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A BMAP|PH|1 Queue With Feedback Operating In A Random Environment

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Abstract—Feedback queues play an important role in real-life service systems, where customers may require repeated services. In this paper, we consider a feedback queue with batch Markovian arrivals and phase type services. We further assume that both the arrival process and service times are influenced by an external finite state Markovian environment. The stationary state distributions of the queue and the sojourn time are calculated and numerical examples are presented.

Keywords—Batch Markovian arrival process, Phase type service, Feedback, Random environment, Algorithm.

1. INTRODUCTION

Feedback queues have been investigated intensively since the work by Takacs [1]. The reason of interest to these queues stems from the fact that these queues well describe real-life situations where the service of a customer may be repeated due to, e.g., a nonsatisfactory quality of a service. Such situations take place, e.g., in computer communication networks where the transmission of a protocol data unit should be sometimes repeated due to the occurrence of an error.

We can refer to the papers [2–10], where the results of Takacs have been developed and generalized. Our paper has two distinguishing features. The first one is that we assume a more general arrival process (compared to the traditional assumption of a stationary Poisson model), namely the BMAP (batch Markovian arrival process). The second feature is that we suppose that the operation of the system is dependent on the state of some external random process, i.e., a random

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environment. Changing the state of this process causes changes of the parameters related to the arrival and service processes and the probability of the service repetition. Such a situation takes place, e.g., in modelling the processes of message exchanges between the network infrastructure and a moving customer in a mobile communication network. Increasing the distance between this user and the base station of his current cell may cause the degradation of the quality of service. In particular, this situation implies the need of a more frequent repetition of message transmissions due to the occurrences of transmission errors.

A related model has been considered by the authors in [11]. That model differs from the present one in the following respects.

(i) A general service time distribution is allowed (while here we are considering the special case of a phase type distribution), but this distribution does not depend on the state of the random environment (while here we are allowing such dependence).

(ii) A synchronous random environment (it means that the epochs of changing the state of the environment are synchronized with the service completion epochs) is considered while here we are considering an asynchronous random environment (absolutely independent on the behavior of a queue).

The rest of this paper is organized as follows. In Section 2, the mathematical model is fixed and the behavior of the system is described in terms of a multidimensional continuous time Markov chain. In Section 3, this Markov chain is investigated via the tool of multidimensional discrete time quasi-Toeplitz Markov chains (see [12,13]). Section 4 contains the results relating to the stationary distribution of the sojourn time of the queue under investigation. Section 5 states some numerical illustration of the obtained results.

2. THE MATHEMATICAL MODEL

The behavior of the queue we deal with depends on the state of a random environment. The random environment is an irreducible continuous time Markov chain $\xi_t, t \geq 0$. It has a finite state space $\{1, \dots, M\}$, $M \geq 2$. The sojourn time of the process ξ_t in state m has an exponential distribution with parameter μ_m, $0 < \mu_m < \infty$, $m = \overline{1, M}$. After this time expires, the process ξ_t jumps into the state m' according to a fixed transition probability matrix $P = \|p_{m,m'}\|_{m,m'=\overline{1,M}}$. We denote by $Q = \operatorname{diag}\{\mu_1, \dots, \mu_M\}(P - I)$ the infinitesimal generator of the process ξ_t, $t \geq 0$. Here, and subsequently, $\operatorname{diag}\{c_1, \dots, c_M\}$ represents the diagonal matrix having diagonal entries $(c_1, \dots, c_M)$. I is an identity matrix of appropriate dimension. If the dimension of this matrix is not clear from the context, it is explicitly indicated as a lower index to the letter I.

The process of the input flow is of the type BMAP. For more details about the BMAP and related questions we refer to [14] and the survey by Chakravarthy [15]. Usually, the BMAP is identified by an irreducible continuous time Markov chain ν_t, $t \geq 0$, which is referred to as the directing (or underlying) process of the BMAP. The state space of this process is given by $\{0, 1, \dots, W\}$. Here, and below, we write $\bar{W} = W + 1$. The sojourn time of the process in state ν is exponentially distributed with parameter λ_ν, $0 < \lambda_\nu < \infty$. When the sojourn time expires, with probability $p_{\nu,\nu'}^{(k)}$ the process ν_t jumps into the state ν' and a batch of k customers arrives, $k \geq 0$, $\nu, \nu' = \overline{0, W}$. It is assumed that

$$p_{\nu,\nu}^{(0)} = 0, \qquad \sum_{k=0}^{\infty} \sum_{\nu'=0}^{W} p_{\nu,\nu'}^{(k)} = 1, \qquad \nu = \overline{0, W}.$$

Lucantoni [14] offered to keep information about the parameters (i.e., $W, \lambda_0, \dots, \lambda_W, p_{\nu,\nu'}^{(k)}$, $k \geq 0$, ν, $\nu' = \overline{0, W}$) in the following matrices $D_k, k \geq 0$:

$$\begin{aligned} (D_k)_{\nu,\nu'} &= \lambda_\nu p_{\nu,\nu'}^{(k)}, \qquad k \geq 1 \quad \text{and} \quad k = 0, \qquad \nu \neq \nu', \\ (D_0)_{\nu,\nu} &= -\lambda_\nu, \qquad \nu = \overline{0, W} \end{aligned}$$

with the matrix generating function,

$$D(z) = \sum_{k=0}^{\infty} D_k z^k, \qquad |z| \leq 1.$$

The set of BMAPs is dense in the set of all point processes, hence, any arrival process can be approximated by a BMAP. This feature explains our choice of the BMAP as a model for the arrival stream under a fixed value of the random environment. It is worth mentioning that the BMAP in general is a correlated process. Hence, it is suitable to model the flows in modern communication networks.

In our model, the parameters of the BMAP are modulated by the random environment ξ_t, $t \geq 0$ in the following way. The state of the directing process ν_t is determined by $\{0, 1, \ldots, W\}$, as in case of the usual BMAP. However, in our case we assume that for a fixed value m of the random environment, the matrices D_k, which characterize the transition of the process ν_t, depend on the state m. Therefore, these matrices are given by $D_k^{(m)}$, $k \geq 0$ and their matrix generating functions look like

$$D^{(m)}(z) = \sum_{k=0}^{\infty} D_k^{(m)} z^k, \qquad |z| \leq 1, \quad m = \overline{1, M}.$$

In our model, the service process is assumed to be of phase (PH) type. The usual PH type service process is described by a continuous time Markov process η_t, $t \geq 0$. The state of this process at the epoch of a service start is defined according to a probabilistic row-vector $\vec{\beta} = (\beta_1, \ldots, \beta_K)$. Further, transitions of the process η_t, $t \geq 0$ are defined by a matrix S of dimension K. The diagonal entries $S_{k,k}, |S_{k,k}| < \infty$, $k = \overline{1, K}$ of the matrix are negative and $-S_{k,k}$ define the parameters of the exponentially distributed sojourn times of the process in the states $k = \overline{1, K}$. The nondiagonal entries of the matrix S define the intensities of transitions of the process η_t, $t \geq 0$ in the state space $\{1, \ldots, K\}$. The value $-\sum_{k'=1}^{K} S_{k,k'}$ defines the intensity of the transition of the process η_t, $t \geq 0$ from state k into the absorbing state. The epoch of the transition of the process η_t, $t \geq 0$ into this absorbing state defines the service completion epoch. We set $S_0 = -S\mathbf{e}$. Here, and subsequently, $\mathbf{e}$ denotes a column vector of appropriate size consisting of all ones. It is assumed that all the entries of the column vector S_0 are nonnegative and at least one of them is positive. For more details on PH type distributions, see the presentation in [16].

In our case, we assume that the random environment has the following impact on the service process. Under a fixed value m of the random environment ξ_t, $t \geq 0$, the initial state of the process η_t, $t \geq 0$ at the epoch of a service start is defined by the row vector $\vec{\beta}^{(m)}$ and transitions of the process η_t, $t \geq 0$ are defined by the subgenerator $S^{(m)}$, $m = \overline{1, M}$. If the state of the process ξ_t, $t \geq 0$ changes during the service time, the intensity of the corresponding sojourn time instantaneously changes in a corresponding manner.

In feedback models, it is usually supposed that a customer forever leaves the system after the service with a fixed probability p while it returns to the server with probability $1 - p$. Different variants of customer behavior are dealt with in the literature. Following Takacs, we assume here that this customer immediately returns to the server to be serviced again. The next customer enters the service only when the previous one completely finishes the service and leaves the system.

In our model, we assume that for a fixed value m of the process ξ_t, $t \geq 0$, the probability that a customer leaves the system after the service completion epoch is equal to p_m, $m = \overline{1, M}$. With supplementary probability, $1 - p_m$, the customer is immediately returned to the server and is processed again. We do not impose any restrictions relating to the number of possible service repetitions. Thus, if the random environment ξ_t, $t \geq 0$ maintains the state m during all time of the customer processing, then the total number of services, which are provided to this customer, is a geometrically distributed variable with a parameter p_m. Note that because the random environment may change the state during a customer processing, actual distribution of a number of services, which are provided to this customer, is much more complicated than the geometrical one.

Now, the model is defined. Our aim is to present the stability condition for the queue of this system and algorithms for calculating the stationary state distribution of the queue and sojourn time distribution.

To this end, we consider the following four-dimensional continuous time process $\{i_t, \xi_t, \nu_t, \eta_t\}$, $t \geq 0$, where the components $\{\xi_t, \nu_t, \eta_t\}$ are defined previously while i_t denotes the number of customers in the system at the epoch t, $i_t \geq 0$, $t \geq 0$. It is obvious that this process is a multi-dimensional continuous time Markov chain which has one denumerable component (i_t, $t \geq 0$) and three finite components ($\xi_t = \overline{1, M}$, $\nu_t = \overline{0, W}$, $\eta_t = \overline{1, K}$, $t \geq 0$).

Denote

$$\begin{gathered} p\,(i, m, \nu, n) = \lim_{t\to\infty} P\left\{i_t = i,\ \xi_t = m,\ \nu_t = \nu,\ \eta_t = n\right\}, \\ i \geq 0, \qquad m = \overline{1, M}, \qquad \nu = \overline{0, W}, \qquad n = \overline{1, K}. \end{gathered} \tag{1}$$

The problem of establishing conditions for the existence of the limits (1) will be discussed a little bit later.

Enumerating the states of the Markov chain $\{i_t, \xi_t, \nu_t, \eta_t\}$, $t \geq 0$ in lexicographic order, we introduce the probability row vectors $\vec{p}_i$, $i \geq 0$ of the probabilities (1) corresponding to the states i of the components i_t, $i \geq 0$. We also use $\vec{p} = (\vec{p}_0, \vec{p}_1, \dots)$.

It is worthwhile to mention that in the general notation (1) of the stationary state probabilities of the process $\{i_t, \xi_t, \nu_t, \eta_t\}$, $t \geq 0$ we should pay attention that the state of the service process, η_t, $t \geq 0$, is not defined for $i_t = 0$. Hence, it is more correct to write

$$p\,(0, m, \nu) = \lim_{t\to\infty} P\left\{i_t = 0,\ \xi_t = m,\ \nu_t = \nu\right\}.$$

This implies that the row vectors $\vec{p}_i$, $i \geq 1$ are of size $M\bar{W}K$, while the dimension of the vector $\vec{p}_0$ is only $M\bar{W}$.

It is well known that the stationary state distribution of a conservative continuous time Markov chain is defined by the infinite system of linear algebraic equations of the form,

$$\vec{p}A = \vec{0}, \qquad \vec{p}\mathbf{e} = 1, \tag{2}$$

where matrix A is the (infinitesimal) generator of the Markov chain.

LEMMA. *For the Markov chain $\{i_t, \xi_t, \nu_t, \eta_t\}$, $t \geq 0$ the infinitesimal generator A has the following form,*

$$A = \begin{pmatrix} \tilde{\mathcal{C}} & \tilde{\mathcal{D}}_1 & \tilde{\mathcal{D}}_2 & \tilde{\mathcal{D}}_3 & \dots \\ \tilde{\mathcal{H}} & \mathcal{C} & \mathcal{D}_1 & \mathcal{D}_2 & \dots \\ 0 & \mathcal{H} & \mathcal{C} & \mathcal{D}_1 & \dots \\ 0 & 0 & \mathcal{H} & \mathcal{C} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3}$$

where

$$\begin{aligned} \mathcal{D}_k &= \operatorname{diag}\left\{D_k^{(m)} \otimes I_K, m = \overline{1, M}\right\},\ k \geq 0, \\ \tilde{\mathcal{D}}_k &= \operatorname{diag}\left\{D_k^{(m)} \otimes \vec{\beta}^{(m)}, m = \overline{1, M}\right\},\ k \geq 0, \\ \mathcal{C} &= Q \otimes I_{\bar{W}K} + \mathcal{D}_0 + \mathcal{S} - \mathcal{H}, \\ \tilde{\mathcal{C}} &= Q \otimes I_{\bar{W}} + \tilde{\tilde{\mathcal{D}}}_0, \\ \mathcal{H} &= \operatorname{diag}\left\{I_{\bar{W}} \otimes S_0^{(m)}\vec{\beta}^{(m)}p_m, m = \overline{1, M}\right\}, \\ \tilde{\mathcal{H}} &= \operatorname{diag}\left\{I_{\bar{W}} \otimes S_0^{(m)}p_m, m = \overline{1, M}\right\}, \\ \tilde{\tilde{\mathcal{D}}}_0 &= \operatorname{diag}\left\{D_0^{(m)}, m = \overline{1, M}\right\}, \\ \mathcal{S} &= \operatorname{diag}\left\{I_{\bar{W}} \otimes \left(S^{(m)} + S_0^{(m)}\vec{\beta}^{(m)}\right),\ m = \overline{1, M}\right\}, \end{aligned} \tag{4}$$

$\otimes$ denotes the Kronecker product of matrices, see [17].

The proof of the given lemma requires the analysis of all possible transitions of the Markov chain $\{i_t, \xi_t, \nu_t, \eta_t\}$, $t \geq 0$ during a small period of time.

Note that there is some irregularity in the first two block-rows of generator A. All other block rows are obtained by shifting the block row $\mathcal{H}$, $\mathcal{C}$, $\mathcal{D}_1$, $\mathcal{D}_2, \dots$ to the right (with supplementing by a zero block from the left). These entries of the block row have the dimension $M\bar{W}K \times M\bar{W}K$.

Matrix $\tilde{\mathcal{C}}$ is also a square matrix, but of size $M\bar{W}$. Matrix $\tilde{\mathcal{H}}$ is of dimension $M\bar{W}K \times M\bar{W}$ and matrices $\tilde{\mathcal{D}}_k$, $k \geq 0$ have dimension $M\bar{W} \times M\bar{W}K$.

We have to decide when the Markov chain $\{i_t, \xi_t, \nu_t, \eta_t\}$, $t \geq 0$ possesses a stationary state distribution and to calculate the vector $\vec{p}$ of the stationary distribution by (2)–(4) if it exists.

3. ANALYSIS OF THE STATIONARY DISTRIBUTION OF THE MARKOV CHAIN

To calculate the stationary probability vector $\vec{p}$ by means of known theoretical tools, we reduce the investigation of the continuous time Markov chain $\{i_t, \xi_t, \nu_t, \eta_t\}$, $t \geq 0$ to the analysis of the related discrete time Markov chain $\{i_{t_m}, \xi_{t_m}, \nu_{t_m}, \eta_{t_m}\}$, $m \geq 1$ embedded into the chain $\{i_t, \xi_t, \nu_t, \eta_t\}$, $t \geq 0$ at all those epochs t_m, $m \geq 1$ of its transitions.

We denote the stationary probability vector of the chain $\{i_{t_m}, \xi_{t_m}, \nu_{t_m}, \eta_{t_m}\}$, $m \geq 1$ by

$$\vec{\pi} = (\vec{\pi}_0, \vec{\pi}_1, \vec{\pi}_2, \dots).$$

It is easy to show (for more details see, e.g., [12]) that for our queueing model the following holds,

(i) the conditions for existence of the stationary distributions $\vec{p}$ and $\vec{\pi}$ coincide,

(ii) the vector $\vec{\pi}$ is the unique solution to the following system of linear algebraic equations,

$$\vec{\pi} = \vec{\pi} Y, \qquad \vec{\pi}\mathbf{e} = 1, \tag{5}$$

where the matrix Y of the one-step transition probabilities of the discrete time Markov chain is calculated by

$$Y = \mathcal{R}^{-1} A + I. \tag{6}$$

Here, the matrix $\mathcal{R}$ is a diagonal matrix having diagonal entries equal to the modules of the diagonal entries of generator A.

The concrete form of matrix $\mathcal{R}$ is as follows,

$$\mathcal{R} = \operatorname{diag}\left\{\tilde{R}, R, R, R, \dots\right\}, \tag{7}$$

where

$$\begin{aligned}
\tilde{R} &= \operatorname{diag}\left\{\mu_m \left(1 - p_{m,m'}\right), m = \overline{1,M}\right\} \otimes I_{\bar{W}} \\
&\quad + \operatorname{diag}\left\{\operatorname{diag}\left\{\lambda_\nu^{(m)}, \nu = \overline{0,W}\right\}, m = \overline{1,M}\right\}, \\
R &= \operatorname{diag}\left\{\mu_m \left(1 - p_{m,m'}\right), m = \overline{1,M}\right\} \otimes I_{\bar{W}K} \\
&\quad + \operatorname{diag}\left\{\operatorname{diag}\left\{\lambda_\nu^{(m)}, \nu = \overline{0,W}\right\} \oplus \operatorname{diag}\left\{s_k^{(m)}, k = \overline{1,K}\right\}, m = \overline{1,M}\right\} \\
&\quad + \operatorname{diag}\left\{(1 - p_m)\, I_{\bar{W}} \otimes \operatorname{diag}\left\{s_k^{(m)} \vec{\beta}_k^{(m)}, k = \overline{1,K}\right\}, m = \overline{1,M}\right\}.
\end{aligned}$$

Here, $-\lambda_\nu^{(m)}$ is the diagonal entry of the matrix $D_0^{(m)}$, $-s_k^{(m)}$ is the diagonal entry of the matrix $S^{(m)}$, $\nu = \overline{0,W}$, $k = \overline{1,K}$, and $\oplus$ denotes the Kronecker sum of matrices, see [17].

As it was mentioned above, for an analogous reduction of a multidimensional continuous time Markov chain to a discrete time Markov chain and its related investigation, we refer to [12].

REMARK. Because the reduction to the corresponding discrete time Markov chain can be performed in different ways, it is possible to use, instead of matrix $\mathcal{R}$, any matrix $\bar{\mathcal{R}} = rI$ where r is greater or equal to the maximal diagonal entry of matrix $\mathcal{R}$. It can simplify some further results. However, we prefer to use matrix $\mathcal{R}$ in form (7) because in this case the constructed discrete time Markov chain is exactly the embedded (or jump) Markov chain for an original multidimensional continuous time Markov chain.

Thus, if we derive a criterion for the existence of the stationary distribution $\vec{\pi}$, we get the same for the stationary distribution $\vec{p}$. If we calculate the entries $\vec{\pi}_i$ of the vector $\vec{\pi}$, we calculate the entries $\vec{p}_i$ of the vector $\vec{p}$ by,

$$\vec{p}_0 = c\vec{\pi}_0\tilde{R}^{-1}, \quad \vec{p}_i = c\vec{\pi}_i R^{-1}, \qquad i \geq 1, \tag{8}$$

where

$$c = \left(\vec{\pi}_0\tilde{R}^{-1}\mathbf{e} + \sum_{i=1}^{\infty}\vec{\pi}_i R^{-1}\mathbf{e}\right)^{-1}. \tag{9}$$

Matrix Y, defined by formula (6), has the following structure,

$$Y = \begin{pmatrix} \tilde{R}^{-1}\tilde{C}+I & \tilde{R}^{-1}\tilde{\mathcal{D}}_1 & \tilde{R}^{-1}\tilde{\mathcal{D}}_2 & \tilde{R}^{-1}\tilde{\mathcal{D}}_3 & \dots \\ R^{-1}\tilde{\mathcal{H}} & R^{-1}C+I & R^{-1}\mathcal{D}_1 & R^{-1}\mathcal{D}_2 & \dots \\ 0 & R^{-1}\mathcal{H} & R^{-1}C+I & R^{-1}\mathcal{D}_1 & \dots \\ 0 & 0 & R^{-1}\mathcal{H} & R^{-1}C+I & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{10}$$

As mentioned above, the vectors $\vec{p}_0$ (and $\vec{\pi}_0$) have smaller dimension compared to the rest $\vec{p}_i$ (and $\vec{\pi}_i$), $i \geq 1$. To avoid dealing with vectors of different dimensions, we eliminate the vector $\vec{\pi}_0$ from (5),(10) by

$$\vec{\pi}_0 = \vec{\pi}_1 R^{-1}\mathcal{F}\tilde{R}, \tag{11}$$

where $\mathcal{F} = -\tilde{\mathcal{H}}\tilde{\mathcal{C}}^{-1}$.

We denote

$$\hat{\pi} = (\vec{\pi}_1, \vec{\pi}_2, \dots).$$

As a result of the elimination procedure, we get the following system of linear algebraic equations for the vector $\hat{\pi}$,

$$\hat{\pi}\hat{A} = \hat{\pi}, \qquad \hat{\pi}\mathbf{e} + \vec{\pi}_1 R^{-1}\mathcal{F}\tilde{R}\mathbf{e} = 1, \tag{12}$$

where matrix $\hat{A}$ has the form,

$$\hat{A} = \begin{pmatrix} R^{-1}\mathcal{F}\tilde{\mathcal{D}}_1 + R^{-1}\mathcal{C}+I & R^{-1}\mathcal{F}\tilde{\mathcal{D}}_2 + R^{-1}\mathcal{D}_1 & R^{-1}\mathcal{F}\tilde{\mathcal{D}}_3 + R^{-1}\mathcal{D}_2 & \dots \\ R^{-1}\mathcal{H} & R^{-1}\mathcal{C}+I & R^{-1}\mathcal{D}_1 & \dots \\ 0 & R^{-1}\mathcal{H} & R^{-1}\mathcal{C}+I & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{13}$$

By (13), we see that the Markov chain with the stationary probability vector $\hat{\pi}$ belongs to the class of $M|G|1$ type Markov chains [18] and to the class of multidimensional quasi-Toeplitz Markov chains [13]. Hence, the problem of solving system (12) can be solved at least in two ways.

Following the first way, which is sometimes called the transform approach, we exploit the tools of vector and matrix generating functions.

We use the notation

$$\vec{\Pi}(z) = \sum_{i=1}^{\infty}\vec{\pi}_i z^i, \qquad Y(z) = \sum_{l=0}^{\infty} Y_l z^l, \qquad V(z) = \sum_{l=0}^{\infty} V_l z^l,$$

where

$$\begin{gathered} Y_0 = R^{-1}\mathcal{H}, \qquad Y_1 = R^{-1}\mathcal{C} + I, \qquad Y_l = R^{-1}\mathcal{D}_{l-1}, \qquad l \geq 2, \\ V_0 = R^{-1}\mathcal{F}\tilde{\mathcal{D}}_1 + R^{-1}\mathcal{C} + I, \qquad V_l = R^{-1}\mathcal{F}\tilde{\mathcal{D}}_{l+1} + R^{-1}\mathcal{D}_l, \qquad l \geq 1. \end{gathered} \tag{14}$$

Multiplying the equations of system (13) by corresponding degrees of z and summing up, we get the following functional equation for the vector generating function $\vec{\Pi}(z)$,

$$\vec{\Pi}(z)\left(Y(z) - zI\right) = \vec{\pi}_1 H(z), \tag{15}$$

where

$$\begin{gathered} H(z) = R^{-1} z \left(\mathcal{H} - \mathcal{F}\tilde{\mathcal{D}}(z)\right) = z\left(Y(z) - zV(z)\right), \\ Y(z) = Iz + R^{-1}\left(\mathcal{H} + \mathcal{C}z + z\mathcal{D}(z)\right), \\ \mathcal{D}(z) = \sum_{k=1}^{\infty} \mathcal{D}_k z^k, \qquad \tilde{\mathcal{D}}(z) = \sum_{k=1}^{\infty} \tilde{\mathcal{D}}_k z^k. \end{gathered}$$

As follows from [13], the necessary and sufficient condition for the existence of the stationary state distribution $\hat{\pi}$ (and $\vec{\pi}$, and $\vec{p}$) is given by

$$\vec{X} \left. \frac{dY(z)}{dz} \right|_{z=1} \mathbf{e} < 1. \tag{16}$$

Here, the row vector $\vec{X}$ is the unique solution of the following system,

$$\vec{X} Y(1) = \vec{X}, \qquad \vec{X}\mathbf{e} = 1. \tag{17}$$

Conditions (16),(17) are constructive ones. First, we solve system (17) of linear algebraic equations for the entries of the vector $\vec{X}$ of dimension $M\bar{W}K$. Then, we check whether condition (16) is fulfilled. Conditions (16),(17) can be easily verified computationally. We assume in the following that these conditions are fulfilled.

Let z_k be the roots of equation,

$$\det\left(Y(z) - zI\right) = 0, \tag{18}$$

in the unit disk $|z| < 1$ of a complex plane with multiplicities $n_k, n_k \geq 1$, respectively. Denote K' as the number of different roots. It is known (see [13]), that $\sum_{k=1}^{K'} n_k = \bar{W} + \bar{W}MK - 1$.

Exploiting the analyticity of the vector generating function $\vec{\Pi}(z)$ in the region $|z| < 1$, we get the following system of linear algebraic equations for the entries of the vector $\vec{\pi}_1$,

$$\vec{\pi}_1 \frac{d^n}{dz^n} \left\{ H(z) \operatorname{Adj}\left(Y(z) - zI\right) \right\} \Big|_{z=z_k} = 0, \qquad n = 0, \dots, n_k - 1, \qquad k = 1, \dots, K', \tag{19}$$

$$\vec{\pi}_1 H(z) \operatorname{Adj}\left(Y(z) - zI\right) \left(\det\left(Y(z) - zI\right)\right)^{-1} \Big|_{z=1} \mathbf{e} + \vec{\pi}_1 R^{-1} \mathcal{F} \tilde{R} \mathbf{e} = 1, \tag{20}$$

which has the unique solution (see [13]). Here, Adj denotes the symbol of the adjoint matrix.

Having obtained the vector $\vec{\pi}_1$, we calculate the probability vector $\vec{\pi}_0$ by (11) and all the other vectors $\vec{\pi}_i$, $i \geq 2$ sequentially by (12).

Hence, the problem of calculating the stationary state probability vectors is solved by means of the transform approach.

This approach has three disadvantages. The first one is the necessity to solve equation (18). Theoretically, this problem is trivial, but in practice problems arise sometimes, especially when the dimension of the vector $\vec{\pi}_1$ is rather high. The second problem arises in case of multiple roots. It consists of the necessity to take the derivatives in (19) analytically or by means of some specialized software. The last problem consists of the unstable work of the recursion for the calculation of the vectors $\vec{\pi}_i, i \geq 2$ based on equations (12). The subtraction operation is implemented here recurrently and the recursion can work (depending on the dimension of the vectors and the load of the system) in an unstable manner.

Thus, another way of finding the stationary distribution $\vec{\pi}_i, i \geq 0$ shall be exploited. Essentially, this way stems from a recursion by Ramaswami [19], which is described in [18]. Here, we present only the resulting formulas, but not their derivation. This is based on censored Markov chains (see [20]).

In this way, the probability vectors $\vec{\pi}_l$ are calculated as follows,

$$\vec{\pi}_l = \vec{\pi}_1 \Phi_l, \qquad l \geq 2, \tag{19}$$

where matrices Φ_l are calculated recurrently,

$$\Phi_k = \left(\bar{V}_{k-1} + \sum_{i=2}^{k-1} \Phi_i \bar{Y}_{k+1-i} \right) \left(I - \bar{Y}_1\right)^{-1}, \qquad k \geq 2. \tag{20}$$

Here,

$$\bar{V}_n = \sum_{i=n}^{\infty} V_i G^{i-n}, \qquad \bar{Y}_n = \sum_{i=n}^{\infty} Y_i G^{i-n}, \tag{21}$$

where the matrix G satisfies the equation,

$$G = \sum_{i=0}^{\infty} Y_i G^i, \tag{22}$$

and the vector $\vec{\pi}_1$ is calculated as the unique solution of the system,

$$\vec{\pi}_1 \left(I - \bar{V}_0\right) = \vec{0}, \tag{23}$$

$$\vec{\pi}_1 \left[\sum_{l=1}^{\infty} \Phi_l \mathbf{e} + R^{-1} \mathcal{F} \tilde{R} \mathbf{e} \right] = 1. \tag{24}$$

Note that equation (24) can be replaced by the equation for the vector $\vec{\pi}_1$ that is obtained from the normalization condition (20) using the first way. This procedure, defined by formulas (19)–(24), has proven to be rather stable and accurate.

Since the problem of calculating the vectors $\vec{\pi}_l$, $l \geq 0$ is solved, due to the formulas (8),(9) the problem of calculating the stationary state distribution of the investigated queueing model is solved as well. Different performance characteristics of the model can be calculated using the stationary state probability vectors $\vec{p}_l$, $l \geq 0$.

For instance, the probability P_0 to have an empty system is computed by $P_0 = \vec{p}_0 \mathbf{e}$, the average number of customers in the system is computed by $L = \sum_{i=1}^{\infty} i \vec{p}_i \mathbf{e}$, etc.

Note that in the case when the input flow is ordinary, i.e., no batch arrivals are allowed (the input is coded as MAP), the problem of calculating the vectors $\vec{p}_i$, $i \geq 0$ can be solved much easier. In this case, generator A has only three block diagonals. Therefore, the process $\{i_t, \xi_t, \nu_t, \eta_t\}$, $t \geq 0$ belongs to the class of quasi-birth-and-death processes and we get the following statement as an evident corollary of results derived by Neuts [16].

ASSERTION. *In the case of a MAP input flow in all states of the random environment, the stationary state probability vectors $\vec{p}_i$, $i \geq 0$ are calculated by,*

$$\vec{p}_0 = \vec{c} T \mathcal{F}, \qquad \vec{p}_i = \vec{c} T^i, \quad i \geq 1, \tag{25}$$

where vector $\vec{c}$ is the solution to the system,

$$\vec{c} \left(T \mathcal{F} \tilde{\mathcal{D}}_1 + \mathcal{D}_1 \right) = \vec{0}, \qquad \vec{c} \left(T \mathcal{F} + T (I - T)^{-1} \right) \mathbf{e} = 1,$$

and matrix T is the solution of the equation,

$$T^2 \mathcal{H} + T \mathcal{C} + \mathcal{D}_1 = 0. \tag{26}$$

4. SOJOURN TIME DISTRIBUTION

The sojourn time is one of the most important characteristics of queueing models.

We denote the Laplace-Stieltjes transform of the sojourn time distribution for our model by $w(s)$, $\mathrm{Re}(s) > 0$, and assume that the service discipline is FIFO.

THEOREM. *The Laplace-Stieltjes transform $w(s)$ is calculated as follows,*

$$w(s) = \bar{c}\left[\vec{p}_0 \sum_{k=1}^{\infty} \tilde{\mathcal{D}}_k \sum_{l=1}^{k} \vec{W}_{l-1}(s) + \sum_{i=1}^{\infty} \vec{p}_i \sum_{k=1}^{\infty} \mathcal{D}_k \sum_{l=1}^{k} \vec{W}_{i+l-1}(s)\right], \tag{27}$$

where

$$\vec{W}_i(s) = (\mathcal{Z}(s))^{i+1}\mathbf{e}, \quad i \geq 0, \qquad \mathcal{Z}(s) = (sI - \Gamma)^{-1}\mathcal{H}, \qquad \Gamma = \mathcal{C} + \mathcal{D}(1), \tag{28}$$

$$\bar{c} = \left[\vec{p}_0 \tilde{\mathcal{D}}'(1)\,\mathbf{e} + \sum_{i=1}^{\infty} \vec{p}_i \mathcal{D}'(1)\,\mathbf{e}\right]^{-1}. \tag{29}$$

The proof is based on the probabilistic interpretation of the Laplace-Stieltjes transform. The value $w(s)$ is the probability of no arrivals of catastrophes during the sojourn time where catastrophes constitute a stationary Poisson input with intensity s, $s > 0$.

Formula (27) represents essentially the formula of total probability. The probabilistic meaning of the involved terms is the following.

Customers arriving in a batch of size k meet i customers waiting for service with probability $\bar{c}\vec{p}_i k\mathcal{D}_k$, $i \geq 0$, $k \geq 1$.

The probability for a tagged customer arriving in a batch of size k to be the l^{th} among the customers of the batch is equal to $1/k$ (i.e., we assume uniform distribution of the place of a tagged customer in a batch).

The entries $(\vec{W}_i(s))_{m,\nu,\eta}$ of the vector $\vec{W}_i(s)$ are the probabilities of no catastrophes arriving during the virtual sojourn time with i customers in the system and a given state (m, ν, η) of the process $\{\xi_t, \nu_t, \eta_t\}$, $t \geq 0$ at the virtual customer arrival epoch. The vector is calculated by

$$\vec{W}_i(s) = \int_0^{\infty} e^{-st} P_i(t, 0)\,\mathcal{H}\,dt\mathbf{e}, \tag{30}$$

where the matrices $P_i(t, l)$ have entries $(P_i(t, l))_{m,\nu,\eta;m',\nu',\eta'}$ defined as the probability to have l customers in the system before the tagged customer and being in state (m', ν', η') of the process $\{\xi_\tau, \nu_\tau, \eta_\tau\}$, $\tau \geq 0$, at epoch t provided that i customers were before this customer and the state of the process $(\xi_\tau, \nu_\tau, \eta_\tau)$, $\tau \geq 0$, is given by (m, ν, η) at epoch 0, $0 \leq l \leq i$.

For the matrices $P_i(t, l)$ combined into the row block vector,

$$\mathcal{P}_i(t) = (P_i(t, 0), \ldots, P_i(t, i)),$$

the differential equation,

$$\frac{d\mathcal{P}_i(t)}{dt} = \mathcal{P}_i(t)\,\Psi_i,$$

can be derived. Here, the matrix Ψ_i has all zero blocks except for the diagonal blocks that are equal to Γ and the subdiagonal blocks that are equal to $\mathcal{H}$. Solving this equation with an obvious initial condition,

$$\mathcal{P}_i(0) = (0, \ldots, 0, I),$$

and exploiting (30), we obtain

$$W_i(s) = (0, \ldots, 0, I)\,(sI - \Psi_i)^{-1}\,(I, 0, \ldots, 0)^{\top}\,\mathcal{H}.$$

After some calculations, we get (28).

The determinant in (28) is not equal to zero for $s \geq 0$ due to Tausska's theorem, (see [21]). Formula (29) is derived from the normalization condition. This completes the brief outline of the proof.

COROLLARY. *The mean sojourn time W is calculated as*

$$W = \bar{c}\left[\vec{p}_0 \sum_{k=1}^{\infty} \tilde{\mathcal{D}}_k \sum_{l=1}^{k} \vec{W}_{l-1} + \sum_{i=1}^{\infty} \vec{p}_i \sum_{k=1}^{\infty} \mathcal{D}_k \sum_{l=1}^{k} \vec{W}_{i+l-1}\right], \tag{31}$$

where

$$\vec{W}_n = -\sum_{i=0}^{n} \left(-\Gamma^{-1}\mathcal{H}\right)^i \Gamma^{-1}\mathrm{e}, \qquad n \geq 0. \tag{32}$$

The proof follows from the evident formula $W = -w'(s)|_{s=0}$ taking into account the relations,

$$\vec{W}_i(s) = \mathcal{Z}(s)\,\vec{W}_{i-1}(s), \qquad i \geq 1.$$

REMARK. If we omit a random environment and assume a stationary Poisson input with intensity λ and an exponentially distributed service time with parameter μ, we get from our formulas that

$$p_i = (1-\rho)\,\rho^i, \qquad i \geq 0, \quad \rho = \frac{\lambda}{p\mu}, \tag{33}$$

Here, p is the probability of a customer to leave the system after the service completion, and

$$w(s) = \frac{\mu p (1-\rho)}{s + \mu p (1-\rho)}. \tag{34}$$

The formulas (33),(34), agree with Takacs [1].

5. NUMERICAL RESULTS

To illustrate the feasibility and results of the presented algorithms, we consider the following examples.

Let the random environment have two states and be defined by intensities $\mu_1 = 2$, $\mu_2 = 3$, and the transition matrix $P = [0.3, 0.7; 0.6, 0.4]$. The stationary probability vector of the states of the random environment is given by $[0.46153846, 0.53846154]$.

The BMAP input is characterized by the matrices,

$$D_0^{(1)} = \begin{pmatrix} -86 & 0.01 \\ 0.01 & -2.75 \end{pmatrix}, \qquad D_1^{(1)} = \begin{pmatrix} 42.5 & 0.495 \\ 0.1 & 1.27 \end{pmatrix}, \qquad D_2^{(1)} = \begin{pmatrix} 42.5 & 0.495 \\ 0.1 & 1.27 \end{pmatrix}$$

when the random environment stays in state 1 and by the matrices,

$$D_0^{(2)} = \begin{pmatrix} -8 & 1 \\ 1 & -11 \end{pmatrix}, \qquad D_1^{(2)} = \begin{pmatrix} 1.5 & 4 \\ 2.5 & 3.5 \end{pmatrix}, \qquad D_2^{(2)} = \begin{pmatrix} 0.5 & 1 \\ 2.5 & 1.5 \end{pmatrix}$$

when the state of the random environment is 2.

The given BMAPs have fundamental rates $\lambda^{(1)} = 25.783$, $\lambda^{(2)} = 11.25$, variation coefficients $c_v^{(1)} = 0.032$, $c_v^{(2)} = 0.015$, as well as correlation coefficients $c_c^{(1)} = 0.408$, $c_c^{(2)} = -0.04$.

The service time distribution of PH type is characterized by the vectors $\beta^{(1)} = (0.2, 0.8)$, $\beta^{(2)} = (0.9, 0.1)$ and the subgenerators,

$$S^{(1)} = \begin{pmatrix} -170 & 15 \\ 40 & -210 \end{pmatrix}, \qquad S^{(2)} = \begin{pmatrix} -110 & 80 \\ 10 & -150 \end{pmatrix}.$$

The average service times are $b_1^{(1)} = 0.00607$ and $b_1^{(2)} = 0.01395$.

In the first experiment, we fix the values of the probabilities p_m, $m = 1, 2$ to leave the system after the service completion when the random environment is in state m as $p_1 = 0.5$, $p_2 = 0.7$. So, if at the service completion epoch the random environment stays in state 1, then the customer leaves the system with probability $p_1 = 0.5$ or returns for one more service with probability $1 - p_1 = 0.5$. If the random environment stays in state 2, then the customer leaves the system with probability $p_2 = 0.7$ or returns for one more service with probability $1 - p_2 = 0.3$.

The key point of both ways to calculate the stationary state distribution is to formulate the system of linear algebraic equations for the entries of the unknown vector $\vec{\pi}_1$. When we use equation (20), we get the roots $z_1 = 0.677513$, $z_2 = 0.787391$, $z_3 = 0.95018$, $z_4 = 1$, and $z_5 = 0$. The last root has multiplicity $n_5 = 4$. Note that for the root $z_5 = 0$, the matrix $H(z_5)\mathrm{Adj}(Y(z_5) - z_5 I)$ has only zero entries as well as its derivatives of order $1, 2, 3$. Only the derivative of order 4 gives us four linear independent equations for the entries of the vector $\vec{\pi}_1$. Finally, the system has the form,

$$\vec{\pi}_1 \begin{pmatrix} 0.79871 & 0.00003 & 0 & 0 & -0.10000 & 0 & -0.00044 & 0 \\ 0.92373 & 0.00004 & 0 & 0 & 0.04616 & 0 & -0.00051 & 0 \\ 26.28086 & 0 & 0 & 0 & 0 & -0.09458 & 0 & -0.00064 \\ 31.28946 & 0 & 0 & 0 & 0 & 0.05397 & 0 & -0.00076 \\ 2.14717 & -0.00001 & 0 & 0 & 0.00034 & 0 & -0.10216 & -0.00014 \\ 7.19310 & -0.00003 & -0.00001 & 0 & 0.00115 & 0 & 0.12647 & -0.00047 \\ 2.86168 & 0 & 0.00000 & 0 & 0 & 0.00031 & -0.00013 & -0.10174 \\ 9.65802 & 0.00001 & 0.00001 & 0 & -0.00001 & 0.00104 & -0.00045 & 0.12880 \end{pmatrix} = (1, 0, \dots, 0)\,. \tag{35}$$

When we apply the second approach, system (23),(24) is reduced to the equation,

$$\vec{\pi}_1 \begin{pmatrix} 43.6002 & -0.6908 & -0.0225 & -0.0897 & -0.0554 & -0.0055 & -0.0386 & -0.0043 \\ 46.3518 & 0.5834 & -0.0239 & -0.0958 & -0.0531 & -0.012 & -0.0413 & -0.0046 \\ 3.4597 & -0.0206 & 0.9339 & -0.7498 & -0.0633 & -0.007 & -0.08 & -0.0079 \\ 3.9054 & -0.0245 & -0.4684 & 0.6864 & -0.0753 & -0.0084 & -0.0847 & -0.019 \\ 2.9227 & -0.0239 & -0.0017 & -0.0067 & 0.9385 & -0.7311 & -0.1386 & -0.0144 \\ 6.3894 & -0.09 & -0.0042 & -0.01696 & -0.4659 & 0.9824 & -0.3412 & -0.0443 \\ 2.3653 & -0.0048 & -0.02 & -0.0172 & -0.1142 & -0.0117 & 0.8870 & -0.7179 \\ 3.7719 & -0.0127 & -0.0127 & -0.0623 & -0.25895 & -0.03508 & -0.5837 & 0.9686 \end{pmatrix} = (1, 0, ..., 0). \tag{36}$$

Although systems (35),(36) have different matrices, they have the same solution defined by

$$\vec{\pi}_1 = (0.00477, 0.0099, 0.0078, 0.0132, 0.0169, 0.0138, 0.022, 0.0177)\,.$$

The aim of the second experiment is to illustrate the dependence of probability $P_0 = \vec{p}_0\mathbf{e}$ to have an empty system, the average number of customers in the system $L = \sum_{i=0}^{\infty} i\vec{p}_i\mathbf{e}$, and the average sojourn time W on the value of the probability p_2 (for several fixed values of the probability p_1).

Figure 1 illustrates the dependence of the value P_0 on the probability p_2. As can be seen in Figure 1, the probability of an idle system increases monotonously as the probabilities p_1 and p_2 of leaving the system increase.

Figures 2–5 show the dependence of L and W on p_2. For convenience of presentation, the dependence for small ($p_2 \in [0, 0.2]$) and large values of p_2 are given on separate figures. One should pay attention that there are only three curves in Figures 4 and 5 while four curves are presented in Figures 2 and 3. It is explained by the fact that the stationary state distribution of the system does not exist for small values of probability p_2 that a customer leaves the system after the service completion at the epoch when the state of the random environment is 2 while the value of the corresponding probability in the state 1 is also small: $p_1 = 0.3$. As it can be seen from Figures 2–5, the average queue length and the average sojourn time decrease monotonously as the probabilities p_1 and p_2 of leaving the system increase.

In the third experiment, we would like to refine the dependence of the probability of an idle system and the average queue length on the probabilities p_1 and p_2 of leaving the system after

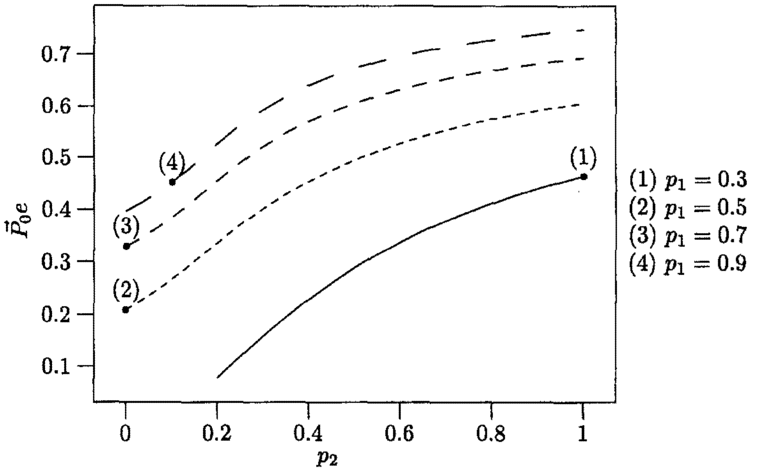

Figure 1. Dependence of the probability of idle state $\vec{p}_0\mathbf{e}$ of the system on probability p_2.

service. To this end, we keep the parameters of the random environment the same as in the previous experiments. However, we assume that the parameters of the service process and the BMAP input do not depend on the state of the random environment and correspond to the parameters of arrival and service processes considered for state 1 of the random environment in the first experiment. Thus, the random environment influences only the value of the feedback probability. For example, such a situation takes place in modelling the access of a mobile user of a broadband wireless mobile telecommunication network to the base station. A change of the feedback probability is caused by a change of the radio environment depending on the distance between the user and the corresponding base station.

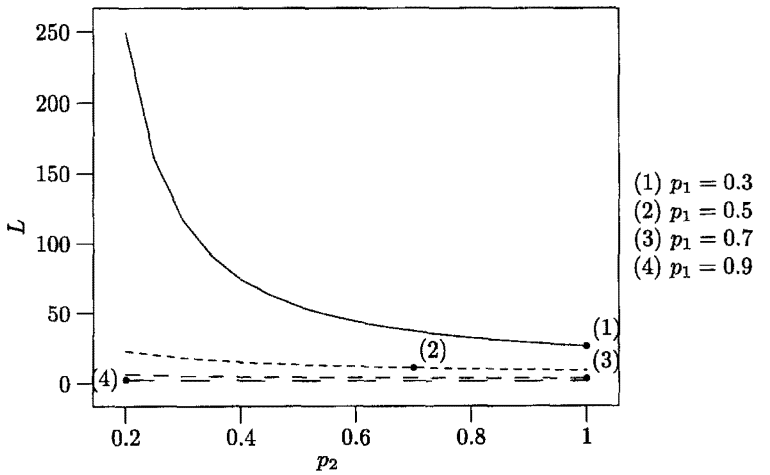

Figure 2. Dependence of the average queue length of the system on large values of the probability p_2.

As can be seen from Figures 6 and 7, the probability of an idle system increases and the average queue length decreases as p_1 and p_2 increase. Note that the value of L sharply increases when the values of p_1 and p_2 become small and it tends to infinity near the origin of coordinates due to failure of the stability condition.

Looking at Figures 2–5, one may note the similarity of behavior of the values of L and W. The well-known Little's law, which is valid for variety of queueing systems, states that $\lambda = L/W$, where L is the average amount of customers in the system, W is the mean sojourn time, and λ is the intensity of arrivals. So, the following natural question arises whether or not a constant $\tilde{c}$ exists for our model, such that the ratio L/W is equal to $\tilde{c}$?

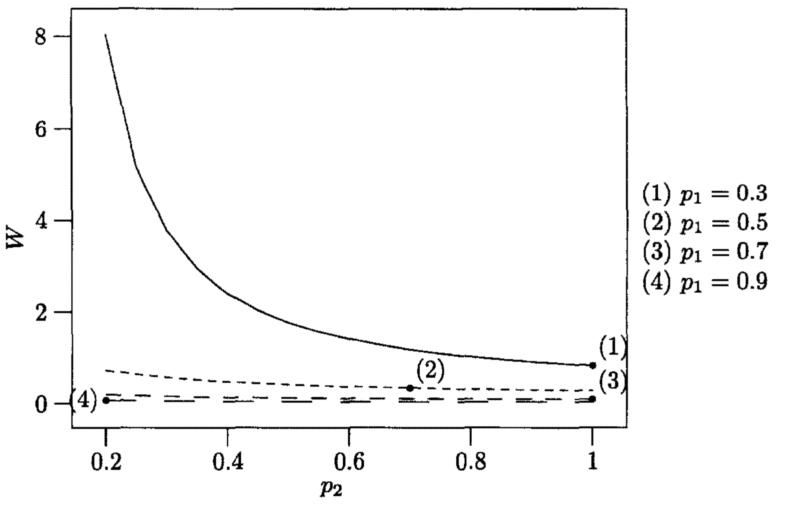

Figure 3. Dependence of the average sojourn time in the system W on large values of the probability p_2.

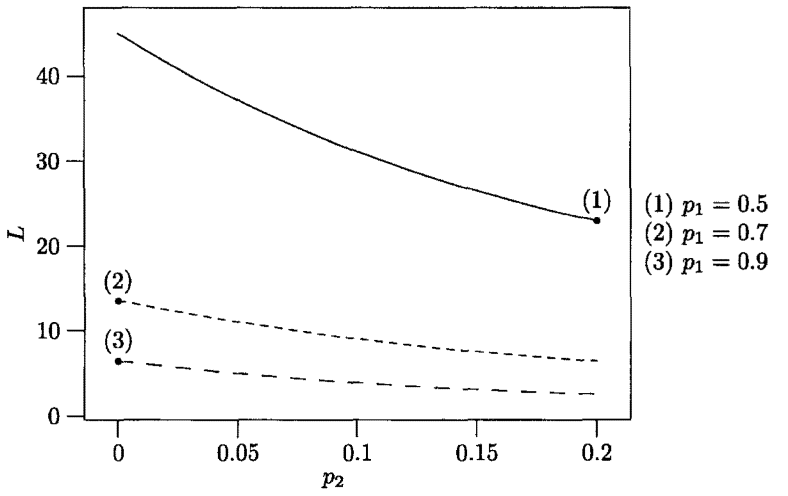

Figure 4. Dependence of the average queue length of the system on small values of the probability p_2.

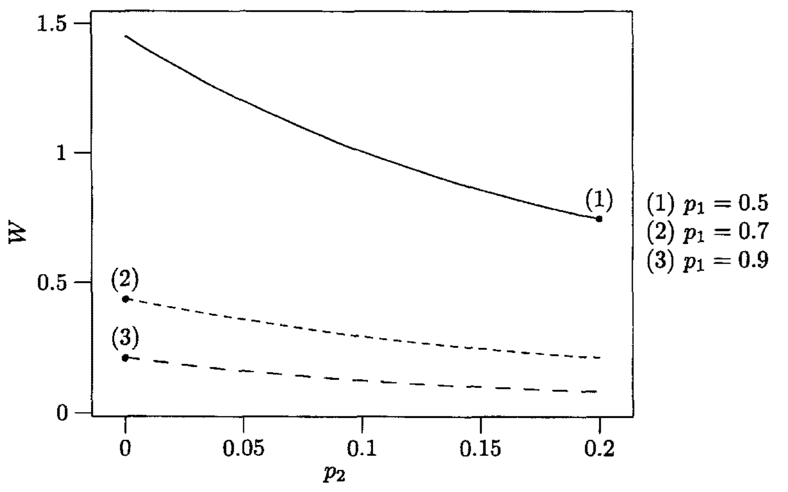

Figure 5. Dependence of the average sojourn time in the system W on small values of the probability p_2.

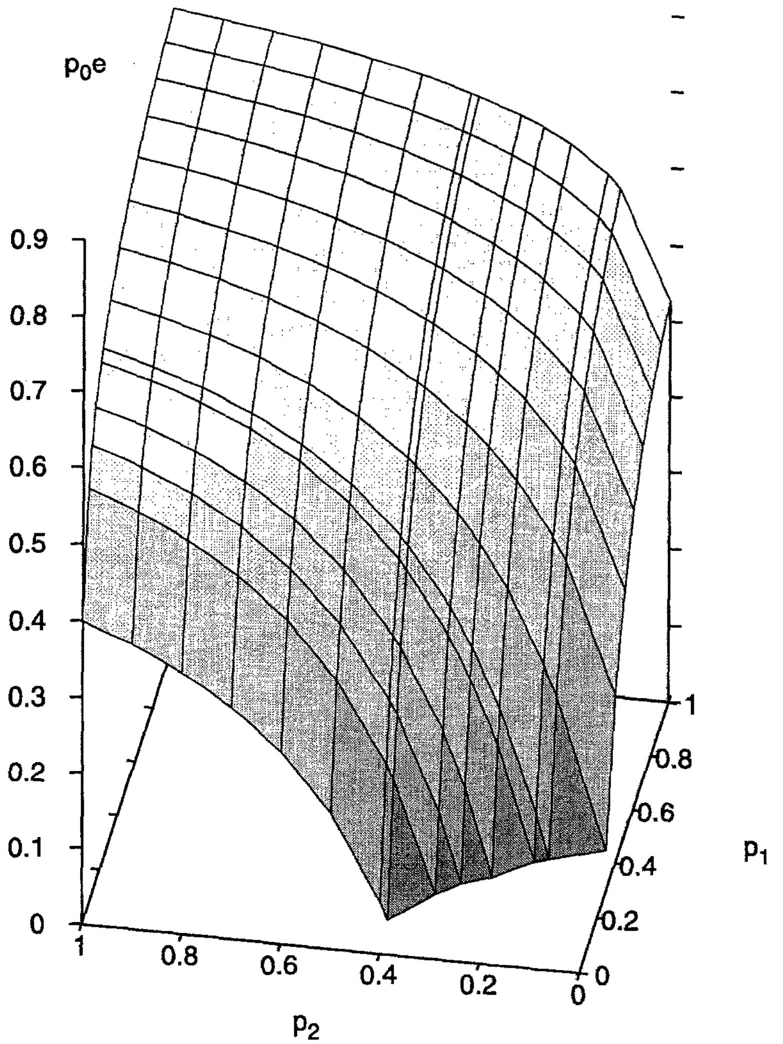

Figure 6. Dependence of the probability of idle state $\vec{p}_0\mathbf{e}$ of the system on probabilities p_1 and p_2.

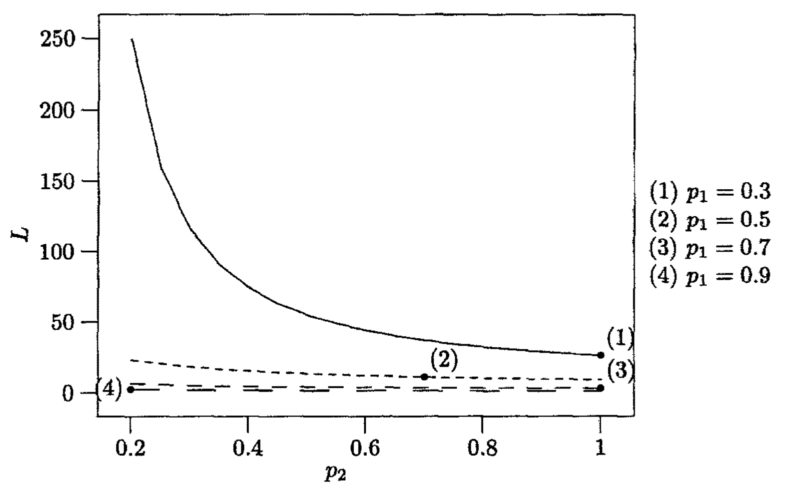

Figure 7. Dependence of the average queue length of the system on on probabilities p_1 and p_2.

To answer this question, we continue the second experiment and calculate the ratio L/W. This ratio does not depend on the values of p_1 and p_2. It is equal to $(\bar{c})^{-1}$, with $\bar{c}$ given by formula (31). Because the constant $(\bar{c})^{-1}$ has the meaning of an averaged arrival rate, we can conclude that Little's law holds for the system under study. This conclusion follows from the results of the experiment and agrees with intuitive reasonings. To the best of our knowledge, the question whether Little's law is valid for systems with BMAP input is not clearly answered and documented in the literature.

It is worth to mention that the presented results are valid and the algorithms work also for the case when some probabilities $1 - p_m$ that the customer should be served again (but not all) are equal to 1. Therefore, the case where temporarily message transmission is not possible at all (e.g., a user of a mobile network temporarily enters an area where he is not covered by any base station) is investigated in this paper as well. Our model is more general than models with server breakdowns.

6. CONCLUSION

The feedback queueing model of the type BMAP|PH|1 operating in a Markovian random environment is investigated. The random environment has a finite state space. Changing its state causes instantaneous changes of the parameters of the BMAP input, the PH service processes, and the probability of repeated service.

The stationary distribution of the multidimensional continuous time Markov chain describing the behavior of the system is calculated by means of reduction to the discrete time Markov chain at transition epochs. The Laplace-Stieltjes transform of the customer sojourn time distribution is calculated.

Moreover, practicability of the derived algorithms is illustrated by several numerical examples. The ratio of the average queue length and the mean sojourn time for the system working in a random environment is constant and does not depend on the feedback probabilities. The correctness of Little's law for the system under study is verified numerically.

The presented results can be applied to the capacity planning of real-life objects and the performance evaluation of all those situations where a repeated service is required and the operation of the object is subject to some external influence.

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