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On the emergence and properties of weird quasiperiodic attractors

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ABSTRACT

We recently described a specific type of attractors of two-dimensional discontinuous piecewise linear maps, characterized by two discontinuity lines dividing the phase plane into three partitions, related to economic applications. To our knowledge, this type of attractor, which we call a weird quasiperiodic attractor, has not yet been studied in detail. It has a rather complex geometric structure and other interesting properties that are worth exploring in more depth. To this end, we consider a simpler map that can also possess weird quasiperiodic attractors, namely, a 2D discontinuous piecewise linear map F with a single discontinuity line dividing the phase plane into two partitions, where two different homogeneous linear maps are defined. Map F depends on four parameters — the traces and determinants of the two Jacobian matrices. In the parameter space of map F , we obtain specific regions associated with the existence of weird quasiperiodic attractors; describe some characteristic properties of these attractors; and explain one of the possible mechanisms of their appearance.

1. Introduction

Two-dimensional (2D) *piecewise linear continuous* maps generating rotational dynamics have been extensively studied in recent decades. In particular, many researchers have considered the family of conservative maps defined as $M = M_L : (x, y) \rightarrow (\tau_L x - y, x)$ for $x \leq 0$, and $M = M_R : (x, y) \rightarrow (\tau_R x - y, x)$ for $x \geq 0$, depending on two real parameters, τ_R and τ_L , which are the traces of the two Jacobian matrices (see [1–7]). Since straight lines through the origin (also called rays) are mapped by map M into straight lines through the origin, the dynamics of map M can be described by a circle map with well-defined rotation number $\theta(\tau_R, \tau_L)$. It was shown that in the (τ_R, τ_L) -parameter plane, there exist sausage-like resonance regions, corresponding to a rational rotation number $\theta(\tau_R, \tau_L)$ of the associated circle map. For parameters in those regions, each point of the (x, y) -phase plane is periodic. When $\theta(\tau_R, \tau_L)$ is irrational, there is a topological conjugacy between the circle map and a rigid rotation, leading to quasiperiodic trajectories, that is, each point of the (x, y) -phase plane belongs to a quasiperiodic trajectory (see, e.g., [2,6]).

A more general *linearly homogeneous* map satisfying $g(\alpha X) = \alpha g(X)$ for all $X \in \mathbb{R}^d$ and $\alpha \geq 0$ is considered in [8]. In the 2D case, this class of maps is associated with the well-known border collision form G_μ for

$\mu = 0$. We denote this map as G_0 . Recall that the 2D border collision normal form is defined as

$$G_\mu = \begin{cases} G_L : (x, y) \rightarrow (\tau_L x + y + \mu, -\delta_L x) & \text{if } x \leq 0, \\ G_R : (x, y) \rightarrow (\tau_R x + y + \mu, -\delta_R x) & \text{if } x \geq 0. \end{cases} \quad (1)$$

This map depends on parameters τ_L , δ_L , τ_R , and δ_R , which are the trace and determinant of the Jacobian matrix in partition L (with $x \leq 0$) and partition R (with $x \geq 0$), respectively; parameter $\mu \neq 0$ is usually scaled out by fixing it as $\mu = 1$. Since map G_μ is quite important not only for various applications, but also for the bifurcation theory of nonsmooth maps, its dynamics has been studied by many researchers (see, e.g., [9–14]). In [8], map G_0 is considered, whose dynamics can be characterized by the values on the unit circle $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$. Introducing the polar coordinates $(x, y) = (\rho \cos(\theta), \rho \sin(\theta))$, the action of map G_0 can be split into a radial expansion described by a scalar function ρ and a rotation on circle S^1 . For $\delta_R \delta_L \leq 0$, the map is noninvertible, and one of the aims of [8] is to show that even in this low-dimensional case, it is not a trivial task to determine the stability of the fixed point in the origin. For $\delta_R \delta_L > 0$, the map is invertible, and the stability of the fixed point is not guaranteed even if this fixed point is attracting for both linear maps G_L and G_R . As a result, there are

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regions in the parameter space associated with divergence and so-called *dangerous bifurcation* (see, e.g., [15–19]).

The dynamics of 2D piecewise linear *homogeneous discontinuous* maps, which can be quite intricate, is little studied as yet. Such maps appear in some applications (see, e.g., [20–23]). For instance, in [21] an economic model is considered where the phase space is separated into three partitions by two vertical discontinuity lines. One linear homogeneous map acts in the middle partition, while the other acts in the two external partitions. The origin is a fixed point for both maps, but since it belongs to the middle partition, it is a virtual fixed point for the external linear map. In the cited work, the existence of attractors that have a special *weird* structure – not observed in other maps – is highlighted. In [21,22] these attractors are called *weird quasiperiodic attractors*.

Moreover, in [24], it is shown that weird quasiperiodic attractors emerge quite naturally in linear discrete dynamic economic models of dimension two or higher when policymakers use simple piecewise linear feedback rules to stabilize their dynamics. These economic models describe financial market fluctuations and business cycle dynamics, thereby also offering a novel perspective about their origins. The only requirement for these feedback rules is that they preserve the underlying fixed point of the system – a requirement that holds for many popular feedback rules discussed in economics (e.g., those that react to a change in a state variable or to the deviation from its fixed point value).

The goal of our paper is twofold. First, we analyze a mechanism for the appearance of weird quasiperiodic attractors. Second, we describe their basic properties. To simplify the analysis, one could consider the 2D border collision normal form G_0 . However, in order to introduce discontinuity, some offset must be added to G_0 that destroys the property of homogeneity. Instead, we consider the 2D piecewise linear map F defined by the same linear functions as in map G_0 , but with the border line shifted from $x = 0$ to $x = h$, $h \neq 0$. In this way, map F remains homogeneous but becomes discontinuous:

$$F = \begin{cases} F_L : (x, y) \rightarrow (\tau_L x + y, -\delta_L x) & \text{if } x < h, \\ F_R : (x, y) \rightarrow (\tau_R x + y, -\delta_R x) & \text{if } x > h. \end{cases} \quad (2)$$

Examples of the attractors of map F for $h = -1$, shown in Fig. 1, justify the use of the word *weird*. Moreover, weird quasiperiodic attractors can coexist, and in such cases their basins may also appear weird, as shown in Fig. 2. At first glance, it is hard to believe that these attractors are not chaotic. However, the characteristic property of map F is that both its components, maps F_L and F_R , are homogeneous, with the same fixed point in the origin, and it is quite straightforward to prove that map F cannot have hyperbolic cycles. This property facilitates the analysis of its dynamics; in particular, one can immediately exclude the possibility that its attractors are chaotic.¹ Obviously, a similar property holds also for a 2D piecewise linear map defined by several nonhomogeneous linear maps having the same (nonzero) fixed point, which, via a change of variables, can be translated to the origin. In our companion paper [25], we discuss other classes of piecewise linear discontinuous maps with weird quasiperiodic attractors. We think that similar attractors may also be observed in other discontinuous maps, not only piecewise linear ones.

In the present paper, we consider map F given in (2) as the simplest representative of a class of 2D discontinuous piecewise linear maps, defined by a finite number of linear homogeneous functions in different partitions of the phase plane, separated by smooth discontinuity curves.

A weird quasiperiodic attractor (WQA for short) \mathcal{A} of map F is an attractor (a closed invariant attracting set with a dense trajectory) with

¹ Among several known definitions of chaos, we use the following one: a 2D map $F : I \rightarrow I$, $I \subseteq \mathbb{R}^2$, is said to be chaotic on a closed invariant set $X \subseteq I$ if (a) periodic points of F are dense in X , and (b) F is topologically transitive, i.e., there is a dense orbit on X .

a rather complicated, say, weird, structure, which does not include periodic points. Thus, \mathcal{A} is neither an attracting cycle nor a chaotic attractor: it is the closure of quasiperiodic trajectories. This attractor can be defined as follows:

Definition 1. A weird quasiperiodic attractor (WQA) \mathcal{A} of a 2D discontinuous map F is a set satisfying the following properties: (a) \mathcal{A} is a topological attractor; (b) any point of \mathcal{A} is a point of a dense quasiperiodic trajectory; (c) the restriction of F to \mathcal{A} cannot be reduced to a 1D map.

To clarify this definition, it is worth adding a few comments:

- We use the standard (topological) definition of an attracting set, namely, a closed invariant set A is *attracting* if there exists a neighborhood $U(A)$ of A such that $T(U(A)) \subset U(A)$ and $\bigcap_{i=0}^{\infty} T^i(U(A)) = A$. An attractor \mathcal{A} is an attracting set with a dense trajectory.

- Property (b) means that there are no smaller invariant subsets of \mathcal{A} , in particular, no periodic points. As already mentioned, for map F given in (2), it is easy to show that it cannot have hyperbolic cycles (see Property 5 in the next section). This property is straightforward also for other piecewise linear homogeneous maps.

- In some special cases, map F can have an attractor with a rather simple structure (e.g., consisting of a finite number of segments), which can be studied by means of a 1D map (a first return map on one of the segments). We do not call such an attractor a WQA (the property (c) of the above definition is not satisfied). For example, such a nongeneric case may be observed for parameter values $\delta_L = 0$ or $\delta_R = 0$, when the entire partition defined by $x < h$ or $x > h$, respectively, is mapped into the straight line $y = 0$. As shown in our companion paper [23], if a 1D first return map to a suitable segment of this straight line exists, with a finite number of discontinuity points, then this map is topologically conjugated to a piecewise linear circle map with a rational or irrational rotation number.

It is interesting to compare a WQA with other known kinds of attractors that are also associated with quasiperiodic dynamics such as a *Cantor set attractor* (it can be observed, e.g., in a Lorenz map when it is a gap map, see [26]), or a *critical attractor* (as, e.g., in the logistic map at the Feigenbaum accumulation points, see [27]), or a *strange nonchaotic attractor* (SNA for short, observed in maps with quasiperiodic forcing, as described, e.g., in [28,29] for smooth maps or in [30–34] for piecewise smooth maps). All these attractors have a *fractal structure*. As for the structure of a WQA, the examples shown in Figs. 1 and 2 have no fractal structure (see also other examples of WQAs in the next sections). We think that it is a characteristic property of the attractors of the considered class of maps, although it is not an easy task to prove it in a generic case. Moreover, Cantor set attractors and critical attractors are not robust to parameter perturbations, since they exist for parameter sets of measure zero. On the contrary, as we will see, there are regions in the parameter space of the map F where WQAs represent the generic case, i.e., an arbitrarily chosen parameter point leads to WQA.

To some extent, the closest to WQAs are SNAs. Describing the similarities and distinctive properties of these types of attractors is an interesting and non-trivial task. It is not our concern in this paper, and we leave it for future work, however, it is important to note the following. Apart from the fractal structure, which is a distinctive property of SNAs, these invariant sets are often not topological attractors, but attractors only in the Milnor sense. The fractal structure of SNA is often related to the existence of invariant tori, which may behave as a chaotic saddle or may undergo a cascade of period doubling bifurcations, or other bifurcations. In contrast, the WQA is a topological attractor, and it does not contain any smaller invariant subsets: as mentioned above, the WQA is the closure of the quasiperiodic trajectory of any of its points.

The paper is organized as follows. In the next section, several properties of map F are stated that are useful for describing its dynamics. In Section 3, we present two examples of the bifurcation structure

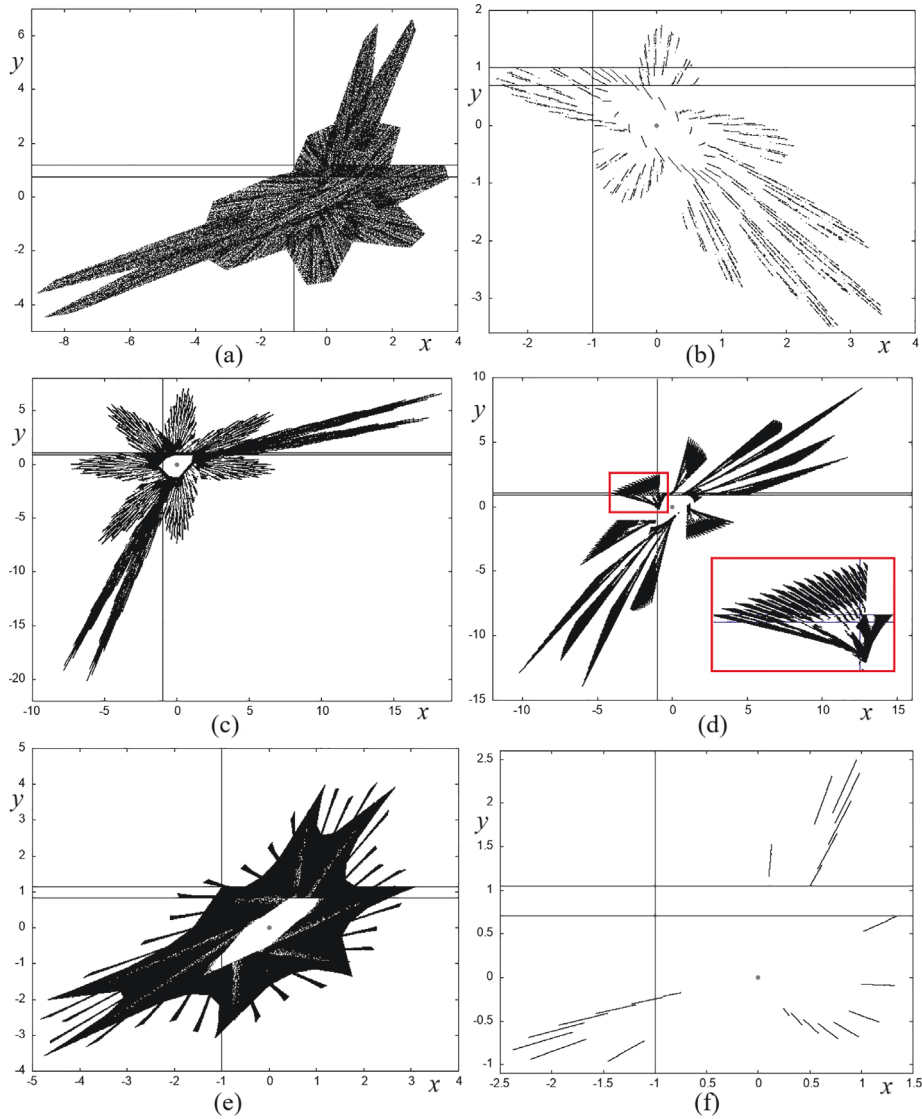


Fig. 1. Attractors of map F for (a) $\delta_L = 0.75$, $\delta_R = 1.2$, $\tau_L = -0.7$, $\tau_R = -2.5$; (b) $\delta_L = 0.7$, $\delta_R = 1.001$, $\tau_L = 0.3$, $\tau_R = 0.71$; (c) $\delta_L = 0.9$, $\delta_R = 1.1$, $\tau_L = -2.5$, $\tau_R = -0.7$; (d) $\delta_L = 0.9$, $\delta_R = 1.1$, $\tau_L = -2.5$, $\tau_R = -1.2$; (e) $\delta_L = 0.84$, $\delta_R = 1.15$, $\tau_L = -1$, $\tau_R = -1.9$; (f) $\delta_L = 1.05$, $\delta_R = 0.7$, $\tau_L = -0.75$, $\tau_R = -1.6$. Discontinuity line $x = -1$ and its images, $y = \delta_L$ and $y = \delta_R$, by maps F_L and F_R , respectively, are also shown.

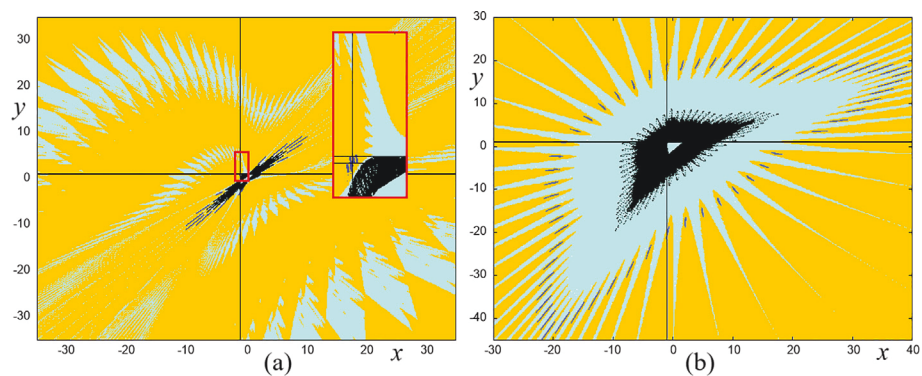


Fig. 2. Two coexisting weird quasiperiodic attractors (shown in black and dark blue) of map F and their basins (in light blue and yellow) for (a) $\delta_L = 0.9$, $\delta_R = 1.1$, $\tau_L = 0.3$, $\tau_R = 0.71$; (b) $\delta_L = 0.9$, $\delta_R = 1.11$, $\tau_L = -2$, $\tau_R = -1.91$.

of the parameter space of map F , in the (τ_L, τ_R) -parameter plane for $\delta_L = 0.9$, $\delta_R = 0.7$, and in the (δ_R, τ_R) -parameter plane for $\delta_L = 0.9$, $\tau_L = -2.5$, with the regions related to the attracting fixed point in the origin, weird quasiperiodic attractors, and divergence. We obtain

analytical expressions for the boundaries of the divergence regions in the simplest case related to the basic and complementary to basic symbolic sequences. In Section 4, we consider in detail the bifurcation structure near two specific divergence regions and discuss, by means

of several examples of the phase portrait of map F , a mechanism of appearance of WQAs for parameters values taken near those divergence regions. Some concluding remarks are presented in Section 5.

2. Preliminaries

We consider a 2D discontinuous piecewise linear map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined as follows:

$$F = \begin{cases} F_L : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow J_L \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x < -1, \\ F_R : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow J_R \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x > -1, \end{cases} \quad (3)$$

where

$$J_L = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}, \quad J_R = \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}. \quad (4)$$

Here, $\delta_i = \det J_i$, $\tau_i = \text{tr} J_i$, $i = L, R$, are real parameters. Note that we could choose any vertical line defined by $x = h$, $h \neq 0$, as a discontinuity line, as parameter h can be scaled out. Indexes L and R in (3) and (4) refer to the left and right partitions of the phase plane,

$$D_L = \{(x, y) : x < -1\}, \quad D_R = \{(x, y) : x > -1\},$$

separated by the discontinuity line

$$C_{-1} = \{(x, y) : x = -1\}.$$

The images of C_{-1} are denoted as

$$C^L = F_L(C_{-1}) = \{(x, y) : y = \delta_L\}, \quad C^R = F_R(C_{-1}) = \{(x, y) : y = \delta_R\}.$$

We call the discontinuity line C_{-1} , as well as its images and preimages, *critical lines* (of corresponding rank), following [35].

Let us state several properties of map F that are easy to verify:

Property 1. *The fixed point $O = (0, 0) \in D_R$ of map F_R is the unique fixed point of map F , provided no eigenvalue of F_R equals 1, i.e., $1 - \tau_R + \delta_R \neq 0$.*

Property 2. *Invertibility of map F is of*

- (a) $Z_1 - Z_0 - Z_1$ type for $0 < \delta_R < \delta_L$, or $\delta_L < \delta_R < 0$;
- (b) $Z_1 - Z_2 - Z_1$ type for $\delta_R < \delta_L < 0$, or $0 < \delta_L < \delta_R$;
- (c) $Z_0 - Z_1 - Z_2$ type for $\delta_L \delta_R < 0$;
- (d) $Z_1 - Z_\infty - Z_1 - Z_0$ type for $\delta_L = 0$, $\delta_R \neq 0$;
- (e) $Z_0 - Z_\infty - Z_0 - Z_1$ type for $\delta_R = 0$, $\delta_L \neq 0$;
- (f) $Z_1 - Z_2 - Z_1$ type for $\delta_R = \delta_L$.

This property can be verified by considering the images of half-planes D_L and D_R , which define regions called *zones* Z_j , where index j indicates the number of preimages of a point in zone Z_j . In generic cases (a), (b), or (c), zones Z_j are separated by the critical lines C^L and C^R . In the nongeneric cases (d) and (e), Z_∞ is the critical line C^L and C^R , respectively. Each point of these lines has an infinite number of preimages (a straight line), since the complete half-plane D_L or D_R is mapped into one straight line, C^L or C^R , respectively. In the nongeneric case (f), Z_2 is the critical line $C^L = C^R$, whose points have two distinct preimages. For example, map F has $Z_1 - Z_2 - Z_1$ type of invertibility in Fig. 1(a-e) and $Z_1 - Z_0 - Z_1$ type of invertibility in Fig. 1(f).

Property 3. *In the (δ_i, τ_i) -parameter plane, $i = L, R$, the eigenvalues $\lambda_{1,2}^i = \frac{1}{2}(\tau_i \pm \sqrt{\tau_i^2 - 4\delta_i})$ of matrix J_i satisfy $|\lambda_{1,2}^i| < 1$ in the stability triangle T^i , defined as*

$$T^i = \{(\delta_i, \tau_i) : \delta_i < 1, -1 - \delta_i < \tau_i < 1 + \delta_i\}. \quad (5)$$

This triangle is bounded by the segments of the straight lines $\tau_i = 1 + \delta_i$ (related to $\lambda_1^i = 1$), $\tau_i = -1 - \delta_i$ (related to $\lambda_2^i = -1$), and $\delta_i = 1$ (related to complex-conjugate eigenvalues $|\lambda_{1,2}^i| = 1$).

Property 4. *For the fixed point O , the boundary of the stability domain T^R defined by $\tau_R = 1 + \delta_R$ corresponds to a degenerate +1 bifurcation, at which any point of half-line $S^R = \{(x, y) : x > -1, y = -\delta_R x\}$ is fixed, while the boundary defined by $\tau_R = -1 - \delta_R$ corresponds to a degenerate flip bifurcation, at which any point (except the fixed point O) of segment $S^{R^2} = \{(x, y) : -1 < x < 1, y = \delta_R x\}$ is 2-periodic. The boundary defined by $\delta_R = 1$ corresponds to a center bifurcation of the fixed point O , at which for a rational rotation number $\rho = m/n$ of matrix J_R , i.e., for $\tau_R = 2 \cos(2\pi m/n)$, there exists an invariant polygon with n sides filled with nonhyperbolic cycles having rotation number m/n , while if ρ is irrational, there then exists an invariant region filled with quasiperiodic trajectories and bounded by an invariant ellipse tangent to the discontinuity line C_{-1} .*

For more details on degenerate bifurcations in piecewise linear maps, we refer to [36].

Let F_σ denote a composite map, $F_\sigma = F_{\sigma_0} \circ F_{\sigma_1} \dots \circ F_{\sigma_{n-1}}$, where $\sigma = \sigma_0 \sigma_1 \dots \sigma_{n-1}$ is a symbolic sequence with $\sigma_j \in \{L, R\}$, $n \geq 2$. The corresponding Jacobian matrix is denoted by J_σ , where $J_\sigma = \prod_{j=0}^{n-1} J_{\sigma_j}$, and its characteristic polynomial is $P_\sigma(\lambda) = \lambda^2 - \text{tr} J_\sigma \lambda + \det J_\sigma = 0$. It is easy to see that the fixed point equation $F_\sigma(x, y) = (x, y)$ has a unique solution $(x, y) = (0, 0)$, provided that $P_\sigma(1) \neq 0$ (i.e., matrix J_σ has no eigenvalue 1), so that the unique hyperbolic fixed point of F_σ for any σ is the fixed point O . Thus, the following property holds:

Property 5. *Map F cannot have cycles of period n for any $n \geq 2$, except for nonhyperbolic n -cycles with an eigenvalue equal to 1.*

Nonhyperbolic n -cycles with one eigenvalue equal to 1 (not related to the fixed point O), associated with condition $P_\sigma(1) = 0$, fill densely specific segments, bounded or one-side unbounded. More precisely, let the eigenvalues of matrix J_σ be real and distinct (i.e., $(\text{tr} J_\sigma)^2 - 4 \det J_\sigma > 0$), and let the corresponding eigenvectors be denoted by V^σ and W^σ . Consider eigenvector V^σ and its $n - 1$ images by map F (all these sets obviously belong to straight lines through the origin). Suppose the following *admissibility* property is satisfied: there exists a subset of V^σ located in partition D_{σ_0} such that its $n - 1$ images by F are all located in the proper partitions, according to the symbolic sequence σ . Let $\{V_j^\sigma\}_{j=0}^{n-1}$ be a *maximal admissible set*. This means that there is a subset of V^σ denoted by V_0^σ belonging to D_{σ_0} such that $F^j(V_0^\sigma) =: V_j^\sigma \subset D_{\sigma_j}$ for all $j = \overline{1, n-1}$, moreover, one boundary point of (at least) one of the sets V_j^σ belongs to the discontinuity line C_{-1} . The latter property implies that all the other boundary points of the sets V_j^σ are images of this point (or these points). Now, let $P_\sigma(1) = 0$, i.e., one eigenvalue of matrix J_σ equals 1, and let V^σ be the corresponding eigenvector (for matrix J_σ it is a straight line through the origin filled with the fixed points). Then we can state the following

Property 6. *Map F has a maximal admissible set of n cyclic segments (bounded or one-side unbounded), say, $S^\sigma = \{S_j^\sigma\}_{j=0}^{n-1}$, $n \geq 2$, filled with nonhyperbolic n -cycles (with eigenvalue +1) having symbolic sequence σ , provided that $P_\sigma(1) = 0$, $\det J_\sigma \neq 1$, and $S_j^\sigma = V_j^\sigma \subset D_{\sigma_j}$ for all $j = \overline{0, n-1}$.*

Obviously, set S^σ is not robust, i.e., it does not persist under parameter perturbations. Put differently, the Lebesgue measure of the parameter set satisfying $P_\sigma(1) = 0$ is zero. Below, for short, we denote a nonhyperbolic cycle having symbolic sequence σ as σ -cycle. Examples of one-side unbounded segments $\{S_j^\sigma\}_{j=0}^{n-1}$ for various σ can be seen in Figs. 6(b),(e), 7(a), and 9(a), while bounded segments are shown in Figs. 10(a) and 13(a).

Since map F cannot have hyperbolic cycles, it cannot have chaotic attractors either. Thus, we can state that in a generic case, a bounded attractor of map F is either the fixed point O or a WQA. Note that set S^σ , consisting of n cyclic segments filled with nonhyperbolic σ -cycles (see Property 6), may be an attracting set, but it is not an attractor since it does not include a dense trajectory. As mentioned in the Introduction, an example of a nongeneric case may also be related to parameter

values $\delta_R = 0$ or $\delta_L = 0$, when an attractor of map F may be studied by means of a 1D first return map (see [23] for details).

With regard to the possible coexistence of attractors of map F , examples of two coexisting WQAs are presented in Figs. 2 and 12(b), while examples of an attracting fixed point O coexisting with a WQA are shown in Figs. 6(c),(f), 9(b), and 10(b).

As we will demonstrate in the next section, some boundaries of a region in the parameter space, associated with WQAs of map F , are related to the existence of specific sets S^σ . As an example, we will consider *basic symbolic sequences*, $\sigma = LR^{n-1}$ and $\sigma = RL^{n-1}$, $n \geq 3$, assuming the simplest case of rotation number $\rho = 1/n$, as well as symbolic sequences that are *complementary* to these sequences, namely, $\sigma = L^2R^{n-2}$ and $\sigma = R^2L^{n-2}$, respectively. Here, considering a (nonhyperbolic) cycle with a symbolic sequence σ , by its rotation number $\rho = m/n$, we mean that along the cycle the trajectory makes m turns around the origin in n iterations.

3. Bifurcation structure of map F : weird quasiperiodic attractors and divergence

In Fig. 3(a), we present the bifurcation structure of map F in the (τ_L, τ_R) -parameter plane for $\delta_L = 0.9$, $\delta_R = 0.7$. Similarly, Fig. 3(b) shows the bifurcation structure of map F in the (δ_R, τ_R) -parameter plane for $\delta_L = 0.9$, $\tau_L = -2.5$. In these figures, blue and yellow regions indicate convergence of an initial point to the fixed point O and to a WQA, respectively, while gray regions indicate divergence. For parameter values as in Fig. 3(a), map F has $Z_1 - Z_0 - Z_1$ invertibility type since $0 < \delta_R < \delta_L$ (see Property 2(a)). In Fig. 3(b), map F has $Z_0 - Z_1 - Z_2$ invertibility type for $\delta_R < 0$ (Property 2(c)), $Z_1 - Z_0 - Z_1$ invertibility type for $0 < \delta_R < 0.9$ (Property 2(a)), $Z_1 - Z_2 - Z_1$ invertibility type for $\delta_R > 0.9$ (Property 2(b)), and in special cases, it is of $Z_0 - Z_\infty - Z_0 - Z_1$ invertibility type for $\delta_R = 0$ (Property 2(e)), and $Z_1 - Z_2 - Z_1$ invertibility type for $\delta_R = \delta_L = 0.9$ (Property 2(f)).

For $\delta_R = 0.7$, as in Fig. 3(a), the fixed point O is attracting in the horizontal strip $-1.7 < \tau_R < 1.7$, while in Fig. 3(b), it is attracting for parameter values belonging to the indicated stability triangle T^R (see (5)). In both figures, the fixed point O can be globally attracting or coexisting either with a WQA, with divergent trajectories, or with segments filled with nonhyperbolic cycles.

In fact, the boundaries of the divergence regions that satisfy conditions $P_\sigma(1) = 0$ for specific σ are related to the existence of n cyclic half-lines, $S^\sigma = \{S_j^\sigma\}_{j=0}^{n-1}$, $n \geq 2$, filled with points of nonhyperbolic σ -cycles (see Property 6). Below we obtain the boundaries of the divergence regions, denoted by $D_{1/n}^R$, associated with the basic and complementary to basic symbolic sequences $\sigma = LR^{n-1}$ and $\sigma = L^2R^{n-2}$, as well as the boundaries of the divergence regions, denoted by $D_{1/n}^L$, associated with symbolic sequences $\sigma = RL^{n-1}$ and $\sigma = R^2L^{n-2}$. The lower index $1/n$ means that in the considered case, the rotation number is $\rho = 1/n$. The boundaries of regions $D_{1/n}^R$ and $D_{1/n}^L$ in Fig. 3(a) and the boundaries of regions $D_{1/n}^R$ in Fig. 3(b) are shown for $n = 2, \dots, 9$. In Fig. 4, the 1D bifurcation diagrams, showing x versus τ_L in (a) and y versus τ_L in (b), illustrate the dynamics of map F for fixed $\tau_R = -2$ and increasing τ_L . It can be seen that the divergence regions $D_{1/n}^L$ (with no bounded dynamics) are crossed, and the regions between them correspond to WQAs. Note that for $\tau_R = -2$, the fixed point O is a saddle.

The divergence regions mentioned above are closely related to the so-called *dangerous bifurcations* occurring in the 2D border collision normal form G_μ . These bifurcations were studied quite intensively due to their importance in the applied context (see, e.g., [17,37], [15,16,18,19]). Recall that in the 2D border collision normal form G_μ defined in (1), a dangerous bifurcation occurs at $\mu = 0$ if for $\mu < 0$ and for $\mu > 0$ an attractor or several attractors of map G_μ coexist with divergent trajectories (i.e., with an attractor at infinity). In this case, as μ tends to 0, any attractor and its basin decrease in size linearly with respect to μ , and at $\mu = 0$, they shrink to point $(x, y) = (0, 0)$. At $\mu = 0$, the

generic trajectory thus diverges. As a result, there exists one or several attractors before and after the bifurcation, i.e., for $\mu < 0$ and $\mu > 0$, but near $\mu = 0$, the basins of bounded trajectories become quite small, so that even small perturbations may lead to divergence, motivating the name *dangerous bifurcation*.

A necessary condition for a dangerous bifurcation to occur is the coexistence of a bounded attractor with an attractor at infinity, which can be studied using compactification of the phase plane (see, e.g., [38]). Recall that in the parameter space of map G_μ , the boundaries of the divergence regions related to dangerous bifurcations satisfy conditions $P_\sigma(1) = 0$ for specific symbolic sequences σ , as well as the corresponding admissibility conditions. Crossing the boundary of a divergence region defined by $P_\sigma(1) = 0$ for some fixed $\mu \neq 0$, an n -cycle of map G_μ with symbolic sequence σ undergoes a *degenerate transcritical bifurcation*: its points tend to infinity while an eigenvalue tends to 1. For $\mu = 0$, a kind of *degenerate +1 bifurcation* occurs: there are n stable and n unstable invariant half-lines issuing from the fixed point in the origin, and these half-lines are filled with points of nonhyperbolic σ -cycles. We refer to [17,18], and [19] for more details related to the dangerous bifurcation in map G_μ , occurring at $\mu = 0$. In any case, condition $P_\sigma(1) = 0$ depends only on the parameters of matrices J_L and J_R , i.e., from τ_i and δ_i , $i = L, R$. These matrices are defined in the same way for both maps G_μ and F , confirming an intuitive idea about the similarity of the dynamics of maps G_μ and F at infinity. However, map F is discontinuous, and its border line is not $x = 0$ but $x = -1$, which leads to some specific new properties mentioned below.

Let us obtain the boundaries of the divergence regions (satisfying conditions $P_\sigma(1) = 0$) in the parameter space of map F by considering the simplest case, namely, regions $D_{1/n}^R$ associated with symbolic sequences $\sigma = LR^{n-1}$ and $\sigma = L^2R^{n-2}$, $n \geq 3$, and regions $D_{1/n}^L$ associated with $\sigma = RL^{n-1}$ and $\sigma = R^2L^{n-2}$.

3.1. $\sigma = LR^{n-1}$, $n \geq 3$

Consider first a composite map F_σ with $\sigma = LR^{n-1}$, $n \geq 3$. Matrix $J_{LR^{n-1}} = J_R^{n-1}J_L$ can be defined as follows:

$$J_{LR^{n-1}} = \begin{pmatrix} a_{n-1} & a_{n-2} \\ -\delta_R a_{n-2} & -\delta_R a_{n-3} \end{pmatrix} \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} =$$

$$\begin{pmatrix} \tau_L a_{n-1} - \delta_L a_{n-2} & a_{n-1} \\ -\tau_L \delta_R a_{n-2} + \delta_L \delta_R a_{n-3} & -\delta_R a_{n-2} \end{pmatrix},$$

where a_k is the solution of the second-order homogeneous linear difference equation

$$a_k = \tau_R a_{k-1} - \delta_R a_{k-2}, \quad a_0 = 1, \quad a_1 = \tau_R, \quad k = 2, 3, \dots$$

The eigenvalues

$$\lambda_{1,2}^{LR^{n-1}} = 0.5(\tau_L a_{n-1} - (\delta_L + \delta_R) a_{n-2}) \pm \sqrt{(\tau_L a_{n-1} - (\delta_L + \delta_R) a_{n-2})^2 - 4\delta_L \delta_R^{n-1}}$$

of matrix $J_{LR^{n-1}}$ are real if $(\tau_L a_{n-1} - (\delta_L + \delta_R) a_{n-2})^2 - 4\delta_L \delta_R^{n-1} > 0$, and the corresponding eigenvectors have slopes

$$K_i^{LR^{n-1}} = \frac{\delta_R(\delta_L a_{n-3} - \tau_L a_{n-2})}{\delta_R a_{n-2} + \lambda_i}, \quad i = 1, 2. \tag{6}$$

Transition from real to complex-conjugate eigenvalues occurs crossing a parameter set denoted $E_{LR^{n-1}}$ and defined as follows:

$$E_{LR^{n-1}} : (\tau_L a_{n-1} - (\delta_L + \delta_R) a_{n-2})^2 - 4\delta_L \delta_R^{n-1} = 0. \tag{7}$$

The characteristic polynomial of $J_{LR^{n-1}}$ can be written as follows:

$$P_{LR^{n-1}}(\lambda) = \lambda^2 + (-\tau_L a_{n-1} + (\delta_L + \delta_R) a_{n-2}) \lambda + \delta_L \delta_R^{n-1}. \tag{8}$$

Considering the linear map $F_{LR^{n-1}}$, i.e., map $F_{LR^{n-1}} : (x, y) \rightarrow J_{LR^{n-1}}(x, y)$, without its relation to the piecewise linear map F , condition $P_{LR^{n-1}}(1) = 0$ (for $\delta_L \delta_R^{n-1} \neq 1$) means that map $F_{LR^{n-1}}$ has an

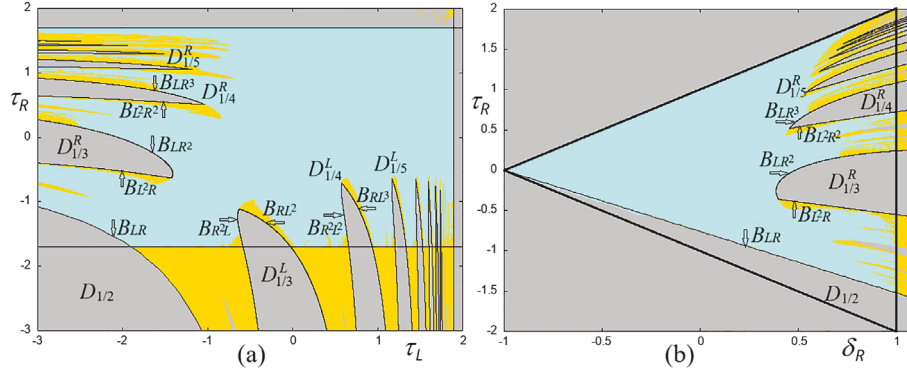


Fig. 3. Bifurcation structure of map F (a) in the (τ_L, τ_R) -parameter plane for $\delta_L = 0.9$, $\delta_R = 0.7$, and (b) in the (δ_R, τ_R) -parameter plane for $\delta_L = 0.9$, $\tau_L = -2.5$. The boundaries $B_{LR^{n-1}}$ and $B_{L^2R^{n-2}}$ of the divergence region $D_{1/n}^R$, defined in (12) and (13), respectively, and the boundaries $B_{RL^{n-1}}$ and $B_{R^2L^{n-2}}$ of the divergence region $D_{1/n}^L$, defined in (16) and (18), respectively, are shown for $n = 2, \dots, 9$. Blue and yellow parameter regions indicate convergence to the fixed point O and to a WQA, respectively. Gray regions indicate divergence. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

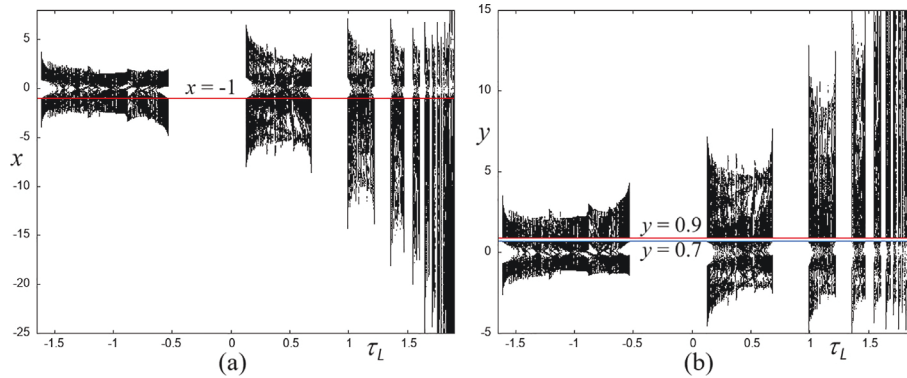


Fig. 4. A 1D bifurcation diagram of map F showing x versus τ_L in (a), and y versus τ_L in (b) for $\delta_L = 0.9$, $\delta_R = 0.7$, and $\tau_R = -2$.

eigenvalue equal to 1 and the straight line through the origin, say $y = K^{LR^{n-1}}x$, associated with the corresponding eigenvector, is filled with fixed points, where

$$K^{LR^{n-1}} = \frac{\delta_R(\delta_L a_{n-3} - \tau_L a_{n-2})}{\delta_R a_{n-2} + 1}. \quad (9)$$

However, considering the piecewise linear map F , condition

$$P_{LR^{n-1}}(1) = 1 - \tau_L a_{n-1} + (\delta_L + \delta_R)a_{n-2} + \delta_L \delta_R^{n-1} = 0 \quad (10)$$

is necessary but not sufficient for the existence of set $S^{LR^{n-1}}$ consisting of n segments filled with points of nonhyperbolic LR^{n-1} -cycles. According to Property 6, the admissibility conditions must be satisfied, that is, these segments must be located in the proper partitions. Here we are interested in the boundaries of the divergence regions, so these segments are *half-lines* and their “end points at infinity” are, for the compactified map, periodic as well, with symbolic sequence LR^{n-1} . In the considered case, denoting the starting half-line belonging to D_L as $S_0^{LR^{n-1}}$, so that

$$S_0^{LR^{n-1}} = \{(x, y) : y = K^{LR^{n-1}}x, x < -1\}$$

(assuming $K^{LR^{n-1}} \neq \infty$, that is, $\delta_R a_{n-2} + 1 \neq 0$, see (9), that holds for $\delta_R \neq 1$), its images $S_j^{LR^{n-1}} = F(S_{j-1}^{LR^{n-1}})$ for $j = 1, n-1$ must be completely located in partition D_R . For short, we denote these admissibility conditions as $(A^{LR^{n-1}})$, that is,

$$(A^{LR^{n-1}}) : S_0^{LR^{n-1}} \subset D_L, S_j^{LR^{n-1}} \subset D_R, j = \overline{1, n-1} \quad (11)$$

(an example of admissible set S^{LR^4} consisting of five half-lines filled with nonhyperbolic LR^4 -cycles is shown in Fig. 6(a)).

In contrast to the map G_0 , map F can also have a set $S^{LR^{n-1}}$ satisfying condition $P_{LR^{n-1}}(1) = 0$ and admissibility conditions $(A^{LR^{n-1}})$,

but consisting of *bounded segments* (see Fig. 7(a) for an example), and in such a case the corresponding parameter set is not associated with the boundary of a divergence region (see the point marked g in Fig. 5(a) corresponding to Fig. 7(a)). In fact, comparing with map G_0 , in map F the border line is shifted to the left, from $x = 0$ to $x = -1$, so that partition D_R is enlarged, providing additional space for the segments of set $S^{LR^{n-1}}$ to be still admissible, which in the case of map G_0 would be not admissible (or, in other words, virtual).

To summarize, we can state that among the parameter values satisfying $P_{LR^{n-1}}(1) = 0$ given in (10), there may be a subset denoted $B_{LR^{n-1}}$ that satisfies the admissibility conditions $(A^{LR^{n-1}})$ given in (11):

$$B_{LR^{n-1}} : 1 - \tau_L a_{n-1} + (\delta_L + \delta_R)a_{n-2} + \delta_L \delta_R^{n-1} = 0 \text{ and } (A^{LR^{n-1}}) \text{ holds,} \quad (12)$$

where segments $S_j^{LR^{n-1}}$, $j = \overline{1, n-1}$, in $(A^{LR^{n-1}})$ can be either one-side unbounded (then $B_{LR^{n-1}}$ belongs to a boundary of the divergence region $D_{1/n}^R$), or bounded.

3.2. $\sigma = L^2R^{n-2}$, $n \geq 3$

Following analogous steps for the complementary symbolic sequence $\sigma = L^2R^{n-2}$, we obtain a set of parameter values, denoted by $B_{L^2R^{n-2}}$, satisfying $P_{L^2R^{n-2}}(1) = 0$, and the corresponding admissibility conditions:

$$B_{L^2R^{n-2}} : 1 + (\tau_L(\delta_L + \delta_R) - \delta_L \tau_R)a_{n-3} - (\tau_L^2 - 2\delta_L)a_{n-2} + \delta_L^2 \delta_R^{n-2} = 0 \text{ and } (A^{L^2R^{n-2}}) \text{ holds,} \quad (13)$$

where

$$(A^{L^2R^{n-2}}) : S_0^{L^2R^{n-2}} \subset D_L, S_1^{L^2R^{n-2}} \subset D_L, S_j^{L^2R^{n-2}} \subset D_R, j = \overline{2, n-1},$$

and

$$S_0^{L^2 R^{n-2}} = \{(x, y) : y = K^{L^2 R^{n-2}} x, x < -1\}$$

with

$$K^{L^2 R^{n-2}} = \frac{\delta_R(\delta_L a_{n-3} + \tau_L(\delta_L a_{n-4} - \tau_L a_{n-3}))}{\delta_R(\tau_L a_{n-3} - \delta_L a_{n-4}) + 1}$$

(see an example of set $S^{L^2 R^3}$ consisting of five half-lines filled with nonhyperbolic $L^2 R^3$ -cycles in Fig. 6(d)).

In this way, two boundaries, $B_{L R^{n-1}}$ and $B_{L^2 R^{n-2}}$, of the divergence region $D_{1/n}^R$, $n \geq 3$, are obtained (see Fig. 3).

Note that for a parameter point belonging to region $D_{1/n}^R$, map F has an attracting n -cycle at infinity with symbolic sequence either LR^{n-1} or $L^2 R^{n-2}$, associated with the unstable eigenvector of matrix $J_{L R^{n-1}}$ or $J_{L^2 R^{n-2}}$, respectively, and its $n - 1$ images by map F (see, e.g., vectors $V_j^{LR^4}$ and $V_j^{L^2 R^3}$, $j = 0, 4$, in Fig. 6(a) and (d), respectively). A boundary, say $H_{L R^{n-1}}$, separating the corresponding subregions of region $D_{1/n}^R$ can be defined by the condition that the eigenvector of matrix $J_{L R^{n-1}}$ has slope 0 (and, thus, the slope of its preimage by F_R is equal to infinity). Then, for a parameter point belonging to region $D_{1/n}^R$, we have the following:

- on one side of boundary $H_{L R^{n-1}}$, it holds that the unstable eigenvectors $V_j^{LR^{n-1}}$ are admissible, while eigenvectors $V_j^{L^2 R^{n-2}}$ are not admissible (so that an attracting LR^{n-1} -cycle at infinity exists, while the $L^2 R^{n-2}$ -cycle is virtual);

- on the other side of boundary $H_{L R^{n-1}}$, the unstable eigenvectors $V_j^{L^2 R^{n-2}}$ are admissible, while eigenvectors $V_j^{LR^{n-1}}$ are not admissible (so that an attracting $L^2 R^{n-2}$ -cycle at infinity exists, while the LR^{n-1} -cycle is virtual).

So, assuming that in (6) $\delta_R \neq 0$, from $K_i^{LR^{n-1}} = 0$ boundary $H_{L R^{n-1}}$ can be defined as follows:

$$H_{L R^{n-1}} : \delta_L a_{n-3} - \tau_L a_{n-2} = 0, \quad H_{L R^{n-1}} \subset D_{1/n}^R. \quad (14)$$

Transition from real to complex-conjugate eigenvalues of matrix $J_{L^2 R^{n-2}}$ (when map F has no cycles at infinity) occurs crossing a parameter set

$$E_{L^2 R^{n-2}} : ((\delta_L \tau_R - \tau_L(\delta_L + \delta_R))a_{n-3} + (\tau_L^2 - 2\delta_L)a_{n-2}) - 4\delta_L^2 \delta_R^{n-2} = 0. \quad (15)$$

As an example, region $D_{1/5}^R$ is shown magnified in Fig. 5(a), where besides its boundaries B_{LR^4} and $B_{L^2 R^3}$, also curves H_{LR^4} , E_{LR^4} , and $E_{L^2 R^3}$ are shown, given in (14), (7), and (15), respectively.

3.3. $\sigma = RL^{n-1}$ and $\sigma = R^2 L^{n-2}$, $n \geq 3$

Analogous boundaries associated with the divergence regions $D_{1/n}^L$ and symbolic sequences $\sigma = RL^{n-1}$ and $\sigma = R^2 L^{n-2}$ can be obtained by changing $L \leftrightarrow R$ in all the above notations. In particular, the boundary $B_{RL^{n-1}}$ of divergence regions $D_{1/n}^L$ can be defined as

$$B_{RL^{n-1}} : 1 - \tau_R b_{n-1} + (\delta_R + \delta_L)b_{n-2} + \delta_R \delta_L^{n-1} = 0 \text{ and } (A^{RL^{n-1}}) \text{ holds,} \quad (16)$$

where the admissibility conditions $(A^{RL^{n-1}})$ are as follows:

$$(A^{RL^{n-1}}) : S_0^{RL^{n-1}} \subset D_R, S_j^{RL^{n-1}} \subset D_L, \quad j = \overline{1, n-1},$$

and b_k is a solution of the second-order homogeneous linear difference equation

$$b_k = \tau_L b_{k-1} - \delta_L b_{k-2}, \quad b_0 = 1, \quad b_1 = \tau_L, \quad k = 2, 3, \dots$$

Transition from real to complex-conjugate eigenvalues of matrix J_{RL^n} is associated with crossing a parameter set $E_{RL^{n-1}}$ satisfying

$$E_{RL^{n-1}} : (\tau_R b_{n-1} - (\delta_L + \delta_R)b_{n-2})^2 - 4\delta_R \delta_L^{n-1} = 0. \quad (17)$$

The boundary $B_{R^2 L^{n-2}}$ of divergence regions $D_{1/n}^L$ can be defined as

$$B_{R^2 L^{n-2}} : 1 + (\tau_R(\delta_L + \delta_R) - \delta_R \tau_L)b_{n-3} - (\tau_R^2 - 2\delta_R)b_{n-2} + \delta_R^2 \delta_L^{n-2} = 0 \text{ and } (A^{R^2 L^{n-2}}) \text{ holds,} \quad (18)$$

where

$$(A^{R^2 L^{n-2}}) : S_0^{R^2 L^{n-2}} \subset D_R, S_1^{R^2 L^{n-2}} \subset D_R, S_j^{R^2 L^{n-2}} \subset D_L, \quad j = \overline{2, n-1}.$$

Transition from real to complex-conjugate eigenvalues of matrix $J_{R^2 L^{n-2}}$ are associated with a parameter set

$$E_{R^2 L^{n-2}} : ((\delta_R \tau_L - \tau_R(\delta_R + \delta_L))b_{n-3} + (\tau_R^2 - 2\delta_R)b_{n-2}) - 4\delta_R^2 \delta_L^{n-2} = 0, \quad (19)$$

and boundary $H_{RL^{n-1}}$ can be defined as follows:

$$H_{RL^{n-1}} : \delta_R b_{n-3} - \tau_R b_{n-2} = 0, \quad H_{RL^{n-1}} \subset D_{1/n}^L. \quad (20)$$

Fig. 8(a) presents an example of the divergence region $D_{1/5}^L$ together with its boundaries B_{RL^4} and $B_{R^2 L^3}$, as well as the curves H_{RL^4} , E_{RL^4} , and $E_{R^2 L^3}$, given in (16), (18), (17), (19), and (20), respectively.

Note that substituting $n = 2$ either to (12) or (16) the boundary B_{LR} of the divergence region $D_{1/2}$ can be obtained:

$$B_{LR} : 1 - \tau_R \tau_L + \delta_R + \delta_L + \delta_R \delta_L = 0, \quad S_0^{LR} \in D_L, S_1^{LR} \in D_R,$$

(see Fig. 3), where $S_0^{LR} = \{(x, y) : y = -\frac{\tau_L \delta_R}{1 + \delta_R} x, x < -1\}$ and $S_1^{LR} = \{(x, y) : y = -\frac{\tau_R \delta_L}{1 + \delta_L} x, x > -\frac{\tau_L}{1 + \delta_R}\}$.

In the next section, the dynamics near regions $D_{1/5}^R$ and $D_{1/5}^L$, associated with WQAs, is described in more detail.

4. Appearance of weird quasiperiodic attractors near divergence regions

4.1. Dynamics of map F near the divergence region $D_{1/5}^R$

Let us describe the dynamics of map F near the divergence region $D_{1/5}^R$ by considering this region in the (τ_L, τ_R) -parameter plane for $\delta_L = 0.9$, $\delta_R = 0.7$ to see how its boundaries, B_{LR^4} given in (12) and $B_{L^2 R^3}$ given in (13), are related to the appearance of WQAs. In Fig. 5(a), the bifurcation structure near the divergence region $D_{1/5}^R$ is shown magnified. Besides curves B_{LR^4} and $B_{L^2 R^3}$, also curves H_{LR^4} , E_{LR^4} and $E_{L^2 R^3}$ given in (14), (7), and (15), respectively, are shown. Note that in the considered case, the complete region $D_{1/5}^R$ is inside the stability domain of the fixed point O , so that in all the examples below the fixed point O is attracting. Note also that map F has $Z_1 - Z_0 - Z_1$ type of invertibility (see Property 2(a)), and it is not difficult to show that any invariant set of map F cannot have points in zone Z_0 (i.e., in the strip between the critical lines C^R and C^L), or in any image of Z_0 .

Fig. 5(b) illustrates the dynamics by means of a 1D bifurcation diagram showing x versus τ_R for fixed $\tau_L = -2$. At first glance, this bifurcation diagram can be misleading because the attractors here seem chaotic. However, as already mentioned, map F cannot have chaotic attractors, and the limit sets we observe here are WQAs born when either boundary B_{LR^4} (for increasing τ_R from a point inside the region) or boundary $B_{L^2 R^3}$ (for decreasing τ_R) is crossed. The mechanism of appearance of such attractors is illustrated in Fig. 6(a),(b), ..., (f), presenting phase portraits of map F at the parameter values marked in Fig. 5(a) by a, b, \dots, f , respectively. One more example is shown in Fig. 7(a), associated with the parameter point marked in Fig. 5(a) by g . In all these figures, the basin of the fixed point O is shown in light blue, the basin of a WQA in dark yellow, the basin of a set of segments filled with nonhyperbolic cycles in light yellow, and the basin of diverging trajectories in gray.

- Let us begin with the parameter point a . The corresponding phase portrait of map F is shown in Fig. 6(a). The parameter point a is located below curve B_{LR^4} but above curve H_{LR^4} . In this subregion of $D_{1/5}^R$, the attracting fixed point O coexists with divergent trajectories. More precisely, it holds that $P_{LR^4}(1) < 0$, i.e., one eigenvalue of matrix J_{LR^4} is larger than 1 (the second eigenvalue is obviously positive and less than 1); moreover, the corresponding invariant unstable half-lines of map F (shown in dark blue in Fig. 6(a)) are located in the proper partitions: $V_0^{LR^4} \subset D_L, V_j^{LR^4} \subset D_R, j = \overline{1, 4}$.

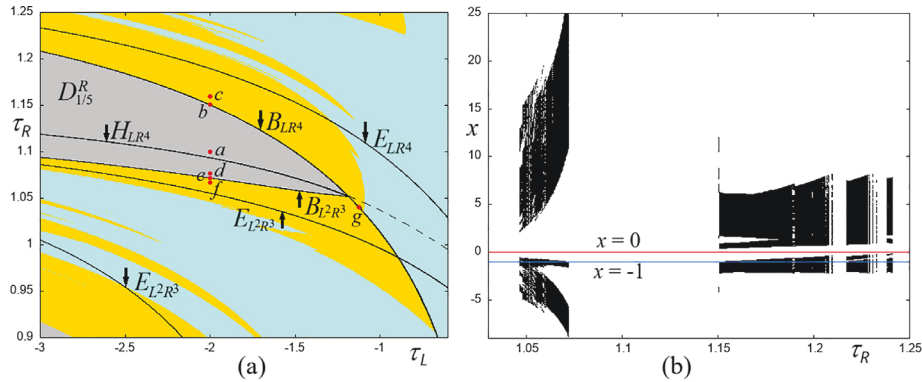


Fig. 5. (a) The existence regions (in yellow) of the WQAs near the divergence region $D_{1/5}^R$ (in gray) bounded by curves B_{LR^4} and $B_{L^2R^3}$ in the (τ_L, τ_R) -parameter plane for $\delta_L = 0.9$, $\delta_R = 0.7$; (b) 1D bifurcation diagram x versus τ_R for $\tau_L = -2$. The phase portrait of map F at parameter points marked in (a) by a, b, \dots, f is shown in Fig. 6(a),(b), \dots ,(f), respectively, and by g in Fig. 7(a).

The left column in Fig. 6 illustrates how the phase portrait of map F changes for increasing values of τ_R , when the parameter point approaches boundary B_{LR^4} (at which $P_{LR^4}(1) = 0$) and crosses it (so that $P_{LR^4}(1) > 0$ holds).

- **Parameter point b.** Fig. 6(b) shows the corresponding phase portrait of map F , where besides the attracting fixed point O there exists a set S^{LR^4} consisting of five cyclic half-lines, $S_0^{LR^4} \subset D_L$ and $S_j^{LR^4} \subset D_R$, $j = \overline{1, 4}$, filled with nonhyperbolic LR^4 -cycles, $S_j^{LR^4} = \overline{V_j^{LR^4}}$ (see Property 6).

- **Parameter point c.** Fig. 6(c) presents the phase portrait after crossing boundary B_{LR^4} , where the attracting fixed point O coexists with a WQA denoted \mathcal{A} , which appears after crossing B_{LR^4} . The mechanism of creation of attractor \mathcal{A} can be outlined as follows. First, note that the parameter point c is above curve B_{LR^4} but below curve E_{LR^4} . Hence, the eigenvalues of matrix J_{LR^4} are both real, positive, and less than 1. Eigenvectors $V_j^{LR^4}$, $j = \overline{0, 4}$, which were forward invariant and repelling before the crossing B_{LR^4} , become attracting after the crossing. Consider half-line $V_0^{LR^4} \subset D_L$ (shown in blue in the inset in Fig. 6(c)). Its image $V_5^{LR^4} = F_{LR^4}(V_0^{LR^4})$ necessarily intersects the discontinuity line. Then, under further iterations, a part of half-line $V_5^{LR^4}$ located in D_L , say, half-line $V_5^{LR^4}|_{D_L}$, is mapped as before by F_L into a half-line issuing from C^L in D_R , while a segment of half-line $V_5^{LR^4}$ located in D_R , say segment $V_5^{LR^4}|_{D_R}$, is mapped by F_R into a new segment issuing from C^R , which in this example belongs to D_L . Next iterations of segment $V_5^{LR^4}|_{D_R}$ lead to further new segments approaching $V_j^{LR^4}$, $j = \overline{0, 4}$, as the number of iterations tends to infinity (but they are never mapped exactly into these half-lines). The resulting attractor \mathcal{A} is a WQA, and it can be seen as a limit set of this iteration process. Note that it could happen that immediately after crossing the curve B_{LR^4} , segment $V_5^{LR^4}|_{D_R}$ belongs to the basin of the attracting fixed point O , and in this case a WQA is not created.

- Let us now turn to the parameter point d located above curve $B_{L^2R^3}$ but below curve H_{LR^4} . The related phase portrait is shown in Fig. 6(d). In this case, the fixed point O coexists with divergent trajectories; more precisely, it holds that $P_{L^2R^3}(1) < 0$, that is, one eigenvalue of matrix $J_{L^2R^3}$ is larger than 1, and the corresponding forward invariant repelling eigenvectors of map F are located in the proper partitions: $V_{0,1}^{L^2R^3} \subset D_L$, $V_j^{L^2R^3} \subset D_R$, $j = \overline{2, 4}$ (in Fig. 6(d), these eigenvectors are shown in dark blue).

The right column in Fig. 6 illustrates how the phase portrait of map F changes for decreasing values of τ_R when the parameter point crosses boundary $B_{L^2R^3}$.

- **Parameter point e.** Fig. 6(e) shows the related phase portrait. It holds that $(\tau_L, \tau_R) \in B_{L^2R^3}$, thus $P_{L^2R^3}(1) = 0$, and besides the attracting fixed point O , there exists a set $S^{L^3R^3}$ consisting of 5-cyclic half-lines,

$S_{0,1}^{L^3R^3} \subset D_L$ and $S_j^{L^2R^3} \subset D_R$, $j = \overline{2, 4}$ ($S^{L^3R^3} = V_j^{L^2R^3}$), filled with nonhyperbolic L^2R^3 -cycles (see Property 6);

- **Parameter point f.** Fig. 6(f) presents the corresponding phase portrait. The parameter point f is below $B_{L^2R^3}$, so that $P_{L^2R^3}(1) > 0$, and the attracting fixed point O coexists with a WQA denoted \mathcal{A} , born after crossing boundary $B_{L^2R^3}$. The mechanism of its creation is similar to the one described in the previous example. Note that in this case, the parameter point is below curve $B_{L^2R^3}$ but above curve $E_{L^2R^3}$. Hence, the eigenvalues of matrix $J_{L^2R^3}$ are both real, positive, and less than 1.

It is interesting to compare the WQAs shown in Fig. 6(c) and (f). Considering map F^5 , it can be determined that the attractor in Fig. 6(f) consists of five cyclic blocks, unlike the attractor shown in Fig. 6(c), which is not cyclic. In fact, the complete block shown magnified in the inset in Fig. 6(f) is mapped in one block in partition D_L , while the block of the attractor shown magnified in the inset in Fig. 6(c) is mapped partially in D_L and partially in D_R , breaking the cyclicity of this attractor.

- **Parameter point g.** This example is related to the case when the (τ_L, τ_R) -parameter point belongs to curve B_{LR^4} . Since this point is below curve H_{LR^4} , the LR^4 -cycle at infinity does not exist and curve B_{LR^4} no longer serves as a boundary of the divergence region $D_{1/5}^R$. However, an admissible set S^{LR^4} still exists: it consists of five cyclic bounded segments, $S_0^{LR^4} \subset D_L$ and $S_j^{LR^4} \subset D_R$, $j = \overline{1, 4}$, filled with nonhyperbolic LR^4 -cycles. Fig. 7(a) presents the phase portrait of map F , where the attracting fixed point O coexists with segments $S_j^{LR^4}$, $j = \overline{0, 4}$, shown in dark blue.

Starting from the case show in Fig. 7(a), a WQA denoted \mathcal{A} appears for both increasing and decreasing values of τ_R , as in Fig. 7(b) and Fig. 7(c), respectively. In particular, after crossing B_{LR^4} for increasing τ_R (see Fig. 7(b)), eigenvectors $V_j^{LR^4}$ become attracting, so that segment $F^5(V_0^{LR^4})$ necessarily intersects the discontinuity line C_{-1} and the mechanism of creation of attractor \mathcal{A} is similar to the one illustrated in Fig. 6(b) and (c). Note that for further increasing τ_R , attractor \mathcal{A} disappears quite quickly due to a contact with the basin boundary of the fixed point O . In contrast, crossing B_{LR^4} for decreasing τ_R (see Fig. 7(c)), eigenvectors $V_j^{LR^4}$ become repelling, so that segment $F^5(V_4^{LR^4})$ necessarily intersects the discontinuity line C_{-1} , and the mechanism of creation of a WQA is associated with further images of segment $F^5(V_4^{LR^4})|_{D_L}$.

4.2. Dynamics of map F near the divergence region $D_{1/5}^L$

We now turn to an example of the bifurcation structure near the divergence region $D_{1/5}^L$. Fig. 8(a) shows the (τ_L, τ_R) -parameter plane for $\delta_L = 0.9$, $\delta_R = 0.7$, where besides curves B_{RL^4} and $B_{R^2L^3}$, given in (16) and (18), also curves H_{RL^4} , E_{RL^4} , and $E_{R^2L^3}$, given in (20), (17), and

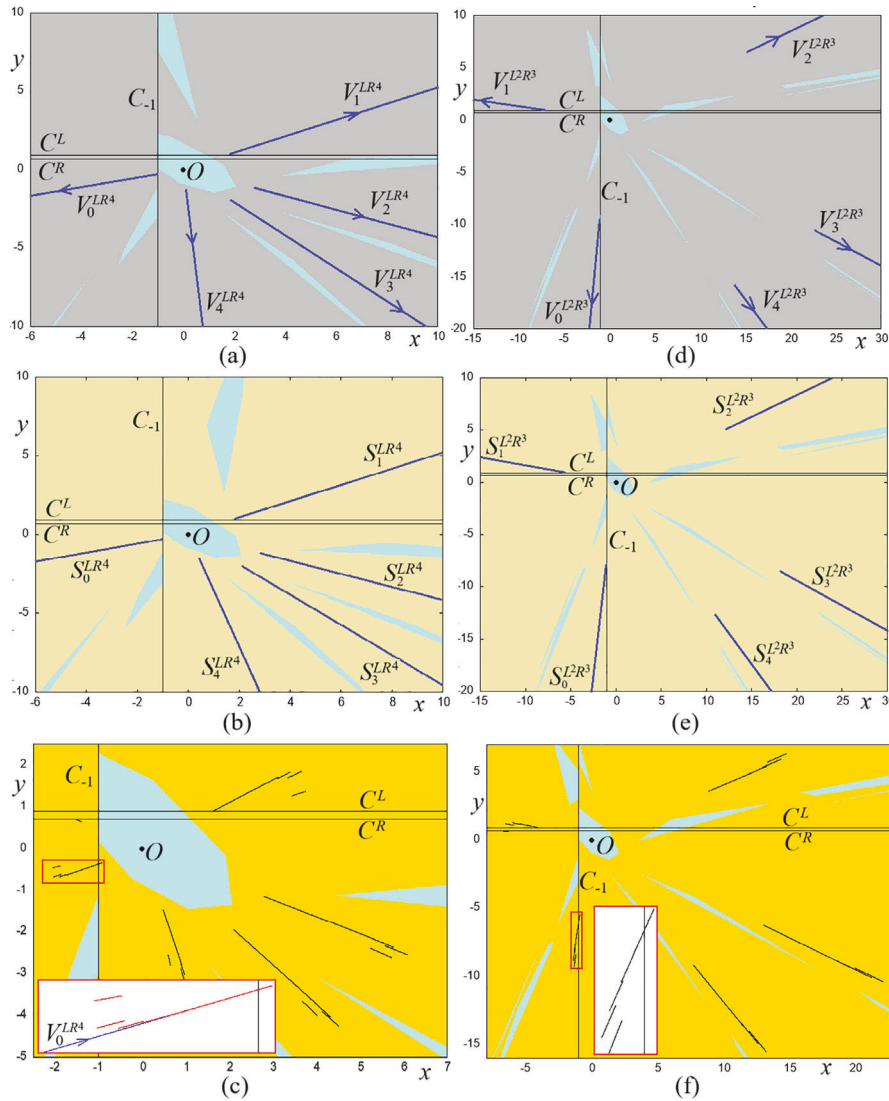


Fig. 6. Phase portrait of map F at $\delta_L = 0.9$, $\delta_R = 0.7$, $\tau_L = -2$ and (a) $\tau_R = 1.1$; (b) $\tau_R = 1.15045$; (c) $\tau_R = 1.16$; (d) $\tau_R = 1.075$; (e) $\tau_R = 1.0719$; (f) $\tau_R = 1.067$. The related parameter points are marked in Fig. 5 by a, b, \dots, f , respectively. The insets in (c) and (f) show the marked windows magnified.

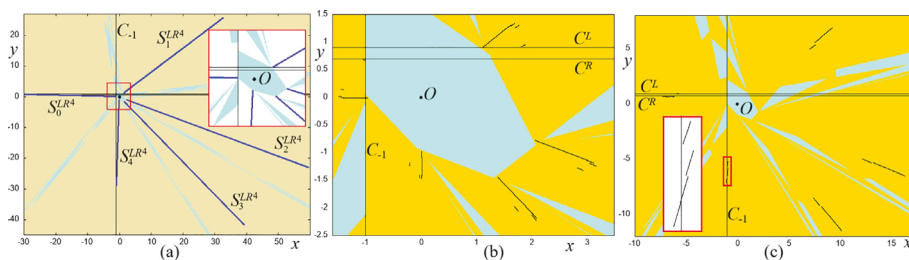


Fig. 7. Phase portrait of map F at $\delta_L = 0.9$, $\delta_R = 0.7$, $\tau_L = -1.125$ and (a) $\tau_R = 1.04053$, (b) $\tau_R = 1.05$, (c) $\tau_R = 1.02$. The parameter point related to (a) is marked in Fig. 5(a) by g . In (c), the inset shows the indicated window magnified.

(19), respectively, are shown. Comparing with the divergence region $D_{1/5}^R$ in Fig. 5(a), region $D_{1/5}^L$ in Fig. 8(a) is located not entirely in the stability domain of the fixed point O , but also in the region below the straight line $\tau_R = -1.7$, where the fixed point O is a saddle. Fig. 8(b) and its magnified part in the right panel present a 1D bifurcation diagram for y versus τ_L , setting $\tau_R = -2$. The corresponding parameter path is indicated in Fig. 8(a) by the red horizontal segment. This diagram confirms, in particular, that attractors of map F have no points in the strip between the critical lines C^R and C^L (zone Z_0 in Property 2(a)),

defined in the considered case by $y = 0.7$ and $y = 0.9$, respectively (see two horizontal straight lines in Fig. 8(b)). Below, we give a few examples related to Fig. 8(b).

To illustrate the dynamics of map F near the divergence region $D_{1/5}^L$, we present the phase portraits of map F for parameter values indicated in Fig. 8(a) by a, b, c, \dots, j . The following should be noted:

- *Parameter points a and b.* In Fig. 9(a), related to the parameter point a belonging to curve $B_{R^2L^3}$, besides the attracting fixed point O , there exists a set $S^{R^2L^3}$ consisting of five cyclic half-lines, $S_{0,1}^{R^2L^3} \subset D_R$

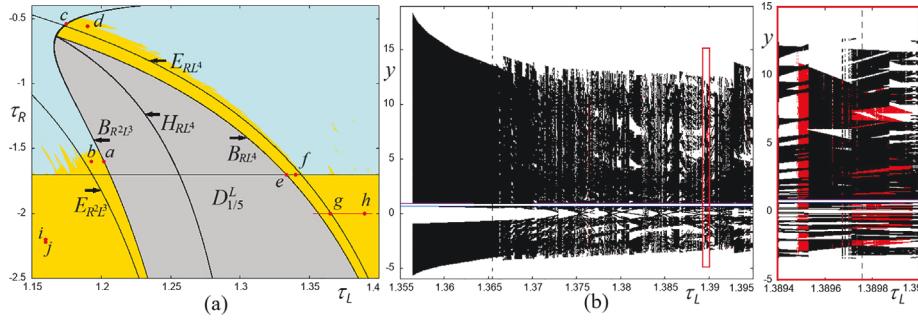


Fig. 8. (a) The existence regions (in yellow) of the WQAs near the divergence region $D_{1/5}^L$ (in gray) bounded by curves B_{RL^4} and $B_{R^2L^3}$ in the (τ_L, τ_R) -parameter plane for $\delta_L = 0.9$, $\delta_R = 0.7$; (b) 1D bifurcation diagram y versus τ_L for $\tau_R = -2$. The rightmost figure presents a magnified window indicated in (b), where coexisting WQAs (shown in red and black) are better visible. The phase portraits of map F at parameter points marked by a, b, \dots, j are shown in Figs. 9(a),(b), 10(a),(b), 11 (a),(b), 12(a),(b), and 13(a),(b), respectively.

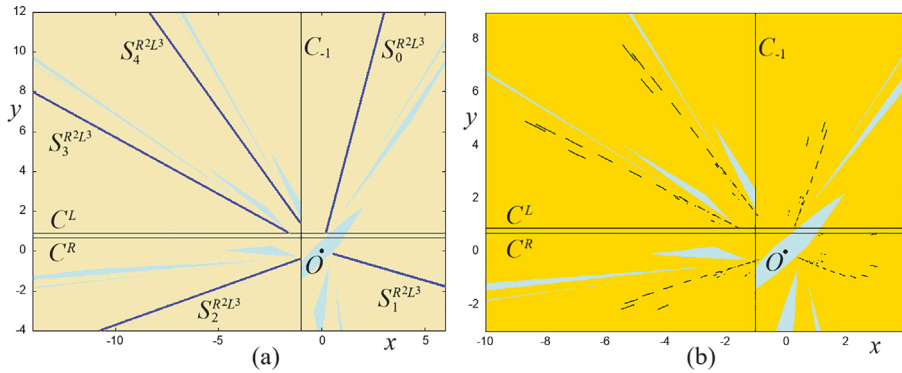


Fig. 9. Phase portrait of map F at $\delta_L = 0.9$, $\delta_R = 0.7$, $\tau_R = -1.6$ and (a) $\tau_L = 1.201945$, (b) $\tau_L = 1.193$. In Fig. 8(a), the related parameter points are marked by a and b , respectively.

and $S_j^{R^2L^3} \subset D_L$, $j = \overline{2,4}$ ($S_j^{R^2L^3} = V_j^{R^2L^3}$), filled with nonhyperbolic R^2L^3 -cycles (see Property 6). Unlike crossing boundary B_{RL^4} , crossing boundary $B_{R^2L^3}$ may not lead to a WQA. In fact, after crossing B_{RL^4} , eigenvectors $V_j^{R^2L^3}$, $j = \overline{0,4}$, become attracting, thus after five iterations, half-line $F^5(V_4^{R^2L^3})$ and/or $F^5(V_2^{R^2L^3})$ necessarily intersects partition D_R . It may happen that the segment(s) of $F^5(V_4^{R^2L^3})$ and/or $F^5(V_2^{R^2L^3})$ located in D_R immediately after the bifurcation enter(s) the basin of the attracting fixed point O , and thus no WQA appears. It can be seen in Fig. 9(a) that segments $S_j^{R^2L^3}$ are relatively far from the basin of the fixed point O , and for decreasing τ_L a WQA, as, e.g., in Fig. 9(b), is created. Note that the related parameter point b in Fig. 8(a) is located between curves $B_{R^2L^3}$ and $E_{R^2L^3}$, where the eigenvalues of the matrix $J_{R^2L^3}$ are both real, positive, and less than 1.

• **Parameter points c and d .** In Fig. 10(a), the phase portrait of map F is shown for the parameter point c belonging to curve $B_{R^2L^3}$. In contrast to the case in Fig. 9(a), the parameter point is above curve H_{RL^4} . In this region, an R^2L^3 -cycle at infinity does not exist, and this part of curve $B_{R^2L^3}$ is not at the boundary of the divergence region $D_{1/5}^L$. However, an admissible set $S^{R^2L^3}$ still exists, consisting of five cyclic bounded segments, $S_{0,1}^{R^2L^3} \subset D_R$ and $S_j^{R^2L^3} \subset D_L$, $j = \overline{2,4}$ ($S_j^{R^2L^3} = V_j^{R^2L^3}$) filled with nonhyperbolic R^2L^3 -cycles (see Property 6). An example of a WQA existing for the parameter values on the right of curve $B_{R^2L^3}$ is shown in Fig. 10(b). Note that the related parameter point d in Fig. 8(a) belongs to the region where eigenvectors $V_j^{R^2L^3}$, $j = \overline{0,4}$, are repelling. Comparing with the example in Fig. 9, boundary $B_{R^2L^3}$ is crossed in the opposite direction: segment $F^5(V_0^{R^2L^3})$ necessarily intersects partition D_L (see the inset in Fig. 10(b)). Further iterations of segment $F^5(V_0^{R^2L^3})|_{D_L}$ lead to new segments, and the WQA can be seen as a limit set of this process.

• **Parameter points e and f .** In Fig. 11(a) a special case is illustrated where parameter point e belongs to curve B_{RL^4} and $\tau_R = -1.7$. Recall that at $\tau_R = -1.7$, the fixed point O undergoes a degenerate flip

bifurcation (see Property 4): there is a segment $S^{R^2} = \{(x, y) : -1 < x < 1, y = 0.7x\}$ filled with points of nonhyperbolic 2-cycles (except the fixed point O), and this segment coexists with a set S^{RL^4} consisting of five cyclic half-lines, $S_0^{RL^4} \subset D_R$ and $S_j^{RL^4} \subset D_L$, $j = \overline{1,4}$, filled with nonhyperbolic RL^4 -cycles. An example of a WQA born on the right side of curve B_{RL^3} and coexisting with segment S^{R^2} is shown in Fig. 11(b). The related parameter point is marked f in Fig. 8(a).

• **Parameter points g and h .** In Fig. 12, two examples of the phase portrait of map F are shown related to the 1D bifurcation diagram in Fig. 8(b). It is easy to verify that the WQA shown in Fig. 12(a) related to the parameter point g , consists of five cyclic blocks, that is, each block is F^5 -invariant. In this case, only one block intersects the discontinuity line, and this entire block is mapped into the same partition D_R . The cyclicity is not lost immediately but soon after crossing curve E_{RL^4} (at $\tau_L \approx 1.36540484$, see the dashed vertical line in Fig. 8(b)), when the eigenvalues of matrix J_{RL^4} become complex-conjugate. Fig. 12(b) related to the parameter point h , illustrates two coexisting WQAs and their basins at $\tau_L = 1.38975$ (see the dashed vertical line in the right panel of Fig. 8(b)). As can be seen in the right panel of Fig. 8(b), there are several intervals of bistability, confirming that the coexistence of two WQAs may be persistent under parameter perturbations.

• **Parameter points i and j .** Finally, it is worth noting that in the (τ_L, τ_R) -parameter plane in Fig. 8(a), there are parameter values associated with segments filled with nonhyperbolic cycles having symbolic sequences that are different from those considered above, i.e., from the basic and complementary symbolic sequences. An example is shown in Fig. 13(a), in the case when map F has a set of seven cyclic segments $S_j^{R^4L^3}$, $j = \overline{0,6}$, filled with nonhyperbolic R^4L^3 -cycles (the related point is marked i). A WQA appearing for decreasing τ_R , e.g., at the parameter point j , is shown in Fig. 13(b). A mechanism of creation of this attractor is associated with segment $V_6^{R^4L^3} \subset D_L$, which now, after seven iterations, intersects the discontinuity line that leads to a new segment in D_R , whose images approach, as $n \rightarrow \infty$, segments $V_j^{R^4L^3}$.

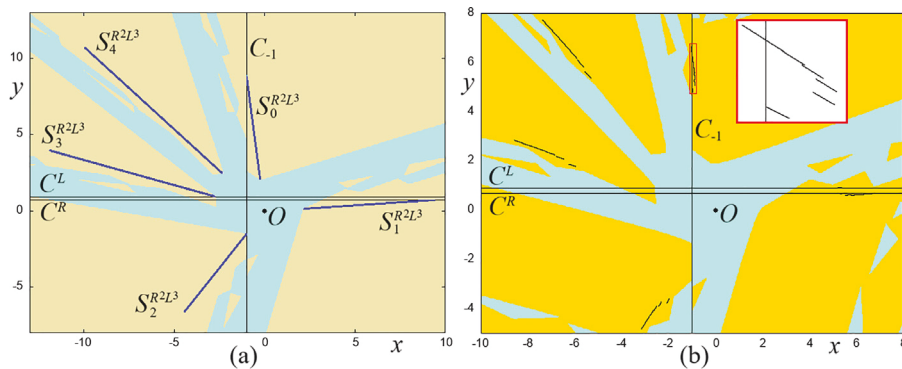


Fig. 10. Phase portrait of map F at $\delta_L = 0.9$, $\delta_R = 0.7$ and (a) $\tau_L = 1.175$, $\tau_R = -0.543$, (b) $\tau_L = 1.19$, $\tau_R = -0.56$. In Fig. 8(a), the related parameter points are marked by c and d , respectively.

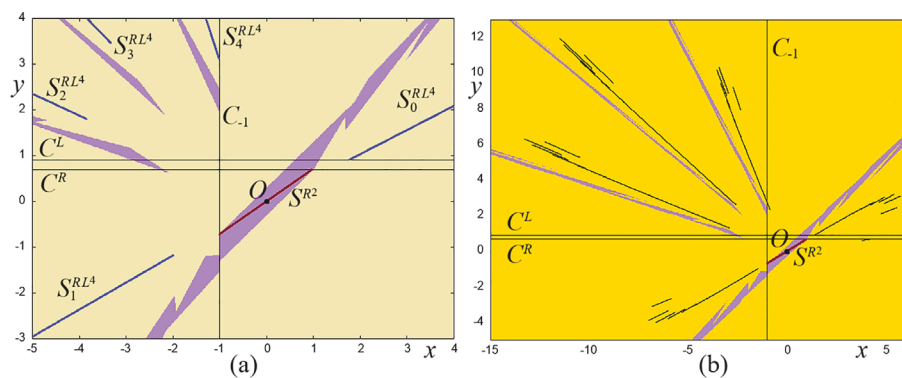


Fig. 11. Phase portrait of map F at $\delta_L = 0.9$, $\delta_R = 0.7$, $\tau_R = -1.7$ and (a) $\tau_L = 1.333153$, (b) $\tau_L = 1.34$. In Fig. 8(a), the related parameter points are marked by e and f , respectively.

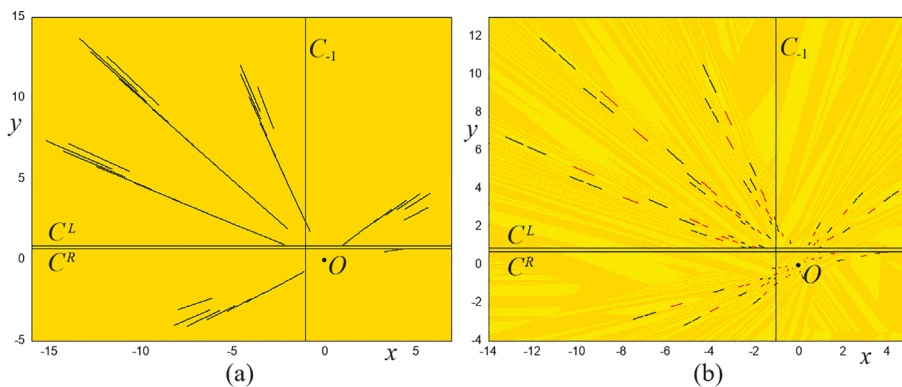


Fig. 12. Phase portrait of map F at $\delta_L = 0.9$, $\delta_R = 0.7$, $\tau_R = -2$ and (a) $\tau_L = 1.365$, (b) $\tau_L = 1.38975$. In Fig. 8(a), the related parameter points are marked by g and h , respectively.

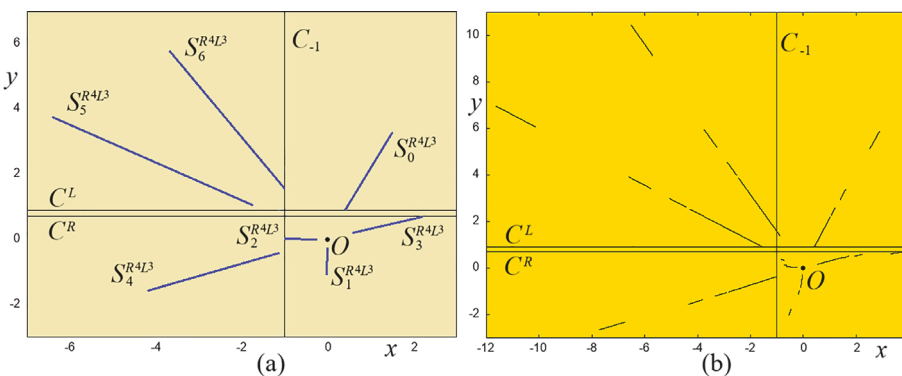


Fig. 13. Phase portrait of map F at $\delta_L = 0.9$, $\delta_R = 0.7$, $\tau_L = 1.16$ and (a) $\tau_R = -2.199528$, (b) $\tau_R = -2.219$. In Fig. 8(a), the related parameter points are marked by i and j , respectively.

The WQA is a limit set of this process. This is not surprising, since we have shown only a few curves in the parameter space associated with $P_\sigma(1) = 0$, but clearly there are infinitely many such curves (although they all have zero Lebesgue measure), and they may be associated with the appearance of WQAs.

5. Conclusions

We considered a 2D piecewise linear map defined by two linear homogeneous functions in two partitions of the phase plane separated by a vertical discontinuity line. This map can have a new type of attractor, which we call a weird quasiperiodic attractor. This type of attractor was first observed in applied models (see, e.g., [20–22]). A WQA may consist of an infinite number of segments, fill some 2D subsets of the phase plane in a (seemingly) dense manner, or have a mixed structure. At first glance, a WQA may be misinterpreted as a chaotic attractor due to its intricate (weird) structure. However, chaos cannot be observed in the considered map since it cannot have hyperbolic cycles. The complex structure of the attractor is primarily related to the discontinuity of the map and the homogeneity of its linear components.

The present paper can be seen as a first step towards understanding the basic properties of WQAs and the mechanisms of their emergence. We described the bifurcation structure of the parameter space of map F and analytically obtained some boundaries of the divergence regions, crossing which a WQA can appear. Special attention was paid to the case when map F has $Z_1 - Z_0 - Z_1$ type of invertibility (an analog of a 1D gap map). To illustrate how a WQA can appear/disappear, we described in detail the corresponding transformations of the phase portrait of the map.

However, many issues remain unresolved. The first set of questions concerns the class of maps that can have WQAs. It is clear that this class includes discontinuous piecewise linear maps of dimension two or higher, defined by two or more homogeneous linear functions, or linear functions with the same fixed point (which can be translated to the origin, leading to the homogeneous case). It is also clear that discontinuity is a necessary element for the existence of a WQA. However, it remains an open problem whether linearity of the map components is a necessary condition or whether the functions defining the map can be nonlinear. There are also questions related to the structure of WQAs. For example, we conjecture that the structure of a WQA cannot be fractal, but it is not easy to prove this statement in the general case. One of the main questions is related to the quasiperiodicity of the trajectory on the attractor. It is well known that for the standard quasiperiodic attractor, such as an attracting closed invariant curve in the case of irrational rotation, the maximum Lyapunov exponent is zero, and there is no sensitive dependence on initial conditions. In the case of WQA, dependence on initial conditions is inevitable due to the discontinuity of the map. Taking two adjacent points belonging to the attractor but on opposite sides of the discontinuity line will necessarily lead to two images of these points that are distant from each other. In [20], this property is called weak sensitive dependence on initial conditions. The question is how weak this dependence is and whether the maximum Lyapunov exponent remains zero. It is also worth mentioning an interesting problem related to the mechanism of the emergence of WQAs and their properties when the map is noninvertible of $Z_1 - Z_2 - Z_1$ type. Some of the aforementioned problems are addressed in our companion paper [25], while others are left for further research.

CRedit authorship contribution statement

Laura Gardini: Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Davide Radi:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Noemi Schmitt:** Writing – review & editing, Writing – original draft,

Formal analysis. **Iryna Sushko:** Writing – review & editing, Writing – original draft, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Frank Westerhoff:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

References

- [1] Beardon AF, Bullett SR, Rippon PJ. Periodic orbits of difference equations. *Proc Roy Soc Edinb* 1995;125:657–74.
- [2] Sivak AG. Periodicity of recursive sequences and the dynamics of homogeneous piecewise linear maps of the plane. *J Difference Equ Appl* 2002;8(6):513–35.
- [3] Lagarias JC, Rains E. Dynamics of a family of piecewise-linear area-preserving plane maps I. Rational rotation numbers. *J Difference Equ Appl* 2005;11:1089–108.
- [4] Lagarias JC, Rains E. Dynamics of a family of piecewise-linear area-preserving plane maps II. Invariant circles. *J Difference Equ Appl* 2005;11:1137–63.
- [5] Lagarias JC, Rains E. Dynamics of a family of piecewise-linear area-preserving plane maps III. Cantor set spectra. *J Difference Equ Appl* 2005;11:1205–24.
- [6] Garcia-Morato LB, Freire Macias E, Ponce Nuñez E, Torres Perat F. Bifurcation patterns in homogeneous area-preserving piecewise-linear map. *Qual Theory Dyn Syst* 2019;18:547–82.
- [7] Roberts JAG, Saito A, Vivaldi F. Critical curves of rotations. *Indag Math* 2024;35:989–1008.
- [8] Simpson DJW. The stability of fixed points on switching manifolds of piecewise-smooth continuous maps. *J Dyn Differ Equ* 2020;32:1527–52.
- [9] Nusse HE, Yorke JA. Border-collision bifurcations including ‘period two to period three’ bifurcation for piecewise smooth systems. *Phys D* 1992;57:39–57.
- [10] Sushko I, Gardini L. Center bifurcation for two-dimensional border-collision normal form. *Int J Bifur Chaos* 2008;18:1029–50.
- [11] di Bernardo M, Budd CJ, Champneys AR, Kowalczyk P. Piecewise-smooth dynamical systems: Theory and applications. *Appl Math Sci* 2008;163. New York: Springer.
- [12] Simpson D, Meiss J. Neimark–Sacker bifurcations in planar, piecewise-smooth, continuous maps. *SIAM J Appl Dyn Syst* 2008;7:795–824.
- [13] Simpson D. Bifurcations in piecewise-smooth continuous systems. *World Sci Ser Nonlinear Sci* 2010;70. Singapore: World Scientific.
- [14] Zhusubaliyev ZhT, Mosekilde E. Bifurcations and chaos in piecewise-smooth dynamical systems, vol. 44, Hackensack, NJ: World Scientific; 2003, World Sci Ser Nonlinear Sci Ser A Monogr Treatises.
- [15] Do Y, Baek HK. Dangerous border-collision bifurcations of a piecewise-smooth map. *Commun Pure Appl Anal* 2006;5(3):493–503.
- [16] Do Y. A mechanism for dangerous border collision bifurcations. *Chaos Solit Fract* 2007;32:352–62.
- [17] Ganguli A, Banerjee S. Dangerous bifurcation at border collision: When does it occur? *Phys Rev E* 2005;71:057202-1–057202-4.

- [18] Gardini L, Avrutin V, Schanz M. Connection between bifurcations on the Poincaré equator and dangerous bifurcations. In: Sharkovsky A, Sushko I, editors. Iteration theory. Grazer Math. Ber.; 2009, p. 53–72.
- [19] Avrutin V, Zhusubaliyev ZhT, Saha A, Banerjee S, Sushko I, Gardini L. Dangerous bifurcations revisited. *Int J Bifur Chaos* 2016;26(14):1630040, (24 pages).
- [20] Kollar LE, Stepan G, Turi J. Dynamics of piecewise linear discontinuous maps. *Int J Bif Chaos* 2004;14:2341–51.
- [21] Gardini L, Radi D, Schmitt N, Sushko I, Westerhoff F. On the limits of informationally efficient stock markets: New insights from a chartist-fundamentalist model. *Int Rev Financ Anal* 2025;105:104436. <http://dx.doi.org/10.1016/j.irfa.2025.104436>.
- [22] Gardini L, Radi D, Schmitt N, Sushko I, Westerhoff F. How risk aversion may shape the dynamics of stock markets: A chartist-fundamentalist approach. In: Working paper. University of Bamberg; 2025.
- [23] Gardini L, Radi D, Schmitt N, Sushko I, Westerhoff F. Dynamics of 1D discontinuous maps with multiple partitions and linear functions having the same fixed point. In: An application to financial market modeling. 2025, <https://arxiv.org/abs/2503.20449>.
- [24] Gardini L, Radi D, Schmitt N, Sushko I, Westerhoff F. Piecewise linear feedback rules and weird quasiperiodic dynamics: Some general observations and two economic applications. In: Working paper. University of Bamberg; 2025.
- [25] Gardini L, Radi D, Schmitt N, Sushko I, Westerhoff F. Abundance of weird quasiperiodic attractors in PWL maps. 2025, <https://arxiv.org/abs/2504.04778>.
- [26] Avrutin V, Gardini L, Sushko I, Tramontana F. Continuous and discontinuous piecewise-smooth one-dimensional maps: Invariant sets and bifurcation structures. Singapore: World Scientific; 2019.
- [27] Sharkovsky A, Kolyada S, Sivak A, Fedorenko V. Dynamics of one-dimensional maps. Dordrecht: Kluwer Academic; 1997.
- [28] Grebogi C, Ott E, Pelikan S, Yorke JA. Strange attractors that are not chaotic. *Phys D* 1984;13:261–8.
- [29] Feudel U, Kuznetsov S, Pikovsky A. Strange nonchaotic attractors: Dynamics between order and chaos in quasiperiodically forced systems. Singapore: World Scientific; 2006.
- [30] Li G, Yue Y, Xie J, Grebogi C. Strange nonchaotic attractors in a nonsmooth dynamical system. *Commun Nonlinear Sci Numer Simul* 2019;78:104858.
- [31] Li G, Yue Y, Xie J, Grebogi C. Multistability in a quasiperiodically forced piecewise smooth dynamical system. *Commun Nonlinear Sci Numer Simul* 2020;84:105165.
- [32] Duan J, Wei Z, Li G, Li D, Grebogi C. Strange nonchaotic attractors in a class of quasiperiodically forced piecewise smooth systems. *Nonlinear Dynam* 2024;112:12565–77. <http://dx.doi.org/10.1007/s11071-024-09678-6>.
- [33] Zhao Y, Zhang Y. Border-collision bifurcation route to strange nonchaotic attractors in the piecewise linear normal form map. *Chaos Solit Fract* 2023;171:113491.
- [34] Zhang Y, Shen Y. A new route to strange nonchaotic attractors in an interval map. *Int J Bifur Chaos* 2020;30(4):2050063.
- [35] Mira C, Gardini L, Barugola A, Cathala JC. Chaotic dynamics in two-dimensional noninvertible maps. Singapore: World Scientific; 1996.
- [36] Sushko I, Gardini L. Degenerate bifurcations and border collisions in piecewise smooth 1D and 2D maps. *Int J Bifur Chaos* 2010;20:2045–70.
- [37] Hassouneh MA, Abed EH, Nusse HE. Robust dangerous border-collision bifurcations in piecewise smooth systems. *Phys Rev Lett* 2004;92:070201.
- [38] Simpson DJW. Border-collision bifurcations in \mathbb{R}^n . *SIAM Rev* 2016;58(2):177–226.