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A bootstrap functional central limit theorem for time-varying linear processes

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We provide a functional central limit theorem for a broad class of smooth functions for possibly non-causal multivariate linear processes with time-varying coefficients. Since the limiting processes depend on unknown quantities, we propose a local block bootstrap procedure to circumvent this inconvenience in practical applications. In particular, we prove bootstrap validity for a very large class of processes. Our results are illustrated by some numerical examples.

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1 Introduction

Unifying asymptotic theory is a powerful tool to develop statistical test procedures or to quantify the uncertainty of parameter estimators. Still, in many applications the limiting random objects of interest depend on unknown quantities that rely on the data generating process, e.g. its variance or the underlying dependence structure. Therefore, they cannot be used directly to construct (asymptotically) valid confidence sets or critical values of hypothesis tests. The bootstrap offers a convenient way to overcome these difficulties and is therefore the key to enable practical use of asymptotic results. From a statistical perspective, (functional) central limit theorems (CLT) and their bootstrap counterparts are particularly appealing as they can be applied to approximate confidence sets for parameters or critical values of L_2-test statistics. While there exists a large body of literature on bootstrap validity for empirical processes based stationary processes, see e.g. Künsch (1989); Naik-Nimbalkar and Rajarshi (1994); Bühlmann (1995); Doukhan, Lang, Leucht and Neumann (2015); Wieczorek (2016), there are no comparable results for locally stationary processes. The goal of the present paper is to fill this gap as the assumption of a gradually changing probabilistic structure over time is much more realistic than a stationary setting

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in many applications. Typical examples of use are medical or economical data, see Dette, Preuß and Vetter (2011); Anderson and Sandsten (2019); Jentsch, Leucht, Meyer and Beering (2020), for instance. The idea of approximating non-stationary time series on segments by stationary ones can be found in Priestley (1965), whereas the concept of local stationarity, on which this paper is based, goes back to Dahlhaus (1997). An overview of the state of the art is to be found in Dahlhaus (2012). More recently, Dahlhaus, Richter and Wu (2019) and Phandoidaen and Richter (2022) developed a broad asymptotic theory including laws of large numbers and (functional) CLTs for non-linear, causal locally stationary processes. However, on the bootstrap side methods and validity checks for locally stationary data are mostly tailor-made for specific applications so far: Sergides and Paparoditis (2009) used a semiparametric bootstrap in a testing framework, while wild bootstrap methods have been considered for instance by Vogt (2015), Brunotte (2022) and Karmakar, Richter and Wu (2022). Sergides and Paparoditis (2008) established a frequency domain bootstrap for statistics of the local periodogram, and Kreiss and Paparoditis (2015) developed a time-frequency domain bootstrap for a broad class of periodogram based statistics. In the present paper, we verify that a local version of the block bootstrap can be applied successfully to mimic the distributional behaviour of a large class of empirical processes indexed by smooth functions. This local block bootstrap was initially proposed by Paparoditis and Politis (2002) for the mean and by Dowla, Paparoditis and Politis (2013) for heteroscedastic time series with trend. Moreover, it was used by Franzier and Koo (2021) for indirect estimation in locally stationary structured models. In our case, we investigate possibly two-sided time-varying linear processes. The reasons are two-fold: On the one hand, to the best of our knowledge, there is no functional central limit theorem (FCLT) for non-causal locally stationary linear processes. However, non-causal models play an important role in the context of economic data. Examples for the application of non-causal AR models to stock volume data can be found in Breidt, Davis and Trindade (2001); Andrews, Calder and Davis (2009) and in Lanne and Saikkonen (2014) for non-causal VAR models. On the other hand, a generalization to non-linear processes would rely on high-level assumptions. Since the technical notation in the proofs would blow up, we restrict ourselves to linear processes here and leave an adaptation of the proof to causal non-linear processes for future research. It is important to note that the method of proof used by Phandoidaen and Richter (2022) to verify a FCLT for locally stationary Bernoulli shifts cannot be adapted in a straight-forward manner to prove validity of block bootstrap methods as it relies on martingale difference approaches for Bernoulli shifts. However, the bootstrap process does not inherit this structure.

The rest of the paper is organized as follows: In Section 2, we describe the setting under consideration and provide a FCLT for smooth functions of locally stationary linear processes. Following this, Section 3 is devoted to a local block bootstrap procedure for empirical processes. Afterwards, we illustrate the finite-sample performance of our method in Section 4. The proof of the main asymptotic result is carried out in Section 5, while all remaining proofs as well as several auxiliary results including their proofs are deferred to an Appendix in an online supplement to this paper.

Notation Let $|\cdot|_1$ denote the max column sum matrix norm, i.e. $|M|_1 = \max_{1\le j\le r}\sum_{i=1}^{d}|m^{(i,j)}|$ for some $(d\times r)$ matrix $M=(m^{(i,j)})$. Note that this matrix norm is submultiplicative. For a d-dimensional vector $\underline{v}=(v_1,\dots,v_d)'$, its ℓ_p-norm is denoted by $|\underline{v}|_p$, $p\in[1,\infty]$, whereas the $\mathscr{L}^p$-norm with $p\in[1,\infty)$ for d-dimensional random vectors $\underline{X}$ is signified by $\|\underline{X}\|_p := \left(E\,|\underline{X}|_p\right)^{1/p}$. Additionally, said notation is transferred to the bootstrap world by defining $\|\underline{X}^\star\|_{p,\star} := \left(E^\star\,|\underline{X}^\star|_p\right)^{1/p}$ as the (conditional) bootstrap $\mathscr{L}^p$-norm of $\underline{X}^\star$. Besides, the Lipschitz seminorm $|\cdot|_{\text{Lip}}$ of a function f as above signifies

$$|\mathsf{f}|_{\text{Lip}} := \sup_{\underline{\mathsf{x}},\underline{\mathsf{y}}\in\mathsf{T},\ \underline{\mathsf{x}}\neq\underline{\mathsf{y}}} |f(\underline{\mathsf{x}})-f(\underline{\mathsf{y}})|\,/\,|\underline{\mathsf{x}}-\underline{\mathsf{y}}|_1.$$

2 A functional CLT for locally stationary linear processes

Let $(\underline{\varepsilon}_t)_{t\in\mathbb{Z}}$ be a sequence of i.i.d. centred $\mathbb{R}^d$-valued random vectors and $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ a d-variate (possibly) two-sided linear process

$$\underline{X}_{t,T} = \underline{\mu}\left(\frac{t}{T}\right) + \sum_{j\in\mathbb{Z}} A_{t,T}(j)\,\underline{\varepsilon}_{t-j}, \tag{2.1}$$

where $\underline{\mu}=(\mu_1,\dots,\mu_d)'$ is a d-variate time-varying mean function and $(A_{t,T}(j))_{j\in\mathbb{Z}}$ are coefficient matrices of dimension $(d\times d)$. Note that the sequence $(\underline{\varepsilon}_t)_{t\in\mathbb{Z}}$ is not necessarily a white noise because we do not assume $E\underline{\varepsilon}_t^2<\infty$. Hence, $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ is only having a MA(∞)-representation but is not automatically a MA(∞)-process. To ensure existence of the afore-defined process and to exclude rapid changes in the coefficients over time leading to a meaningful statistical methodology, we impose the following conditions concerning $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$:

Assumption 1 (Locally stationary linear processes)**.**
The process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ is of form (2.1) with the following specifications:

(i) The innovations $(\underline{\varepsilon}_t)_{t\in\mathbb{Z}}$ are i.i.d., centred and $E|\varepsilon_1|_1<\infty$.

(ii) For some $\vartheta\in(0,1)$ and a constant $B<\infty$

$$\sup_{t,T}|A_{t,T}(j)|_1 \le B\,\vartheta^{j}. \tag{2.2}$$

Further, there exists an entry-wise continuously differentiable function $A(\cdot,j)\colon[0,1]\to\mathbb{R}^{d\times d}$ with $A(u,j)=\left(a^{(p,q)}(u,j)\right)_{p,q=1,\dots,d}$ such that for all $p,q=1,\dots,d$ and $s\le k\in\{0,1\}$, it holds

$$\sup_u\left|\frac{\partial^s a^{(p,q)}(u,j)}{\partial u^s}\right| \le B\,\vartheta^j \quad \textit{and} \quad \sup_{t,T} T\left|A_{t,T}(j)-A\left(\frac{t}{T},j\right)\right|_1 \le B\,\vartheta^j, \quad j\in\mathbb{Z}. \tag{2.3}$$

(iii) Each component of the mean function $\underline{\mu}$ is continuously differentiable.

Remark 2.1. *(i) This kind of assumptions represents a classical framework for statistical inference based on locally stationary processes, see Dahlhaus (2012) and Jentsch et al. (2020). Still, many papers, e.g. Dahlhaus and Subba Rao (2006) and Cardinali and Nason (2010), only require a polynomial decay instead of a geometric one as in* (2.2) *and* (2.3)*. In fact, a polynomial decay is* *sufficient in the present context as well. However, the degree of decay depends on the presumed order of absolute moments of the function f introduced later in a complicated manner, see Beering (2021) for details in a comparable context. For sake of notational simplicity, we stick to the exponential decay here.*

(ii) *Other and more general definitions of local stationarity are invoked for example in Vogt (2012) and Dahlhaus et al. (2019). They do not require a linear representation of the process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ to the price of citet causality.*

Having introduced nonstationarity as in Assumption 1, the process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ can be approximated locally by a (strictly) stationary linear process, its so-called *companion process* $\left(\underline{\widetilde{X}}_t(u)\right)_{t\in\mathbb{Z}}$

$$\underline{\widetilde{X}}_t(u) = \underline{\mu}(u) + \sum_{j\in\mathbb{Z}} A(u,j)\,\underline{\varepsilon}_{t-j}, \tag{2.4}$$

as long as u is close to the rescaled time t/T. Hence, compared to the original process the mean function $\underline{\mu}$ stays the same for $u = t/T$, whereas the function $A_{t,T}(j)$ is replaced by $A(u,j)$. From Assumption 1(ii), we can conclude

$$\sup_{u\in[0,1]} |A(u,j)|_1 \leq \widetilde{B}\,\vartheta^j \tag{2.5}$$

for some finite constant $\widetilde{B}$. This inequality connotes that (2.4) possesses a strictly stationary solution for each fixed u while Assumption 1 is satisfied.

Remark 2.2. (i) *Closeness of the locally stationary process and its companion process as well as closeness of companion processes for nearby rescaled time points can be specified. We obtain*

$$\sup_{1\leq t\leq T} \left\| \underline{X}_{t,T} - \underline{\widetilde{X}}_t\left(\frac{t}{T}\right) \right\|_1 = O(T^{-1})$$

if Assumption 1 holds true for $k=0$ and $\|\underline{\widetilde{X}}_0(u_1) - \underline{\widetilde{X}}_0(u_2)\|_1 \leq C|u_1-u_2|_1\ \forall u_1,u_2 \in [0,1]$ for some $C<\infty$ if Assumption 1 holds true for $k=1$; see Jentsch et al. (2020, Lemma 2.1).

(ii) *Although the construction with $A_{t,T}(j)$ and $A(t/T,j)$ appears to be unnecessarily complicated, it is required to include time-varying ARMA-processes, see Dahlhaus (2012) for details.*

Statistical methods for locally stationary processes can either rely on *local* or *global* characteristics of the process. Local quantities of interest are, for instance, the local variance $\mathrm{Var}(\widetilde{X}_0(u))$ for any fixed $u\in(0,1)$ or the local characteristic function, introduced in Jentsch et al. (2020) as $\varphi(u,\underline{s}) := E\left(e^{i\langle \underline{s},\underline{\widetilde{X}}_1(u)\rangle}\right)$, $\underline{s}\in\mathbb{R}^d$. These quantities can be estimated using kernel estimates based on the observations $X_{1,T},\ldots,X_{T,T}$; e.g. $\varphi(u,\underline{s})$ can be estimated by the local empirical characteristic function (ECF)

$$\widehat{\varphi}(u,\underline{s}) = \frac{1}{b_T\,T}\sum_{t=1}^{T} K\left(\frac{t/T-u}{b_T}\right) e^{i\langle \underline{s},\underline{X}_{t,T}\rangle}, \quad \underline{s}\in\mathbb{R}^d, \tag{2.6}$$

using a suitable kernel function K and an appropriate bandwidth b_T. A simple and prominent example for a global quantity of interest is the integrated volatility in high-frequency finance, see Feng (2015, sec. 2) and references therein. Consider a discrete-time model for the intraday log returns $Y_{t,T} = \sqrt{T}^{-1}\sigma(t/T)\,\varepsilon_t$ for a smooth deterministic spot volatility function σ and a centred stationary process $(\varepsilon_t)_t$. Then, a natural estimator for the integrated volatility is given by the realized volatility

$$RV = \sum_{t=1}^{T} Y_{t,T}^2. \tag{2.7}$$

Our goal is to derive a FCLT that is flexible enough to deduce the asymptotic distribution of both quantities (2.6) and (2.7). Therefore, we consider function classes changing with T

$$\mathscr{F}_T := \left\{ \sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}_{t,T}\right) : s \in \mathscr{S} \subseteq \overline{\mathbb{R}}^d \,\middle|\, (w_{t,T})_t \text{ satisfying Ass. 2, } f \text{ satisfying Ass. 4} \right\}$$

where the underlying Assumptions 2 and 4 are specified below. Obviously, the local ECF is included considering its real and imaginary part separately by setting $w_{t,T} := (b_T T)^{-1/2} K\left(\frac{t/T-u}{b_T}\right)$ and $f\left(\underline{s}, \underline{X}_{t,T}\right) = \cos(\underline{s}'\underline{X}_{t,T})$ or $f\left(\underline{s}, \underline{X}_{t,T}\right) = \sin(\underline{s}'\underline{X}_{t,T})$, respectively. Secondly, if we choose $w_{t,T} := T^{-1/2}$, $X_{t,T} = \sigma(t/T)\,\varepsilon_t$ and $f(\underline{s}, \underline{X}_{t,T}) = X_{t,T}^2$, we end up with RV. The first example perfectly illustrates that several weights may be zero. We control for the number of zero weights and the magnitude of the non-zero weights as follows:

Assumption 2 (Weights)**.**
The sequence of non-negative weights $(w_{t,T})_{t=1}^T$ fulfils $\sup_{1\le t\le T} w_{t,T} \le C_w\, d_T^{-1/2}$ for some finite constant C_w, where $d_T \underset{T\to\infty}{\longrightarrow} \infty$ denotes the number of non-zero weights in $(w_{t,T})_{t=1}^T$.

Before we specify the class of functions for our Donsker-type result, we need to introduce some auxiliary quantities to properly handle the dependence structure of the observed and the companion processes within the proofs relying on truncation arguments. Considering a truncation parameter $M \in \mathbb{N}$, we set

$$\underline{X}_{t,T}^{(M)} := \underline{\mu}\left(\frac{t}{T}\right) + \sum_{|j|<M} A_{t,T}(j)\,\underline{\varepsilon}_{t-j} \quad \text{and} \quad \widetilde{\underline{X}}_t^{(M)}(u) := \underline{\mu}(u) + \sum_{|j|<M} A(u,j)\,\underline{\varepsilon}_{t-j}. \tag{2.8}$$

Assumption 3 (Function I)**.**
Let $(\mathscr{S}, \rho)$ be a compact semimetric space with $\mathscr{S} \subseteq \overline{\mathbb{R}}^d$. The function $f\colon \mathscr{S} \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\sup_{\underline{s}\in\mathscr{S}} |f(\underline{s},\underline{x}) - f(\underline{s},\underline{x}^\circ)| \le C_{Lip}\, |\underline{x}-\underline{x}^\circ|_1, \quad \underline{x}, \underline{x}^\circ \in \mathbb{R}^d, \tag{2.9}$$

for some $C_{Lip} < \infty$. Additionally, for some $\delta \in (0, 1/2)$ it holds

$$\sup_{t\le T,\, \underline{s}\in\mathscr{S},\, M\in\mathbb{N}} E[|f(\underline{s},\underline{X}_{t,T})|^{2+\delta} + |f(\underline{s},\underline{X}_{t,T}^{(M)})|^{2+\delta}] < \infty \qquad \textit{and}$$

$$\sup_{u\in[0,1],\, \underline{s}\in\mathscr{S},\, M\in\mathbb{N}} E[|f(\underline{s},\widetilde{\underline{X}}_0(u))|^{2+\delta} + |f(\underline{s},\widetilde{\underline{X}}_0^{(M)}(u))|^{2+\delta}] < \infty.$$

Remark 2.3. *(i) If $\|\underline{\varepsilon}_0\|_{2+\delta} < \infty$, the moment conditions in Assumption 3 follow from* (2.9)*. However, only $\|\underline{\varepsilon}_0\|_1 < \infty$ is presumed in Assumption 1. In doing so, we can allow for data generating processes with infinite variance as long as the function applied to it is regular enough, see Section 4.2 for an example.*

(ii) Obviously, assuming validity of Assumption 3 additionally to Assumption 1 allows for a generalization of the results stated in Remark 2.2(i) towards

$$\sup_{1\le t\le T}\left\| f\left(\underline{s},\underline{X}_{t,T}\right)-f\left(\underline{s},\widetilde{\underline{X}}_t\left(\frac{t}{T}\right)\right)\right\|_1 = O(T^{-1}).$$

(iii) Our assumptions are slightly different compared to those in Phandoidaen and Richter (2022), where they allow for Hölder continuity of f with respect to (w.r.t.) x. We expect that it is possible to relax our assumption in a similar way, which, however, would have an effect on the choice of tuning parameters of the bootstrap procedure in Section 3. For sake of notational simplicity, we stick to Lipschitz continuity here. Moreover, note that we work under weaker moment constraints regarding the data generating process in their case $s=1$.

We abbreviate the centred version of f by

$$\bar{f}\left(\underline{s},\cdot\right) := f\left(\underline{s},\cdot\right) - Ef\left(\underline{s},\cdot\right) \tag{2.10}$$

and state a CLT for the finite-dimensional distributions first. Note that this result is sufficient to deduce asymptotic normality of the realized volatility defined in (2.7).

Theorem 2.4 (Central Limit Theorem)**.**
Suppose Assumption 1 holds true for $k=1$ and Assumptions 2 as well as 3 are valid. Then, for any $J\in\mathbb{N}$ and $\underline{s}_j\in\mathscr{S}$, $j=1,\dots,J$, we have

$$\left(\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s}_j,\underline{X}_{t,T}\right), j=1,\dots,J\right) \xrightarrow{d} \mathscr{N}\left(\underline{0},\mathbf{V}\right)$$

as $T\to\infty$, where $\mathbf{V} := \left(\mathbf{V}\left(\underline{s}_{j_1},\underline{s}_{j_2}\right)\right)_{j_1,j_2=1,\dots,J}$ is a $(J\times J)$ covariance matrix with

$$\mathbf{V}\left(\underline{s}_{j_1},\underline{s}_{j_2}\right) = \sum_{h\in\mathbb{Z}}\lim_{T\to\infty}\sum_{t=1}^{T} w_{t,T}\,w_{t+h,T}\,\mathrm{Cov}\left(f\left(\underline{s}_{j_1},\widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2},\widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right)\right).$$

We aim at deriving a FCLT. For that purpose, some additional conditions on f are imposed to assure tightness. To this end, let $D(u,\mathscr{S},\rho) = \max\{\#\mathscr{T}_0 \,|\, \mathscr{T}_0\subseteq\mathscr{S},\ \rho\left(\underline{s}_1,\underline{s}_2\right) > u\ \forall\, \underline{s}_1\neq\underline{s}_2\in\mathscr{T}_0\}$ denote the usual packing number defined e.g. in van der Vaart and Wellner (2000, Def. 2.2.3).

Assumption 4 (Function II)**.**

(i) Additionally to Assumption 3, it holds $\|f(\cdot,\underline{x})\|_\infty < \infty$ for any $\underline{x}\in\mathbb{R}^d$.

(ii) It holds $|f\left(\underline{s},\mathbf{x}\right) - f\left(\underline{s}^\circ,\mathbf{x}\right)| \le g(\mathbf{x})\,\rho\left(\underline{s},\underline{s}^\circ\right)$ with some function $g\colon\mathbb{R}^d\to\mathbb{R}_{\ge 0}$ satisfying one of the following conditions:

(a) $E[g^{2+\delta}(Y)] < K$ for some $K<\infty$ with $Y\in\left\{\widetilde{X}_t(u),\widetilde{X}_t^{(M)}(u)\right\}_{t\in\mathbb{Z},M\in\mathbb{N}}$,

(b) $\|f\|_\infty<\infty$ and $E[g^{1+\delta/2}(Y)] < K$ for some $K<\infty$ with $Y\in\{\widetilde{X}_t(u),\widetilde{X}_t^{(M)}(u)\}_{t\in\mathbb{Z},M\in\mathbb{N}}$.

(iii) For any $u>0$, let $D(u,\mathscr{S},\rho)\le C_D\,(1+u^{-1})^d$.

Note that in Assumption 4(ii), we distinguish between bounded and unbounded functions f. At first glance, boundedness seems much more restrictive as unboundedness, but the different assumptions concerning the moments of the function g open up the field of applications. As an example, consider the ECF case in Jentsch et al. (2020), where the function g is equal to the $|\cdot|_1$-norm. Combined with an α-stable distribution with $\alpha = 1.5$, which we will use in our simulation study later on, we are not able to fulfil case (a) of the second part of Assumption 4 due to the lack of second absolute moments. Especially in finance, the absence of those moments is quite common. Thus, instead of being mostly excluding, the separate handling of bounded functions broadens the scope.

Assumption 4(iii) holds, for instance, for $(\mathscr{S},\rho) = \left([-S,S]^d, \rho\left(\underline{s}_1,\underline{s}_2\right) = |\underline{s}_1 - \underline{s}_2|_1\right)$ for any $S \in (0,\infty)$ as well as for $(\mathscr{S},\rho) = \left(\overline{\mathbb{R}}^d, \rho\left(\underline{s}_1,\underline{s}_2\right) = \sum_{i=1}^d \left|\arctan\left(\underline{s}_{1,i}\right) - \arctan\left(\underline{s}_{2,i}\right)\right|_1\right)$.

Theorem 2.5 (Functional Central Limit Theorem).
Suppose Assumption 1 holds true for $k = 1$ and Assumptions 2 as well as 4 are valid. Then,

$$\left(\sum_{t=1}^{T} w_{t,T}\, \bar{f}\left(\underline{s}, \underline{X}_{t,T}\right)\right)_{\underline{s}\in\mathscr{S}} \xrightarrow{d} \left(G\left(\underline{s}\right)\right)_{\underline{s}\in\mathscr{S}}$$

as T tends to ∞, where $\left(G\left(\underline{s}\right)\right)_{\underline{s}\in\mathscr{S}}$ is a centred Gaussian process with continuous sample paths w.r.t. ρ and covariance function $V\left(\underline{s},\underline{s}^\circ\right)$ originating from Theorem 2.4.

3 Locally blockwise bootstrapped empirical processes

As already pointed out in Paparoditis and Politis (2002), the classical block bootstrap algorithm for stationary time series has to be modified in the case of locally stationary time series to capture not only the dependence structure but also the time-changing characteristics of the process. More precisely, given $X_{1,T},\ldots,X_{T,T}$, a block of a bootstrap analogue starting at time point t should only consist of a stretch of the original time series with time index close to t. This is achieved by the introduction of an additional tuning parameter, the so-called window parameter, that controls for the range of observations a certain bootstrap block can be drawn from. An adaption of the local block bootstrap (LBB) proposed by Dowla et al. (2013) to the present setting reads as follows:

Algorithm 1 (Bootstrap Algorithm).

(a) Consider a blocklength L_T depending on T.

(b) Select a window parameter $D_T \in (0,1)$ such that $TD_T \in \mathbb{N}$.

(c) Generate i.i.d. integers $k_0,\ldots,k_{\lfloor T/L_T\rfloor-1}$ using a discrete uniform distribution on $[-TD_T, TD_T]$.

(d) For $i = 0,\ldots\lfloor T/L_T\rfloor - 1$, define $\underline{X}^\star_{1,T},\ldots,\underline{X}^\star_{T,T}$ by $\underline{X}^\star_{j+iL_T,T} := \underline{X}_{j+iL_T+k_i,T}$ for $j = 1,\ldots,L_T$, if the resulting set of indices is in $[1,T]$ and use $-k_i$ instead of k_i otherwise.

(e) Construct the bootstrap estimator by replacing $\underline{X}_{t,T}$ with $\underline{X}^\star_{t,T}$, that is

$$\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^\star_{t,T}\right).$$

Remark 3.1. *(i) The distribution used to generate $k_0, \dots, k_{\lfloor T/L_T \rfloor -1}$ does not need to assign uniform weights to every choice of k_i, see Paparoditis and Politis (2002). Here, we choose the discrete uniform distribution as it is easy to handle, analogously to Dowla et al. (2013). Besides, it matches the choice made for the ordinary moving block bootstrap algorithm designed for stationary processes.*

(ii) The case differentiation in part (d) of Algorithm 1 ensures that if a block is in danger of going over the edge, there is a sound way out. By adjusting the sign of k_i for the whole block, the interrelated dependence structure is preserved, and moreover, no observation is used twice in the same block.

To establish asymptotic validity, we have to modify our assumptions towards more restrictive moment conditions in the case of unbounded f. Especially when it comes to covariance results, finite $(2+\delta)$-th absolute moments of f are not always sufficient but we require the following:

Assumption 5 (Function III)**.**
Assumption 1 for $k = 1$ plus Assumptions 2 and 3 are satisfied and

$$\sup_{u\in[0,1],\underline{s}\in\mathscr{S},M\in\mathbb{N}} E[|f(\underline{s},\widetilde{X}_0(u))|^{4+\delta} + |f(\underline{s},\widetilde{X}_0^{(M)}(u))|^{4+\delta}] < \infty.$$

With the blocklength L_T and the window parameter D_T, two new parameters are involved, which need to behave good-naturedly in combination with the number of non-zero weights d_T:

Assumption 6 (Bootstrap Rates)**.**
For the blocklength $L_T \in \mathbb{N}$ and the window parameter D_T with $TD_T \in \mathbb{N}$, it holds

$$L_T \to \infty, \quad L_T = o\left(d_T^{\frac{\delta}{2(1+\delta)}}\right), \quad d_T^{\frac{2\delta}{2+\delta}} = \mathscr{O}(TD_T) \quad \text{and} \quad TD_T = \mathscr{O}\left(d_T^{\frac{1}{2+\delta}}\right).$$

Remark 3.2. *(i) In particular, these assumptions imply $L_T = o(TD_T)$.*

(ii) To the best of our knowledge, optimal choices of the tuning parameters L_T and D_T of the local block bootstrap have not been derived in the literature so far. As already mentioned in Dowla et al. (2013), an adaptation of the subsampling cross-validation approach proposed in Hall, Horowitz and Jing (1995) might be a feasible option. A deep consideration of this issue would go far beyond the scope of the paper and is left for future research.

In the style of (2.10), we define

$$\bar{f}^\star\left(\underline{s},\underline{X}_{t,T}^\star\right) := f\left(\underline{s},\underline{X}_{t,T}^\star\right) - E^\star f\left(\underline{s},\underline{X}_{t,T}^\star\right) \tag{3.1}$$

for $t = 1,\dots,T$ and $\underline{s} \in \mathbb{R}^d$ as the bootstrap version of the centred function $\bar{f}$, which enables us to state the bootstrap counterpart to Theorem 2.4 in a comprehensive way:

Theorem 3.3 (Bootstrap Central Limit Theorem)**.**
Under Assumptions 5 and 6, it holds for any $J \in \mathbb{N}$ and $\underline{s}_j \in \mathscr{S}$, $j = 1,\dots,J$,

$$\left(\sum_{t=1}^T w_{t,T}\,\bar{f}^\star\left(\underline{s}_j,\underline{X}_{t,T}^\star\right),\ j=1,\dots,J\right) \xrightarrow{d} Z \sim \mathscr{N}(0,\mathbf{V})$$

in probability, where $\mathbf{V}$ *is defined in Theorem 2.4. If additionally* $\mathbf{V}(\underline{s}_1, \underline{s}_1) > 0$*, then*

$$\sup_{v \in \mathbb{R}} \left| P^\star \left(\sum_{t=1}^{T} w_{t,T} \bar{f}^\star \left(\underline{s}_1, \underline{X}^\star_{t,T} \right) \leq v \right) - \Phi \left(\frac{v}{\mathbf{V}(\underline{s}_1, \underline{s}_1)} \right) \right| \xrightarrow{P} 0.$$

Note that this result is in line with Paparoditis and Politis (2002, Th. 3.1), where the mean of locally stationary time series $X_{t,T} = \mu + V(t/T)\varepsilon_t$ with $\mu \in \mathbb{R}$, a smooth function V, and α-mixing stationary innovations $(\varepsilon_t)_t$ satisfying the stronger moment assumption $E|\varepsilon_t|^6 < \infty$ were considered.

For the more general case of a functional CLT, stronger assumptions concerning the function g introduced in Assumption 4 are required, too:

Assumption 7 (Function IV)**.**
The function g originating from Assumption 4 satisfies one of the following conditions:

(a) $Eg^{4+\delta}(Y) < K$ *for some* $K < \infty$ *with* $Y \in \{\widetilde{X}_t(u), \widetilde{X}_t^{(M)}(u)\}_{t \in \mathbb{Z}, M \in \mathbb{N}}$ *and for some constant* $C_{g,a} < \infty$

$$\sup_{t \in \{1,\ldots,T\}} \left\| g\left(\underline{X}_{t,T}\right) - g\left(\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right) \right\|_{4+\delta} \leq \frac{C_{g,a}}{T},$$

(b) $\|f\|_\infty < \infty$*,* $Eg^{1+\delta/2}(Y) < K$ *for some* $K < \infty$ *with* $Y \in \{\widetilde{X}_t(u), \widetilde{X}_t^{(M)}(u)\}_{t \in \mathbb{Z}, M \in \mathbb{N}}$ *and for some constant* $C_{g,b} < \infty$

$$\sup_{t \in \{1,\ldots,T\}} \left\| g\left(\underline{X}_{t,T}\right) - g\left(\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right) \right\|_{\frac{2+\delta}{2}} \leq \frac{C_{g,b}}{T}.$$

Finally, we use the previously established bootstrap CLT in combination with a tightness result to prove the desired bootstrap FCLT:

Theorem 3.4 (Bootstrap Functional Central Limit Theorem)**.**
Let Assumptions 4 to 7 be true. Then, for $\bar{f}^\star$ *as in (3.1) it holds*

$$\left(\sum_{t=1}^{T} w_{t,T} \bar{f}^\star \left(\underline{s}, \underline{X}^\star_{t,T} \right) \right)_{\underline{s} \in \mathscr{S}} \xrightarrow{d} \left(G\left(\underline{s}\right) \right)_{\underline{s} \in \mathscr{S}}$$

in P-probability as T tends to ∞*, where* $\left(G\left(\underline{s}\right) \right)_{\underline{s} \in \mathscr{S}}$ *is defined in Theorem 2.5.*

We expect similar results to hold for a suitable version of the dependent wild bootstrap. Results on its validity in related settings can be found in Brunotte (2022). For our example in Subsection 4.1, the local block bootstrap outperformed the dependent wild bootstrap in a simulation study in Feng (2015). A detailed comparison of both methods is beyond the scope of the paper.

4 Numerical results

We illustrate the finite sample performance of the proposed local bootstrap procedure by two small numerical examples. In both cases, we replicate the simulations $N = 500$ times each with $B = 500$ bootstrap resamplings. The implementations are carried out in `R`; see R Core Team (2022).

4.1 An application to realized volatility

We investigate the coverage of bootstrap-based confidence sets for the realized volatility introduced in (2.7) as a global characteristic of the data generating process. As a straight-forward consequence of Theorem 2.4 and Theorem 3.3, we obtain the following result:

Corollary 4.1.
Assume that $(\varepsilon_t)_t$ *is a linear process with exponentially decaying coefficients and innovations that have finite absolute* $8+\delta$ *moments. Further, suppose that the long-run variance of* $(\varepsilon_t)_t$ *is positive and that* σ *is continuously differentiable on* $[0,1]$. *Then, under Assumption 6,*

$$\sup_{v\in\mathbb{R}}\left|P^\star\Big(\sqrt{T}\sum_{t=1}^{T}(\underline{X}_{t,T}^\star - E^*\underline{X}_{t,T}^\star)\le v\Big) - P\Big(\sqrt{T}\,\Big(RV-\sigma_\varepsilon^2\int_0^1\sigma^2(u)\,du\Big)\le v\Big)\right| \xrightarrow{P} 0.$$

In order to illustrate the finite-sample performance of the local bootstrap for RV, we revisit a scenario similar to Feng (2015). We consider the coverage of symmetric 90% confidence intervals derived by local bootstrap for samples of size $T=1000$ and $T=2000$ with $\sigma(u)=0.32(u-0.5)^2+0.04,\ u\in[0,1]$, reflecting a volatility smile, and $\varepsilon_t=0.5\,\eta_{t-1}+\eta_t$. Here, $(\eta_t)_t$ is a sequence of i.i.d. innovations satisfying $\eta_t\sim\mathcal{N}(0,0.8)$. As it can be seen from Figure 1, the coverage of the confidence intervals for RV obtained by the local bootstrap is close to the desired level. In particular, the results are robust w.r.t. appropriate choices of blocklength L_T and half window size $DT=D_T T$. Note that the optimal blocklengths of the ordinary moving block bootstrap in the sense of Lahiri (2003, Cor. 7.1, $k=2$) are $L_{1000}=3$ and $L_{2000}=4$ in this example.

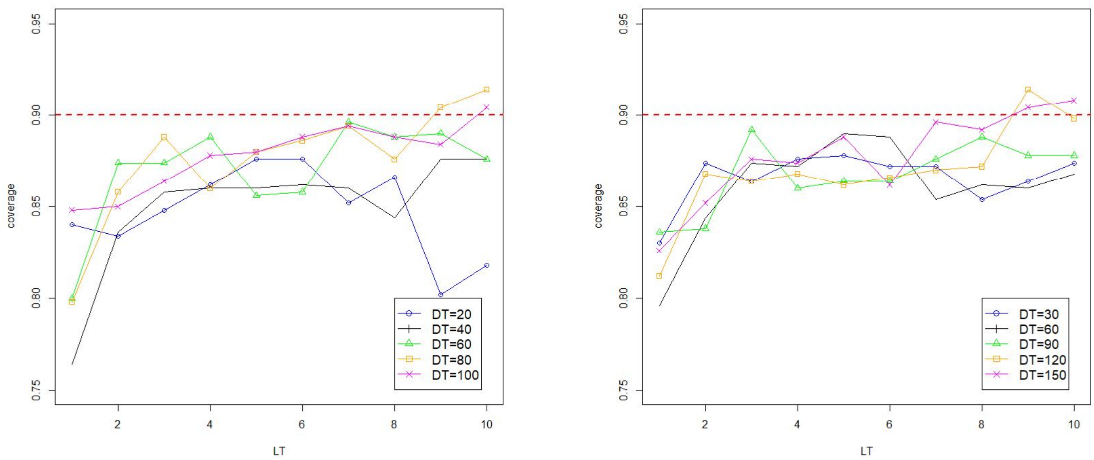

Figure 1: Coverage for $T=1000$ (left) and $T=2000$ (right) for different L_T and DT.

4.2 An application to local empirical characteristic functions

We consider local ECFs as introduced in (2.6). Therefore, we need to specify our requirements for both kernel and bandwidth.

Assumption 8 (Kernel and Bandwidth)**.**

(i) The function $K\colon \mathbb{R} \to [0,\infty)$ *is non-negative, symmetric and Lipschitz continuous. Additionally, it integrates up to 1 and has compact support* $[-1,1]$.

(ii) The sequence of bandwidths $(b_T)_{T\in\mathbb{N}}$ *is non-negative and fulfils* $b_T \to 0$, $b_T^2 T \to \infty$ *and* $b_T^3 T = o(1)$ *as* $T \to \infty$.

Now we are able to state the following corollary:

Corollary 4.2.
Suppose Assumptions 1 and 4 are satisfied for $k=1$ *plus Assumptions 6 and 8. Additionally, let*

$$\sum_{h_1,h_2\in\mathbb{Z}} \Big(\operatorname{Cov}\Big(\cos\Big(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\Big), \cos\Big(\langle \underline{s}, \underline{\widetilde{X}}_{h_2}(u)\rangle\Big)\Big) \operatorname{Cov}\Big(\sin\Big(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\Big), \sin\Big(\langle \underline{s}, \underline{\widetilde{X}}_{h_1}(u)\rangle\Big)\Big)$$
$$- \Big(\operatorname{Cov}\Big(\cos\Big(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\Big), \sin\Big(\langle \underline{s}, \underline{\widetilde{X}}_{h_1}(u)\rangle\Big)\Big) \operatorname{Cov}\Big(\cos\Big(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\Big), \sin\Big(\langle \underline{s}, \underline{\widetilde{X}}_{h_2}(u)\rangle\Big)\Big)\Big)\Big) > 0 \quad (4.1)$$

be fulfilled. Then, it holds for every $\underline{s} \in \mathbb{R}^d$,

$$\sup_{v\in\mathbb{R}} \left| P^\star\left(\left|(b_T T)^{1/2}\left(\widehat{\varphi}^\star(u;\underline{s}) - E^\star \widehat{\varphi}^\star(u;\underline{s})\right)\right| \le v\right) - P\left(\left|(b_T T)^{1/2}\left(\widehat{\varphi}(u;\underline{s}) - \varphi(u;\underline{s})\right)\right| \le v\right)\right| \xrightarrow{P} 0$$

as T *tends to* ∞.

We aim to examine the finite-sample impact of different sample and window size choices with regard to coverage results in the ECF setup addressed in Jentsch et al. (2020, sec. 5). That is, we generate a locally stationary process $(X_{t,T})_{t=1}^T$ by

$$X_{t,T} = \begin{cases} 0.9\sin\left(2\pi\frac{1}{T}\right)\varepsilon_0 + \varepsilon_1, & t = 1, \\ 0.9\sin\left(2\pi\frac{t}{T}\right)X_{t-1,T} + \varepsilon_t, & t = 2,\dots,T, \end{cases}$$

with $(\varepsilon_t)_{t\in\mathbb{Z}}$ forming an i.i.d. sequence and following an α-stable marginal distribution with parameters $\mu = 0$, $\alpha = 1.5$, $\beta = 0$ and $\gamma = 0.5$. The innovations bequeaths the α-stable distribution to the companion process with slightly different parameters:

$$\widetilde{\mu}(u) = 0, \quad \widetilde{\alpha}(u) = 1.5, \quad \widetilde{\beta}(u) = 0 \quad \text{and} \quad \widetilde{\gamma}(u) = \frac{0.5}{1 - |0.9\sin(2\pi u)|^{1.5}}.$$

This leads to

$$\varphi(u;s) = \exp\left(-\frac{0.5|s|^{1.5}}{1 - |0.9\sin(2\pi u)|^{1.5}}\right)$$

as belonging characteristic function for the companion process. Remembering (2.6), we need to specify some other parameters, which are $s = 6$, $u = 0.4$ as well as $b_T = T^{-0.35}$ and $b_T = T^{-0.4}$, respectively. Moreover, we choose the blocklength L_T equal to 15 as well as to 25. Regarding the sample size, we look at both $T = 2000$ and $T = 5000$. Because of these choices, there is no need to consider endpoints as they are filtered out by the kernel function. Furthermore, our simulations are based on a significance level of 0.05. Figure 2 shows the increase of the coverage results for higher choices of the bootstrap window size towards the aimed 0.95. Overall, the coverage results for $T = 2000$ grow faster than those for $T = 5000$. In most cases, they also exceed the target value of 0.95 around $TD_T = 200$ and decline again for larger values of TD_T. For the larger sample size, in turn, there is only an upward trend discernible, but with more variation within for $b_T = T^{-0.35}$. As to $b_T = T^{-0.4}$, the variation within the upward trend is higher for $T = 2000$. Concerning the blocklengths, the coverage results for $L_T = 15$ start closer to the target value than those for $L_T = 25$. On the other hand, the differences between the values for $T = 2000$ and $T = 5000$ are smaller for the larger blocklength. A wider simulation study to examine parameter choice impact can be found in Beering (2021, ch. 5).

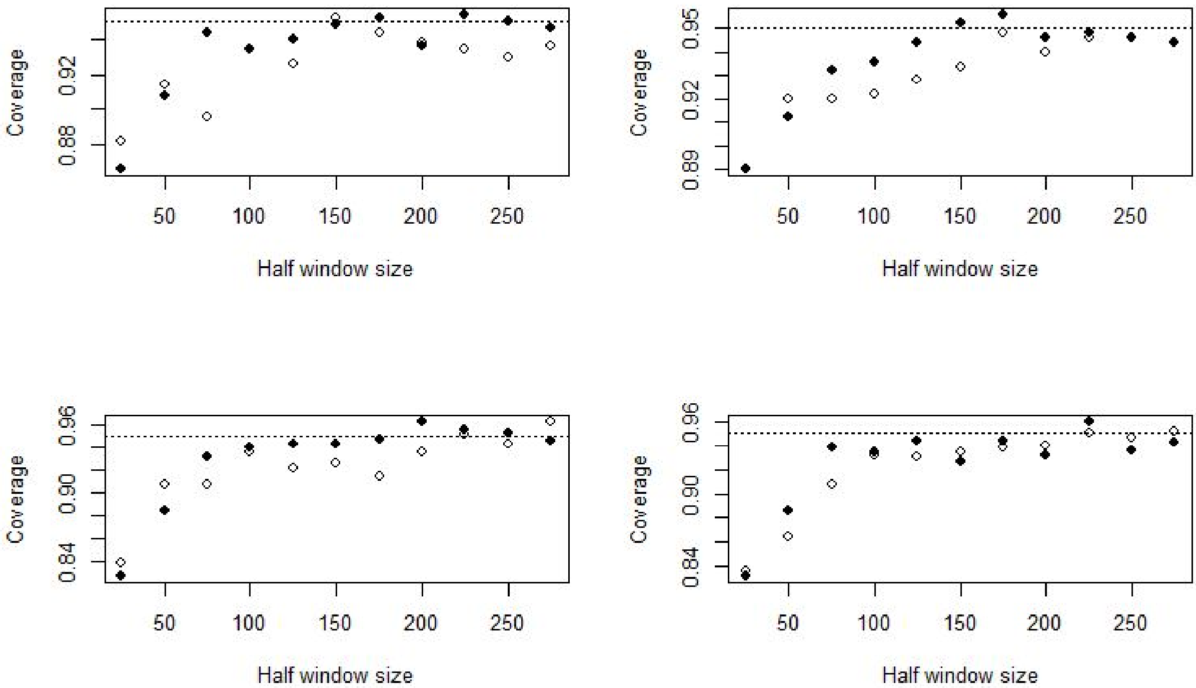

Figure 2: Coverage for different half window sizes TD_T with $L_T = 15$ in the first and $L_T = 25$ in the second row, both with $b_T = T^{-0.35}$ on the left and $b_T = T^{-0.4}$ on the right and for $T = 2000$ ($\bullet$) and $T = 5000$ ($\circ$).

For a more involved example in the field of hypothesis testing based on L_2 statistics and relying on the FCLTs stated in Theorems 2.5 and 3.4 instead of the CLTs, see Beering (2021, ch. 7), where a characteristic function based test for local independence is established.

5 Proof of Theorem 2.5

Throughout this section, C denotes a generic constant that may change its value from line to line.

Proof of Theorem 2.5. Following van der Vaart and Wellner (2000, Th. 1.5.4), we need to show convergence of the fidis and asymptotic tightness in order to prove process convergence. Finally, the continuity of the sample path of the limiting process can be concluded with the help of van der Vaart and Wellner (2000, Add. 1.5.8). Theorem 2.4 gives the required convergence of the fidis. Using van der Vaart and Wellner (2000, Th. 1.5.7), we show uniform equicontinuity. In view of

$$\lim_{T\to\infty} P\Big(\sup_{\underline{s}\in\mathscr{S}}\Big|\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\underline{X}_{t,T}\right)\Big| > \lambda\Big) = \lim_{T\to\infty} P\Big(\sup_{\underline{s}\in\mathscr{S}}\Big|\sum_{t=1}^{T} w_{t,T}\,\bar{f}\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\Big| > \frac{\lambda}{2}\Big),$$

which follows straightforwardly from

$$E\Big(\sup_{\underline{s}\in\mathscr{S}}\Big|\sum_{t=1}^{T} w_{t,T}\left(f\left(\underline{s},\underline{X}_{t,T}\right) - f\left(\underline{s},\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right)\Big|\Big) \le C\sum_{t=1}^{T} w_{t,T} E\left|\underline{X}_{t,T} - \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right|_1 \le C d_T^{-1/2},$$

it remains to show

$$\lim_{r\to 0}\limsup_{T\to\infty} P\Big(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\Big|\sum_{t=1}^{T} w_{t,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right) - \bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right)\Big| > \lambda\Big) = 0 \tag{5.1}$$

for any $\lambda > 0$. For this purpose, we define

$$\kappa_T := \left\lfloor d_T^{\frac{1}{2m}}\right\rfloor \quad \text{and} \quad \mu_T := \left\lfloor \frac{d_T}{2\kappa_T}\right\rfloor$$

for some case-specific $m > 1$ which will be particularized for the cases (a) and (b) in Assumption 4 later on. In the style of Arcones and Yu (1994), we divide our set of indices into blocks H_t, T_t and R in such a way that the indices of the first κ_T non-negative weights are in H_1, the indices of the second κ_T non-negative weights in T_1, the indices of the second κ_T non-negative weights in H_2 and so on until we have eventually μ_T H-blocks and μ_T T-blocks each. The remaining indices are arranged in block R. We establish an upper bound for the RHS of (5.1) considering the H-blocks, T-blocks and the R-block separately. Regarding the last one, we obtain from Assumption 2 and Assumption 4

$$\begin{aligned}
&\lim_{r\to 0}\limsup_{T\to\infty} P\Big(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\sum_{i\in R} w_{i,T}\left|\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right)\right| > \frac{\lambda}{3}\Big)\\
&\le \lim_{r\to 0}\limsup_{T\to\infty} C \sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\sum_{i\in R} w_{i,T} E\left[g\left(\underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right)\right]\rho\left(\underline{s}_1,\underline{s}_2\right)\\
&\le \limsup_{T\to\infty} C d_T^{-1/2}\kappa_T = 0.
\end{aligned}$$

As the sum of the T-blocks can be treated analogously to the one containing the H-blocks, we focus on the latter. To make use of the block structure such that the involved random variables whose indices are situated in different blocks $H_1,\ldots,H_{\mu_T}$ are independent, we rely on $\underline{\widetilde{X}}_i^{(M)}(\frac{i}{T})$ with $M = \lceil \kappa_T/2\rceil$:

$$\begin{aligned}
&\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right)\right)\\
&= \sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\\
&\qquad + \sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\\
&\qquad\qquad + \sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)
\end{aligned}$$

$=: \underline{\mathrm{I}}\mathrm{a} + \underline{\mathrm{I}}\mathrm{b} + \underline{\mathrm{I}}\mathrm{c}.$

First, similar arguments as used in the proof of Lemma A.2 yield for $\underline{\mathrm{I}}\mathrm{a}$ (and similarly $\underline{\mathrm{I}}\mathrm{b}$)

$$\lim_{r\to 0}\limsup_{T\to\infty} E\Bigg(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left|\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|\Bigg)\le C\limsup_{T\to\infty}\sum_{t=1}^{T} w_{t,T}\,\vartheta^{\lceil\kappa_T/2\rceil},$$

which is zero. Hence, it remains to show asymptotic negligibility of

$$\lim_{r\to 0}\limsup_{T\to\infty} P\Bigg(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\left|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right|>\frac{\lambda}{9}\Bigg). \tag{5.2}$$

Since the summands with different indices t are independent we can make use of standard empirical process theory. To this end, consider $\underline{s},\underline{s}_1,\underline{s}_2\in\mathscr{S}$ and define

$$\nu_T(\underline{s}) := \sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\,\bar{f}\left(\underline{s},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\quad\text{and}\quad \nu_T(\underline{s}_1,\underline{s}_2) := \nu_T(\underline{s}_1)-\nu_T(\underline{s}_2).$$

Inspired by Arcones and Yu (1994), we use a classical chaining argument. For this purpose, let

$$r_k := r\,2^{-k},\quad k=0,\dots,k_T, \tag{5.3}$$

for some r, k_T which will be specified thereinafter. Moreover, let $\mathscr{F}_k\subseteq\mathscr{S}$ be an index set satisfying

$$\#\mathscr{F}_k = D(k) = D(r_k,\mathscr{S},\rho)\quad\text{and}\quad \sup_{\underline{s}_1\in\mathscr{S}}\min_{\underline{s}_2\in\mathscr{F}_k}\rho(\underline{s}_1,\underline{s}_2)<r_k,\quad k\in\{0,\dots,k_T\}.$$

By Assumption 4(iii), it holds $D(k)\le r_k^{-d}$ for $r>0$ chosen sufficiently small. This gives us the existence of maps $\pi_k\colon\mathscr{S}\to\mathscr{F}_k$ for $k=0,\dots,k_T$ such that $|\underline{s}-\pi_k\underline{s}|_1\le r_k\,\forall\underline{s}\in\mathscr{S}$. Subsequently, we get the following two inequalities for $\underline{s}$, $\underline{s}_1,\underline{s}_2\in\mathscr{S}$ with $\rho(\underline{s}_1,\underline{s}_2)<r$:

$$\rho(\pi_0\underline{s}_1,\pi_0\underline{s}_2)\le 3r\quad\text{and}\quad\rho(\pi_k\underline{s},\pi_{k-1}\underline{s})\le 3r_k,\quad k\in\{1,\dots,k_T\}.$$

Thus, we get

$$\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S}\\ \rho(\underline{s}_1,\underline{s}_2)<r}}|\nu_T(\underline{s}_1,\underline{s}_2)|\le 2\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S}\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}|\nu_T(\underline{s}_1,\underline{s}_2)|+\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\le 3r}}|\nu_T(\underline{s}_1,\underline{s}_2)|+2\sum_{k=1}^{k_T}\sup_{\substack{\underline{s}_1\in\mathscr{F}_k,\underline{s}_2\in\mathscr{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\le 3r_k}}|\nu_T(\underline{s}_1,\underline{s}_2)|.$$

Again, we introduce some auxiliary quantities. Let

$$\lambda_k := r_k^{\frac{1}{4(1+\delta)}}\vee\left(\frac{4}{\bar{C}}\,r_k^{\frac{1}{1+\delta}}\log D(k)\right)^{1/2},\quad k\in\{1,\dots,k_T\},$$

for a constant $\bar{C}\in(0,\infty)$ specified later, which may take different values in cases (a) and (b). We get

$$\log D(k)\le\lambda_k^2\,\frac{\bar{C}}{4}\,r_k^{-\frac{1}{1+\delta}}. \tag{5.4}$$

Additionally, let r be small enough to allow for

$$4\sum_{k\in\mathbb{N}}\lambda_k\le\frac{\lambda}{27}. \tag{5.5}$$

Since we have $D(k)=\mathscr{O}\left(r_k^{-d}\right)$, summability of $(\lambda_k)_{k=1}^{k_T}$ for $T\to\infty$ is assured. At this point, we come back to (5.2). With the preassigned notation and (5.5), we can split up as follows:

$$\begin{aligned}
&P\Big(\sup_{\underline{s}_1,\underline{s}_2\in\mathscr{S},\,\rho(\underline{s}_1,\underline{s}_2)<r}\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left|\bar f\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar f\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|>\frac{\lambda}{9}\Big)\\
&\le P\Big(2\sup_{\underline{s}_1,\underline{s}_2\in\mathscr{S},\,\rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|>\frac{\lambda}{27}\Big)\\
&\qquad+P\Big(2\sum_{k=1}^{k_T}\sup_{\underline{s}_1\in\mathscr{F}_k,\underline{s}_2\in\mathscr{F}_{k-1},\,\rho(\underline{s}_1,\underline{s}_2)\le 3r_k}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|>4\sum_{k=1}^{k_T}\lambda_k\Big)+P\Big(\sup_{\underline{s}_1,\underline{s}_2\in\mathscr{F}_0\ \rho(\underline{s}_1,\underline{s}_2)\le 3r}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|>\frac{\lambda}{27}\Big)\\
&=:\mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}\tag{5.6}$$

In order to show asymptotic negligibility of terms II and III, we want to make use of Bernstein's inequality for sums of independent random variables exerted on the outer sum of ν_T. Term I, however, will be discussed by using a symmetrization lemma at the end of the proof.

The remaining part of the proof presumes Assumption 4(a) to hold. For (b) see Lemma A.6 in the Appendix. Before starting with the examination of term II in (5.6), we specify the lower bound of m as $m>\frac{20+15\delta-4\delta^2-3\delta^3}{2\delta(1-\delta^2)}$. Moreover, we need r_{k_T} in (5.3) to meet the following bounding conditions:

$$d_T^{-\frac{(2m\delta-4-3\delta)(1+\delta)}{2m(4+3\delta)}}\le r_{k_T}\le d_T^{-\frac{2}{m(1-\delta)}}.\tag{5.7}$$

Note that our choice of m guaranties that the left-hand side (LHS) is strictly smaller than the RHS. Now we turn our attention to the second summand in (5.6). To be able to apply Bernstein's inequality, we need to establish an upper bound for the variance of the inner sum of ν_T. Consider $l:=|i_1-i_2|$. Then,

$$\begin{aligned}
&\operatorname{Var}\left(\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right)\\
&\le\sum_{t=1}^{\mu_T}\sum_{i_1,i_2\in H_t} w_{i_1,T}\,w_{i_2,T}\Big|\operatorname{Cov}\Big(\Big(f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\Big)\\
&\qquad-\Big(f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\Big),\,f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\Big)\Big|\\
&\qquad\qquad+\sum_{t=1}^{\mu_T}\sum_{i_1,i_2\in H_t} w_{i_1,T}\,w_{i_2,T}\Big|\operatorname{Cov}\Big(f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\\
&\qquad\Big(f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\Big)-\Big(f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_2}^{(M(l))}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_2}^{(M(l))}\left(\frac{i_2}{T}\right)\right)\Big)\Big)\Big|
\end{aligned}\tag{5.8}$$

for $M(l):=\lceil\min\{M,l/2\}\rceil$. For the first covariance of (5.8) (and similarly for the second), we have

$$\begin{aligned}
&\Big|\operatorname{Cov}\Big(\Big(f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\Big)-\Big(f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\Big),\\
&\qquad\qquad f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\Big)\Big|\\
&\le\Big|\operatorname{Cov}\Big(f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\,f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\Big)\Big|\\
&\qquad+\Big|\operatorname{Cov}\Big(f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),\,f\left(\underline{s}_1,\underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\Big)\Big|,
\end{aligned}\tag{5.9}$$

and, again, we only examine the first covariance of (5.9). Similarly to Lemma A.2 we have

$$\begin{aligned}
&\left|\operatorname{Cov}\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right),f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|\\
&\le E\left|\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right)\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|\\
&\quad+C\rho(\underline{s}_1,\underline{s}_2)\sum_{M>|j|\ge M(l)}\vartheta^j. \qquad (5.10)
\end{aligned}$$

In the later following calculations to bound the variance of ν_T, we will need two suitable but different bounds. Therefore, we establish two alternative bounds for the first summand of (5.10):

i) Using Hölder's inequality, we get

$$\begin{aligned}
&E\left|\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right)\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|\\
&\qquad\le C\left\|\underline{\varepsilon}_0\right\|_{\frac{2+\delta}{1+\delta}}\sum_{M>|j|\ge M(l)}\vartheta^j.
\end{aligned}$$

ii) On the other hand, using similar arguments as in the proof of Lemma A.3, we get

$$\sup_{M\in\mathbb{N}}\left\|f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\right\|_{2+\delta}\left\|f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right\|_{\frac{2+\delta}{1+\delta}}\le C\rho\left(\underline{s}_1,\underline{s}_2\right).$$

These two bounds for (5.10) and similarly for (5.9) allow us to bound the covariance in (5.8) via

$$\begin{aligned}
&\left|\operatorname{Cov}\left(\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_1}^{(M)}\left(\frac{i_1}{T}\right)\right)\right)-\left(f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_1}^{(M(l))}\left(\frac{i_1}{T}\right)\right)\right),\right.\right.\\
&\qquad\qquad\left.\left.f\left(\underline{s}_1,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_{i_2}^{(M)}\left(\frac{i_2}{T}\right)\right)\right)\right|\\
&\le C\min\left\{\rho\left(\underline{s}_1,\underline{s}_2\right),\sum_{M>|j|\ge\left\lceil\frac{|i_1-i_2|}{2}\right\rceil}\vartheta^j\right\}\le C\min\left\{\rho\left(\underline{s}_1,\underline{s}_2\right),|i_1-i_2|^{-\frac{1+\delta}{\delta}}\right\}.
\end{aligned}$$

Thus, we obtain for any $R_0\ge 2$

$$\operatorname{Var}\left(\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right)\le C\mu_T\,\kappa_T\,d_T^{-1}\sum_{t=0}^{d_T}\min\left\{\rho\left(\underline{s}_1,\underline{s}_2\right),t^{-\frac{1+\delta}{\delta}}\right\}\le C\left(R_0\,\rho\left(\underline{s}_1,\underline{s}_2\right)+R_0^{-1/\delta}\right).$$

With $R_0:=\left\lfloor\rho\left(\underline{s}_1,\underline{s}_2\right)^{-\frac{\delta}{1+\delta}}\right\rfloor$ and for any r chosen sufficiently small, we get

$$\operatorname{Var}\left(\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right)\le C\rho\left(\underline{s}_1,\underline{s}_2\right)^{\frac{1}{1+\delta}}. \qquad (5.11)$$

Since we aim at the application of Bernstein's inequality to bound II in (5.6), we first provide a suitable approximation of II by a sum of bounded random variables. To this end, we define

$$\breve{f}\left(\underline{s},\widetilde{\underline{X}}_t^{(M)}\left(\frac{t}{T}\right)\right):=f\left(\underline{s},\widetilde{\underline{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)\mathbb{1}_{\Omega_{\sup,t}},\quad \breve{f}^c\left(\underline{s},\widetilde{\underline{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)=\breve{f}\left(\underline{s},\widetilde{\underline{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)-E\breve{f}\left(\underline{s},\widetilde{\underline{X}}_t^{(M)}\left(\frac{t}{T}\right)\right)$$

with

$$\Omega_{\sup,t}:=\left\{\omega\in\Omega\,\middle|\,\sup_{\underline{s}\in\mathscr{S},\,i\in H_t}w_{i,T}\left|f\left(\underline{s},\widetilde{\underline{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|\le d_T^{-\frac{\delta}{4+3\delta}}\right\}.$$

Note that in view of compactness of $(\mathscr{S},\rho)$, we have

$$P\left(\Omega\backslash\Omega_{\sup,t}\right) \leq P\bigg(\sup_{\underline{s}\in\mathscr{S},\, i\in H_t} w_{t,T}\left|f\left(\underline{s},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-f\left(\underline{0},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right| > \frac{1}{2}\,d_T^{-\frac{\delta}{4+3\delta}}\bigg)$$
$$+P\bigg(\sup_{\underline{s}\in\mathscr{S},\, i\in H_t} w_{t,T}\left|f\left(\underline{0},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right| > \frac{1}{2}\,d_T^{-\frac{\delta}{4+3\delta}}\bigg)$$
$$\leq C\sum_{i\in H_t} w_{i,T}^{2+\delta}\, d_T^{\frac{(2+\delta)\delta}{4+3\delta}}\left[E\left[g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right]^{2+\delta}+E\left|f\left(\underline{0},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|^{2+\delta}\right]$$
$$\leq C\,d_T^{-\frac{\delta^2}{2(4+3\delta)}-1+\frac{1}{2m}} = o(1).$$

Hence, II in (5.6) can be bounded from above by

$$P\Bigg(2\sum_{k=1}^{k_T}\sup_{\substack{\underline{s}_1\in\mathscr{F}_k,\underline{s}_2\in\mathscr{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}}|\nu_T(\underline{s}_1,\underline{s}_2)| > 4\sum_{k=1}^{k_T}\lambda_k\Bigg)$$
$$\leq P\Bigg(\sum_{k=1}^{k_T}\sup_{\substack{\underline{s}_1\in\mathscr{F}_k,\underline{s}_2\in\mathscr{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}}\left|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left[\check{f}^c\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\check{f}^c\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right]\right| > \sum_{k=1}^{k_T}\lambda_k\Bigg)$$
$$+\sum_{k=1}^{k_T}\mathbb{1}_{C\,r_k\sum_{t=}^{\mu_T}\sum_{i\in H_t} w_{i,T}\,E\left(g\left(\underline{\widetilde{X}}_t^{(M)}\left(\frac{i}{T}\right)\right)\mathbb{1}_{\Omega\backslash\Omega_{\sup,t}}\right)\geq\frac{\lambda_k}{2}}+o(1). \tag{5.12}$$

Note that asymptotic negligibility of the middle term on the RHS of (5.12) follows from

$$\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\,E\left[g\left(\underline{\widetilde{X}}_t^{(M)}\left(\frac{i}{T}\right)\right)\mathbb{1}_{\Omega\backslash\Omega_{\sup,t}}\right] \leq C\,d_T^{1/2}\left(d_T^{-\frac{\delta^2}{2(4+3\delta)}-1+\frac{1}{2m}}\right)^{\frac{1+\delta}{2+\delta}}$$

since $(r_k/\lambda_k)_k$ is uniformly bounded. To bound the first summand on the RHS of (5.12), we apply Bernstein's inequality and get

$$\sum_{k=1}^{k_T}\sum_{\substack{\underline{s}_1\in\mathscr{F}_k,\underline{s}_2\in\mathscr{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}} P\Bigg(\left|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\check{f}^c\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\check{f}^c\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right| > \lambda_k\Bigg)$$
$$\leq 2\sum_{k=1}^{k_T} D(k)\,D(k-1)\,\exp\left(-\frac{1}{2}\cdot\frac{\lambda_k^2}{V_{II,k}+\frac{\check{M}\lambda_k}{3}}\right), \tag{5.13}$$

where, by definition of H_t and $\check{f}$,

$$\check{M} := 4\,d_T^{\frac{4+\delta(3-2m)}{2m(4+3\delta)}} \geq \sup_{\underline{s}_1,\underline{s}_2\in\mathscr{S},\, t\in\{1,\ldots,\mu_T\}}\left|\sum_{i\in H_t} w_{i,T}\left(\check{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\check{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right|$$

and, in view of (5.11), for some appropriately chosen $C<\infty$

$$V_{II,k} := C\,r_k^{\frac{1}{1+\delta}} \geq \sup_{\substack{\underline{s}_1\in\mathscr{F}_k,\underline{s}_2\in\mathscr{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}}\operatorname{Var}\left(\nu_T(\underline{s}_1,\underline{s}_2)\right)$$
$$\geq \sup_{\substack{\underline{s}_1\in\mathscr{F}_k,\underline{s}_2\in\mathscr{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\,r_k}}\operatorname{Var}\Bigg(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\check{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\check{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\Bigg).$$

Having in mind that $r_{k_T} \leq r_k$ is fulfilled by construction, we take up on (5.13) to get

$$\begin{aligned} \mathrm{II} &\leq 2 \sum_{k=1}^{k_T} \exp\left(2\log\left(D(k)\right) - \frac{1}{2} \cdot \frac{\lambda_k^2}{C_1\, r_k^{\frac{1}{1+\delta}} + C_2\, r_{k_T}^{\frac{1}{1+\delta}}\, \lambda} \right) + o(1) \\ &\leq 2 \sum_{k=1}^{k_T} \exp\left(-\frac{\bar{C}}{2}\, \lambda_k^2\, r_k^{-\frac{1}{1+\delta}} \right) + o(1) \end{aligned} \tag{5.14}$$

which vanishes with $r \to 0$ in view of (5.4) for suitably chosen constants $C_1,\ C_2,\ \bar{C} < \infty$.

Concerning term III of (5.6), we can follow the same steps with $V_{III} := C r^{\frac{1}{1+\delta}}$ for some expediently chosen $C < \infty$ and obtain

$$\begin{aligned} \mathrm{III} &\leq \sum_{\substack{\underline{s}_1 \in \mathscr{F}_k, \underline{s}_2 \in \mathscr{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3\, r}} P\left(\left| \sum_{t=1}^{\mu_T} \sum_{i \in H_t} w_{i,T} \left(\breve{f}\left(\underline{s}_1, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) - \breve{f}\left(\underline{s}_2, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right) \right| > \frac{\lambda}{27} \right) + o(1) \\ &\leq 2 D^2(0) \exp\left(-\frac{1}{2} \cdot \frac{C_1 \lambda^2}{V_{III} + \frac{\breve{M} C_2 \lambda}{3}} \right) + o(1) \end{aligned} \tag{5.15}$$

which vanishes with $r \to 0$ with the same arguments as before for suited constants $C_1,\ C_2 < \infty$.

Now we move on with the remaining first summand in (5.6) and aim at verifying

$$\lim_{r \to 0} \limsup_{T \to \infty} E\left(\sup_{\underline{s}_1, \underline{s}_2 \in \mathscr{S},\, \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}} \left| \nu_T\left(\underline{s}_1, \underline{s}_2\right) \right| \right) = 0. \tag{5.16}$$

Once again, we need some further notation. For $\underline{s} \in \mathscr{S}$ let

$$L_{t,T}\left(\underline{s}\right) := \sum_{i \in H_t} w_{i,T}\, f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}\left(\frac{i}{T}\right)\right) \quad \text{and} \quad L_{t,T}^0\left(\underline{s}\right) := \zeta_t\, L_{t,T}\left(\underline{s}\right),$$

where $(\zeta_t)_{t=1}^{\mu_T}$ are i.i.d. Rademacher variables independent of $(\underline{\varepsilon}_t)_{t \in \mathbb{Z}}$. As $(L_{t,T}(\underline{s}))_{t=1}^{\mu_T}$ consists of independent random variables by construction, we can apply a standard symmetrization lemma (see e.g. van der Vaart and Wellner (2000, Lem. 2.3.1)) to get

$$E\left(\sup_{\underline{s}_1, \underline{s}_2 \in \mathscr{S},\, \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}} \left| \nu_T\left(\underline{s}_1, \underline{s}_2\right) \right| \right) \leq 2 E\left(\sup_{\underline{s}_1, \underline{s}_2 \in \mathscr{S},\, \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}} \left| \sum_{t=1}^{\mu_T} \left(L_{t,T}^0\left(\underline{s}_1\right) - L_{t,T}^0\left(\underline{s}_2\right) \right) \right| \right). \tag{5.17}$$

Note that $\sum_{t=1}^{\mu_T} L_{t,T}^0$ has sub-Gaussian increments conditionally on $L_{1,T}, \ldots, L_{\mu_T,T}$. This is the case since for $\underline{s}_1, \underline{s}_2 \in \mathscr{S}$ and $\eta > 0$, we get by applying Hoeffding's inequality

$$\begin{aligned} P\left(\left| \sum_{t=1}^{\mu_T} L_{t,T}^0\left(\underline{s}_1\right) - L_{t,T}^0\left(\underline{s}_2\right) \right| > \widehat{\rho}_{T,2}\left(\underline{s}_1, \underline{s}_2\right) \eta \,\middle|\, L_{1,T}, \ldots, L_{\mu_T,T} \right) &\leq 2 \exp\left(-\frac{\widehat{\rho}_{T,2}\left(\underline{s}_1, \underline{s}_2\right)^2 \eta^2}{2 \sum_{t=1}^{\mu_T} \left(L_{t,T}\left(\underline{s}_1\right) - L_{t,T}\left(\underline{s}_2\right)\right)^2} \right) \\ &= 2 \exp\left(-\frac{\eta^2}{2} \right) \end{aligned} \tag{5.18}$$

with the random semimetric

$$\widehat{\rho}_{T,2}\left(\underline{s}_1, \underline{s}_2\right) := \left(\sum_{t=1}^{\mu_T} \left(L_{t,T}\left(\underline{s}_1\right) - L_{t,T}\left(\underline{s}_2\right)\right)^2 \right)^{1/2} \tag{5.19}$$

on $\mathscr{S}$. We aim at verifying (5.16) with the help of a maximal inequality for sub-Gaussian processes, which will be more convenient with a different semimetric. To obtain this new semimetric, we note that

$$\left(L_{t,T}\left(\underline{s}_1\right) - L_{t,T}\left(\underline{s}_2\right)\right)^2 \leq 2^{\frac{2+\delta}{3}}\ \left| L_{t,T} \right|_\infty^{\frac{2+\delta}{3}}\ \left| L_{t,T} \right|_{\mathrm{Lip}}^{\frac{4-\delta}{3}}\ \rho\left(\underline{s}_1, \underline{s}_2\right)^{\frac{4-\delta}{3}} \tag{5.20}$$

holds on $(\mathscr{S},\rho)$ with $|L_{t,T}|_{\mathrm{Lip}}$ denoting the Lipschitz constant of $L_{t,T}$. By defining

$$Q_T := 2^{\frac{2+\delta}{6}} \left(\sum_{t=1}^{\mu_T} |L_{t,T}|_{\infty}^{\frac{2+\delta}{3}} \; |L_{t,T}|_{\mathrm{Lip}}^{\frac{4-\delta}{3}} \right)^{1/2},$$

we get

$$EQ_T \le C \Big(\sum_{t=1}^{\mu_T} \left(E\,|L_{t,T}|_{\infty}^{2+\delta} \right)^{1/3} \left(E\,|L_{t,T}|_{\mathrm{Lip}}^{\frac{4-\delta}{2}} \right)^{2/3} \Big)^{1/2}. \tag{5.21}$$

Regarding the expectations on the RHS, we obtain

$$\begin{aligned}
E\,|L_{t,T}|_{\infty}^{2+\delta} &\le \left\| \sum_{i\in H_t} w_{i,T} \left(C\, g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right) + \left| f\left(\underline{0}, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right| \right) \right\|_{2+\delta}^{2+\delta} \\
&\le C \left(\sum_{i\in H_t} d_T^{1/2} \left(\left\| g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right\|_{2+\delta} + \left\| f\left(\underline{0}, \underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right\|_{2+\delta} \right) \right)^{2+\delta} \\
&\le C\, \kappa_T^{2+\delta}\, d_T^{-\frac{2+\delta}{2}}, \\
E\,|L_{t,T}|_{\mathrm{Lip}}^{\frac{4-\delta}{2}} &\le \Big(\sum_{i\in H_t} w_{i,T} \Big\| g\Big(\underline{\widetilde{X}}_i^{(M)}\Big(\frac{i}{T}\Big)\Big)\Big\|_{\frac{4-\delta}{2}} \Big)^{\frac{4-\delta}{2}} \le C\, \kappa_T^{\frac{4-\delta}{2}}\, d_T^{\frac{4-\delta}{4}}.
\end{aligned}$$

Together, we have for (5.21)

$$EQ_T \le C \Big(\sum_{t=1}^{\mu_T} \kappa_T^{(2+\delta)\frac{1}{3}+\frac{4-\delta}{2}\cdot\frac{2}{3}}\, d_T^{\frac{1}{2}\left((2+\delta)\frac{1}{3}+\frac{4-\delta}{2}\cdot\frac{2}{3}\right)} \Big)^{1/2} = C\left(\mu_T\, \kappa_T^2\, d_T^{-1}\right)^{1/2} \le C\, d_T^{\frac{1}{4m}}.$$

From $\widehat{\rho}_{T,2}(\underline{s}_1,\underline{s}_2) \le Q_T\, \rho(\underline{s}_1,\underline{s}_2)^{\frac{4-\delta}{6}} =: \breve{\rho}_T(\underline{s}_1,\underline{s}_2)$, we see that $\breve{\rho}_T$ is again a random semimetric as $\frac{4-\delta}{6} \in (0,1)$. Now we make use of van der Vaart and Wellner (2000, Cor. 2.2.8) to get

$$\begin{aligned}
&E\Big(\sup_{\underline{s}_1,\underline{s}_2\in\mathscr{S},\, \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}} \Big| \sum_{t=1}^{\mu_T} \left(L_{t,T}^0(\underline{s}_1) - L_{t,T}^0(\underline{s}_2) \right) \Big| \,\Big|\, L_{1,T},\ldots,L_{\mu_T,T} \Big) \\
&\le C_3 \int_0^{Q_T r_{k_T}^{\frac{4-\delta}{6}}} \Big(\log\Big(D(u,\mathscr{S},\rho_T) \Big) \Big)^{1/2} du \\
&\le C_3 \int_0^{Q_T r_{k_T}^{\frac{4-\delta}{6}}} \Big(\log\Big(D\Big(\Big(\frac{u}{Q_T}\Big)^{\frac{6}{4-\delta}}, \mathscr{S},\rho \Big) \Big) \Big)^{1/2} du \\
&\le C\, Q_T \int_0^{r_{k_T}^{\frac{4-\delta}{6}}} u^{-\frac{3}{4-\delta}}\, du \\
&\le C\, Q_T\, r_{k_T}^{\frac{1-\delta}{6}}.
\end{aligned}$$

Returning to (5.17), we get from (5.7) and (5.22)

$$E\Big(\sup_{\underline{s}_1,\underline{s}_2\in\mathscr{S},\, \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}} \Big| \sum_{t=1}^{\mu_T} \left(L_{t,T}^0(\underline{s}_1) - L_{t,T}^0(\underline{s}_2) \right) \Big| \Big) \le C\, EQ_T\, r_{k_T}^{\frac{1-\delta}{6}} \le C\, d_T^{-\frac{1}{12m}},$$

which tends to 0 as $T\to\infty$, and the proof is completed. □

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Supplemental material for
A bootstrap functional central limit theorem for time-varying linear processes

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A. Appendix

While the proof of 2.5 is given in Section 5 of the main document, all remaining proofs as well as several auxiliary results are contained in this appendix. Throughout, C denotes a generic constant that may change its value from line to line.

A.1. Proof of Theorem 2.4

Throughout the proof, we use the following notation based on (2.10):

$$Y_T\left(\underline{s}\right) := \sum_{t=1}^{T} w_{t,T}\, \bar{f}\left(\underline{s}, \underline{X}_{t,T}\right) \quad \text{and} \quad Y_{t,T} := \underline{c}'\left(w_{t,T}\, \bar{f}\left(\underline{s}_j, \underline{X}_{t,T}^{(M)}\right), j=1,\ldots,J\right)$$

for $t = 1, \ldots, T$ and any $c \in \mathbb{R}^J$. In addition to that, we consider

$$Z_T := \underline{c}'\left(Y_T\left(\underline{s}_j\right), j=1,\ldots,J\right) \quad \text{and} \quad Z_T^{(M)} := \sum_{t=1}^{T} Y_{t,T}$$

and define the corresponding truncated version $\mathbf{V}_M$ of the covariance matrix $\mathbf{V}$ as

$$\mathbf{V}_M := \left(\mathbf{V}_M\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right)\right)_{j_1, j_2 = 1, \ldots, J}$$

with

$$\mathbf{V}_M\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right) := \sum_{h=-2(M-1)}^{2(M-1)} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T}\, \mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right).$$

Using the Cramér-Wold device, we are to prove

$$Z_T \xrightarrow{d} \mathcal{N}\left(0, \underline{c}'\mathbf{V}\underline{c}\right), \quad \forall c \in \mathbb{R}^J. \tag{A.1}$$

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Concerning the variance, we distinguish between two cases, namely

$$\text{(A)}\ \underline{c}'\mathbf{V}\underline{c} = 0 \quad \text{and} \quad \text{(B)}\ \underline{c}'\mathbf{V}\underline{c} > 0.$$

Regarding case (A), we have $\mathrm{Var}(Z_T) \longrightarrow \underline{c}'\mathbf{V}\underline{c} = 0$ by Lemma A.5, which gives $Z_T \xrightarrow{d} \mathscr{N}(0,0)$ as required. In the sequel, let $\underline{c}'\mathbf{V}\underline{c} > 0$ (case (B)). In order to show (A.1), Brockwell and Davis (1991, Prop. 6.3.9) impose the verification of the following conditions:

(1) $\forall M \in \mathbb{N} : Z_T^{(M)} \underset{T\to\infty}{\xrightarrow{d}} \mathscr{N}\left(0, \underline{c}'\mathbf{V}_M\underline{c}\right)$,

(2) $\underline{c}'\mathbf{V}_M\underline{c} \underset{M\to\infty}{\longrightarrow} \underline{c}'\mathbf{V}\underline{c}$,

(3) $\forall \varepsilon > 0 : \lim_{M\to\infty} \limsup_{T\to\infty} P\left(\left|Z_T - Z_T^{(M)}\right| \geq \varepsilon\right) = 0.$

In terms of constraint (2), it suffices to show

$$\mathbf{V}\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right) - \mathbf{V}_M\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right) \longrightarrow 0, \quad M \to \infty,$$

for $j_1, j_2 = 1, \ldots, J$. The difference above can be bounded by

$$\begin{aligned}
&\left|\mathbf{V}\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right) - \mathbf{V}_M\left(\underline{s}_{j_1}, \underline{s}_{j_2}\right)\right| \\
&\leq \left|\sum_{|h|>2(M-1)} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T}\, \mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\
&\qquad + \left|\sum_{h=-2(M-1)}^{2(M-1)} \lim_{T\to\infty} \sum_{t=1}^{T} w_{t,T}\, w_{t+h,T} \left(\mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right.\right. \\
&\qquad\qquad \left.\left. - \mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right)\right| \\
&=: \underline{\mathrm{I}} + \underline{\mathrm{II}}.
\end{aligned}$$

Using Lemma A.4, we obtain asymptotic negligibility of $\underline{\mathrm{I}}$ as $M \to \infty$ under our weight assumptions. Regarding $\underline{\mathrm{II}}$, we obtain

$$\begin{aligned}
&\left|\mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) - \mathrm{Cov}\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right), f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right| \\
&\leq \left|E\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right) - E\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right| \\
&\qquad + \left|E\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right) E\left(f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right. \\
&\qquad\qquad \left. - E\left(f\left(\underline{s}_{j_1}, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right) E\left(f\left(\underline{s}_{j_2}, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right| \\
&=: \underline{\mathrm{II}}\mathrm{a} + \underline{\mathrm{II}}\mathrm{b}.
\end{aligned} \tag{A.2}$$

Starting with the second summand on the right-hand side (RHS), it holds

$$\underline{\mathrm{II}}\mathrm{b} \leq C \left\|\underline{\widetilde{X}}_0\left(\frac{t}{T}\right) - \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right\|_1 \leq C\, \vartheta^M$$

due to Assumption 3 and Lemma A.2. The remaining summand $\underline{\text{II}}$a of (A.2) can be bounded with the use of Lemma A.3

$$\begin{aligned}
\text{IIa} \le &\Bigg| E\Bigg(\Big(f\Big(\underline{s}_{j_1},\underline{\widetilde{X}}_0\Big(\frac{t}{T}\Big)\Big)-f\Big(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\Big(\frac{t}{T}\Big)\Big)\Big)f\Big(\underline{s}_{j_2},\underline{\widetilde{X}}_h\Big(\frac{t}{T}\Big)\Big) \\
&\qquad + f\Big(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\Big(\frac{t}{T}\Big)\Big)\Big(f\Big(\underline{s}_{j_2},\underline{\widetilde{X}}_h\Big(\frac{t}{T}\Big)\Big)-f\Big(\underline{s}_{j_2},\underline{\widetilde{X}}_h^{(M)}\Big(\frac{t}{T}\Big)\Big)\Big)\Bigg)\Bigg| \\
\le &\Bigg| E\Bigg(\Big(f\Big(\underline{s}_{j_1},\underline{\widetilde{X}}_0\Big(\frac{t}{T}\Big)\Big)-f\Big(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\Big(\frac{t}{T}\Big)\Big)\Big)f\Big(\underline{s}_{j_2},\underline{\widetilde{X}}_h\Big(\frac{t}{T}\Big)\Big)\Bigg)\Bigg| \\
&\qquad + \Bigg| E\Bigg(f\Big(\underline{s}_{j_1},\underline{\widetilde{X}}_0^{(M)}\Big(\frac{t}{T}\Big)\Big)\Big(f\Big(\underline{s}_{j_2},\underline{\widetilde{X}}_h\Big(\frac{t}{T}\Big)\Big)-f\Big(\underline{s}_{j_2},\underline{\widetilde{X}}_h^{(M)}\Big(\frac{t}{T}\Big)\Big)\Big)\Bigg)\Bigg| \\
\le & C\,\vartheta^{\frac{M\delta}{1+\delta}},
\end{aligned}$$

which yields asymptotic negligibility of $\underline{\text{II}}$ and, thus, finishes the verification of condition (2).

Below, we focus on condition (1). By (2), it holds $\underline{c}'\mathbf{V}_M\underline{c} > 0$ for sufficiently large M. Hence, it is adequate to show

$$\frac{Z_T^{(M)}}{\left(\operatorname{Var}\left(Z_T^{(M)}\right)\right)^{1/2}} \xrightarrow{d} \mathscr{N}(0,1)$$

for $T \to \infty$ because

$$\operatorname{Var}\left(Z_T^{(M)}\right) = \underline{c}'\mathbf{V}_M\underline{c} + o(1) \tag{A.3}$$

can be demonstrated analogously to Lemma A.5. As the number of non-zero weights equals d_T, $Z_T^{(M)}$ has only d_T non-vanishing summands denoted by $Y_{t_1,T},\ldots,Y_{t_{d_T},T}$. Consequently, we have

$$Z_T^{(M)} = \sum_{h=1}^{d_T} Y_{t_h,T}, \tag{A.4}$$

and $(Y_{t_h,T}, h=1,\ldots,d_T)$ form a triangular array of centred $(2(M-1))$-dependent random variables such that the CLT in Romano and Wolf (2000, Th. 2.1) can be applied if the requirements listed therein can be fulfilled (putting their $\gamma = 0$). These conditions read for some $\delta > 0$ as well as finite constants $\Delta_T, K_{1,T}$ and $K_{2,T}$ depending on T as follows:

(i) $E\left|Y_{t_h,T}\right|^{2+\delta} \le \Delta_T \quad \forall h \in \{1,\ldots,d_T\}$, (ii) $\dfrac{\operatorname{Var}\left(\sum_{h=a}^{a+k-1} Y_{t_h,T}\right)}{k} \le K_{1,T} \quad \forall a \forall k \ge 2(M-1)$,

(iii) $\dfrac{\operatorname{Var}\left(\sum_{h=1}^{d_T} Y_{t_h,T}\right)}{d_T} \ge K_{2,T}$, (iv) $\dfrac{K_{1,T}}{K_{2,T}} = \mathscr{O}(1)$,

(v) $\dfrac{\Delta_T}{K_{2,T}^{1+\delta/2}} = \mathscr{O}(1)$, (vi) $\dfrac{(2(M-1))^{2+2/\delta}}{d_T} \underset{T\to\infty}{\longrightarrow} 0$.

Now we verify the validity of the conditions stated above starting with (i):

$$E\left|Y_{t_h,T}\right|^{2+\delta} \le C d_T^{-\frac{2+\delta}{2}} \left(\sum_{j=1}^{J} \left|c_j\right|\right)^{2+\delta} =: \Delta_T.$$

Going on to requirement (ii), we make use of the upper bounds for the weights as presupposed in Assumption 2

$$\operatorname{Var}\left(\sum_{h=a}^{a+k-1} Y_{t_h,T}\right) = \sum_{h=\max\{-(k-1),-2M\}}^{\min\{k-1,2M\}} \sum_{s=\max\{1,1-h\}}^{\min\{k,k-h\}} \operatorname{Cov}\left(Y_{t_{s+h+a-1},T}, Y_{t_{s+a-1},T}\right) \le (2M+1)Ckd_T^{-1},$$

which proves the validity of (ii) with $K_{1,T}=(2M+1)C\,d_T^{-1}$. For (iii), we obtain from (A.3), (A.4) and (2) that for T large enough there exists some $C_L>0$ such that

$$\operatorname{Var}\Big(\sum_{h=1}^{d_T} Y_{t_h,T}\Big)=\underline{c}'\mathbf{V}_M\underline{c}+o(1)\geq C_L$$

for T large enough. Hence, (iii) holds with $K_{2,T}=C_L\,d_T^{-1}$, which obviously satisfies (iv) and (v), too. Lastly, we see that requirement (vi) holds trivially due to the fact that M is fixed.

Finally, we verify the remaining constraint (3) proving

$$\lim_{M\to\infty}\limsup_{T\to\infty}\operatorname{Var}\left(Z_T-Z_T^{(M)}\right)=\lim_{M\to\infty}\limsup_{T\to\infty}\operatorname{Var}\Big(\sum_{t=1}^{T} c\,w_{t,T}\left(f\left(\underline{s},\underline{X}_{t,T}\right)-f\left(\underline{s},\underline{X}_{t,T}^{(M)}\right)\right)\Big)=0.$$

We start by inserting another truncated version of the companion process

$$\underline{\widetilde{X}}_h^{(M,v)}\left(\frac{t}{T}\right):=\underline{\mu}\left(\frac{t}{T}\right)+\sum_{|j|<\min\{M,v\}}A\left(\frac{t}{T},j\right)\underline{\varepsilon}_{t-j}$$

with $v:=\lceil |h|/2\rceil$. Using Lemma A.4 and similar arguments, we obtain

$$\begin{aligned}
&\left|\operatorname{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right),f\left(\underline{s},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&\leq\left|\operatorname{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right.\right.\\
&\qquad\qquad\left.\left.-f\left(\underline{s},\underline{\widetilde{X}}_0^{(v)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M,v)}\left(\frac{t}{T}\right)\right),f\left(\underline{s},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&\qquad\qquad+\left|\operatorname{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_0^{(v)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M,v)}\left(\frac{t}{T}\right)\right),\right.\right.\\
&\qquad\qquad\left.\left.f\left(\underline{s},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_h^{(v)}\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_h^{(M,v)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&\leq C\,\vartheta^{\frac{v\delta}{1+\delta}}.
\end{aligned}$$

This bound is independent of M and can be totalled over h. So, Lebesgue's theorem can be used to justify the following result based on the same argumentation as in the proof of Lemma A.5 as well as in the proof of Lemma A.4:

$$\begin{aligned}
&\lim_{M\to\infty}\limsup_{T\to\infty}\operatorname{Var}\Big(\sum_{t=1}^{T}c\,w_{t,T}\left(f\left(\underline{s},\underline{X}_{t,T}\right)-f\left(\underline{s},\underline{X}_{t,T}^{(M)}\right)\right)\Big)\\
&\leq c^2\sum_{h=-\infty}^{\infty}\lim_{M\to\infty}\limsup_{T\to\infty}\sum_{t=1}^{T}w_{t,T}\,w_{t+h,T}\left|\operatorname{Cov}\left(f\left(\underline{s},\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right),\right.\right.\\
&\qquad\qquad\left.\left.f\left(\underline{s},\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s},\underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|\\
&=0,
\end{aligned}$$

which concludes (c) and, hence, finishes the proof.

A.2. Proofs of the results in Section 3

As already indicated in the Bootstrap Algorithm 1, some indices can be responsible for a change in the sign of k_i. The following definition groups those indices for easier handling:

Definition A.1 (Endpoints)**.**
Considering Algorithm 1, all indices $t \in 1,\dots,T$ which might cause a sign switch are called *endpoints* $EP = EP_1 \cup EP_2$ with

$$EP_1 := \{t \in \{1,\dots,T\} \,|\, 1 \le t \le TD_T\} \quad \text{and} \quad EP_2 := \{t \in \{1,\dots,T\} \,|\, T - TD_T < t \le T\}.$$

Proof of Theorem 3.3. This proof is inspired by Dowla, Paparoditis and Politis (2013) but enhanced to meet the demands imposed by our assumptions. For sake of notational simplicity, we consider $J = 1$ only. We split the sum up in one sum containing all indices belonging to whole bootstrap blocks without including endpoints and two with the remaining indices in the way that we have

$$\begin{aligned}\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) &= \sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right) + \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right)\\ &\qquad + \sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{T} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right)\\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{A.5}$$

Recall that $L_T = o(TD_T)$. In view of Lemma A.7 and Assumption 6, we can bound

$$E^{\star}|\mathrm{I}| \le E^{\star}\left|\sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\, f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right)\right| = \mathscr{O}_P\left(TD_T\, d_T^{-1/2}\right) = \mathscr{O}_P\left(d_T^{-\frac{\delta}{2(2+\delta)}}\right).$$

Due to the similar structure of III, we obtain $\mathrm{I} + \mathrm{III} = o_{P^{\star}}(1)$. The remaining term II of (A.5) can be rewritten to make the single blocks visible, to wit

$$\mathrm{II} = \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \sum_{j=1}^{L_T} w_{tL_T+j,T}\, \bar{f}^{\star}\left(\underline{s}, \underline{X}^{\star}_{tL_T+j,T}\right) =: \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \xi^{\star}_{t,T}.$$

Since the bootstrap blocks are independent, the newly defined $\left(\xi^{\star}_{t,T}\right)$ are independent as well. Lemma A.12 gives us

$$P - \lim_{T\to\infty} \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \mathrm{Var}^{\star}\left(\xi^{\star}_{t,T}\right) = \mathbf{V}\left(\underline{s}, \underline{s}\right). \tag{A.6}$$

Thus, it suffices to consider the case $\mathbf{V}(\underline{s}, \underline{s}) > 0$ only. We aim for applying the classical central limit theorem for independent data using Lyapunov's condition. To this end, note that

$$E\left(E^{\star}\left|\xi^{\star}_{t,T}\right|^{2+\delta}\right) \le C\, d_T^{-\frac{2+\delta}{2}} \left(\sum_{j=1}^{L_T} \mathbb{1}_{\{w_{tL_T+j,T}>0\}}\right)^{2+\delta}. \tag{A.7}$$

Furthermore, it holds

$$\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \left(\sum_{j=1}^{L_T} \mathbb{1}_{\{w_{tL_T+j,T}>0\}}\right)^{2+\delta} \le d_T\, L_T^{1+\delta}. \tag{A.8}$$

Now, combining (A.6) and (A.7) and using (A.8), we obtain

$$\frac{\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} E^\star \left|\xi_{t,T}^\star\right|^{2+\delta}}{\left(\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1} \operatorname{Var}^\star\left(\xi_{t,T}^\star\right)\right)^{\frac{2+\delta}{2}}} = \frac{d_T L_T^{1+\delta}\mathscr{O}_P\left(d_T^{-\frac{2+\delta}{2}}\right)}{\left(V\left(\underline{s},\underline{s}\right)\right)^{\frac{2+\delta}{2}} + o_P(1)} = o_P(1).$$

Thereby, Lyapunov's condition is fulfilled, which implies asymptotic normality as desired. The second assertion of the theorem is an immediate consequence of the first one due to continuity of the Gaussian distribution function, see van der Vaart (1998, Lem. 2.11). □

Within the next proof and the corresponding auxiliary results in the Appendix, we will use some additional notation. In particular, we abbreviate $\mathbb{X}_T := (X_{1,T}', \ldots, X_{T,T}')'$ and $P(\cdot \mid \mathbb{X}_T = \underline{x}_T)$ by $P_T^\star(\cdot)$. Corresponding expectations and variances are denoted by $E_T^\star$ and $\operatorname{Var}_T^\star$.

Proof of Theorem 3.4. With the same arguments as in the proof of Theorem 2.5, it is sufficient to show that there exist sets $(\Omega_T)_{T\in\mathbb{N}}$ with $P(\mathbb{X}_T \in \Omega_T) \to 1$ as $T\to\infty$ such that for any $(\underline{x}_T)_T$ with $x_T \in \Omega_T$ for all T it holds

$$\lim_{r\to 0}\limsup_{T\to\infty} P\left(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\left|\sum_{t=1}^T w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}_{t,T}^\star\right) - \bar{f}^\star\left(\underline{s}_2,\underline{X}_{t,T}^\star\right)\right)\right| > \lambda \,\middle|\, \mathbb{X}_T = \underline{x}_T\right) = 0 \tag{A.9}$$

with $\lambda > 0$, whereas $\bar{f}^\star$ is defined in (3.1).

Here, we only consider the case (a) in Assumption 4 and 7. Part (b) is deferred to Lemma A.18 in the Appendix. First, we define

$$G_T = \left\{\underline{x}_T \,\middle|\, \sum_{t=L_T\lfloor T/L_T\rfloor+1}^T E_T^\star[g\left(\underline{X}_{t,T}^\star\right)] \le L_T\, d_T^{\delta/(2(1+\delta))}\right\}.$$

From $\sup_{1\le t\le T} E|E^\star g\left(\underline{X}_{t,T}^\star\right)| \le C$, we get $P(G_T)\to 1$ as $T\to\infty$ applying Markov's inequality. With the sets A_T, B_T and K_T established in Lemmata A.13, A.14 and A.15, respectively, we set

$$\Omega_T := A_T \cap B_T \cap G_T \cap K_T.$$

In view of the above mentioned results, it holds $\lim_{T\to\infty} P(\mathbb{X}_T \in \Omega_T) = 1$.

After these preparations, we start by splitting the LHS of (A.9) such that we get one sum containing the indices of whole independent blocks and a second one containing the remaining indices:

$$\begin{aligned}
&P_T^\star\left(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\left|\sum_{t=1}^T w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}_{t,T}^\star\right) - \bar{f}^\star\left(\underline{s}_2,\underline{X}_{t,T}^\star\right)\right)\right| > \lambda\right)\\
&\le P_T^\star\left(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\left|\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}_{t,T}^\star\right) - \bar{f}^\star\left(\underline{s}_2,\underline{X}_{t,T}^\star\right)\right)\right| > \frac{\lambda}{2}\right)\\
&\qquad + P_T^\star\left(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\left|\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}_{t,T}^\star\right) - \bar{f}^\star\left(\underline{s}_2,\underline{X}_{t,T}^\star\right)\right)\right| > \frac{\lambda}{2}\right).
\end{aligned} \tag{A.10}$$

By construction of G_T, the second sum on the RHS of (A.10) can be bounded by

$$\frac{4}{\lambda}\sum_{t=L_T\lfloor T/L_T\rfloor+1}^T w_{t,T}\, E_T^\star\left[\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\left|f\left(\underline{s}_1,\underline{X}_{t,T}^\star\right) - f\left(\underline{s}_2,\underline{X}_{t,T}^\star\right)\right|\right] \le C\,r\,d_T^{-1/2}\sum_{t=L_T\lfloor T/L_T\rfloor+1}^T E_T^\star g\left(\underline{X}_{t,T}^\star\right)$$

$$= \mathscr{O}\left(d_T^{-\frac{1-\delta}{2(1+\delta)}}\right),$$

which tends to 0 as $T \to \infty$ leaving us with the first sum on the RHS of (A.10) to deal with. Similarly to the proof of Theorem 2.5, we define $\nu_T^\star(\underline{s})$ as well as $\nu_T^\star(\underline{s}_1, \underline{s}_2)$ by

$$\nu_T^\star(\underline{s}) := \sum_{t=0}^{\lfloor T/L_T \rfloor - 1} \sum_{j=1}^{L_T} w_{tL_T+j,T}\, \bar{f}^\star\left(\underline{s}_1, \underline{X}^\star_{tL_T+j,T}\right) \quad \text{and} \quad \nu_T^\star(\underline{s}_1, \underline{s}_2) := \nu_T^\star(\underline{s}_1) - \nu_T^\star(\underline{s}_2),$$

respectively. We will use the same notation regarding the sequence $(r_k)_k$, the index sets $(\mathscr{F}_k)_{k=0}^{k_T}$ and the maps $(\pi_k)_{k=0}^{k_T}$ as in the proof of Theorem 2.5. As before, we split the left over sum of (A.10) up into

$$\begin{aligned}
&P_T^\star\left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathscr{S} \\ \rho(\underline{s}_1, \underline{s}_2) < r}} \left| \sum_{t=1}^{L_T \lfloor T/L_T \rfloor} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1, \underline{X}^\star_{t,T}\right) - \bar{f}^\star\left(\underline{s}_2, \underline{X}^\star_{t,T}\right)\right)\right| > \frac{\lambda}{2}\right) \\
&\leq P_T^\star\left(2 \sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathscr{S} \\ \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}}} \left|\nu_T^\star(\underline{s}_1, \underline{s}_2)\right| > \frac{\lambda}{6}\right) + P_T^\star\left(2 \sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1 \in \mathscr{F}_k, \underline{s}_2 \in \mathscr{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3 r_k}} \left|\nu_T^\star(\underline{s}_1, \underline{s}_2)\right| > \frac{\lambda}{6}\right) \\
&\qquad + P_T^\star\left(\sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathscr{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3 r}} \left|\nu_T^\star(\underline{s}_1, \underline{s}_2)\right| > \frac{\lambda}{6}\right) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}
\end{aligned} \tag{A.11}$$

postponing the magnitude of k_T. One can use Bernstein's inequality for the discussion of terms II and III. Here, we will carry out the details for II, only. To this end, we consider the bootstrap variance of $\nu_T^\star(\underline{s}_1, \underline{s}_2)$ first. With the help of Lemmata A.14 and A.16, we obtain (with $(c_T)_T$ as in Lemma A.14)

$$\operatorname{Var}_T^\star\left(\nu_T^\star(\underline{s}_1, \underline{s}_2)\right) \leq C\left\{\left|\underline{s}_1 - \underline{s}_2\right|_1^{1/2} + L_T^{-1} + c_T L_T\left(T^{-\frac{3+\delta}{2(4+\delta)}} + D_T^{\frac{3+\delta}{2(4+\delta)}} + (T D_T)^{-\frac{1}{2}}\right)\right\}.$$

Next, Lemma A.13 allows us to bound

$$\left|\sum_{j=1}^{L_T} w_{tL_T+j,T}\left(\bar{f}^\star\left(\underline{s}_1, \underline{X}^\star_{tL_T+j,T}\right) - \bar{f}^\star\left(\underline{s}_2, \underline{X}^\star_{tL_T+j,T}\right)\right)\right| \leq C L_T\, d_T^{-\frac{2+\delta^2}{2(4+\delta)}} < C\, d_T^{-\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}} \tag{A.12}$$

due to Assumption 6. Abbreviating $y_T := L_T^{-1} + c_T L_T\left(T^{-\frac{3+\delta}{2(4+\delta)}} + D_T^{\frac{3+\delta}{2(4+\delta)}} + (T D_T)^{-1/2}\right)$, we specify k_T such that

$$\left(\max\left\{y_T, d_T^{-\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}\right\}\right)^{\frac{(1+\delta)(2+\delta)(4+\delta)}{\delta^2(\delta-\delta^2)}} \leq r_{k_T} \leq L_T^{-\frac{1+\delta}{\delta^2}}. \tag{A.13}$$

Tedious straightforward calculations similar to Beering (2021, proof of Th. 3.13) show that the upper bound is strictly larger than the lower bound for all sufficiently large T. The specific choice of the upper bound in (A.13) will become relevant during the examination of term I of equation (A.11), while the lower bound is essential for a successful application of Bernstein's inequality. In particular for II in (A.13), we will make us of $\rho(\underline{s}_1, \underline{s}_2) \leq 3 r_k$ to obtain

$$\operatorname{Var}^\star\left(\nu_T^\star(\underline{s}_1, \underline{s}_2)\right) \leq C\left(r_k^{1/2} + r_{k_T}^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}}\right) \leq C\, r_k^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}} \tag{A.14}$$

for $T > \bar{T}$ as it holds $r_{k_T} < r_k$ for all $k = 0, 1, \dots, k_T$ and because of

$$\frac{1}{2} - \frac{\delta^2\left(\delta-\delta^2\right)}{(1+\delta)(2+\delta)(4+\delta)} > 0 \quad \forall \delta \in (0,1).$$

After having determined the bounds to be used in Bernstein's inequality, we turn our attention to λ and, as in the proof of Theorem 2.5, define a sequence $(\lambda_k)_{k\in\mathbb{N}}$ satisfying

$$\sum_{k\in\mathbb{N}} \lambda_k \leq \frac{\lambda}{3}$$

for r sufficiently small by setting

$$\lambda_k := r_k^{\frac{\delta^2\left(\delta-\delta^2\right)}{4(1+\delta)(2+\delta)(4+\delta)}} \vee \left(\frac{4}{\bar{C}} \log\left(D(k)\right) r_k^{\frac{\delta^2\left(\delta-\delta^2\right)}{(1+\delta)(2+\delta)(4+\delta)}}\right)^{1/2}, \quad k \in \mathbb{N},$$

for a finite constant $\bar{C} > 0$, which will be specified further during the upcoming calculations. Again, $D(k) = \mathscr{O}\left(r_k^{-d}\right)$ ensures the summability of $(\lambda_k)_k$.

At this point, we return to II in (A.11) and use both the definition of $(\lambda_k)_{k=1}^{k_T}$ and Bernstein's inequality with the previously established upper bounds in (A.12) and (A.14) to get

$$\begin{aligned}
&\limsup_{T\to\infty} P_T^\star\left(2\sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1\in\mathscr{F}_k, \underline{s}_2\in\mathscr{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3\, r_k}} \left|\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right| > \frac{\lambda}{6}\right)\\
&\leq \limsup_{T\to\infty} 2\sum_{k=1}^{k_T} D(k)D(k-1)\exp\left(-\frac{1}{2}\cdot\frac{\lambda_k^2}{C_1\, r_k^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}} + C_2\, d_T^{\frac{2-2\delta+\delta^3}{2(1+\delta)(4+\delta)}}\lambda_k/3}\right)\\
&\leq \limsup_{T\to\infty} 2\sum_{k=1}^{k_T} \exp\left(2\log(D(k)) - \bar{C}\frac{\lambda_k^2}{r_k^{\frac{\delta^2(\delta-\delta^2)}{(1+\delta)(2+\delta)(4+\delta)}}}\right)\\
&\leq 2\sum_{k\in\mathbb{N}} \exp\left(-C\, 2^{\frac{k\delta^2\left(\delta-\delta^2\right)}{2(1+\delta)(2+\delta)(4+\delta)}}\, r^{-\frac{\delta^2\left(\delta-\delta^2\right)}{2(1+\delta)(2+\delta)(4+\delta)}}\right)\\
&\xrightarrow[r\to 0]{} 0.
\end{aligned}$$

Now term I of (A.11) is left, and, as in the proof of Lemma 2.5, we aim for making use of a symmetrization lemma. To this end, let

$$L_{t,T}^\star\left(\underline{s}\right) := \sum_{j=1}^{L_T} w_{tL_T+j,T}\, f\left(\underline{s}, \underline{X}_{tL_T+j,T}^\star\right) \quad \text{and} \quad L_{t,T}^{\star,0}\left(\underline{s}\right) := \zeta_t\, L_{t,T}^\star\left(\underline{s}\right), \quad \underline{s} \in \mathscr{S}, \tag{A.15}$$

where $(\zeta_t)_{t=0}^{\lfloor T/L_T\rfloor -1}$ are again i.i.d. Rademacher variables, but this time, they are independent of the series of uniformly distributed random variables used in Algorithm 1 and $\mathbb{X}_T$. This gives

$$\begin{aligned}
\mathrm{I} &= P_T^\star\Big(\sup_{\underline{s}_1,\underline{s}_2\in\mathscr{S},\, \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}} \left|\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right| > \frac{\lambda}{12}\Big)\\
&\leq \frac{12}{\lambda} E_T^\star\Big(\sup_{\underline{s}_1,\underline{s}_2\in\mathscr{S},\, \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}} \left|\nu_T^\star\left(\underline{s}_1,\underline{s}_2\right)\right|\Big)
\end{aligned}$$

$$\leq \frac{24}{\lambda} E_T^\star \Big(\sup_{\underline{s}_1,\underline{s}_2 \in \mathscr{S},\, \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}} \Big| \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(L_{t,T}^{\star,0}(\underline{s}_1) - L_{t,T}^{\star,0}(\underline{s}_2) \right) \Big| \Big). \tag{A.16}$$

Next, we define

$$\widehat{\rho}_{T,2}(\underline{s}_1,\underline{s}_2) := \left(\sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(L_{t,T}^\star(\underline{s}_1) - L_{t,T}^\star(\underline{s}_2) \right)^2 \right)^{1/2}, \quad \underline{s}_1,\underline{s}_2 \in \mathscr{S}. \tag{A.17}$$

For any $\eta > 0$, Hoeffding's inequality gives

$$\begin{aligned}
&P^\star \Bigg(\sup_{\substack{\underline{s}_1,\underline{s}_2 \in \mathscr{S} \\ \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}}} \Big| \sum_{t=0}^{\lfloor T/L_T \rfloor -1} L_{t,T}^{\star,0}(\underline{s}_1) - L_{t,T}^{\star,0}(\underline{s}_2) \Big| > \widehat{\rho}_{T,2}(\underline{s}_1,\underline{s}_2)\, \eta \,\Bigg|\, L_{1,T}^\star, \ldots, L_{L_T \lfloor T/L_T \rfloor, T}^\star \Bigg) \\
&\leq 2 \exp \left(- \frac{\widehat{\rho}_{T,2}(\underline{s}_1,\underline{s}_2)^2 \eta^2}{2 \sum_{t=1}^{L_T \lfloor T/L_T \rfloor} \left(L_{t,T}^\star(\underline{s}_1) - L_{t,T}^\star(\underline{s}_2) \right)^2} \right) \\
&= 2 \exp \left(- \frac{\eta^2}{2} \right).
\end{aligned}$$

Thus, $\sum_{t=0}^{\lfloor T/L_T \rfloor -1} L_{t,T}^{\star,0}$ possesses sub-Gaussian increments conditionally on $L_{0,T}^\star, \ldots, L_{\lfloor T/L_T \rfloor -1,T}^\star$. We continue by establishing an upper bound for the difference in (A.17) as follows:

$$\left(L_{t,T}^\star(\underline{s}_1) - L_{t,T}^\star(\underline{s}_2) \right)^2 \leq 2^{1-\delta} \left| L_{t,T}^\star \right|_\infty^{1-\delta} \left| L_{t,T}^\star \right|_{\mathrm{Lip}}^{1+\delta} \rho(\underline{s}_1,\underline{s}_2)^{1+\delta}, \quad \underline{s}_1,\underline{s}_2 \in \mathscr{S}. \tag{A.18}$$

In order to create a more suitable semimetric during the further course of this proof, we define

$$Q_T := 2^{\frac{1-\delta}{2}} \left(\sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left| L_{t,T}^\star \right|_\infty^{1-\delta} \left| L_{t,T}^\star \right|_{\mathrm{Lip}}^{1+\delta} \right)^{1/2}$$

and introduce a new semimetric using

$$\widehat{\rho}_{T,2}(\underline{s}_1,\underline{s}_2) \leq Q_T \rho(\underline{s}_1,\underline{s}_2)^{\frac{1+\delta}{2}} =: \check{\rho}_T(\underline{s}_1,\underline{s}_2).$$

Now we come back to (A.16). Using van der Vaart and Wellner (2000, Cor. 2.2.8), we obtain

$$\begin{aligned}
\mathrm{I} &\leq \frac{24}{\lambda} E_T^\star \left[E^* \Big(\sup_{\underline{s}_1,\underline{s}_2 \in \mathscr{S},\, \rho(\underline{s}_1,\underline{s}_2) \leq r_{k_T}} \Big| \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(L_{t,T}^{\star,0}(\underline{s}_1) - L_{t,T}^{\star,0}(\underline{s}_2) \right) \Big| \,\Big|\, L_{1,T}^\star, \ldots, L_{L_T \lfloor T/L_T \rfloor, T}^\star \Big) \right] \\
&= \frac{24}{\lambda} E_T^\star \left[E^* \Big(\sup_{\underline{s}_1,\underline{s}_2 \in \mathscr{S},\, \check{\rho}(\underline{s}_1,\underline{s}_2) \leq Q_T r_{k_T}^{\frac{1+\delta}{2}}} \Big| \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(L_{t,T}^{\star,0}(\underline{s}_1) - L_{t,T}^{\star,0}(\underline{s}_2) \right) \Big| \,\Big|\, L_{1,T}^\star, \ldots, L_{L_T \lfloor T/L_T \rfloor, T}^\star \Big) \right] \\
&\leq C E_T^\star \left(\int_0^{Q_T r_{k_T}^{\frac{1+\delta}{2}}} \left(\log D(u, \mathscr{S}, \check{\rho}_T) \right)^{1/2} du \right) \\
&\leq C E_T^\star \left(\int_0^{Q_T r_{k_T}^{\frac{1+\delta}{2}}} \left(\log \left(\left(\frac{u}{Q_T} \right)^{-\frac{2}{1+\delta}} + 1 \right)^d \right)^{1/2} du \right) \\
&= C E_T^\star[Q_T] \int_0^{r_{k_T}^{\frac{1+\delta}{2}}} \left(\log \left(u^{-\frac{2}{1+\delta}} + 1 \right)^d \right)^{1/2} du.
\end{aligned}$$

Moreover, Jensen's inequality yields

$$E_T^\star[Q_T] \leq C_1 \Bigg(\sum_{t=0}^{\lfloor T/L_T \rfloor - 1} E^\star \left(\left| L_{t,T}^\star \right|_\infty^{1-\delta} \left| L_{t,T}^\star \right|_{\mathrm{Lip}}^{1+\delta} \right) \Bigg)^{1/2} \leq C L_T^{\frac{1+\delta}{2}},$$

where the latter inequality follows from Lemma A.15. Plugging this result into the previous calculations and remembering the upper bound of r_{k_T} in (A.13) gives

$$\mathrm{I} \leq \limsup_{T\to\infty} C L_T^{\frac{1+\delta}{2}} \int_0^{r_{k_T}^{\frac{1+\delta}{2}}} \left(\log \left(u^{-\frac{2}{1+\delta}} + 1 \right)^d \right)^{1/2} du \leq \limsup_{T\to\infty} C L_T^{\frac{1+\delta}{2}} L_T^{-\frac{1+\delta}{2\delta}} = 0.$$

This terminates the proof. □

A.3. Proofs of the results in Section 4

Proof of Corollary 4.1. First of all, note that compared to our main results, we face an additional bias term

$$\frac{\sigma_\varepsilon^2}{\sqrt{T}} \left(\sum_{t=1}^{T} \sigma^2 \left(\frac{t}{T} \right) - \int_0^1 \sigma^2(u)\, du \right)$$

here. As its order of magnitude is $O(T^{-1/2})$, it is negligible as $T \to \infty$. Straight-forward calculations yield the validity of Assumptions 1, 2, 3 and 4 besides the Lipschitz condition in 3 which does not hold true. Therefore, we have to use an additional truncation step to deduce the result. To this end, let

$$f^{(N)}(X_{t,T}) = \min \left\{ X_{t,T}^2, N \right\}$$

which the satisfies all the conditions in Theorems 2.4 and 3.3. Thus, the desired result follows from Brockwell and Davis (1991, Prop. 6.3.9) if its prerequisites are met on the original as well as on the bootstrap side. While the proof of their condition (i) is obvious, (ii) can be proven along the lines of the proof of Theorem 2.4. Finally, for (iii) we have to show

$$\lim_{N\to\infty} \limsup_{T\to\infty} E \left[\sum_{t=1}^{T} T^{-1/2} \left(X_{t,T}^2 - f^{(N)}(X_{t,T}) - E \left[X_{t,T}^2 - f^{(N)}(X_{t,T}) \right] \right) \right]^2 = 0$$

which, in turn, can be verified under the moment assumptions stated in the corollary by slightly adapting the proof of Theorem 2.4. The bootstrap counterpart can be proven analogously. □

Proof of Corollary 4.2. We get from Theorem 2.4 combined with Beering (2021, Lem. 4.3), which ensures the negligibility of the bias,

$$(b_T T)^{1/2} \begin{pmatrix} \Re\left(\widehat{\varphi}(u;\underline{s}) - \varphi(u;\underline{s})\right) \\ \Im\left(\widehat{\varphi}(u;\underline{s}) - \varphi(u;\underline{s})\right) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} Z^{(1)}(u;\underline{s}) \\ Z^{(2)}(u;\underline{s}) \end{pmatrix} \sim \mathscr{N}\left(\underline{0}, \mathbf{V}(u;\underline{s})\right),$$

where

$$\mathbf{V}(u;\underline{s}) = \begin{pmatrix} V_\Re(u;\underline{s}) & V_{\Re\Im}(u;\underline{s}) \\ V_{\Re\Im}(u;\underline{s}) & V_\Im(u;\underline{s}) \end{pmatrix}$$

with

$$V_{\Re}(u;\underline{s}) = \int_{-1}^{1} K^2(x)\,dx \sum_{h\in\mathbb{Z}} \mathrm{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right), \cos\left(\langle \underline{s}, \underline{\widetilde{X}}_h(u)\rangle\right)\right),$$

$$V_{\Im}(u;\underline{s}) = \int_{-1}^{1} K^2(x)\,dx \sum_{h\in\mathbb{Z}} \mathrm{Cov}\left(\sin\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right), \sin\left(\langle \underline{s}, \underline{\widetilde{X}}_h(u)\rangle\right)\right)$$

and

$$V_{\Re\Im}(u;\underline{s}) = \int_{-1}^{1} K^2(x)\,dx \sum_{h\in\mathbb{Z}} \mathrm{Cov}\left(\cos\left(\langle \underline{s}, \underline{\widetilde{X}}_0(u)\rangle\right), \sin\left(\langle \underline{s}, \underline{\widetilde{X}}_h(u)\rangle\right)\right),$$

see Beering (2021, Th. 5.4). Analogously, we obtain from Theorem 3.3

$$(b_T T)^{1/2} \begin{pmatrix} \Re\left(\widehat{\varphi}^\star(u;\underline{s}) - E^\star \varphi^\star(u;\underline{s})\right) \\ \Im\left(\widehat{\varphi}^\star(u;\underline{s}) - E^\star \varphi^\star(u;\underline{s})\right) \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\underline{0}, \mathbf{V}(u;\underline{s})\right)$$

in P-probability. The covariance matrix $\mathbf{V}$ is positive definite due to condition (4.1). Because of the continuous mapping theorem, $\left|(b_T T)^{1/2}\left(\widehat{\varphi}(u;\underline{s}) - \varphi(u;\underline{s})\right)\right|$ and $\left|(b_T T)^{1/2}\left(\widehat{\varphi}^\star(u;\underline{s}) - E^\star \widehat{\varphi}^\star(u;\underline{s})\right)\right|$ converge to the same limit distribution (in P-probability) $P^{|Z(u;\underline{s})|}$. Note that $\mathrm{Var}(|Z(u;\underline{s})|) > 0$ since it holds

$$\begin{aligned}
\mathrm{Var}\left(|Z(u;\underline{s})|^2\right) &= \mathrm{Var}\left(\left(Z^{(1)}(u;\underline{s})\right)^2\right) + \mathrm{Var}\left(\left(Z^{(2)}(u;\underline{s})\right)^2\right) + 2\,\mathrm{Cov}\left(\left(Z^{(1)}(u;\underline{s})\right)^2, \left(Z^{(2)}(u;\underline{s})\right)^2\right) \\
&= V_{\Re}^2 + V_{\Im}^2 - V_{\Re}V_{\Im} + E\left(\left(Z^{(1)}(u;\underline{s})\right)^2 \left(Z^{(2)}(u;\underline{s})\right)^2\right) \\
&> 0.
\end{aligned}$$

Hence, the desired result follows immediately. □

A.4. Auxiliary results required for Section 2

Lemma A.2.

Under Assumptions 1 valid for $k = 0$ and Assumption 3, we have for every $M \in \mathbb{N}$ and some $C_{trunc} < \infty$

$$\sup_{u\in[0,1]} \left\| f\left(\underline{s}, \underline{\widetilde{X}}_t(u)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}(u)\right) \right\|_1 \le C_{trunc}\, \vartheta^M.$$

Proof. With the use of (2.5) and the Lipschitz condition (2.9), it holds for some $C_{trunc} < \infty$

$$\begin{aligned}
\sup_{u\in[0,1]} \left\| f\left(\underline{s}, \underline{\widetilde{X}}_t(u)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}(u)\right) \right\|_1 &\le C_{Lip} \sup_{u\in[0,1]} \left\| \underline{\widetilde{X}}_t(u) - \underline{\widetilde{X}}_t^{(M)}(u) \right\|_1 \\
&\le C_{Lip} \sup_{u\in[0,1]} \left\| \sum_{|j|\ge M} A(u,j)\, \underline{\varepsilon}_{t-j} \right\|_1 \\
&\le C_{trunc}\, \vartheta^M.
\end{aligned}$$

□

Lemma A.3.
Suppose Assumptions 1 for $k=0$ and Assumption 3 are fulfilled. Then, it holds uniformly for all $t_1, t_2 \in \mathbb{Z}$, $(\underline{s}_1, u_1), (\underline{s}_2, u_2) \in \mathscr{S} \times [0,1]$

$$\left\| \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right)\right) f\left(\underline{s}_2, \underline{\widetilde{X}}_{t_2}(u_2)\right) \right\|_1 \leq C_{DP}\, \vartheta^{\frac{M\delta}{1+\delta}}$$

for any $M \in \mathbb{N}$ with some positive constant $C_{DP} < \infty$ being independent of M.

Proof. By an iterative application of Hölder's inequality, we obtain

$$\begin{aligned}
&\left\| \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right)\right) f\left(\underline{s}_2, \underline{\widetilde{X}}_{t_2}(u_2)\right) \right\|_1 \\
&\leq C \left\| f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right\|_{\frac{2+\delta}{1+\delta}} \\
&\leq C \left(E\left| f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_{t_1}^{(M)}(u_1)\right) \right| \right)^{\frac{\delta}{1+\delta}} \\
&\leq C\, \vartheta^{\frac{M\delta}{1+\delta}},
\end{aligned}$$

where the latter inequality follows from Lemma A.2. □

Lemma A.4 (Covariance Bounds)**.**
Suppose Assumption 1 holds true for $k=0$ and Assumption 3 is valid. Then, we have for some $C_{Cov} < \infty$ and every $h \in \mathbb{Z}$

$$\sup_{\underline{s}_1, \underline{s}_2 \in \mathscr{S}, u_1, u_2 \in [0,1]} \left| \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right)\right)\right| \leq C_{Cov}\, \vartheta^{\frac{|h|\delta}{2(1+\delta)}}.$$

Proof. W.l.o.g. consider $h \geq 1$ and let $M := \lceil h/2 \rceil$. We split the covariance up and get

$$\begin{aligned}
&\left| \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right)\right)\right| \\
&\leq \left| \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right)\right)\right| \\
&\qquad + \left| \mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h^{(M)}(u_2)\right)\right)\right| \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Obviously, it suffices to verify the result for the first summand. We start by splitting term I up further using Lemmata A.2 and A.3

$$\begin{aligned}
I &\leq \left| E\left(\left[f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right)\right] f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right)\right)\right| \\
&\qquad + E\left| f\left(\underline{s}_1, \underline{\widetilde{X}}_0(u_1)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}(u_1)\right)\right| E\left| f\left(\underline{s}_2, \underline{\widetilde{X}}_h(u_2)\right)\right| \\
&\leq C_{DP}\, \vartheta^{\frac{M\delta}{1+\delta}} + C_{trunc}\, \vartheta^M,
\end{aligned}$$

which can obviously be bounded by $C\, \vartheta^{\frac{h\delta}{2(1+\delta)}}$. □

Lemma A.5 (Convergence of covariances)**.**
Suppose Assumption 1 is satisfied for $k = 1$ and Assumptions 2 as well as 3 hold. Then, for any $\underline{s}_1, \underline{s}_2 \in \mathscr{S}$ we have

$$\operatorname{Cov}\Big(\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}_1, \underline{X}_{t,T}\right), \sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s}_2, \underline{X}_{t,T}\right)\Big) \underset{T\to\infty}{\longrightarrow} \mathbf{V}\left(\underline{s}_1, \underline{s}_2\right),$$

where $\mathbf{V}(\underline{s}_1, \underline{s}_2)$ is defined in Theorem 2.4.

Proof. First, we rewrite the covariance and obtain

$$\begin{aligned}
&\lim_{T\to\infty} \sum_{t_1,t_2=1}^{T} \operatorname{Cov}\left(w_{t_1,T}\, f\left(\underline{s}_1, \underline{X}_{t_1,T}\right), w_{t_2,T}\, f\left(\underline{s}_2, \underline{X}_{t_2,T}\right)\right) \\
&\qquad = \lim_{T\to\infty} \sum_{h=-(T-1)}^{T-1} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right).
\end{aligned}$$

Note that

$$\sup_t \left|\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right)\right| \leq C \rho^{\frac{|h|\delta}{2+2\delta}}$$

can be verified along the lines of the proof of Lemma A.4. Hence, these covariances are absolutely summable w.r.t. h. Due to the assumptions regarding the weights, Lebesgue's dominated convergence theorem is applicable, and we obtain

$$\begin{aligned}
&\lim_{T\to\infty} \sum_{h=-(T-1)}^{T-1} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right) \\
&\qquad = \sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\, w_{t+h,T} \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right) \mathbb{1}_{|h|<T}. \qquad \text{(A.19)}
\end{aligned}$$

Next, we incorporate the companion process into the covariance terms

$$\begin{aligned}
&\operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right) \\
&= \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right) \\
&\qquad + \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{X}_{t+h,T}\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_{t+h}\left(\frac{t+h}{T}\right)\right)\right) \\
&\qquad\qquad + \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right), f\left(\underline{s}_2, \underline{\widetilde{X}}_{t+h}\left(\frac{t+h}{T}\right)\right)\right) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}
\end{aligned}$$

and show asymptotic negligibility of I and II. Using Remark 2.3, we get

$$\mathrm{I} \leq E\left(\left|f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right| \left|f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right|\right) + O(T^{-1}).$$

Inspired by the proof of Lemma A.4, we obtain for the first summand from above with double use of Hölder's inequality

$$E\left(\left|f\left(\underline{s}_1, \underline{X}_{t,T}\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right| \left|f\left(\underline{s}_2, \underline{X}_{t+h,T}\right)\right|\right)$$

$$\leq C\left(\left(E\left|f\left(\underline{s}_1,\underline{X}_{t,T}\right)-f\left(\underline{s}_1,\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right|\right)^{\frac{\delta}{1+\delta}}\left(E\left|f\left(\underline{s}_1,\underline{X}_{t,T}\right)-f\left(\underline{s}_1,\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)\right|^{2+\delta}\right)^{\frac{1}{(1+\delta)(2+\delta)}}\right)$$
$$\leq C T^{-\frac{\delta}{1+\delta}}$$

and, hence, $\mathrm{I}=\mathscr{O}\left(T^{-\frac{\delta}{1+\delta}}\right)$. Analogously, we have $\mathrm{II}=\mathscr{O}\left(T^{-\frac{\delta}{1+\delta}}\right)$. Thus, we can write (A.19) as follows:

$$\sum_{h\in\mathbb{Z}}\lim_{T\to\infty}\sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\,w_{t+h,T}\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{X}_{t,T}\right),f\left(\underline{s}_2,\underline{X}_{t+h,T}\right)\right)\mathbb{1}_{|h|<T}$$
$$=\sum_{h\in\mathbb{Z}}\lim_{T\to\infty}\sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\,w_{t+h,T}\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right)\mathbb{1}_{|h|<T}.$$

Now we introduce some notational adaptions and set

$$\underline{\widetilde{X}}_t(z)=\begin{cases}\underline{\widetilde{X}}_t(1), & z>1,\\ \underline{\widetilde{X}}_t(0), & z<0,\end{cases}\quad\text{for all } t\in\mathbb{Z}\text{ and}\quad w_{t,T}=\begin{cases}w_{T,T}, & t>T,\\ w_{1,T}, & t<1.\end{cases}$$

This allows us to consider

$$\left|\sum_{h\in\mathbb{Z}}\lim_{T\to\infty}\left\{\sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\,w_{t+h,T}\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right)\right.\right.$$
$$\left.\left.-\sum_{t=1}^{T} w_{t,T}\,w_{t+h,T}\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)\right)\right\}\mathbb{1}_{|h|<T}\right|.$$

Using Lemma A.4, the absolute value of the term in curly brackets can be bounded by $C d_T^{-1}\, h\vartheta^{h\delta/(2\delta+2)}$, which converges to 0 as $T\to\infty$ for fixed h. The next step will be to change the argument of $\underline{\widetilde{X}}_h(\cdot)$ to eliminate h. To this end, we show asymptotic negligibility of

$$\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right).$$

With the help of the Lipschitz condition (2.9) in Assumption 3 and Remark 2.2(i), it holds

$$\left|E\left(f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right|\leq C_1\frac{|h|}{T}.$$

Next, remembering the use of Hölder's inequality in the proof of Lemma A.3, we obtain

$$E\left(\left|f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right|\left|f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|\right)\leq C_2\left(\frac{|h|}{T}\right)^{\frac{\delta}{1+\delta}}.$$

Thus, we have

$$\left|\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t+h}{T}\right)\right)-f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right|\leq\mathscr{O}\left(\left(\frac{|h|}{T}\right)^{\frac{\delta}{1+\delta}}\right)+\mathscr{O}\left(\frac{|h|}{T}\right).$$

For fixed h, this converges to 0 as $T\to\infty$. Summing up, we get

$$\lim_{T\to\infty}\sum_{h=-(T-1)}^{T-1}\sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} w_{t,T}\,w_{t+h,T}\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{X}_{t,T}\right),f\left(\underline{s}_2,\underline{X}_{t+h,T}\right)\right)$$
$$=\sum_{h\in\mathbb{Z}}\lim_{T\to\infty}\sum_{t=1}^{T} w_{t,T}\,w_{t+h,T}\operatorname{Cov}\left(f\left(\underline{s}_1,\underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right),f\left(\underline{s}_2,\underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right).$$

Finally, we obtain existence of the RHS using Lemma A.4 again. □

Lemma A.6.

Under the prerequisites of Theorem 2.5 combined with Assumption 4(ii)(b), it holds

$$\lim_{r\to 0}\limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S}\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left|\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|>\frac{\lambda}{9}\right)=0$$

using the notation of the proof of Theorem 2.5.

Proof. We bound the quantity of interest as in (5.6) and start by specifying our choice for m and demanding both a lower and an upper bound for r_{k_T}. Here, we assume $m>1+\frac{2}{\delta(1+\delta)}$, and that the bounds for r_{k_T} are of the following form:

$$d_T^{-\frac{(m-1)(1+\delta)}{2m}}\le r_{k_T}\le d_T^{-\frac{1}{m\delta}}. \tag{A.20}$$

Again, by our choice of m, we guarantee for the existence of a $k_T\in\mathbb{N}$ fulfilling the aforesaid requirements. At this point, we return to the three summands on the RHS of (5.6). The treatment of the individual terms will be carried out analogously to part (a). On account of this, we start by establishing the necessary bounds for the application of Bernstein's inequality on term II. First, we constitute an upper bound for the variance of the inner sum of ν_T. With $l:=|i_1-i_2|$ and the very same arguments as in (a), we obtain

$$\operatorname{Var}\left(\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right)\le C\rho\left(\underline{s}_1,\underline{s}_2\right)^{\frac{1}{1+\delta}} \tag{A.21}$$

using the boundedness of f appropriately. Going back to term II of equation (5.6), we use Bernstein's inequality and the notation we introduced in (5.5) to get

$$\mathrm{II}\le 2\sum_{k=1}^{k_T}D(k)D(k-1)\exp\left(-\frac{1}{2}\cdot\frac{\lambda_k^2}{V_{II,k}+\frac{\breve{M}\lambda_k}{3}}\right).$$

Here, using again the boundedness of f, we can set

$$\breve{M}:=Cd_T^{-\frac{m-1}{2m}}\ge\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S}\\ t\in\{1,\dots,\mu_T\}}}\left|\sum_{i\in H_t}w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right|$$

and, in view of (A.21),

$$V_{II,k}:=Cr_k^{\frac{1}{1+\delta}}\ge\sup_{\substack{\underline{s}_1\in\mathscr{F}_k,\underline{s}_2\in\mathscr{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\le 3r_k}}\operatorname{Var}\left(\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right).$$

Similarly to (5.14), we obtain

$$\mathrm{II}\le 2\sum_{k=1}^{k_T}\exp\left(2\log\left(D(k)\right)-\frac{1}{2}\cdot\frac{\lambda_k^2}{Cr_k^{\frac{1}{1+\delta}}}\right)\le 2\sum_{k\in\mathbb{N}}\exp\left(-\frac{\overline{C}}{2}2^{\frac{k}{2(1+\delta)}}r^{-\frac{1}{2(1+\delta)}}\right)\xrightarrow[r\to 0]{}0.$$

By the same arguments as before, we get

$$V_{III}:=Cr^{\frac{1}{1+\delta}}\ge\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\le 3r}}\operatorname{Var}\left(\sum_{t=1}^{\mu_T}\sum_{i\in H_t}w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right).$$

Using the same $\breve{M}$ as while treating term II, Bernstein's inequality applied to equation (5.15) gives us

$$\begin{aligned}
&P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3r}}\left|\sum_{t=1}^{\mu_T}\sum_{i\in H_t} w_{i,T}\left(\bar{f}\left(\underline{s}_1,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)-\bar{f}\left(\underline{s}_2,\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)\right|>\frac{\lambda}{27}\right)\\
&\leq 2D^2(0)\exp\left(-\frac{1}{2}\cdot\frac{C\lambda^2}{V_{III}+\frac{\breve{M}C\lambda}{3}}\right)\\
&\leq 2\exp\left(2\log\left(D(0)\right)-\overline{C}\,r^{-\frac{1}{1+\delta}}\right)\\
&\underset{r\to 0}{\longrightarrow} 0.
\end{aligned}$$

Now term I is left. As in part (a), by Markov's inequality it is sufficient to show

$$\lim_{r\to 0}\limsup_{T\to\infty} E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S}\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|\right)=0. \tag{A.22}$$

Again, using the notation of the proof of Theorem 2.5 (case a), we get

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S}\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\nu_T\left(\underline{s}_1,\underline{s}_2\right)\right|\right)\leq 2E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S}\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L_{t,T}^0\left(\underline{s}_1\right)-L_{t,T}^0\left(\underline{s}_2\right)\right)\right|\right)$$

similar to (5.17). By Hoeffding's inequality we obtain

$$P\left(\left|\sum_{t=1}^{\mu_T}L_{t,T}^0\left(\underline{s}_1\right)-L_{t,T}^0\left(\underline{s}_2\right)\right|>\widehat{\rho}_{T,2}\left(\underline{s}_1,\underline{s}_2\right)\eta\,\middle|\,L_{1,T},\ldots,L_{\mu_T,T}\right)\leq 2\exp\left(-\frac{\eta^2}{2}\right)$$

for any $\underline{s}_1,\underline{s}_2\in\mathscr{S}$ and $\eta>0$ comparable to (5.18) with $\widehat{\rho}_{T,2}$ as in (5.19). Thus, we can check for (A.22) with the use of a maximal inequality for sub-Gaussian processes again. To ease the following part, we use a slightly different semimetric as above. In order to get the said new semimetric, we notice that

$$\left(L_{t,T}\left(\underline{s}_1\right)-L_{t,T}\left(\underline{s}_2\right)\right)^2\leq 2^{\frac{2-\delta}{2}}\left|L_{t,T}\right|_\infty^{\frac{2-\delta}{2}}\left|L_{t,T}\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}\,\rho\left(\underline{s}_1,\underline{s}_2\right)^{\frac{2+\delta}{2}}$$

holds for $\underline{s}_1,\underline{s}_2\in\mathscr{S}$, comparable to (5.20). Next, set

$$Q_T:=2^{\frac{2-\delta}{4}}\left(\sum_{t=1}^{\mu_T}\left|L_{t,T}\right|_\infty^{\frac{2-\delta}{2}}\left|L_{t,T}\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}\right)^{1/2}. \tag{A.23}$$

We can use the bounds

$$\left|L_{t,T}\right|_\infty^{\frac{2-\delta}{2}}=\left(\sup_{\underline{s}\in\mathscr{S}}\left|\sum_{i\in H_t}w_{i,T}\,f\left(\underline{s},\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right|\right)^{\frac{2-\delta}{2}}\leq C_1\,\kappa_T^{\frac{2-\delta}{2}}\,d_T^{-\frac{2-\delta}{4}}$$

and

$$\left|L_{t,T}\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}\leq\left(\sum_{i\in H_t}w_{i,T}\,g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)^{\frac{2+\delta}{2}}\leq C_2\,d_T^{\frac{2+\delta}{4}}\left(\sum_{i\in H_t}g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)^{\frac{2+\delta}{2}}$$

to obtain

$$Q_T\leq C_3\left(\kappa_T^{\frac{2-\delta}{2}}\,d_T\sum_{t=1}^{\mu_T}\left(\sum_{i\in H_t}g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right)^{\frac{2+\delta}{2}}\right)^{1/2}. \tag{A.24}$$

Now we can define the new semimetric using the definition of Q_T:

$$\widehat{\rho}_{T,2}\left(\underline{s}_1,\underline{s}_2\right) \le Q_T\, \rho\left(\underline{s}_1,\underline{s}_2\right)^{\frac{2+\delta}{4}} =: \breve{\rho}_T\left(\underline{s}_1,\underline{s}_2\right).$$

As in part (a), we have

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d \\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L_{t,T}^0\left(\underline{s}_1\right)-L_{t,T}^0\left(\underline{s}_2\right)\right)\right|\,\middle|\, L_{1,T},\ldots,L_{\mu_T,T}\right) \le C\int_0^{Q_T r_{k_T}^{\frac{2+\delta}{4}}} \left(\log\left(D\left(u,\mathscr{S},\breve{\rho}_T\right)\right)\right)^{1/2} du. \tag{A.25}$$

By Assumption 4, it holds

$$D\left(u,\mathscr{S},\breve{\rho}_T\right) = D\left(\left(\frac{u}{Q_T}\right)^{\frac{4}{2+\delta}},\mathscr{S},\rho\right) \le C\left(\left(\frac{u}{Q_T}\right)^{-\frac{4}{2+\delta}}+1\right)^d.$$

Now we can insert this bound into (A.25) and obtain

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S} \\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L_{t,T}^0\left(\underline{s}_1\right)-L_{t,T}^0\left(\underline{s}_2\right)\right)\right|\right) \le C E[Q_T]\int_0^{r_{k_T}^{\frac{2+\delta}{4}}} u^{-\frac{2}{2+\delta}}\, du \le C E[Q_T]\, r_{k_T}^{\delta/4}. \tag{A.26}$$

Next, we focus on $E[Q_T]$ and get with the use of (A.24)

$$E[Q_T] \le C_3\left(\kappa_T^{\frac{2-\delta}{2}} d_T^{-1}\sum_{t=1}^{\mu_T}\left\|\sum_{i\in H_t} g\left(\underline{\widetilde{X}}_i^{(M)}\left(\frac{i}{T}\right)\right)\right\|_{\frac{2+\delta}{2}}^{\frac{2+\delta}{2}}\right)^{1/2} \le C_3\, d_T^{\frac{1}{4m}}. \tag{A.27}$$

Finally, combining (A.26) and (A.27) and using the upper bound of r_{k_T} in (A.20) we have

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S} \\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}\left(L_{t,T}^0\left(\underline{s}_1\right)-L_{t,T}^0\left(\underline{s}_2\right)\right)\right|\right) = C d_T^{-\frac{1}{4m}},$$

which tends to 0 as $T\to\infty$, and the proof is completed. □

A.5. Auxiliary results required for Section 3

Lemma A.7.

Assume the validity of Assumption 5. Then, it holds

$$\sup_{s\in\mathscr{S}}\sup_{t\le T} E\left[E^\star\left|f\left(\underline{s},\underline{X}_{t,T}^\star\right)\right|^{2+\delta}\right] < \infty.$$

Proof. We distinguish whether t is an endpoint or not. Starting with $t\notin EP$, we have

$$\sup_{s\in\mathscr{S}}\sup_{t\le T} E\left[E^\star\left|f\left(\underline{s},\underline{X}_{t,T}^\star\right)\right|^{2+\delta}\right] = \sup_{s\in\mathscr{S}}\sup_{t\le T}\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} E\left|f\left(\underline{s},\underline{X}_{t+r,T}\right)\right|^{2+\delta} < \infty$$

by Assumption 3. The cases $t\in EP_1$ and $t\in EP_2$ can be treated similarly, and we only examine the latter case further. There, we obtain

$$E^\star\left|f\left(\underline{s},\underline{X}_{t,T}^\star\right)\right|^{2+\delta} = \frac{1}{2TD_T+1}\left(\sum_{r=-TD_T}^{T-t}\left|f\left(\underline{s},\underline{X}_{t+r,T}\right)\right|^{2+\delta} + \sum_{r=T-t+1}^{TD_T}\left|f\left(\underline{s},\underline{X}_{t-r,T}\right)\right|^{2+\delta}\right).$$

Consequently, the desired result follows with the same arguments as above. □

Lemma A.8.

Under Assumption 5, we have for all $t_1, t_2 \in \{1, \ldots, T\}$ and $\underline{s} \in \mathscr{S}$

$$f\left(\underline{s}, \underline{X}_{t_1,T}\right) f\left(\underline{s}, \underline{X}_{t_2,T}\right) = f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right) + \mathscr{O}_P\left(T^{-\frac{3+\delta}{2(4+\delta)}}\right).$$

Here, the $\mathscr{O}_P$-term does not depend on the choices for t_1 and t_2 or $\underline{s}$.

Proof. We want to make use of the closeness between the process $\left(\underline{X}_{t,T}\right)_{t=1}^{T}$ and its companion process $\left(\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right)_{t\in\mathbb{Z}}$. So the first step is to rewrite the difference between the two products:

$$\begin{aligned}
&E\left|f\left(\underline{s}, \underline{X}_{t_1,T}\right) f\left(\underline{s}, \underline{X}_{t_2,T}\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right)\right| \\
&\leq E\left|\left(f\left(\underline{s}, \underline{X}_{t_1,T}\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right) f\left(\underline{s}, \underline{X}_{t_2,T}\right)\right| + E\left|f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\left(f\left(\underline{s}, \underline{X}_{t_2,T}\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{t_2}\left(\frac{t_2}{T}\right)\right)\right)\right| \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Starting with I, we get using Remark 2.3

$$\begin{aligned}
\mathrm{I} &\leq C\left(E\left|f\left(\underline{s}, \underline{X}_{t_1,T}\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right|\right)^{\frac{3+\delta}{2(4+\delta)}} \left(E\left|f\left(\underline{s}, \underline{X}_{t_1,T}\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{t_1}\left(\frac{t_1}{T}\right)\right)\right|^{\frac{2(5+\delta)}{6+\delta}}\right)^{\frac{3+\delta}{2(4+\delta)}} \\
&\leq C T^{-\frac{3+\delta}{2(4+\delta)}}
\end{aligned}$$

and, in complete analogy, $\mathrm{II} = \mathscr{O}\left(T^{\frac{3+\delta}{2(4+\delta)}}\right)$. □

Lemma A.9.

Suppose Assumptions 5 and 6 are fulfilled. Then, for all $h, k \in \{1, \ldots, T\}$, $-TD_T \leq r, l \leq TD_T$ and $\underline{s} \in \mathscr{S}$ it holds

$$f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) = f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) + \mathscr{O}_P\left(D_T\right)$$

and

$$f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k+l}{T}\right)\right) = f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right) + \mathscr{O}_P\left(D_T^{\frac{3+\delta}{2(4+\delta)}}\right).$$

Both $\mathscr{O}_P$-terms are independent of the choices for h, k, r, l and $\underline{s}$.

Proof. (i) Applying Remark 2.2(ii) after using the Lipschitz condition (2.9), we get part (i) straightforwardly as it holds for $-TD_T \leq r \leq TD_T$

$$E\left|f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right| \leq C_{Lip}\frac{|r|}{T} \leq CD_T.$$

(ii) We split up

$$\begin{aligned}
&E\left|f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k+l}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right| \\
&\leq E\left|\left(f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right)\right) f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right| \\
&\qquad + E\left|f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right)\left(f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k+l}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right)\right)\right|
\end{aligned}$$

$=: \mathrm{I} + \mathrm{II}.$

It suffices to investigate I. Using Hölder's inequality, we get

$$
\begin{aligned}
\mathrm{I} &\leq \left\| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) \right\|_{\frac{4+\delta}{3+\delta}} \left\| f\left(\underline{s}, \underline{\widetilde{X}}_{k+l}\left(\frac{k}{T}\right)\right) \right\|_{4+\delta} \\
&\leq C \Bigg(E \Bigg(\left| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) \right|^{\frac{1}{2}} \\
&\qquad\qquad \cdot \left| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) \right|^{\frac{4+\delta}{3+\delta} - \frac{1}{2}} \Bigg) \Bigg)^{\frac{3+\delta}{4+\delta}} \\
&\leq C \left(\frac{|r|}{T}\right)^{\frac{3+\delta}{2(4+\delta)}} \left(E \left| f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h+r}{T}\right)\right) - f\left(\underline{s}, \underline{\widetilde{X}}_{h+r}\left(\frac{h}{T}\right)\right) \right|^{\frac{2(5+\delta)}{6+\delta}} \right)^{\frac{3+\delta}{2(4+\delta)}} \\
&= \mathscr{O}\left(D_T^{\frac{3+\delta}{2(4+\delta)}} \right).
\end{aligned}
$$

□

Lemma A.10 (Product Covariance Bound I)**.**
Suppose Assumption 5 is satisfied. Then, for all $u_1, \ldots, u_4 \in [0,1]$, $\underline{s} \in \mathscr{S}$

(i) and $t_1 \in \mathbb{N}$ and $t_2, r \in \mathbb{N}_0$ fulfilling $t_1 > t_2$ we have for some $\widetilde{\rho} \in (0,1)$

$$\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u_1)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u_2)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u_3)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_0(u_4)\right)\right)\right| \leq C_{Cov,2i}\, \widetilde{\rho}^{\,t_1-t_2}.$$

(ii) and $t_1, t_2 \in \mathbb{N}$ with $t_1 < t_2$ it holds for some $\widetilde{\rho} \in (0,1)$

$$\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+t_2}(u_1)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u_2)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u_3)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_0(u_4)\right)\right)\right| \leq C_{Cov,2ii}\, \max\{\widetilde{\rho}^{\,t_2-t_1}, \widetilde{\rho}^{\,t_1}\}.$$

Here, the constants $C_{Cov,2i}$, $C_{Cov,2ii} < \infty$ are independent of $\underline{s}$, $u_1, \ldots, u_4$ as well as of t_1, t_2 and r.

This lemma's proof uses the abbreviation

$$\bar{f}_M\left(\underline{s}, \underline{\widetilde{X}}_t(u)\right) := f\left(\underline{s}, \underline{\widetilde{X}}_t^{(M)}(u)\right) - E f\left(\underline{s}, \underline{\widetilde{X}}_t(u)\right) \tag{A.28}$$

for $t \in \mathbb{Z}$, $\underline{s} \in \mathbb{R}^d$, $u \in [0,1]$ and some $M \in \mathbb{N}$.

Proof. (i) Since $t_1 + r \geq t_1 > t_2 > 0$, we set $M := \lceil \frac{t_1 - t_2}{2} \rceil$. Now we are going to follow the proof of Lemma A.4 but modified for the covariance of products. Hence, we start by inserting the truncated version of the companion process as introduced in (2.8) with the above defined truncation parameter M and obtain

$$\left|\operatorname{Cov}\left(\bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1+r}(u_1)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_1}(u_2)\right), \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_{t_2}(u_3)\right) \bar{f}\left(\underline{s}, \underline{\widetilde{X}}_0(u_4)\right)\right)\right|$$

$$
\begin{aligned}
&\leq \Big|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u_2)\right),\right.\\
&\qquad\qquad\left.\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)-\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right)\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\right)\Big|\\
&\quad+\Big|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u_2)\right)-\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u_1)\right)\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u_2)\right),\right.\\
&\qquad\qquad\qquad\qquad\left.\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\right)\Big|\\
&=: \mathrm{I}+\mathrm{II}
\end{aligned}
\tag{A.29}
$$

using (A.28). As the terms I and II have a similar structure, we focus on term I and split up further

$$
\begin{aligned}
\mathrm{I}&\leq \Big|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u_2)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right)\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)-\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\right)\right)\Big|\\
&\quad+\Big|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u_2)\right)\ \bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right)-\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right)\right)\right)\Big|\\
&=: \mathrm{Ia}+\mathrm{Ib}.
\end{aligned}
\tag{A.30}
$$

For the same reasons as above, we limit ourselves to the investigation of the first subterm of (A.30) and obtain by an iterative application of Hölder's inequality and Lemma A.2

$$
\begin{aligned}
\mathrm{Ia}&\leq E\left|\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+r}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u_2)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right)\left[\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)-\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\right]\right|\\
&\qquad\qquad+CE\left|\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right)\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)-\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\right)\right|\\
&\leq C\left\|\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)-\bar{f}_M\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\right\|_{\frac{4+\delta}{1+\delta}}\\
&\leq C\left(E\left|f\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)-f\left(\underline{s},\underline{\widetilde{X}}_{0}^{(M)}(u_4)\right)\right|\right)^{\frac{\delta}{(3+\delta)}}\\
&\leq C\left(\rho^M\right)^{\frac{\delta}{3+\delta}}.
\end{aligned}
$$

In complete analogy, we get the same result as upper bound for Ib. In conclusion, this upper bound is valid for term I and II in (A.29) as well. Setting $\widetilde{\rho}=\rho^{\frac{\delta}{2(3+\delta)}}$ finishes the proof.

(ii) Applying part (i) and Lemma A.4 leads to

$$
\begin{aligned}
&\Big|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+t_2}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u_2)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\right)\Big|\\
&\leq \Big|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+t_2}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u_2)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\right)\Big|\\
&\qquad+\Big|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+t_2}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u_2)\right)\right)\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\right)\Big|\\
&\qquad\qquad+\Big|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+t_2}(u_1)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2}(u_3)\right)\right)\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1}(u_2)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_4)\right)\right)\Big|\\
&\leq C\max\{\widetilde{\rho}^{t_2-t_1},\widetilde{\rho}^{t_1}\}
\end{aligned}
$$

for some $\widetilde{\rho}\in(0,1)$.

□

Lemma A.11.

Suppose Assumptions 5 and 6 are valid. Then, for all indices $t_1,t_2\in\{1,\dots,T\}\backslash EP$ as well as for all

$\underline{s} \in \mathscr{S}$ *and* $u_1, u_2 \in [0,1]$ *it holds*

$$\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\Bigg(\Big(f\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u_1)\right)-\frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}f\left(\underline{s},\underline{\widetilde{X}}_{t_1+l}(u_1)\right)\Big)\cdot\Big(f\left(\underline{s},\underline{\widetilde{X}}_{t_2+r}(u_2)\right)-\frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}f\left(\underline{s},\underline{\widetilde{X}}_{t_2+l}(u_2)\right)\Big)\Bigg)$$

$$=\mathrm{Cov}\left(f\left(\underline{s},\widetilde{X}_{t_1}(u_1)\right),f\left(\underline{s},\widetilde{X}_{t_2}(u_2)\right)\right)+\mathscr{O}_P\left((TD_T)^{-1/2}\right).$$

At that, the occurring $\mathscr{O}_P$*-term is independent of* $t_1, t_2, \underline{s}$ *and* u_1, u_2.

Proof. First, note that by Lemma A.4 we have

$$(2TD_T+1)^{-1}\sum_{l=-TD_T}^{TD_T}\bar{f}(\underline{s},\underline{\widetilde{X}}_{t_2+l}(u))=\mathscr{O}_P((TD_T)^{-1/2})$$

for all $u \in [0,1]$. Thus, we obtain

$$\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\Bigg(\Big(f\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u_1)\right)-\frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}f\left(\underline{s},\underline{\widetilde{X}}_{t_1+l}(u_1)\right)\Big)\cdot\Big(f\left(\underline{s},\underline{\widetilde{X}}_{t_2+r}(u_2)\right)-\frac{1}{2TD_T+1}\sum_{l=-TD_T}^{TD_T}f\left(\underline{s},\underline{\widetilde{X}}_{t_2+l}(u_2)\right)\Big)\Bigg)$$

$$=\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+r}(u_2)\right)+\mathscr{O}_P((TD_T)^{-1}).$$

W.l.o.g. we consider $v := t_1 - t_2 > 0$ only. It remains to bound

$$E\Bigg(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_1+r}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t_2+r}(u_2)\right)-\mathrm{Cov}\left(f\left(\underline{s},\widetilde{X}_{t_1}(u_1)\right),f\left(\underline{s},\widetilde{X}_{t_2}(u_2)\right)\right)\Bigg)^2$$

$$\leq\frac{1}{(2TD_T+1)}\sum_{t=-2TD_T}^{2TD_T}\cdot\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t+v}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u_2)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_2)\right)\right)\right|$$

$$\leq\frac{2}{2TD_T+1}\sum_{t=1}^{v-1}\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t+v}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u_2)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_2)\right)\right)\right|$$
$$+\frac{2}{2TD_T+1}\sum_{t=v+1}^{2TD_T}\left|\mathrm{Cov}\left(\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t+v}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{t}(u_2)\right),\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{v}(u_1)\right)\bar{f}\left(\underline{s},\underline{\widetilde{X}}_{0}(u_2)\right)\right)\right|+\mathscr{O}((TD_T)^{-1})$$

$$=:\mathrm{I}+\mathrm{II}+\mathscr{O}((TD_T)^{-1}).$$

Applying Lemma A.10(i) to term I and Lemma A.10(ii) to II, we obtain the desired result, namely $I + II = \mathscr{O}((TD_T)^{-1})$. $\square$

Lemma A.12 (*P*-Convergence of the Bootstrap Variance)**.**
Suppose Assumptions 5 and 6 are true. Then, we have for all $\underline{s} \in \mathscr{S}$

$$\mathrm{Var}^{\star}\Bigg(\sum_{t=1}^{T}w_{t,T}\,f\left(\underline{s},\underline{X}^{\star}_{t,T}\right)\Bigg)\xrightarrow{P}\mathbf{V}\left(\underline{s},\underline{s}\right)$$

as $T \to \infty$*, where* $\mathbf{V}(\underline{s},\underline{s})$ *has its roots in Theorem 2.4.*

Proof. We start by dividing the quantity of interest into three sums such that the middle one does not contain endpoints and only full bootstrap blocks:

$$\begin{aligned}
&\operatorname{Var}^\star\Big(\sum_{t=1}^{T} w_{t,T}\, f\left(\underline{s},\underline{X}^\star_{t,T}\right)\Big)\\
&= \operatorname{Var}^\star\Big(\sum_{t=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t,T}\, f\left(\underline{s},\underline{X}^\star_{t,T}\right)\Big) + \operatorname{Var}^\star\Big(\sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t,T}\, f\left(\underline{s},\underline{X}^\star_{t,T}\right)\Big)\\
&\qquad + \operatorname{Var}^\star\Big(\sum_{t=L_T\lfloor (T-TD_T)/L_T\rfloor+1}^{T} w_{t,T}\, f\left(\underline{s},\underline{X}^\star_{t,T}\right)\Big)\\
&=: \mathrm{I}+\mathrm{II}+\mathrm{III}. \qquad\qquad (A.31)
\end{aligned}$$

Note that Assumption 6 and independence of the bootstrap blocks guarantee validity of this partition for any sufficiently large T. Since terms I and III from (A.31) are of a similar type, we focus on the first. From Lemma A.7 and the rates in Assumption 6, we obtain

$$\mathrm{I} = \sum_{t_1,t_2=1}^{L_T\lceil (TD_T+1)/L_T\rceil} w_{t_1,T}\, w_{t_2,T}\, \operatorname{Cov}^\star\left(f\left(\underline{s},\underline{X}^\star_{t_1,T}\right), f\left(\underline{s},\underline{X}^\star_{t_2,T}\right)\right) = \mathcal{O}_P\left(d_T^{-\frac{\delta}{2+\delta}}\right) = o_P(1). \qquad (A.32)$$

Thus, we only need to set our focus on the second bootstrap variance term of (A.31):

$$\begin{aligned}
\mathrm{II} = \sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \sum_{j=1}^{L_T}\sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T} \Big(&\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} f\left(\underline{s},\underline{X}_{tL_T+j+r,T}\right) f\left(\underline{s},\underline{X}_{tL_T+l+r,T}\right)\\
&- \frac{1}{(2TD_T+1)^2}\sum_{r=-TD_T}^{TD_T} f\left(\underline{s},\underline{X}_{tL_T+j+r,T}\right)\sum_{k=-TD_T}^{TD_T} f\left(\underline{s},\underline{X}_{tL_T+l+k,T}\right)\Big).
\end{aligned}$$

We aim for transforming the bootstrap covariance into the real world one with negligible error. To do so, the first step is to change the process $\left(\underline{X}_{t,T}\right)$ to the companion process with the aid of Lemma A.8:

$$\begin{aligned}
\mathrm{II} = &\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor-1} \sum_{j=1}^{L_T}\sum_{l=1}^{L_T} w_{tL_T+j,T}\, w_{tL_T+l,T}\\
&\cdot\Bigg(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} f\left(\underline{s},\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right) f\left(\underline{s},\underline{\widetilde{X}}_{tL_T+l+r}\left(\frac{tL_T+l+r}{T}\right)\right)\\
&\quad - \frac{1}{(2TD_T+1)^2}\Bigg(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s},\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)\\
&\quad\cdot\sum_{k=-TD_T}^{TD_T} f\left(\underline{s},\underline{\widetilde{X}}_{tL_T+l+k}\left(\frac{tL_T+l+k}{T}\right)\right)\Bigg)\Bigg) + \mathcal{O}_P\left(L_T T^{-\frac{3+\delta}{2(4+\delta)}}\right). \qquad (A.33)
\end{aligned}$$

Recalling that $L_T = o(d_T^{\frac{\delta}{2(1+\delta)}})$, straight-forward calculations give $\mathcal{O}_P\left(L_T T^{-\frac{3+\delta}{2(4+\delta)}}\right) = o_P(1)$. Next, we change the argument of $\left(\underline{\widetilde{X}}_t(u)\right)$ so that it loses the dependence of the inner summation index. With the aid of Lemma A.9, the first summand of (A.33) becomes

$$\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1}\sum_{h=-(L_T-1)}^{L_T-1}\sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$

$$\cdot\Bigg(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} f\left(\underline{s},\widetilde{\underline{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right) f\left(\underline{s},\widetilde{\underline{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)$$

$$-\frac{1}{(2TD_T+1)^2}\left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s},\widetilde{\underline{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h+r}{T}\right)\right)\right.$$

$$\left.\cdot\sum_{k=-TD_T}^{TD_T} f\left(\underline{s},\widetilde{\underline{X}}_{tL_T+j+k}\left(\frac{tL_T+j+k}{T}\right)\right)\right)\Bigg)$$

$$=\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1}\sum_{h=-(L_T-1)}^{L_T-1}\sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$

$$\cdot\Bigg(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T} f\left(\underline{s},\widetilde{\underline{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right) f\left(\underline{s},\widetilde{\underline{X}}_{tL_T+j+r}\left(\frac{tL_T+j}{T}\right)\right)$$

$$-\frac{1}{(2TD_T+1)^2}\left(\sum_{r=-TD_T}^{TD_T} f\left(\underline{s},\widetilde{\underline{X}}_{tL_T+j+h+r}\left(\frac{tL_T+j+h}{T}\right)\right)\right.$$

$$\left.\cdot\sum_{k=-TD_T}^{TD_T} f\left(\underline{s},\widetilde{\underline{X}}_{tL_T+j+k}\left(\frac{tL_T+j}{T}\right)\right)\right)\Bigg)+\mathcal{O}_P\left(L_T D_T^{\frac{3+\delta}{2(4+\delta)}}\right). \tag{A.34}$$

Note that under Assumption 6, it holds $\mathcal{O}_P\left(L_T D_T^{\frac{3+\delta}{2(4+\delta)}}\right)=o_P(1)$. Hence, we focus again on the first summand on the RHS of (A.34) and obtain its equivalence to

$$\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1}\sum_{h=-(L_T-1)}^{L_T-1}\sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$

$$\cdot\operatorname{Cov}\left(f\left(\underline{s},\widetilde{\underline{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right),f\left(\underline{s},\widetilde{\underline{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right)+\mathcal{O}_P\left(\frac{L_T}{\sqrt{TD_T}}\right) \tag{A.35}$$

by Lemma A.11. Comparably to the proof of Lemma A.5, we would like to rewrite the inner sum in order to eliminate the minimum and maximum determining the index bounds. Therefore, we show asymptotic negligibility of the difference between the first summand of (A.35) and

$$\sum_{t=\lceil (TD_T+1)/L_T\rceil}^{\lfloor (T-TD_T)/L_T\rfloor -1}\sum_{h=-(L_T-1)}^{L_T-1}\sum_{j=1}^{L_T} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$

$$\cdot\operatorname{Cov}\left(f\left(\underline{s},\widetilde{\underline{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right),f\left(\underline{s},\widetilde{\underline{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right) \tag{A.36}$$

T tends to infinity. Note that (A.36) is well-defined for sufficiently large T as we skipped endpoints within the summation. Regarding the difference in question, we get with the very same arguments as in the proof of Lemma A.5

$$\Bigg| \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=\max\{1,1-h\}}^{\min\{L_T,L_T-h\}} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right)$$
$$- \sum_{t=\lceil (TD_T+1)/L_T \rceil}^{\lfloor (T-TD_T)/L_T \rfloor -1} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{j=1}^{L_T} w_{tL_T+j+h,T}\, w_{tL_T+j,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{tL_T+j+h}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{tL_T+j}{T}\right)\right)\right) \Bigg|$$
$$= \mathscr{O}\left(L_T^{-1}\right).$$

Thus, we can proceed with (A.36). The next step will be to incorporate the sum over j back into the sum over t. Thereby, we obtain equivalence of (A.36) and

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil +1}^{L_T\lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{t+h}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right)\right) \tag{A.37}$$

for all sufficiently large T. By following exactly the lines of the proof of Lemma A.5, we can rewrite (A.37) as

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil +1}^{L_T\lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \left(\operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right)\right) + \mathscr{O}\left(\left(\frac{|h|}{T}\right)^{\frac{\delta}{1+\delta}}\right)\right)$$
$$= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil +1}^{L_T\lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right)\right) + \mathscr{O}\left(\left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}}\right). \tag{A.38}$$

Since the last term of (A.38) tends to 0 as $T \to \infty$, it remains to show

$$\lim_{T\to\infty} \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil +1}^{L_T\lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right)\right) = \mathbf{V}\left(\underline{s}, \underline{s}\right). \tag{A.39}$$

In view of Lebesgue's dominated convergence theorem, we obtain that the LHS of (A.39) equals

$$\sum_{h\in\mathbb{Z}} \lim_{T\to\infty} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil +1}^{L_T\lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right)\right).$$

By definition of $V_h(\underline{s}, \underline{s})$, it suffices to show asymptotic negligibility of

$$\sum_{h\in\mathbb{Z}} \Bigg| \sum_{t=1}^{T} w_{t+h,T}\, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right)\right)$$
$$- \sum_{h\in\mathbb{Z}} \sum_{t=L_T\lceil (TD_T+1)/L_T \rceil +1}^{L_T\lfloor (T-TD_T)/L_T \rfloor} w_{t+h,T}\, w_{t,T} \operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right)\right) \Bigg|$$
$$\leq \sum_{h\in\mathbb{Z}} \sum_{t=1}^{L_T\lceil (TD_T+1)/L_T \rceil} w_{t+h,T}\, w_{t,T} \left| \operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right)\right) \right|$$
$$+ \sum_{h\in\mathbb{Z}} \sum_{t=L_T\lfloor (T-TD_T)/L_T \rfloor +1}^{T} w_{t+h,T}\, w_{t,T} \left| \operatorname{Cov}\left(f\left(\underline{s}, \widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}, \widetilde{\underline{X}}_0\left(\frac{t}{T}\right)\right)\right) \right|,$$

which is, indeed, of order $\mathscr{O}\left(d_T^{-\frac{\delta}{2+\delta}}\right)$. $\square$

Lemma A.13.

Suppose Assumptions 5 to 7 hold true. Then, $\lim_{T\to\infty} P(\mathbb{X}_T \in A_T) = 1$, *where*

$$A_T := \left\{ \underline{x}_T \in \mathbb{R}^{dT} \,\middle|\, \sup_{\substack{\underline{s}\in\mathscr{S}\\ t=1,\ldots,T}} w_{t,T} \left| f\left(\underline{s}, \underline{X}^{\star}_{t,T}\right)\right| \le d_T^{-\frac{2+\delta^2}{2(4+\delta)}} \text{ for } \mathbb{X}_T = \underline{x}_T \right\}.$$

Proof. Recalling the Lipschitz condition in Assumption 4, Assumption 2, and the moment conditions on g, we obtain asymptotic negligibility of

$$P\left(\mathbb{X}_T \in A_T^C\right) \le C d_T^{\frac{2+\delta^2}{2}} \sum_{t=1}^{T} w_{t,T}^{4+\delta} \left[E\left[g\left(\underline{X}^{\star}_{t,T}\right)\right]^{4+\delta} + E\left|f\left(\underline{0}, \underline{X}^{\star}_{t,T}\right)\right|^{4+\delta}\right] = \mathscr{O}\left(d_T^{-\frac{\delta-\delta^2}{2}}\right).$$

$\square$

Lemma A.14.

Suppose that Assumptions 5 and 6 are fulfilled. Then, there exist sets $(B_T)_{T\in\mathbb{N}}$ *with* $P\left(\mathbb{X}_T \in B_T\right) \to 1$ *as* $T\to\infty$ *such that for any sequence* $(\underline{x}_T)_T$ *with* $\underline{x}_T \in B_T$ *for all* T *and for any deterministic sequence* $(c_T)_T$ *with* $c_T \underset{T\to\infty}{\longrightarrow} \infty$

$$\begin{aligned}
&\operatorname{Var}\left(\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(f\left(\underline{s}_1, \underline{X}^{\star}_{t,T}\right) - f\left(\underline{s}_2, \underline{X}^{\star}_{t,T}\right)\right) \,\middle|\, \mathbb{X}_T = \underline{x}_T \right)\\
&= \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}\\
&\qquad \cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \widetilde{\underline{X}}_h\left(\tfrac{t}{T}\right)\right) - f\left(\underline{s}_2, \widetilde{\underline{X}}_h\left(\tfrac{t}{T}\right)\right), f\left(\underline{s}_1, \widetilde{\underline{X}}_0\left(\tfrac{t}{T}\right)\right) - f\left(\underline{s}_2, \widetilde{\underline{X}}_0\left(\tfrac{t}{T}\right)\right)\right)\\
&\qquad\qquad + \mathscr{O}\left(L_T^{-1} + c_T L_T\left(T^{-\frac{3+\delta}{2(4+\delta)}} + D_T^{\frac{3+\delta}{2(4+\delta)}} + (TD_T)^{-1/2}\right)\right)
\end{aligned}$$

holds on B_T *for* $\underline{s}_1, \underline{s}_2 \in \mathscr{S}$, *where both* $\underline{s}_1$ *and* $\underline{s}_2$ *have no influence on the* $\mathscr{O}$*-term.*

Proof. The proof follows the lines of the proof of Lemma A.12 aiming to replace the $\mathscr{O}_P$-terms in the former proof by $\mathscr{O}$-terms with slightly modified rates to assure them to hold uniformly on $(B_T)_T$. Let $(c_T)_T$ be a deterministic, monotonically non-decreasing sequence tending to infinity at an arbitrary (slow) rate. Starting from (A.31), we first argue that the effect of the endpoints is of negligible order. To this end, define

$$B_{T,1} := \left\{ \underline{x}_T \in \mathbb{R}^{dT} \,\middle|\, [\text{ I+III in (A.31) with } \operatorname{Var}^{\star} \text{ substituted by } \operatorname{Var}^{\star}_T] \le d_T^{-\frac{\delta}{2+\delta}} c_T \right\}.$$

From the proof of (A.32), we obtain $P(\mathbb{X}_T \in B_{T,1}) \to 1$ as $T\to\infty$ by Markov's inequality. At this point, we come back to the remaining term II of (A.31) and define II_T as II in (A.31) with $\operatorname{Var}^{\star}$ substituted by $\operatorname{Var}^{\star}_T$. Further, we set

$$R_T := \mathrm{II}_T - \sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T}$$
$$\cdot \operatorname{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)$$

and

$$B_{T,2} := \left\{ \underline{x}_T \,\middle|\, |R_T| \leq C\left(L_T^{-1} + \left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}} + c_T L_T \left(T^{-\frac{3+\delta}{2(4+\delta)}} + D_T^{\frac{3+\delta}{2(4+\delta)}} + (TD_T)^{-1/2}\right)\right)\right\}.$$

With the arguments in the proof of Lemma A.12 and Markov's inequality, we get

$$P(\mathbb{X}_T \in B_{T,2}) \underset{T\to\infty}{\longrightarrow} 1.$$

Additionally, Assumption 6 yields (together with $T \geq d_T$)

$$\mathscr{O}\left(d_T^{\frac{\delta}{2+\delta}}\right) \subseteq \mathscr{O}\left(L_T^{-1}\right) \quad \text{and} \quad \mathscr{O}\left(\left(\frac{L_T}{T}\right)^{\frac{\delta}{1+\delta}}\right) \subseteq \mathscr{O}\left(L_T^{-1}\right).$$

Thus, the desired result follows from $P(\mathbb{X}_T \in B_{T,1} \cap B_{T,2}) \to 1$ as $T \to \infty$. □

Lemma A.15.
Provided the validity of Assumptions 6 and 7 (case (a)), there exists a sequence of sets $(K_T)_{T\in\mathbb{N}}$ with $P(\mathbb{X}_T \in K_T) \underset{T\to\infty}{\longrightarrow} 1$ such that for any sequence $(\underline{x}_T)_T$ with $\underline{x}_T \in K_T$ for all T

$$\sum_{t=0}^{\lfloor T/L_T\rfloor -1} E\left(\left|L_{t,T}^{\star}\right|_{\infty}^{1-\delta} \left|L_{t,T}^{\star}\right|_{\mathrm{Lip}}^{1+\delta} \,\middle|\, \mathbb{X}_t = x_t\right) \leq L_T^{1+\delta}$$

with $L_{t,T}^{\star}(\underline{s})$ being defined as in (A.15)*.*

Proof. As $P(\mathbb{X}_T \in K_T) \leq L_T^{-1-\delta} \sum_{t=0}^{\lfloor T/L_T\rfloor -1} E\left[E^{\star}\left(\left|L_{t,T}^{\star}\right|_{\infty}^{1-\delta} \left|L_{t,T}^{\star}\right|_{\mathrm{Lip}}^{1+\delta}\right)\right]$, it is enough to show that the sum on the RHS is of lower order than $(L_T^{1+\delta})_T$. An iterative application of both Hölder's and Jensen's inequality yields

$$\begin{aligned}
&\sum_{t=0}^{\lfloor T/L_T\rfloor -1} E\left[E^{\star}\left(\left|L_{t,T}^{\star}\right|_{\infty}^{1-\delta} \left|L_{t,T}^{\star}\right|_{\mathrm{Lip}}^{1+\delta}\right)\right] \\
&\leq \sum_{t=0}^{\lfloor T/L_T\rfloor -1} \left(EE^{\star}\left|L_{t,T}^{\star}\right|_{\infty}^{4+\delta}\right)^{\frac{1-\delta}{4+\delta}} \left(EE^{\star}\left|L_{t,T}^{\star}\right|_{\mathrm{Lip}}^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}}\right)^{\frac{2+3\delta}{2(4+\delta)}} \\
&\leq C \sum_{t=0}^{\lfloor T/L_T\rfloor -1} \left(E\left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}_{tL_T+j,T}^{\star}\right)\right\|_{4+\delta,\star}\right)^{4+\delta}\right. \\
&\qquad\qquad \left. + E\left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| f\left(\underline{0}, \underline{X}_{tL_T+j,T}^{\star}\right)\right\|_{4+\delta,\star}\right)^{4+\delta}\right)^{\frac{1-\delta}{4+\delta}} \\
&\qquad\qquad\qquad \cdot \left(E\left(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}_{tL_T+j,T}^{\star}\right)\right\|_{\frac{2(1+\delta)(4+\delta)}{2+3\delta},\star}\right)^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}}\right)^{\frac{2+3\delta}{2(4+\delta)}}
\end{aligned}$$

$$
\leq C \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \left(\left(E\Big(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \Big)^{4+\delta} \right)^{\frac{1-\delta}{4+\delta}} \right.
$$

$$
\left. + \left(E\Big(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| f\left(\underline{0}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \Big)^{4+\delta} \right)^{\frac{1-\delta}{4+\delta}} \right)
$$

$$
\cdot \left(E\Big(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| g\left(\underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{\frac{2(1+\delta)(4+\delta)}{2+3\delta},\star} \Big)^{\frac{2(1+\delta)(4+\delta)}{2+3\delta}} \right)^{\frac{2+3\delta}{2(4+\delta)}}
$$

$$
=: C \sum_{t=0}^{\lfloor T/L_T \rfloor -1} (\mathrm{I} + \mathrm{II}) \cdot \mathrm{III} \tag{A.40}
$$

with an obvious definition of I, II, and III. In order to show that the RHS is of order $o(L_T^{1+\delta})$, we investigate the three newly defined terms one by one beginning with the second. From Assumption 5, we obtain

$$
\mathrm{II} \leq \Big(\sum_{j=1}^{L_T} w_{tL_T+j,T} \left\| \left\| f\left(\underline{0}, \underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \right\|_{4+\delta} \Big)^{1-\delta} \leq C \Big(\sum_{j=1}^{L_T} w_{tL_T+j,T} \Big)^{1-\delta}. \tag{A.41}
$$

Proceeding with the first subterm on the RHS of (A.40), we play on Assumption 7 and obtain for $tL_T + j \notin EP$ (and similarly for endpoints)

$$
\left\| \left\| g\left(\underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} \right\|_{4+\delta}
$$

$$
\leq \left\| \left\| g\left(\underline{X}^{\star}_{tL_T+j,T} \right) \right\|_{4+\delta,\star} - \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right) \right) \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \right\|_{4+\delta}
$$

$$
+ \left\| \left(\frac{1}{2TD_T+1} \sum_{r=-TD_T}^{TD_T} \left| g\left(\underline{\widetilde{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T} \right) \right) \right|^{4+\delta} \right)^{\frac{1}{4+\delta}} \right\|_{4+\delta}
$$

$$
\leq C \left(\frac{1}{T} + 1 \right).
$$

This leads to

$$
\mathrm{I} \leq C \Big(\sum_{j=1}^{L_T} w_{tL_T+j,T} \Big)^{1-\delta}. \tag{A.42}
$$

At this point, only term III of equation (A.40) is left to be examined. Because said term has the same building type as term I, we can repeat the belonging procedure to get

$$
\mathrm{III} \leq C \Big(\sum_{j=1}^{L_T} w_{tL_T+j,T} \Big)^{1+\delta}. \tag{A.43}
$$

Joining equations (A.41), (A.42) and (A.43), we obtain

$$
C \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \Big(\sum_{j=1}^{L_T} w_{tL_T+j,T} \Big)^{1-\delta} \cdot \Big(\sum_{j=1}^{L_T} w_{tL_T+j,T} \Big)^{1+\delta} \leq C \sum_{t=0}^{\lfloor T/L_T \rfloor -1} \Big(\sum_{j=1}^{L_T} w_{tL_T+j,T} \Big)^{2} = C L_T
$$

as an upper bound for equation (A.40), which is of order $o(L_T^{1+\delta})$. □

Lemma A.16.

Suppose Assumptions 5, 6 and 7 are satisfied. Then, for all $\underline{s}_1, \underline{s}_2 \in \mathscr{S}$ it holds

$$\sum_{h=-(L_T-1)}^{L_T-1} \sum_{t=L_T\lceil (TD_T+1)/L_T\rceil+1}^{L_T\lfloor (T-TD_T)/L_T\rfloor} w_{t+h,T}\, w_{t,T} \cdot \left|\mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right)\right|$$
$$\leq C_{DC}\left|\underline{s}_1 - \underline{s}_2\right|_1^{1/2} \tag{A.44}$$

for some positive constant $C_{DC} < \infty$ neither depending on $\underline{s}_1, \underline{s}_2$ nor on t.

Proof. First, we split the covariance up inserting the truncated version of the companion process with truncation parameter $M := \lceil |h|/2 \rceil$ like in (2.8) and obtain

$$\begin{aligned}
&\left|\mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right), f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\
&\leq \left|\mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) + f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right),\right.\right. \\
&\qquad\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right)\right| \\
&\qquad + \left|\mathrm{Cov}\left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right),\right.\right. \\
&\qquad \left.\left. f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_1, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right) + f\left(\underline{s}_2, \underline{\widetilde{X}}_h^{(M)}\left(\frac{t}{T}\right)\right)\right)\right| \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

For symmetry reasons, we examine only term I. With the use of the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\mathrm{I} &\leq \left(E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|^2\right)^{1/2} \\
&\qquad \cdot \left(E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_h\left(\frac{t}{T}\right)\right)\right|^2\right)^{1/2} \\
&=: \mathrm{Ia} \cdot \mathrm{Ib}.
\end{aligned} \tag{A.45}$$

Continuing again with the first factor, we apply Hölder's inequality iteratively and similar arguments as in the proof of Lemma A.3 to obtain

$$\begin{aligned}
\mathrm{Ia} &\leq \left(E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right.\right. \\
&\qquad \left.\left. - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|^{\frac{2+\delta}{1+\delta}}\right)^{\frac{1+\delta}{2+\delta}\cdot\frac{1}{2}} \\
&\qquad \cdot \left(E\left|f\left(\underline{s}_1, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0\left(\frac{t}{T}\right)\right)\right.\right. \\
&\qquad \left.\left. - \left(f\left(\underline{s}_1, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right) - f\left(\underline{s}_2, \underline{\widetilde{X}}_0^{(M)}\left(\frac{t}{T}\right)\right)\right)\right|^{2+\delta}\right)^{\frac{1}{2+\delta}\cdot\frac{1}{2}} \\
&\leq C\rho^{\frac{M\delta}{2(1+\delta)}}.
\end{aligned}$$

For Ib in equation (A.45), we use Hölder's inequality anew and get in case (a) of Assumption 7

$$\mathrm{Ib} \le C\left(E\left|f\left(\underline{s}_1,\widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right)-f\left(\underline{s}_2,\widetilde{\underline{X}}_h\left(\frac{t}{T}\right)\right)\right|^{\frac{2+\delta}{1+\delta}}\right)^{\frac{1+\delta}{2+\delta}\cdot\frac{1}{2}} \le C\left|\underline{s}_1-\underline{s}_2\right|_1^{1/2}.$$

Part (b) can be treated in a similar manner using boundedness of f instead of Hölder's inequality. Consequently, we have

$$\mathrm{I} \le C\rho^{\frac{M\delta}{2(1+\delta)}}\left|\underline{s}_1-\underline{s}_2\right|_1^{1/2}.$$

Hence, we can bound (A.44) by

$$C\sum_{h=-(L_T-1)}^{L_T-1}\;\sum_{t=L_T\lceil(TD_T+1)/L_T\rceil+1}^{L_T\lfloor(T-TD_T)/L_T\rfloor} w_{t+h,T}\,w_{t,T}\,\rho^{\frac{M\delta}{2(1+\delta)}}\left|\underline{s}_1-\underline{s}_2\right|_1^{1/2} \le C_{DC}\left|\underline{s}_1-\underline{s}_2\right|_1^{1/2}.$$

□

Lemma A.17.

Under Assumptions 6 and 7 (case (b)), there exists a sequence of sets $(\bar{K}_T)_{T\in\mathbb{N}}$ with $P(\mathbb{X}_T\in\bar{K}_T)\underset{T\to\infty}{\longrightarrow}1$, such that for any sequence $(\underline{x}_T)_T$ with $\underline{x}_T\in\bar{K}_T$ for all T

$$\sum_{t=0}^{\lfloor T/L_T\rfloor-1} E\left(\left|L^{\star}_{t,T}\right|_{\infty}^{\frac{2-\delta}{2}}\left|L^{\star}_{t,T}\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}\,\middle|\,\mathbb{X}_t=x_t\right) \le C L_T^{1+\delta}$$

holds with $L^{\star}_{t,T}$ being defined in (A.15).

Proof. This proof models itself on the proof of Lemma A.15, i.e. we show that

$$E\left[\sum_{t=0}^{\lfloor T/L_T\rfloor-1} E^{\star}\left(\left|L^{\star}_{t,T}\right|_{\infty}^{\frac{2-\delta}{2}}\left|L^{\star}_{t,T}\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}\right)\right]=O(L_T).$$

First, note that

$$\left|L^{\star}_{t,T}\right|_{\infty} \le C\sum_{j=1}^{L_T} w_{tL_T+j,T} \tag{A.46}$$

and

$$\left|L^{\star}_{t,T}\right|_{\mathrm{Lip}} = \sum_{j=1}^{L_T} w_{tL_T+j,T}\, g\left(\underline{X}^{\star}_{tL_T+j,T}\right). \tag{A.47}$$

Hence, we have

$$E^{\star}\left(\left|L^{\star}_{t,T}\right|_{\mathrm{Lip}}\right)^{\frac{2+\delta}{2}} \le \left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\left\|g\left(\underline{X}^{\star}_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star}\right)^{\frac{2+\delta}{2}}.$$

For sake of notational simplicity, we consider non-endpoints only in the sequel. We get

$$\begin{aligned}
&\left\|\left\|g\left(\underline{X}^{\star}_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star}\right\|_{\frac{2+\delta}{2}}\\
&\le \left\|\left\|g\left(\underline{X}^{\star}_{tL_T+j,T}\right)\right\|_{\frac{2+\delta}{2},\star}-\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left|g\left(\widetilde{\underline{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}}\right\|_{\frac{2+\delta}{2}}\\
&\qquad+\left\|\left(\frac{1}{2TD_T+1}\sum_{r=-TD_T}^{TD_T}\left|g\left(\widetilde{\underline{X}}_{tL_T+j+r}\left(\frac{tL_T+j+r}{T}\right)\right)\right|^{\frac{2+\delta}{2}}\right)^{\frac{2}{2+\delta}}\right\|_{\frac{2+\delta}{2}}
\end{aligned}$$

$$=: \mathrm{I}+\mathrm{II}. \tag{A.48}$$

Term I can be bounded by C/T similarly to the respective term in the aforementioned proof. Straightforward arguments yield II$= O(1)$. Taken all together, from (A.46), (A.47) and (A.48) we obtain

$$E\Big[\sum_{t=0}^{\lfloor T/L_T\rfloor-1} E^\star\left(\left|L^\star_{t,T}\right|_\infty^{\frac{2-\delta}{2}}\left|L^\star_{t,T}\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}\right)\Big] \le C\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2-\delta}{2}}\left(\sum_{j=1}^{L_T} w_{tL_T+j,T}\right)^{\frac{2+\delta}{2}} \le CL_T.$$

□

Lemma A.18.

Suppose that the set of assumptions of Theorem 3.4 holds with part (b) in Assumption 4 and 7. Then, there exist sets $\left(\bar{\Omega}_T\right)_{T\in\mathbb{N}}$ with $P\left(\mathbb{X}_T\in\bar{\Omega}_T\right)\to 1$ as $T\to\infty$ such that for any $(\underline{x}_T)_T$ with $x_T\in\bar{\Omega}_T$ for all T

$$\lim_{r\to 0}\limsup_{T\to\infty} P\Big(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\Big|\sum_{t=1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\Big|>\lambda\,\Big|\,\mathbb{X}_T=\underline{x}_T\Big)=0$$

with $\lambda>0$, whereas $\bar{f}^\star$ is defined in (3.1).

Proof. First, we define subsets $\left(\bar{\Omega}_T\right)_{T\in\mathbb{N}}$ of Ω as

$$\bar{\Omega}_T = A_T\cap B_T\cap\bar{K}_T$$

with A_T as in Lemma A.13, B_T as in Lemma A.14 and $\bar{K}_T$ being defined in Lemma A.17. Then, we obtain $\lim_{T\to\infty}P\left(\bar{\Omega}_T\right)=1$. As in the proof of part (a), we split up

$$\begin{aligned}
&P^\star_T\Big(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\Big|\sum_{t=1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\Big|>\lambda\Big)\\
&\le P^\star_T\Big(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\Big|\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\Big|>\frac{\lambda}{2}\Big)\\
&\qquad + P^\star_T\Big(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\Big|\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\Big|>\frac{\lambda}{2}\Big).
\end{aligned}\tag{A.49}$$

With the use of Markov's inequality, asymptotic negligibility of the second sum of the RHS of (A.49) can be verified since it can be bounded by

$$\frac{2}{\lambda}E^\star_T\Big(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\Big|\sum_{t=L_T\lfloor T/L_T\rfloor+1}^{T} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\Big|\Big)=o\left(d_T^{-\frac{1}{2(1+\delta)}}\right).$$

It remains to consider

$$P^\star_T\Big(\sup_{\rho(\underline{s}_1,\underline{s}_2)<r}\Big|\sum_{t=1}^{L_T\lfloor T/L_T\rfloor} w_{t,T}\left(\bar{f}^\star\left(\underline{s}_1,\underline{X}^\star_{t,T}\right)-\bar{f}^\star\left(\underline{s}_2,\underline{X}^\star_{t,T}\right)\right)\Big|>\frac{\lambda}{2}\Big)$$

$$\leq P_T^\star\Bigg(2 \sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S}\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}} |\nu_T^\star(\underline{s}_1,\underline{s}_2)| > \frac{\lambda}{6}\Bigg) + P_T^\star\Bigg(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3r}} |\nu_T^\star(\underline{s}_1,\underline{s}_2)| > \frac{\lambda}{6}\Bigg)$$

$$+ P_T^\star\Bigg(2\sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1\in\mathscr{F}_k,\underline{s}_2\in\mathscr{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3r_k}} |\nu_T^\star(\underline{s}_1,\underline{s}_2)| > \frac{\lambda}{6}\Bigg)$$

$$=: \mathrm{I}+\mathrm{II}+\mathrm{III}. \tag{A.50}$$

While I and II can be treated as in the proof of Theorem 3.4, we have to adapt the investigation of term III. Using the notation of the proof of Theorem 3.4, it is enough to verify asymptotic negligibility of

$$E_T^\star\Bigg(\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathscr{S}\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\Bigg|\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\Big(L_{t,T}^{\star,0}(\underline{s}_1)-L_{t,T}^{\star,0}(\underline{s}_2)\Big)\Bigg|\Bigg). \tag{A.51}$$

To this end, first recall that $\sum_{t=0}^{\lfloor T/L_T\rfloor-1} L_{t,T}^{\star,0}$ comes with sub-Gaussian increments w.r.t. $\widehat{\rho}_{T,2}$ defined in (A.17) conditionally on $L_{0,T}^\star,\ldots,L_{\lfloor T/L_T\rfloor-1,T}^\star$. Second, it holds

$$\big(L_{t,T}^\star(\underline{s}_1)-L_{t,T}^\star(\underline{s}_2)\big)^2 \leq 2^{1-\delta}\left|L_{t,T}^\star\right|_\infty^{\frac{2-\delta}{2}}\left|L_{t,T}^\star\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}\rho(\underline{s}_1,\underline{s}_2)^{\frac{2+\delta}{2}}, \quad \underline{s}_1,\underline{s}_2\in\mathscr{S},$$

as seen comparably in (A.18). The next step is again to establish a semimetric which suits us more than ρ and $\widehat{\rho}_{T,2}$, respectively. To this end, let

$$Q_T := 2^{\frac{2-\delta}{4}}\Bigg(\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\left|L_{t,T}^\star\right|_\infty^{\frac{2-\delta}{2}}\left|L_{t,T}^\star\right|_{\mathrm{Lip}}^{\frac{2+\delta}{2}}\Bigg)^{1/2}$$

and define

$$\widehat{\rho}_{T,2}(\underline{s}_1,\underline{s}_2) \leq Q_T\rho(\underline{s}_1,\underline{s}_2)^{\frac{2+\delta}{4}} =: \breve{\rho}_T(\underline{s}_1,\underline{s}_2).$$

Returning to (A.51), we obtain from the maximal inequality for sub-Gaussian processes in van der Vaart and Wellner (2000, Cor. 2.2.8) that

$$\begin{aligned}
&E_T^\star\Bigg[E^\star\Bigg(\sup_{\rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}\Bigg|\sum_{t=0}^{\lfloor T/L_T\rfloor-1}\Big(L_{t,T}^{\star,0}(\underline{s}_1)-L_{t,T}^{\star,0}(\underline{s}_2)\Big)\Bigg|\,\Bigg|\,L_{1,T}^\star,\ldots,L_{L_T\lfloor T/L_T\rfloor,T}^\star\Bigg)\Bigg]\\
&\leq C E^\star\Bigg(\int_0^{Q_T r_{k_T}^{\frac{2+\delta}{4}}}\big(\log D(u,\mathscr{S},\breve{\rho}_T)\big)^{1/2}du\Bigg)\\
&\leq C E_T^\star\Bigg[\int_0^{Q_T r_{k_T}^{\frac{2+\delta}{4}}}\Bigg(\log\Bigg(\Bigg(\frac{u}{Q_T}\Bigg)^{-\frac{4}{2+\delta}}+1\Bigg)^d\Bigg)^{1/2}du\Bigg]\\
&= C E_T^\star[Q_T]\int_0^{r_{k_T}^{\frac{2+\delta}{4}}}\Big(\log\big(u^{-\frac{4}{2+\delta}}+1\big)^d\Big)^{1/2}du\\
&\leq C L_T^{\frac{1+\delta}{2}} L_T^{-\frac{1+\delta}{4\delta}}\\
&= C L_T^{-\frac{1-\delta-\delta^2}{4\delta}} \underset{T\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where the last inequality follows from Lemma A.17 and

$$\int_0^{r_{k_T}^{\frac{2+\delta}{4}}}\Big(\log\big(u^{\frac{4}{2+\delta}}+1\big)^d\Big)^{1/2}du \leq C_2 L_T^{-\frac{1+\delta}{4\delta}}.$$

Hence, applying Markov's inequality we have proven asymptotic negligibility of III in (A.50). □

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