol{\alpha}$

$$
\begin{aligned}
f(\mathrm{vec}(\boldsymbol{\alpha})^{\bar{\Delta}}|\boldsymbol{\Theta}, \boldsymbol{\Sigma}_\varepsilon, \mathbf{Y}^*, \mathbf{X}, \boldsymbol{\beta}) \propto \exp & \left\{-\frac{1}{2}\left(\mathrm{vec}(\boldsymbol{\alpha})^{\bar{\Delta}} - \boldsymbol{\mu}_\alpha^\diamond\right)'(\boldsymbol{\Sigma}_\alpha^\diamond)^{-1}\right.\\
& \left.\left(\mathrm{vec}(\boldsymbol{\alpha})^{\bar{\Delta}} - \boldsymbol{\mu}_\alpha^\diamond\right)\right\}
\end{aligned}
$$

with

- $\boldsymbol{\Sigma}_\alpha^\diamond = \left\{\mathbf{V}_\alpha'\,(\mathbf{I}_M \otimes \boldsymbol{\Theta})'\,\mathbf{E}'\,(\boldsymbol{\Sigma}_\varepsilon \otimes \mathbf{I}_N)^{-1}\,\mathbf{E}\,(\mathbf{I}_M \otimes \boldsymbol{\Theta})\,\mathbf{V}_\alpha + \left(\boldsymbol{\Sigma}_\beta^0\right)^{-1}\right\}^{-1}$
- $\boldsymbol{\mu}_\alpha^\diamond = \boldsymbol{\Sigma}_\alpha^\diamond\left\{\mathbf{V}_\alpha'\,(\mathbf{I}_M \otimes \boldsymbol{\Theta})'\,\mathbf{E}'\,(\boldsymbol{\Sigma}_\varepsilon \otimes \mathbf{I}_N)^{-1}\,\mathbf{E}\mathrm{vec}\left(\tilde{\mathbf{Y}}_X^*\right) + \left(\boldsymbol{\Sigma}_\alpha^0\right)^{-1}\boldsymbol{\mu}_\alpha^0\right\}$

Conditional distribution for latent variables $\ddot{\boldsymbol{\Theta}}$ and $\boldsymbol{\Theta}$ respectively

(Given $\ddot{\boldsymbol{\Theta}} = \boldsymbol{\Theta}\boldsymbol{\Lambda}^{\frac{1}{2}}$, $\ddot{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\boldsymbol{\Lambda}^{\frac{1}{2}}$, $\ddot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\boldsymbol{\Lambda}^{\frac{1}{2}}$ and $\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{\Lambda}^{-\frac{1}{2}} = \mathbf{I}_K$)

Marginal specification

$$
\begin{aligned}
f(\boldsymbol{\Theta}|\mathbf{Z}, \boldsymbol{\Psi}) \propto & |\mathbf{R}|^{-\frac{N}{2}}\exp\left[-\frac{1}{2}\mathrm{tr}\left\{(\boldsymbol{\Theta} - \mathbf{Z}\boldsymbol{\gamma})\mathbf{R}^{-1}(\boldsymbol{\Theta} - \mathbf{Z}\boldsymbol{\gamma})'\right\}\right]\\
= & |\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-\frac{1}{2}}|^{-\frac{N}{2}}\exp\left[-\frac{1}{2}\mathrm{tr}\left\{(\boldsymbol{\Theta} - \mathbf{Z}\boldsymbol{\gamma})\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}^{\frac{1}{2}}(\boldsymbol{\Theta} - \mathbf{Z}\boldsymbol{\gamma})'\right\}\right]\\
\propto & \exp\left[-\frac{1}{2}\mathrm{tr}\left\{(\boldsymbol{\Theta}\boldsymbol{\Lambda}^{\frac{1}{2}} - \mathbf{Z}\boldsymbol{\gamma}\boldsymbol{\Lambda}^{\frac{1}{2}})\boldsymbol{\Omega}^{-1}(\boldsymbol{\Theta}\boldsymbol{\Lambda}^{\frac{1}{2}} - \mathbf{Z}\boldsymbol{\gamma}\boldsymbol{\Lambda}^{\frac{1}{2}})'\right\}\right]\\
= & \exp\left[-\frac{1}{2}\mathrm{tr}\left\{(\ddot{\boldsymbol{\Theta}} - \mathbf{Z}\ddot{\boldsymbol{\gamma}})\boldsymbol{\Omega}^{-1}(\ddot{\boldsymbol{\Theta}} - \mathbf{Z}\ddot{\boldsymbol{\gamma}})'\right\}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
f(\text{vec}(\boldsymbol{\Theta})|\mathbf{Z},\boldsymbol{\Psi}) \propto & \exp\left[-\frac{1}{2}\text{vec}(\boldsymbol{\Theta}-\mathbf{Z}\boldsymbol{\gamma})'\left(\mathbf{R}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\boldsymbol{\Theta}-\mathbf{Z}\boldsymbol{\gamma})\right] \\
= & \exp\left[-\frac{1}{2}\left\{\text{vec}(\boldsymbol{\Theta})-\text{vec}(\mathbf{Z}\boldsymbol{\gamma})\right\}'\left\{\left(\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-\frac{1}{2}}\right)^{-1}\otimes\mathbf{I}_N\right\}\right. \\
& \left.\left\{\text{vec}(\boldsymbol{\Theta})-\text{vec}(\mathbf{Z}\boldsymbol{\gamma})\right\}\right] \\
= & \exp\left[-\frac{1}{2}\left\{\text{vec}(\boldsymbol{\Theta})'\left(\boldsymbol{\Lambda}^{\frac{1}{2}}\otimes\mathbf{I}_N\right)\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\left(\boldsymbol{\Lambda}^{\frac{1}{2}}\otimes\mathbf{I}_N\right)\text{vec}(\boldsymbol{\Theta})-\right.\right. \\
& 2\text{vec}(\mathbf{I}_N\boldsymbol{\Theta}\boldsymbol{\Lambda}^{\frac{1}{2}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\mathbf{I}_N\mathbf{Z}\boldsymbol{\gamma}\boldsymbol{\Lambda}^{\frac{1}{2}})+ \\
& \left.\left.\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})\right\}\right] \\
= & \exp\left[-\frac{1}{2}\left\{\text{vec}(\ddot{\boldsymbol{\Theta}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\ddot{\boldsymbol{\Theta}})-\right.\right. \\
& 2\text{vec}(\ddot{\boldsymbol{\Theta}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})+ \\
& \left.\left.\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})\right\}\right]
\end{aligned}
$$

Joint distribution with respect to $\mathbf{Y}^*$

$$
\begin{aligned}
f_{y^*}(\mathbf{Y}^*|\cdot)f(\boldsymbol{\Theta}|\mathbf{Z},\boldsymbol{\Psi}) \propto & \exp\left[-\frac{1}{2}\left\{\text{vec}\left(\tilde{\mathbf{Y}}_X^*\right)-\text{vec}\left(\boldsymbol{\Theta}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\alpha}'\right)\right\}'\tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1}\right. \\
& \left.\left\{\text{vec}\left(\tilde{\mathbf{Y}}_X^*\right)-\left(\ddot{\boldsymbol{\alpha}}\otimes\mathbf{I}_N\right)\text{vec}\left(\ddot{\boldsymbol{\Theta}}\right)\right\}\right] \\
& \exp\left[-\frac{1}{2}\left\{\text{vec}(\ddot{\boldsymbol{\Theta}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\ddot{\boldsymbol{\Theta}})-\right.\right. \\
& 2\text{vec}(\ddot{\boldsymbol{\Theta}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})+ \\
& \left.\left.\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})\right\}\right] \\
= & \exp\left[-\frac{1}{2}\left\{\text{vec}\left(\ddot{\boldsymbol{\Theta}}\right)'\left(\ddot{\boldsymbol{\alpha}}\otimes\mathbf{I}_N\right)'\tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1}\left(\ddot{\boldsymbol{\alpha}}\otimes\mathbf{I}_N\right)\text{vec}\left(\ddot{\boldsymbol{\Theta}}\right)+\right.\right. \\
& \text{vec}(\ddot{\boldsymbol{\Theta}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\ddot{\boldsymbol{\Theta}})- \\
& 2\text{vec}(\ddot{\boldsymbol{\Theta}})'\left(\ddot{\boldsymbol{\alpha}}\otimes\mathbf{I}_N\right)'\tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1}\text{vec}\left(\tilde{\mathbf{Y}}_X^*\right)- \\
& 2\text{vec}(\ddot{\boldsymbol{\Theta}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})+\text{vec}\left(\tilde{\mathbf{Y}}_X^*\right)'\tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1}\text{vec}\left(\tilde{\mathbf{Y}}_X^*\right)+ \\
& \left.\left.\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\text{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})\right\}\right]
\end{aligned}
$$

Updated Conditional distribution for expanded latent variables

$$f(\text{vec}(\ddot{\boldsymbol{\Theta}})|\boldsymbol{\Omega}, \boldsymbol{\Sigma}_\varepsilon, \ddot{\boldsymbol{\alpha}}, \mathbf{Y}^*, \boldsymbol{\beta}, \mathbf{X}, \mathbf{Z}) \propto \exp\left[-\frac{1}{2}\left\{\text{vec}\left(\ddot{\boldsymbol{\Theta}}\right)' (\ddot{\boldsymbol{\alpha}} \otimes \mathbf{I}_N)' \tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1} \cdot \right.\right.$$
$$(\ddot{\boldsymbol{\alpha}} \otimes \mathbf{I}_N) \text{vec}\left(\ddot{\boldsymbol{\Theta}}\right) + \text{vec}(\ddot{\boldsymbol{\Theta}})'\mathbf{R}^{-1}\text{vec}(\ddot{\boldsymbol{\Theta}}) -$$
$$2\text{vec}\left(\ddot{\boldsymbol{\Theta}}\right)' (\ddot{\boldsymbol{\alpha}} \otimes \mathbf{I}_N)' \tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1} \text{vec}\left(\tilde{\mathbf{Y}}_X^*\right) -$$
$$\left.\left. 2\text{vec}(\ddot{\boldsymbol{\Theta}})'\mathbf{R}^{-1}\text{vec}(\mathbf{Z}\ddot{\gamma})\right\}\right]$$
$$= \exp\left\{-\frac{1}{2}\left(\text{vec}(\ddot{\boldsymbol{\Theta}}) - \boldsymbol{\mu}_\theta^\diamond\right)' (\boldsymbol{\Sigma}_\theta^\diamond)^{-1}\right.$$
$$\left.\left(\text{vec}(\ddot{\boldsymbol{\Theta}}) - \boldsymbol{\mu}_\theta^\diamond\right)\right\}$$

with

- $\boldsymbol{\Sigma}_\theta^\diamond = \left\{(\ddot{\boldsymbol{\alpha}} \otimes \mathbf{I}_N)' \tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1} (\ddot{\boldsymbol{\alpha}} \otimes \mathbf{I}_N) + \left(\boldsymbol{\Omega}^{-1} \otimes \mathbf{I}_N\right)^{-1}\right\}^{-1}$
- $\boldsymbol{\mu}_\theta^\diamond = \boldsymbol{\Sigma}_\theta^\diamond \left\{(\ddot{\boldsymbol{\alpha}} \otimes \mathbf{I}_N)' \tilde{\boldsymbol{\Sigma}}_\varepsilon^{-1} \text{vec}\left(\tilde{\mathbf{Y}}_X^*\right) + \left(\boldsymbol{\Omega}^{-1} \otimes \mathbf{I}_N\right)^{-1} \text{vec}(\mathbf{Z}\ddot{\gamma})\right\}$

Posterior for $\boldsymbol{\Omega}$ including update of $\mathbf{S}_\Omega^0$

Jacobian transform for $\boldsymbol{\Omega}$with respect to $\boldsymbol{\Lambda}$ and $\mathbf{R}$

This derivation tries to recreate the short result for the differential of $\boldsymbol{\Omega}$ and the Jacobian transform given by Zhang et al. (2006, esp. pp. 884 f.), and picked up by Conti et al. (2014), in order to depict all necessary steps that were left out.

Let $\boldsymbol{\Omega} = \boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{R}\boldsymbol{\Lambda}^{\frac{1}{2}}$ with $\mathbf{R} = \mathbf{R}_0 + \mathbf{I}_K$, where $\mathbf{R}_0$ is the skew-symmetric matrix of the latent variables' correlation off-diagonal elements and $\mathbf{I}_K$ is the identity matrix of dimension $K \times K$. Expand

$$\begin{aligned}
\boldsymbol{\Omega} &= \boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{R}\boldsymbol{\Lambda}^{\frac{1}{2}} = \boldsymbol{\Lambda}^{\frac{1}{2}}(\mathbf{R}_0 + \mathbf{I}_K)\boldsymbol{\Lambda}^{\frac{1}{2}} \\
&= \boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{R}_0\boldsymbol{\Lambda}^{\frac{1}{2}} + \boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{\Lambda}^{\frac{1}{2}} \\
&= \boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{R}_0\boldsymbol{\Lambda}^{\frac{1}{2}} + \boldsymbol{\Lambda} \\
&= \boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{R}_0\boldsymbol{\Lambda}^{\frac{1}{2}} + \sum_{k=1}^{K} \mathbf{E}_k\boldsymbol{\Omega}\mathbf{E}_k
\end{aligned}$$

and using $\mathbf{E}_k$ as suited elimination matrices, constructing the variance $\mathbf{\Lambda}$ as sum of the diagonal elements of $\mathbf{\Omega}$, with $|\mathbf{E}_k| = 0, \forall k$. According to Eaton (2007, p. 171, proposition 5.12), for the differential $d\mathbf{\Omega}$ it can be followed that

$$\begin{aligned} d\mathbf{\Omega} &= (|\mathbf{\Lambda}^{\frac{1}{2}}|^{K-1} + 0)d(\mathbf{R}_0 + \mathbf{I}_K) \\ &= |\mathbf{\Lambda}^{\frac{1}{2}}|^{K-1} d\mathbf{R}. \end{aligned}$$

Hence, the Jacobian of the transformation $\mathbf{\Omega} \rightarrow \mathbf{\Lambda}, \mathbf{R}$ can be described with $\mathcal{J}(\mathbf{\Omega} \rightarrow \mathbf{\Lambda}, \mathbf{R}) = |\mathbf{\Lambda}^{\frac{1}{2}}|^{K-1} = \prod_{k=1}^{K} \lambda_i^{\frac{K-1}{2}} = \prod_{k=1}^{K} (\omega_{ii}{}^2)^{\frac{K-1}{2}}$ Thus, the distribution of $\mathbf{\Lambda}$ and $\mathbf{R}$ can be written in terms of $\mathbf{\Omega}$ and a scale matrix $\mathbf{S}_\Omega^0$ using an inverse-Wishart density, i.e.

$$\begin{aligned} f(\mathbf{\Lambda}, \mathbf{R}|\mathbf{S}_\Omega^0) &= \mathbf{J}_{\mathbf{\Omega}\rightarrow\mathbf{R},\mathbf{\Lambda}} f_{\mathbf{\Omega}}(g^{-1}(\mathbf{\Lambda}, \mathbf{R})) \\ &\propto |\mathbf{\Lambda}|^{\frac{K-1}{2}} |\mathbf{S}_\Omega^0|^{\frac{\nu}{2}} |\mathbf{\Lambda}^{\frac{1}{2}} \mathbf{R} \mathbf{\Lambda}^{\frac{1}{2}}|^{-\frac{\nu+K+1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_\Omega^0 \mathbf{\Lambda}^{-\frac{1}{2}} \mathbf{R}^{-1} \mathbf{\Lambda}^{-\frac{1}{2}}\right)\right\} \\ &= |\mathbf{S}_\Omega^0|^{\frac{\nu}{2}} |\mathbf{\Lambda}|^{\frac{K-1}{2}} |\mathbf{\Lambda}|^{-\frac{\nu+K+1}{2}} |\mathbf{R}|^{-\frac{\nu+K+1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_\Omega^0 \mathbf{\Lambda}^{-1} \mathbf{R}^{-1}\right)\right\} \\ &= |\mathbf{S}_\Omega^0|^{\frac{\nu}{2}} |\mathbf{\Lambda}|^{-\frac{\nu+2}{2}} |\mathbf{R}|^{-\frac{\nu+K+1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}_\Omega^0 \mathbf{\Lambda}^{-1} \mathbf{R}^{-1}\right)\right\}. \end{aligned}$$

Update of $\mathbf{S}_\Omega^0$respect to $\mathbf{\Lambda}$ and $\mathbf{R}$, and thus $\mathbf{\Omega}$

Let

$$\pi(\mathbf{S}_\Omega^0) \propto \prod_{k=1}^{K} s_k^{\frac{1}{2}-1} \exp\left\{-s_k(2\nu^* A_k^2)^{-1}\right\}$$

then

$$
\begin{aligned}
f(\mathbf{S}^0_\Omega|\boldsymbol{\Lambda},\mathbf{R}) \propto & f(\boldsymbol{\Lambda},\mathbf{R}|\mathbf{S}^0_\Omega)\pi(\mathbf{S}^0_\Omega)\\
=&|\mathbf{S}^0_\Omega|^{\frac{\nu}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}^0_\Omega\boldsymbol{\Lambda}^{-1}\mathbf{R}^{-1}\right)\right\}\cdot\\
&\prod_{k=1}^{K} s_k^{\frac{1}{2}-1}\exp\left\{-s_k(2\nu^0_\Omega A_k^2)^{-1}\right\}\\
=&|\mathbf{S}^0_\Omega|^{\frac{\nu}{2}} \exp\left\{-\frac{1}{2}\sum_{k=1}^{K} s_k \underbrace{\left(\boldsymbol{\Lambda}^{-1}\mathbf{R}^{-1}\right)_{kk}}_{\Rightarrow(\boldsymbol{\Omega})_{kk}}\right\}\\
&\prod_{k=1}^{K} s_k^{\frac{1}{2}-1}\exp\left\{-s_k(2\nu^0_\Omega A_k^2)^{-1}\right\}\\
=&|\mathbf{S}^0_\Omega|^{\frac{\nu}{2}}\prod_{k=1}^{K} s_k^{\frac{1}{2}-1}\\
&\exp\left\{-\frac{1}{2}s_k\left((\boldsymbol{\Omega})_{kk}+(\nu^0_\Omega A_k^2)^{-1}\right)\right\}\cdot\\
=&\prod_{k=1}^{K} s_k^{\frac{\nu+1}{2}-1}\exp\left\{-\frac{1}{2}s_k\left((\boldsymbol{\Omega})_{kk}+(\nu^0_\Omega A_k^2)^{-1}\right)\right\},
\end{aligned}
$$

resulting in a Gamma distribution for every conditional s_k.

Prior for $\boldsymbol{\Omega}$

$$
\pi(\boldsymbol{\Omega}|\mathbf{S}^0_\Omega) \propto |\boldsymbol{\Omega}|^{-\frac{\nu^0+K+1}{2}}|\mathbf{S}^0_\Omega|^{\frac{\nu^0}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}^0_\Omega\boldsymbol{\Omega}^{-1}\right)\right\}
$$

Joint distribution

$$
\begin{aligned}
f(\ddot{\boldsymbol{\Theta}}|\mathbf{Z},\boldsymbol{\Omega},\ddot{\gamma})\pi(\boldsymbol{\Omega}) \propto & |\boldsymbol{\Omega}|^{-\frac{N}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\left(\ddot{\boldsymbol{\theta}}_i-\ddot{\gamma}'\mathbf{z}_i\right)'\boldsymbol{\Omega}^{-1}\left(\ddot{\boldsymbol{\theta}}_i-\ddot{\gamma}'\mathbf{z}_i\right)\right\}\\
&|\boldsymbol{\Omega}|^{-\frac{\nu^0+K+1}{2}}|\mathbf{S}^0_\Omega|^{\frac{\nu^0}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\left(\mathbf{S}^0_\Omega\boldsymbol{\Omega}^{-1}\right)\right\}
\end{aligned}
$$

Posterior of $\boldsymbol{\Omega}$

$$
\begin{aligned}
f(\boldsymbol{\Omega}|\ldots) \propto & |\mathbf{S}_{\Omega}^{0}|^{\frac{\nu^0}{2}} |\boldsymbol{\Omega}|^{-\frac{\nu^0+K+1}{2}} |\boldsymbol{\Omega}|^{-\frac{N}{2}} \\
& \exp\left[-\frac{1}{2}\left\{\sum_{i=1}^{N}\left(\ddot{\boldsymbol{\theta}}_i - \ddot{\gamma}'\mathbf{z}_i\right)' \boldsymbol{\Omega}^{-1}\left(\ddot{\boldsymbol{\theta}}_i - \ddot{\gamma}'\mathbf{z}_i\right)\right.\right. \\
& \left.\left. \mathrm{tr}\left(\mathbf{S}_{\Omega}^{0}\boldsymbol{\Omega}^{-1}\right)\right\}\right] \\
= & |\mathbf{S}|^{\frac{\nu}{2}} |\boldsymbol{\Omega}|^{-\frac{\nu+N+K+1}{2}} \cdot \\
& \exp\left[-\frac{1}{2}\mathrm{tr}\left\{\left(\left(\ddot{\boldsymbol{\Theta}} - \mathbf{Z}\ddot{\gamma}\right)'\left(\ddot{\boldsymbol{\Theta}} - \mathbf{Z}\ddot{\gamma}\right) + \mathbf{S}_{\Omega}^{0}\right)\boldsymbol{\Omega}^{-1}\right\}\right]
\end{aligned}
$$

Posterior for $\ddot{\gamma}$ to retrieve γ

Prior for $\ddot{\gamma}$

$$
\pi(\mathrm{vec}(\ddot{\boldsymbol{\gamma}})) \propto \exp\left\{-\frac{1}{2}\left(\mathrm{vec}(\ddot{\boldsymbol{\gamma}}) - \boldsymbol{\mu}_{\ddot{\gamma}}^{0}\right)'\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{0}\right)^{-1}\left(\mathrm{vec}(\ddot{\boldsymbol{\gamma}}) - \boldsymbol{\mu}_{\ddot{\gamma}}^{0}\right)\right\}
$$

Joint distribution with respect to $\ddot{\boldsymbol{\Theta}}$ and Prior

$$
\begin{aligned}
f(\mathrm{vec}(\ddot{\boldsymbol{\Theta}})|\mathbf{Z},\boldsymbol{\Omega},\ddot{\gamma})\pi(\mathrm{vec}(\ddot{\boldsymbol{\gamma}})) \propto & \exp\left[-\frac{1}{2}\left\{\mathrm{vec}(\ddot{\boldsymbol{\Theta}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\mathrm{vec}(\ddot{\boldsymbol{\Theta}}) - \right.\right. \\
& 2\mathrm{vec}(\ddot{\boldsymbol{\gamma}})'\left(\mathbf{I}_K\otimes\mathbf{Z}\right)'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\mathrm{vec}(\ddot{\boldsymbol{\Theta}}) + \\
& \left.\left. \mathrm{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\mathrm{vec}(\mathbf{Z}\ddot{\boldsymbol{\gamma}})\right\}\right] \\
& \exp\left[-\frac{1}{2}\left\{\mathrm{vec}(\ddot{\boldsymbol{\gamma}})'\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{0}\right)^{-1}\mathrm{vec}(\ddot{\boldsymbol{\gamma}})\right.\right. \\
& \left. -2\mathrm{vec}(\ddot{\gamma})'\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{0}\right)^{-1}\boldsymbol{\mu}_{\ddot{\gamma}}^{0} + \boldsymbol{\mu}_{\ddot{\gamma}}^{0,\prime}\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{0}\right)^{-1}\boldsymbol{\mu}_{\ddot{\gamma}}^{0}\right\} \\
\propto & \exp\left[-\frac{1}{2}\left\{\mathrm{vec}(\ddot{\boldsymbol{\gamma}})'\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{0}\right)^{-1}\mathrm{vec}(\ddot{\boldsymbol{\gamma}}) + \right.\right. \\
& \mathrm{vec}(\ddot{\boldsymbol{\gamma}})'\left(\mathbf{I}_K\otimes\mathbf{Z}\right)'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\left(\mathbf{I}_K\otimes\mathbf{Z}\right)\mathrm{vec}(\ddot{\boldsymbol{\gamma}}) - \\
& \mathrm{vec}(\ddot{\boldsymbol{\gamma}})'\left(\mathbf{I}_K\otimes\mathbf{Z}\right)'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\mathrm{vec}(\ddot{\boldsymbol{\Theta}}) - \\
& 2\mathrm{vec}(\ddot{\boldsymbol{\gamma}})'\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{0}\right)^{-1}\boldsymbol{\mu}_{\ddot{\gamma}}^{0} + \boldsymbol{\mu}_{\ddot{\gamma}}^{0,\prime}\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{0}\right)^{-1}\boldsymbol{\mu}_{\ddot{\gamma}}^{0} + \\
& \left.\left. \mathrm{vec}(\ddot{\boldsymbol{\Theta}})'\left(\boldsymbol{\Omega}^{-1}\otimes\mathbf{I}_N\right)\mathrm{vec}(\ddot{\boldsymbol{\Theta}})\right\}\right]
\end{aligned}
$$

Posterior for $\ddot{\gamma}$

$$f(\mathrm{vec}(\ddot{\boldsymbol{\gamma}})|\ddot{\boldsymbol{\Theta}},\boldsymbol{\Omega}) \quad \propto \quad \exp\left\{-\frac{1}{2}\left(\mathrm{vec}(\ddot{\boldsymbol{\gamma}})-\boldsymbol{\mu}_{\ddot{\gamma}}^{\diamond}\right)'\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{\diamond}\right)^{-1}\left(\mathrm{vec}(\ddot{\boldsymbol{\gamma}})-\boldsymbol{\mu}_{\ddot{\gamma}}^{\diamond}\right)\right\}$$

with

- $\boldsymbol{\Sigma}_{\ddot{\gamma}}^{\diamond}=\left\{(\mathbf{I}_K\otimes\mathbf{Z})'(\boldsymbol{\Omega}\otimes\mathbf{I}_N)^{-1}(\mathbf{I}_K\otimes\mathbf{Z})+\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{0}\right)^{-1}\right\}^{-1}$
- $\boldsymbol{\mu}_{\ddot{\gamma}}^{\diamond}=\boldsymbol{\Sigma}_{\ddot{\gamma}}^{\diamond}\left\{(\mathbf{I}_K\otimes\mathbf{Z})'(\boldsymbol{\Omega}\otimes\mathbf{I}_N)^{-1}\mathrm{vec}(\boldsymbol{\Theta})+\left(\boldsymbol{\Sigma}_{\ddot{\gamma}}^{0}\right)^{-1}\boldsymbol{\mu}_{\ddot{\gamma}}^{0}\right\}$

Appendix C – R Implementation and LRZ Environment Details

Main Estimation Routine and R-Version used

The main estimation routine contains steps for pre-processing input, all Gibbs-sampling steps and the Metropolis-Hastings step described above, and processing preparing the output. In order to give an understanding of possible inputs and resulting outputs, the list below describes all currently implemented possibilities. Note, due to the changes of the LRZ the estimation based upon R-3.3.1 was carried out for scenarios (F1a), (F1c), (F2b), (F3b), (F4a) and (F4b), while for all other settings–including the empirical application–R-3.3.2 was used.

Input parameters

`Y`: `data.frame` object containing outcomes / measurements.

`X`: `data.frame` object containing covariates, *Default*=`NULL`.

`IRT_log`: `logical` value indicating, whether IRT-version (`TRUE`) or not should get estimated, *Default*=`FALSE`.

`restrictIRT_log`: `logical` value indicating, whether the IRT-versions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ should be restricted, *Default*=`FALSE`. Allows interactive user input if set to `TRUE`.

`indicatorLoadingMat`: `matrix` object indicating the dependence structure in $\boldsymbol{\alpha}$, *Default*=`NULL`.

`numberLatentVars`: `integer` specifying the number of latent variables, if `indicatorLoadingMat` is not defined and interactive user input will be requested, *Default*=`NULL`.

`indicatorIRTBetaMat`: `matrix` object, one-dimensional indicating the difficulty structure in IRT, *Default*=`NULL`.

`thetaStore_log`: `logical` value indicating, whether additional estimates for $\boldsymbol{\Theta}$ (`TRUE`), encompassing the column means and covariance estimates for each draw, or not should get stored, *Default*=`FALSE`.

`orthoFactor_log`: `logical` value indicating, whether orthogonal factor model (`TRUE`) or not should get estimated, *Default*=`FALSE`.

`binary_set`: `logical vector` indicating column position of dichotomous variables in $\mathbf{Y}$, *Default*=`NULL`.

`ordered_set`: `logical vector` indicating column position of ordered variables in $\mathbf{Y}$, *Default*=`NULL`.

`hyperbeta1`: `vector` specifying the mean hyperparameters for $\boldsymbol{\beta}$, *Default*=`NULL`.

`hyperbeta2`: `matrix` specifying the variance hyperparameters for $\boldsymbol{\beta}$, *Default*=`NULL`.

`hyperalpha1`: `vector` specifying the mean hyperparameters for $\boldsymbol{\alpha}$, *Default*=`NULL`.

`hyperalpha2`: `matrix` specifying the variance hyperparameters for $\boldsymbol{\alpha}$, *Default*=`NULL`.

`hypersigma1`: `vector` specifying the shape hyperparameters for $\boldsymbol{\Sigma}_{\varepsilon}$, *Default*=`NULL`.

`hypersigma2`: `vector` specifying the scale hyperparameters for $\boldsymbol{\Sigma}_{\varepsilon}$, *Default*=`NULL`.

`hyperScaleMatTheta1`: `numeric` value specifying the degrees of freedom of the scale matrix of $\boldsymbol{\Theta}$'s correlation, *Default*=`NULL`.

`hyperScaleMatTheta2`: `logical` specifying the scaling matrix of the scale matrix of $\boldsymbol{\Theta}$'s correlation, *Default*=`NULL`.

`hyperGamma1`: `vector` specifying the mean hyperparameters for $\boldsymbol{\gamma}$, *Default*=`NULL`.

`hyperGamma2`: `matrix` specifying the variance hyperparameters for $\boldsymbol{\gamma}$, *Default*=`NULL`.

`hyperOmegaShape1`: `vector` specifying the shape hyperparameter for $\boldsymbol{\Omega}$ in an orthogonal factor model, *Default*=`NULL`.

`hyperOmegaScale2`: `matrix` specifying the scale hyperparameter for $\boldsymbol{\Omega}$ in an orthogonal factor model, *Default*=`NULL`.

`startbeta`: `matrix` specifying the start values for $\boldsymbol{\beta}$, *Default*=`NULL`.

`starttheta`: `matrix` specifying the start values for $\boldsymbol{\Theta}$, *Default*=`NULL`.

`startR`: `matrix` specifying the start values for $\boldsymbol{\Theta}$'s correlation, *Default*=`NULL`.

`startalpha`: `matrix` specifying the start values for $\boldsymbol{\alpha}$, *Default*=`NULL`.

`startYlatent`: `matrix` specifying the start values for the latent $\mathbf{Y}^*$s, *Default*=`NULL`.

`startsigma`: `vector` specifying the start values for the variances in $\boldsymbol{\Sigma}_\varepsilon$, *Default*=`NULL`.

`startxis`: `vector` or `list` specifying the start values for the cut-off parameters $\boldsymbol{\xi}$ of the ordered variables, *Default*=`NULL`.

`startgamma`: `matrix` specifying the start values for $\boldsymbol{\gamma}$, *Default*=`NULL`.

`ITERATIONS_int`: `integer` value specifying the numbers of iterations of the sampler, *Default* = `10000`.

`useintercept`: `logical` value specifying, whether an intercept should get estimated when doing the factor analysis with covariates, *Default* = `TRUE`.

`metroHaUserDF`: `integer` value specifying the degrees of freedom within the Metropolis-Hastings algorithm for ordered variables, *Default* = `10`.

`metroHaMaxIter`: `integer` value specifying the maximal number of iterations within the optimisation step during the Metropolis-Hastings algorithm for ordered variables, *Default* = `500`.

`metroHaUserSDzeta`: `integer` value specifying the standard deviation of transformed cut-off parameters, i.e., $\boldsymbol{\zeta}$, within the Metropolis-Hastings algorithm for ordered variables, *Default* = `100`. Changes should be made with caution.

`metroHaUserSDxi`: `integer` value specifying the standard deviation for cut-off parameters, i.e., $\boldsymbol{\xi}$, within the Metropolis-Hastings algorithm for ordered variables, *Default* = `1`. Changes should be made with caution.

`naaction`: `character` value specifying the handling of missing values. The *Default* = `"imputeX"` (or simple no specification), currently results in ignoring missing values in $\mathbf{Y}$ and imputation in $\mathbf{X}$. Other possibilities are list-wise deletion of missing values in either $\mathbf{Y}$ (`"naomitX"`, and imputing $\mathbf{X}$) or $\mathbf{X}$ (`"naomitX"`, and currently ignoring missing values in $\mathbf{Y}$), or deleting missing values in both kinds of data (`"naomitall"`).

`thetaStoreBurnIn`: `integer` value specifying the additional burn.in used to create plausible values $\hat{\boldsymbol{\Theta}}$, *Default* = `1`.

`seed`: `integer` value specifying the starting seed of the sampler, *Default* = `100000001`, if set to `NULL` no seed will be used.

`iterationInfo`: `integer` value specifying when to print information on the current time used (every `iterationInfo`-th iteration), *Default* = `50`.

`impXevery`: `integer` value specifying during which `impXevery`-th iteration to impute covariates, *Default* = `1`.

`imputeInfoUsed`: `character` value specifying during which information to use for imputation. $\mathbf{Y}$ and $\mathbf{X}$ (`"YData"`),, $\mathbf{Y}^*$, $\mathbf{\Theta}$ and $\mathbf{X}$ (`"YStarTheta"`), $\mathbf{\Theta}$ and $\mathbf{X}$ (`"Theta"`), $\mathbf{Y}^*$ and $\mathbf{X}$ only (`"YStar"`) or only $\mathbf{X}$ (`"Xonly"`). *Default* = `"YData"` (in factor analysis) and is always set to `"Theta"` in the case of the IRT model.

`impMiBu`: `integer` value specifying the minimum number of buckets for the CART imputation of covariates, *Default* = `5`.

`impSplitDev`: `numeric` value specifying the splitting deviance for the tree within the CART imputation of covariates, *Default* = `1e-04`.

`impStartIter`: `integer` value specifying number of iterations used for the initial imputation creating the starting solution, *Default* = `1`.

`impStartOnly`: `logical` value specifying whether only initial imputation should be used, *Default* = `FALSE`.

`thinStoreEvery`: `integer` value specifying during which `thinStoreEvery`-th iteration use to store parameters, *Default* = `1`.

`warnInput`: `logical` value activates warnings with respect to input parameters, *Default*=`TRUE`.

`inputADelta`: `logical` value whether indicator matrix for $\boldsymbol{\alpha}$ should be explicitly specified, *Default*=`FALSE`. Allows interactive user input if set to `TRUE`.

`inputBDelta`: `logical` value whether indicator for $\boldsymbol{\beta}$ should be explicitly specified, *Default*=`FALSE`. Allows interactive user input if set to `TRUE`.

`estimSigmaFree`: `logical` value whether variances in $\mathbf{\Sigma}_{\varepsilon}$ should be free estimated (ignoring restriction for categorical variables, i.e., fixed variance at 1), *Default*=`FALSE`.

`TESTX`: `data.frame` object, used for testing purposes only(!) with respect to known covariates (no missings, no imputation), i.e., imputes true covariates, *Default*=`NULL`.

`verbose`: `logical` value specifying a verbose mode, printing information on the input parameters, *Default* = `FALSE`.

Outputs

Is a list with several elements and sub-lists:

`OrdThresholds`: Contains draws of the cut-off parameters $\boldsymbol{\xi}$.

`Alpha`: Contains draws of the loading or item discrimination parameters $\boldsymbol{\alpha}$.

`Beta`: Contains draws of the covariate influences (in factor analysis) or item discrimination parameters (in IRT) $\boldsymbol{\beta}$.

`Sigma`: Contains draws of the diagonal elements of idiosyncratic variances of $\boldsymbol{\Sigma}_{\varepsilon}$.

`R`: Contains draws of the latent variables' correlations $\mathbf{R}$.

`Gamma`: Contains draws of the covariate influences (in IRT) $\boldsymbol{\gamma}$, otherwise it's NULL.

`infos`: List containing additional information (see below).

`infosXTheta`: List containing additional outputs for $\boldsymbol{\Theta}$ and $\mathbf{X}$ (see below).

Elements stored in `infos`:

`ITERATIONS_int`: Number of iterations carried out.

`N_obs`: Number of rows observations.

`M_measure`: Number columns of measurements.

`Q_indep`: Number columns of covariates,

`K_facto`: Number of latent variables.

`idx_categoric`: Index for all categorical measurements.

`delta_int_matA`: Indicator matrix of non-zeroαs.

`delta_int_matB`: Indicator matrix of non-zeroαs.

`ordered_set`: Logical vector of position of ordered scaled variables.

`noX`: Logical indicator whether covariates were present (`TRUE`) or not (`FALSE`).

`isIRT`: Logical indicator whether the model estimate was an IRT model (`TRUE`) or not (`FALSE`).

`isOrthogonal`: Logical indicator whether the model had orthogonal latent variables (`TRUE`) or not (`FALSE`).

`setHyper`: Logical indicator whether the hyper parameters were not set by the user and therefore from the function (`TRUE`) or not (`FALSE`),

`oldnames`: List containing measurements' and covariates' column names.

Elements stored in `infosXTheta`:

`thetaHat`: The average of all $\mathbf{\Theta}$ draws (except burn-in specified via `thetaStoreBurnIn`) , i.e., plausible values $\hat{\mathbf{\Theta}}$.

`thetaMeans`: Column mean estimates of all $\mathbf{\Theta}$ draws.

`thetaCovs`: Covariance estimates of all $\mathbf{\Theta}$ draws.

`covarMeans`: Column mean estimates of all covariate $\mathbf{X}$s draws.

`covarCovs`: Covariance estimates of all covariate $\mathbf{X}$s draws.

Additional R-Packages in Use and LRZ-Environment Details

As pointed out, estimations were carried out at the SLURM-based Linux cluster of the Leibniz Rechenzentrum. As stated, it allows to conduct up to 28 parallel estimations running R-3.3.1 (up until October 2018) and, due to changes made by the LRZ, R-3.3.2 afterwards. Aside from an altered R-version, these changes had also an effect on necessary versions of auxiliary R-packages needed. The table gives an overview of the packages (and necessary dependent packages) and changes needed.

Package name	Version for R-3.3.1	Version for R-3.3.2	Change needed
BH	1.65.0-1	1.65.0-1	no
bitops	1.0-6	1.0-6	no
caTools	1.17	1.17.1.1	yes
coda	0.19-1	0.19-1	no
doParallel	1.0.10	1.0.10	no
e1071	1.6-8	1.6-8	no
emdbook	1.3.9	1.3.9	no
evaluate	0.10.1	0.10.1	no
foreach	1.4.3	1.4.3	no
Formula	1.2-2	1.2-2	no
ggplot2	2.2.0	2.2.0	no
gridExtra	2.2.1	2.2.1	no
hdrcde	3.1	3.1	no
highr	0.6	0.6	no
Hmisc	3.14-6	3.14-6	no
htmlTable	1.7	1.7	no
htmltools	0.3.5	0.3.5	no
htmlwidgets	0.8	0.8	no
httpuv	1.3.3	1.3.3	no
iterators	1.0.8	1.0.10	yes
KernSmooth	2.23-15	2.23-15	no
knitr	1.15.1	1.15.1	no
lazyeval	0.2.0	0.2.0	no
markdown	0.7.7	0.7.7	no
MASS	7.3-47	7.3-51	yes
Matrix	1.2-11	1.2-14	yes
MatrixModels	0.4-1	0.4-1	no
mcmc	0.9-5	0.9-5	no
MCMCPack	1.3-8	1.3-8	no
mime	0.5	0.5	no

Package name	Version for R-3.3.1	Version for R-3.3.2	Change needed
latticeExtra	0.6-28	0.6-28	no
misc3d	0.8-4	0.8-4	no
mvnfast	0.2.2	0.2.5	yes
mvtnorm	1.0-6	1.0-8	yes
numDeriv	2016.8-1	2016.8-1	no
pracma	2.0.7	2.0.7	no
quadprog	1.5-5	1.5-5	no
quantreg	5.33	5.33	no
R6	2.2.0	2.2.0	no
Rcpp	0.12.13	0.12.19	yes
RcppArmadillo	0.7.960.1.2	0.7.960.1.2	no
rgl	0.96.0	0.96.0	no
rpart	4.1-11	4.1-13	yes
scales	0.5.0	0.5.0	no
shiny	0.14.2	0.14.2	no
snow	0.4-2	0.4-2	no
sourcetools	0.1.6	0.1.6	no
SparseM	1.77	1.77	no
stringi	1.1.5	1.2.4	yes
survival	2.41-3	2.41-3	no
tibble	1.2	1.2	no
truncnorm	1.0.7(*)	1.0.7(*)	no
ucminf	1.1-4	1.1-4	no
viridis	0.4.0	0.4.0	no
xtable	1.8-2	1.8-2	no
yaml	2.1.14	2.1.14	no

Table 7.1: R-packages used, their version and statement, whether they needed a change. (*) Note that the package truncnorm is not the official CRAN-version. A modification was done to protect the output, because several calls lead because garbage collection to missing returns and therefore crashes. But, no changes to the actual algorithm were made.

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Scenario (F1a), FA, BD

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.506	-0.026	0.063	0.039	0.085	0.013	0.110	0.028	1.057	0.984	0.992
$\sigma^2_{\varepsilon,6}$	0.490	0.478	-0.013	0.060	0.040	0.083	0.012	0.108	0.025	1.014	0.976	0.984
$\sigma^2_{\varepsilon,9}$	0.528	0.512	-0.015	0.079	0.051	0.112	0.015	0.144	0.029	1.011	0.936	0.968

Table 7.2: Scenario (F1a): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.480	0.026	0.067	0.049	0.087	0.009	0.116	0.031	1.051	0.960	0.960
$\alpha_{2,1}$	0.674	0.660	-0.014	0.078	0.055	0.108	0.017	0.139	0.040	1.011	0.968	0.976
$\alpha_{3,1}$	0.601	0.621	0.020	0.050	0.035	0.070	0.008	0.090	0.022	1.052	0.968	0.976
$\alpha_{4,2}$	-0.683	-0.678	0.005	0.079	0.065	0.104	0.015	0.138	0.048	1.000	0.936	0.936
$\alpha_{5,2}$	-0.603	-0.598	0.004	0.068	0.047	0.080	0.011	0.111	0.034	0.988	0.960	0.952
$\alpha_{6,2}$	0.700	0.706	0.007	0.050	0.036	0.062	0.006	0.084	0.023	1.000	0.936	0.968
$\alpha_{7,3}$	0.540	0.526	-0.015	0.068	0.048	0.096	0.014	0.124	0.032	1.012	0.976	0.984
$\alpha_{8,3}$	-0.378	-0.383	-0.005	0.050	0.037	0.065	0.008	0.086	0.026	1.000	0.976	0.968
$\alpha_{9,3}$	-0.709	-0.706	0.002	0.057	0.044	0.082	0.008	0.106	0.027	1.000	0.960	0.968

Table 7.3: Scenario (F1a): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.668	0.654	-0.014	0.042	0.033	0.057	0.003	0.076	0.020	1.019	0.984	0.984
$\beta_{2,1}$	0.554	0.556	0.002	0.048	0.042	0.058	0.005	0.081	0.030	0.984	0.904	0.904
$\beta_{3,1}$	-0.505	-0.501	0.003	0.027	0.019	0.030	0.001	0.042	0.013	1.000	0.944	0.944
$\beta_{4,1}$	-0.560	-0.552	0.008	0.048	0.039	0.062	0.005	0.084	0.027	1.016	0.952	0.944
$\beta_{5,1}$	0.554	0.550	-0.004	0.043	0.031	0.053	0.003	0.072	0.021	1.000	0.960	0.960
$\beta_{6,1}$	-0.432	-0.431	0.001	0.026	0.018	0.031	0.001	0.043	0.012	1.000	0.960	0.960
$\beta_{7,1}$	0.676	0.667	-0.009	0.044	0.031	0.059	0.004	0.077	0.021	1.019	0.976	0.968
$\beta_{8,1}$	0.536	0.536	0.000	0.035	0.024	0.045	0.002	0.060	0.015	1.000	0.984	0.984
$\beta_{9,1}$	0.486	0.489	0.002	0.026	0.020	0.032	0.001	0.043	0.014	1.000	0.944	0.936
$\beta_{1,2}$	-0.619	-0.598	0.021	0.051	0.040	0.066	0.003	0.089	0.025	1.049	0.952	0.952
$\beta_{2,2}$	0.788	0.783	-0.005	0.054	0.041	0.068	0.008	0.092	0.030	1.000	0.952	0.944
$\beta_{3,2}$	0.661	0.660	-0.002	0.023	0.021	0.035	0.001	0.045	0.014	1.000	0.968	0.968
$\beta_{4,2}$	-0.644	-0.632	0.012	0.061	0.051	0.074	0.007	0.102	0.037	1.000	0.928	0.936
$\beta_{5,2}$	-0.744	-0.732	0.012	0.054	0.041	0.059	0.005	0.086	0.029	1.015	0.896	0.896
$\beta_{6,2}$	0.711	0.700	-0.010	0.029	0.024	0.036	0.001	0.050	0.017	1.027	0.912	0.912
$\beta_{7,2}$	0.735	0.725	-0.010	0.049	0.038	0.071	0.005	0.091	0.024	1.016	0.976	0.976
$\beta_{8,2}$	0.308	0.307	-0.001	0.037	0.029	0.045	0.002	0.061	0.020	1.000	0.920	0.920
$\beta_{9,2}$	0.562	0.556	-0.006	0.028	0.021	0.037	0.001	0.049	0.014	1.000	0.976	0.976

Table 7.4: Scenario (F1a): β estimates (part 1).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,3}$	-0.632	-0.616	0.016	0.059	0.042	0.072	0.004	0.098	0.028	1.014	0.944	0.944
$\beta_{2,3}$	0.769	0.752	-0.017	0.052	0.042	0.071	0.007	0.093	0.028	1.031	0.936	0.928
$\beta_{3,3}$	0.794	0.791	-0.003	0.030	0.025	0.038	0.001	0.052	0.017	1.000	0.936	0.936
$\beta_{4,3}$	-0.250	-0.246	0.004	0.051	0.040	0.072	0.004	0.093	0.026	1.000	0.960	0.960
$\beta_{5,3}$	-0.622	-0.614	0.008	0.053	0.036	0.061	0.004	0.085	0.024	1.000	0.960	0.968
$\beta_{6,3}$	-0.693	-0.685	0.008	0.030	0.023	0.040	0.001	0.053	0.015	1.027	0.968	0.968
$\beta_{7,3}$	-0.435	-0.436	-0.001	0.053	0.041	0.069	0.004	0.092	0.028	1.000	0.952	0.952
$\beta_{8,3}$	-0.470	-0.469	0.000	0.042	0.032	0.050	0.002	0.070	0.022	0.981	0.944	0.944
$\beta_{9,3}$	-0.693	-0.687	0.006	0.030	0.025	0.041	0.001	0.054	0.017	1.000	0.952	0.952
$\beta_{1,4}$	0.764	0.739	-0.026	0.058	0.040	0.070	0.004	0.096	0.027	1.061	0.952	0.952
$\beta_{2,4}$	-0.407	-0.389	0.018	0.044	0.037	0.057	0.004	0.077	0.024	1.036	0.952	0.952
$\beta_{3,4}$	-0.505	-0.499	0.005	0.030	0.020	0.035	0.001	0.048	0.013	1.029	0.968	0.968
$\beta_{4,4}$	-0.556	-0.561	-0.006	0.056	0.041	0.072	0.006	0.096	0.029	1.000	0.976	0.968
$\beta_{5,4}$	0.764	0.750	-0.014	0.055	0.042	0.060	0.005	0.086	0.029	1.015	0.904	0.904
$\beta_{6,4}$	-0.523	-0.525	-0.001	0.030	0.022	0.036	0.001	0.050	0.015	0.974	0.952	0.952
$\beta_{7,4}$	-0.592	-0.586	0.006	0.051	0.040	0.067	0.005	0.089	0.028	1.000	0.960	0.960
$\beta_{8,4}$	0.339	0.341	0.001	0.037	0.027	0.045	0.002	0.061	0.019	1.000	0.952	0.952
$\beta_{9,4}$	0.241	0.241	-0.000	0.033	0.021	0.037	0.001	0.052	0.014	1.000	0.960	0.960

Table 7.5: Scenario (F1a): β estimates (part 2).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.107	0.004	0.043	0.028	0.056	0.003	0.074	0.018	1.000	0.976	0.976
$r_{1,3}$	-0.128	-0.123	0.004	0.044	0.034	0.060	0.005	0.078	0.022	0.982	0.968	0.960
$r_{2,3}$	0.199	0.197	-0.002	0.044	0.031	0.057	0.005	0.076	0.020	1.000	0.984	0.976

Table 7.6: Scenario (F1a): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.750	-0.000	0.049	0.037	0.060	0.007	0.082	0.027	0.984	0.936	0.936
$\xi_{2,3}$	1.250	1.249	-0.001	0.069	0.050	0.083	0.013	0.114	0.037	0.988	0.944	0.952
$\xi_{5,2}$	0.750	0.750	0.000	0.044	0.032	0.053	0.004	0.072	0.022	1.000	0.944	0.952
$\xi_{5,3}$	1.250	1.252	0.002	0.057	0.042	0.069	0.007	0.095	0.030	1.000	0.960	0.960
$\xi_{8,2}$	0.750	0.751	0.001	0.036	0.027	0.044	0.002	0.060	0.019	1.000	0.960	0.960
$\xi_{8,3}$	1.250	1.255	0.005	0.045	0.032	0.056	0.004	0.076	0.021	1.000	0.944	0.944

Table 7.7: Scenario (F1a): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\|bias\|}$	$\hat{\sigma}(\|bias\|)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.509	-0.023	0.065	0.048	0.094	0.014	0.120	0.032	1.039	0.976	0.976
$\sigma^2_{\varepsilon,6}$	0.490	0.477	-0.013	0.065	0.044	0.094	0.013	0.120	0.026	1.013	0.960	0.984
$\sigma^2_{\varepsilon,9}$	0.528	0.529	0.001	0.080	0.055	0.122	0.015	0.154	0.029	0.990	0.984	0.992

Table 7.8: Scenario (F1b): σ^2_{ε} estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\|bias\|}$	$\hat{\sigma}(\|bias\|)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.480	0.025	0.079	0.055	0.101	0.012	0.135	0.037	1.032	0.960	0.960
$\alpha_{2,1}$	0.674	0.658	-0.015	0.079	0.063	0.123	0.021	0.154	0.045	1.010	0.968	0.984
$\alpha_{3,1}$	0.601	0.616	0.015	0.054	0.040	0.079	0.008	0.101	0.024	1.015	0.968	0.968
$\alpha_{4,2}$	-0.683	-0.676	0.007	0.090	0.073	0.121	0.019	0.160	0.053	0.991	0.952	0.944
$\alpha_{5,2}$	-0.603	-0.596	0.007	0.079	0.063	0.093	0.015	0.130	0.047	1.000	0.920	0.928
$\alpha_{6,2}$	0.700	0.704	0.005	0.053	0.038	0.071	0.007	0.094	0.023	1.000	0.952	0.968
$\alpha_{7,3}$	0.540	0.526	-0.015	0.074	0.059	0.111	0.018	0.142	0.039	1.011	0.976	0.984
$\alpha_{8,3}$	-0.378	-0.390	-0.012	0.061	0.043	0.078	0.013	0.104	0.034	1.000	0.984	0.984
$\alpha_{9,3}$	-0.709	-0.694	0.014	0.065	0.045	0.092	0.010	0.118	0.028	1.013	0.952	0.984

Table 7.9: Scenario (F1b): α^{Δ} estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.668	0.649	-0.019	0.052	0.039	0.067	0.005	0.090	0.026	1.032	0.952	0.952
$\beta_{2,1}$	0.554	0.552	-0.002	0.059	0.045	0.068	0.006	0.095	0.032	1.000	0.928	0.904
$\beta_{3,1}$	-0.505	-0.500	0.005	0.029	0.022	0.035	0.001	0.048	0.015	1.000	0.952	0.952
$\beta_{4,1}$	-0.560	-0.550	0.010	0.060	0.044	0.073	0.007	0.099	0.032	1.000	0.944	0.936
$\beta_{5,1}$	0.554	0.544	-0.010	0.051	0.037	0.062	0.004	0.085	0.025	1.000	0.944	0.944
$\beta_{6,1}$	-0.432	-0.431	0.002	0.029	0.019	0.037	0.001	0.049	0.012	1.000	0.968	0.968
$\beta_{7,1}$	0.676	0.665	-0.011	0.049	0.038	0.069	0.006	0.090	0.025	1.016	0.968	0.968
$\beta_{8,1}$	0.536	0.538	0.002	0.047	0.033	0.053	0.003	0.074	0.024	1.000	0.944	0.936
$\beta_{9,1}$	0.486	0.487	0.000	0.030	0.022	0.038	0.001	0.050	0.015	1.000	0.952	0.952
$\beta_{1,2}$	-0.619	-0.598	0.021	0.057	0.048	0.078	0.005	0.103	0.031	1.042	0.928	0.928
$\beta_{2,2}$	0.788	0.784	-0.004	0.062	0.047	0.079	0.009	0.107	0.032	1.000	0.936	0.936
$\beta_{3,2}$	0.661	0.660	-0.001	0.031	0.025	0.041	0.002	0.054	0.017	1.000	0.960	0.960
$\beta_{4,2}$	-0.644	-0.632	0.012	0.066	0.056	0.086	0.009	0.116	0.040	1.000	0.952	0.952
$\beta_{5,2}$	-0.744	-0.724	0.020	0.060	0.046	0.069	0.007	0.097	0.033	1.041	0.912	0.912
$\beta_{6,2}$	0.711	0.699	-0.012	0.036	0.027	0.042	0.002	0.059	0.018	1.047	0.936	0.936
$\beta_{7,2}$	0.735	0.727	-0.009	0.059	0.048	0.083	0.007	0.108	0.031	1.013	0.952	0.952
$\beta_{8,2}$	0.308	0.306	-0.002	0.041	0.034	0.053	0.003	0.071	0.024	1.000	0.928	0.928
$\beta_{9,2}$	0.562	0.553	-0.009	0.030	0.023	0.043	0.002	0.056	0.015	1.027	0.984	0.984

Table 7.10: Scenario (F1b): β estimates (part 1).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,3}$	-0.632	-0.606	0.026	0.076	0.053	0.084	0.005	0.119	0.037	1.045	0.944	0.936
$\beta_{2,3}$	0.769	0.746	-0.024	0.059	0.047	0.082	0.008	0.107	0.030	1.056	0.952	0.960
$\beta_{3,3}$	0.794	0.785	-0.009	0.035	0.028	0.045	0.002	0.060	0.020	1.023	0.920	0.920
$\beta_{4,3}$	-0.250	-0.252	-0.001	0.066	0.047	0.084	0.007	0.112	0.033	1.000	0.976	0.984
$\beta_{5,3}$	-0.622	-0.611	0.011	0.057	0.040	0.072	0.005	0.097	0.026	1.014	0.960	0.968
$\beta_{6,3}$	-0.693	-0.686	0.007	0.035	0.026	0.047	0.001	0.061	0.017	1.000	0.968	0.968
$\beta_{7,3}$	-0.435	-0.441	-0.006	0.066	0.048	0.081	0.005	0.110	0.033	1.000	0.968	0.968
$\beta_{8,3}$	-0.470	-0.466	0.003	0.051	0.036	0.059	0.003	0.082	0.024	1.000	0.944	0.936
$\beta_{9,3}$	-0.693	-0.682	0.011	0.039	0.033	0.048	0.002	0.066	0.024	1.020	0.944	0.944
$\beta_{1,4}$	0.764	0.730	-0.035	0.075	0.051	0.081	0.006	0.117	0.034	1.084	0.936	0.936
$\beta_{2,4}$	-0.407	-0.384	0.023	0.054	0.045	0.066	0.005	0.091	0.032	1.045	0.928	0.936
$\beta_{3,4}$	-0.505	-0.497	0.008	0.035	0.024	0.041	0.001	0.056	0.017	1.024	0.960	0.960
$\beta_{4,4}$	-0.556	-0.560	-0.004	0.063	0.048	0.084	0.008	0.111	0.034	1.000	0.968	0.960
$\beta_{5,4}$	0.764	0.747	-0.017	0.059	0.049	0.070	0.006	0.098	0.035	1.027	0.896	0.896
$\beta_{6,4}$	-0.523	-0.525	-0.002	0.038	0.028	0.042	0.001	0.060	0.020	1.000	0.928	0.928
$\beta_{7,4}$	-0.592	-0.579	0.013	0.058	0.046	0.079	0.006	0.104	0.031	1.014	0.976	0.968
$\beta_{8,4}$	0.339	0.339	-0.001	0.040	0.032	0.053	0.002	0.070	0.021	1.000	0.944	0.944
$\beta_{9,4}$	0.241	0.236	-0.005	0.032	0.024	0.043	0.002	0.057	0.016	1.000	0.952	0.952

Table 7.11: Scenario (F1b): β estimates (part 2).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.104	0.007	0.049	0.035	0.067	0.005	0.087	0.023	1.000	0.976	0.976
$r_{1,3}$	-0.128	-0.123	0.005	0.046	0.037	0.071	0.007	0.089	0.025	1.000	0.984	0.984
$r_{2,3}$	0.199	0.197	-0.002	0.055	0.039	0.067	0.006	0.091	0.027	1.000	0.952	0.952

Table 7.12: Scenario (F1b): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.744	-0.006	0.055	0.037	0.070	0.008	0.093	0.025	1.000	0.968	0.960
$\xi_{2,3}$	1.250	1.246	-0.004	0.071	0.056	0.096	0.015	0.126	0.039	0.989	0.968	0.944
$\xi_{5,2}$	0.750	0.748	-0.002	0.055	0.038	0.062	0.006	0.087	0.027	1.000	0.944	0.944
$\xi_{5,3}$	1.250	1.245	-0.005	0.070	0.049	0.081	0.010	0.113	0.034	1.000	0.952	0.944
$\xi_{8,2}$	0.750	0.757	0.007	0.042	0.034	0.052	0.004	0.071	0.024	1.000	0.952	0.936
$\xi_{8,3}$	1.250	1.262	0.012	0.056	0.039	0.066	0.006	0.091	0.029	1.000	0.968	0.944

Table 7.13: Scenario (F1b): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.672	0.141	0.141	0.036	0.048	0.003	0.149	0.035	4.056	0.056	0.056
$\sigma^2_{\varepsilon,6}$	0.490	0.678	0.188	0.188	0.041	0.056	0.004	0.196	0.039	4.683	0.016	0.008
$\sigma^2_{\varepsilon,9}$	0.528	0.736	0.208	0.208	0.051	0.067	0.005	0.219	0.048	4.196	0.056	0.048

Table 7.14: Scenario (F1c): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.552	-0.047	0.079	0.063	0.101	0.012	0.135	0.048	1.122	0.976	0.968
$\alpha_{2,1}$	0.674	0.712	0.038	0.069	0.056	0.102	0.009	0.130	0.039	1.099	0.992	0.976
$\alpha_{3,1}$	0.601	0.486	-0.115	0.115	0.036	0.048	0.002	0.126	0.033	3.361	0.264	0.256
$\alpha_{4,2}$	-0.683	-0.765	-0.082	0.112	0.090	0.120	0.017	0.172	0.075	1.212	0.944	0.912
$\alpha_{5,2}$	-0.603	-0.651	-0.049	0.076	0.058	0.083	0.008	0.119	0.045	1.171	0.944	0.912
$\alpha_{6,2}$	0.700	0.595	-0.104	0.104	0.041	0.049	0.002	0.117	0.036	2.732	0.432	0.424
$\alpha_{7,3}$	0.540	0.523	-0.017	0.071	0.054	0.099	0.010	0.129	0.036	1.011	0.960	0.968
$\alpha_{8,3}$	-0.378	-0.458	-0.080	0.086	0.059	0.075	0.008	0.119	0.047	1.552	0.896	0.832
$\alpha_{9,3}$	-0.709	-0.567	0.141	0.141	0.047	0.060	0.004	0.155	0.043	3.170	0.336	0.328

Table 7.15: Scenario (F1c): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.668	0.662	-0.006	0.044	0.034	0.061	0.005	0.080	0.022	1.000	0.984	0.984
$\beta_{2,1}$	0.554	0.561	0.006	0.048	0.040	0.058	0.003	0.081	0.029	1.000	0.920	0.928
$\beta_{3,1}$	-0.505	-0.502	0.003	0.026	0.020	0.031	0.001	0.042	0.014	1.000	0.928	0.936
$\beta_{4,1}$	-0.560	-0.574	-0.014	0.052	0.047	0.067	0.007	0.090	0.035	1.015	0.960	0.960
$\beta_{5,1}$	0.554	0.552	-0.002	0.046	0.031	0.054	0.003	0.074	0.022	1.000	0.952	0.960
$\beta_{6,1}$	-0.432	-0.431	0.001	0.028	0.019	0.033	0.001	0.045	0.013	1.000	0.944	0.944
$\beta_{7,1}$	0.676	0.648	-0.027	0.049	0.034	0.058	0.004	0.080	0.022	1.135	0.976	0.976
$\beta_{8,1}$	0.536	0.546	0.010	0.038	0.026	0.047	0.002	0.063	0.018	1.022	0.960	0.952
$\beta_{9,1}$	0.486	0.488	0.002	0.026	0.021	0.033	0.001	0.044	0.015	1.000	0.912	0.912
$\beta_{1,2}$	-0.619	-0.578	0.041	0.057	0.043	0.068	0.004	0.095	0.027	1.224	0.936	0.944
$\beta_{2,2}$	0.788	0.763	-0.025	0.058	0.039	0.066	0.004	0.092	0.026	1.061	0.928	0.920
$\beta_{3,2}$	0.661	0.649	-0.013	0.029	0.024	0.036	0.001	0.049	0.017	1.057	0.936	0.936
$\beta_{4,2}$	-0.644	-0.633	0.011	0.065	0.056	0.077	0.008	0.108	0.041	1.000	0.920	0.912
$\beta_{5,2}$	-0.744	-0.684	0.060	0.072	0.048	0.059	0.004	0.099	0.036	1.381	0.760	0.776
$\beta_{6,2}$	0.711	0.625	-0.085	0.086	0.035	0.038	0.001	0.096	0.031	2.514	0.352	0.344
$\beta_{7,2}$	0.735	0.650	-0.085	0.092	0.057	0.068	0.004	0.120	0.043	1.612	0.704	0.720
$\beta_{8,2}$	0.308	0.276	-0.032	0.047	0.034	0.046	0.002	0.070	0.025	1.204	0.856	0.856
$\beta_{9,2}$	0.562	0.509	-0.053	0.055	0.033	0.038	0.001	0.070	0.026	1.778	0.744	0.744

Table 7.16: Scenario (F1c): β estimates (part 1).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\|bias\|}$	$\hat{\sigma}(\|bias\|)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,3}$	-0.632	-0.587	0.045	0.070	0.048	0.073	0.005	0.107	0.033	1.181	0.904	0.912
$\beta_{2,3}$	0.769	0.718	-0.051	0.065	0.051	0.069	0.004	0.102	0.035	1.262	0.888	0.888
$\beta_{3,3}$	0.794	0.774	-0.020	0.036	0.028	0.039	0.001	0.056	0.021	1.098	0.888	0.888
$\beta_{4,3}$	-0.250	-0.254	-0.004	0.056	0.040	0.075	0.005	0.098	0.028	1.000	0.968	0.960
$\beta_{5,3}$	-0.622	-0.610	0.012	0.055	0.038	0.062	0.003	0.088	0.026	1.015	0.960	0.960
$\beta_{6,3}$	-0.693	-0.627	0.066	0.068	0.037	0.042	0.001	0.082	0.030	1.878	0.664	0.664
$\beta_{7,3}$	-0.435	-0.379	0.056	0.073	0.050	0.067	0.004	0.106	0.036	1.290	0.824	0.824
$\beta_{8,3}$	-0.470	-0.432	0.038	0.055	0.037	0.052	0.002	0.080	0.027	1.200	0.880	0.880
$\beta_{9,3}$	-0.693	-0.629	0.064	0.066	0.038	0.042	0.001	0.081	0.032	1.767	0.696	0.688
$\beta_{1,4}$	0.764	0.712	-0.053	0.073	0.046	0.071	0.005	0.107	0.032	1.246	0.904	0.904
$\beta_{2,4}$	-0.407	-0.353	0.055	0.065	0.042	0.057	0.003	0.090	0.032	1.400	0.824	0.832
$\beta_{3,4}$	-0.505	-0.476	0.029	0.039	0.026	0.036	0.001	0.056	0.019	1.270	0.872	0.872
$\beta_{4,4}$	-0.556	-0.598	-0.043	0.066	0.054	0.078	0.007	0.108	0.040	1.149	0.936	0.912
$\beta_{5,4}$	0.764	0.723	-0.041	0.063	0.047	0.060	0.004	0.092	0.035	1.164	0.832	0.840
$\beta_{6,4}$	-0.523	-0.529	-0.006	0.031	0.023	0.038	0.001	0.051	0.015	1.000	0.944	0.944
$\beta_{7,4}$	-0.592	-0.571	0.021	0.051	0.038	0.066	0.004	0.088	0.026	1.050	0.928	0.944
$\beta_{8,4}$	0.339	0.334	-0.005	0.039	0.028	0.046	0.002	0.063	0.019	1.021	0.944	0.944
$\beta_{9,4}$	0.241	0.227	-0.014	0.035	0.025	0.038	0.001	0.055	0.018	1.049	0.904	0.904

Table 7.17: Scenario (F1c): β estimates (part 2).

	ψ	$\bar{\bar{\check{\psi}}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.091	0.019	0.053	0.033	0.068	0.003	0.090	0.021	1.051	0.992	0.992
$r_{1,3}$	-0.128	-0.130	-0.002	0.057	0.042	0.075	0.005	0.099	0.028	1.000	0.984	0.984
$r_{2,3}$	0.199	0.211	0.012	0.054	0.038	0.069	0.004	0.093	0.026	1.015	0.976	0.976

Table 7.18: Scenario (F1c): R estimates.

	ψ	$\bar{\bar{\check{\psi}}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.756	0.006	0.045	0.035	0.060	0.004	0.079	0.024	1.000	0.960	0.952
$\xi_{2,3}$	1.250	1.259	0.009	0.059	0.045	0.081	0.007	0.105	0.031	1.000	0.968	0.960
$\xi_{5,2}$	0.750	0.751	0.001	0.044	0.031	0.054	0.004	0.073	0.021	1.000	0.960	0.968
$\xi_{5,3}$	1.250	1.254	0.004	0.058	0.043	0.071	0.006	0.097	0.031	1.000	0.960	0.960
$\xi_{8,2}$	0.750	0.765	0.015	0.040	0.030	0.046	0.003	0.065	0.021	1.042	0.944	0.936
$\xi_{8,3}$	1.250	1.278	0.028	0.053	0.038	0.060	0.005	0.085	0.028	1.102	0.936	0.928

Table 7.19: Scenario (F1c): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.500	-0.032	0.070	0.055	0.100	0.014	0.130	0.036	1.060	0.944	0.944
$\sigma^2_{\varepsilon,6}$	0.490	0.497	0.007	0.070	0.044	0.095	0.012	0.124	0.026	1.000	0.968	0.968
$\sigma^2_{\varepsilon,9}$	0.528	0.515	-0.013	0.079	0.053	0.139	0.016	0.167	0.027	1.011	1.000	1.000

Table 7.20: Scenario (F2a): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.498	0.008	0.081	0.062	0.112	0.017	0.145	0.044	1.000	0.968	0.952
$\alpha_{2,1}$	0.674	0.645	-0.029	0.088	0.060	0.124	0.020	0.160	0.039	1.029	0.944	0.960
$\alpha_{3,1}$	0.601	0.628	0.027	0.057	0.044	0.082	0.009	0.106	0.028	1.075	0.952	0.960
$\alpha_{4,2}$	-0.683	-0.670	0.013	0.090	0.073	0.123	0.017	0.162	0.050	1.009	0.952	0.944
$\alpha_{5,2}$	-0.603	-0.636	-0.034	0.082	0.070	0.104	0.022	0.140	0.058	1.049	0.960	0.944
$\alpha_{6,2}$	0.700	0.696	-0.004	0.051	0.036	0.073	0.007	0.094	0.023	1.000	0.968	0.984
$\alpha_{7,3}$	0.540	0.507	-0.033	0.081	0.064	0.121	0.016	0.155	0.040	1.051	0.952	0.968
$\alpha_{8,3}$	-0.378	-0.385	-0.007	0.066	0.053	0.092	0.016	0.119	0.039	1.000	0.960	0.952
$\alpha_{9,3}$	-0.709	-0.711	-0.002	0.067	0.050	0.102	0.013	0.128	0.034	0.988	0.976	0.984

Table 7.21: Scenario (F2a): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.709	0.708	-0.002	0.058	0.045	0.074	0.007	0.100	0.032	0.986	0.936	0.928
$\beta_{2,1}$	0.486	0.471	-0.015	0.058	0.047	0.069	0.005	0.096	0.033	1.014	0.896	0.896
$\beta_{3,1}$	-0.523	-0.525	-0.001	0.028	0.022	0.035	0.001	0.048	0.015	1.000	0.936	0.936
$\beta_{4,1}$	-0.254	-0.259	-0.005	0.051	0.040	0.071	0.005	0.093	0.026	1.000	0.984	0.984
$\beta_{5,1}$	0.507	0.518	0.012	0.046	0.038	0.062	0.005	0.082	0.027	1.017	0.944	0.936
$\beta_{6,1}$	-0.619	-0.609	0.009	0.029	0.021	0.037	0.001	0.049	0.014	1.059	0.952	0.952
$\beta_{7,1}$	0.672	0.660	-0.011	0.056	0.041	0.080	0.005	0.103	0.027	1.014	0.968	0.968
$\beta_{8,1}$	0.789	0.782	-0.007	0.059	0.044	0.068	0.004	0.095	0.031	1.000	0.944	0.944
$\beta_{9,1}$	0.679	0.675	-0.004	0.029	0.022	0.038	0.001	0.050	0.015	1.000	0.920	0.928
$\beta_{1,2}$	-0.217	-0.394	-0.177	0.179	0.079	0.152	0.002	0.241	0.057	2.361	0.968	0.960
$\beta_{2,2}$	0.601	0.553	-0.048	0.077	0.054	0.149	0.003	0.174	0.027	1.160	1.000	1.000
$\beta_{3,2}$	0.736	0.757	0.021	0.056	0.043	0.130	0.001	0.146	0.022	1.044	1.000	1.000
$\beta_{4,2}$	-0.724	-0.760	-0.036	0.079	0.052	0.161	0.006	0.185	0.027	1.080	1.000	1.000
$\beta_{5,2}$	-0.298	-0.524	-0.227	0.227	0.069	0.145	0.003	0.272	0.058	3.386	0.840	0.840
$\beta_{6,2}$	0.428	0.272	-0.156	0.157	0.065	0.131	0.001	0.209	0.049	2.500	0.968	0.968
$\beta_{7,2}$	0.787	0.695	-0.092	0.104	0.071	0.159	0.004	0.199	0.041	1.465	1.000	1.000
$\beta_{8,2}$	0.250	0.132	-0.118	0.119	0.065	0.143	0.001	0.193	0.040	2.015	0.984	0.992
$\beta_{9,2}$	0.768	0.541	-0.227	0.227	0.065	0.133	0.001	0.265	0.054	3.631	0.656	0.648

Table 7.22: Scenario (F2a): β estimates (part 1).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,3}$	-0.690	-0.229	0.461	0.461	0.084	0.332	0.002	0.571	0.067	5.583	0.992	0.992
$\beta_{2,3}$	0.401	0.490	0.089	0.113	0.075	0.330	0.002	0.356	0.028	1.320	1.000	1.000
$\beta_{3,3}$	0.738	0.670	-0.067	0.114	0.098	0.309	0.002	0.340	0.046	1.119	1.000	1.000
$\beta_{4,3}$	-0.527	-0.386	0.142	0.144	0.084	0.334	0.002	0.372	0.035	1.898	1.000	1.000
$\beta_{5,3}$	-0.525	0.048	0.573	0.573	0.108	0.325	0.002	0.661	0.093	5.398	0.720	0.728
$\beta_{6,3}$	-0.569	-0.222	0.347	0.348	0.138	0.310	0.002	0.476	0.099	2.641	0.976	0.976
$\beta_{7,3}$	-0.625	-0.454	0.170	0.172	0.084	0.335	0.002	0.384	0.039	2.170	1.000	1.000
$\beta_{8,3}$	-0.501	-0.206	0.295	0.295	0.099	0.327	0.002	0.447	0.064	3.141	1.000	1.000
$\beta_{9,3}$	-0.239	0.319	0.558	0.558	0.138	0.313	0.002	0.643	0.119	4.167	0.688	0.688
$\beta_{1,4}$	0.798	0.587	-0.212	0.212	0.084	0.171	0.003	0.278	0.063	2.682	0.912	0.912
$\beta_{2,4}$	-0.784	-0.796	-0.012	0.063	0.044	0.171	0.005	0.186	0.020	1.000	1.000	1.000
$\beta_{3,4}$	-0.609	-0.573	0.036	0.065	0.045	0.148	0.001	0.167	0.022	1.113	1.000	1.000
$\beta_{4,4}$	-0.318	-0.389	-0.071	0.089	0.069	0.172	0.004	0.202	0.040	1.284	1.000	1.000
$\beta_{5,4}$	0.427	0.151	-0.276	0.276	0.066	0.160	0.002	0.321	0.056	4.303	0.688	0.688
$\beta_{6,4}$	-0.746	-0.895	-0.149	0.151	0.073	0.149	0.001	0.219	0.051	2.211	0.960	0.960
$\beta_{7,4}$	-0.695	-0.754	-0.059	0.074	0.056	0.177	0.003	0.198	0.029	1.292	1.000	1.000
$\beta_{8,4}$	0.582	0.443	-0.139	0.142	0.069	0.163	0.002	0.222	0.043	2.135	0.992	0.992
$\beta_{9,4}$	0.623	0.357	-0.266	0.266	0.074	0.151	0.001	0.308	0.063	3.730	0.632	0.632

Table 7.23: Scenario (F2a): β estimates (part 2).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,5}$	0.223	0.587	0.364	0.364	0.085	0.269	0.003	0.456	0.067	4.400	1.000	1.000
$\beta_{2,5}$	-0.569	-0.487	0.081	0.101	0.067	0.264	0.002	0.289	0.027	1.344	1.000	1.000
$\beta_{3,5}$	-0.256	-0.308	-0.052	0.094	0.078	0.246	0.002	0.272	0.038	1.109	1.000	1.000
$\beta_{4,5}$	-0.564	-0.445	0.119	0.127	0.071	0.269	0.002	0.304	0.031	1.726	1.000	1.000
$\beta_{5,5}$	-0.647	-0.189	0.457	0.457	0.090	0.259	0.002	0.528	0.079	5.178	0.760	0.752
$\beta_{6,5}$	-0.666	-0.394	0.272	0.274	0.116	0.247	0.002	0.377	0.083	2.496	0.968	0.968
$\beta_{7,5}$	0.304	0.426	0.123	0.126	0.074	0.269	0.002	0.305	0.034	1.825	1.000	1.000
$\beta_{8,5}$	-0.355	-0.122	0.233	0.233	0.085	0.261	0.002	0.356	0.056	2.918	1.000	1.000
$\beta_{9,5}$	-0.765	-0.322	0.443	0.443	0.110	0.249	0.002	0.511	0.095	4.145	0.680	0.672
$\beta_{1,6}$	0.337	0.240	-0.097	0.104	0.066	0.101	0.003	0.152	0.046	1.597	0.912	0.920
$\beta_{2,6}$	0.691	0.662	-0.029	0.061	0.046	0.103	0.005	0.125	0.026	1.070	0.984	0.984
$\beta_{3,6}$	0.781	0.794	0.012	0.039	0.032	0.077	0.001	0.090	0.017	1.042	0.992	0.992
$\beta_{4,6}$	0.257	0.202	-0.055	0.071	0.056	0.105	0.004	0.134	0.035	1.250	0.992	0.992
$\beta_{5,6}$	0.228	0.099	-0.129	0.132	0.062	0.089	0.002	0.164	0.048	2.164	0.768	0.768
$\beta_{6,6}$	0.399	0.324	-0.075	0.078	0.046	0.078	0.001	0.115	0.031	1.800	0.944	0.944
$\beta_{7,6}$	0.586	0.527	-0.058	0.078	0.057	0.110	0.004	0.142	0.035	1.247	0.976	0.976
$\beta_{8,6}$	-0.772	-0.818	-0.046	0.069	0.047	0.099	0.004	0.126	0.030	1.200	0.992	0.992
$\beta_{9,6}$	-0.351	-0.457	-0.106	0.106	0.049	0.079	0.001	0.136	0.038	2.388	0.848	0.840

Table 7.24: Scenario (F2a): β estimates (part 3).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,7}$	0.376	0.245	-0.132	0.133	0.067	0.122	0.003	0.186	0.047	2.129	0.944	0.944
$\beta_{2,7}$	0.636	0.588	-0.048	0.073	0.055	0.121	0.004	0.149	0.031	1.167	0.984	0.992
$\beta_{3,7}$	0.728	0.740	0.013	0.043	0.030	0.101	0.001	0.113	0.014	1.020	1.000	1.000
$\beta_{4,7}$	-0.739	-0.768	-0.029	0.067	0.053	0.135	0.006	0.157	0.030	1.062	0.992	0.992
$\beta_{5,7}$	0.716	0.558	-0.158	0.161	0.072	0.116	0.004	0.203	0.056	2.256	0.792	0.792
$\beta_{6,7}$	0.227	0.119	-0.108	0.110	0.054	0.102	0.001	0.154	0.039	2.140	0.952	0.952
$\beta_{7,7}$	-0.677	-0.696	-0.019	0.060	0.047	0.133	0.004	0.151	0.025	1.027	1.000	1.000
$\beta_{8,7}$	-0.727	-0.821	-0.093	0.100	0.066	0.120	0.004	0.164	0.044	1.600	0.984	0.984
$\beta_{9,7}$	0.617	0.445	-0.172	0.172	0.055	0.103	0.001	0.202	0.046	3.273	0.704	0.704
$\beta_{1,8}$	0.573	0.510	-0.063	0.074	0.050	0.083	0.005	0.117	0.033	1.413	0.920	0.920
$\beta_{2,8}$	-0.370	-0.377	-0.007	0.052	0.038	0.074	0.004	0.095	0.025	1.016	0.968	0.968
$\beta_{3,8}$	-0.306	-0.295	0.010	0.034	0.026	0.053	0.001	0.066	0.017	1.024	0.968	0.968
$\beta_{4,8}$	-0.371	-0.373	-0.002	0.052	0.040	0.085	0.005	0.105	0.023	1.000	1.000	1.000
$\beta_{5,8}$	0.237	0.167	-0.069	0.073	0.046	0.066	0.003	0.103	0.033	1.686	0.864	0.872
$\beta_{6,8}$	-0.682	-0.724	-0.042	0.049	0.033	0.054	0.001	0.077	0.023	1.405	0.912	0.920
$\beta_{7,8}$	0.705	0.673	-0.032	0.073	0.052	0.093	0.006	0.124	0.036	1.060	0.952	0.952
$\beta_{8,8}$	-0.745	-0.773	-0.029	0.056	0.040	0.077	0.004	0.100	0.027	1.095	0.992	0.976
$\beta_{9,8}$	-0.550	-0.615	-0.065	0.067	0.036	0.055	0.001	0.090	0.027	1.900	0.840	0.848

Table 7.25: Scenario (F2a): β estimates (part 4).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.106	0.004	0.046	0.036	0.067	0.005	0.086	0.023	1.000	0.984	0.976
$r_{1,3}$	-0.128	-0.119	0.009	0.057	0.044	0.073	0.009	0.098	0.031	1.000	0.960	0.960
$r_{2,3}$	0.199	0.192	-0.008	0.054	0.037	0.070	0.008	0.093	0.026	1.000	0.968	0.952

Table 7.26: Scenario (F2a): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.735	-0.015	0.049	0.042	0.072	0.008	0.093	0.028	1.016	0.952	0.960
$\xi_{2,3}$	1.250	1.230	-0.020	0.073	0.054	0.098	0.013	0.129	0.036	1.022	0.952	0.952
$\xi_{5,2}$	0.750	0.764	0.014	0.050	0.040	0.063	0.008	0.085	0.030	1.016	0.944	0.960
$\xi_{5,3}$	1.250	1.277	0.027	0.069	0.058	0.085	0.015	0.116	0.046	1.047	0.952	0.944
$\xi_{8,2}$	0.750	0.744	-0.006	0.057	0.044	0.066	0.005	0.093	0.030	1.000	0.920	0.920
$\xi_{8,3}$	1.250	1.246	-0.004	0.070	0.053	0.084	0.007	0.116	0.036	1.000	0.936	0.936

Table 7.27: Scenario (F2a): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.680	0.149	0.149	0.038	0.051	0.003	0.158	0.036	4.026	0.120	0.112
$\sigma^2_{\varepsilon,6}$	0.490	0.512	0.022	0.055	0.040	0.076	0.007	0.099	0.024	1.062	0.960	0.968
$\sigma^2_{\varepsilon,9}$	0.528	0.719	0.191	0.191	0.060	0.078	0.006	0.207	0.055	3.333	0.288	0.304

Table 7.28: Scenario (F2b): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.572	-0.066	0.098	0.079	0.111	0.015	0.156	0.062	1.168	0.960	0.928
$\alpha_{2,1}$	0.674	0.760	0.086	0.107	0.075	0.117	0.012	0.166	0.057	1.323	0.976	0.912
$\alpha_{3,1}$	0.601	0.500	-0.101	0.101	0.035	0.050	0.002	0.114	0.031	3.057	0.424	0.416
$\alpha_{4,2}$	-0.683	-0.667	0.016	0.081	0.062	0.104	0.013	0.139	0.043	1.010	0.920	0.944
$\alpha_{5,2}$	-0.603	-0.631	-0.029	0.065	0.045	0.085	0.012	0.111	0.035	1.068	0.992	0.984
$\alpha_{6,2}$	0.700	0.687	-0.013	0.042	0.033	0.060	0.004	0.077	0.022	1.019	0.960	0.968
$\alpha_{7,3}$	0.540	0.549	0.009	0.077	0.061	0.113	0.009	0.145	0.039	1.000	0.976	0.976
$\alpha_{8,3}$	-0.378	-0.442	-0.064	0.085	0.060	0.088	0.011	0.129	0.047	1.268	0.936	0.896
$\alpha_{9,3}$	-0.709	-0.577	0.132	0.132	0.054	0.069	0.006	0.150	0.048	2.630	0.544	0.544

Table 7.29: Scenario (F2b): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.709	0.723	0.014	0.052	0.040	0.067	0.007	0.090	0.028	1.015	0.944	0.944
$\beta_{2,1}$	0.486	0.494	0.008	0.054	0.043	0.063	0.004	0.088	0.029	1.015	0.936	0.928
$\beta_{3,1}$	-0.523	-0.522	0.001	0.026	0.018	0.031	0.001	0.043	0.012	1.000	0.952	0.952
$\beta_{4,1}$	-0.254	-0.248	0.006	0.043	0.031	0.060	0.003	0.077	0.020	1.000	0.968	0.960
$\beta_{5,1}$	0.507	0.516	0.010	0.042	0.030	0.052	0.003	0.070	0.020	1.020	0.960	0.960
$\beta_{6,1}$	-0.619	-0.612	0.006	0.024	0.019	0.031	0.001	0.042	0.012	1.000	0.944	0.952
$\beta_{7,1}$	0.672	0.666	-0.006	0.045	0.036	0.070	0.005	0.088	0.022	1.000	0.976	0.976
$\beta_{8,1}$	0.789	0.797	0.008	0.052	0.038	0.060	0.004	0.084	0.027	1.016	0.936	0.936
$\beta_{9,1}$	0.679	0.676	-0.003	0.026	0.019	0.033	0.001	0.044	0.013	1.000	0.944	0.944
$\beta_{1,2}$	-0.217	-0.329	-0.112	0.118	0.070	0.092	0.004	0.156	0.054	1.756	0.800	0.800
$\beta_{2,2}$	0.601	0.497	-0.104	0.109	0.064	0.088	0.004	0.147	0.047	1.775	0.808	0.816
$\beta_{3,2}$	0.736	0.652	-0.085	0.087	0.050	0.058	0.003	0.109	0.041	1.836	0.728	0.736
$\beta_{4,2}$	-0.724	-0.694	0.030	0.090	0.065	0.102	0.006	0.144	0.044	1.037	0.936	0.936
$\beta_{5,2}$	-0.298	-0.419	-0.121	0.124	0.060	0.075	0.003	0.149	0.050	2.060	0.664	0.656
$\beta_{6,2}$	0.428	0.147	-0.281	0.281	0.038	0.052	0.002	0.286	0.037	7.447	0.000	0.000
$\beta_{7,2}$	0.787	0.476	-0.311	0.311	0.088	0.100	0.006	0.328	0.081	3.670	0.088	0.104
$\beta_{8,2}$	0.250	0.060	-0.190	0.190	0.058	0.078	0.003	0.207	0.052	3.431	0.288	0.296
$\beta_{9,2}$	0.768	0.474	-0.294	0.294	0.050	0.059	0.003	0.300	0.049	5.960	0.000	0.000

Table 7.30: Scenario (F2b): β estimates (part 1).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,3}$	-0.690	-0.346	0.344	0.344	0.131	0.188	0.008	0.398	0.108	2.802	0.560	0.568
$\beta_{2,3}$	0.401	0.593	0.193	0.197	0.112	0.176	0.008	0.275	0.083	1.899	0.928	0.928
$\beta_{3,3}$	0.738	0.872	0.134	0.146	0.090	0.126	0.007	0.201	0.069	1.607	0.856	0.856
$\beta_{4,3}$	-0.527	-0.473	0.054	0.122	0.101	0.194	0.008	0.242	0.062	1.060	0.984	0.984
$\beta_{5,3}$	-0.525	-0.181	0.344	0.344	0.107	0.153	0.007	0.380	0.094	3.364	0.352	0.336
$\beta_{6,3}$	-0.569	0.049	0.618	0.618	0.074	0.111	0.006	0.628	0.073	8.405	0.000	0.000
$\beta_{7,3}$	-0.625	0.062	0.687	0.687	0.124	0.195	0.009	0.715	0.119	5.629	0.000	0.000
$\beta_{8,3}$	-0.501	-0.028	0.474	0.474	0.114	0.164	0.008	0.503	0.107	4.272	0.136	0.128
$\beta_{9,3}$	-0.239	0.423	0.662	0.662	0.104	0.126	0.007	0.674	0.103	6.442	0.000	0.000
$\beta_{1,4}$	0.798	0.632	-0.166	0.168	0.094	0.111	0.006	0.210	0.075	1.979	0.672	0.688
$\beta_{2,4}$	-0.784	-0.846	-0.062	0.090	0.070	0.108	0.006	0.149	0.050	1.200	0.960	0.960
$\beta_{3,4}$	-0.609	-0.643	-0.034	0.054	0.044	0.067	0.003	0.092	0.032	1.129	0.928	0.928
$\beta_{4,4}$	-0.318	-0.355	-0.037	0.084	0.063	0.112	0.005	0.147	0.043	1.071	0.976	0.976
$\beta_{5,4}$	0.427	0.251	-0.176	0.176	0.065	0.086	0.004	0.199	0.055	2.892	0.456	0.456
$\beta_{6,4}$	-0.746	-1.016	-0.271	0.271	0.048	0.062	0.003	0.278	0.046	5.729	0.000	0.000
$\beta_{7,4}$	-0.695	-0.993	-0.298	0.298	0.089	0.124	0.007	0.325	0.081	3.494	0.248	0.224
$\beta_{8,4}$	0.582	0.369	-0.213	0.213	0.067	0.093	0.004	0.234	0.060	3.328	0.288	0.288
$\beta_{9,4}$	0.623	0.328	-0.295	0.295	0.061	0.068	0.003	0.303	0.060	4.934	0.000	0.000

Table 7.31: Scenario (F2b): β estimates (part 2).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,5}$	0.223	0.513	0.290	0.291	0.115	0.157	0.007	0.336	0.097	2.698	0.584	0.584
$\beta_{2,5}$	-0.569	-0.448	0.120	0.137	0.086	0.145	0.007	0.210	0.058	1.486	0.944	0.944
$\beta_{3,5}$	-0.256	-0.156	0.100	0.113	0.072	0.102	0.005	0.159	0.053	1.506	0.880	0.880
$\beta_{4,5}$	-0.564	-0.515	0.049	0.108	0.083	0.160	0.007	0.204	0.051	1.071	0.968	0.968
$\beta_{5,5}$	-0.647	-0.372	0.275	0.275	0.090	0.126	0.006	0.306	0.078	3.211	0.368	0.360
$\beta_{6,5}$	-0.666	-0.179	0.487	0.487	0.068	0.091	0.005	0.495	0.067	7.235	0.000	0.000
$\beta_{7,5}$	0.304	0.839	0.535	0.535	0.112	0.166	0.008	0.561	0.108	4.884	0.008	0.008
$\beta_{8,5}$	-0.355	0.020	0.375	0.375	0.094	0.134	0.006	0.400	0.088	4.117	0.096	0.096
$\beta_{9,5}$	-0.765	-0.241	0.524	0.524	0.089	0.102	0.005	0.534	0.087	5.966	0.000	0.000
$\beta_{1,6}$	0.337	0.257	-0.079	0.088	0.057	0.075	0.004	0.122	0.041	1.544	0.816	0.832
$\beta_{2,6}$	0.691	0.684	-0.007	0.051	0.039	0.080	0.006	0.100	0.025	1.000	0.984	0.976
$\beta_{3,6}$	0.781	0.765	-0.016	0.038	0.027	0.044	0.001	0.062	0.018	1.068	0.912	0.920
$\beta_{4,6}$	0.257	0.219	-0.038	0.061	0.047	0.078	0.004	0.105	0.032	1.149	0.960	0.960
$\beta_{5,6}$	0.228	0.145	-0.082	0.086	0.050	0.059	0.002	0.109	0.040	1.786	0.728	0.736
$\beta_{6,6}$	0.399	0.267	-0.133	0.133	0.036	0.043	0.001	0.140	0.034	3.806	0.088	0.088
$\beta_{7,6}$	0.586	0.428	-0.157	0.158	0.067	0.083	0.005	0.182	0.057	2.478	0.552	0.560
$\beta_{8,6}$	-0.772	-0.878	-0.107	0.110	0.062	0.073	0.004	0.137	0.052	1.853	0.752	0.712
$\beta_{9,6}$	-0.351	-0.472	-0.121	0.121	0.041	0.046	0.001	0.130	0.038	3.122	0.232	0.232

Table 7.32: Scenario (F2b): β estimates (part 3).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,7}$	0.376	0.282	-0.094	0.100	0.062	0.081	0.004	0.135	0.048	1.662	0.840	0.848
$\beta_{2,7}$	0.636	0.582	-0.053	0.070	0.049	0.081	0.005	0.113	0.032	1.269	0.912	0.912
$\beta_{3,7}$	0.728	0.677	-0.051	0.055	0.038	0.049	0.002	0.078	0.028	1.523	0.848	0.848
$\beta_{4,7}$	-0.739	-0.720	0.019	0.061	0.052	0.093	0.006	0.119	0.032	1.026	0.968	0.968
$\beta_{5,7}$	0.716	0.627	-0.089	0.093	0.052	0.069	0.004	0.121	0.040	1.814	0.760	0.776
$\beta_{6,7}$	0.227	0.038	-0.189	0.189	0.041	0.046	0.002	0.195	0.039	4.707	0.000	0.000
$\beta_{7,7}$	-0.677	-0.872	-0.194	0.195	0.078	0.098	0.006	0.221	0.069	2.658	0.504	0.488
$\beta_{8,7}$	-0.727	-0.892	-0.164	0.164	0.078	0.079	0.005	0.186	0.069	2.333	0.464	0.456
$\beta_{9,7}$	0.617	0.408	-0.209	0.209	0.046	0.050	0.002	0.215	0.044	4.652	0.008	0.008
$\beta_{1,8}$	0.573	0.530	-0.043	0.058	0.040	0.068	0.005	0.095	0.025	1.250	0.928	0.936
$\beta_{2,8}$	-0.370	-0.395	-0.025	0.052	0.039	0.059	0.004	0.083	0.028	1.083	0.936	0.912
$\beta_{3,8}$	-0.306	-0.307	-0.001	0.028	0.020	0.034	0.001	0.047	0.013	1.000	0.960	0.960
$\beta_{4,8}$	-0.371	-0.367	0.004	0.047	0.034	0.066	0.004	0.085	0.022	1.000	0.992	0.992
$\beta_{5,8}$	0.237	0.194	-0.043	0.052	0.035	0.048	0.002	0.074	0.026	1.370	0.824	0.824
$\beta_{6,8}$	-0.682	-0.760	-0.078	0.078	0.033	0.034	0.001	0.087	0.028	2.576	0.360	0.360
$\beta_{7,8}$	0.705	0.613	-0.092	0.100	0.063	0.075	0.005	0.132	0.046	1.573	0.728	0.768
$\beta_{8,8}$	-0.745	-0.811	-0.067	0.072	0.046	0.061	0.005	0.099	0.037	1.593	0.872	0.832
$\beta_{9,8}$	-0.550	-0.623	-0.073	0.073	0.035	0.036	0.001	0.083	0.029	2.314	0.448	0.448

Table 7.33: Scenario (F2b): β estimates (part 4).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.094	0.017	0.046	0.032	0.063	0.003	0.082	0.020	1.057	0.984	0.984
$r_{1,3}$	-0.128	-0.188	-0.061	0.081	0.060	0.077	0.007	0.118	0.046	1.250	0.864	0.864
$r_{2,3}$	0.199	0.198	-0.002	0.053	0.039	0.067	0.006	0.090	0.028	1.000	0.952	0.952

Table 7.34: Scenario (F2b): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.766	0.016	0.048	0.043	0.067	0.006	0.088	0.033	1.032	0.960	0.952
$\xi_{2,3}$	1.250	1.283	0.033	0.074	0.054	0.092	0.010	0.125	0.039	1.070	0.976	0.976
$\xi_{5,2}$	0.750	0.761	0.011	0.042	0.030	0.052	0.004	0.070	0.021	1.020	0.984	0.984
$\xi_{5,3}$	1.250	1.269	0.019	0.059	0.042	0.070	0.008	0.096	0.030	1.029	0.960	0.944
$\xi_{8,2}$	0.750	0.756	0.006	0.051	0.040	0.059	0.005	0.083	0.028	1.000	0.936	0.928
$\xi_{8,3}$	1.250	1.264	0.014	0.062	0.052	0.075	0.007	0.104	0.038	1.012	0.912	0.896

Table 7.35: Scenario (F2b): ξ estimates.

Before deletion for (F2a) and (F2b); Scenario (F2c)

19 parameter coverage rates lie below the lower bound and 31 lie above the upper bound. For 36 parameters the RRMSE lies above 1.5 (maximum: 5.185 for one of the $\boldsymbol{\beta}$s)and for 34 one can find an absolute bias above 0.1 (maximum: 0.550 for one of the $\boldsymbol{\beta}$s).

	$\overline{\overline{bias}}$	$\overline{\overline{\lvert bias \rvert}}$	$\overline{\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}}$	$\overline{\overline{RMSE\left(\check{\boldsymbol{\psi}}\right)}}$	$\overline{RRMSE}$	$\overline{Cover}_{CI}$	$\overline{Cover}_{Quant}$
σ^2_ε	-0.018	0.070	0.100	0.128	1.039	0.963	0.973
(range)	(-0.033, 0.003)	(0.062, 0.084)	(0.083, 0.128)	(0.110, 0.159)	(1.000, 1.096)	(0.944, 0.984)	(0.960, 0.992)
α^Δ	-0.005	0.067	0.090	0.119	1.026	0.952	0.962
(range)	(-0.030, 0.028)	(0.046, 0.086)	(0.063, 0.109)	(0.083, 0.147)	(1.000, 1.103)	(0.912, 0.984)	(0.928, 1.000)
β	0.010	0.136	0.147	0.211	1.939	0.934	0.934
(range)	(-0.262, 0.550)	(0.024, 0.550)	(0.030, 0.331)	(0.042, 0.638)	(1.000, 5.185)	(0.624, 1.000)	(0.624, 1.000)
r	0.000	0.047	0.059	0.079	1.000	0.971	0.965
(range)	(-0.009, 0.007)	(0.043, 0.050)	(0.057, 0.061)	(0.075, 0.084)	(1.000, 1.000)	(0.952, 0.984)	(0.944, 0.984)
ξ	-0.004	0.055	0.066	0.092	1.022	0.936	0.931
(range)	(-0.020, 0.020)	(0.044, 0.066)	(0.053, 0.085)	(0.072, 0.116)	(1.000, 1.041)	(0.904, 0.984)	(0.912, 0.968)

	$\overline{\overline{(\hat{T})}}_{eff}$	$\overline{\overline{\hat{\sigma}_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{\overline{RMSE_{eff}\left(\check{\boldsymbol{\psi}}_{eff}\right)}}$	$\overline{Cover}_{CI,eff}$
σ^2_ε	508.746	0.100	0.128	0.960
(range)	(395.836, 693.114)	(0.095, 0.139)	(0.124, 0.167)	(0.944, 1.000)
α^Δ	906.184	0.090	0.119	0.956
(range)	(453.603, 1557.031)	(0.073, 0.124)	(0.094, 0.162)	(0.944, 0.976)
β	17769.283	0.147	0.211	0.933
(range)	(2520.532, 34776.232)	(0.035, 0.335)	(0.048, 0.661)	(0.632, 1.000)
r	4121.833	0.059	0.080	0.968
(range)	(3510.380, 5084.858)	(0.067, 0.073)	(0.086, 0.098)	(0.960, 0.976)
ξ	2958.963	0.066	0.092	0.935
(range)	(1520.613, 5582.982)	(0.063, 0.098)	(0.086, 0.129)	(0.920, 0.952)

Table 7.36: Scenario (F2c): Aggregated estimates and effective sample-size estimates. Before deletion, 1000 observations, dedicated, 9 measurements, dichotomous measurements: 1, 4, 7, ordered measurements: 2,5,8, with 7 covariates plus intercept, 3 latent variables.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.499	-0.033	0.064	0.047	0.090	0.014	0.116	0.033	1.096	0.944	0.968
$\sigma^2_{\varepsilon,6}$	0.490	0.493	0.003	0.062	0.046	0.083	0.011	0.110	0.029	1.000	0.960	0.960
$\sigma^2_{\varepsilon,9}$	0.528	0.505	-0.023	0.084	0.051	0.128	0.014	0.159	0.027	1.021	0.984	0.992

Table 7.37: Scenario (F2c): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.496	0.010	0.072	0.056	0.095	0.012	0.126	0.039	1.000	0.936	0.928
$\alpha_{2,1}$	0.674	0.644	-0.030	0.086	0.058	0.109	0.016	0.147	0.035	1.040	0.928	0.968
$\alpha_{3,1}$	0.601	0.629	0.028	0.053	0.036	0.073	0.008	0.095	0.024	1.103	0.968	0.984
$\alpha_{4,2}$	-0.683	-0.670	0.013	0.085	0.063	0.106	0.014	0.143	0.045	1.010	0.912	0.936
$\alpha_{5,2}$	-0.603	-0.630	-0.027	0.069	0.052	0.087	0.014	0.117	0.040	1.049	0.984	0.968
$\alpha_{6,2}$	0.700	0.696	-0.003	0.046	0.037	0.063	0.006	0.083	0.025	1.000	0.952	0.952
$\alpha_{7,3}$	0.540	0.515	-0.025	0.069	0.061	0.108	0.014	0.136	0.041	1.034	0.944	0.960
$\alpha_{8,3}$	-0.378	-0.379	-0.001	0.056	0.046	0.079	0.013	0.103	0.033	1.000	0.960	0.960
$\alpha_{9,3}$	-0.709	-0.715	-0.007	0.064	0.042	0.092	0.011	0.118	0.027	1.000	0.984	1.000

Table 7.38: Scenario (F2c): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.709	0.704	-0.006	0.048	0.037	0.063	0.005	0.084	0.024	1.000	0.944	0.944
$\beta_{2,1}$	0.486	0.473	-0.013	0.052	0.042	0.059	0.004	0.084	0.030	1.015	0.904	0.896
$\beta_{3,1}$	-0.523	-0.522	0.001	0.026	0.018	0.030	0.001	0.042	0.012	1.000	0.960	0.960
$\beta_{4,1}$	-0.254	-0.251	0.003	0.043	0.032	0.060	0.003	0.078	0.021	1.000	0.968	0.968
$\beta_{5,1}$	0.507	0.517	0.010	0.043	0.030	0.052	0.003	0.071	0.021	1.019	0.968	0.952
$\beta_{6,1}$	-0.619	-0.612	0.006	0.024	0.019	0.031	0.001	0.042	0.012	1.033	0.952	0.952
$\beta_{7,1}$	0.672	0.666	-0.006	0.047	0.036	0.069	0.005	0.088	0.024	1.000	0.976	0.976
$\beta_{8,1}$	0.789	0.780	-0.009	0.050	0.037	0.057	0.003	0.081	0.025	1.016	0.912	0.920
$\beta_{9,1}$	0.679	0.675	-0.004	0.025	0.019	0.032	0.001	0.043	0.013	1.000	0.936	0.936
$\beta_{1,2}$	-0.217	-0.395	-0.179	0.179	0.075	0.145	0.002	0.236	0.056	2.519	0.936	0.944
$\beta_{2,2}$	0.601	0.556	-0.045	0.073	0.052	0.143	0.003	0.167	0.026	1.154	1.000	1.000
$\beta_{3,2}$	0.736	0.760	0.024	0.056	0.043	0.125	0.001	0.142	0.022	1.060	1.000	1.000
$\beta_{4,2}$	-0.724	-0.762	-0.038	0.069	0.050	0.153	0.004	0.174	0.024	1.118	1.000	1.000
$\beta_{5,2}$	-0.298	-0.516	-0.218	0.218	0.063	0.138	0.002	0.261	0.052	3.603	0.832	0.816
$\beta_{6,2}$	0.428	0.289	-0.139	0.141	0.063	0.126	0.001	0.194	0.045	2.333	0.952	0.952
$\beta_{7,2}$	0.787	0.699	-0.088	0.098	0.066	0.152	0.003	0.189	0.039	1.494	0.992	0.992
$\beta_{8,2}$	0.250	0.141	-0.109	0.113	0.060	0.138	0.001	0.185	0.037	1.910	1.000	1.000
$\beta_{9,2}$	0.768	0.552	-0.216	0.216	0.067	0.127	0.001	0.254	0.056	3.373	0.688	0.696

Table 7.39: Scenario (F2c): β estimates (part 1).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\|bias\|}$	$\hat{\sigma}(\|bias\|)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,3}$	-0.690	-0.236	0.454	0.454	0.093	0.328	0.002	0.563	0.074	4.978	0.984	0.976
$\beta_{2,3}$	0.401	0.491	0.091	0.117	0.079	0.324	0.002	0.352	0.030	1.306	1.000	1.000
$\beta_{3,3}$	0.738	0.672	-0.066	0.124	0.101	0.299	0.002	0.335	0.050	1.089	1.000	1.000
$\beta_{4,3}$	-0.527	-0.383	0.144	0.150	0.087	0.330	0.002	0.371	0.038	1.794	1.000	1.000
$\beta_{5,3}$	-0.525	0.025	0.550	0.550	0.108	0.318	0.002	0.638	0.094	5.185	0.760	0.768
$\beta_{6,3}$	-0.569	-0.253	0.316	0.319	0.142	0.301	0.002	0.449	0.101	2.327	0.968	0.976
$\beta_{7,3}$	-0.625	-0.453	0.171	0.173	0.095	0.331	0.002	0.382	0.046	2.010	1.000	1.000
$\beta_{8,3}$	-0.501	-0.208	0.293	0.293	0.104	0.322	0.002	0.442	0.068	2.990	1.000	1.000
$\beta_{9,3}$	-0.239	0.297	0.536	0.536	0.148	0.304	0.002	0.621	0.127	3.757	0.704	0.704
$\beta_{1,4}$	0.798	0.586	-0.212	0.213	0.074	0.165	0.002	0.273	0.057	3.000	0.904	0.904
$\beta_{2,4}$	-0.784	-0.803	-0.019	0.068	0.049	0.164	0.004	0.183	0.023	1.024	1.000	1.000
$\beta_{3,4}$	-0.609	-0.575	0.034	0.068	0.047	0.143	0.001	0.163	0.024	1.092	1.000	1.000
$\beta_{4,4}$	-0.318	-0.386	-0.068	0.082	0.061	0.166	0.003	0.192	0.032	1.342	1.000	1.000
$\beta_{5,4}$	0.427	0.165	-0.262	0.262	0.059	0.155	0.001	0.306	0.050	4.542	0.768	0.752
$\beta_{6,4}$	-0.746	-0.884	-0.139	0.142	0.073	0.144	0.001	0.209	0.050	2.000	0.968	0.968
$\beta_{7,4}$	-0.695	-0.764	-0.069	0.083	0.064	0.171	0.003	0.197	0.036	1.329	1.000	1.000
$\beta_{8,4}$	0.582	0.442	-0.140	0.142	0.068	0.158	0.001	0.218	0.043	2.211	1.000	1.000
$\beta_{9,4}$	0.623	0.370	-0.253	0.253	0.079	0.145	0.001	0.295	0.067	3.354	0.624	0.624

Table 7.40: Scenario (F2c): β estimates (part 2).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,5}$	0.223	0.579	0.356	0.356	0.086	0.264	0.002	0.446	0.068	4.256	0.976	0.968
$\beta_{2,5}$	-0.569	-0.489	0.079	0.104	0.070	0.259	0.002	0.287	0.029	1.289	1.000	1.000
$\beta_{3,5}$	-0.256	-0.310	-0.054	0.100	0.081	0.238	0.002	0.267	0.040	1.094	1.000	1.000
$\beta_{4,5}$	-0.564	-0.442	0.122	0.128	0.075	0.265	0.002	0.302	0.034	1.762	1.000	1.000
$\beta_{5,5}$	-0.647	-0.207	0.440	0.440	0.089	0.254	0.002	0.510	0.077	5.034	0.800	0.800
$\beta_{6,5}$	-0.666	-0.419	0.248	0.250	0.121	0.239	0.002	0.357	0.086	2.206	0.968	0.968
$\beta_{7,5}$	0.304	0.430	0.126	0.129	0.083	0.265	0.002	0.304	0.039	1.750	1.000	1.000
$\beta_{8,5}$	-0.355	-0.124	0.231	0.231	0.088	0.257	0.001	0.352	0.058	2.807	1.000	1.000
$\beta_{9,5}$	-0.765	-0.342	0.423	0.423	0.119	0.242	0.002	0.491	0.102	3.689	0.648	0.640
$\beta_{1,6}$	0.337	0.240	-0.097	0.100	0.059	0.093	0.002	0.143	0.041	1.812	0.896	0.896
$\beta_{2,6}$	0.691	0.658	-0.033	0.054	0.041	0.094	0.004	0.114	0.022	1.133	0.992	0.992
$\beta_{3,6}$	0.781	0.795	0.014	0.039	0.030	0.072	0.001	0.086	0.017	1.042	0.992	0.992
$\beta_{4,6}$	0.257	0.209	-0.048	0.062	0.051	0.096	0.003	0.121	0.032	1.250	0.984	0.984
$\beta_{5,6}$	0.228	0.105	-0.122	0.123	0.050	0.083	0.001	0.152	0.039	2.558	0.808	0.808
$\beta_{6,6}$	0.399	0.326	-0.073	0.075	0.044	0.073	0.001	0.109	0.031	1.851	0.912	0.904
$\beta_{7,6}$	0.586	0.534	-0.052	0.069	0.051	0.101	0.004	0.128	0.031	1.265	0.952	0.952
$\beta_{8,6}$	-0.772	-0.823	-0.051	0.065	0.044	0.091	0.003	0.117	0.028	1.300	0.992	0.984
$\beta_{9,6}$	-0.351	-0.454	-0.103	0.103	0.045	0.074	0.001	0.130	0.034	2.489	0.840	0.840

Table 7.41: Scenario (F2c): β estimates (part 3).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,7}$	0.376	0.238	-0.138	0.139	0.063	0.115	0.002	0.185	0.045	2.338	0.928	0.936
$\beta_{2,7}$	0.636	0.591	-0.045	0.065	0.044	0.115	0.003	0.137	0.024	1.219	1.000	1.000
$\beta_{3,7}$	0.728	0.741	0.013	0.043	0.033	0.096	0.001	0.109	0.016	1.038	1.000	1.000
$\beta_{4,7}$	-0.739	-0.765	-0.026	0.062	0.048	0.126	0.004	0.146	0.027	1.054	1.000	1.000
$\beta_{5,7}$	0.716	0.561	-0.155	0.156	0.064	0.109	0.002	0.194	0.050	2.585	0.824	0.824
$\beta_{6,7}$	0.227	0.133	-0.094	0.096	0.050	0.097	0.001	0.141	0.034	2.000	0.976	0.976
$\beta_{7,7}$	-0.677	-0.711	-0.033	0.062	0.048	0.125	0.004	0.145	0.027	1.097	1.000	1.000
$\beta_{8,7}$	-0.727	-0.815	-0.088	0.094	0.064	0.113	0.003	0.154	0.043	1.592	0.976	0.976
$\beta_{9,7}$	0.617	0.453	-0.164	0.164	0.055	0.098	0.001	0.194	0.047	3.145	0.696	0.712
$\beta_{1,8}$	0.573	0.508	-0.065	0.072	0.045	0.074	0.003	0.108	0.030	1.545	0.920	0.920
$\beta_{2,8}$	-0.370	-0.377	-0.007	0.047	0.034	0.066	0.003	0.085	0.022	1.000	1.000	0.992
$\beta_{3,8}$	-0.306	-0.297	0.009	0.031	0.024	0.048	0.001	0.060	0.015	1.026	0.984	0.984
$\beta_{4,8}$	-0.371	-0.376	-0.005	0.044	0.032	0.076	0.004	0.091	0.019	1.000	1.000	0.992
$\beta_{5,8}$	0.237	0.172	-0.065	0.069	0.039	0.059	0.002	0.095	0.029	1.756	0.872	0.880
$\beta_{6,8}$	-0.682	-0.722	-0.040	0.047	0.029	0.049	0.001	0.071	0.020	1.447	0.944	0.944
$\beta_{7,8}$	0.705	0.676	-0.028	0.064	0.047	0.083	0.005	0.111	0.031	1.067	0.952	0.952
$\beta_{8,8}$	-0.745	-0.770	-0.026	0.045	0.033	0.068	0.003	0.086	0.022	1.120	0.976	0.976
$\beta_{9,8}$	-0.550	-0.613	-0.063	0.064	0.035	0.050	0.001	0.084	0.027	2.028	0.840	0.848

Table 7.42: Scenario (F2c): β estimates (part 4).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.109	0.002	0.043	0.029	0.057	0.004	0.075	0.019	1.000	0.984	0.984
$r_{1,3}$	-0.128	-0.121	0.007	0.050	0.038	0.061	0.006	0.084	0.026	1.000	0.976	0.968
$r_{2,3}$	0.199	0.190	-0.009	0.048	0.034	0.059	0.006	0.079	0.024	1.000	0.952	0.944

Table 7.43: Scenario (F2c): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.734	-0.016	0.047	0.042	0.062	0.007	0.084	0.030	1.033	0.920	0.912
$\xi_{2,3}$	1.250	1.230	-0.020	0.066	0.055	0.085	0.012	0.116	0.036	1.024	0.912	0.920
$\xi_{5,2}$	0.750	0.762	0.012	0.044	0.031	0.053	0.005	0.072	0.022	1.019	0.984	0.968
$\xi_{5,3}$	1.250	1.270	0.020	0.063	0.044	0.071	0.009	0.100	0.033	1.041	0.952	0.944
$\xi_{8,2}$	0.750	0.740	-0.010	0.050	0.038	0.056	0.004	0.080	0.026	1.000	0.904	0.912
$\xi_{8,3}$	1.250	1.237	-0.013	0.060	0.046	0.071	0.006	0.099	0.031	1.013	0.944	0.928

Table 7.44: Scenario (F2c): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,5}$	0.481	0.472	-0.010	0.028	0.021	0.036	0.002	0.048	0.014	1.029	0.976	0.984
$\sigma^2_{\varepsilon,10}$	0.302	0.306	0.004	0.012	0.009	0.015	0.001	0.020	0.006	1.071	0.968	0.968
$\sigma^2_{\varepsilon,15}$	0.658	0.655	-0.002	0.029	0.023	0.037	0.002	0.050	0.016	1.000	0.944	0.936

Table 7.45: Scenario (F3a): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	0.577	0.582	0.005	0.049	0.037	0.056	0.003	0.079	0.025	1.000	0.936	0.936
$\alpha_{2,1}$	-0.773	-0.755	0.017	0.063	0.044	0.069	0.005	0.099	0.030	1.027	0.928	0.928
$\alpha_{3,1}$	-0.499	-0.499	-0.000	0.043	0.034	0.052	0.003	0.072	0.023	1.000	0.944	0.944
$\alpha_{4,1}$	-0.783	-0.775	0.007	0.055	0.040	0.061	0.005	0.087	0.029	1.000	0.920	0.936
$\alpha_{5,1}$	0.676	0.679	0.003	0.023	0.016	0.031	0.001	0.040	0.011	1.000	0.968	0.968
$\alpha_{6,2}$	0.757	0.748	-0.010	0.058	0.045	0.081	0.007	0.106	0.031	1.014	0.968	0.976
$\alpha_{7,2}$	0.385	0.383	-0.002	0.046	0.036	0.055	0.002	0.076	0.025	1.000	0.944	0.944
$\alpha_{8,2}$	0.646	0.648	0.001	0.048	0.039	0.068	0.005	0.088	0.027	1.000	0.960	0.952
$\alpha_{9,2}$	0.796	0.813	0.017	0.062	0.050	0.074	0.008	0.102	0.036	1.026	0.952	0.944
$\alpha_{10,2}$	-0.344	-0.342	0.001	0.018	0.014	0.022	0.001	0.030	0.009	1.000	0.936	0.936
$\alpha_{11,3}$	-0.488	-0.498	-0.010	0.041	0.031	0.052	0.003	0.070	0.021	1.020	0.960	0.944
$\alpha_{12,3}$	0.748	0.730	-0.018	0.059	0.047	0.069	0.005	0.097	0.032	1.027	0.928	0.928
$\alpha_{13,3}$	0.712	0.699	-0.013	0.054	0.038	0.066	0.005	0.090	0.025	1.015	0.936	0.944
$\alpha_{14,3}$	0.732	0.737	0.005	0.047	0.035	0.055	0.004	0.077	0.025	1.000	0.944	0.944
$\alpha_{15,3}$	-0.634	-0.637	-0.003	0.027	0.020	0.032	0.001	0.044	0.013	1.000	0.960	0.960

Table 7.46: Scenario (F3a): α^{Δ} estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	-0.672	-0.675	-0.003	0.031	0.024	0.042	0.002	0.055	0.016	1.000	0.968	0.968
$\beta_{2,1}$	0.789	0.783	-0.006	0.040	0.028	0.050	0.003	0.067	0.019	1.021	0.968	0.968
$\beta_{3,1}$	0.679	0.679	-0.000	0.031	0.022	0.040	0.001	0.053	0.014	1.000	0.976	0.968
$\beta_{4,1}$	-0.217	-0.213	0.004	0.030	0.020	0.040	0.001	0.052	0.012	1.000	0.992	0.992
$\beta_{5,1}$	-0.601	-0.602	-0.001	0.019	0.015	0.023	0.000	0.032	0.010	1.000	0.904	0.912
$\beta_{6,1}$	-0.736	-0.725	0.011	0.041	0.031	0.053	0.004	0.071	0.021	1.020	0.944	0.936
$\beta_{7,1}$	-0.724	-0.718	0.006	0.035	0.025	0.041	0.002	0.057	0.017	1.024	0.944	0.936
$\beta_{8,1}$	-0.298	-0.299	-0.001	0.037	0.022	0.041	0.001	0.057	0.015	1.000	0.968	0.968
$\beta_{9,1}$	-0.428	-0.422	0.006	0.034	0.024	0.043	0.002	0.058	0.016	1.000	0.960	0.960
$\beta_{10,1}$	0.787	0.786	-0.002	0.012	0.009	0.015	0.000	0.020	0.006	1.000	0.968	0.968
$\beta_{11,1}$	-0.250	-0.253	-0.003	0.026	0.020	0.036	0.001	0.047	0.013	1.000	0.968	0.960
$\beta_{12,1}$	-0.768	-0.757	0.011	0.037	0.029	0.048	0.003	0.064	0.019	1.022	0.936	0.944
$\beta_{13,1}$	-0.690	-0.682	0.008	0.039	0.029	0.045	0.002	0.063	0.021	1.021	0.952	0.952
$\beta_{14,1}$	0.401	0.398	-0.003	0.029	0.025	0.038	0.001	0.051	0.017	1.000	0.944	0.944
$\beta_{15,1}$	-0.738	-0.738	-0.000	0.017	0.013	0.024	0.000	0.031	0.008	0.955	0.976	0.976
$\beta_{1,2}$	-0.527	-0.524	0.003	0.034	0.026	0.048	0.002	0.062	0.017	1.000	0.992	0.992
$\beta_{2,2}$	0.525	0.522	-0.003	0.041	0.034	0.053	0.003	0.071	0.023	1.000	0.944	0.944
$\beta_{3,2}$	0.569	0.558	-0.010	0.037	0.028	0.046	0.001	0.063	0.019	1.022	0.936	0.936
$\beta_{4,2}$	0.625	0.611	-0.014	0.040	0.027	0.046	0.002	0.065	0.019	1.043	0.944	0.944
$\beta_{5,2}$	-0.501	-0.501	0.000	0.021	0.014	0.026	0.001	0.036	0.010	1.000	0.952	0.952
$\beta_{6,2}$	-0.239	-0.236	0.003	0.040	0.032	0.052	0.002	0.070	0.021	1.000	0.944	0.944
$\beta_{7,2}$	-0.798	-0.787	0.011	0.039	0.029	0.051	0.002	0.068	0.020	1.021	0.968	0.968
$\beta_{8,2}$	-0.784	-0.786	-0.002	0.045	0.033	0.055	0.003	0.075	0.023	0.982	0.944	0.952
$\beta_{9,2}$	-0.609	-0.611	-0.002	0.033	0.026	0.049	0.003	0.062	0.017	1.000	0.984	0.984
$\beta_{10,2}$	0.318	0.317	-0.001	0.015	0.011	0.018	0.000	0.025	0.008	1.000	0.920	0.920
$\beta_{11,2}$	-0.427	-0.426	0.001	0.036	0.028	0.044	0.002	0.060	0.019	0.978	0.944	0.944
$\beta_{12,2}$	-0.746	-0.735	0.010	0.040	0.028	0.056	0.003	0.072	0.018	1.021	0.992	0.992
$\beta_{13,2}$	0.695	0.678	-0.017	0.039	0.028	0.053	0.003	0.070	0.017	1.067	0.952	0.952
$\beta_{14,2}$	0.582	0.579	-0.002	0.031	0.023	0.042	0.002	0.055	0.015	1.000	0.976	0.976
$\beta_{15,2}$	0.623	0.625	0.002	0.019	0.015	0.028	0.001	0.036	0.010	1.000	0.952	0.952

Table 7.47: Scenario (F3a): β estimates (part 1).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,3}$	0.223	0.222	-0.001	0.037	0.025	0.049	0.002	0.064	0.016	1.000	0.992	0.992
$\beta_{2,3}$	0.569	0.558	-0.010	0.045	0.034	0.058	0.003	0.078	0.022	1.000	0.952	0.952
$\beta_{3,3}$	0.256	0.258	0.002	0.039	0.030	0.047	0.001	0.065	0.021	1.000	0.944	0.944
$\beta_{4,3}$	0.564	0.554	-0.009	0.038	0.034	0.049	0.002	0.067	0.023	1.020	0.936	0.936
$\beta_{5,3}$	-0.647	-0.639	0.007	0.022	0.015	0.029	0.001	0.038	0.010	1.038	0.968	0.968
$\beta_{6,3}$	-0.666	-0.661	0.005	0.059	0.047	0.064	0.004	0.093	0.034	1.000	0.904	0.896
$\beta_{7,3}$	0.304	0.300	-0.004	0.034	0.026	0.048	0.001	0.062	0.017	1.000	0.960	0.960
$\beta_{8,3}$	0.355	0.351	-0.005	0.037	0.030	0.052	0.002	0.068	0.019	1.000	0.976	0.976
$\beta_{9,3}$	0.765	0.763	-0.001	0.041	0.031	0.055	0.004	0.073	0.021	0.981	0.960	0.960
$\beta_{10,3}$	-0.337	-0.337	0.000	0.015	0.011	0.019	0.000	0.026	0.007	1.000	0.960	0.960
$\beta_{11,3}$	0.691	0.690	-0.001	0.040	0.031	0.051	0.002	0.068	0.022	1.000	0.952	0.952
$\beta_{12,3}$	0.781	0.759	-0.022	0.050	0.037	0.061	0.003	0.083	0.025	1.069	0.936	0.936
$\beta_{13,3}$	-0.257	-0.255	0.002	0.041	0.031	0.052	0.002	0.070	0.020	1.000	0.968	0.968
$\beta_{14,3}$	-0.228	-0.235	-0.007	0.029	0.022	0.042	0.001	0.054	0.014	1.000	0.984	0.984
$\beta_{15,3}$	0.399	0.403	0.004	0.024	0.019	0.031	0.001	0.041	0.012	1.000	0.944	0.944
$\beta_{1,4}$	0.586	0.585	-0.001	0.037	0.029	0.048	0.002	0.065	0.019	1.000	0.952	0.952
$\beta_{2,4}$	-0.772	-0.756	0.016	0.049	0.040	0.058	0.003	0.081	0.028	1.016	0.920	0.920
$\beta_{3,4}$	-0.351	-0.346	0.005	0.036	0.026	0.044	0.001	0.060	0.017	1.023	0.944	0.952
$\beta_{4,4}$	-0.376	-0.361	0.016	0.040	0.028	0.043	0.002	0.062	0.020	1.043	0.904	0.896
$\beta_{5,4}$	0.636	0.625	-0.010	0.022	0.016	0.026	0.001	0.036	0.011	1.038	0.936	0.936
$\beta_{6,4}$	-0.728	-0.718	0.010	0.048	0.034	0.061	0.004	0.082	0.022	1.017	0.952	0.960
$\beta_{7,4}$	0.739	0.733	-0.006	0.038	0.026	0.050	0.002	0.066	0.017	1.000	0.960	0.976
$\beta_{8,4}$	-0.716	-0.715	0.001	0.034	0.025	0.054	0.003	0.067	0.016	1.000	0.992	0.992
$\beta_{9,4}$	-0.227	-0.226	0.001	0.040	0.030	0.044	0.002	0.063	0.021	1.000	0.904	0.896
$\beta_{10,4}$	0.677	0.676	-0.001	0.014	0.010	0.018	0.000	0.024	0.007	1.000	0.960	0.960
$\beta_{11,4}$	-0.727	-0.733	-0.005	0.039	0.029	0.048	0.002	0.065	0.021	1.000	0.936	0.944
$\beta_{12,4}$	-0.617	-0.599	0.018	0.047	0.033	0.053	0.003	0.075	0.022	1.056	0.928	0.936
$\beta_{13,4}$	-0.573	-0.562	0.011	0.043	0.031	0.051	0.002	0.071	0.021	1.019	0.936	0.936
$\beta_{14,4}$	-0.370	-0.367	0.003	0.031	0.026	0.040	0.002	0.054	0.018	1.000	0.952	0.944
$\beta_{15,4}$	0.306	0.303	-0.003	0.023	0.018	0.028	0.001	0.039	0.013	1.000	0.912	0.912

Table 7.48: Scenario (F3a): β estimates (part 2).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.112	-0.001	0.031	0.023	0.040	0.001	0.053	0.015	1.000	0.944	0.944
$r_{1,3}$	-0.128	-0.134	-0.006	0.031	0.020	0.037	0.001	0.050	0.013	1.000	0.984	0.984
$r_{2,3}$	0.199	0.192	-0.007	0.030	0.023	0.040	0.001	0.053	0.015	1.000	0.968	0.968

Table 7.49: Scenario (F3a): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{4,2}$	0.750	0.747	-0.003	0.034	0.025	0.041	0.002	0.056	0.017	1.000	0.928	0.920
$\xi_{4,3}$	1.250	1.243	-0.007	0.047	0.038	0.055	0.004	0.077	0.028	1.000	0.936	0.936
$\xi_{9,2}$	0.750	0.754	0.004	0.033	0.024	0.044	0.003	0.058	0.016	1.000	0.968	0.960
$\xi_{9,3}$	1.250	1.262	0.012	0.049	0.034	0.063	0.005	0.084	0.023	1.017	0.976	0.984
$\xi_{14,2}$	0.750	0.749	-0.001	0.031	0.024	0.037	0.002	0.051	0.016	1.000	0.920	0.912
$\xi_{14,3}$	1.250	1.254	0.004	0.041	0.030	0.048	0.003	0.067	0.021	1.000	0.944	0.960

Table 7.50: Scenario (F3a): ξ estimates.

Scenario (F3b)

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,5}$	0.481	0.557	0.076	0.076	0.021	0.026	0.001	0.081	0.019	3.762	0.120	0.096
$\sigma^2_{\varepsilon,10}$	0.302	0.318	0.016	0.017	0.011	0.013	0.000	0.023	0.008	1.667	0.760	0.744
$\sigma^2_{\varepsilon,15}$	0.658	0.683	0.026	0.032	0.023	0.031	0.001	0.047	0.018	1.333	0.864	0.848

Table 7.51: Scenario (F3b): σ^2_{ε} estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	0.577	0.587	0.010	0.043	0.032	0.050	0.002	0.069	0.022	1.019	0.936	0.944
$\alpha_{2,1}$	-0.773	-0.801	-0.029	0.061	0.045	0.064	0.005	0.093	0.033	1.086	0.928	0.888
$\alpha_{3,1}$	-0.499	-0.519	-0.020	0.042	0.033	0.046	0.002	0.066	0.025	1.060	0.912	0.912
$\alpha_{4,1}$	-0.783	-0.797	-0.014	0.045	0.029	0.053	0.003	0.073	0.021	1.038	0.968	0.968
$\alpha_{5,1}$	0.676	0.627	-0.050	0.050	0.021	0.025	0.000	0.056	0.018	2.571	0.472	0.464
$\alpha_{6,2}$	0.757	0.749	-0.008	0.054	0.041	0.070	0.005	0.093	0.028	1.000	0.968	0.976
$\alpha_{7,2}$	0.385	0.397	0.012	0.040	0.030	0.047	0.002	0.065	0.021	1.020	0.928	0.936
$\alpha_{8,2}$	0.646	0.657	0.011	0.043	0.038	0.058	0.004	0.077	0.027	1.018	0.944	0.944
$\alpha_{9,2}$	0.796	0.802	0.006	0.048	0.035	0.060	0.004	0.081	0.025	1.000	0.952	0.952
$\alpha_{10,2}$	-0.344	-0.342	0.002	0.015	0.011	0.018	0.000	0.025	0.007	1.000	0.968	0.968
$\alpha_{11,3}$	-0.488	-0.484	0.004	0.031	0.024	0.044	0.002	0.057	0.015	1.000	0.976	0.976
$\alpha_{12,3}$	0.748	0.678	-0.070	0.075	0.048	0.055	0.003	0.098	0.036	1.618	0.720	0.752
$\alpha_{13,3}$	0.712	0.707	-0.005	0.045	0.031	0.057	0.003	0.076	0.020	1.000	0.976	0.960
$\alpha_{14,3}$	0.732	0.742	0.010	0.044	0.031	0.048	0.003	0.068	0.022	1.019	0.936	0.928
$\alpha_{15,3}$	-0.634	-0.616	0.018	0.027	0.018	0.027	0.001	0.040	0.012	1.185	0.880	0.880

Table 7.52: Scenario (F3b): α^{Δ} estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	-0.672	-0.669	0.003	0.028	0.021	0.035	0.001	0.048	0.014	1.000	0.952	0.952
$\beta_{2,1}$	0.789	0.789	0.000	0.039	0.026	0.044	0.003	0.062	0.018	1.000	0.952	0.960
$\beta_{3,1}$	0.679	0.677	-0.002	0.026	0.020	0.034	0.001	0.045	0.013	1.000	0.968	0.968
$\beta_{4,1}$	-0.217	-0.214	0.003	0.025	0.018	0.034	0.001	0.045	0.012	1.000	0.968	0.968
$\beta_{5,1}$	-0.601	-0.601	0.000	0.015	0.011	0.020	0.000	0.026	0.008	1.000	0.960	0.960
$\beta_{6,1}$	-0.736	-0.727	0.009	0.038	0.029	0.045	0.003	0.063	0.020	1.021	0.920	0.928
$\beta_{7,1}$	-0.724	-0.707	0.017	0.031	0.024	0.035	0.001	0.050	0.017	1.114	0.904	0.904
$\beta_{8,1}$	-0.298	-0.296	0.002	0.029	0.021	0.035	0.001	0.048	0.014	1.000	0.952	0.952
$\beta_{9,1}$	-0.428	-0.413	0.015	0.029	0.022	0.036	0.001	0.049	0.015	1.091	0.912	0.920
$\beta_{10,1}$	0.787	0.785	-0.003	0.010	0.008	0.013	0.000	0.018	0.005	1.083	0.976	0.976
$\beta_{11,1}$	-0.250	-0.246	0.004	0.023	0.016	0.030	0.001	0.040	0.010	1.000	0.984	0.984
$\beta_{12,1}$	-0.768	-0.715	0.053	0.056	0.033	0.039	0.002	0.071	0.025	1.757	0.672	0.688
$\beta_{13,1}$	-0.690	-0.677	0.013	0.035	0.030	0.038	0.002	0.056	0.021	1.045	0.880	0.880
$\beta_{14,1}$	0.401	0.398	-0.003	0.027	0.022	0.032	0.001	0.045	0.015	1.000	0.936	0.928
$\beta_{15,1}$	-0.738	-0.739	-0.001	0.016	0.011	0.021	0.000	0.028	0.007	1.000	0.976	0.976
$\beta_{1,2}$	-0.527	-0.472	0.056	0.057	0.033	0.040	0.001	0.072	0.026	1.886	0.688	0.688
$\beta_{2,2}$	0.525	0.500	-0.025	0.041	0.034	0.045	0.002	0.065	0.025	1.128	0.888	0.888
$\beta_{3,2}$	0.569	0.528	-0.040	0.047	0.031	0.039	0.001	0.064	0.022	1.436	0.832	0.832
$\beta_{4,2}$	0.625	0.593	-0.031	0.043	0.029	0.039	0.001	0.061	0.021	1.238	0.840	0.848
$\beta_{5,2}$	-0.501	-0.487	0.015	0.021	0.016	0.023	0.000	0.033	0.012	1.182	0.920	0.920
$\beta_{6,2}$	-0.239	-0.200	0.039	0.046	0.035	0.044	0.002	0.068	0.026	1.349	0.856	0.864
$\beta_{7,2}$	-0.798	-0.710	0.088	0.089	0.041	0.042	0.001	0.101	0.034	2.333	0.400	0.408
$\beta_{8,2}$	-0.784	-0.724	0.060	0.069	0.042	0.045	0.002	0.086	0.034	1.500	0.696	0.712
$\beta_{9,2}$	-0.609	-0.518	0.091	0.091	0.034	0.039	0.001	0.100	0.030	2.882	0.296	0.304
$\beta_{10,2}$	0.318	0.283	-0.036	0.036	0.016	0.015	0.000	0.040	0.014	2.438	0.376	0.376
$\beta_{11,2}$	-0.427	-0.371	0.057	0.060	0.034	0.036	0.001	0.072	0.028	1.789	0.616	0.640
$\beta_{12,2}$	-0.746	-0.637	0.109	0.109	0.040	0.044	0.002	0.119	0.036	2.900	0.280	0.296
$\beta_{13,2}$	0.695	0.648	-0.047	0.057	0.034	0.044	0.002	0.076	0.025	1.404	0.768	0.768
$\beta_{14,2}$	0.582	0.553	-0.028	0.036	0.025	0.035	0.001	0.053	0.018	1.294	0.880	0.888
$\beta_{15,2}$	0.623	0.623	0.000	0.018	0.014	0.024	0.001	0.032	0.009	1.000	0.960	0.960

Table 7.53: Scenario (F3b): β estimates (part 1).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,3}$	0.223	0.199	-0.024	0.036	0.027	0.041	0.001	0.058	0.018	1.158	0.944	0.944
$\beta_{2,3}$	0.569	0.522	-0.047	0.057	0.035	0.049	0.002	0.079	0.026	1.396	0.856	0.856
$\beta_{3,3}$	0.256	0.243	-0.012	0.034	0.027	0.040	0.001	0.056	0.019	1.024	0.936	0.936
$\beta_{4,3}$	0.564	0.513	-0.051	0.054	0.038	0.041	0.001	0.072	0.030	1.571	0.776	0.776
$\beta_{5,3}$	-0.647	-0.610	0.037	0.038	0.023	0.025	0.001	0.047	0.018	1.833	0.712	0.712
$\beta_{6,3}$	-0.666	-0.698	-0.032	0.061	0.047	0.055	0.003	0.087	0.038	1.100	0.872	0.864
$\beta_{7,3}$	0.304	0.234	-0.070	0.071	0.038	0.041	0.001	0.084	0.032	2.000	0.616	0.616
$\beta_{8,3}$	0.355	0.280	-0.075	0.077	0.037	0.043	0.001	0.090	0.031	2.073	0.600	0.608
$\beta_{9,3}$	0.765	0.664	-0.101	0.101	0.043	0.043	0.002	0.112	0.037	2.558	0.336	0.344
$\beta_{10,3}$	-0.337	-0.293	0.044	0.044	0.016	0.017	0.000	0.047	0.015	2.765	0.272	0.280
$\beta_{11,3}$	0.691	0.611	-0.080	0.081	0.038	0.041	0.001	0.094	0.032	2.195	0.544	0.544
$\beta_{12,3}$	0.781	0.655	-0.127	0.127	0.049	0.047	0.002	0.137	0.043	2.776	0.216	0.216
$\beta_{13,3}$	-0.257	-0.218	0.039	0.046	0.032	0.043	0.001	0.067	0.023	1.366	0.840	0.840
$\beta_{14,3}$	-0.228	-0.203	0.025	0.033	0.021	0.036	0.001	0.051	0.015	1.300	0.936	0.936
$\beta_{15,3}$	0.399	0.400	0.001	0.022	0.018	0.026	0.001	0.036	0.013	1.000	0.912	0.912
$\beta_{1,4}$	0.586	0.577	-0.009	0.031	0.026	0.041	0.001	0.055	0.017	1.025	0.936	0.936
$\beta_{2,4}$	-0.772	-0.733	0.039	0.054	0.038	0.049	0.003	0.078	0.028	1.245	0.832	0.840
$\beta_{3,4}$	-0.351	-0.327	0.024	0.033	0.026	0.037	0.001	0.054	0.018	1.200	0.888	0.896
$\beta_{4,4}$	-0.376	-0.333	0.043	0.049	0.031	0.037	0.001	0.064	0.025	1.487	0.776	0.776
$\beta_{5,4}$	0.636	0.604	-0.032	0.034	0.020	0.023	0.000	0.042	0.017	1.696	0.736	0.744
$\beta_{6,4}$	-0.728	-0.719	0.008	0.040	0.030	0.052	0.003	0.069	0.020	1.020	0.952	0.952
$\beta_{7,4}$	0.739	0.730	-0.009	0.033	0.025	0.042	0.001	0.057	0.016	1.025	0.944	0.952
$\beta_{8,4}$	-0.716	-0.694	0.023	0.036	0.027	0.045	0.002	0.061	0.018	1.154	0.936	0.944
$\beta_{9,4}$	-0.227	-0.201	0.025	0.038	0.031	0.036	0.001	0.057	0.023	1.167	0.840	0.832
$\beta_{10,4}$	0.677	0.665	-0.012	0.017	0.013	0.015	0.000	0.024	0.010	1.235	0.824	0.824
$\beta_{11,4}$	-0.727	-0.693	0.034	0.042	0.028	0.040	0.001	0.061	0.021	1.351	0.864	0.864
$\beta_{12,4}$	-0.617	-0.554	0.063	0.067	0.042	0.043	0.002	0.084	0.034	1.667	0.648	0.664
$\beta_{13,4}$	-0.573	-0.568	0.005	0.035	0.025	0.044	0.002	0.059	0.017	1.000	0.936	0.928
$\beta_{14,4}$	-0.370	-0.373	-0.003	0.026	0.021	0.034	0.001	0.046	0.015	1.000	0.952	0.952
$\beta_{15,4}$	0.306	0.310	0.004	0.021	0.016	0.024	0.001	0.034	0.011	1.038	0.928	0.928

Table 7.54: Scenario (F3b): β estimates (part 2).

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.115	-0.004	0.028	0.020	0.035	0.001	0.047	0.013	1.000	0.960	0.968
$r_{1,3}$	-0.128	-0.142	-0.015	0.029	0.021	0.033	0.001	0.046	0.014	1.094	0.928	0.928
$r_{2,3}$	0.199	0.178	-0.021	0.032	0.023	0.035	0.001	0.050	0.016	1.182	0.920	0.904

Table 7.55: Scenario (F3b): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{4,2}$	0.750	0.752	0.002	0.028	0.021	0.035	0.002	0.048	0.015	1.000	0.952	0.960
$\xi_{4,3}$	1.250	1.249	-0.001	0.037	0.029	0.047	0.002	0.064	0.019	1.000	0.960	0.960
$\xi_{9,2}$	0.750	0.743	-0.007	0.029	0.021	0.036	0.002	0.049	0.013	1.000	0.952	0.960
$\xi_{9,3}$	1.250	1.240	-0.010	0.041	0.028	0.051	0.003	0.069	0.018	1.020	0.984	0.992
$\xi_{14,2}$	0.750	0.745	-0.005	0.028	0.020	0.031	0.001	0.044	0.014	1.029	0.920	0.920
$\xi_{14,3}$	1.250	1.249	-0.001	0.036	0.028	0.041	0.002	0.058	0.019	1.000	0.952	0.936

Table 7.56: Scenario (F3b): ξ estimates.

Scenario (F4a)

Note, that this scenario is uses solely binary measurements and encompasses two covariates without intercept. Therefore estimation of $\boldsymbol{\Sigma}_{\varepsilon}$ and $\mathbf{R}$ was not necessary.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	0.601	0.580	-0.021	0.057	0.046	0.072	0.004	0.097	0.032	1.043	0.928	0.928
$\alpha_{2,1}$	0.683	0.666	-0.017	0.065	0.053	0.079	0.006	0.110	0.036	1.012	0.904	0.928
$\alpha_{3,1}$	-0.603	-0.572	0.031	0.060	0.047	0.071	0.004	0.098	0.033	1.086	0.936	0.936
$\alpha_{4,1}$	-0.700	-0.657	0.042	0.072	0.052	0.077	0.005	0.112	0.036	1.141	0.888	0.912
$\alpha_{5,1}$	0.540	0.534	-0.007	0.057	0.039	0.071	0.004	0.096	0.025	1.000	0.960	0.960
$\alpha_{6,1}$	0.378	0.369	-0.009	0.053	0.036	0.059	0.002	0.084	0.025	1.000	0.968	0.968
$\alpha_{7,1}$	-0.709	-0.692	0.016	0.065	0.047	0.078	0.006	0.108	0.032	1.025	0.944	0.944
$\alpha_{8,1}$	-0.754	-0.719	0.035	0.069	0.049	0.081	0.006	0.113	0.033	1.104	0.904	0.936
$\alpha_{9,1}$	-0.608	-0.586	0.022	0.064	0.044	0.071	0.004	0.101	0.029	1.027	0.920	0.928
$\alpha_{10,1}$	0.455	0.424	-0.030	0.054	0.038	0.064	0.003	0.088	0.025	1.119	0.936	0.944

Table 7.57: Scenario (F4a): α^{Δ} estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	-0.488	-0.479	0.009	0.046	0.034	0.056	0.002	0.077	0.022	1.000	0.944	0.952
$\beta_{2,1}$	0.399	0.384	-0.015	0.048	0.037	0.057	0.003	0.079	0.025	1.017	0.920	0.920
$\beta_{3,1}$	-0.426	-0.394	0.032	0.053	0.041	0.054	0.002	0.081	0.029	1.136	0.896	0.896
$\beta_{4,1}$	0.453	0.435	-0.017	0.052	0.040	0.058	0.003	0.083	0.028	1.032	0.896	0.896
$\beta_{5,1}$	0.628	0.617	-0.012	0.048	0.031	0.059	0.003	0.080	0.020	1.018	0.968	0.952
$\beta_{6,1}$	0.438	0.411	-0.028	0.049	0.037	0.049	0.002	0.074	0.027	1.127	0.880	0.880
$\beta_{7,1}$	-0.338	-0.307	0.031	0.055	0.043	0.054	0.003	0.082	0.032	1.111	0.864	0.872
$\beta_{8,1}$	0.422	0.404	-0.018	0.053	0.042	0.057	0.003	0.084	0.029	1.046	0.888	0.880
$\beta_{9,1}$	0.631	0.615	-0.017	0.052	0.039	0.057	0.003	0.082	0.028	1.032	0.896	0.904
$\beta_{10,1}$	-0.408	-0.391	0.018	0.047	0.037	0.053	0.002	0.076	0.026	1.034	0.912	0.912
$\beta_{1,2}$	-0.564	-0.510	0.054	0.068	0.042	0.057	0.003	0.093	0.031	1.379	0.816	0.832
$\beta_{2,2}$	0.654	0.671	0.016	0.060	0.044	0.063	0.004	0.092	0.033	1.028	0.920	0.920
$\beta_{3,2}$	0.785	0.734	-0.051	0.065	0.050	0.060	0.003	0.095	0.036	1.281	0.808	0.808
$\beta_{4,2}$	0.668	0.615	-0.053	0.067	0.052	0.061	0.003	0.097	0.039	1.269	0.824	0.824
$\beta_{5,2}$	0.554	0.569	0.015	0.051	0.043	0.059	0.003	0.083	0.032	1.031	0.928	0.920
$\beta_{6,2}$	-0.505	-0.460	0.045	0.056	0.043	0.050	0.002	0.080	0.031	1.296	0.816	0.816
$\beta_{7,2}$	0.560	0.494	-0.066	0.069	0.044	0.057	0.003	0.094	0.032	1.673	0.832	0.832
$\beta_{8,2}$	0.554	0.496	-0.058	0.069	0.043	0.059	0.003	0.095	0.031	1.421	0.816	0.824
$\beta_{9,2}$	-0.432	-0.438	-0.005	0.048	0.032	0.054	0.002	0.076	0.023	1.000	0.936	0.936
$\beta_{10,2}$	-0.676	-0.642	0.034	0.057	0.039	0.057	0.002	0.085	0.027	1.150	0.872	0.888

Table 7.58: Scenario (F4a): β estimates.

Scenario (F4b)

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	0.601	0.579	-0.022	0.056	0.046	0.072	0.004	0.097	0.032	1.043	0.920	0.928
$\alpha_{2,1}$	0.683	0.661	-0.022	0.065	0.053	0.079	0.006	0.110	0.036	1.037	0.904	0.912
$\alpha_{3,1}$	-0.603	-0.568	0.035	0.060	0.047	0.071	0.004	0.099	0.033	1.118	0.928	0.920
$\alpha_{4,1}$	-0.700	-0.656	0.043	0.072	0.052	0.077	0.005	0.112	0.036	1.141	0.896	0.912
$\alpha_{5,1}$	0.540	0.530	-0.010	0.057	0.040	0.071	0.004	0.096	0.026	1.000	0.960	0.960
$\alpha_{6,1}$	0.378	0.367	-0.011	0.052	0.037	0.059	0.002	0.084	0.025	1.016	0.952	0.960
$\alpha_{7,1}$	-0.709	-0.690	0.019	0.067	0.046	0.078	0.006	0.108	0.031	1.025	0.944	0.944
$\alpha_{8,1}$	-0.754	-0.720	0.034	0.068	0.049	0.082	0.006	0.113	0.033	1.091	0.912	0.912
$\alpha_{9,1}$	-0.608	-0.574	0.034	0.065	0.045	0.071	0.004	0.102	0.030	1.113	0.888	0.888
$\alpha_{10,1}$	0.455	0.423	-0.031	0.055	0.037	0.064	0.003	0.089	0.025	1.119	0.936	0.936

Table 7.59: Scenario (F4b): α^{Δ} estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	-0.488	-0.476	0.012	0.044	0.033	0.057	0.002	0.077	0.021	1.019	0.968	0.968
$\beta_{2,1}$	0.399	0.373	-0.027	0.049	0.038	0.057	0.003	0.080	0.025	1.107	0.920	0.912
$\beta_{3,1}$	-0.426	-0.386	0.040	0.055	0.041	0.055	0.002	0.083	0.029	1.232	0.864	0.864
$\beta_{4,1}$	0.453	0.430	-0.023	0.053	0.038	0.059	0.003	0.084	0.027	1.065	0.904	0.904
$\beta_{5,1}$	0.628	0.616	-0.012	0.045	0.030	0.059	0.003	0.078	0.019	1.038	0.976	0.984
$\beta_{6,1}$	0.438	0.402	-0.037	0.052	0.038	0.050	0.002	0.077	0.027	1.208	0.872	0.872
$\beta_{7,1}$	-0.338	-0.303	0.035	0.055	0.042	0.054	0.002	0.083	0.031	1.150	0.856	0.864
$\beta_{8,1}$	0.422	0.398	-0.024	0.052	0.039	0.058	0.003	0.084	0.027	1.066	0.888	0.896
$\beta_{9,1}$	0.631	0.601	-0.030	0.055	0.040	0.058	0.002	0.085	0.028	1.115	0.896	0.912
$\beta_{10,1}$	-0.408	-0.384	0.024	0.047	0.036	0.054	0.002	0.076	0.025	1.093	0.904	0.912
$\beta_{1,2}$	-0.564	-0.501	0.063	0.072	0.044	0.058	0.003	0.097	0.032	1.518	0.792	0.792
$\beta_{2,2}$	0.654	0.674	0.019	0.060	0.044	0.064	0.004	0.092	0.033	1.028	0.920	0.920
$\beta_{3,2}$	0.785	0.722	-0.063	0.071	0.053	0.061	0.003	0.100	0.039	1.413	0.792	0.792
$\beta_{4,2}$	0.668	0.603	-0.065	0.074	0.055	0.062	0.003	0.103	0.041	1.394	0.776	0.784
$\beta_{5,2}$	0.554	0.562	0.008	0.046	0.040	0.060	0.003	0.081	0.029	1.000	0.952	0.952
$\beta_{6,2}$	-0.505	-0.449	0.055	0.062	0.044	0.051	0.002	0.085	0.033	1.462	0.784	0.792
$\beta_{7,2}$	0.560	0.484	-0.076	0.078	0.045	0.057	0.003	0.101	0.034	1.875	0.792	0.792
$\beta_{8,2}$	0.554	0.487	-0.067	0.074	0.044	0.060	0.003	0.100	0.032	1.593	0.800	0.808
$\beta_{9,2}$	-0.432	-0.430	0.003	0.044	0.030	0.055	0.002	0.074	0.021	1.000	0.976	0.976
$\beta_{10,2}$	-0.676	-0.638	0.038	0.059	0.042	0.058	0.002	0.088	0.030	1.161	0.872	0.880

Table 7.60: Scenario (F4b): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.533	0.002	0.043	0.033	0.051	0.004	0.070	0.023	1.000	0.912	0.912
$\sigma^2_{\varepsilon,6}$	0.490	0.497	0.007	0.041	0.031	0.048	0.003	0.067	0.022	1.000	0.952	0.936
$\sigma^2_{\varepsilon,9}$	0.528	0.536	0.008	0.050	0.039	0.063	0.004	0.085	0.027	1.016	0.936	0.936

Table 7.61: Scenario (I1a): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.501	0.005	0.047	0.033	0.059	0.005	0.079	0.021	1.000	0.968	0.976
$\alpha_{2,1}$	0.674	0.659	-0.015	0.053	0.041	0.069	0.007	0.093	0.028	1.015	0.920	0.928
$\alpha_{3,1}$	0.601	0.594	-0.007	0.034	0.024	0.041	0.002	0.056	0.016	1.000	0.944	0.952
$\alpha_{4,2}$	-0.683	-0.672	0.011	0.053	0.035	0.058	0.005	0.083	0.023	1.016	0.952	0.952
$\alpha_{5,2}$	-0.603	-0.595	0.007	0.043	0.029	0.047	0.004	0.067	0.021	1.000	0.928	0.928
$\alpha_{6,2}$	0.700	0.692	-0.008	0.033	0.022	0.036	0.001	0.051	0.015	1.000	0.936	0.936
$\alpha_{7,3}$	0.540	0.528	-0.012	0.043	0.031	0.049	0.004	0.069	0.022	1.019	0.928	0.936
$\alpha_{8,3}$	-0.378	-0.375	0.003	0.030	0.024	0.035	0.002	0.050	0.017	1.000	0.928	0.928
$\alpha_{9,3}$	-0.709	-0.697	0.012	0.040	0.035	0.045	0.002	0.064	0.027	1.019	0.904	0.904

Table 7.62: Scenario (I1a): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.628	-0.630	-0.001	0.055	0.041	0.062	0.004	0.088	0.029	0.986	0.928	0.928
$\gamma_{1,2}$	0.438	0.444	0.006	0.050	0.039	0.062	0.003	0.085	0.027	1.000	0.952	0.936
$\gamma_{1,3}$	0.338	0.347	0.009	0.047	0.037	0.055	0.003	0.077	0.027	1.017	0.960	0.960
$\gamma_{2,1}$	-0.422	-0.426	-0.004	0.044	0.030	0.053	0.003	0.072	0.021	1.000	0.952	0.952
$\gamma_{2,2}$	-0.631	-0.639	-0.008	0.042	0.036	0.061	0.004	0.079	0.025	1.000	0.960	0.960
$\gamma_{2,3}$	-0.408	-0.419	-0.010	0.043	0.032	0.052	0.003	0.072	0.023	1.019	0.952	0.944
$\gamma_{3,1}$	-0.564	-0.580	-0.016	0.056	0.047	0.063	0.007	0.091	0.035	1.014	0.928	0.904
$\gamma_{3,2}$	-0.654	-0.673	-0.019	0.059	0.050	0.071	0.008	0.098	0.039	1.027	0.936	0.936
$\gamma_{3,3}$	-0.785	-0.803	-0.018	0.063	0.056	0.073	0.010	0.103	0.045	1.024	0.936	0.936

Table 7.63: Scenario (I1a): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.385	0.382	-0.003	0.039	0.028	0.047	0.002	0.065	0.019	1.000	0.944	0.944
$\beta_{2,1}$	0.646	0.640	-0.007	0.048	0.035	0.056	0.003	0.078	0.024	1.000	0.944	0.944
$\beta_{3,1}$	-0.796	-0.790	0.006	0.022	0.017	0.030	0.001	0.039	0.011	1.000	0.952	0.952
$\beta_{4,1}$	-0.344	-0.340	0.004	0.041	0.030	0.053	0.002	0.070	0.020	1.000	0.968	0.968
$\beta_{5,1}$	0.488	0.488	-0.001	0.047	0.037	0.052	0.002	0.074	0.027	1.000	0.912	0.912
$\beta_{6,1}$	-0.748	-0.749	-0.002	0.027	0.021	0.031	0.001	0.044	0.015	1.000	0.928	0.928
$\beta_{7,1}$	0.712	0.707	-0.004	0.043	0.034	0.057	0.003	0.075	0.023	1.000	0.976	0.968
$\beta_{8,1}$	0.732	0.729	-0.003	0.041	0.033	0.051	0.002	0.069	0.022	1.000	0.952	0.952
$\beta_{9,1}$	0.634	0.628	-0.006	0.024	0.018	0.032	0.001	0.042	0.011	1.034	0.952	0.952

Table 7.64: Scenario (I1a): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.108	0.003	0.041	0.029	0.058	0.003	0.074	0.019	1.000	0.992	0.992
$r_{1,3}$	-0.128	-0.124	0.003	0.050	0.037	0.061	0.004	0.084	0.025	1.000	0.960	0.960
$r_{2,3}$	0.199	0.194	-0.005	0.046	0.035	0.057	0.004	0.078	0.025	1.000	0.952	0.952

Table 7.65: Scenario (I1a): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.757	0.007	0.042	0.035	0.057	0.003	0.074	0.025	1.000	0.928	0.928
$\xi_{2,3}$	1.250	1.257	0.007	0.062	0.047	0.080	0.006	0.108	0.032	1.000	0.944	0.944
$\xi_{5,2}$	0.750	0.751	0.001	0.044	0.035	0.052	0.003	0.073	0.024	1.000	0.928	0.912
$\xi_{5,3}$	1.250	1.253	0.003	0.057	0.044	0.071	0.004	0.096	0.032	1.000	0.952	0.952
$\xi_{8,2}$	0.750	0.757	0.007	0.045	0.033	0.054	0.003	0.074	0.022	1.000	0.952	0.960
$\xi_{8,3}$	1.250	1.260	0.010	0.055	0.047	0.077	0.004	0.101	0.032	1.000	0.960	0.960

Table 7.66: Scenario (I1a): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.533	0.002	0.047	0.036	0.060	0.005	0.081	0.024	1.000	0.944	0.944
$\sigma^2_{\varepsilon,6}$	0.490	0.500	0.010	0.045	0.036	0.057	0.003	0.077	0.025	1.018	0.944	0.920
$\sigma^2_{\varepsilon,9}$	0.528	0.534	0.006	0.061	0.048	0.074	0.004	0.102	0.034	1.000	0.952	0.952

Table 7.67: Scenario (I1b): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.502	0.004	0.056	0.041	0.070	0.006	0.094	0.028	1.000	0.952	0.936
$\alpha_{2,1}$	0.674	0.656	-0.018	0.059	0.045	0.081	0.009	0.106	0.030	1.028	0.952	0.960
$\alpha_{3,1}$	0.601	0.595	-0.006	0.037	0.027	0.048	0.003	0.065	0.018	1.000	0.952	0.952
$\alpha_{4,2}$	-0.683	-0.668	0.015	0.058	0.045	0.069	0.007	0.096	0.029	1.014	0.888	0.896
$\alpha_{5,2}$	-0.603	-0.593	0.010	0.047	0.032	0.056	0.005	0.077	0.022	1.018	0.944	0.960
$\alpha_{6,2}$	0.700	0.690	-0.009	0.036	0.025	0.042	0.002	0.059	0.017	1.023	0.960	0.960
$\alpha_{7,3}$	0.540	0.528	-0.012	0.056	0.038	0.057	0.006	0.085	0.028	1.015	0.888	0.896
$\alpha_{8,3}$	-0.378	-0.376	0.002	0.037	0.030	0.042	0.003	0.059	0.022	1.000	0.912	0.920
$\alpha_{9,3}$	-0.709	-0.699	0.010	0.046	0.038	0.054	0.004	0.075	0.029	1.017	0.912	0.912

Table 7.68: Scenario (I1b): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.628	-0.624	0.004	0.059	0.042	0.073	0.006	0.099	0.029	1.000	0.936	0.944
$\gamma_{1,2}$	0.438	0.440	0.002	0.062	0.048	0.073	0.005	0.102	0.035	1.000	0.944	0.944
$\gamma_{1,3}$	0.338	0.349	0.012	0.051	0.043	0.065	0.004	0.088	0.031	1.015	0.968	0.968
$\gamma_{2,1}$	-0.422	-0.429	-0.007	0.053	0.040	0.062	0.004	0.087	0.027	1.000	0.936	0.936
$\gamma_{2,2}$	-0.631	-0.642	-0.011	0.052	0.039	0.073	0.005	0.094	0.027	1.000	0.984	0.976
$\gamma_{2,3}$	-0.408	-0.421	-0.012	0.049	0.042	0.062	0.004	0.084	0.029	1.016	0.952	0.952
$\gamma_{3,1}$	-0.564	-0.585	-0.021	0.070	0.060	0.076	0.012	0.109	0.049	1.022	0.904	0.896
$\gamma_{3,2}$	-0.654	-0.670	-0.016	0.074	0.061	0.085	0.013	0.119	0.048	1.011	0.952	0.928
$\gamma_{3,3}$	-0.785	-0.802	-0.017	0.080	0.065	0.087	0.016	0.125	0.055	1.010	0.928	0.920

Table 7.69: Scenario (I1b): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.385	0.377	-0.008	0.043	0.032	0.055	0.002	0.074	0.021	1.000	0.952	0.952
$\beta_{2,1}$	0.646	0.640	-0.006	0.054	0.043	0.066	0.005	0.090	0.030	1.000	0.904	0.904
$\beta_{3,1}$	-0.796	-0.791	0.005	0.027	0.020	0.035	0.001	0.047	0.013	1.000	0.976	0.968
$\beta_{4,1}$	-0.344	-0.346	-0.002	0.054	0.041	0.062	0.003	0.087	0.029	1.000	0.904	0.896
$\beta_{5,1}$	0.488	0.484	-0.004	0.048	0.036	0.061	0.003	0.082	0.024	1.000	0.960	0.960
$\beta_{6,1}$	-0.748	-0.746	0.002	0.033	0.025	0.037	0.001	0.053	0.018	0.976	0.912	0.912
$\beta_{7,1}$	0.712	0.703	-0.008	0.049	0.042	0.067	0.004	0.088	0.030	1.000	0.928	0.928
$\beta_{8,1}$	0.732	0.725	-0.008	0.054	0.040	0.059	0.003	0.085	0.029	1.015	0.904	0.904
$\beta_{9,1}$	0.634	0.624	-0.010	0.028	0.018	0.038	0.001	0.049	0.011	1.065	0.984	0.984

Table 7.70: Scenario (I1b): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.106	0.005	0.050	0.032	0.068	0.004	0.088	0.020	1.000	1.000	1.000
$r_{1,3}$	-0.128	-0.124	0.004	0.057	0.038	0.072	0.006	0.096	0.026	1.000	0.984	0.984
$r_{2,3}$	0.199	0.188	-0.011	0.052	0.040	0.068	0.006	0.090	0.028	1.000	0.960	0.960

Table 7.71: Scenario (I1b): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.756	0.006	0.052	0.042	0.066	0.005	0.089	0.031	1.000	0.944	0.944
$\xi_{2,3}$	1.250	1.258	0.008	0.073	0.052	0.095	0.008	0.126	0.034	1.000	0.960	0.960
$\xi_{5,2}$	0.750	0.749	-0.001	0.053	0.041	0.061	0.004	0.086	0.029	1.000	0.952	0.928
$\xi_{5,3}$	1.250	1.253	0.003	0.065	0.051	0.084	0.005	0.112	0.035	0.988	0.960	0.960
$\xi_{8,2}$	0.750	0.761	0.011	0.052	0.040	0.064	0.003	0.087	0.028	1.016	0.960	0.960
$\xi_{8,3}$	1.250	1.270	0.020	0.070	0.054	0.091	0.006	0.120	0.039	1.023	0.960	0.952

Table 7.72: Scenario (I1b): ξ estimates.

	ψ	$\bar{\bar{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.531	-0.001	0.044	0.035	0.053	0.004	0.073	0.024	1.000	0.928	0.928
$\sigma^2_{\varepsilon,6}$	0.490	0.497	0.007	0.043	0.032	0.049	0.003	0.069	0.022	1.000	0.936	0.928
$\sigma^2_{\varepsilon,9}$	0.528	0.535	0.007	0.052	0.038	0.064	0.004	0.087	0.026	1.000	0.928	0.928

Table 7.73: Scenario (I1c): σ^2_ε estimates.

	ψ	$\bar{\bar{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.505	0.000	0.048	0.034	0.060	0.005	0.081	0.022	1.000	0.976	0.976
$\alpha_{2,1}$	0.674	0.664	-0.010	0.055	0.042	0.071	0.008	0.095	0.029	1.015	0.936	0.928
$\alpha_{3,1}$	0.601	0.601	0.000	0.034	0.023	0.042	0.002	0.057	0.015	1.000	0.968	0.976
$\alpha_{4,2}$	-0.683	-0.673	0.010	0.053	0.036	0.059	0.005	0.084	0.024	1.000	0.944	0.952
$\alpha_{5,2}$	-0.603	-0.597	0.005	0.043	0.030	0.048	0.004	0.067	0.022	1.000	0.944	0.944
$\alpha_{6,2}$	0.700	0.693	-0.006	0.034	0.022	0.036	0.001	0.052	0.015	1.000	0.952	0.952
$\alpha_{7,3}$	0.540	0.531	-0.009	0.044	0.031	0.049	0.004	0.069	0.022	1.000	0.936	0.944
$\alpha_{8,3}$	-0.378	-0.376	0.002	0.030	0.024	0.036	0.002	0.050	0.017	1.000	0.912	0.928
$\alpha_{9,3}$	-0.709	-0.700	0.009	0.041	0.035	0.046	0.002	0.065	0.027	1.000	0.904	0.904

Table 7.74: Scenario (I1c): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.628	-0.604	0.025	0.061	0.043	0.064	0.004	0.093	0.030	1.057	0.888	0.896
$\gamma_{1,2}$	0.438	0.414	-0.025	0.056	0.043	0.063	0.003	0.090	0.031	1.061	0.928	0.928
$\gamma_{1,3}$	0.338	0.351	0.013	0.047	0.036	0.055	0.003	0.077	0.026	1.017	0.952	0.952
$\gamma_{2,1}$	-0.422	-0.421	0.000	0.047	0.034	0.055	0.003	0.076	0.024	1.000	0.936	0.936
$\gamma_{2,2}$	-0.631	-0.643	-0.012	0.048	0.041	0.064	0.004	0.086	0.028	1.016	0.944	0.952
$\gamma_{2,3}$	-0.408	-0.414	-0.006	0.042	0.032	0.053	0.003	0.071	0.023	1.000	0.952	0.944
$\gamma_{3,1}$	-0.564	-0.574	-0.010	0.063	0.048	0.065	0.007	0.096	0.037	1.013	0.880	0.856
$\gamma_{3,2}$	-0.654	-0.675	-0.021	0.062	0.051	0.074	0.008	0.102	0.040	1.038	0.936	0.944
$\gamma_{3,3}$	-0.785	-0.795	-0.010	0.061	0.054	0.073	0.010	0.101	0.042	1.000	0.944	0.936

Table 7.75: Scenario (I1c): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\|bias\|}$	$\hat{\sigma}(\|bias\|)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.385	0.382	-0.003	0.039	0.027	0.047	0.002	0.065	0.018	1.000	0.960	0.960
$\beta_{2,1}$	0.646	0.640	-0.007	0.049	0.035	0.056	0.004	0.079	0.025	1.000	0.952	0.952
$\beta_{3,1}$	-0.796	-0.790	0.006	0.021	0.017	0.030	0.001	0.039	0.011	1.000	0.960	0.960
$\beta_{4,1}$	-0.344	-0.339	0.005	0.042	0.031	0.053	0.002	0.071	0.021	1.000	0.960	0.960
$\beta_{5,1}$	0.488	0.488	0.000	0.047	0.036	0.052	0.002	0.074	0.027	1.000	0.904	0.904
$\beta_{6,1}$	-0.748	-0.750	-0.002	0.028	0.022	0.031	0.001	0.045	0.016	1.000	0.928	0.928
$\beta_{7,1}$	0.712	0.707	-0.005	0.043	0.034	0.057	0.003	0.075	0.023	1.000	0.976	0.976
$\beta_{8,1}$	0.732	0.729	-0.003	0.042	0.032	0.051	0.002	0.070	0.022	1.000	0.960	0.960
$\beta_{9,1}$	0.634	0.628	-0.006	0.024	0.017	0.032	0.001	0.042	0.011	1.034	0.952	0.952

Table 7.76: Scenario (I1c): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.106	0.004	0.043	0.030	0.058	0.003	0.076	0.019	1.000	0.992	0.992
$r_{1,3}$	-0.128	-0.122	0.006	0.050	0.037	0.061	0.004	0.084	0.025	1.000	0.960	0.960
$r_{2,3}$	0.199	0.199	0.000	0.044	0.035	0.058	0.004	0.077	0.025	0.982	0.952	0.952

Table 7.77: Scenario (I1c): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.757	0.007	0.042	0.034	0.057	0.004	0.075	0.025	1.000	0.936	0.944
$\xi_{2,3}$	1.250	1.256	0.006	0.062	0.046	0.081	0.006	0.108	0.031	1.000	0.968	0.944
$\xi_{5,2}$	0.750	0.751	0.001	0.045	0.035	0.052	0.003	0.073	0.025	0.982	0.920	0.928
$\xi_{5,3}$	1.250	1.254	0.004	0.058	0.044	0.071	0.004	0.097	0.031	0.986	0.952	0.952
$\xi_{8,2}$	0.750	0.757	0.007	0.045	0.032	0.054	0.002	0.074	0.022	1.000	0.952	0.960
$\xi_{8,3}$	1.250	1.259	0.009	0.055	0.047	0.077	0.004	0.101	0.032	1.000	0.960	0.944

Table 7.78: Scenario (I1c): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.532	0.001	0.050	0.036	0.060	0.005	0.082	0.025	1.000	0.944	0.928
$\sigma^2_{\varepsilon,6}$	0.490	0.501	0.011	0.047	0.037	0.056	0.003	0.078	0.026	1.000	0.928	0.928
$\sigma^2_{\varepsilon,9}$	0.528	0.535	0.007	0.059	0.048	0.073	0.005	0.100	0.034	1.000	0.928	0.936

Table 7.79: Scenario (I1d): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.500	0.006	0.057	0.041	0.069	0.007	0.094	0.028	1.000	0.936	0.952
$\alpha_{2,1}$	0.674	0.650	-0.024	0.062	0.044	0.081	0.009	0.108	0.027	1.056	0.936	0.936
$\alpha_{3,1}$	0.601	0.594	-0.007	0.040	0.030	0.048	0.003	0.067	0.020	1.000	0.944	0.936
$\alpha_{4,2}$	-0.683	-0.670	0.013	0.062	0.040	0.068	0.007	0.097	0.027	1.014	0.920	0.920
$\alpha_{5,2}$	-0.603	-0.600	0.002	0.047	0.037	0.057	0.005	0.078	0.027	1.000	0.936	0.920
$\alpha_{6,2}$	0.700	0.689	-0.011	0.037	0.026	0.042	0.002	0.059	0.018	1.023	0.928	0.928
$\alpha_{7,3}$	0.540	0.525	-0.015	0.045	0.037	0.056	0.004	0.077	0.026	1.035	0.928	0.920
$\alpha_{8,3}$	-0.378	-0.375	0.003	0.036	0.029	0.043	0.003	0.059	0.021	1.000	0.936	0.936
$\alpha_{9,3}$	-0.709	-0.695	0.014	0.049	0.044	0.053	0.003	0.077	0.034	1.031	0.912	0.912

Table 7.80: Scenario (I1d): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.628	-0.626	0.002	0.064	0.046	0.073	0.006	0.102	0.032	1.000	0.912	0.920
$\gamma_{1,2}$	0.438	0.445	0.007	0.060	0.044	0.072	0.005	0.099	0.030	1.000	0.952	0.952
$\gamma_{1,3}$	0.338	0.349	0.012	0.062	0.046	0.068	0.004	0.097	0.034	1.013	0.920	0.920
$\gamma_{2,1}$	-0.422	-0.432	-0.010	0.053	0.041	0.061	0.004	0.086	0.030	1.015	0.920	0.920
$\gamma_{2,2}$	-0.631	-0.643	-0.011	0.056	0.039	0.071	0.005	0.095	0.028	1.000	0.976	0.968
$\gamma_{2,3}$	-0.408	-0.416	-0.007	0.052	0.040	0.064	0.004	0.088	0.028	1.000	0.936	0.928
$\gamma_{3,1}$	-0.564	-0.586	-0.022	0.067	0.055	0.075	0.013	0.106	0.044	1.024	0.944	0.928
$\gamma_{3,2}$	-0.654	-0.680	-0.025	0.069	0.063	0.084	0.015	0.115	0.052	1.033	0.952	0.928
$\gamma_{3,3}$	-0.785	-0.810	-0.025	0.077	0.071	0.090	0.018	0.125	0.061	1.020	0.936	0.912

Table 7.81: Scenario (I1d): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.385	0.383	-0.002	0.044	0.034	0.055	0.003	0.075	0.023	1.000	0.944	0.944
$\beta_{2,1}$	0.646	0.639	-0.007	0.052	0.039	0.064	0.004	0.087	0.027	1.000	0.960	0.960
$\beta_{3,1}$	-0.796	-0.792	0.004	0.024	0.021	0.035	0.001	0.045	0.014	1.000	0.960	0.960
$\beta_{4,1}$	-0.344	-0.343	0.001	0.043	0.033	0.062	0.003	0.080	0.021	1.000	0.984	0.984
$\beta_{5,1}$	0.488	0.491	0.002	0.056	0.043	0.060	0.003	0.087	0.032	1.000	0.920	0.920
$\beta_{6,1}$	-0.748	-0.750	-0.002	0.031	0.023	0.037	0.001	0.051	0.016	1.000	0.920	0.920
$\beta_{7,1}$	0.712	0.704	-0.008	0.048	0.041	0.066	0.004	0.087	0.027	1.016	0.960	0.960
$\beta_{8,1}$	0.732	0.730	-0.002	0.047	0.036	0.058	0.003	0.078	0.026	1.000	0.928	0.928
$\beta_{9,1}$	0.634	0.624	-0.010	0.030	0.022	0.038	0.001	0.051	0.014	1.028	0.976	0.976

Table 7.82: Scenario (I1d): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.111	-0.001	0.046	0.033	0.068	0.004	0.086	0.020	0.982	1.000	1.000
$r_{1,3}$	-0.128	-0.127	0.000	0.056	0.042	0.071	0.006	0.096	0.028	1.000	0.968	0.968
$r_{2,3}$	0.199	0.193	-0.007	0.053	0.043	0.067	0.006	0.091	0.031	1.000	0.944	0.944

Table 7.83: Scenario (I1d): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.750	0.000	0.048	0.039	0.066	0.005	0.086	0.027	1.000	0.960	0.968
$\xi_{2,3}$	1.250	1.254	0.004	0.073	0.055	0.094	0.007	0.126	0.037	1.000	0.960	0.952
$\xi_{5,2}$	0.750	0.756	0.006	0.050	0.039	0.062	0.004	0.084	0.027	1.000	0.936	0.936
$\xi_{5,3}$	1.250	1.265	0.015	0.065	0.051	0.085	0.005	0.114	0.035	1.012	0.968	0.968
$\xi_{8,2}$	0.750	0.757	0.007	0.051	0.039	0.065	0.004	0.087	0.027	1.000	0.944	0.944
$\xi_{8,3}$	1.250	1.260	0.010	0.073	0.058	0.093	0.007	0.125	0.041	1.000	0.952	0.952

Table 7.84: Scenario (I1d): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,3}$	0.531	0.531	-0.001	0.045	0.033	0.053	0.004	0.074	0.023	1.000	0.920	0.912
$\sigma^2_{\varepsilon,6}$	0.490	0.498	0.008	0.041	0.031	0.049	0.003	0.068	0.022	1.020	0.952	0.936
$\sigma^2_{\varepsilon,9}$	0.528	0.535	0.007	0.051	0.040	0.064	0.004	0.086	0.027	1.000	0.944	0.944

Table 7.85: Scenario (I1e): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.506	-0.505	0.001	0.048	0.033	0.060	0.005	0.081	0.022	0.983	0.976	0.976
$\alpha_{2,1}$	0.674	0.665	-0.009	0.055	0.041	0.071	0.008	0.095	0.027	1.015	0.952	0.936
$\alpha_{3,1}$	0.601	0.602	0.001	0.035	0.025	0.042	0.002	0.058	0.016	0.977	0.952	0.960
$\alpha_{4,2}$	-0.683	-0.674	0.009	0.054	0.034	0.059	0.005	0.084	0.023	1.016	0.944	0.944
$\alpha_{5,2}$	-0.603	-0.598	0.004	0.042	0.029	0.048	0.004	0.067	0.021	1.000	0.944	0.944
$\alpha_{6,2}$	0.700	0.694	-0.005	0.033	0.023	0.036	0.001	0.052	0.016	1.000	0.936	0.936
$\alpha_{7,3}$	0.540	0.531	-0.009	0.043	0.032	0.049	0.004	0.069	0.023	1.000	0.936	0.928
$\alpha_{8,3}$	-0.378	-0.377	0.001	0.030	0.024	0.035	0.002	0.050	0.017	0.974	0.912	0.912
$\alpha_{9,3}$	-0.709	-0.701	0.008	0.041	0.036	0.045	0.002	0.065	0.027	1.000	0.912	0.912

Table 7.86: Scenario (I1e): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.628	-0.603	0.026	0.056	0.046	0.064	0.004	0.091	0.031	1.059	0.888	0.896
$\gamma_{1,2}$	0.438	0.414	-0.025	0.053	0.041	0.063	0.003	0.088	0.027	1.065	0.936	0.944
$\gamma_{1,3}$	0.338	0.350	0.012	0.048	0.037	0.055	0.003	0.077	0.027	1.017	0.960	0.960
$\gamma_{2,1}$	-0.422	-0.422	-0.000	0.048	0.034	0.055	0.003	0.077	0.024	1.000	0.936	0.928
$\gamma_{2,2}$	-0.631	-0.639	-0.007	0.050	0.038	0.064	0.004	0.086	0.026	1.000	0.968	0.960
$\gamma_{2,3}$	-0.408	-0.415	-0.007	0.043	0.033	0.053	0.003	0.072	0.023	1.000	0.952	0.944
$\gamma_{3,1}$	-0.564	-0.573	-0.009	0.060	0.044	0.065	0.007	0.094	0.032	1.000	0.928	0.936
$\gamma_{3,2}$	-0.654	-0.673	-0.019	0.062	0.051	0.073	0.008	0.102	0.040	1.025	0.936	0.936
$\gamma_{3,3}$	-0.785	-0.795	-0.010	0.062	0.056	0.072	0.009	0.102	0.044	1.000	0.912	0.912

Table 7.87: Scenario (I1e): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.385	0.382	-0.003	0.040	0.028	0.047	0.002	0.065	0.019	1.000	0.944	0.936
$\beta_{2,1}$	0.646	0.640	-0.006	0.048	0.035	0.056	0.004	0.078	0.024	1.000	0.920	0.936
$\beta_{3,1}$	-0.796	-0.790	0.007	0.022	0.017	0.030	0.001	0.039	0.011	1.037	0.944	0.944
$\beta_{4,1}$	-0.344	-0.339	0.005	0.042	0.031	0.053	0.002	0.071	0.021	1.000	0.968	0.968
$\beta_{5,1}$	0.488	0.489	0.001	0.047	0.037	0.052	0.002	0.075	0.027	1.000	0.912	0.912
$\beta_{6,1}$	-0.748	-0.750	-0.002	0.028	0.020	0.031	0.001	0.044	0.014	1.000	0.936	0.936
$\beta_{7,1}$	0.712	0.706	-0.005	0.044	0.034	0.057	0.003	0.076	0.023	1.000	0.960	0.968
$\beta_{8,1}$	0.732	0.730	-0.003	0.041	0.033	0.051	0.002	0.070	0.022	1.000	0.960	0.960
$\beta_{9,1}$	0.634	0.629	-0.005	0.024	0.018	0.032	0.001	0.043	0.012	1.000	0.960	0.960

Table 7.88: Scenario (I1e): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.106	0.005	0.041	0.032	0.058	0.003	0.075	0.020	1.000	0.976	0.984
$r_{1,3}$	-0.128	-0.121	0.007	0.052	0.037	0.061	0.004	0.085	0.025	1.000	0.944	0.952
$r_{2,3}$	0.199	0.202	0.002	0.047	0.035	0.058	0.004	0.078	0.025	0.983	0.944	0.944

Table 7.89: Scenario (I1e): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{2,2}$	0.750	0.757	0.007	0.042	0.035	0.057	0.004	0.075	0.025	1.000	0.936	0.944
$\xi_{2,3}$	1.250	1.257	0.007	0.062	0.046	0.081	0.006	0.108	0.031	1.000	0.952	0.944
$\xi_{5,2}$	0.750	0.751	0.001	0.044	0.034	0.052	0.003	0.073	0.024	1.000	0.928	0.912
$\xi_{5,3}$	1.250	1.253	0.003	0.057	0.045	0.071	0.004	0.096	0.032	0.986	0.952	0.952
$\xi_{8,2}$	0.750	0.757	0.007	0.045	0.032	0.054	0.003	0.074	0.022	1.000	0.952	0.952
$\xi_{8,3}$	1.250	1.260	0.010	0.055	0.047	0.077	0.004	0.101	0.032	1.014	0.952	0.952

Table 7.90: Scenario (I1e): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,1}$	0.385	0.386	0.001	0.024	0.020	0.033	0.001	0.043	0.013	1.000	0.952	0.960
$\sigma^2_{\varepsilon,4}$	0.234	0.268	0.034	0.043	0.037	0.052	0.004	0.072	0.027	1.239	0.928	0.928
$\sigma^2_{\varepsilon,7}$	0.687	0.695	0.007	0.041	0.031	0.053	0.004	0.071	0.021	1.000	0.960	0.960

Table 7.91: Scenario (I2a): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.683	-0.680	0.003	0.018	0.014	0.027	0.001	0.034	0.009	1.045	0.968	0.968
$\alpha_{2,1}$	0.603	0.596	-0.006	0.036	0.030	0.047	0.004	0.063	0.021	1.000	0.928	0.928
$\alpha_{3,1}$	0.700	0.693	-0.006	0.035	0.026	0.047	0.003	0.062	0.017	1.000	0.960	0.968
$\alpha_{5,1}$	-0.540	-0.534	0.007	0.034	0.026	0.048	0.004	0.062	0.017	1.000	0.984	0.992
$\alpha_{8,1}$	-0.378	-0.379	-0.001	0.035	0.026	0.043	0.006	0.059	0.019	1.000	0.952	0.936
$\alpha_{9,1}$	0.709	0.702	-0.007	0.046	0.036	0.061	0.006	0.081	0.026	1.000	0.936	0.936
$\alpha_{4,2}$	0.754	0.731	-0.024	0.036	0.028	0.040	0.003	0.056	0.021	1.184	0.896	0.896
$\alpha_{5,2}$	-0.608	-0.610	-0.002	0.048	0.036	0.064	0.005	0.084	0.024	1.000	0.976	0.960
$\alpha_{6,2}$	-0.455	-0.452	0.002	0.035	0.024	0.044	0.002	0.059	0.016	1.000	0.960	0.960
$\alpha_{7,3}$	-0.577	-0.568	0.009	0.037	0.026	0.042	0.002	0.059	0.018	1.000	0.936	0.920
$\alpha_{8,3}$	0.773	0.768	-0.004	0.071	0.057	0.088	0.015	0.120	0.042	1.000	0.928	0.920
$\alpha_{9,3}$	0.499	0.497	-0.003	0.061	0.044	0.076	0.007	0.103	0.030	1.000	0.976	0.968

Table 7.92: Scenario (I2a): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.564	-0.565	-0.001	0.039	0.034	0.051	0.002	0.069	0.024	1.000	0.936	0.936
$\gamma_{1,2}$	-0.654	-0.661	-0.006	0.043	0.034	0.056	0.002	0.075	0.023	1.019	0.952	0.952
$\gamma_{1,3}$	-0.785	-0.787	-0.002	0.043	0.035	0.054	0.002	0.074	0.024	1.000	0.960	0.944
$\gamma_{2,1}$	0.668	0.692	0.024	0.047	0.040	0.057	0.005	0.079	0.030	1.088	0.920	0.912
$\gamma_{2,2}$	-0.554	-0.575	-0.021	0.046	0.042	0.057	0.005	0.078	0.031	1.069	0.912	0.912
$\gamma_{2,3}$	-0.505	-0.527	-0.023	0.045	0.036	0.052	0.004	0.073	0.026	1.075	0.968	0.968
$\gamma_{3,1}$	-0.560	-0.573	-0.014	0.051	0.036	0.062	0.004	0.085	0.025	1.016	0.944	0.944
$\gamma_{3,2}$	0.554	0.560	0.006	0.053	0.037	0.067	0.004	0.090	0.025	1.000	0.960	0.968
$\gamma_{3,3}$	-0.432	-0.442	-0.010	0.052	0.035	0.060	0.004	0.083	0.024	1.016	0.984	0.960

Table 7.93: Scenario (I2a): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.712	0.709	-0.002	0.024	0.017	0.029	0.001	0.040	0.012	1.000	0.984	0.976
$\beta_{2,1}$	0.732	0.722	-0.011	0.046	0.037	0.059	0.003	0.079	0.025	1.017	0.952	0.960
$\beta_{3,1}$	0.634	0.631	-0.003	0.046	0.033	0.060	0.002	0.080	0.022	1.000	0.968	0.968
$\beta_{4,1}$	-0.579	-0.583	-0.004	0.024	0.019	0.028	0.001	0.039	0.013	1.000	0.928	0.928
$\beta_{5,1}$	0.713	0.712	-0.001	0.059	0.043	0.067	0.004	0.094	0.031	1.000	0.944	0.928
$\beta_{6,1}$	0.755	0.756	0.001	0.041	0.031	0.050	0.002	0.069	0.020	1.000	0.968	0.968
$\beta_{7,1}$	-0.406	-0.409	-0.003	0.025	0.019	0.032	0.001	0.043	0.013	1.000	0.944	0.944
$\beta_{8,1}$	-0.474	-0.465	0.008	0.048	0.036	0.057	0.004	0.079	0.025	1.000	0.936	0.936
$\beta_{9,1}$	0.488	0.476	-0.012	0.049	0.039	0.064	0.004	0.085	0.027	1.016	0.928	0.936

Table 7.94: Scenario (I2a): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.108	0.003	0.034	0.028	0.047	0.002	0.061	0.019	1.000	0.936	0.936
$r_{1,3}$	-0.128	-0.124	0.003	0.046	0.034	0.060	0.003	0.080	0.022	1.000	0.952	0.952
$r_{2,3}$	0.199	0.194	-0.005	0.039	0.032	0.055	0.003	0.071	0.022	1.000	0.944	0.944

Table 7.95: Scenario (I2a): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{3,2}$	0.750	0.749	-0.001	0.051	0.032	0.057	0.003	0.080	0.022	1.000	0.952	0.952
$\xi_{3,3}$	1.250	1.257	0.007	0.061	0.045	0.075	0.003	0.103	0.030	1.000	0.952	0.936
$\xi_{6,2}$	0.750	0.748	-0.002	0.042	0.034	0.055	0.003	0.073	0.023	1.000	0.944	0.936
$\xi_{6,3}$	1.250	1.259	0.009	0.064	0.046	0.080	0.005	0.108	0.032	1.000	0.960	0.960
$\xi_{8,2}$	0.750	0.746	-0.004	0.045	0.034	0.056	0.006	0.076	0.023	1.000	0.960	0.944
$\xi_{8,3}$	1.250	1.246	-0.004	0.065	0.052	0.078	0.011	0.108	0.037	1.000	0.920	0.920

Table 7.96: Scenario (I2a): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,1}$	0.385	0.387	0.003	0.027	0.020	0.038	0.002	0.049	0.013	1.000	0.984	0.976
$\sigma^2_{\varepsilon,4}$	0.234	0.285	0.051	0.057	0.044	0.059	0.006	0.087	0.034	1.412	0.928	0.928
$\sigma^2_{\varepsilon,7}$	0.687	0.695	0.007	0.048	0.035	0.062	0.005	0.083	0.025	1.000	0.936	0.936

Table 7.97: Scenario (I2b): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.683	-0.679	0.004	0.023	0.017	0.031	0.001	0.041	0.011	1.000	0.976	0.976
$\alpha_{2,1}$	0.603	0.596	-0.007	0.042	0.030	0.054	0.004	0.072	0.020	1.000	0.976	0.976
$\alpha_{3,1}$	0.700	0.693	-0.006	0.040	0.033	0.054	0.004	0.072	0.022	1.000	0.952	0.952
$\alpha_{5,1}$	-0.540	-0.534	0.007	0.042	0.030	0.058	0.005	0.075	0.019	1.000	0.952	0.992
$\alpha_{8,1}$	-0.378	-0.380	-0.002	0.040	0.028	0.051	0.008	0.068	0.020	0.980	0.944	0.960
$\alpha_{9,1}$	0.709	0.698	-0.011	0.050	0.040	0.070	0.007	0.092	0.026	1.016	0.952	0.944
$\alpha_{4,2}$	0.754	0.718	-0.036	0.044	0.032	0.046	0.004	0.067	0.024	1.341	0.896	0.896
$\alpha_{5,2}$	-0.608	-0.608	-0.001	0.056	0.044	0.076	0.007	0.100	0.031	1.000	0.960	0.960
$\alpha_{6,2}$	-0.455	-0.453	0.001	0.043	0.030	0.053	0.003	0.072	0.020	0.981	0.960	0.960
$\alpha_{7,3}$	-0.577	-0.566	0.011	0.043	0.029	0.049	0.004	0.069	0.019	1.020	0.928	0.944
$\alpha_{8,3}$	0.773	0.769	-0.003	0.082	0.070	0.103	0.020	0.140	0.054	1.000	0.936	0.920
$\alpha_{9,3}$	0.499	0.499	-0.001	0.067	0.049	0.087	0.009	0.116	0.034	1.000	0.968	0.952

Table 7.98: Scenario (I2b): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.564	-0.562	0.002	0.047	0.036	0.058	0.003	0.079	0.025	1.000	0.952	0.952
$\gamma_{1,2}$	-0.654	-0.662	-0.007	0.049	0.038	0.065	0.003	0.086	0.025	1.000	0.960	0.960
$\gamma_{1,3}$	-0.785	-0.787	-0.002	0.054	0.048	0.066	0.003	0.091	0.036	1.000	0.936	0.928
$\gamma_{2,1}$	0.668	0.701	0.034	0.056	0.046	0.067	0.007	0.092	0.035	1.125	0.944	0.928
$\gamma_{2,2}$	-0.554	-0.585	-0.031	0.056	0.054	0.068	0.006	0.095	0.041	1.099	0.912	0.904
$\gamma_{2,3}$	-0.505	-0.536	-0.031	0.056	0.039	0.065	0.005	0.090	0.028	1.115	0.984	0.984
$\gamma_{3,1}$	-0.560	-0.574	-0.014	0.062	0.045	0.073	0.006	0.100	0.032	1.013	0.960	0.944
$\gamma_{3,2}$	0.554	0.556	0.002	0.061	0.043	0.077	0.006	0.104	0.029	1.000	0.984	0.968
$\gamma_{3,3}$	-0.432	-0.441	-0.008	0.054	0.040	0.074	0.005	0.097	0.026	1.000	0.976	0.968

Table 7.99: Scenario (I2b): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.712	0.711	-0.001	0.028	0.020	0.034	0.001	0.047	0.013	1.000	0.952	0.944
$\beta_{2,1}$	0.732	0.722	-0.011	0.054	0.042	0.069	0.004	0.093	0.027	1.015	0.960	0.968
$\beta_{3,1}$	0.634	0.629	-0.005	0.053	0.038	0.069	0.003	0.092	0.026	1.000	0.968	0.960
$\beta_{4,1}$	-0.579	-0.582	-0.003	0.029	0.023	0.033	0.001	0.047	0.016	1.000	0.912	0.912
$\beta_{5,1}$	0.713	0.713	0.000	0.065	0.050	0.075	0.005	0.105	0.036	1.000	0.936	0.944
$\beta_{6,1}$	0.755	0.753	-0.002	0.047	0.038	0.058	0.002	0.079	0.026	1.000	0.936	0.936
$\beta_{7,1}$	-0.406	-0.412	-0.006	0.030	0.024	0.037	0.001	0.051	0.017	1.026	0.960	0.960
$\beta_{8,1}$	-0.474	-0.459	0.014	0.053	0.042	0.066	0.006	0.090	0.029	1.015	0.936	0.952
$\beta_{9,1}$	0.488	0.472	-0.016	0.056	0.043	0.074	0.005	0.098	0.028	1.014	0.944	0.944

Table 7.100: Scenario (I2b): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.109	0.002	0.043	0.030	0.055	0.003	0.073	0.020	1.000	0.968	0.968
$r_{1,3}$	-0.128	-0.125	0.003	0.055	0.036	0.069	0.004	0.092	0.023	1.000	0.968	0.968
$r_{2,3}$	0.199	0.199	-0.001	0.044	0.036	0.064	0.004	0.083	0.023	1.000	0.968	0.968

Table 7.101: Scenario (I2b): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{3,2}$	0.750	0.752	0.002	0.050	0.036	0.063	0.003	0.085	0.024	1.000	0.952	0.952
$\xi_{3,3}$	1.250	1.257	0.007	0.062	0.042	0.084	0.004	0.109	0.027	1.000	0.976	0.976
$\xi_{6,2}$	0.750	0.753	0.003	0.054	0.038	0.065	0.004	0.089	0.026	1.000	0.960	0.952
$\xi_{6,3}$	1.250	1.269	0.019	0.078	0.055	0.096	0.007	0.129	0.040	1.022	0.976	0.976
$\xi_{8,2}$	0.750	0.751	0.001	0.056	0.039	0.066	0.009	0.091	0.029	0.986	0.936	0.936
$\xi_{8,3}$	1.250	1.249	-0.001	0.080	0.058	0.091	0.015	0.128	0.043	1.000	0.920	0.920

Table 7.102: Scenario (I2b): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,1}$	0.385	0.385	0.001	0.024	0.020	0.034	0.001	0.044	0.013	1.000	0.960	0.960
$\sigma^2_{\varepsilon,4}$	0.234	0.272	0.038	0.046	0.036	0.054	0.004	0.075	0.026	1.289	0.944	0.936
$\sigma^2_{\varepsilon,7}$	0.687	0.694	0.006	0.041	0.033	0.056	0.005	0.074	0.021	1.000	0.944	0.944

Table 7.103: Scenario (I2c): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.683	-0.681	0.002	0.019	0.014	0.027	0.001	0.035	0.009	1.000	0.968	0.968
$\alpha_{2,1}$	0.603	0.598	-0.005	0.036	0.030	0.048	0.004	0.064	0.021	1.000	0.928	0.944
$\alpha_{3,1}$	0.700	0.695	-0.004	0.035	0.027	0.048	0.003	0.063	0.018	1.000	0.960	0.968
$\alpha_{5,1}$	-0.540	-0.535	0.006	0.034	0.025	0.048	0.004	0.062	0.016	1.000	0.992	0.992
$\alpha_{8,1}$	-0.378	-0.382	-0.004	0.037	0.027	0.045	0.007	0.061	0.021	1.000	0.952	0.960
$\alpha_{9,1}$	0.709	0.703	-0.005	0.045	0.038	0.062	0.006	0.081	0.027	1.000	0.952	0.944
$\alpha_{4,2}$	0.754	0.739	-0.015	0.032	0.025	0.041	0.003	0.055	0.017	1.079	0.952	0.952
$\alpha_{5,2}$	-0.608	-0.620	-0.012	0.050	0.039	0.066	0.006	0.087	0.028	1.016	0.976	0.968
$\alpha_{6,2}$	-0.455	-0.460	-0.005	0.035	0.027	0.045	0.002	0.060	0.018	1.000	0.952	0.944
$\alpha_{7,3}$	-0.577	-0.579	-0.002	0.037	0.026	0.044	0.003	0.060	0.018	1.000	0.944	0.944
$\alpha_{8,3}$	0.773	0.787	0.014	0.074	0.065	0.096	0.018	0.129	0.051	1.010	0.952	0.936
$\alpha_{9,3}$	0.499	0.508	0.009	0.064	0.043	0.079	0.008	0.106	0.030	1.013	0.984	0.976

Table 7.104: Scenario (I2c): α^Δ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.564	-0.562	0.002	0.042	0.034	0.053	0.002	0.072	0.024	1.000	0.928	0.928
$\gamma_{1,2}$	-0.654	-0.668	-0.013	0.044	0.036	0.058	0.002	0.078	0.025	1.018	0.968	0.944
$\gamma_{1,3}$	-0.785	-0.779	0.006	0.046	0.034	0.055	0.002	0.075	0.024	1.000	0.936	0.936
$\gamma_{2,1}$	0.668	0.654	-0.014	0.046	0.037	0.058	0.005	0.079	0.026	1.017	0.912	0.904
$\gamma_{2,2}$	-0.554	-0.529	0.026	0.052	0.037	0.059	0.004	0.082	0.027	1.103	0.912	0.936
$\gamma_{2,3}$	-0.505	-0.534	-0.030	0.049	0.036	0.053	0.004	0.076	0.027	1.151	0.936	0.928
$\gamma_{3,1}$	-0.560	-0.522	0.038	0.061	0.045	0.063	0.003	0.093	0.032	1.152	0.904	0.912
$\gamma_{3,2}$	0.554	0.505	-0.049	0.067	0.046	0.066	0.003	0.100	0.032	1.266	0.872	0.872
$\gamma_{3,3}$	-0.432	-0.423	0.010	0.051	0.036	0.060	0.003	0.083	0.024	1.000	0.936	0.936

Table 7.105: Scenario (I2c): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.712	0.710	-0.001	0.024	0.018	0.029	0.001	0.040	0.012	1.000	0.968	0.968
$\beta_{2,1}$	0.732	0.721	-0.011	0.046	0.037	0.059	0.003	0.080	0.026	1.017	0.944	0.952
$\beta_{3,1}$	0.634	0.631	-0.003	0.047	0.033	0.060	0.003	0.080	0.022	1.000	0.968	0.968
$\beta_{4,1}$	-0.579	-0.583	-0.005	0.024	0.019	0.029	0.001	0.040	0.013	1.033	0.912	0.912
$\beta_{5,1}$	0.713	0.716	0.003	0.058	0.044	0.067	0.004	0.094	0.032	1.000	0.920	0.920
$\beta_{6,1}$	0.755	0.757	0.001	0.041	0.031	0.051	0.002	0.069	0.021	1.000	0.960	0.960
$\beta_{7,1}$	-0.406	-0.409	-0.003	0.026	0.020	0.032	0.001	0.044	0.013	1.000	0.936	0.936
$\beta_{8,1}$	-0.474	-0.465	0.008	0.050	0.036	0.058	0.005	0.081	0.025	1.016	0.936	0.952
$\beta_{9,1}$	0.488	0.476	-0.012	0.048	0.038	0.064	0.004	0.085	0.026	1.016	0.936	0.936

Table 7.106: Scenario (I2c): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.105	0.005	0.036	0.028	0.047	0.002	0.063	0.018	1.000	0.944	0.944
$r_{1,3}$	-0.128	-0.117	0.010	0.045	0.035	0.059	0.003	0.079	0.022	1.018	0.960	0.960
$r_{2,3}$	0.199	0.154	-0.045	0.055	0.039	0.054	0.003	0.082	0.029	1.314	0.904	0.912

Table 7.107: Scenario (I2c): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{3,2}$	0.750	0.748	-0.002	0.051	0.032	0.057	0.003	0.080	0.022	1.000	0.944	0.952
$\xi_{3,3}$	1.250	1.256	0.006	0.060	0.045	0.075	0.003	0.102	0.030	1.000	0.944	0.944
$\xi_{6,2}$	0.750	0.750	-0.000	0.043	0.035	0.055	0.003	0.074	0.024	0.982	0.928	0.928
$\xi_{6,3}$	1.250	1.260	0.010	0.064	0.046	0.080	0.005	0.108	0.032	1.013	0.960	0.960
$\xi_{8,2}$	0.750	0.748	-0.002	0.046	0.035	0.059	0.008	0.079	0.024	1.000	0.960	0.928
$\xi_{8,3}$	1.250	1.250	-0.000	0.070	0.051	0.083	0.014	0.114	0.038	1.000	0.936	0.920

Table 7.108: Scenario (I2c): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,1}$	0.385	0.388	0.004	0.029	0.022	0.037	0.002	0.050	0.015	1.000	0.944	0.944
$\sigma^2_{\varepsilon,4}$	0.234	0.285	0.051	0.055	0.045	0.059	0.005	0.086	0.034	1.420	0.912	0.920
$\sigma^2_{\varepsilon,7}$	0.687	0.699	0.012	0.056	0.042	0.068	0.007	0.094	0.029	1.014	0.952	0.952

Table 7.109: Scenario (I3a): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.683	-0.679	0.004	0.021	0.016	0.031	0.001	0.039	0.010	1.000	0.984	0.984
$\alpha_{2,1}$	0.603	0.590	-0.013	0.046	0.032	0.053	0.004	0.073	0.022	1.037	0.952	0.936
$\alpha_{3,1}$	0.700	0.693	-0.006	0.039	0.031	0.053	0.004	0.070	0.021	1.000	0.960	0.960
$\alpha_{5,1}$	-0.540	-0.532	0.009	0.041	0.031	0.059	0.005	0.075	0.020	1.000	0.984	0.984
$\alpha_{8,1}$	0.709	0.695	-0.014	0.045	0.032	0.051	0.004	0.072	0.022	1.019	0.944	0.944
$\alpha_{9,1}$	0.709	0.695	-0.014	0.045	0.032	0.051	0.004	0.072	0.022	1.019	0.944	0.944
$\alpha_{4,2}$	0.754	0.720	-0.034	0.042	0.034	0.046	0.004	0.066	0.025	1.286	0.920	0.920
$\alpha_{5,2}$	-0.608	-0.610	-0.002	0.061	0.046	0.076	0.007	0.103	0.033	0.987	0.960	0.952
$\alpha_{6,2}$	-0.455	-0.455	-0.001	0.043	0.033	0.053	0.003	0.073	0.023	0.982	0.944	0.936
$\alpha_{7,3}$	-0.577	-0.561	0.016	0.048	0.034	0.055	0.004	0.077	0.023	1.035	0.928	0.944
$\alpha_{8,3}$	0.773	0.753	-0.020	0.076	0.060	0.092	0.010	0.127	0.041	1.021	0.928	0.920
$\alpha_{9,3}$	0.499	0.490	-0.009	0.059	0.041	0.080	0.006	0.105	0.026	1.000	0.976	0.976

Table 7.110: Scenario (I3a): α^Δ estimates with cross-loading and equality restriction.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.564	-0.568	-0.004	0.049	0.039	0.058	0.003	0.081	0.028	1.000	0.920	0.928
$\gamma_{1,2}$	-0.654	-0.658	-0.004	0.048	0.034	0.063	0.003	0.084	0.021	1.000	0.984	0.984
$\gamma_{1,3}$	-0.785	-0.790	-0.005	0.059	0.047	0.069	0.003	0.096	0.033	1.000	0.936	0.944
$\gamma_{2,1}$	0.668	0.703	0.035	0.059	0.047	0.066	0.007	0.094	0.037	1.119	0.936	0.920
$\gamma_{2,2}$	-0.554	-0.586	-0.032	0.057	0.048	0.067	0.006	0.093	0.037	1.088	0.928	0.904
$\gamma_{2,3}$	-0.505	-0.533	-0.028	0.063	0.044	0.068	0.006	0.097	0.033	1.069	0.952	0.944
$\gamma_{3,1}$	-0.560	-0.582	-0.022	0.062	0.049	0.079	0.008	0.106	0.036	1.039	0.968	0.936
$\gamma_{3,2}$	0.554	0.563	0.009	0.064	0.049	0.082	0.008	0.111	0.033	1.000	0.960	0.960
$\gamma_{3,3}$	-0.432	-0.460	-0.027	0.068	0.051	0.085	0.008	0.115	0.036	1.062	0.952	0.952

Table 7.111: Scenario (I3a): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.712	0.709	-0.003	0.031	0.022	0.035	0.001	0.049	0.016	1.000	0.928	0.928
$\beta_{2,1}$	0.732	0.717	-0.016	0.057	0.045	0.068	0.004	0.094	0.032	1.014	0.952	0.952
$\beta_{3,1}$	0.634	0.630	-0.004	0.054	0.042	0.069	0.003	0.093	0.029	1.000	0.960	0.936
$\beta_{4,1}$	-0.579	-0.584	-0.006	0.029	0.021	0.034	0.001	0.048	0.015	1.000	0.952	0.952
$\beta_{5,1}$	0.713	0.712	-0.001	0.064	0.047	0.072	0.005	0.102	0.034	0.988	0.936	0.944
$\beta_{6,1}$	0.755	0.753	-0.002	0.044	0.033	0.056	0.002	0.076	0.022	0.982	0.960	0.960
$\beta_{7,1}$	-0.406	-0.412	-0.006	0.033	0.023	0.038	0.001	0.053	0.016	1.026	0.944	0.944
$\beta_{8,1}$	-0.474	-0.463	0.011	0.057	0.040	0.070	0.004	0.095	0.026	1.015	0.968	0.968
$\beta_{9,1}$	0.488	0.472	-0.016	0.053	0.040	0.069	0.003	0.092	0.027	1.031	0.960	0.968

Table 7.112: Scenario (I3a): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.109	0.002	0.036	0.027	0.054	0.003	0.068	0.017	1.000	0.968	0.976
$r_{1,3}$	-0.128	-0.117	0.010	0.048	0.038	0.070	0.006	0.090	0.024	1.017	0.984	0.992
$r_{2,3}$	0.199	0.197	-0.003	0.048	0.036	0.066	0.005	0.086	0.024	1.000	0.968	0.968

Table 7.113: Scenario (I3a): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{3,2}$	0.750	0.750	-0.000	0.054	0.037	0.060	0.003	0.085	0.026	0.985	0.928	0.920
$\xi_{3,3}$	1.250	1.259	0.009	0.064	0.045	0.080	0.004	0.107	0.030	1.000	0.968	0.976
$\xi_{6,2}$	0.750	0.757	0.007	0.051	0.040	0.065	0.004	0.088	0.027	1.000	0.936	0.944
$\xi_{6,3}$	1.250	1.278	0.028	0.075	0.059	0.097	0.007	0.130	0.043	1.043	0.952	0.952
$\xi_{8,2}$	0.750	0.744	-0.006	0.050	0.038	0.065	0.005	0.087	0.026	1.000	0.920	0.936
$\xi_{8,3}$	1.250	1.250	0.000	0.059	0.046	0.084	0.007	0.109	0.030	1.000	0.976	0.968

Table 7.114: Scenario (I3a): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,1}$	0.385	0.390	0.005	0.026	0.019	0.034	0.001	0.045	0.012	1.000	0.976	0.976
$\sigma^2_{\varepsilon,4}$	0.234	0.267	0.033	0.043	0.036	0.054	0.005	0.073	0.026	1.244	0.944	0.944
$\sigma^2_{\varepsilon,7}$	0.687	0.697	0.010	0.050	0.040	0.065	0.006	0.087	0.028	1.000	0.960	0.952

Table 7.115: Scenario (I3b): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.683	-0.684	-0.001	0.019	0.015	0.028	0.001	0.036	0.009	1.000	0.984	0.992
$\alpha_{2,1}$	0.603	0.600	-0.002	0.038	0.030	0.048	0.004	0.064	0.021	1.000	0.952	0.960
$\alpha_{3,1}$	0.700	0.700	0.000	0.035	0.030	0.048	0.003	0.063	0.021	1.000	0.952	0.952
$\alpha_{5,1}$	-0.540	-0.537	0.003	0.033	0.026	0.049	0.004	0.062	0.016	1.000	0.992	0.984
$\alpha_{8,1}$	0.709	0.703	-0.005	0.036	0.028	0.046	0.003	0.061	0.020	1.022	0.944	0.928
$\alpha_{9,1}$	0.709	0.703	-0.005	0.036	0.028	0.046	0.003	0.061	0.020	1.022	0.944	0.928
$\alpha_{4,2}$	0.754	0.744	-0.011	0.030	0.024	0.040	0.003	0.053	0.016	1.027	0.976	0.976
$\alpha_{5,2}$	-0.608	-0.620	-0.012	0.050	0.035	0.066	0.005	0.086	0.025	1.017	0.968	0.968
$\alpha_{6,2}$	-0.455	-0.462	-0.007	0.035	0.026	0.045	0.002	0.060	0.018	1.023	0.960	0.960
$\alpha_{7,3}$	-0.577	-0.574	0.003	0.042	0.032	0.052	0.004	0.071	0.022	1.000	0.936	0.928
$\alpha_{8,3}$	0.773	0.774	0.001	0.068	0.054	0.087	0.009	0.117	0.039	1.000	0.944	0.952
$\alpha_{9,3}$	0.499	0.503	0.004	0.058	0.042	0.076	0.005	0.100	0.028	0.986	0.976	0.984

Table 7.116: Scenario (I3b): α^Δ estimates with cross-loading and equality restriction.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.564	-0.562	0.002	0.044	0.035	0.054	0.002	0.074	0.025	1.000	0.928	0.928
$\gamma_{1,2}$	-0.654	-0.653	0.001	0.049	0.034	0.061	0.002	0.082	0.022	1.000	0.952	0.960
$\gamma_{1,3}$	-0.785	-0.781	0.004	0.047	0.037	0.056	0.002	0.078	0.026	1.000	0.936	0.936
$\gamma_{2,1}$	0.668	0.648	-0.020	0.046	0.037	0.057	0.005	0.079	0.025	1.054	0.928	0.928
$\gamma_{2,2}$	-0.554	-0.521	0.033	0.052	0.040	0.060	0.004	0.085	0.027	1.158	0.896	0.904
$\gamma_{2,3}$	-0.505	-0.536	-0.032	0.048	0.037	0.053	0.003	0.076	0.027	1.173	0.944	0.928
$\gamma_{3,1}$	-0.560	-0.531	0.029	0.060	0.043	0.069	0.006	0.097	0.029	1.072	0.920	0.936
$\gamma_{3,2}$	0.554	0.502	-0.052	0.071	0.048	0.072	0.005	0.107	0.032	1.265	0.888	0.896
$\gamma_{3,3}$	-0.432	-0.423	0.010	0.054	0.036	0.065	0.005	0.089	0.024	1.016	0.944	0.952

Table 7.117: Scenario (I3b): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\|bias\|}$	$\hat{\sigma}(\|bias\|)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.712	0.710	-0.002	0.025	0.018	0.029	0.001	0.041	0.012	1.000	0.952	0.952
$\beta_{2,1}$	0.732	0.720	-0.012	0.046	0.036	0.059	0.003	0.080	0.025	1.017	0.944	0.936
$\beta_{3,1}$	0.634	0.632	-0.002	0.048	0.033	0.060	0.002	0.081	0.022	1.000	0.968	0.968
$\beta_{4,1}$	-0.579	-0.581	-0.002	0.024	0.018	0.029	0.001	0.040	0.013	0.968	0.928	0.928
$\beta_{5,1}$	0.713	0.713	-0.000	0.058	0.044	0.067	0.004	0.094	0.032	1.000	0.936	0.936
$\beta_{6,1}$	0.755	0.755	-0.000	0.040	0.031	0.051	0.002	0.069	0.020	0.980	0.968	0.968
$\beta_{7,1}$	-0.406	-0.407	-0.002	0.025	0.020	0.032	0.001	0.043	0.013	1.000	0.936	0.936
$\beta_{8,1}$	-0.474	-0.468	0.006	0.049	0.039	0.062	0.003	0.084	0.025	1.000	0.936	0.936
$\beta_{9,1}$	0.488	0.474	-0.014	0.046	0.039	0.061	0.002	0.082	0.026	1.017	0.944	0.944

Table 7.118: Scenario (I3b): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.101	0.010	0.034	0.025	0.048	0.002	0.061	0.016	1.024	0.976	0.976
$r_{1,3}$	-0.128	-0.120	0.008	0.046	0.036	0.062	0.004	0.081	0.024	1.000	0.976	0.976
$r_{2,3}$	0.199	0.151	-0.048	0.058	0.039	0.057	0.003	0.085	0.029	1.373	0.920	0.920

Table 7.119: Scenario (I3b): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{3,2}$	0.750	0.750	0.000	0.050	0.033	0.057	0.003	0.079	0.023	1.000	0.936	0.936
$\xi_{3,3}$	1.250	1.259	0.009	0.061	0.046	0.075	0.003	0.103	0.031	1.013	0.936	0.936
$\xi_{6,2}$	0.750	0.750	0.000	0.042	0.035	0.055	0.003	0.074	0.024	1.000	0.936	0.928
$\xi_{6,3}$	1.250	1.261	0.011	0.063	0.047	0.080	0.005	0.108	0.032	1.013	0.960	0.960
$\xi_{8,2}$	0.750	0.748	-0.002	0.044	0.036	0.060	0.004	0.079	0.024	1.000	0.960	0.952
$\xi_{8,3}$	1.250	1.255	0.005	0.056	0.045	0.078	0.006	0.102	0.031	1.000	0.960	0.960

Table 7.120: Scenario (I3b): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,1}$	0.385	0.392	0.007	0.046	0.036	0.057	0.004	0.078	0.025	1.000	0.944	0.944
$\sigma^2_{\varepsilon,4}$	0.234	0.317	0.083	0.086	0.056	0.080	0.010	0.122	0.047	1.729	0.896	0.888
$\sigma^2_{\varepsilon,7}$	0.687	0.707	0.020	0.089	0.062	0.106	0.017	0.146	0.044	1.009	0.928	0.920

Table 7.121: Scenario (I4a): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.683	-0.679	0.004	0.041	0.032	0.047	0.003	0.067	0.023	1.000	0.944	0.944
$\alpha_{2,1}$	0.603	0.597	-0.005	0.072	0.047	0.083	0.011	0.115	0.034	0.988	0.952	0.952
$\alpha_{3,1}$	0.700	0.688	-0.012	0.069	0.053	0.082	0.010	0.114	0.039	1.012	0.936	0.928
$\alpha_{5,1}$	-0.540	-0.542	-0.002	0.063	0.049	0.086	0.013	0.113	0.034	1.000	0.960	0.976
$\alpha_{8,1}$	0.709	0.683	-0.026	0.064	0.050	0.078	0.010	0.107	0.036	1.052	0.928	0.944
$\alpha_{9,1}$	0.709	0.683	-0.026	0.064	0.050	0.078	0.010	0.107	0.036	1.052	0.928	0.944
$\alpha_{4,2}$	0.754	0.701	-0.053	0.063	0.047	0.066	0.007	0.096	0.037	1.345	0.928	0.920
$\alpha_{5,2}$	-0.608	-0.600	0.008	0.080	0.067	0.111	0.015	0.146	0.048	1.000	0.976	0.976
$\alpha_{6,2}$	-0.455	-0.454	0.000	0.058	0.048	0.079	0.007	0.105	0.033	1.000	0.968	0.960
$\alpha_{7,3}$	-0.577	-0.536	0.041	0.076	0.063	0.089	0.012	0.125	0.044	1.101	0.912	0.920
$\alpha_{8,3}$	0.773	0.717	-0.055	0.131	0.091	0.144	0.026	0.207	0.063	1.067	0.872	0.896
$\alpha_{9,3}$	0.499	0.460	-0.039	0.111	0.079	0.123	0.017	0.176	0.057	1.038	0.888	0.896

Table 7.122: Scenario (I4a): α^Δ estimates with cross-loading and equality restriction.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.564	-0.568	-0.004	0.070	0.057	0.088	0.007	0.120	0.041	1.000	0.928	0.936
$\gamma_{1,2}$	-0.654	-0.663	-0.009	0.082	0.060	0.098	0.008	0.134	0.043	1.000	0.968	0.952
$\gamma_{1,3}$	-0.785	-0.799	-0.014	0.092	0.070	0.109	0.009	0.151	0.051	1.009	0.952	0.952
$\gamma_{2,1}$	0.668	0.735	0.068	0.096	0.082	0.104	0.018	0.150	0.066	1.189	0.928	0.880
$\gamma_{2,2}$	-0.554	-0.618	-0.064	0.095	0.076	0.106	0.016	0.150	0.062	1.163	0.944	0.936
$\gamma_{2,3}$	-0.505	-0.538	-0.033	0.082	0.066	0.109	0.013	0.143	0.050	1.050	0.976	0.944
$\gamma_{3,1}$	-0.560	-0.605	-0.045	0.119	0.128	0.157	0.217	0.207	0.243	1.030	0.976	0.928
$\gamma_{3,2}$	0.554	0.555	0.001	0.115	0.095	0.151	0.129	0.199	0.148	0.993	0.976	0.960
$\gamma_{3,3}$	-0.432	-0.447	-0.015	0.117	0.099	0.157	0.151	0.205	0.169	1.000	0.976	0.968

Table 7.123: Scenario (I4a): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\|bias\|}$	$\hat{\sigma}(\|bias\|)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.712	0.718	0.006	0.046	0.031	0.050	0.002	0.071	0.022	1.000	0.936	0.928
$\beta_{2,1}$	0.732	0.706	-0.026	0.084	0.062	0.098	0.008	0.137	0.042	1.030	0.904	0.904
$\beta_{3,1}$	0.634	0.618	-0.016	0.080	0.055	0.099	0.007	0.134	0.036	1.010	0.952	0.952
$\beta_{4,1}$	-0.579	-0.570	0.008	0.040	0.029	0.049	0.002	0.067	0.020	1.020	0.944	0.952
$\beta_{5,1}$	0.713	0.700	-0.013	0.086	0.063	0.108	0.010	0.145	0.045	1.000	0.960	0.968
$\beta_{6,1}$	0.755	0.734	-0.021	0.065	0.050	0.084	0.005	0.113	0.033	1.038	0.944	0.944
$\beta_{7,1}$	-0.406	-0.440	-0.034	0.056	0.035	0.054	0.002	0.081	0.026	1.158	0.896	0.904
$\beta_{8,1}$	-0.474	-0.417	0.057	0.086	0.073	0.101	0.008	0.143	0.051	1.153	0.912	0.920
$\beta_{9,1}$	0.488	0.479	-0.009	0.079	0.067	0.100	0.007	0.136	0.048	1.000	0.928	0.920

Table 7.124: Scenario (I4a): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.119	-0.009	0.064	0.040	0.083	0.006	0.109	0.026	1.000	0.984	0.984
$r_{1,3}$	-0.128	-0.111	0.016	0.074	0.043	0.113	0.018	0.139	0.030	1.012	0.992	0.992
$r_{2,3}$	0.199	0.196	-0.003	0.075	0.059	0.106	0.015	0.138	0.041	1.000	0.968	0.976

Table 7.125: Scenario (I4a): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{3,2}$	0.750	0.748	-0.002	0.078	0.054	0.093	0.007	0.128	0.037	1.000	0.944	0.936
$\xi_{3,3}$	1.250	1.261	0.011	0.104	0.075	0.124	0.010	0.171	0.053	1.000	0.968	0.968
$\xi_{6,2}$	0.750	0.767	0.017	0.081	0.061	0.096	0.010	0.132	0.045	1.010	0.936	0.936
$\xi_{6,3}$	1.250	1.293	0.043	0.110	0.087	0.141	0.015	0.188	0.066	1.045	0.952	0.944
$\xi_{8,2}$	0.750	0.740	-0.010	0.081	0.062	0.096	0.012	0.133	0.046	1.000	0.936	0.936
$\xi_{8,3}$	1.250	1.242	-0.008	0.102	0.089	0.126	0.019	0.173	0.068	1.000	0.928	0.928

Table 7.126: Scenario (I4a): ξ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\sigma^2_{\varepsilon,1}$	0.385	0.389	0.004	0.027	0.020	0.036	0.001	0.047	0.013	1.000	0.968	0.968
$\sigma^2_{\varepsilon,4}$	0.234	0.279	0.045	0.051	0.040	0.058	0.005	0.082	0.030	1.383	0.920	0.928
$\sigma^2_{\varepsilon,7}$	0.687	0.699	0.012	0.053	0.045	0.070	0.008	0.094	0.031	1.014	0.944	0.936

Table 7.127: Scenario (I4b): σ^2_ε estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\psi})}$	$\hat{\sigma}(\hat{\sigma}(\check{\psi}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\alpha_{1,1}$	-0.683	-0.683	0.000	0.021	0.016	0.029	0.001	0.037	0.010	1.000	0.968	0.968
$\alpha_{2,1}$	0.603	0.600	-0.002	0.039	0.034	0.050	0.004	0.068	0.024	1.000	0.928	0.928
$\alpha_{3,1}$	0.700	0.698	-0.001	0.039	0.032	0.051	0.004	0.068	0.023	1.000	0.952	0.952
$\alpha_{5,1}$	-0.540	-0.537	0.003	0.034	0.025	0.051	0.004	0.065	0.016	0.977	0.984	0.984
$\alpha_{8,1}$	0.709	0.703	-0.005	0.038	0.031	0.048	0.004	0.066	0.022	1.000	0.928	0.944
$\alpha_{9,1}$	0.709	0.703	-0.005	0.038	0.031	0.048	0.004	0.066	0.022	1.000	0.928	0.944
$\alpha_{4,2}$	0.754	0.734	-0.021	0.035	0.027	0.044	0.003	0.059	0.019	1.128	0.952	0.952
$\alpha_{5,2}$	-0.608	-0.612	-0.004	0.051	0.037	0.071	0.006	0.092	0.024	1.000	0.968	0.976
$\alpha_{6,2}$	-0.455	-0.459	-0.005	0.038	0.031	0.049	0.003	0.066	0.022	1.000	0.952	0.944
$\alpha_{7,3}$	-0.577	-0.573	0.004	0.045	0.036	0.056	0.005	0.076	0.026	1.000	0.936	0.936
$\alpha_{8,3}$	0.773	0.778	0.005	0.074	0.061	0.095	0.011	0.128	0.044	0.990	0.952	0.960
$\alpha_{9,3}$	0.499	0.503	0.004	0.065	0.046	0.082	0.006	0.110	0.031	1.000	0.968	0.968

Table 7.128: Scenario (I4b): α^Δ estimates with cross-loading and equality restriction.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\gamma_{1,1}$	-0.564	-0.559	0.005	0.044	0.035	0.054	0.002	0.074	0.025	1.000	0.952	0.952
$\gamma_{1,2}$	-0.654	-0.659	-0.004	0.045	0.033	0.059	0.003	0.079	0.022	1.000	0.968	0.960
$\gamma_{1,3}$	-0.785	-0.774	0.011	0.048	0.037	0.056	0.003	0.079	0.025	1.000	0.944	0.936
$\gamma_{2,1}$	0.668	0.660	-0.008	0.044	0.036	0.061	0.005	0.080	0.025	1.018	0.952	0.960
$\gamma_{2,2}$	-0.554	-0.540	0.014	0.047	0.041	0.062	0.005	0.083	0.029	1.016	0.936	0.944
$\gamma_{2,3}$	-0.505	-0.534	-0.030	0.049	0.035	0.056	0.004	0.078	0.026	1.132	0.944	0.936
$\gamma_{3,1}$	-0.560	-0.527	0.033	0.068	0.048	0.072	0.007	0.106	0.033	1.078	0.888	0.888
$\gamma_{3,2}$	0.554	0.493	-0.061	0.079	0.050	0.075	0.006	0.115	0.034	1.310	0.864	0.880
$\gamma_{3,3}$	-0.432	-0.425	0.007	0.056	0.039	0.069	0.006	0.093	0.028	1.000	0.952	0.952

Table 7.129: Scenario (I4b): γ estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias \rvert}$	$\hat{\sigma}(\lvert bias \rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\beta_{1,1}$	0.712	0.702	-0.010	0.027	0.019	0.029	0.001	0.042	0.013	1.065	0.904	0.904
$\beta_{2,1}$	0.732	0.725	-0.008	0.050	0.041	0.062	0.003	0.085	0.029	1.000	0.952	0.936
$\beta_{3,1}$	0.634	0.642	0.008	0.049	0.036	0.063	0.003	0.084	0.024	1.000	0.952	0.952
$\beta_{4,1}$	-0.579	-0.572	0.007	0.026	0.020	0.030	0.001	0.042	0.014	1.031	0.912	0.912
$\beta_{5,1}$	0.713	0.698	-0.015	0.061	0.044	0.070	0.005	0.098	0.031	1.014	0.952	0.960
$\beta_{6,1}$	0.755	0.748	-0.007	0.043	0.031	0.053	0.002	0.072	0.020	1.019	0.960	0.968
$\beta_{7,1}$	-0.406	-0.403	0.002	0.027	0.020	0.034	0.001	0.046	0.013	1.000	0.968	0.968
$\beta_{8,1}$	-0.474	-0.467	0.007	0.054	0.042	0.066	0.004	0.091	0.028	1.000	0.920	0.904
$\beta_{9,1}$	0.488	0.475	-0.014	0.050	0.041	0.064	0.003	0.086	0.028	1.016	0.952	0.952

Table 7.130: Scenario (I4b): β estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$r_{1,2}$	-0.111	-0.107	0.003	0.037	0.026	0.050	0.002	0.065	0.016	1.000	1.000	1.000
$r_{1,3}$	-0.128	-0.122	0.006	0.047	0.037	0.065	0.005	0.085	0.024	1.000	0.984	0.984
$r_{2,3}$	0.199	0.151	-0.049	0.059	0.041	0.063	0.004	0.091	0.029	1.365	0.952	0.952

Table 7.131: Scenario (I4b): R estimates.

	ψ	$\bar{\check{\psi}}$	$\overline{bias}$	$\overline{\lvert bias\rvert}$	$\hat{\sigma}(\lvert bias\rvert)$	$\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$	$\hat{\sigma}(\hat{\sigma}(\check{\boldsymbol{\psi}}))$	$\overline{RMSE}$	$\hat{\sigma}(RMSE)$	RRMSE	$Cover_{CI}$	$Cover_{Quant}$
$\xi_{3,2}$	0.750	0.749	-0.001	0.052	0.037	0.060	0.003	0.084	0.025	1.000	0.928	0.936
$\xi_{3,3}$	1.250	1.260	0.010	0.066	0.050	0.080	0.004	0.110	0.033	1.012	0.936	0.936
$\xi_{6,2}$	0.750	0.752	0.002	0.044	0.037	0.058	0.003	0.078	0.026	1.000	0.920	0.912
$\xi_{6,3}$	1.250	1.264	0.014	0.068	0.049	0.085	0.005	0.114	0.034	1.012	0.960	0.968
$\xi_{8,2}$	0.750	0.750	-0.000	0.050	0.040	0.064	0.005	0.086	0.027	1.000	0.952	0.960
$\xi_{8,3}$	1.250	1.256	0.006	0.064	0.054	0.084	0.008	0.112	0.038	1.000	0.936	0.936

Table 7.132: Scenario (I4b): ξ estimates.

Appendix E – Comparisions of Results of the Simulation Studies

Dedicated, CC CART BD, $\overline{\hat{\sigma}(\check{\psi})}$ differences

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\sigma^2_{\varepsilon,1}$	0.009	0.002	0.009	0.002
$\sigma^2_{\varepsilon,4}$	0.009	0.001	0.008	0.001
$\sigma^2_{\varepsilon,7}$	0.011	0.001	0.010	0.001

Table 7.133: CC CART comparison, dedicated IRT, w.r.t. BD, σ^2_ε, $\overline{\hat{\sigma}(\check{\psi})}$ differences.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\alpha_{1,1}$	0.011	0.001	0.010	0.001
$\alpha_{2,1}$	0.012	0.002	0.012	0.002
$\alpha_{3,1}$	0.007	0.001	0.007	0.001
$\alpha_{4,2}$	0.011	0.001	0.010	0.001
$\alpha_{5,2}$	0.009	0.001	0.010	0.001
$\alpha_{6,2}$	0.006	0.000	0.006	0.000
$\alpha_{7,3}$	0.008	0.000	0.007	0.000
$\alpha_{8,3}$	0.007	0.001	0.008	0.000
$\alpha_{9,3}$	0.009	0.001	0.008	0.000

Table 7.134: CC CART comparison, dedicated IRT, w.r.t. BD, α^Δ, $\overline{\hat{\sigma}(\check{\psi})}$ differences.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\gamma_{1,1}$	0.011	0.002	0.011	0.002
$\gamma_{1,2}$	0.011	0.001	0.010	0.001
$\gamma_{1,3}$	0.010	0.000	0.013	0.000
$\gamma_{2,1}$	0.009	0.002	0.008	0.002
$\gamma_{2,2}$	0.012	0.003	0.010	0.003
$\gamma_{2,3}$	0.010	0.001	0.012	0.001
$\gamma_{3,1}$	0.013	0.002	0.012	0.002
$\gamma_{3,2}$	0.014	0.003	0.013	0.002
$\gamma_{3,3}$	0.014	0.000	0.017	-0.001

Table 7.135: CC CART comparison, dedicated IRT, w.r.t. BD, γ, $\overline{\hat{\sigma}(\check{\psi})}$ differences.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\beta_{1,1}$	0.008	0.000	0.008	0.000
$\beta_{2,1}$	0.010	0.000	0.008	0.000
$\beta_{3,1}$	0.005	0.000	0.005	0.000
$\beta_{4,1}$	0.009	0.000	0.009	0.000
$\beta_{5,1}$	0.009	0.000	0.008	0.000
$\beta_{6,1}$	0.006	0.000	0.006	0.000
$\beta_{7,1}$	0.010	0.000	0.009	0.000
$\beta_{8,1}$	0.008	0.000	0.007	0.000
$\beta_{9,1}$	0.006	0.000	0.006	0.000

Table 7.136: CC CART comparison, dedicated IRT, w.r.t. BD, β, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$r_{1,2}$	0.010	0.000	0.010	0.000
$r_{1,3}$	0.011	0.000	0.010	0.000
$r_{2,3}$	0.011	0.001	0.010	0.001

Table 7.137: CC CART comparison, dedicated IRT, w.r.t. BD, R, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\xi_{2,2}$	0.009	0.000	0.009	0.000
$\xi_{2,3}$	0.015	0.001	0.014	0.001
$\xi_{5,2}$	0.009	0.000	0.010	0.000
$\xi_{5,3}$	0.013	0.000	0.014	0.000
$\xi_{8,2}$	0.010	0.000	0.011	0.000
$\xi_{8,3}$	0.014	0.000	0.016	0.000

Table 7.138: CC CART comparison, dedicated IRT, w.r.t. BD, ξ, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences.

Dedicated, CC CART BD, $\overline{\hat{\sigma}(\check{\psi})}$ fractions

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\sigma^2_{\varepsilon,1}$	1.176	1.039	1.176	1.039
$\sigma^2_{\varepsilon,4}$	1.188	1.021	1.167	1.021
$\sigma^2_{\varepsilon,7}$	1.175	1.016	1.159	1.016

Table 7.139: CC CART comparison, dedicated IRT, w.r.t. BD, σ^2_ε, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\alpha_{1,1}$	1.186	1.017	1.169	1.017
$\alpha_{2,1}$	1.174	1.029	1.174	1.029
$\alpha_{3,1}$	1.171	1.024	1.171	1.024
$\alpha_{4,2}$	1.190	1.017	1.172	1.017
$\alpha_{5,2}$	1.191	1.021	1.213	1.021
$\alpha_{6,2}$	1.167	1.000	1.167	1.000
$\alpha_{7,3}$	1.163	1.000	1.143	1.000
$\alpha_{8,3}$	1.200	1.029	1.229	1.000
$\alpha_{9,3}$	1.200	1.022	1.178	1.000

Table 7.140: CC CART comparison, dedicated IRT, w.r.t. BD, α^Δ, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\gamma_{1,1}$	1.177	1.032	1.177	1.032
$\gamma_{1,2}$	1.177	1.016	1.161	1.016
$\gamma_{1,3}$	1.182	1.000	1.236	1.000
$\gamma_{2,1}$	1.170	1.038	1.151	1.038
$\gamma_{2,2}$	1.197	1.049	1.164	1.049
$\gamma_{2,3}$	1.192	1.019	1.231	1.019
$\gamma_{3,1}$	1.206	1.032	1.190	1.032
$\gamma_{3,2}$	1.197	1.042	1.183	1.028
$\gamma_{3,3}$	1.192	1.000	1.233	0.986

Table 7.141: CC CART comparison, dedicated IRT, w.r.t. BD, γ, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\beta_{1,1}$	1.170	1.000	1.170	1.000
$\beta_{2,1}$	1.179	1.000	1.143	1.000
$\beta_{3,1}$	1.167	1.000	1.167	1.000
$\beta_{4,1}$	1.170	1.000	1.170	1.000
$\beta_{5,1}$	1.173	1.000	1.154	1.000
$\beta_{6,1}$	1.194	1.000	1.194	1.000
$\beta_{7,1}$	1.175	1.000	1.158	1.000
$\beta_{8,1}$	1.157	1.000	1.137	1.000
$\beta_{9,1}$	1.188	1.000	1.188	1.000

Table 7.142: CC CART comparison, dedicated IRT, w.r.t. BD, β, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$r_{1,2}$	1.172	1.000	1.172	1.000
$r_{1,3}$	1.180	1.000	1.164	1.000
$r_{2,3}$	1.193	1.018	1.175	1.018

Table 7.143: CC CART comparison, dedicated IRT, w.r.t. BD, R, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\xi_{2,2}$	1.158	1.000	1.158	1.000
$\xi_{2,3}$	1.188	1.012	1.175	1.012
$\xi_{5,2}$	1.173	1.000	1.192	1.000
$\xi_{5,3}$	1.183	1.000	1.197	1.000
$\xi_{8,2}$	1.185	1.000	1.204	1.000
$\xi_{8,3}$	1.182	1.000	1.208	1.000

Table 7.144: CC CART comparison, dedicated IRT, w.r.t. BD, ξ, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions.

Dedicated, CC CART BD, $\overline{RMSE}$ differences

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\sigma^2_{\varepsilon,1}$	0.011	0.003	0.012	0.004
$\sigma^2_{\varepsilon,4}$	0.010	0.002	0.011	0.001
$\sigma^2_{\varepsilon,7}$	0.017	0.002	0.015	0.001

Table 7.145: CC CART comparison, dedicated IRT, w.r.t. BD, σ^2_{ε}, $\overline{RMSE}$ differences.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\alpha_{1,1}$	0.015	0.002	0.015	0.002
$\alpha_{2,1}$	0.013	0.002	0.015	0.002
$\alpha_{3,1}$	0.009	0.001	0.011	0.002
$\alpha_{4,2}$	0.013	0.001	0.014	0.001
$\alpha_{5,2}$	0.010	0.000	0.011	0.000
$\alpha_{6,2}$	0.008	0.001	0.008	0.001
$\alpha_{7,3}$	0.016	0.000	0.008	0.000
$\alpha_{8,3}$	0.009	0.000	0.009	0.000
$\alpha_{9,3}$	0.011	0.001	0.013	0.001

Table 7.146: CC CART comparison, dedicated IRT, w.r.t. BD, α^{Δ}, $\overline{RMSE}$ differences.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\gamma_{1,1}$	0.011	0.005	0.014	0.003
$\gamma_{2,1}$	0.017	0.005	0.014	0.003
$\gamma_{3,1}$	0.011	0.000	0.020	0.000
$\gamma_{1,2}$	0.015	0.004	0.014	0.005
$\gamma_{2,2}$	0.015	0.007	0.016	0.007
$\gamma_{3,2}$	0.012	-0.001	0.016	0.000
$\gamma_{1,3}$	0.018	0.005	0.015	0.003
$\gamma_{2,3}$	0.021	0.004	0.017	0.004
$\gamma_{3,3}$	0.022	-0.002	0.022	-0.001

Table 7.147: CC CART comparison, dedicated IRT, w.r.t. BD, γ, $\overline{RMSE}$ differences.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\beta_{1,1}$	0.009	0.000	0.010	0.000
$\beta_{2,1}$	0.012	0.001	0.009	0.000
$\beta_{3,1}$	0.008	0.000	0.006	0.000
$\beta_{4,1}$	0.017	0.001	0.010	0.001
$\beta_{5,1}$	0.008	0.000	0.013	0.001
$\beta_{6,1}$	0.009	0.001	0.007	0.000
$\beta_{7,1}$	0.013	0.000	0.012	0.001
$\beta_{8,1}$	0.016	0.001	0.009	0.001
$\beta_{9,1}$	0.007	0.000	0.009	0.001

Table 7.148: CC CART comparison, dedicated IRT, w.r.t. BD, β, $\overline{RMSE}$ differences.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$r_{1,2}$	0.014	0.002	0.012	0.001
$r_{1,3}$	0.012	0.000	0.012	0.001
$r_{2,3}$	0.012	-0.001	0.013	0.000

Table 7.149: CC CART comparison, dedicated IRT, w.r.t. BD, R, $\overline{RMSE}$ differences.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\xi_{2,2}$	0.015	0.001	0.012	0.001
$\xi_{2,3}$	0.018	0.000	0.018	0.000
$\xi_{5,2}$	0.013	0.000	0.011	0.000
$\xi_{5,3}$	0.016	0.001	0.018	0.000
$\xi_{8,2}$	0.013	0.000	0.013	0.000
$\xi_{8,3}$	0.019	0.000	0.024	0.000

Table 7.150: CC CART comparison, dedicated IRT, w.r.t. BD, ξ, $\overline{RMSE}$ differences.

Dedicated, CC CART BD, $\overline{RMSE}$ fractions

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\sigma^2_{\varepsilon,1}$	1.157	1.043	1.171	1.057
$\sigma^2_{\varepsilon,4}$	1.149	1.030	1.164	1.015
$\sigma^2_{\varepsilon,7}$	1.200	1.024	1.176	1.012

Table 7.151: CC CART comparison, dedicated IRT, w.r.t. BD, σ^2_ε, $\overline{RMSE}$ fractions.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\alpha_{1,1}$	1.190	1.025	1.190	1.025
$\alpha_{2,1}$	1.140	1.022	1.161	1.022
$\alpha_{3,1}$	1.161	1.018	1.196	1.036
$\alpha_{4,2}$	1.157	1.012	1.169	1.012
$\alpha_{5,2}$	1.149	1.000	1.164	1.000
$\alpha_{6,2}$	1.157	1.020	1.157	1.020
$\alpha_{7,3}$	1.232	1.000	1.116	1.000
$\alpha_{8,3}$	1.180	1.000	1.180	1.000
$\alpha_{9,3}$	1.172	1.016	1.203	1.016

Table 7.152: CC CART comparison, dedicated IRT, w.r.t. BD, α^Δ, $\overline{RMSE}$ fractions.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\gamma_{1,1}$	1.125	1.057	1.159	1.034
$\gamma_{1,2}$	1.200	1.059	1.165	1.035
$\gamma_{1,3}$	1.143	1.000	1.260	1.000
$\gamma_{2,1}$	1.208	1.056	1.194	1.069
$\gamma_{2,2}$	1.190	1.089	1.203	1.089
$\gamma_{2,3}$	1.167	0.986	1.222	1.000
$\gamma_{3,1}$	1.198	1.055	1.165	1.033
$\gamma_{3,2}$	1.214	1.041	1.173	1.041
$\gamma_{3,3}$	1.214	0.981	1.214	0.990

Table 7.153: CC CART comparison, dedicated IRT, w.r.t. BD, γ, $\overline{RMSE}$ fractions.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\beta_{1,1}$	1.138	1.000	1.154	1.000
$\beta_{2,1}$	1.154	1.013	1.115	1.000
$\beta_{3,1}$	1.205	1.000	1.154	1.000
$\beta_{4,1}$	1.243	1.014	1.143	1.014
$\beta_{5,1}$	1.108	1.000	1.176	1.014
$\beta_{6,1}$	1.205	1.023	1.159	1.000
$\beta_{7,1}$	1.173	1.000	1.160	1.013
$\beta_{8,1}$	1.232	1.014	1.130	1.014
$\beta_{9,1}$	1.167	1.000	1.214	1.024

Table 7.154: CC CART comparison, dedicated IRT, w.r.t. BD, β, $\overline{RMSE}$ fractions.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$r_{1,2}$	1.189	1.027	1.162	1.014
$r_{1,3}$	1.143	1.000	1.143	1.012
$r_{2,3}$	1.154	0.987	1.167	1.000

Table 7.155: CC CART comparison, dedicated IRT, w.r.t. BD, R, $\overline{RMSE}$ fractions.

	(I1b)&(I1a)	(I1c)&(I1a)	(I1d)&(I1a)	(I1e)&(I1a)
$\xi_{2,2}$	1.203	1.014	1.162	1.014
$\xi_{2,3}$	1.167	1.000	1.167	1.000
$\xi_{5,2}$	1.178	1.000	1.151	1.000
$\xi_{5,3}$	1.167	1.010	1.188	1.000
$\xi_{8,2}$	1.176	1.000	1.176	1.000
$\xi_{8,3}$	1.188	1.000	1.238	1.000

Table 7.156: CC CART comparison, dedicated IRT, w.r.t. BD, ξ, $\overline{RMSE}$ fractions.

Dedicated, CC w/ CART, $\overline{\hat{\sigma}(\check{\psi})}$ and $\overline{RMSE}$ fractions

	(I1b)/(I1c) ($\overline{\hat{\sigma}(\check{\psi})}$)	(I1d)/(I1e) ($\overline{\hat{\sigma}(\check{\psi})}$)	(I1b)/(I1c) ($\overline{RMSE}$)	(I1d)/(I1e) ($\overline{RMSE}$)
$\sigma^2_{\varepsilon,3}$	1.132	1.132	1.110	1.108
$\sigma^2_{\varepsilon,6}$	1.163	1.143	1.116	1.147
$\sigma^2_{\varepsilon,9}$	1.156	1.141	1.172	1.163

Table 7.157: CC to CART comparison, dedicated IRT, σ^2_{ε}, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I1b)/(I1c) ($\overline{\hat{\sigma}(\check{\psi})}$)	(I1d)/(I1e) ($\overline{\hat{\sigma}(\check{\psi})}$)	(I1b)/(I1c) ($\overline{RMSE}$)	(I1d)/(I1e) ($\overline{RMSE}$)
$\alpha_{1,1}$	1.167	1.150	1.160	1.160
$\alpha_{2,1}$	1.141	1.141	1.116	1.137
$\alpha_{3,1}$	1.143	1.143	1.140	1.155
$\alpha_{4,2}$	1.169	1.153	1.143	1.155
$\alpha_{5,2}$	1.167	1.188	1.149	1.164
$\alpha_{6,2}$	1.167	1.167	1.135	1.135
$\alpha_{7,3}$	1.163	1.143	1.232	1.116
$\alpha_{8,3}$	1.167	1.229	1.180	1.180
$\alpha_{9,3}$	1.174	1.178	1.154	1.185

Table 7.158: CC to CART comparison, dedicated IRT, α^{Δ}, dedicated IRT, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I1b)/(I1c) ($\overline{\hat{\sigma}(\check{\psi})}$)	(I1d)/(I1e) ($\overline{\hat{\sigma}(\check{\psi})}$)	(I1b)/(I1c) ($\overline{RMSE}$)	(I1d)/(I1e) ($\overline{RMSE}$)
$\gamma_{1,1}$	1.141	1.141	1.065	1.121
$\gamma_{1,2}$	1.159	1.143	1.133	1.125
$\gamma_{1,3}$	1.182	1.236	1.143	1.260
$\gamma_{2,1}$	1.127	1.109	1.145	1.117
$\gamma_{2,2}$	1.141	1.109	1.093	1.105
$\gamma_{2,3}$	1.170	1.208	1.183	1.222
$\gamma_{3,1}$	1.169	1.154	1.135	1.128
$\gamma_{3,2}$	1.149	1.151	1.167	1.127
$\gamma_{3,3}$	1.192	1.250	1.238	1.225

Table 7.159: CC to CART comparison, dedicated IRT, γ, dedicated IRT, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I1b)/(I1c) $(\overline{\hat{\sigma}(\check{\psi})})$	(I1d)/(I1e) $(\overline{\hat{\sigma}(\check{\psi})})$	(I1b)/(I1c) $(\overline{RMSE})$	(I1d)/(I1e) $(\overline{RMSE})$
$\beta_{1,1}$	1.170	1.170	1.138	1.154
$\beta_{2,1}$	1.179	1.143	1.139	1.115
$\beta_{3,1}$	1.167	1.167	1.205	1.154
$\beta_{4,1}$	1.170	1.170	1.225	1.127
$\beta_{5,1}$	1.173	1.154	1.108	1.160
$\beta_{6,1}$	1.194	1.194	1.178	1.159
$\beta_{7,1}$	1.175	1.158	1.173	1.145
$\beta_{8,1}$	1.157	1.137	1.214	1.114
$\beta_{9,1}$	1.188	1.188	1.167	1.186

Table 7.160: CC to CART comparison, dedicated IRT, β, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I1b)/(I1c) $(\overline{\hat{\sigma}(\check{\psi})})$	(I1d)/(I1e) $(\overline{\hat{\sigma}(\check{\psi})})$	(I1b)/(I1c) $(\overline{RMSE})$	(I1d)/(I1e) $(\overline{RMSE})$
$r_{1,2}$	1.172	1.172	1.158	1.147
$r_{1,3}$	1.180	1.164	1.143	1.129
$r_{2,3}$	1.172	1.155	1.169	1.167

Table 7.161: CC to CART comparison, dedicated IRT, R, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I1b)/(I1c) $(\overline{\hat{\sigma}(\check{\psi})})$	(I1d)/(I1e) $(\overline{\hat{\sigma}(\check{\psi})})$	(I1b)/(I1c) $(\overline{RMSE})$	(I1d)/(I1e) $(\overline{RMSE})$
$\xi_{2,2}$	1.158	1.158	1.187	1.147
$\xi_{2,3}$	1.173	1.160	1.167	1.167
$\xi_{5,2}$	1.173	1.192	1.178	1.151
$\xi_{5,3}$	1.183	1.197	1.155	1.188
$\xi_{8,2}$	1.185	1.204	1.176	1.176
$\xi_{8,3}$	1.182	1.208	1.188	1.238

Table 7.162: CC to CART comparison, dedicated IRT, ξ, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

Cross-Loading, CC CART BD, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences, (I2b) – (I3b)

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\sigma^2_{\varepsilon,1}$	0.005	0.001	0.004	0.001
$\sigma^2_{\varepsilon,4}$	0.007	0.002	0.007	0.002
$\sigma^2_{\varepsilon,7}$	0.009	0.003	0.015	0.012

Table 7.163: CC CART comparison, cross-loading IRT, w.r.t. BD, σ^2_{ε}, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\alpha_{1,1}$	0.004	0.000	0.004	0.001
$\alpha_{2,1}$	0.007	0.001	0.006	0.001
$\alpha_{3,1}$	0.007	0.001	0.006	0.001
$\alpha_{5,1}$	0.010	0.000	0.011	0.001
$\alpha_{8,1}$	0.008	0.002	0.008	0.003
$\alpha_{9,1}$	0.009	0.001	0.008	0.003
$\alpha_{4,2}$	0.006	0.001	0.006	0.000
$\alpha_{5,2}$	0.012	0.002	0.012	0.002
$\alpha_{6,2}$	0.009	0.001	0.009	0.001
$\alpha_{7,3}$	0.007	0.002	0.013	0.010
$\alpha_{8,3}$	0.015	0.008	0.004	-0.001
$\alpha_{9,3}$	0.011	0.003	0.004	0.000

Table 7.164: CC CART comparison, cross-loading IRT, w.r.t. BD, α^{Δ}, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\gamma_{1,1}$	0.007	0.002	0.007	0.003
$\gamma_{1,2}$	0.009	0.002	0.007	0.005
$\gamma_{1,3}$	0.012	0.001	0.015	0.002
$\gamma_{2,1}$	0.010	0.001	0.009	0.000
$\gamma_{2,2}$	0.011	0.002	0.010	0.003
$\gamma_{2,3}$	0.013	0.001	0.016	0.001
$\gamma_{3,1}$	0.011	0.001	0.017	0.007
$\gamma_{3,2}$	0.010	-0.001	0.015	0.005
$\gamma_{3,3}$	0.014	0.000	0.025	0.005

Table 7.165: CC CART comparison, cross-loading IRT, w.r.t. BD, γ, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\beta_{1,1}$	0.005	0.000	0.006	0.000
$\beta_{2,1}$	0.010	0.000	0.009	0.000
$\beta_{3,1}$	0.009	0.000	0.009	0.000
$\beta_{4,1}$	0.005	0.001	0.006	0.001
$\beta_{5,1}$	0.008	0.000	0.005	0.000
$\beta_{6,1}$	0.008	0.001	0.006	0.001
$\beta_{7,1}$	0.005	0.000	0.006	0.000
$\beta_{8,1}$	0.009	0.001	0.013	0.005
$\beta_{9,1}$	0.010	0.000	0.005	-0.003

Table 7.166: CC CART comparison, cross-loading IRT, w.r.t. BD, β, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$r_{1,2}$	0.008	0.000	0.007	0.001
$r_{1,3}$	0.009	-0.001	0.010	0.002
$r_{2,3}$	0.009	-0.001	0.011	0.002

Table 7.167: CC CART comparison, cross-loading IRT, w.r.t. BD, R, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\xi_{3,2}$	0.006	0.000	0.003	0.000
$\xi_{3,3}$	0.009	0.000	0.005	0.000
$\xi_{6,2}$	0.010	0.000	0.010	0.000
$\xi_{6,3}$	0.016	0.000	0.017	0.000
$\xi_{8,2}$	0.010	0.003	0.009	0.004
$\xi_{8,3}$	0.013	0.005	0.006	0.000

Table 7.168: CC CART comparison, cross-loading IRT, w.r.t. BD, ξ, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences.

Cross-Loading, CC CART BD, $\overline{\hat{\sigma}(\check{\psi})}$ differences, (I4a) and (I4b)

	(I4a)&(I2a)	(I4b)&(I2a)
$\sigma^2_{\varepsilon,1}$	0.024	0.003
$\sigma^2_{\varepsilon,4}$	0.028	0.006
$\sigma^2_{\varepsilon,7}$	0.053	0.017

Table 7.169: CC CART comparison, cross-loading IRT, w.r.t. BD, σ^2_{ε}, $\overline{\hat{\sigma}(\check{\psi})}$ differences, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\alpha_{1,1}$	0.020	0.002
$\alpha_{2,1}$	0.036	0.003
$\alpha_{3,1}$	0.035	0.004
$\alpha_{5,1}$	0.038	0.003
$\alpha_{8,1}$	0.035	0.005
$\alpha_{9,1}$	0.017	-0.013
$\alpha_{4,2}$	0.026	0.004
$\alpha_{5,2}$	0.047	0.007
$\alpha_{6,2}$	0.035	0.005
$\alpha_{7,3}$	0.047	0.014
$\alpha_{8,3}$	0.056	0.007
$\alpha_{9,3}$	0.047	0.006

Table 7.170: CC CART comparison, cross-loading IRT, w.r.t. BD, α^{Δ}, $\overline{\hat{\sigma}(\check{\psi})}$ differences, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\gamma_{1,1}$	0.037	0.003
$\gamma_{1,2}$	0.042	0.003
$\gamma_{1,3}$	0.055	0.002
$\gamma_{2,1}$	0.047	0.004
$\gamma_{2,2}$	0.049	0.005
$\gamma_{2,3}$	0.057	0.004
$\gamma_{3,1}$	0.095	0.010
$\gamma_{3,2}$	0.084	0.008
$\gamma_{3,3}$	0.097	0.009

Table 7.171: CC CART comparison, cross-loading IRT, w.r.t. BD, γ, $\overline{\hat{\sigma}(\check{\psi})}$ differences.

	(I4a)&(I2a)	(I4b)&(I2a)
$\beta_{1,1}$	0.021	0.000
$\beta_{2,1}$	0.039	0.003
$\beta_{3,1}$	0.039	0.003
$\beta_{4,1}$	0.021	0.002
$\beta_{5,1}$	0.041	0.003
$\beta_{6,1}$	0.034	0.003
$\beta_{7,1}$	0.022	0.002
$\beta_{8,1}$	0.044	0.009
$\beta_{9,1}$	0.036	0.000

Table 7.172: CC CART comparison, cross-loading IRT, w.r.t. BD, β, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$r_{1,2}$	0.036	0.003
$r_{1,3}$	0.053	0.005
$r_{2,3}$	0.051	0.008

Table 7.173: CC CART comparison, cross-loading IRT, w.r.t. BD, R, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\xi_{3,2}$	0.036	0.003
$\xi_{3,3}$	0.049	0.005
$\xi_{6,2}$	0.041	0.003
$\xi_{6,3}$	0.061	0.005
$\xi_{8,2}$	0.040	0.008
$\xi_{8,3}$	0.048	0.006

Table 7.174: CC CART comparison, cross-loading IRT, w.r.t. BD, ξ, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ differences, (I4a) and (I4b).

Cross-Loading, CC CART BD, $\overline{\hat{\sigma}(\check{\psi})}$ fractions

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\sigma^2_{\varepsilon,1}$	1.152	1.030	1.121	1.030
$\sigma^2_{\varepsilon,4}$	1.135	1.038	1.135	1.038
$\sigma^2_{\varepsilon,7}$	1.170	1.057	1.283	1.226

Table 7.175: CC CART comparison, cross-loading IRT, w.r.t. BD, σ^2_ε, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\alpha_{1,1}$	1.148	1.000	1.148	1.037
$\alpha_{2,1}$	1.149	1.021	1.128	1.021
$\alpha_{3,1}$	1.149	1.021	1.128	1.021
$\alpha_{5,1}$	1.208	1.000	1.229	1.021
$\alpha_{8,1}$	1.186	1.047	1.186	1.070
$\alpha_{9,1}$	1.148	1.016	1.186	1.070
$\alpha_{4,2}$	1.150	1.025	1.150	1.000
$\alpha_{5,2}$	1.188	1.031	1.188	1.031
$\alpha_{6,2}$	1.205	1.023	1.205	1.023
$\alpha_{7,3}$	1.167	1.048	1.310	1.238
$\alpha_{8,3}$	1.170	1.091	1.045	0.989
$\alpha_{9,3}$	1.145	1.039	1.053	1.000

Table 7.176: CC CART comparison, cross-loading IRT, w.r.t. BD, α^Δ, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\gamma_{1,1}$	1.137	1.039	1.137	1.059
$\gamma_{1,2}$	1.161	1.036	1.125	1.089
$\gamma_{1,3}$	1.222	1.019	1.278	1.037
$\gamma_{2,1}$	1.175	1.018	1.158	1.000
$\gamma_{2,2}$	1.193	1.035	1.175	1.053
$\gamma_{2,3}$	1.250	1.019	1.308	1.019
$\gamma_{3,1}$	1.177	1.016	1.274	1.113
$\gamma_{3,2}$	1.149	0.985	1.224	1.075
$\gamma_{3,3}$	1.233	1.000	1.417	1.083

Table 7.177: CC CART comparison, cross-loading IRT, w.r.t. BD, γ, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\beta_{1,1}$	1.172	1.000	1.207	1.000
$\beta_{2,1}$	1.169	1.000	1.153	1.000
$\beta_{3,1}$	1.150	1.000	1.150	1.000
$\beta_{4,1}$	1.179	1.036	1.214	1.036
$\beta_{5,1}$	1.119	1.000	1.075	1.000
$\beta_{6,1}$	1.160	1.020	1.120	1.020
$\beta_{7,1}$	1.156	1.000	1.188	1.000
$\beta_{8,1}$	1.158	1.018	1.228	1.088
$\beta_{9,1}$	1.156	1.000	1.078	0.953

Table 7.178: CC CART comparison, cross-loading IRT, w.r.t. BD, β, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$r_{1,2}$	1.170	1.000	1.149	1.021
$r_{1,3}$	1.150	0.983	1.167	1.033
$r_{2,3}$	1.164	0.982	1.200	1.036

Table 7.179: CC CART comparison, cross-loading IRT, w.r.t. BD, R, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\xi_{3,2}$	1.105	1.000	1.053	1.000
$\xi_{3,3}$	1.120	1.000	1.067	1.000
$\xi_{6,2}$	1.182	1.000	1.182	1.000
$\xi_{6,3}$	1.200	1.000	1.212	1.000
$\xi_{8,2}$	1.179	1.054	1.161	1.071
$\xi_{8,3}$	1.167	1.064	1.077	1.000

Table 7.180: CC CART comparison, cross-loading IRT, w.r.t. BD, ξ, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions.

Cross-Loading, CC CART BD, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions, (I4a) and (I4b)

	(I4a)&(I2a)	(I4b)&(I2a)
$\sigma^2_{\varepsilon,1}$	1.727	1.091
$\sigma^2_{\varepsilon,4}$	1.538	1.115
$\sigma^2_{\varepsilon,7}$	2.000	1.321

Table 7.181: CC CART comparison, cross-loading IRT, w.r.t. BD, σ^2_{ε}, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\alpha_{1,1}$	1.741	1.074
$\alpha_{2,1}$	1.766	1.064
$\alpha_{3,1}$	1.745	1.085
$\alpha_{5,1}$	1.792	1.062
$\alpha_{8,1}$	1.814	1.116
$\alpha_{9,1}$	1.279	0.787
$\alpha_{4,2}$	1.650	1.100
$\alpha_{5,2}$	1.734	1.109
$\alpha_{6,2}$	1.795	1.114
$\alpha_{7,3}$	2.119	1.333
$\alpha_{8,3}$	1.636	1.080
$\alpha_{9,3}$	1.618	1.079

Table 7.182: CC CART comparison, cross-loading IRT, w.r.t. BD, α^{Δ}, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\gamma_{1,1}$	1.725	1.059
$\gamma_{1,2}$	1.750	1.054
$\gamma_{1,3}$	2.019	1.037
$\gamma_{2,1}$	1.825	1.070
$\gamma_{2,2}$	1.860	1.088
$\gamma_{2,3}$	2.096	1.077
$\gamma_{3,1}$	2.532	1.161
$\gamma_{3,2}$	2.254	1.119
$\gamma_{3,3}$	2.617	1.150

Table 7.183: CC CART comparison, cross-loading IRT, w.r.t. BD, γ, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\beta_{1,1}$	1.724	1.000
$\beta_{2,1}$	1.661	1.051
$\beta_{3,1}$	1.650	1.050
$\beta_{4,1}$	1.750	1.071
$\beta_{5,1}$	1.612	1.045
$\beta_{6,1}$	1.680	1.060
$\beta_{7,1}$	1.688	1.062
$\beta_{8,1}$	1.772	1.158
$\beta_{9,1}$	1.562	1.000

Table 7.184: CC CART comparison, cross-loading IRT, w.r.t. BD, β, $\overline{\hat{\sigma}(\check{\psi})}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$r_{1,2}$	1.766	1.064
$r_{1,3}$	1.883	1.083
$r_{2,3}$	1.927	1.145

Table 7.185: CC CART comparison, cross-loading IRT, w.r.t. BD, R, $\overline{\hat{\sigma}(\check{\psi})}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\xi_{3,2}$	1.632	1.053
$\xi_{3,3}$	1.653	1.067
$\xi_{6,2}$	1.745	1.055
$\xi_{6,3}$	1.762	1.062
$\xi_{8,2}$	1.714	1.143
$\xi_{8,3}$	1.615	1.077

Table 7.186: CC CART comparison, cross-loading IRT, w.r.t. BD, ξ, $\overline{\hat{\sigma}(\check{\psi})}$ fractions, (I4a) and (I4b).

Cross-Loading, CC CART BD, $\overline{RMSE}$ differences

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\sigma^2_{\varepsilon,1}$	0.006	0.001	0.007	0.002
$\sigma^2_{\varepsilon,4}$	0.015	0.003	0.014	0.001
$\sigma^2_{\varepsilon,7}$	0.012	0.003	0.023	0.016

Table 7.187: CC CART comparison, cross-loading IRT, w.r.t. BD, σ^2_ε, $\overline{RMSE}$ differences.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\alpha_{1,1}$	0.007	0.001	0.005	0.002
$\alpha_{2,1}$	0.009	0.001	0.010	0.001
$\alpha_{3,1}$	0.010	0.001	0.008	0.001
$\alpha_{5,1}$	0.013	0.000	0.013	0.000
$\alpha_{8,1}$	0.009	0.002	0.013	0.002
$\alpha_{9,1}$	0.011	0.000	0.013	0.002
$\alpha_{4,2}$	0.011	-0.001	0.010	-0.003
$\alpha_{5,2}$	0.016	0.003	0.019	0.002
$\alpha_{6,2}$	0.013	0.001	0.014	0.001
$\alpha_{7,3}$	0.010	0.001	0.018	0.012
$\alpha_{8,3}$	0.020	0.009	0.007	-0.003
$\alpha_{9,3}$	0.013	0.003	0.002	-0.003

Table 7.188: CC CART comparison, cross-loading IRT, w.r.t. BD, α^Δ, $\overline{RMSE}$ differences.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\gamma_{1,1}$	0.010	0.003	0.012	0.005
$\gamma_{2,1}$	0.011	0.003	0.009	0.007
$\gamma_{3,1}$	0.017	0.001	0.022	0.004
$\gamma_{1,2}$	0.013	0.000	0.015	0.000
$\gamma_{2,2}$	0.017	0.004	0.015	0.007
$\gamma_{3,2}$	0.017	0.003	0.024	0.003
$\gamma_{1,3}$	0.015	0.008	0.021	0.012
$\gamma_{2,3}$	0.014	0.010	0.021	0.017
$\gamma_{3,3}$	0.014	0.000	0.032	0.006

Table 7.189: CC CART comparison, cross-loading IRT, w.r.t. BD, γ, $\overline{RMSE}$ differences.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\beta_{1,1}$	0.007	0.000	0.009	0.001
$\beta_{2,1}$	0.014	0.001	0.015	0.001
$\beta_{3,1}$	0.012	0.000	0.013	0.001
$\beta_{4,1}$	0.008	0.001	0.009	0.001
$\beta_{5,1}$	0.011	0.000	0.008	0.000
$\beta_{6,1}$	0.010	0.000	0.007	0.000
$\beta_{7,1}$	0.008	0.001	0.010	0.000
$\beta_{8,1}$	0.011	0.002	0.016	0.005
$\beta_{9,1}$	0.013	0.000	0.007	-0.003

Table 7.190: CC CART comparison, cross-loading IRT, w.r.t. BD, β, $\overline{RMSE}$ differences.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$r_{1,2}$	0.012	0.002	0.007	0.000
$r_{1,3}$	0.012	-0.001	0.010	0.001
$r_{2,3}$	0.012	0.011	0.015	0.014

Table 7.191: CC CART comparison, cross-loading IRT, w.r.t. BD, R, $\overline{RMSE}$ differences.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\xi_{3,2}$	0.005	0.000	0.005	-0.001
$\xi_{3,3}$	0.006	-0.001	0.004	0.000
$\xi_{6,2}$	0.016	0.001	0.015	0.001
$\xi_{6,3}$	0.021	0.000	0.022	0.000
$\xi_{8,2}$	0.015	0.003	0.011	0.003
$\xi_{8,3}$	0.020	0.006	0.001	-0.006

Table 7.192: CC CART comparison, cross-loading IRT, w.r.t. BD, ξ, $\overline{RMSE}$ differences.

Cross-Loading, CC CART BD, $\overline{RMSE}$ differences, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\sigma^2_{\varepsilon,1}$	0.035	0.004
$\sigma^2_{\varepsilon,4}$	0.050	0.010
$\sigma^2_{\varepsilon,7}$	0.075	0.023

Table 7.193: CC CART comparison, cross-loading IRT, w.r.t. BD, σ^2_ε, $\overline{RMSE}$ differences, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\alpha_{1,1}$	0.033	0.003
$\alpha_{2,1}$	0.052	0.005
$\alpha_{3,1}$	0.052	0.006
$\alpha_{5,1}$	0.051	0.003
$\alpha_{8,1}$	0.048	0.007
$\alpha_{9,1}$	0.026	-0.015
$\alpha_{4,2}$	0.040	0.003
$\alpha_{5,2}$	0.062	0.008
$\alpha_{6,2}$	0.046	0.007
$\alpha_{7,3}$	0.066	0.017
$\alpha_{8,3}$	0.087	0.008
$\alpha_{9,3}$	0.073	0.007

Table 7.194: CC CART comparison, cross-loading IRT, w.r.t. BD, α^Δ, $\overline{RMSE}$ differences, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\gamma_{1,1}$	0.051	0.005
$\gamma_{2,1}$	0.059	0.004
$\gamma_{3,1}$	0.077	0.005
$\gamma_{1,2}$	0.071	0.001
$\gamma_{2,2}$	0.072	0.005
$\gamma_{3,2}$	0.070	0.005
$\gamma_{1,3}$	0.122	0.021
$\gamma_{2,3}$	0.109	0.025
$\gamma_{3,3}$	0.122	0.010

Table 7.195: CC CART comparison, cross-loading IRT, w.r.t. BD, γ, $\overline{RMSE}$ differences, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\beta_{1,1}$	0.031	0.002
$\beta_{2,1}$	0.058	0.006
$\beta_{3,1}$	0.054	0.004
$\beta_{4,1}$	0.028	0.003
$\beta_{5,1}$	0.051	0.004
$\beta_{6,1}$	0.044	0.003
$\beta_{7,1}$	0.038	0.003
$\beta_{8,1}$	0.064	0.012
$\beta_{9,1}$	0.051	0.001

Table 7.196: CC CART comparison, cross-loading IRT, w.r.t. BD, β, $\overline{RMSE}$ differences, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$r_{1,2}$	0.048	0.004
$r_{1,3}$	0.059	0.005
$r_{2,3}$	0.067	0.020

Table 7.197: CC CART comparison, cross-loading IRT, w.r.t. BD, R, $\overline{RMSE}$ differences, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\xi_{3,2}$	0.048	0.004
$\xi_{3,3}$	0.068	0.007
$\xi_{6,2}$	0.059	0.005
$\xi_{6,3}$	0.080	0.006
$\xi_{8,2}$	0.057	0.010
$\xi_{8,3}$	0.065	0.004

Table 7.198: CC CART comparison, cross-loading IRT, w.r.t. BD, ξ, $\overline{RMSE}$ differences, (I4a) and (I4b).

Cross-Loading, CC CART BD, $\overline{RMSE}$ fractions

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\sigma^2_{\varepsilon,1}$	1.140	1.023	1.163	1.047
$\sigma^2_{\varepsilon,4}$	1.208	1.042	1.194	1.014
$\sigma^2_{\varepsilon,7}$	1.169	1.042	1.324	1.225

Table 7.199: CC CART comparison, cross-loading IRT, w.r.t. BD, σ^2_ε, $\overline{RMSE}$ fractions.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\alpha_{1,1}$	1.206	1.029	1.147	1.059
$\alpha_{2,1}$	1.143	1.016	1.159	1.016
$\alpha_{3,1}$	1.161	1.016	1.129	1.016
$\alpha_{5,1}$	1.210	1.000	1.210	1.000
$\alpha_{8,1}$	1.153	1.034	1.220	1.034
$\alpha_{9,1}$	1.136	1.000	1.220	1.034
$\alpha_{4,2}$	1.196	0.982	1.179	0.946
$\alpha_{5,2}$	1.190	1.036	1.226	1.024
$\alpha_{6,2}$	1.220	1.017	1.237	1.017
$\alpha_{7,3}$	1.169	1.017	1.305	1.203
$\alpha_{8,3}$	1.167	1.075	1.058	0.975
$\alpha_{9,3}$	1.126	1.029	1.019	0.971

Table 7.200: CC CART comparison, cross-loading IRT, w.r.t. BD, α^Δ, $\overline{RMSE}$ fractions.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\gamma_{1,1}$	1.145	1.043	1.174	1.072
$\gamma_{1,2}$	1.147	1.040	1.120	1.093
$\gamma_{1,3}$	1.230	1.014	1.297	1.054
$\gamma_{2,1}$	1.165	1.000	1.190	1.000
$\gamma_{2,2}$	1.218	1.051	1.192	1.090
$\gamma_{2,3}$	1.233	1.041	1.329	1.041
$\gamma_{3,1}$	1.176	1.094	1.247	1.141
$\gamma_{3,2}$	1.156	1.111	1.233	1.189
$\gamma_{3,3}$	1.169	1.000	1.386	1.072

Table 7.201: CC CART comparison, cross-loading IRT, w.r.t. BD, γ, $\overline{RMSE}$ fractions.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\beta_{1,1}$	1.175	1.000	1.225	1.025
$\beta_{2,1}$	1.177	1.013	1.190	1.013
$\beta_{3,1}$	1.150	1.000	1.162	1.012
$\beta_{4,1}$	1.205	1.026	1.231	1.026
$\beta_{5,1}$	1.117	1.000	1.085	1.000
$\beta_{6,1}$	1.145	1.000	1.101	1.000
$\beta_{7,1}$	1.186	1.023	1.233	1.000
$\beta_{8,1}$	1.139	1.025	1.203	1.063
$\beta_{9,1}$	1.153	1.000	1.082	0.965

Table 7.202: CC CART comparison, cross-loading IRT, w.r.t. BD, β, $\overline{RMSE}$ fractions.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$r_{1,2}$	1.197	1.033	1.115	1.000
$r_{1,3}$	1.150	0.988	1.125	1.012
$r_{2,3}$	1.169	1.155	1.211	1.197

Table 7.203: CC CART comparison, cross-loading IRT, w.r.t. BD, R, $\overline{RMSE}$ fractions.

	(I2b)&(I2a)	(I2c)&(I2a)	(I3a)&(I2a)	(I3b)&(I2a)
$\xi_{3,2}$	1.062	1.000	1.062	0.988
$\xi_{3,3}$	1.058	0.990	1.039	1.000
$\xi_{6,2}$	1.219	1.014	1.205	1.014
$\xi_{6,3}$	1.194	1.000	1.204	1.000
$\xi_{8,2}$	1.197	1.039	1.145	1.039
$\xi_{8,3}$	1.185	1.056	1.009	0.944

Table 7.204: CC CART comparison, cross-loading IRT, w.r.t. BD, ξ, $\overline{RMSE}$ fractions.

Cross-Loading, CC CART BD, $\overline{RMSE}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\sigma^2_{\varepsilon,1}$	1.814	1.093
$\sigma^2_{\varepsilon,4}$	1.694	1.139
$\sigma^2_{\varepsilon,7}$	2.056	1.324

Table 7.205: CC CART comparison, cross-loading IRT, w.r.t. BD, σ^2_ε, $\overline{RMSE}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\alpha_{1,1}$	1.971	1.088
$\alpha_{2,1}$	1.825	1.079
$\alpha_{3,1}$	1.839	1.097
$\alpha_{5,1}$	1.823	1.048
$\alpha_{8,1}$	1.814	1.119
$\alpha_{9,1}$	1.321	0.815
$\alpha_{4,2}$	1.714	1.054
$\alpha_{5,2}$	1.738	1.095
$\alpha_{6,2}$	1.780	1.119
$\alpha_{7,3}$	2.119	1.288
$\alpha_{8,3}$	1.725	1.067
$\alpha_{9,3}$	1.709	1.068

Table 7.206: CC CART comparison, cross-loading IRT, w.r.t. BD, α^Δ, $\overline{RMSE}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\gamma_{1,1}$	1.739	1.072
$\gamma_{1,2}$	1.787	1.053
$\gamma_{1,3}$	2.041	1.068
$\gamma_{2,1}$	1.899	1.013
$\gamma_{2,2}$	1.923	1.064
$\gamma_{2,3}$	1.959	1.068
$\gamma_{3,1}$	2.435	1.247
$\gamma_{3,2}$	2.211	1.278
$\gamma_{3,3}$	2.470	1.120

Table 7.207: CC CART comparison, cross-loading IRT, w.r.t. BD, γ, $\overline{RMSE}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\beta_{1,1}$	1.775	1.050
$\beta_{2,1}$	1.734	1.076
$\beta_{3,1}$	1.675	1.050
$\beta_{4,1}$	1.718	1.077
$\beta_{5,1}$	1.543	1.043
$\beta_{6,1}$	1.638	1.043
$\beta_{7,1}$	1.884	1.070
$\beta_{8,1}$	1.810	1.152
$\beta_{9,1}$	1.600	1.012

Table 7.208: CC CART comparison, cross-loading IRT, w.r.t. BD, β, $\overline{RMSE}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$r_{1,2}$	1.787	1.066
$r_{1,3}$	1.738	1.062
$r_{2,3}$	1.944	1.282

Table 7.209: CC CART comparison, cross-loading IRT, w.r.t. BD, R, $\overline{RMSE}$ fractions, (I4a) and (I4b).

	(I4a)&(I2a)	(I4b)&(I2a)
$\xi_{3,2}$	1.600	1.050
$\xi_{3,3}$	1.660	1.068
$\xi_{6,2}$	1.808	1.068
$\xi_{6,3}$	1.741	1.056
$\xi_{8,2}$	1.750	1.132
$\xi_{8,3}$	1.602	1.037

Table 7.210: CC CART comparison, cross-loading IRT, w.r.t. BD, ξ, $\overline{RMSE}$ fractions, (I4a) and (I4b).

Cross-Loading, CC w/ CART, $\overline{\hat{\sigma}(\check{\psi})}$ and $\overline{RMSE}$ fractions

	(I2b)/(I2c) $(\overline{\hat{\sigma}(\check{\psi})})$	(I3a)/(I3b) $(\overline{\hat{\sigma}(\check{\psi})})$	(I2b)/(I2c) $(\overline{RMSE})$	(I3a)/(I3b) $(\overline{RMSE})$
$\sigma^2_{\varepsilon,1}$	1.118	1.088	1.114	1.111
$\sigma^2_{\varepsilon,4}$	1.093	1.093	1.160	1.178
$\sigma^2_{\varepsilon,7}$	1.107	1.046	1.122	1.080

Table 7.211: CC to CART comparison, cross-loading IRT, σ^2_ε, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I2b)/(I2c) $(\overline{\hat{\sigma}(\check{\psi})})$	(I3a)/(I3b) $(\overline{\hat{\sigma}(\check{\psi})})$	(I2b)/(I2c) $(\overline{RMSE})$	(I3a)/(I3b) $(\overline{RMSE})$
$\alpha_{1,1}$	1.148	1.107	1.171	1.083
$\alpha_{2,1}$	1.125	1.104	1.125	1.141
$\alpha_{3,1}$	1.125	1.104	1.143	1.111
$\alpha_{5,1}$	1.208	1.204	1.210	1.210
$\alpha_{8,1}$	1.133	1.109	1.115	1.180
$\alpha_{9,1}$	1.129	1.109	1.136	1.180
$\alpha_{4,2}$	1.122	1.150	1.218	1.245
$\alpha_{5,2}$	1.152	1.152	1.149	1.198
$\alpha_{6,2}$	1.178	1.178	1.200	1.217
$\alpha_{7,3}$	1.114	1.058	1.150	1.085
$\alpha_{8,3}$	1.073	1.057	1.085	1.085
$\alpha_{9,3}$	1.101	1.053	1.094	1.050

Table 7.212: CC to CART comparison, cross-loading IRT, α^Δ, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I2b)/(I2c) $(\overline{\hat{\sigma}(\check{\psi})})$	(I3a)/(I3b) $(\overline{\hat{\sigma}(\check{\psi})})$	(I2b)/(I2c) $(\overline{RMSE})$	(I3a)/(I3b) $(\overline{RMSE})$
$\gamma_{1,1}$	1.094	1.074	1.097	1.095
$\gamma_{1,2}$	1.121	1.033	1.103	1.024
$\gamma_{1,3}$	1.200	1.232	1.213	1.231
$\gamma_{2,1}$	1.155	1.158	1.165	1.190
$\gamma_{2,2}$	1.153	1.117	1.159	1.094
$\gamma_{2,3}$	1.226	1.283	1.184	1.276
$\gamma_{3,1}$	1.159	1.145	1.075	1.093
$\gamma_{3,2}$	1.167	1.139	1.040	1.037
$\gamma_{3,3}$	1.233	1.308	1.169	1.292

Table 7.213: CC to CART comparison, cross-loading IRT, γ, $\overline{\hat{\sigma}(\check{\psi})}$ fractions.

	(I2b)/(I2c) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I3a)/(I3b) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I2b)/(I2c) $(\overline{RMSE})$	(I3a)/(I3b) $(\overline{RMSE})$
$\beta_{1,1}$	1.172	1.207	1.175	1.195
$\beta_{2,1}$	1.169	1.153	1.162	1.175
$\beta_{3,1}$	1.150	1.150	1.150	1.148
$\beta_{4,1}$	1.138	1.172	1.175	1.200
$\beta_{5,1}$	1.119	1.075	1.117	1.085
$\beta_{6,1}$	1.137	1.098	1.145	1.101
$\beta_{7,1}$	1.156	1.188	1.159	1.233
$\beta_{8,1}$	1.138	1.129	1.111	1.131
$\beta_{9,1}$	1.156	1.131	1.153	1.122

Table 7.214: CC to CART comparison, cross-loading IRT, β, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions.

	(I2b)/(I2c) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I3a)/(I3b) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I2b)/(I2c) $(\overline{RMSE})$	(I3a)/(I3b) $(\overline{RMSE})$
$r_{1,2}$	1.170	1.125	1.159	1.115
$r_{1,3}$	1.169	1.129	1.165	1.111
$r_{2,3}$	1.185	1.158	1.012	1.012

Table 7.215: CC to CART comparison, cross-loading IRT, R, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions.

	(I2b)/(I2c) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I3a)/(I3b) $(\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})})$	(I2b)/(I2c) $(\overline{RMSE})$	(I3a)/(I3b) $(\overline{RMSE})$
$\xi_{3,2}$	1.105	1.053	1.062	1.076
$\xi_{3,3}$	1.120	1.067	1.069	1.039
$\xi_{6,2}$	1.182	1.182	1.203	1.189
$\xi_{6,3}$	1.200	1.212	1.194	1.204
$\xi_{8,2}$	1.119	1.083	1.152	1.101
$\xi_{8,3}$	1.096	1.077	1.123	1.069

Table 7.216: CC to CART comparison, cross-loading IRT, ξ, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions.

Cross-Loading, CC w/ CART, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ and $\overline{RMSE}$ fractions, (I4a) and (I4b).

	(I4a)/(I4b) ($\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$)	(I4a)/(I4b) ($\overline{RMSE}$)
$\sigma^2_{\varepsilon,1}$	1.583	1.660
$\sigma^2_{\varepsilon,4}$	1.379	1.488
$\sigma^2_{\varepsilon,7}$	1.514	1.553

Table 7.217: CC to CART comparison, cross-loading IRT, σ^2_ε, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions, (I4a)&(I4b).

	(I4a)/(I4b) ($\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$)	(I4a)/(I4b) ($\overline{RMSE}$)
$\alpha_{1,1}$	1.621	1.811
$\alpha_{2,1}$	1.660	1.691
$\alpha_{3,1}$	1.608	1.676
$\alpha_{5,1}$	1.686	1.738
$\alpha_{8,1}$	1.625	1.621
$\alpha_{9,1}$	1.625	1.621
$\alpha_{4,2}$	1.500	1.627
$\alpha_{5,2}$	1.563	1.587
$\alpha_{6,2}$	1.612	1.591
$\alpha_{7,3}$	1.589	1.645
$\alpha_{8,3}$	1.516	1.617
$\alpha_{9,3}$	1.500	1.600

Table 7.218: CC to CART comparison, cross-loading IRT, α^Δ, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions, (I4a)&(I4b).

	(I4a)/(I4b) ($\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$)	(I4a)/(I4b) ($\overline{RMSE}$)
$\gamma_{1,1}$	1.630	1.622
$\gamma_{1,2}$	1.661	1.696
$\gamma_{1,3}$	1.946	1.911
$\gamma_{2,1}$	1.705	1.875
$\gamma_{2,2}$	1.710	1.807
$\gamma_{2,3}$	1.946	1.833
$\gamma_{3,1}$	2.181	1.953
$\gamma_{3,2}$	2.013	1.730
$\gamma_{3,3}$	2.275	2.204

Table 7.219: CC to CART comparison, cross-loading IRT, γ, $\overline{\hat{\sigma}(\check{\boldsymbol{\psi}})}$ fractions, (I4a)&(I4b).

	(I4a)/(I4b) $(\overline{\hat{\sigma}(\check{\psi})})$	(I4a)/(I4b) $(\overline{RMSE})$
$\beta_{1,1}$	1.724	1.690
$\beta_{2,1}$	1.581	1.612
$\beta_{3,1}$	1.571	1.595
$\beta_{4,1}$	1.633	1.595
$\beta_{5,1}$	1.543	1.480
$\beta_{6,1}$	1.585	1.569
$\beta_{7,1}$	1.588	1.761
$\beta_{8,1}$	1.530	1.571
$\beta_{9,1}$	1.562	1.581

Table 7.220: CC to CART comparison, cross-loading IRT, β, $\overline{\hat{\sigma}(\check{\psi})}$ fractions, (I4a) & (I4b).

	(I4a)/(I4b) $(\overline{\hat{\sigma}(\check{\psi})})$	(I4a)/(I4b) $(\overline{RMSE})$
$r_{1,2}$	1.660	1.677
$r_{1,3}$	1.738	1.635
$r_{2,3}$	1.683	1.516

Table 7.221: CC to CART comparison, cross-loading IRT, R, $\overline{\hat{\sigma}(\check{\psi})}$ fractions, (I4a) & (I4b).

	(I4a)/(I4b) $(\overline{\hat{\sigma}(\check{\psi})})$	(I4a)/(I4b) $(\overline{RMSE})$
$\xi_{3,2}$	1.550	1.524
$\xi_{3,3}$	1.550	1.555
$\xi_{6,2}$	1.655	1.692
$\xi_{6,3}$	1.659	1.649
$\xi_{8,2}$	1.500	1.547
$\xi_{8,3}$	1.500	1.545

Table 7.222: CC to CART comparison, cross-loading IRT, ξ, $\overline{\hat{\sigma}(\check{\psi})}$ fractions, (I4a) & (I4b).

Name	mag9q071_c	mag9v131_c	mag9v13s_c	mag9r261_c	mag9r111_c
Scale	binary	binary	ordered	binary	binary
Mode	1	1	3	0	1
MissPerc	1.1	0.9	7.6	3	2.3
Name	mag9d171_c	mag9d151_c	mag9r051_c	mag9v011_c	mag9v012_c
Scale	binary	binary	binary	binary	binary
Mode	1	1	0	1	1
MissPerc	1.5	0.8	1.2	1.1	2.3
Name	mag9q161_c	mag9d201_c	mag9r191_c	mag9v121_c	mag9q181_c
Scale	binary	binary	binary	binary	binary
Mode	0	0	1	0	1
MissPerc	2.6	1	0.9	1.3	0.5
Name	mag9r25s_c	mag9r061_c	mag9q081_c	mag9q101_c	mag9q021_c
Scale	ordered	binary	binary	binary	binary
Mode	4	0	0	1	0
MissPerc	8.6	21.3	2.7	4.6	5.4
Name	mag9v091_c	mag9q211_c	reg90110_c	reg90120_c	reg90150_c
Scale	binary	binary	binary	binary	binary
Mode	1	0	1	1	1
MissPerc	4.5	4.6	1.1	0.1	1.6
Name	reg9016s_c	reg9017s_c	reg90210_c	reg90220_c	reg90230_c
Scale	ordered	ordered	binary	binary	binary
Mode	5	3	1	1	1
MissPerc	2.8	2.2	1.2	1.2	0.7
Name	reg90240_c	reg90250_c	reg90310_c	reg90320_c	reg9033s_c
Scale	binary	binary	binary	binary	ordered
Mode	1	0	1	1	4
MissPerc	0.5	1.3	0.7	0.9	1.6
Name	reg90340_c	reg90350_c	reg90360_c	reg90370_c	reg90410_c
Scale	binary	binary	binary	binary	binary
Mode	1	1	1	1	1
MissPerc	1.2	1.2	1.3	1.7	4
Name	reg90420_c	reg90430_c	reg90440_c	reg90450_c	reg90460_c
Scale	binary	binary	binary	binary	binary
Mode	1	1	1	1	1
MissPerc	5.7	9.1	7.3	7.6	9.2
Name	reg9047s_c	reg90510_c	reg90520_c	reg90530_c	reg90540_c
Scale	ordered	binary	binary	binary	binary
Mode	3	1	1	1	0
MissPerc	9.5	14.3	14.3	20.8	21.8
Name	reg90550_c	reg90560_c	reg90570_c		
Scale	binary	binary	binary		
Mode	1	1	1		
MissPerc	24.6	25.1	21.7		

Table 7.223: Brief overview of measurements for empirical application, with Name: Variable name, Scale: Scale of measurement, Mode, MissPerc: Percentage of missingness within a variable.

Name	Gender	Repeat	HighISEI	SchoolRS	SchoolGY
O.Name	(t700031)	(t725020)	(t731422_g14, t731472_g14)	(t723080_g1)	(t723080_g1)
Scale	binary	binary	continuous	binary	binary
Mode or Mean	0 (Male)	1 (Not repeated)	51.38	0 (Not Realschule)	0 (Not Gymnasium)
MissPerc	0.2	2.6	20.8	0	0

Table 7.224: Brief overview of covariates for empirical application, with Name: Variable name used, O.Name: Original name in scientific-use-file, Scale: Scale of measurement, Mode (if binary9 or Mean (if continuous), MissPerc: Percentage of missingness within a variable.

Empirical Scenario (M1) – no Covariates

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\alpha_{1,1}$	0.578	2466.586	0.016	0.546	0.609
$\alpha_{2,1}$	0.624	2337.770	0.016	0.592	0.655
$\alpha_{3,1}$	0.588	2671.565	0.014	0.561	0.615
$\alpha_{4,1}$	0.726	776.741	0.023	0.682	0.770
$\alpha_{5,1}$	0.689	1853.497	0.018	0.654	0.725
$\alpha_{6,1}$	0.452	3088.265	0.014	0.425	0.480
$\alpha_{7,1}$	0.875	901.705	0.023	0.830	0.920
$\alpha_{8,1}$	0.533	2732.465	0.015	0.504	0.563
$\alpha_{9,1}$	0.793	1481.331	0.019	0.755	0.832
$\alpha_{10,1}$	0.751	1780.029	0.018	0.716	0.786
$\alpha_{11,1}$	0.409	3074.947	0.014	0.381	0.437
$\alpha_{12,1}$	0.632	2468.533	0.016	0.601	0.663
$\alpha_{13,1}$	0.484	2325.506	0.016	0.454	0.515
$\alpha_{14,1}$	0.526	2167.767	0.016	0.496	0.557
$\alpha_{15,1}$	0.647	886.556	0.023	0.604	0.693
$\alpha_{16,1}$	0.480	3435.672	0.013	0.455	0.506
$\alpha_{17,1}$	0.784	1710.617	0.020	0.744	0.824
$\alpha_{18,1}$	0.602	2245.470	0.016	0.571	0.633
$\alpha_{19,1}$	0.981	1110.152	0.024	0.936	1.028
$\alpha_{20,1}$	0.745	1915.111	0.018	0.711	0.780
$\alpha_{21,1}$	0.844	1760.787	0.019	0.806	0.882
$\alpha_{22,1}$	0.679	1979.330	0.017	0.646	0.713

Table 7.225: Emp. Scenario (M1): α^{Δ} estimates (part 1).

	$\bar{\breve{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\breve{\psi})$	$q_{2.5\%}\left(\breve{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\breve{\boldsymbol{\psi}}\right)$
$\alpha_{23,2}$	0.630	324.802	0.032	0.570	0.693
$\alpha_{24,2}$	0.715	117.076	0.048	0.618	0.810
$\alpha_{25,2}$	0.701	1254.795	0.020	0.662	0.739
$\alpha_{26,2}$	0.698	1656.403	0.015	0.670	0.727
$\alpha_{27,2}$	0.466	955.654	0.020	0.427	0.505
$\alpha_{28,2}$	0.639	979.381	0.021	0.598	0.680
$\alpha_{29,2}$	0.519	2752.430	0.015	0.489	0.548
$\alpha_{30,2}$	0.392	1487.008	0.018	0.357	0.429
$\alpha_{31,2}$	1.170	259.778	0.035	1.103	1.239
$\alpha_{32,2}$	0.332	2984.684	0.014	0.306	0.358
$\alpha_{33,2}$	0.854	553.655	0.026	0.804	0.906
$\alpha_{34,2}$	1.085	224.691	0.037	1.014	1.159
$\alpha_{35,2}$	0.701	787.025	0.020	0.663	0.741
$\alpha_{36,2}$	0.795	680.978	0.025	0.747	0.845
$\alpha_{37,2}$	0.914	395.476	0.030	0.856	0.975
$\alpha_{38,2}$	0.333	2290.394	0.015	0.304	0.363
$\alpha_{39,2}$	0.516	2454.711	0.015	0.486	0.547
$\alpha_{40,2}$	0.615	976.277	0.022	0.572	0.656
$\alpha_{41,2}$	0.790	1244.167	0.020	0.751	0.830
$\alpha_{42,2}$	0.834	1351.830	0.020	0.794	0.874
$\alpha_{43,2}$	0.810	1060.887	0.022	0.767	0.853
$\alpha_{44,2}$	0.890	544.533	0.026	0.840	0.941
$\alpha_{45,2}$	0.604	2199.514	0.016	0.572	0.637
$\alpha_{46,2}$	0.685	1679.915	0.018	0.649	0.720
$\alpha_{47,2}$	0.893	1509.644	0.021	0.853	0.935
$\alpha_{48,2}$	0.707	1947.603	0.018	0.672	0.742
$\alpha_{49,2}$	0.552	2482.088	0.016	0.521	0.584
$\alpha_{50,2}$	0.926	1269.381	0.022	0.884	0.970
$\alpha_{51,2}$	0.399	3296.379	0.015	0.369	0.429
$\alpha_{52,2}$	0.834	1712.218	0.021	0.794	0.874
$\alpha_{53,2}$	0.740	935.537	0.023	0.696	0.785

Table 7.226: Emp. Scenario (M1): α^{Δ} estimates (part 2).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\beta_{1,1}$	-0.387	2055.412	0.014	-0.414	-0.360
$\beta_{2,1}$	-0.042	1773.922	0.014	-0.069	-0.016
$\beta_{3,1}$	-1.414	1251.686	0.017	-1.447	-1.382
$\beta_{4,1}$	1.539	674.263	0.024	1.492	1.586
$\beta_{5,1}$	-0.507	1268.018	0.015	-0.536	-0.478
$\beta_{6,1}$	-0.008	1655.149	0.013	-0.033	0.016
$\beta_{7,1}$	-0.964	875.150	0.019	-1.002	-0.926
$\beta_{8,1}$	0.229	1605.860	0.013	0.203	0.255
$\beta_{9,1}$	-0.620	1475.437	0.016	-0.652	-0.589
$\beta_{10,1}$	-0.141	1084.424	0.015	-0.170	-0.113
$\beta_{11,1}$	0.510	2483.479	0.013	0.484	0.535
$\beta_{12,1}$	0.100	1648.027	0.014	0.073	0.126
$\beta_{13,1}$	-0.501	2371.301	0.013	-0.527	-0.475
$\beta_{14,1}$	0.699	1860.540	0.015	0.670	0.728
$\beta_{15,1}$	-1.304	1012.212	0.020	-1.345	-1.266
$\beta_{16,1}$	-1.551	1137.967	0.018	-1.586	-1.517
$\beta_{17,1}$	0.631	1428.446	0.018	0.597	0.666
$\beta_{18,1}$	0.147	1609.885	0.013	0.121	0.174
$\beta_{19,1}$	-0.591	807.187	0.018	-0.628	-0.556
$\beta_{20,1}$	0.102	1382.262	0.015	0.074	0.131
$\beta_{21,1}$	-0.267	881.606	0.016	-0.298	-0.236
$\beta_{22,1}$	0.002	1370.200	0.014	-0.025	0.029

Table 7.227: Emp. Scenario (M1): β estimates (part 1).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\beta_{23,1}$	-2.082	297.008	0.036	-2.153	-2.014
$\beta_{24,1}$	-2.585	94.505	0.064	-2.716	-2.461
$\beta_{25,1}$	-1.030	809.468	0.018	-1.066	-0.994
$\beta_{26,1}$	-1.999	583.870	0.022	-2.044	-1.958
$\beta_{27,1}$	-1.974	651.143	0.025	-2.025	-1.925
$\beta_{28,1}$	-1.364	927.349	0.020	-1.403	-1.325
$\beta_{29,1}$	-0.404	1394.837	0.013	-0.430	-0.378
$\beta_{30,1}$	-1.273	1561.274	0.017	-1.307	-1.239
$\beta_{31,1}$	-1.853	256.648	0.037	-1.925	-1.783
$\beta_{32,1}$	0.280	4050.805	0.012	0.257	0.305
$\beta_{33,1}$	-1.508	502.455	0.025	-1.559	-1.459
$\beta_{34,1}$	-2.214	182.301	0.043	-2.302	-2.134
$\beta_{35,1}$	-2.466	329.067	0.032	-2.529	-2.405
$\beta_{36,1}$	-1.602	582.800	0.026	-1.654	-1.552
$\beta_{37,1}$	-1.825	324.771	0.033	-1.890	-1.761
$\beta_{38,1}$	-0.792	2539.474	0.014	-0.820	-0.765
$\beta_{39,1}$	-0.485	1871.960	0.014	-0.512	-0.459
$\beta_{40,1}$	-1.339	906.908	0.021	-1.380	-1.299
$\beta_{41,1}$	-0.905	1052.255	0.018	-0.941	-0.870
$\beta_{42,1}$	-0.653	1010.179	0.017	-0.687	-0.619
$\beta_{43,1}$	-1.102	882.722	0.020	-1.141	-1.062
$\beta_{44,1}$	-1.377	481.057	0.025	-1.426	-1.330
$\beta_{45,1}$	-0.449	1592.373	0.015	-0.478	-0.420
$\beta_{46,1}$	-1.251	1091.223	0.019	-1.287	-1.214
$\beta_{47,1}$	-0.148	856.792	0.016	-0.181	-0.116
$\beta_{48,1}$	-0.416	1663.982	0.015	-0.447	-0.385
$\beta_{49,1}$	-0.300	1769.310	0.015	-0.329	-0.271
$\beta_{50,1}$	0.064	967.814	0.018	0.029	0.098
$\beta_{51,1}$	-0.047	2176.948	0.014	-0.075	-0.020
$\beta_{52,1}$	-0.515	913.128	0.018	-0.551	-0.481
$\beta_{53,1}$	-1.197	879.853	0.022	-1.241	-1.154

Table 7.228: Emp. Scenario (M1): β estimates (part 2).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
r_{12}	0.700	1722.609	0.007	0.686	0.713

Table 7.229: Emp. Scenario (M1): R estimates.

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\xi_{3,2}$	0.620	2699.334	0.014	0.594	0.648
$\xi_{3,3}$	1.797	1769.937	0.019	1.761	1.834
$\xi_{16,2}$	1.246	1796.668	0.018	1.211	1.281
$\xi_{16,3}$	1.848	1535.298	0.020	1.811	1.887
$\xi_{26,2}$	0.537	1807.744	0.017	0.504	0.572
$\xi_{26,3}$	1.058	1016.134	0.020	1.019	1.098
$\xi_{26,4}$	2.068	763.100	0.023	2.023	2.114
$\xi_{27,2}$	0.489	1498.299	0.018	0.453	0.525
$\xi_{35,2}$	0.674	710.182	0.026	0.626	0.726
$\xi_{35,3}$	1.225	587.068	0.028	1.171	1.281
$\xi_{46,2}$	0.390	3714.422	0.012	0.367	0.413

Table 7.230: Emp. Scenario (M1): ξ estimates.

Empirical Scenario (M2) – with Covariates

	$\bar{\breve{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\breve{\psi})$	$q_{2.5\%}\left(\breve{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\breve{\boldsymbol{\psi}}\right)$
$\alpha_{1,1}$	0.417	1327.778	0.012	0.395	0.440
$\alpha_{2,1}$	0.443	1388.255	0.011	0.420	0.466
$\alpha_{3,1}$	0.422	2064.695	0.010	0.403	0.441
$\alpha_{4,1}$	0.514	416.693	0.016	0.482	0.545
$\alpha_{5,1}$	0.471	1236.636	0.012	0.448	0.496
$\alpha_{6,1}$	0.317	1608.746	0.010	0.297	0.337
$\alpha_{7,1}$	0.616	776.526	0.016	0.584	0.648
$\alpha_{8,1}$	0.382	1434.508	0.011	0.361	0.403
$\alpha_{9,1}$	0.560	995.442	0.014	0.533	0.588
$\alpha_{10,1}$	0.538	1133.210	0.013	0.513	0.564
$\alpha_{11,1}$	0.295	1524.834	0.010	0.275	0.315
$\alpha_{12,1}$	0.455	1426.406	0.012	0.433	0.478
$\alpha_{13,1}$	0.346	1430.693	0.011	0.325	0.368
$\alpha_{14,1}$	0.375	1290.438	0.011	0.353	0.397
$\alpha_{15,1}$	0.452	721.683	0.016	0.422	0.482
$\alpha_{16,1}$	0.337	2267.738	0.009	0.318	0.355
$\alpha_{17,1}$	0.546	860.982	0.014	0.518	0.574
$\alpha_{18,1}$	0.429	1517.903	0.011	0.407	0.451
$\alpha_{19,1}$	0.677	741.077	0.017	0.644	0.710
$\alpha_{20,1}$	0.528	1201.220	0.012	0.503	0.552
$\alpha_{21,1}$	0.604	1072.832	0.014	0.576	0.632
$\alpha_{22,1}$	0.493	1238.649	0.012	0.469	0.517

Table 7.231: Emp. Scenario (M2): α^{Δ} estimates (part 1).

	$\bar{\breve{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\breve{\psi})$	$q_{2.5\%}\left(\breve{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\breve{\boldsymbol{\psi}}\right)$
$\alpha_{23,2}$	0.486	226.324	0.025	0.440	0.535
$\alpha_{24,2}$	0.534	121.752	0.033	0.467	0.595
$\alpha_{25,2}$	0.521	732.196	0.015	0.491	0.549
$\alpha_{26,2}$	0.540	1485.864	0.011	0.519	0.562
$\alpha_{27,2}$	0.353	612.953	0.015	0.324	0.384
$\alpha_{28,2}$	0.483	570.625	0.016	0.452	0.515
$\alpha_{29,2}$	0.397	1088.564	0.011	0.375	0.419
$\alpha_{30,2}$	0.311	707.492	0.014	0.284	0.338
$\alpha_{31,2}$	0.911	224.424	0.026	0.860	0.965
$\alpha_{32,2}$	0.256	1230.390	0.010	0.235	0.276
$\alpha_{33,2}$	0.672	430.922	0.020	0.634	0.713
$\alpha_{34,2}$	0.832	194.066	0.028	0.779	0.886
$\alpha_{35,2}$	0.540	639.155	0.015	0.511	0.570
$\alpha_{36,2}$	0.619	390.264	0.019	0.581	0.656
$\alpha_{37,2}$	0.700	323.780	0.022	0.656	0.744
$\alpha_{38,2}$	0.257	1004.945	0.012	0.234	0.280
$\alpha_{39,2}$	0.391	1112.807	0.012	0.369	0.415
$\alpha_{40,2}$	0.474	726.024	0.016	0.443	0.507
$\alpha_{41,2}$	0.585	713.767	0.016	0.555	0.617
$\alpha_{42,2}$	0.628	765.925	0.015	0.599	0.659
$\alpha_{43,2}$	0.614	547.422	0.018	0.580	0.650
$\alpha_{44,2}$	0.659	471.240	0.019	0.622	0.695
$\alpha_{45,2}$	0.449	1011.223	0.012	0.425	0.474
$\alpha_{46,2}$	0.517	901.636	0.014	0.489	0.543
$\alpha_{47,2}$	0.681	677.989	0.016	0.651	0.712
$\alpha_{48,2}$	0.539	973.239	0.014	0.513	0.566
$\alpha_{49,2}$	0.427	1120.859	0.012	0.403	0.451
$\alpha_{50,2}$	0.705	592.081	0.016	0.673	0.737
$\alpha_{51,2}$	0.313	1142.435	0.012	0.291	0.336
$\alpha_{52,2}$	0.634	783.598	0.016	0.601	0.666
$\alpha_{53,2}$	0.572	592.459	0.017	0.538	0.606

Table 7.232: Emp. Scenario (M2): α^{Δ} estimates (part 2).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\gamma_{1,1}$	-0.633	1336.341	0.023	-0.677	-0.588
$\gamma_{1,2}$	0.293	306.094	0.029	0.236	0.349
$\gamma_{1,3}$	0.007	213.330	0.001	0.006	0.008
$\gamma_{1,4}$	0.715	3161.184	0.029	0.658	0.772
$\gamma_{1,5}$	2.025	3527.053	0.036	1.956	2.095
$\gamma_{2,1}$	0.254	1877.325	0.021	0.213	0.296
$\gamma_{2,2}$	0.192	358.323	0.027	0.138	0.246
$\gamma_{2,3}$	0.007	121.726	0.001	0.005	0.008
$\gamma_{2,4}$	0.840	3036.306	0.029	0.783	0.896
$\gamma_{2,5}$	1.764	3619.418	0.033	1.699	1.829

Table 7.233: Emp. Scenario (M2): γ estimates.

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\beta_{1,1}$	0.157	208.468	0.023	0.112	0.205
$\beta_{2,1}$	0.537	164.021	0.025	0.490	0.587
$\beta_{3,1}$	-0.866	245.107	0.023	-0.911	-0.820
$\beta_{4,1}$	2.207	272.512	0.043	2.126	2.294
$\beta_{5,1}$	0.117	155.145	0.025	0.069	0.168
$\beta_{6,1}$	0.408	224.106	0.021	0.366	0.449
$\beta_{7,1}$	-0.155	109.401	0.029	-0.210	-0.098
$\beta_{8,1}$	0.729	263.814	0.024	0.683	0.776
$\beta_{9,1}$	0.116	122.018	0.028	0.063	0.172
$\beta_{10,1}$	0.564	132.483	0.027	0.510	0.618
$\beta_{11,1}$	0.897	325.123	0.022	0.853	0.940
$\beta_{12,1}$	0.696	178.689	0.026	0.645	0.745
$\beta_{13,1}$	-0.049	190.215	0.021	-0.091	-0.007
$\beta_{14,1}$	1.190	260.496	0.026	1.142	1.243
$\beta_{15,1}$	-0.705	126.555	0.025	-0.754	-0.657
$\beta_{16,1}$	-1.105	262.529	0.022	-1.148	-1.060
$\beta_{17,1}$	1.344	188.278	0.034	1.279	1.413
$\beta_{18,1}$	0.709	191.865	0.025	0.663	0.759
$\beta_{19,1}$	0.306	132.968	0.032	0.245	0.370
$\beta_{20,1}$	0.796	153.166	0.028	0.740	0.851
$\beta_{21,1}$	0.525	131.679	0.030	0.467	0.586
$\beta_{22,1}$	0.649	157.116	0.027	0.597	0.703

Table 7.234: Emp. Scenario (M2): β estimates (part 1).

	$\bar{\bar{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\beta_{23,1}$	-1.308	210.443	0.031	-1.369	-1.246
$\beta_{24,1}$	-1.712	350.986	0.038	-1.786	-1.637
$\beta_{25,1}$	-0.187	168.443	0.028	-0.238	-0.131
$\beta_{26,1}$	-1.143	146.565	0.027	-1.195	-1.089
$\beta_{27,1}$	-1.406	616.708	0.027	-1.458	-1.352
$\beta_{28,1}$	-0.585	204.124	0.027	-0.636	-0.531
$\beta_{29,1}$	0.232	222.762	0.024	0.186	0.283
$\beta_{30,1}$	-0.782	469.776	0.025	-0.829	-0.732
$\beta_{31,1}$	-0.415	68.416	0.039	-0.487	-0.335
$\beta_{32,1}$	0.689	621.651	0.022	0.646	0.734
$\beta_{33,1}$	-0.451	102.561	0.031	-0.511	-0.388
$\beta_{34,1}$	-0.886	99.135	0.037	-0.955	-0.809
$\beta_{35,1}$	-1.609	269.611	0.033	-1.672	-1.544
$\beta_{36,1}$	-0.622	124.303	0.031	-0.682	-0.558
$\beta_{37,1}$	-0.704	127.610	0.032	-0.764	-0.637
$\beta_{38,1}$	-0.381	558.361	0.022	-0.423	-0.336
$\beta_{39,1}$	0.142	258.040	0.024	0.096	0.190
$\beta_{40,1}$	-0.581	219.170	0.027	-0.633	-0.524
$\beta_{41,1}$	0.048	145.721	0.031	-0.012	0.110
$\beta_{42,1}$	0.362	146.182	0.031	0.301	0.421
$\beta_{43,1}$	-0.109	194.305	0.030	-0.168	-0.048
$\beta_{44,1}$	-0.297	115.287	0.032	-0.358	-0.233
$\beta_{45,1}$	0.279	233.017	0.027	0.227	0.331
$\beta_{46,1}$	-0.413	168.606	0.027	-0.466	-0.360
$\beta_{47,1}$	0.956	152.050	0.037	0.883	1.031
$\beta_{48,1}$	0.457	217.070	0.029	0.402	0.517
$\beta_{49,1}$	0.390	285.908	0.028	0.338	0.447
$\beta_{50,1}$	1.208	146.120	0.040	1.132	1.287
$\beta_{51,1}$	0.460	460.795	0.026	0.411	0.512
$\beta_{52,1}$	0.515	224.349	0.033	0.451	0.582
$\beta_{53,1}$	-0.276	217.636	0.031	-0.336	-0.214

Table 7.235: Emp. Scenario (M2): β estimates (part 2).

	$\bar{\breve{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\breve{\psi})$	$q_{2.5\%}\left(\breve{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\breve{\boldsymbol{\psi}}\right)$
r_{12}	0.552	1658.101	0.010	0.532	0.572

Table 7.236: Emp. Scenario (M2): R estimates.

	$\bar{\breve{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\breve{\psi})$	$q_{2.5\%}\left(\breve{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\breve{\boldsymbol{\psi}}\right)$
$\xi_{3,2}$	0.623	2913.910	0.014	0.595	0.650
$\xi_{3,3}$	1.803	1973.044	0.019	1.766	1.840
$\xi_{16,2}$	1.243	1869.965	0.017	1.209	1.277
$\xi_{16,3}$	1.844	1738.007	0.019	1.806	1.882
$\xi_{26,2}$	0.540	1888.893	0.017	0.507	0.573
$\xi_{26,3}$	1.064	1115.362	0.020	1.026	1.103
$\xi_{26,4}$	2.078	884.976	0.022	2.034	2.121
$\xi_{27,2}$	0.487	1904.984	0.018	0.452	0.522
$\xi_{35,2}$	0.678	681.573	0.025	0.629	0.728
$\xi_{35,3}$	1.231	509.915	0.028	1.177	1.285
$\xi_{46,2}$	0.388	4413.358	0.012	0.366	0.412

Table 7.237: Emp. Scenario (M2): ξ estimates.

Empirical Scenario (M3a) – with Covariates, Hauptschule

Scenario (M3a) *Aggregates*:					
	$\overline{\overline{\check{\psi}}}$	$\overline{(\hat{T})_{eff}}$	$\overline{\hat{\sigma}(\check{\psi})}$	$\overline{q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)}$	$\overline{q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)}$
α_{math}	0.37	2186.96	0.03	0.31	0.43
(range)	(0.071, 0.679)	(648.581, (4170.817)	(0.022, 0.045)	(0.018, 0.609)	(0.124, 0.753)
α_{read}	0.52	1900.55	0.03	0.46	0.59
(range)	(0.043, 0.969)	(348.33, 3582.168)	(0.026, 0.062)	(-0.013, 0.868)	(0.097, 1.074)
β_{math}	0.36	398.10	0.04	0.29	0.44
(range)	(-1.142, 1.914)	(106.092, 2740.69)	(0.026, 0.06)	(-1.2, 1.805)	(-1.085, , 2.028)
β_{read}	-0.31	289.40	0.05	-0.40	-0.23
(range)	(-1.852, 1.004)	(56.73, 4544.126)	(0.028, 0.07)	(-1.985, 0.912)	(-1.728, 1.098)
Scenario (M3a) *Some Single Parameters*:					
	$\overline{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\gamma_{1,1}$	-0.538	1081.352	0.045	-0.626	-0.450
$\gamma_{1,2}$	0.279	586.112	0.047	0.187	0.370
$\gamma_{1,3}$	0.005	151.929	0.001	0.003	0.008
$\gamma_{2,1}$	0.313	653.603	0.040	0.234	0.391
$\gamma_{2,2}$	0.158	294.370	0.042	0.074	0.242
$\gamma_{2,3}$	0.005	78.940	0.001	0.003	0.007
r_{12}	0.595	1782.695	0.018	0.558	0.630
$\xi_{3,2}$	0.627	4590.486	0.022	0.583	0.671
$\xi_{3,3}$	1.646	3931.521	0.032	1.583	1.707
$\xi_{16,2}$	1.144	2946.379	0.029	1.088	1.202
$\xi_{16,3}$	1.788	2892.538	0.033	1.724	1.853
$\xi_{26,2}$	0.654	2664.979	0.027	0.601	0.708
$\xi_{26,3}$	1.187	1882.720	0.032	1.124	1.251
$\xi_{26,4}$	2.070	1591.298	0.039	1.995	2.147
$\xi_{27,2}$	0.470	2864.614	0.027	0.419	0.524
$\xi_{35,2}$	0.716	1234.442	0.036	0.647	0.788
$\xi_{35,3}$	1.253	923.368	0.041	1.174	1.335
$\xi_{46,2}$	0.358	6750.573	0.018	0.323	0.395

Table 7.238: Aggregates and some single parameters for empirical example (M3a) including subsetted data with respect to Hauptschule.

	$\bar{\breve{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\breve{\psi})$	$q_{2.5\%}\left(\breve{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\breve{\boldsymbol{\psi}}\right)$
$\alpha_{1,1}$	0.403	2330.291	0.028	0.348	0.458
$\alpha_{2,1}$	0.286	2329.314	0.027	0.235	0.339
$\alpha_{3,1}$	0.176	3999.891	0.022	0.132	0.219
$\alpha_{4,1}$	0.372	648.581	0.045	0.284	0.459
$\alpha_{5,1}$	0.393	2176.087	0.029	0.337	0.450
$\alpha_{6,1}$	0.219	2714.803	0.026	0.168	0.270
$\alpha_{7,1}$	0.664	1493.504	0.036	0.594	0.735
$\alpha_{8,1}$	0.213	2696.389	0.026	0.162	0.265
$\alpha_{9,1}$	0.636	1354.621	0.035	0.568	0.705
$\alpha_{10,1}$	0.625	1449.651	0.034	0.560	0.694
$\alpha_{11,1}$	0.071	2605.706	0.027	0.018	0.124
$\alpha_{12,1}$	0.189	2609.346	0.026	0.138	0.240
$\alpha_{13,1}$	0.312	2658.437	0.027	0.260	0.365
$\alpha_{14,1}$	0.164	2270.565	0.028	0.110	0.220
$\alpha_{15,1}$	0.501	1510.662	0.034	0.435	0.568
$\alpha_{16,1}$	0.160	4170.817	0.022	0.116	0.204
$\alpha_{17,1}$	0.604	821.289	0.044	0.519	0.693
$\alpha_{18,1}$	0.429	2063.642	0.030	0.370	0.489
$\alpha_{19,1}$	0.679	1512.764	0.036	0.609	0.753
$\alpha_{20,1}$	0.285	2221.434	0.029	0.229	0.342
$\alpha_{21,1}$	0.490	1848.262	0.032	0.429	0.554
$\alpha_{22,1}$	0.261	2627.128	0.028	0.207	0.316

Table 7.239: Emp. Scenario (M3a): α^{Δ} estimates (part 1).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\alpha_{23,2}$	0.501	807.205	0.044	0.420	0.593
$\alpha_{24,2}$	0.652	348.330	0.062	0.537	0.778
$\alpha_{25,2}$	0.573	1874.635	0.031	0.514	0.634
$\alpha_{26,2}$	0.646	2009.675	0.026	0.595	0.698
$\alpha_{27,2}$	0.415	1602.023	0.033	0.350	0.479
$\alpha_{28,2}$	0.675	1309.650	0.036	0.607	0.748
$\alpha_{29,2}$	0.482	2296.605	0.029	0.427	0.539
$\alpha_{30,2}$	0.512	1454.537	0.034	0.448	0.581
$\alpha_{31,2}$	0.803	1170.076	0.039	0.728	0.882
$\alpha_{32,2}$	0.191	2551.123	0.026	0.140	0.242
$\alpha_{33,2}$	0.550	1773.353	0.032	0.488	0.613
$\alpha_{34,2}$	0.969	555.024	0.053	0.868	1.074
$\alpha_{35,2}$	0.722	1282.677	0.034	0.656	0.790
$\alpha_{36,2}$	0.703	1225.303	0.038	0.629	0.780
$\alpha_{37,2}$	0.741	1046.931	0.040	0.663	0.821
$\alpha_{38,2}$	0.285	2547.601	0.027	0.232	0.337
$\alpha_{39,2}$	0.387	2350.424	0.027	0.336	0.439
$\alpha_{40,2}$	0.469	1823.522	0.032	0.405	0.533
$\alpha_{41,2}$	0.624	1920.588	0.033	0.561	0.689
$\alpha_{42,2}$	0.528	2208.372	0.030	0.470	0.589
$\alpha_{43,2}$	0.760	1366.711	0.038	0.687	0.834
$\alpha_{44,2}$	0.710	1453.580	0.038	0.638	0.785
$\alpha_{45,2}$	0.600	2097.196	0.032	0.537	0.664
$\alpha_{46,2}$	0.465	2604.950	0.029	0.407	0.521
$\alpha_{47,2}$	0.477	1754.168	0.033	0.413	0.542
$\alpha_{48,2}$	0.417	2773.275	0.029	0.360	0.474
$\alpha_{49,2}$	0.043	3253.633	0.028	-0.013	0.097
$\alpha_{50,2}$	0.466	1917.332	0.036	0.397	0.537
$\alpha_{51,2}$	0.142	3582.168	0.030	0.083	0.200
$\alpha_{52,2}$	0.332	3164.883	0.032	0.271	0.394
$\alpha_{53,2}$	0.404	2791.359	0.032	0.342	0.468

Table 7.240: Emp. Scenario (M3a): α^{Δ} estimates (part 2).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\gamma_{1,1}$	-0.538	1081.352	0.045	-0.626	-0.450
$\gamma_{1,2}$	0.279	586.112	0.047	0.187	0.370
$\gamma_{1,3}$	0.005	151.929	0.001	0.003	0.008
$\gamma_{2,1}$	0.313	653.603	0.040	0.234	0.391
$\gamma_{2,2}$	0.158	294.370	0.042	0.074	0.242
$\gamma_{2,3}$	0.005	78.940	0.001	0.003	0.007

Table 7.241: Emp. Scenario (M3a): γ estimates.

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\beta_{1,1}$	0.147	138.823	0.036	0.078	0.220
$\beta_{2,1}$	0.469	175.920	0.031	0.409	0.530
$\beta_{3,1}$	-0.848	529.307	0.028	-0.903	-0.794
$\beta_{4,1}$	1.914	353.150	0.057	1.805	2.028
$\beta_{5,1}$	0.089	153.085	0.035	0.020	0.156
$\beta_{6,1}$	0.291	319.513	0.027	0.238	0.344
$\beta_{7,1}$	-0.073	106.092	0.051	-0.171	0.026
$\beta_{8,1}$	0.619	342.221	0.028	0.565	0.675
$\beta_{9,1}$	0.111	110.416	0.050	0.016	0.207
$\beta_{10,1}$	0.643	131.664	0.051	0.546	0.742
$\beta_{11,1}$	0.798	2740.690	0.026	0.748	0.850
$\beta_{12,1}$	0.606	545.916	0.027	0.553	0.660
$\beta_{13,1}$	-0.034	203.588	0.031	-0.095	0.028
$\beta_{14,1}$	0.977	815.855	0.029	0.919	1.035
$\beta_{15,1}$	-0.666	128.951	0.043	-0.750	-0.586
$\beta_{16,1}$	-1.142	795.531	0.029	-1.200	-1.085
$\beta_{17,1}$	1.222	260.072	0.060	1.107	1.340
$\beta_{18,1}$	0.591	156.061	0.039	0.515	0.666
$\beta_{19,1}$	0.327	114.278	0.053	0.226	0.431
$\beta_{20,1}$	0.765	212.644	0.034	0.700	0.833
$\beta_{21,1}$	0.574	124.380	0.043	0.493	0.658
$\beta_{22,1}$	0.641	300.121	0.031	0.581	0.702

Table 7.242: Emp. Scenario (M3a): β estimates (part 1).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\beta_{23,1}$	-1.337	123.485	0.047	-1.428	-1.245
$\beta_{24,1}$	-1.852	205.369	0.065	-1.985	-1.728
$\beta_{25,1}$	-0.218	64.175	0.044	-0.303	-0.131
$\beta_{26,1}$	-1.132	73.445	0.050	-1.230	-1.035
$\beta_{27,1}$	-1.418	174.222	0.042	-1.501	-1.336
$\beta_{28,1}$	-0.583	59.505	0.050	-0.679	-0.486
$\beta_{29,1}$	0.260	89.192	0.039	0.185	0.338
$\beta_{30,1}$	-0.761	104.448	0.042	-0.841	-0.677
$\beta_{31,1}$	-0.405	59.459	0.057	-0.516	-0.293
$\beta_{32,1}$	0.647	499.760	0.030	0.589	0.708
$\beta_{33,1}$	-0.414	67.928	0.043	-0.497	-0.328
$\beta_{34,1}$	-1.021	68.178	0.070	-1.156	-0.883
$\beta_{35,1}$	-1.693	81.866	0.058	-1.813	-1.581
$\beta_{36,1}$	-0.606	68.065	0.052	-0.705	-0.503
$\beta_{37,1}$	-0.767	76.063	0.055	-0.871	-0.661
$\beta_{38,1}$	-0.388	177.391	0.030	-0.447	-0.328
$\beta_{39,1}$	0.090	123.665	0.035	0.024	0.158
$\beta_{40,1}$	-0.588	81.320	0.040	-0.663	-0.511
$\beta_{41,1}$	-0.061	76.998	0.046	-0.151	0.031
$\beta_{42,1}$	0.235	81.803	0.043	0.151	0.318
$\beta_{43,1}$	-0.155	59.511	0.055	-0.262	-0.051
$\beta_{44,1}$	-0.412	56.730	0.053	-0.515	-0.309
$\beta_{45,1}$	0.234	79.131	0.047	0.144	0.326
$\beta_{46,1}$	-0.452	93.312	0.039	-0.529	-0.374
$\beta_{47,1}$	0.827	136.227	0.045	0.744	0.919
$\beta_{48,1}$	0.353	110.836	0.039	0.279	0.430
$\beta_{49,1}$	0.258	4544.126	0.028	0.203	0.314
$\beta_{50,1}$	1.004	194.341	0.047	0.912	1.098
$\beta_{51,1}$	0.407	1055.669	0.031	0.346	0.468
$\beta_{52,1}$	0.377	165.330	0.037	0.307	0.450
$\beta_{53,1}$	-0.184	119.903	0.038	-0.260	-0.109

Table 7.243: Emp. Scenario (M3a): β estimates (part 2).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
r_{12}	0.595	1782.695	0.018	0.558	0.630

Table 7.244: Emp. Scenario (M3a): R estimates.

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\xi_{3,2}$	0.627	4590.486	0.022	0.583	0.671
$\xi_{3,3}$	1.646	3931.521	0.032	1.583	1.707
$\xi_{16,2}$	1.144	2946.379	0.029	1.088	1.202
$\xi_{16,3}$	1.788	2892.538	0.033	1.724	1.853
$\xi_{26,2}$	0.654	2664.979	0.027	0.601	0.708
$\xi_{26,3}$	1.187	1882.720	0.032	1.124	1.251
$\xi_{26,4}$	2.070	1591.298	0.039	1.995	2.147
$\xi_{27,2}$	0.470	2864.614	0.027	0.419	0.524
$\xi_{35,2}$	0.716	1234.442	0.036	0.647	0.788
$\xi_{35,3}$	1.253	923.368	0.041	1.174	1.335
$\xi_{46,2}$	0.358	6750.573	0.018	0.323	0.395

Table 7.245: Emp. Scenario (M3a): ξ estimates.

Empirical Scenario (M3b) – with Covariates, Realschule

Scenario (M3b) *Aggregates*:					
	$\overline{\overline{\check{\psi}}}$	$\overline{(\hat{T})_{eff}}$	$\overline{\hat{\sigma}(\check{\psi})}$	$\overline{q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)}$	$\overline{q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)}$
α_{math}	0.44	2207.57	0.03	0.38	0.49
(range)	(0.24, 0.722)	(635.704, 4338.242)	(0.02, 0.041)	(0.194, 0.654)	(0.286, 0.794)
α_{read}	0.54	1370.00	0.04	0.47	0.62
(range)	(0.222, 0.857)	(189.514, 2622.627)	(0.024, 0.073)	(0.176, 0.746)	(0.269, 0.976)
β_{math}	-0.04	140.63	0.04	-0.11	0.04
(range)	(-1.387, 1.799)	(80.552, 416.281)	(0.027, 0.059)	(-1.451, 1.683)	(-1.323, 1.918)
β_{read}	-0.79	138.80	0.05	-0.88	-0.70
(range)	(-2.229, 0.621)	(62.573, 420.462)	(0.028, 0.076)	(-2.386, 0.524)	(-2.086, 0.722)
Scenario (M3b) *Some Single Parameters*:					
	$\overline{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\gamma_{1,1}$	-0.762	2031.005	0.041	-0.842	-0.682
$\gamma_{1,2}$	0.137	230.932	0.049	0.040	0.233
$\gamma_{1,3}$	0.005	124.465	0.001	0.003	0.007
$\gamma_{2,1}$	0.190	1791.076	0.037	0.118	0.264
$\gamma_{2,2}$	0.093	264.993	0.046	0.004	0.186
$\gamma_{2,3}$	0.006	138.817	0.001	0.004	0.008
r_{12}	0.515	1871.759	0.018	0.479	0.549
$\xi_{3,2}$	0.607	3298.484	0.023	0.563	0.652
$\xi_{3,3}$	1.817	2619.324	0.031	1.755	1.879
$\xi_{16,2}$	1.302	2096.148	0.029	1.245	1.360
$\xi_{16,3}$	1.894	1964.845	0.032	1.831	1.957
$\xi_{26,2}$	0.503	1497.318	0.030	0.446	0.562
$\xi_{26,3}$	1.036	954.143	0.035	0.971	1.106
$\xi_{26,4}$	2.039	712.318	0.039	1.964	2.118
$\xi_{27,2}$	0.483	1535.581	0.032	0.422	0.548
$\xi_{35,2}$	0.712	611.358	0.048	0.620	0.810
$\xi_{35,3}$	1.240	426.611	0.052	1.140	1.346
$\xi_{46,2}$	0.392	4494.754	0.020	0.354	0.432

Table 7.246: Aggregates and some single parameters for empirical example (M3b) including subsetted data with respect to Realschule.

	$\bar{\breve{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\breve{\psi})$	$q_{2.5\%}\left(\breve{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\breve{\boldsymbol{\psi}}\right)$
$\alpha_{1,1}$	0.411	2412.325	0.026	0.361	0.463
$\alpha_{2,1}$	0.394	2835.562	0.025	0.346	0.443
$\alpha_{3,1}$	0.336	3742.424	0.021	0.296	0.377
$\alpha_{4,1}$	0.490	635.704	0.041	0.410	0.573
$\alpha_{5,1}$	0.459	2097.413	0.028	0.406	0.514
$\alpha_{6,1}$	0.253	3089.472	0.023	0.209	0.298
$\alpha_{7,1}$	0.541	1365.930	0.032	0.479	0.606
$\alpha_{8,1}$	0.344	2771.305	0.024	0.297	0.393
$\alpha_{9,1}$	0.564	1529.810	0.030	0.506	0.625
$\alpha_{10,1}$	0.526	2099.614	0.028	0.472	0.580
$\alpha_{11,1}$	0.240	2781.204	0.024	0.194	0.286
$\alpha_{12,1}$	0.377	2202.017	0.025	0.329	0.426
$\alpha_{13,1}$	0.323	2449.323	0.025	0.275	0.374
$\alpha_{14,1}$	0.378	1975.559	0.027	0.326	0.430
$\alpha_{15,1}$	0.389	1199.327	0.033	0.325	0.453
$\alpha_{16,1}$	0.257	4338.242	0.020	0.216	0.297
$\alpha_{17,1}$	0.654	1231.943	0.037	0.582	0.728
$\alpha_{18,1}$	0.521	1969.860	0.028	0.468	0.576
$\alpha_{19,1}$	0.722	1239.411	0.036	0.654	0.794
$\alpha_{20,1}$	0.521	2115.604	0.028	0.464	0.577
$\alpha_{21,1}$	0.560	1817.078	0.029	0.505	0.618
$\alpha_{22,1}$	0.373	2667.309	0.025	0.325	0.423

Table 7.247: Emp. Scenario (M3b): α^{Δ} estimates (part 1).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\alpha_{23,2}$	0.612	285.890	0.062	0.495	0.739
$\alpha_{24,2}$	0.587	189.514	0.073	0.454	0.741
$\alpha_{25,2}$	0.579	1485.236	0.032	0.517	0.644
$\alpha_{26,2}$	0.542	2111.479	0.024	0.494	0.590
$\alpha_{27,2}$	0.389	1166.084	0.034	0.322	0.456
$\alpha_{28,2}$	0.455	1158.430	0.035	0.389	0.527
$\alpha_{29,2}$	0.386	2125.015	0.026	0.335	0.437
$\alpha_{30,2}$	0.304	1144.040	0.033	0.240	0.368
$\alpha_{31,2}$	0.830	422.360	0.052	0.731	0.937
$\alpha_{32,2}$	0.222	2240.716	0.024	0.176	0.269
$\alpha_{33,2}$	0.594	763.458	0.040	0.520	0.676
$\alpha_{34,2}$	0.857	340.468	0.059	0.746	0.976
$\alpha_{35,2}$	0.600	966.015	0.034	0.534	0.669
$\alpha_{36,2}$	0.552	782.124	0.041	0.473	0.633
$\alpha_{37,2}$	0.771	465.087	0.050	0.677	0.872
$\alpha_{38,2}$	0.281	1939.154	0.027	0.227	0.335
$\alpha_{39,2}$	0.428	2083.416	0.027	0.376	0.482
$\alpha_{40,2}$	0.428	1289.955	0.034	0.363	0.496
$\alpha_{41,2}$	0.699	1311.205	0.035	0.631	0.768
$\alpha_{42,2}$	0.703	1295.767	0.034	0.639	0.772
$\alpha_{43,2}$	0.625	1202.085	0.035	0.558	0.695
$\alpha_{44,2}$	0.800	697.858	0.044	0.716	0.890
$\alpha_{45,2}$	0.540	1810.318	0.029	0.484	0.599
$\alpha_{46,2}$	0.552	1746.432	0.031	0.494	0.613
$\alpha_{47,2}$	0.611	1637.825	0.031	0.550	0.672
$\alpha_{48,2}$	0.467	2060.228	0.028	0.412	0.523
$\alpha_{49,2}$	0.407	2411.627	0.027	0.354	0.461
$\alpha_{50,2}$	0.679	1407.243	0.034	0.613	0.746
$\alpha_{51,2}$	0.279	2622.627	0.026	0.229	0.330
$\alpha_{52,2}$	0.606	1829.406	0.032	0.544	0.670
$\alpha_{53,2}$	0.455	1478.981	0.036	0.385	0.526

Table 7.248: Emp. Scenario (M3b): α^{Δ} estimates (part 2).

	$\bar{\bar{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\gamma_{1,1}$	-0.762	2031.005	0.041	-0.842	-0.682
$\gamma_{1,2}$	0.137	230.932	0.049	0.040	0.233
$\gamma_{1,3}$	0.005	124.465	0.001	0.003	0.007
$\gamma_{2,1}$	0.190	1791.076	0.037	0.118	0.264
$\gamma_{2,2}$	0.093	264.993	0.046	0.004	0.186
$\gamma_{2,3}$	0.006	138.817	0.001	0.004	0.008

Table 7.249: Emp. Scenario (M3b): γ estimates.

	$\bar{\bar{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\beta_{1,1}$	-0.283	92.524	0.035	-0.353	-0.216
$\beta_{2,1}$	0.127	105.784	0.034	0.060	0.192
$\beta_{3,1}$	-1.211	143.753	0.034	-1.279	-1.143
$\beta_{4,1}$	1.799	210.493	0.059	1.683	1.918
$\beta_{5,1}$	-0.329	92.145	0.038	-0.404	-0.255
$\beta_{6,1}$	0.187	191.322	0.027	0.134	0.239
$\beta_{7,1}$	-0.863	110.308	0.045	-0.950	-0.777
$\beta_{8,1}$	0.424	119.538	0.031	0.363	0.485
$\beta_{9,1}$	-0.510	96.470	0.045	-0.600	-0.422
$\beta_{10,1}$	-0.008	84.139	0.041	-0.091	0.072
$\beta_{11,1}$	0.619	248.291	0.028	0.565	0.674
$\beta_{12,1}$	0.285	110.037	0.033	0.221	0.350
$\beta_{13,1}$	-0.410	132.080	0.031	-0.471	-0.350
$\beta_{14,1}$	0.889	146.433	0.035	0.820	0.958
$\beta_{15,1}$	-1.211	227.105	0.039	-1.289	-1.136
$\beta_{16,1}$	-1.387	416.281	0.033	-1.451	-1.323
$\beta_{17,1}$	0.928	124.068	0.055	0.820	1.035
$\beta_{18,1}$	0.280	85.587	0.041	0.200	0.361
$\beta_{19,1}$	-0.434	80.552	0.054	-0.542	-0.327
$\beta_{20,1}$	0.284	96.465	0.041	0.204	0.365
$\beta_{21,1}$	-0.175	84.210	0.044	-0.263	-0.090
$\beta_{22,1}$	0.189	96.341	0.033	0.125	0.252

Table 7.250: Emp. Scenario (M3b): β estimates (part 1).

	$\bar{\bar{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\beta_{23,1}$	-1.898	223.522	0.061	-2.019	-1.780
$\beta_{24,1}$	-2.229	152.530	0.076	-2.386	-2.086
$\beta_{25,1}$	-0.718	70.436	0.042	-0.799	-0.632
$\beta_{26,1}$	-1.667	166.265	0.046	-1.757	-1.578
$\beta_{27,1}$	-1.764	223.716	0.045	-1.855	-1.674
$\beta_{28,1}$	-1.121	126.003	0.038	-1.197	-1.046
$\beta_{29,1}$	-0.165	123.917	0.032	-0.226	-0.101
$\beta_{30,1}$	-1.146	267.956	0.033	-1.209	-1.081
$\beta_{31,1}$	-1.348	86.794	0.062	-1.469	-1.232
$\beta_{32,1}$	0.459	420.462	0.028	0.406	0.514
$\beta_{33,1}$	-1.144	83.982	0.047	-1.233	-1.050
$\beta_{34,1}$	-1.689	145.158	0.068	-1.826	-1.553
$\beta_{35,1}$	-2.171	148.403	0.066	-2.301	-2.044
$\beta_{36,1}$	-1.279	100.913	0.045	-1.368	-1.188
$\beta_{37,1}$	-1.403	106.701	0.056	-1.514	-1.294
$\beta_{38,1}$	-0.663	230.980	0.030	-0.719	-0.603
$\beta_{39,1}$	-0.213	127.771	0.035	-0.279	-0.143
$\beta_{40,1}$	-1.082	129.715	0.037	-1.153	-1.007
$\beta_{41,1}$	-0.499	62.573	0.048	-0.587	-0.402
$\beta_{42,1}$	-0.227	69.493	0.048	-0.316	-0.130
$\beta_{43,1}$	-0.754	84.039	0.045	-0.840	-0.663
$\beta_{44,1}$	-0.974	76.947	0.055	-1.075	-0.864
$\beta_{45,1}$	-0.176	90.284	0.039	-0.252	-0.097
$\beta_{46,1}$	-0.904	93.068	0.042	-0.984	-0.822
$\beta_{47,1}$	0.322	77.980	0.044	0.239	0.414
$\beta_{48,1}$	-0.028	113.319	0.036	-0.097	0.047
$\beta_{49,1}$	0.082	131.838	0.035	0.016	0.153
$\beta_{50,1}$	0.621	77.760	0.050	0.524	0.722
$\beta_{51,1}$	0.224	292.132	0.031	0.164	0.286
$\beta_{52,1}$	-0.003	85.521	0.043	-0.085	0.084
$\beta_{53,1}$	-0.904	112.679	0.040	-0.982	-0.827

Table 7.251: Emp. Scenario (M3b): β estimates (part 2).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
r_{12}	0.515	1871.759	0.018	0.479	0.549

Table 7.252: Emp. Scenario (M3b): R estimates.

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\xi_{3,2}$	0.607	3298.484	0.023	0.563	0.652
$\xi_{3,3}$	1.817	2619.324	0.031	1.755	1.879
$\xi_{16,2}$	1.302	2096.148	0.029	1.245	1.360
$\xi_{16,3}$	1.894	1964.845	0.032	1.831	1.957
$\xi_{26,2}$	0.503	1497.318	0.030	0.446	0.562
$\xi_{26,3}$	1.036	954.143	0.035	0.971	1.106
$\xi_{26,4}$	2.039	712.318	0.039	1.964	2.118
$\xi_{27,2}$	0.483	1535.581	0.032	0.422	0.548
$\xi_{35,2}$	0.712	611.358	0.048	0.620	0.810
$\xi_{35,3}$	1.240	426.611	0.052	1.140	1.346
$\xi_{46,2}$	0.392	4494.754	0.020	0.354	0.432

Table 7.253: Emp. Scenario (M3b): ξ estimates.

Empirical Scenario (M3c) – with Covariates, Gymnasium

Scenario (M3c) *Aggregates*:					
	$\bar{\bar{\check{\psi}}}$	$\overline{(\hat{T})_{eff}}$	$\overline{\hat{\sigma}(\check{\psi})}$	$\overline{q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)}$	$\overline{q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)}$
α_{math}	0.54	1372.29	0.03	0.49	0.59
(range)	(0.347, 0.686)	(692.434, 2064.11)	(0.021, 0.035)	(0.301, 0.627)	(0.394, 0.745)
α_{read}	0.52	643.27	0.04	0.45	0.60
(range)	(0.169, 0.893)	(80.612, 1372.223)	(0.022, 0.086)	(0.11, 0.756)	(0.23, 1.034)
β_{math}	-0.30	33.07	0.06	-0.42	-0.19
(range)	(-1.652, 1.509)	(23.844, 66.359)	(0.041, 0.075)	(-1.772, 1.369)	(-1.525, 1.66)
β_{read}	-1.06	101.42	0.06	-1.17	-0.93
(range)	(-2.435, 0.285)	(27.775, 573.768)	(0.035, 0.095)	(-2.582, 0.219)	(-2.285, 0.359)

Scenario (M3c) *Some Single Parameters*:					
	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\gamma_{1,1}$	-0.620	1425.208	0.035	-0.688	-0.550
$\gamma_{1,2}$	0.579	50.460	0.069	0.442	0.712
$\gamma_{1,3}$	0.008	52.395	0.001	0.006	0.010
$\gamma_{2,1}$	0.253	1933.958	0.035	0.186	0.321
$\gamma_{2,2}$	0.406	64.638	0.066	0.282	0.541
$\gamma_{2,3}$	0.007	84.512	0.001	0.006	0.009
r_{12}	0.555	1348.541	0.016	0.524	0.586
$\xi_{3,2}$	0.626	1378.242	0.029	0.570	0.685
$\xi_{3,3}$	1.933	854.874	0.036	1.862	2.002
$\xi_{16,2}$	1.287	897.443	0.034	1.221	1.353
$\xi_{16,3}$	1.881	828.288	0.035	1.811	1.950
$\xi_{26,2}$	0.341	832.397	0.036	0.275	0.415
$\xi_{26,3}$	0.911	390.159	0.044	0.826	0.997
$\xi_{26,4}$	2.072	303.413	0.047	1.982	2.166
$\xi_{27,2}$	0.537	862.437	0.039	0.461	0.615
$\xi_{35,2}$	0.620	315.446	0.062	0.504	0.745
$\xi_{35,3}$	1.295	220.857	0.066	1.172	1.430
$\xi_{46,2}$	0.433	2454.301	0.023	0.387	0.480

Table 7.254: Aggregates and some single parameters for empirical example (M3c) including subsetted data with respect to Gymnasium.

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\alpha_{1,1}$	0.457	1524.948	0.025	0.407	0.507
$\alpha_{2,1}$	0.581	1378.565	0.026	0.530	0.631
$\alpha_{3,1}$	0.554	1853.287	0.022	0.510	0.598
$\alpha_{4,1}$	0.655	739.026	0.031	0.597	0.718
$\alpha_{5,1}$	0.509	1241.916	0.028	0.455	0.564
$\alpha_{6,1}$	0.468	1908.544	0.023	0.425	0.513
$\alpha_{7,1}$	0.550	700.455	0.035	0.482	0.619
$\alpha_{8,1}$	0.501	1799.278	0.023	0.456	0.548
$\alpha_{9,1}$	0.607	1026.962	0.030	0.548	0.668
$\alpha_{10,1}$	0.472	1561.138	0.025	0.424	0.520
$\alpha_{11,1}$	0.473	1717.220	0.023	0.428	0.518
$\alpha_{12,1}$	0.633	1341.821	0.027	0.581	0.685
$\alpha_{13,1}$	0.347	1726.252	0.024	0.301	0.394
$\alpha_{14,1}$	0.594	1272.201	0.025	0.545	0.644
$\alpha_{15,1}$	0.370	692.434	0.035	0.303	0.440
$\alpha_{16,1}$	0.505	2064.110	0.021	0.463	0.548
$\alpha_{17,1}$	0.588	1292.147	0.028	0.535	0.642
$\alpha_{18,1}$	0.554	1568.674	0.025	0.506	0.604
$\alpha_{19,1}$	0.633	818.603	0.033	0.569	0.700
$\alpha_{20,1}$	0.562	1539.749	0.026	0.511	0.614
$\alpha_{21,1}$	0.686	989.607	0.030	0.627	0.745
$\alpha_{22,1}$	0.577	1433.504	0.027	0.525	0.629

Table 7.255: Emp. Scenario (M3c): α^{Δ} estimates (part 1).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\alpha_{23,2}$	0.389	224.807	0.055	0.282	0.496
$\alpha_{24,2}$	0.432	80.612	0.086	0.268	0.594
$\alpha_{25,2}$	0.467	701.326	0.033	0.405	0.532
$\alpha_{26,2}$	0.384	1166.015	0.022	0.343	0.428
$\alpha_{27,2}$	0.288	508.419	0.037	0.217	0.362
$\alpha_{28,2}$	0.356	595.070	0.034	0.290	0.423
$\alpha_{29,2}$	0.325	1337.002	0.024	0.280	0.374
$\alpha_{30,2}$	0.169	668.939	0.031	0.110	0.230
$\alpha_{31,2}$	0.893	122.923	0.072	0.751	1.034
$\alpha_{32,2}$	0.272	1312.022	0.022	0.230	0.315
$\alpha_{33,2}$	0.584	258.743	0.051	0.482	0.686
$\alpha_{34,2}$	0.692	141.847	0.070	0.565	0.831
$\alpha_{35,2}$	0.418	696.291	0.032	0.357	0.481
$\alpha_{36,2}$	0.424	372.212	0.044	0.339	0.516
$\alpha_{37,2}$	0.554	241.716	0.052	0.456	0.659
$\alpha_{38,2}$	0.261	1183.050	0.025	0.211	0.310
$\alpha_{39,2}$	0.411	945.146	0.026	0.358	0.462
$\alpha_{40,2}$	0.497	443.973	0.040	0.419	0.576
$\alpha_{41,2}$	0.661	580.600	0.037	0.591	0.735
$\alpha_{42,2}$	0.805	515.541	0.039	0.730	0.883
$\alpha_{43,2}$	0.623	547.050	0.037	0.553	0.698
$\alpha_{44,2}$	0.764	300.698	0.050	0.666	0.864
$\alpha_{45,2}$	0.473	921.413	0.027	0.421	0.526
$\alpha_{46,2}$	0.561	721.670	0.033	0.497	0.627
$\alpha_{47,2}$	0.766	675.081	0.035	0.699	0.837
$\alpha_{48,2}$	0.676	748.258	0.033	0.611	0.742
$\alpha_{49,2}$	0.607	867.271	0.030	0.547	0.667
$\alpha_{50,2}$	0.789	718.404	0.034	0.722	0.858
$\alpha_{51,2}$	0.321	1372.223	0.024	0.274	0.368
$\alpha_{52,2}$	0.833	499.717	0.041	0.756	0.916
$\alpha_{53,2}$	0.539	473.483	0.041	0.462	0.622

Table 7.256: Emp. Scenario (M3c): α^{Δ} estimates (part 2).

	$\bar{\bar{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\gamma_{1,1}$	-0.620	1425.208	0.035	-0.688	-0.550
$\gamma_{1,2}$	0.579	50.460	0.069	0.442	0.712
$\gamma_{1,3}$	0.008	52.395	0.001	0.006	0.010
$\gamma_{2,1}$	0.253	1933.958	0.035	0.186	0.321
$\gamma_{2,2}$	0.406	64.638	0.066	0.282	0.541
$\gamma_{2,3}$	0.007	84.512	0.001	0.006	0.009

Table 7.257: Emp. Scenario (M3c): γ estimates.

	$\bar{\bar{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\beta_{1,1}$	-0.485	25.881	0.049	-0.581	-0.386
$\beta_{2,1}$	-0.132	28.687	0.061	-0.250	-0.012
$\beta_{3,1}$	-1.652	37.752	0.062	-1.772	-1.525
$\beta_{4,1}$	1.509	45.106	0.075	1.369	1.660
$\beta_{5,1}$	-0.657	30.825	0.055	-0.764	-0.551
$\beta_{6,1}$	-0.044	28.353	0.051	-0.144	0.056
$\beta_{7,1}$	-1.123	27.589	0.060	-1.240	-1.005
$\beta_{8,1}$	0.151	29.925	0.054	0.046	0.258
$\beta_{9,1}$	-0.708	24.603	0.063	-0.829	-0.585
$\beta_{10,1}$	-0.345	28.576	0.052	-0.442	-0.244
$\beta_{11,1}$	0.547	39.317	0.053	0.446	0.650
$\beta_{12,1}$	0.006	25.341	0.065	-0.121	0.132
$\beta_{13,1}$	-0.620	39.827	0.041	-0.698	-0.539
$\beta_{14,1}$	0.772	33.875	0.065	0.649	0.900
$\beta_{15,1}$	-1.442	66.359	0.048	-1.534	-1.345
$\beta_{16,1}$	-1.638	46.238	0.060	-1.756	-1.523
$\beta_{17,1}$	0.496	33.176	0.065	0.370	0.625
$\beta_{18,1}$	0.158	25.471	0.058	0.048	0.273
$\beta_{19,1}$	-0.763	28.153	0.066	-0.889	-0.637
$\beta_{20,1}$	-0.100	30.047	0.059	-0.214	0.016
$\beta_{21,1}$	-0.402	23.844	0.070	-0.535	-0.266
$\beta_{22,1}$	-0.194	28.599	0.061	-0.311	-0.073
$\beta_{23,1}$	-1.961	154.652	0.064	-2.081	-1.828

Table 7.258: Emp. Scenario (M3c): β estimates (part 1).

	$\bar{\check{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\beta_{24,1}$	-2.394	151.216	0.087	-2.570	-2.222
$\beta_{25,1}$	-0.946	54.347	0.050	-1.036	-0.841
$\beta_{26,1}$	-2.047	147.516	0.056	-2.155	-1.937
$\beta_{27,1}$	-1.980	377.126	0.054	-2.082	-1.868
$\beta_{28,1}$	-1.308	97.797	0.047	-1.392	-1.209
$\beta_{29,1}$	-0.427	82.525	0.038	-0.497	-0.345
$\beta_{30,1}$	-1.296	573.768	0.038	-1.369	-1.221
$\beta_{31,1}$	-1.871	50.133	0.095	-2.048	-1.661
$\beta_{32,1}$	0.285	129.028	0.036	0.219	0.359
$\beta_{33,1}$	-1.627	60.040	0.065	-1.744	-1.490
$\beta_{34,1}$	-1.976	68.423	0.083	-2.133	-1.807
$\beta_{35,1}$	-2.435	163.612	0.073	-2.582	-2.285
$\beta_{36,1}$	-1.598	100.368	0.055	-1.697	-1.483
$\beta_{37,1}$	-1.724	78.192	0.063	-1.838	-1.590
$\beta_{38,1}$	-0.731	182.119	0.035	-0.796	-0.657
$\beta_{39,1}$	-0.424	72.720	0.045	-0.504	-0.330
$\beta_{40,1}$	-1.279	57.724	0.055	-1.375	-1.159
$\beta_{41,1}$	-0.725	31.958	0.063	-0.833	-0.588
$\beta_{42,1}$	-0.464	27.775	0.077	-0.594	-0.293
$\beta_{43,1}$	-0.935	31.589	0.062	-1.040	-0.804
$\beta_{44,1}$	-1.141	32.932	0.073	-1.270	-0.979
$\beta_{45,1}$	-0.291	50.990	0.049	-0.378	-0.184
$\beta_{46,1}$	-1.207	49.247	0.057	-1.312	-1.091
$\beta_{47,1}$	-0.092	28.241	0.073	-0.219	0.067
$\beta_{48,1}$	-0.321	35.532	0.066	-0.434	-0.172
$\beta_{49,1}$	-0.237	43.075	0.059	-0.339	-0.107
$\beta_{50,1}$	0.129	30.974	0.076	-0.003	0.297
$\beta_{51,1}$	-0.076	94.115	0.041	-0.150	0.013
$\beta_{52,1}$	-0.449	28.171	0.078	-0.578	-0.274
$\beta_{53,1}$	-1.250	58.008	0.057	-1.353	-1.128

Table 7.259: Emp. Scenario (M3c): β estimates (part 2).

	$\bar{\bar{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
r_{12}	0.555	1348.541	0.016	0.524	0.586

Table 7.260: Emp. Scenario (M3c): R estimates.

	$\bar{\bar{\psi}}$	$(\hat{T})_{eff}$	$\hat{\sigma}(\check{\psi})$	$q_{2.5\%}\left(\check{\boldsymbol{\psi}}\right)$	$q_{97.5\%}\left(\check{\boldsymbol{\psi}}\right)$
$\xi_{3,2}$	0.626	1378.242	0.029	0.570	0.685
$\xi_{3,3}$	1.933	854.874	0.036	1.862	2.002
$\xi_{16,2}$	1.287	897.443	0.034	1.221	1.353
$\xi_{16,3}$	1.881	828.288	0.035	1.811	1.950
$\xi_{26,2}$	0.341	832.397	0.036	0.275	0.415
$\xi_{26,3}$	0.911	390.159	0.044	0.826	0.997
$\xi_{26,4}$	2.072	303.413	0.047	1.982	2.166
$\xi_{27,2}$	0.537	862.437	0.039	0.461	0.615
$\xi_{35,2}$	0.620	315.446	0.062	0.504	0.745
$\xi_{35,3}$	1.295	220.857	0.066	1.172	1.430
$\xi_{46,2}$	0.433	2454.301	0.023	0.387	0.480

Table 7.261: Emp. Scenario (M3c): ξ estimates.

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