Economic Implications of Limited Attention

Dissertation

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Contents

Chapter 1

Introduction

Empirical and experimental research demonstrates that limited attention plays an important role in economic decisions. For example, research shows that consumers do not fully attend to taxes (Chetty, Looney, and Kroft, 2009), odometer mileage of used cars (Busse, Lacetera, Pope, Silva-Risso, and Syndor, 2013; Englmaier, Schmöller, and Stowasser, 2018), age of used cars (Englmaier, Schmöller, and Stowasser, 2018), and overdrafts (Stango and Zinman, 2014). Furthermore, limited attention can explain why people do not hit their saving targets (Karlan, McConnell, Mullainathan, and Zinman, 2016). In addition, attention is necessary for learning, i.e., simply providing more information does not improve performance if the decision-maker does not pay attention to this information (Hanna, Mullainathan, and Schwartzstein, 2014).

In sum, limited attention influences decision-making and, consequently, also influences the welfare of the decision-maker. For example, a consumer who does not perfectly take into account that a good is taxed, perceives the tax-inclusive price as too low and buys too much of the good. To balance her budget, the consumer has to buy less of another good. Consequently, because of the inattention to the tax, the consumption bundle of the consumer might not be optimal. Similarly, evidence shows that some consumers are inattentive to overdrafts and thus incur overdraft fees (Stango and Zinman, 2014). Such overdraft fees are a large source of income for banks (according to Stango and Zinman, 2014, about \$35 billion in the US). Stango and Zinman (2014) show that reminding people about overdrafts decreases their likelihood of incurring overdrafts. On first sight, educating consumers about overdraft fees should increase welfare. However, especially in a competitive market, educating consumers might change the incentive of firms. That means, firms have an incentive to change their fee structure and thus more attention might actually decrease welfare (see, e.g., Heidhues and Köszegi, 2018, for a discussion of these effects). Models of limited attention are helpful in disentangling the different effects of changing attention of consumers or other decision-makers on welfare.

Additionally, models of limited attention offer explanations for a series of systematic biases in decision-making. For instance, Bordalo, Gennaioli, and Shleifer argue that limited attention can explain the decoy effect (Bordalo, Gennaioli, and Shleifer, 2013), the compromise effect (Bordalo, Gennaioli, and Shleifer, 2013), context-dependent willingness to pay (Bordalo, Gennaioli, and Shleifer, 2013), preference reversals (Bordalo, Gennaioli, and Shleifer, 2012), and the Allais Paradox (Bordalo, Gennaioli, and Shleifer, 2012).

This dissertation adds to the economics literature on limited attention by proposing four models of limited attention. Each chapter consists of a self-contained article. The article “Horizontal product differentiation with limited attentive consumers” is joint work with Marc P. Saur and Markus G. Schlatterer. All other articles are single-authored. To show how my models fit into the literature on limited attention I will, in the following, first give a brief overview of existing strands of the economics literature on limited attention. Subsequently, I briefly introduce each of my articles and discuss the economic implications of limited attention. Lastly, I show how my articles add to the different strands of the literature.[1] Each article also contains a more detailed literature review with a specific focus on the topic of that article.

1.1 Models of limited attention

In theoretical economics, a growing literature concentrates on limited attention. Attention is a broad concept and, consequently, models of limited attention cover a range of topics. However, generally this literature can be divided into two parts: First, models that focus on decision problems. These models provide different ways of modeling limited attention and discuss the implications of limited attention on decision-making. Second, models that focus on the competition of (fully rational and fully informed) firms that compete for limited attentive consumers. I will briefly discuss each strand of the literature in turn.

1.1.1 Decision-making and limited attention

In the literature that focuses on modeling attention allocation, the focus is usually on a decision-maker who faces a choice. Consider a decision-maker who has to choose one option from a set of options $X = \{x_1, x_2, ..., x_n\}$. X describes the choice set, i.e., the set of all available options. For example, the decision-maker has to choose a yogurt. Then, X describes the set of all available yogurts. According to psychological research, cognitive resources are limited (Chun and Wolfe, 2005). Therefore, this decision problem includes too much information to process, because too many yogurts are available (each supermarket usually sells a series of different yogurts and decision-makers can visit different supermarkets as well) and each yogurt is described by a range of information.

[1] The purpose of the following literature survey is to provide an overview of existing strands in the literature, not to provide a list of all existing models. For a detailed overview of decision-making under limited attention see Gabaix (2019) and for a detailed overview of behavioral industrial organization models with limited attention see Heidhues and Köszegi (2018).

For instance, each yogurt is characterized by its price, ingredients, and its size. That is, each option $x_i \in X$ actually consists of different dimensions. Therefore, each $x_i \in X$ is actually a vector $x_i = (x_i^1, x_i^2, ..., x_i^l)$. Because resources are limited, not all information can be processed. The task of attention is to select or highlight information for processing (Chun and Wolfe, 2005). The economics literature models attention at different levels. The literature can be divided into three strands: First, attention is modeled as selecting which options to process (consideration set formation). Second, attention can influence the evaluation of the options. Third, attention is modeled as selecting information about the perceived options for costly processing.

Manzini and Mariotti (2018) and Eliaz and Spiegler (2011a), for instance, describe the formation of consideration sets.[2] The consideration set is a subset of the choice set and includes only those elements that the decision-maker actively considers. For example, the decision-maker might only consider the yogurts of a particular brand instead of all existing yogurts. Manzini and Mariotti (2018) assume that the consideration set is formed stochastically. The options differ in their salience and the higher the salience of an option, the more likely that the option enters the consideration set of the decision-maker. In contrast, Eliaz and Spiegler (2011a) assume that the consideration set is formed deterministically. Decision-makers consider an option if that option has a convincing marketing strategy. Manzini and Mariotti (2018) and Eliaz and Spiegler (2011a) assume that once the consideration set is formed, the decision-maker chooses the best option from the consideration set, i.e., the decision-maker fully perceives all options inside the consideration set and knows which option is better.

However, often attention models take an agnostic view on the consideration set formation and discuss other aspects of attention. Bordalo, Gennaioli, and Shleifer (2013) and Köszegi and Szeidl (2013), for instance, analyze how salience influences the evaluation of the perceived options. Following Bordalo, Gennaioli, and Shleifer (2013), assume each good is described by two dimensions, its price and its quality, i.e., $x_i = (p_i, q_i)$, and that the utility function of the decision-maker is

$$u_i = q_i - p_i. \tag{1.1}$$

Bordalo, Gennaioli, and Shleifer (2013) assume that salience induces an overweighting of salient dimensions and an underweighting of less salient dimensions. Thus the decision-maker does not choose an option from the set X using the utility function given in (1.1),

[2]These articles describe methods of how consideration sets might be formed. However, these models are not pure decision-making models but actually consider the competition between the options and thus also fit into the literature on competition for limited attentive consumers.

but a salience-distorted utility:

$$u_i^s = \begin{cases} \frac{2}{1+\delta} q_i - \frac{2\delta}{1+\delta} p_i & \text{if quality is salient} \\ q_i - p_i & \text{if quality and price are equally salient} \\ \frac{2\delta}{1+\delta} q_i - \frac{2}{1+\delta} p_i & \text{if price is salient,} \end{cases} \tag{1.2}$$

where $0 < \delta \leq 1$. Bordalo, Gennaioli, and Shleifer (2013) define salience as standing out compared to the reference good. The reference good is an imaginary good. The price of this reference good is the average price in the market and the quality of this reference good is the average quality in the market: $\bar{x} = (\sum_i q_i/n, \sum_i p_i/n)$. Generally, Bordalo, Gennaioli, and Shleifer (2013) assume that the salience of a dimension of a good is higher, the larger the difference from the average and the closer both values are to zero. Therefore, Bordalo, Gennaioli, and Shleifer (2013) assume that, in the evaluation of a good, salient dimensions are overweighted and less salient dimensions are underweighted, as described in (1.2). δ describes how much a decision-maker is influenced by salience. $\delta = 1$ describes the rational benchmark, where salience plays no role, but as $\delta \longrightarrow 0$, the decision-maker only considers the most salient dimension. This model can explain decoy and compromise effects and context-dependent willingness to pay, but salience theory is applicable to a series of different decision contexts. For example, by applying salience theory to choice under risk, salience theory offers an explanation for the Allais paradox and preference reversals (Bordalo, Gennaioli, and Shleifer, 2012).

Köszegi and Szeidl (2013) build a similar model, where the evaluation of an option is distorted by over-/underweighting some dimensions. Köszegi and Szeidl (2013) define a "focus-weighted utility" that the decision-maker uses to decide between options

$$\tilde{u}_i = \sum_{j=1}^{l} g_j u_j(x_i^j),$$

where $g_j = g(\max_{x_i \in X} u_j(x_i^j) - \min_{x_i \in X} u_j(x_i^j))$. In words, a consumer weights each dimension j of the option x_i with g_j. This weight is an increasing function of the maximal utility difference in this dimension across all options. Dimensions in which options differ more receive a higher weight and dimensions in which options differ less receive a lower weight. For example, if all goods have the same price, but all goods have different quality, then the quality dimension receives a higher weight than the price dimension in the evaluation of all goods. Köszegi and Szeidl (2013) find that consumers prefer goods where the advantages are concentrated rather than balanced across the dimensions. They use this finding to offer predictions about when present-biased behavior and time-inconsistencies are observed.

The critical difference between Bordalo, Gennaioli, and Shleifer (2013) and Köszegi and Szeidl (2013) is that in Bordalo, Gennaioli, and Shleifer (2013) goods might have different dimensions that are salient, for example, for good 1 price is salient and for good 2 quality is salient, whereas in Köszegi and Szeidl (2013), if quality is salient for one good it is salient for all goods.

Importantly, in Bordalo, Gennaioli, and Shleifer (2013) and Köszegi and Szeidl (2013), the realizations of the goods' dimensions are known to the decision-maker. Attention works as a distortion of the weights in the evaluation of the goods. Other models assume that attention is needed to infer the realizations of different dimensions and that this attention is costly. For example, Gabaix (2014) assumes that the optimal choice a depends on a vector of information $y = (y_1, y_2, ..., y_l)$. For example, a rational decision-maker has to choose a to maximize

$$u(a, y) = -\frac{1}{2}\left(a - \sum_{j=1}^{l} \mu_j y_j\right)^2. \tag{1.3}$$

Then, the rational decision-maker chooses $a^r = \sum_{j=1}^{l} \mu_j y_j$.

However, Gabaix (2014) assumes that decision-makers do not automatically know y. The decision-maker has to pay attention to each dimension separately to infer its realization. That means, instead of perceiving the true y_j, the decision-maker perceives $m_j y_j + (1 - m_j) y_j^d$, where $m_j \in [0, 1]$ describes the attention to dimension j and y_j^d is some default. If $m_j = 1$, the decision-maker is fully attentive to dimension j and perceives y_j perfectly. If $m_j = 0$, the decision-maker is fully inattentive to dimension j and focuses on the default y_j^d. For example, if y_j describes a price, then, y_j^d is, for example, the average price of the good of the last two years. Assume in the following that $y_j^d = 0$ for all j. Thus for a given vector $m = (m_1, m_2, ..., m_l)$, the decision-maker maximizes not (1.3) but instead maximizes

$$u(a, y) = -\frac{1}{2}\left(a - \sum_{j=1}^{l} \mu_j m_j y_j\right)^2.$$

Then, instead of choosing a^r, the decision-maker chooses $a^s = \sum_{j=1}^{l} \mu_j m_j y_j$.

Thus the limited attention to the different dimensions distorts the decision. Consequently, the decision-maker incurs a utility loss by not paying full attention to each dimension. Gabaix (2014) assumes that the decision-maker chooses m optimally. That means, the decision-maker compares the costs of better information, i.e., choosing a higher m_j, with the utility loss when the decision-maker is not fully attentive. Gabaix (2014) shows that his model can, for example, explain money illusion (i.e., why behavior changes when both budget and prices increase by the same percentage). However, in his survey, Gabaix (2019) extends the setup from Gabaix (2014) to capture a range of situations. Gabaix (2019) discusses, for example, how to model inattention to taxes: A decision-maker then perceives the tax-inclusive price of a good as $p + m\tau + (1 - m)0$, where p is the tax-exclusive price, τ is the tax, and m the attention to the tax. But this setup can also be applied to capture inattention to add-on costs, to the future, and to probabilities (Gabaix, 2019).

Another prominent example of costly attention is the *rational inattention* approach developed by Sims (2003). In models of rational inattention, the decision-maker's attention allocation is modeled as choosing the joint distribution between the actual state of the world and the signal the decision-maker receives about this state of the world. More attention, i.e., more precise signals are more costly. In particular, the signal reduces uncertainty and this reduction in uncertainty, which is measured via entropy, is costly. Rational inattention has been applied to a range of topics such as discrete choice (Matějka and McKay, 2015), price-setting (Maćkowiak and Wiederholt, 2009), or investments (van Nieuwerburgh and Veldkamp, 2010).

1.1.2 Competition for consumers with limited attention

In a different strand of the literature on limited attention, the focus lies on the strategic interaction of firms that compete for consumers with limited attention. One way to introduce limited attention into competition models is to exogenously divide the population of consumers into a fraction that is fully attentive and a fraction that is inattentive to, for example, prices (see, e.g., Schultz, 2004; Gabaix and Laibson, 2006). The uninformed consumers then make their consumption decision not with information about prices but, for example, form expectations about the prices (see, e.g., Schultz, 2004). These models allow an analysis of the effects of inattention on the market; for example, whether prices decrease as more consumers become attentive. Schultz (2004), for instance, finds that, in a model of horizontal product differentiation where inattentive consumers are inattentive to prices and product characteristics, if attention increases—i.e., more consumers become informed—prices and product differentiation decrease. Consumers benefit from this but profits decrease. In Schultz (2004), the distinction between attentive and inattentive consumers is exogenous and firms cannot influence the fraction of attentive consumers.

However, firms can often influence whether consumers are attentive or inattentive to some information, for example, by advertising. For instance, Gabaix and Laibson (2006) assume that some consumers are uninformed about the existence of add-ons (such as parking fees at a hotel) but that firms can educate these consumers about the existence of such add-ons in the market. Instead of allowing firms to educate consumers, other approaches allow firms to obfuscate information. Ellison and Wolitzky (2012), for example, allow firms to increase the time consumers need to receive information about the good. In both approaches, the decision of shrouding or unshrouding is independent of the decision of how to design the good, which quality to choose, or which price to set.

Yet, products themselves might also influence whether consumers pay attention or to which aspects of the good consumers pay attention. Thus firms can influence whether consumers are attentive or inattentive by choosing the characteristics of the goods accordingly. For example, Allen and Thisse (1992) and Bachi (2016) assume that consumers only notice price differences if the price differences are sufficiently large. That is, by choosing similar prices, firms keep consumers inattentive to price differences.

1.2 This dissertation

This literature overview demonstrates that *attention* is a broad concept. The way attention is modeled in economics encompasses very different approaches from overweighting of salient information to consideration set formation. This dissertation contains four articles on limited attention. In the different articles, I highlight different aspects of attention.

In the article “Vertical product differentiation and consumers with an absolute perception threshold,” I discuss the strategic interactions of two firms that compete for consumers who perceive quality differences imperfectly. I assume that consumers notice quality differences between goods only if the quality difference is sufficiently large, i.e., if the quality difference exceeds the consumers’ absolute perception thresholds. I demonstrate that the equilibria and welfare depend on the perception threshold. Firms want to differentiate their products such that consumers notice the quality difference. If firms would produce goods with indistinguishable qualities, consumers would buy from the firm with the lower price. Bertrand competition would then lead to prices equal to marginal costs and firms would make zero profits. To avoid this price competition, firms produce goods with distinguishable qualities. However, with increasing thresholds, the necessary quality difference to make the difference noticeable increases and, at some point, this becomes too costly. Therefore, for high perception thresholds, both firms produce goods with zero quality. Overall, consumers and firms prefer limited attention to full attention (although different levels).

In “Competition with Constrained Consumer Perception,” I also analyze competition between firms for consumers with an absolute perception threshold on quality differences. In particular, I analyze how the results change if the price competition between firms, when they produce goods with indistinguishable quality, is reduced. I model the increase in firms’ market power via horizontal product differentiation. I demonstrate that horizontal product differentiation changes firms incentives. Because firms do not have to fear Bertrand competition when they produce goods with indistinguishable quality, the incentive of the firms to unnoticeably undercut the quality of the competitor increases. Therefore, in contrast to “Vertical product differentiation and consumers with an absolute perception threshold,” firms now produce goods with indistinguishable qualities.

The article “Horizontal product differentiation with limited attentive consumers” is joint work with Marc P. Saur and Markus G. Schlatterer. In this article, we model limited attention as an attention radius. The attention radius determines which goods a consumer notices. Thus this article proposes a novel method of consideration set formation. The attention radius works as a spotlight that only highlights the goods that are close to a consumer's preferred version of the good; all other goods are ignored. For example, a consumer whose favorite color is blue, notices blue and turquoise goods but does not notice red or yellow goods. We show that equilibria and welfare depend on the size of the attention radii of the consumers. In particular, full attention is not optimal for consumers. Under limited attention, firms differentiate their goods in a way that is beneficial to consumers. In addition, prices may be lower under limited than under full attention.

The article "Rational Allocation of Attention in Decision-Making" models attention as selecting information for costly processing. Decision-makers have to choose between a default option and an alternative option. Decision-makers know the value of their default, but not of the alternative. In addition, some decisions are more important than others. In a more important decision problem more utility is at stake. However, ex-ante decision-makers do not know how important a decision problem is. Decision-makers can either choose between the two options given expectations or pay attention to the missing information to make an informed decision. I show that three optimal attention strategies remain. I argue that these three attention strategies share similarities with bottom-up attention, top-down attention, and inattention—attention mechanisms discussed in the psychological literature. In addition, I demonstrate that, as a consequence of changing attention allocation, choice reversals occur. In an extension, firms are able to influence the processing costs. I show that the firm uses its influence on the attention allocation as a screening device.

All four articles add to both strands of the economics literature on limited attention (decision-making and behavioral industrial organization models). The article "Horizontal product differentiation with limited attentive consumers" proposes a novel formation criterion for consideration sets. The articles "Vertical product differentiation and consumers with an absolute perception threshold" and "Competition with Constrained Consumer Perception" discuss the implications if consumers perceive only "salient" differences in quality. And the article "Rational Allocation of Attention in Decision-Making" assumes that the decision-maker knows which options are available but has to pay attention to receive information about the options.

In addition, the articles "Vertical product differentiation and consumers with an absolute perception threshold," "Competition with Constrained Consumer Perception," and "Horizontal product differentiation with limited attentive consumers" are competition models where firms compete for consumers with limited attention. The article "Rational Allocation of Attention in Decision-Making," although actually a decision-making model, also analyzes the shrouding/highlighting strategies of firms and thus also adds to the industrial organization literature.

Overall, the articles in this dissertation show that attention markedly influences equilibria and welfare. Thus my dissertation underlines the importance to consider not only decision-makers with perfect information in economic models but to consider also decision-makers who are limited in their ability to pay attention to all available information. Limited attention distorts the choices of decision-makers and can thus explain why decision-makers do not choose the best option. By analyzing the underlying mechanism that leads to the decision, it is possible to carve out how the decision context can be improved to help, for example, consumers to make better decisions.

At the same time, attention is a very broad concept. Modeling different aspects of attention can have different implications. For example, in this dissertation in chapters "Vertical product differentiation and consumers with an absolute perception threshold" and "Competition with Constrained Consumer Perception" welfare is highest under perfect perception whereas in chapter "Horizontal product differentiation with limited attentive consumers" welfare is highest under imperfect perception. This highlights the importance to isolate which aspects of attention are relevant for a given situation. In addition, it is necessary to analyze the implications of different aspects of attention to get a complete picture of which implications limited attention can have. My dissertation contributes to the understanding of the economic implications of limited attention.

Chapter 2

Vertical product differentiation and consumers with an absolute perception threshold

Abstract

I analyze the implications of consumers' imperfect perception on equilibrium outcomes and welfare in a model of vertical product differentiation. Consumers are unable to perceive a quality difference between goods if the quality difference is below their absolute perception threshold. I show that the equilibrium and welfare depend on the absolute perception threshold. Consumers and firms benefit from different levels of imperfect attention. But overall welfare is weakly decreasing with decreasing attention.

KEYWORDS: Limited Attention, Perception, Product Differentiation.

JEL CODES: D43, D91, L13.

2.1 Introduction

In this article, I analyze constrained consumer perception in a model of vertical product differentiation. In particular, I discuss the effects of consumers' imperfect perception of quality differences on offered quality and welfare.

Due to cognitive constraints, consumers often do not perceive small differences between goods but do perceive larger differences. This is the case, for example, when a consumer does not notice that one TV uses 3 kWh more than another TV, but does notice a difference of 40 kWh. The purpose of this article is to construct a model that accounts for such perception imperfections: I introduce a perception threshold. Consumers are unable to perceive quality differences that are below their perception thresholds. Two firms compete for consumers by choosing the quality and the price of the good they offer. To examine the influence of the game structure on equilibrium outcomes, I analyze two game structures: One where firms choose their qualities simultaneously and one where firms choose their qualities sequentially.

The results show that an absolute perception threshold influences equilibria and welfare. With simultaneous quality choice, for low and intermediate thresholds (relative to the marginal costs of quality), two pure-strategy asymmetric subgame-perfect equilibria exist: One equilibrium in which firm 1 produces zero and firm 2 a strictly positive quality and one equilibrium in which firm 2 produces zero and firm 1 a strictly positive quality. With sequential quality choice, the first mover can, effectively, choose the equilibrium and thus chooses the equilibrium that maximizes its profit. Therefore, the sequential game structure gives the first mover an advantage.

For high perception thresholds (relative to the marginal costs of quality), with simultaneous as well as with sequential quality-setting, both firms choose zero qualities. With high perception thresholds, firms need to choose very different qualities for consumers to perceive the quality difference. But then the quality costs of the firm with the higher quality are not covered by the revenues. Therefore, the firm produces goods with lower quality. However, as consumers are then unable to discern a quality difference and buy from the firm with the lower price, Bertrand competition yields zero revenues. Thus for any positive quality, the firm has quality costs but no revenue. Therefore, the firm has no incentive to choose a quality above zero.

In this model, high absolute perception thresholds are actually beneficial to consumers. With high perception thresholds, firms choose goods with indistinguishable quality. And although firms then choose zero quality, this leads to Bertrand competition which reduces prices, from which consumers benefit. In contrast, producer surplus is highest under intermediate thresholds. Then, firms choose qualities that are sufficiently different such that all consumers perceive the difference. Consequently, firms set higher prices which increases producer surplus. Overall, the welfare analysis shows that consumers and producers prefer imperfect to perfect perception (although different levels of imperfect perception). In contrast, because the gain to consumers does not balance the loss of the firms and vice versa, overall welfare is higher under perfect (and close to perfect) perception than under imperfect perception.

That perception of differences is imperfect has mainly been analyzed in decision-making models (see, e.g., Luce, 1956; Rubinstein, 1988). Few articles discuss imperfect perception of differences in competition models. Allen and Thisse (1992) and Bachi (2016) discuss a perception constraint on prices. Both articles show that a perception constraint allows firms to set prices above marginal costs and thus that perception constraints markedly influence equilibrium outcomes. Webb (2014) and Webb (2017) are most closely related to this article. Webb (2014, 2017) analyzes two firms that compete (simultaneously and sequentially) for consumers who are constrained in their perception of quality differences. Webb (2014, 2017) assumes that consumers are unable to distinguish qualities when the relative difference is below the perception threshold. Webb (2017) shows that this has a (weakly) negative impact on consumers. In general, Webb (2014) and Webb (2017) are the first articles to analyze a perception threshold for quality. Thus by demonstrating that introducing a perception threshold for quality influences equilibria and has important implications for (consumers') welfare, Webb opens a fruitful avenue for further research.

In contrast to Webb (2014, 2017), who discusses a relative perception threshold, I discuss the implications of an absolute perception threshold. The basis for using a relative perception threshold is Weber's Law. According to Weber's Law, the just noticeable difference between two stimuli is proportional to the level of the stimuli (Hunt, 2007). However, evidence indicates that Weber's Law does not hold at the extremes, for instance, close to zero (Hunt, 2007). This is relevant in models such as the model presented in this article, because under perfect perception, firms have an incentive to (vertically) differentiate their goods in quality. One firm chooses zero quality and the other firm a strictly positive quality. Then, the relative quality difference is infinite and thus always larger than any relative perception threshold. Thus a relative perception threshold has no effect on the equilibrium and no effect on welfare.[1] But, perception is also imperfect close to zero. For example, people do not notice a difference between zero grams and 1 gram of sugar inside a piece of cake. This problem illustrates why the relative perception threshold is unsuitable for extremes, such as values close to zero (Hunt, 2007). Therefore, I focus on the effects of an absolute perception threshold. In contrast to Webb (2017), I show that a high absolute perception threshold is actually beneficial to consumers. In addition, I introduce a marginal cost parameter for quality and show that these costs of quality

[1] As the model setup slightly differs, in Webb (2017) both firms set a strictly positive quality (close to zero) such that the relative difference is large, but not infinite. Consequently, a relative threshold influences the equilibria for high values.

markedly influence the equilibrium outcomes with an absolute perception threshold.

The remainder of this article is structured as follows: Section 2.2 describes the model and introduces the perception threshold. Section 2.3 derives the equilibria and discusses the welfare implications. Section 2.4 summarizes the results and concludes.

2.2 Model

Two firms, firm 1 and firm 2, compete for a unit mass of consumers by choosing the quality and the price of the good they produce. Both firms have identical costs of quality $C(q_i) = cq_i^2$ with $c > 0$, where q_i is the quality of the good of firm $i \in \{1, 2\}$. All other production costs are set to zero. That means, firms have fixed costs for providing quality, for example, firms have costs for research and development. Each firm maximizes its profit

$$\Pi_i(p_i, p_j, q_i, q_j) = x_i p_i - cq_i^2,$$

where x_i is the demand and p_i is the price of firm i.

Each consumer buys exactly one unit of the good. The utility from buying the good from firm $i \in \{1, 2\}$ is

$$u_\theta(i) = v + \theta q_i - p_i, \tag{2.1}$$

where v is the utility consumers receive from consuming a good with zero quality, q_i is the quality, and p_i the price of the good of firm $i \in \{1, 2\}$. θ is a consumer-specific taste parameter that measures the consumer's marginal willingness to pay for quality that is individually and independently drawn from a uniform distribution on $[0, 1]$. I assume that v is large enough such that consumers always buy one unit of the good. For simplicity, denote by h the firm that produces the higher quality, i.e., q_h at price p_h, and by l the firm that produces the lower quality, i.e., q_l at price p_l. Whether firm 1 or firm 2 is the high-quality firm depends on the firms' quality choices.

Consumers decide between buying from firm 1 and firm 2 dependent on the qualities and prices of the goods. However, consumers will only notice a quality difference between the goods if the absolute quality difference exceeds the perception threshold $\tau \in [0, \infty)$, i.e., if $|q_1 - q_2| \geq \tau$, and will assume that the qualities are the same if the difference is below the perception threshold, i.e., if $|q_1 - q_2| < \tau$. Then, the perceived quality $\hat{q}_i$ is

$$\hat{q}_i = \begin{cases} q_i & \text{if } |q_1 - q_2| \geq \tau \\ q(q_1, q_2) & \text{if } |q_1 - q_2| < \tau, \end{cases}$$

where $q(q_1, q_2) > 0$. If the quality difference is below the perception threshold, i.e., if

$|q_1 - q_2| < \tau$, consumers perceive the quality of both firms as identical: $q(q_1, q_2)$. For example, $q(q_1, q_2)$ could be the mean of q_1 and q_2. The actual form of $q(q_1, q_2)$ is not relevant for my results, important is only that consumers perceive q_1 and q_2 as identical. Furthermore, I assume that all consumers have the same perception threshold. An absolute perception threshold of $\tau = 0$ captures the benchmark of perfect perception.

A consumer prefers to buy from firm 1 if her perceived utility from buying the good from firm 1 is higher than her perceived utility from buying the good from firm 2. If the quality difference exceeds the perception threshold, her perceived utility is her true utility. If the quality difference is smaller than the perception threshold, her perceived utility is distorted in the quality. That is, the perceived utility represents the decision utility, whereas the utility as specified in (2.1) represents the experience utility. A consumer thus prefers to buy from firm 1 if $\hat{u}_1 \geq \hat{u}_2 \Leftrightarrow v + \theta \hat{q}_1 - p_1 \geq v + \theta \hat{q}_2 - p_2$. Thus if the quality difference is below the perception threshold, i.e., $|q_1 - q_2| < \tau$, consumers think that the goods only differ in prices and buy from the firm with the lower price. Then, the demand for the good of firm $i \in \{1, 2\}$ depends only on the prices of firm i, p_i, and its competitor, p_j,

$$x_i^{\text{inattentive}}(p_i, p_j) = \begin{cases} 1 & \text{if } p_i < p_j \\ \frac{1}{2} & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j. \end{cases}$$

However, if the quality difference exceeds the perception threshold, i.e., $|q_1 - q_2| \geq \tau$, consumers perceive the qualities perfectly: $\hat{q}_1 = q_1$ and $\hat{q}_2 = q_2$. A consumer then buys from the high-quality firm if $\hat{u}_h \geq \hat{u}_l \Leftrightarrow v + \theta q_h - p_h \geq v + \theta q_l - p_l$. Let $\hat{\theta}$ denote the indifferent consumer:

$$\hat{\theta} \equiv \frac{p_h - p_l}{q_h - q_l}.$$

Then, all consumers with $\theta \leq \hat{\theta}$ buy from the low-quality firm l and all consumers with $\theta > \hat{\theta}$ buy from the high-quality firm h. The demand for the good of the low-quality firm l and the demand for the good of the high-quality firm h are thus[2]

$$x_l^{\text{attentive}}(p_h, p_l, q_h, q_l) = \hat{\theta} \qquad \text{and} \qquad x_h^{\text{attentive}}(p_h, p_l, q_h, q_l) = 1 - \hat{\theta}.$$

[2] In equilibrium, both firms have a positive demand. If both firms set their prices such that one firm captures all consumers, the other firm makes zero revenue. Then, that firm can increase its revenue by decreasing its price.

The profit of firm i is then

$$\Pi_i(p_i, p_j, q_i, q_j) = \begin{cases} -cq_i^2 + p_i x_i^{\text{inattentive}} & \text{if } |q_1 - q_2| < \tau \\ -cq_i^2 + p_i x_i^{\text{attentive}} & \text{if } |q_1 - q_2| \geq \tau. \end{cases}$$

Thus firm i's profit depends on whether the firms choose qualities such that the quality difference exceeds the consumers' perception threshold and on whether the firm is the high- or low-quality firm.

2.3 Results

Firms compete in qualities and prices for consumers: Firms first choose their qualities and, then, after observing the quality of their competitor, independently and simultaneously choose prices. I analyze two different game structures: A game where firms choose their qualities simultaneously (two-stage game) and a game where firm 1 chooses its quality first and firm 2 observes the quality of firm 1 before choosing its own quality (three-stage game). I solve for the subgame-perfect equilibria in pure strategies by backward induction. As the two games differ only in the quality-setting stage(s), the analysis of the price-setting subgame—after the qualities are chosen—is the same for both games.

In the price-setting stage, firms simultaneously and independently set prices given the qualities from the previous stage(s). If the quality difference is below the perception threshold, consumers assume that the goods differ only in prices and buy from the firm with the lowest price. Then, Bertrand competition leads to prices equal to marginal costs: $p_1^* = p_2^* = 0$. If the quality difference exceeds the perception threshold, i.e., $q_h - q_l \geq \tau$, consumers notice that the goods differ in quality. Then, the profits of firm 1 and firm 2 depend on whether they are the firm with the higher (h) or lower (l) quality. All consumers with $\theta \geq \hat{\theta}$ buy from the firm with the higher quality, all others buy from the firm with the lower quality. In equilibrium, firms choose prices such that both firms receive some demand. If firms would choose prices such that one firm receives zero demand, this firm could always increase its profit by reducing its price. The profits of the high- and low-quality firm are then

$$\Pi_h(p_h, p_l, q_h, q_l) = (1 - \hat{\theta})p_h - cq_h^2 = \left(1 - \frac{p_h - p_l}{q_h - q_l}\right) p_h - cq_h^2$$
$$\Pi_l(p_h, p_l, q_h, q_l) = \hat{\theta} p_l - cq_l^2 = \frac{p_h - p_l}{q_h - q_l} p_l - cq_l^2.$$

The first order conditions yield the following best replies:

$$p_h^*(p_l) = \frac{1}{2}\left(p_l + q_h - q_l\right) \qquad \text{and} \qquad p_l^*(p_h) = \frac{p_h}{2}.$$

Thus the equilibrium prices are:

$$p_h^* = \frac{2}{3}(q_h - q_l) \qquad \text{and} \qquad p_l^* = \frac{1}{3}(q_h - q_l).$$

Whether firm 1 or firm 2 is the high-quality firm depends on the firms' choice of quality in the first stage(s).

In the quality-setting stage(s), firms maximize profits by choosing their qualities optimally. Three cases exist. First, firm $i \in \{1, 2\}$ produces a higher quality than firm $j \in \{1, 2\}$ (with $j \neq i$) and consumers notice the quality difference. Second, the firms choose qualities such that the quality difference is below the perception threshold. Then, all consumers assume that the goods have the same quality. In the subsequent price-setting stage, firms set prices equal to marginal costs, i.e., revenues are zero. Third, firm i produces a lower quality than firm j and consumers notice the quality difference. Overall then, the profit of firm i is

$$\Pi_i(q_i, q_j) = -cq_i^2 + \begin{cases} \frac{4}{9}(q_i - q_j) & \text{if } q_i \geq q_j + \tau \\ 0 & \text{if } q_j - \tau < q_i < q_j + \tau \\ \frac{1}{9}(q_j - q_i) & \text{if } q_i \leq q_j - \tau. \end{cases} \tag{2.2}$$

In the sequential game, firms maximize their profits given by (2.2) sequentially and, in the simultaneous game, firms maximize their profits given by (2.2) simultaneously by choosing their qualities. Proposition 1 summarizes the respective subgame-perfect equilibria in pure strategies.

Proposition 1 *Consider the model with an absolute perception threshold and* $i \in \{1, 2\}$, $j \in \{1, 2\}$, *and* $i \neq j$.

(1) If the firms choose quality simultaneously, the subgame-perfect equilibria in pure strategies are

- *if* $\tau \leq \frac{2}{9c} : q_i^* = \frac{2}{9c}$ *and* $q_j^* = 0$ *with* $p_i^* = \frac{4}{27c}$ *and* $p_j^* = \frac{2}{27c}$.
- *if* $\frac{2}{9c} < \tau \leq \frac{4}{9c} : q_i^* = \tau$ *and* $q_j^* = 0$ *with* $p_i^* = \frac{2}{3}\tau$ *and* $p_j^* = \frac{1}{3}\tau$.
- *if* $\frac{4}{9c} < \tau : q_i^* = q_j^* = 0$ *with* $p_i^* = p_j^* = 0$.

(2) If the firms choose quality sequentially, the unique subgame-perfect equilibrium in pure strategies is

- *if* $\tau \leq \frac{2}{9c} : q_1^* = \frac{2}{9c}$ *and* $q_2^* = 0$ *with* $p_1^* = \frac{4}{27c}$ *and* $p_2^* = \frac{2}{27c}$.
- *if* $\frac{2}{9c} < \tau \leq \frac{3}{9c} : q_1^* = \tau$ *and* $q_2^* = 0$ *with* $p_1^* = \frac{2}{3}\tau$ *and* $p_2^* = \frac{1}{3}\tau$.
- *if* $\frac{3}{9c} < \tau \leq \frac{4}{9c} : q_1^* = 0$ *and* $q_2^* = \tau$ *with* $p_1^* = \frac{1}{3}\tau$ *and* $p_2^* = \frac{2}{3}\tau$.
- *if* $\frac{4}{9c} < \tau : q_1^* = q_2^* = 0$ *with* $p_1^* = p_2^* = 0$.

The proof is in the appendix. Proposition 1 shows that in the benchmark of perfect perception, i.e., $\tau = 0$, the model gives rise to product differentiation, where one firm sets zero and the other firm a strictly positive quality. The level of quality depends on the quality costs. With increasing costs, the quality difference decreases. The equilibrium depends on the perception threshold. Figure 2.1 illustrates the equilibrium qualities for the simultaneous move game. Assume a given quality cost. If the absolute threshold is below the benchmark equilibrium quality difference (i.e., $\tau \leq 2/(9c)$), the benchmark equilibrium results. However, as the threshold increases, consumers are unable to discern the quality difference. Then, the firm with the higher quality has costs of quality but consumers are not willing to pay for the higher quality. Therefore, the firm with the higher quality has an incentive to increase its quality such that the quality difference is just noticeable, i.e., to $q_h = \tau$. With increasing τ, the costs of quality increase and for $\tau > 4/(9c)$, the costs of quality exceed the revenues. Then, the firm prefers to set a quality that is indistinguishable from its competitor's and reduces its costs: Both firms set zero quality. However, the cutoffs for the three equilibria also depend on the quality costs. For a given τ, with low costs the benchmark equilibrium occurs. With intermediate costs, the firms differentiate so that the quality difference is just noticeable. With high costs, no firm has an incentive to produce positive quality. Thus which equilibrium occurs depends on the interplay of the quality costs and the threshold. An equilibrium where both firms choose zero quality occurs, when the quality costs to make the product distinguishable are too high. That implies that either a high threshold forces a high quality which is costly even if the unit costs are low or high unit costs make even low qualities costly.

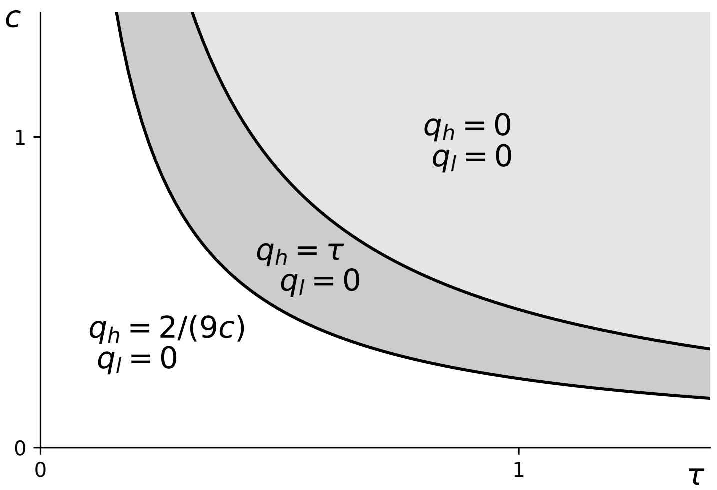

Figure 2.1: Equilibrium qualities with an absolute perception threshold for c and τ. q_h (q_l) is the quality that the firm with the higher (lower) quality chooses in equilibrium.

In the simultaneous game, for a sufficiently low threshold, two asymmetric subgame-perfect equilibria exist: One in which firm 1 and one in which firm 2 produces the strictly positive quality. The other firm always produces zero quality. In the sequential game, firm 1 as the first mover can influence which of these two equilibria exists to maximize its profit. Thus firm 1 has a first-mover advantage. For $\tau \leq 3/(9c)$, the firm that chooses the higher quality makes higher profits and, for $\tau > 3/(9c)$, the firm that chooses the lower quality makes higher profits. Thus firm 1 chooses to be the high-quality firm for $\tau \leq 3/(9c)$ and chooses to be the low-quality firm for $3/(9c) < \tau \leq 4/(9c)$. For $\tau > 4/(9c)$, both firms set the same, zero, quality.

The absolute threshold influences the equilibria and thus also the welfare. Figure 2.2 illustrates the consumer surplus, producer surplus, and welfare as a function of τ. Consumer surplus, producer surplus, and welfare are the same when firms choose quality simultaneously and when they choose quality sequentially. The only difference between simultaneous and sequential quality-setting is that in the simultaneous quality-setting game another equilibrium exists in which the roles of the firms are reversed.

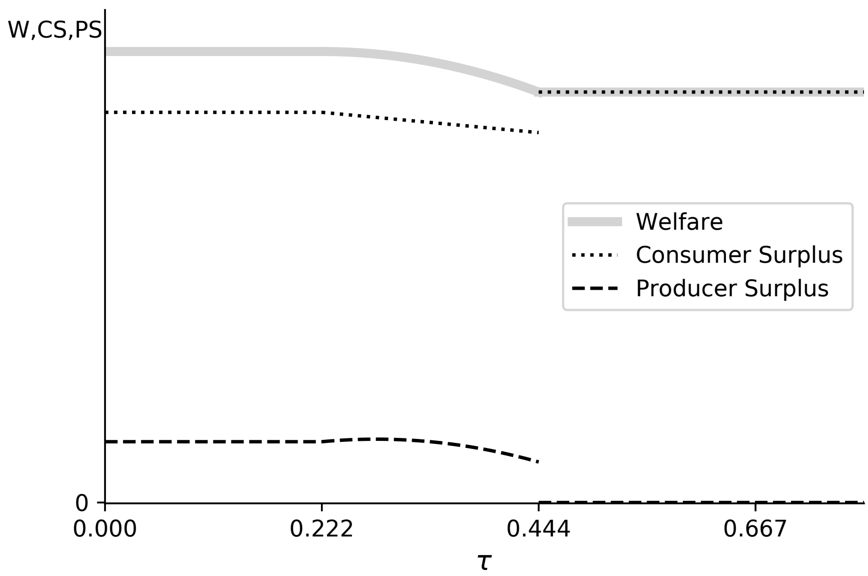

Figure 2.2: Welfare (W, solid), consumer surplus (CS, dotted), and producer surplus (PS, dashed) as a function of the absolute perception threshold τ for $c = 1$ and $v = 1/2$.

Consumer surplus, producer surplus, and welfare are constant for low perception thresholds as the equilibrium is independent of the threshold. For a higher perception threshold, the threshold influences consumer surplus, producer surplus, and welfare. Producer surplus is higher under imperfect than under perfect perception. With intermediate thresholds, one firm produces higher quality than under perfect perception. This allows firms to set higher prices which increases producer surplus. Figure 2.3 illustrates the profits of the high- and the low-quality firm in equilibrium dependent on the perception threshold τ. For low thresholds, the benchmark equilibrium occurs. Thus the profits and the producer surplus are constant. For intermediate thresholds, i.e., $2/(9c) \leq \tau \leq 4/(9c)$, the high-quality firm chooses $q_h = \tau$ and the low-quality firm chooses $q_l = 0$. However, both prices depend on the quality difference and thus both firms set a strictly positive price that increases in τ. As the low quality firm has zero costs but increasing prices and constant demand, its profit increases with τ. In other words, the low-quality firm benefits from the quality of its competitor without incurring the same costs. In contrast, the profit of the high-quality firm strictly decreases in τ. Although, the firm also charges higher prices with higher τ and thus has increasing revenue, the firm also has increasing (quadratic) costs. The additional revenue is absorbed by the additional costs. Thus the profit decreases. For $0 < \tau < 3/(9c)$, the high-quality firm receives a higher profit than the low-quality firm. For $3/(9c) < \tau < 4/(9c)$, the low-quality firm receives a higher profit. For $\tau > 4/(9c)$, both firms receive the same profit. Overall therefore, the high-quality firm prefers better and the low-quality firm prefers worse perception of the consumers (but below $\tau \leq 4/(9c)$). For high thresholds, both firms set zero quality and thus make zero profits. In sum, the producer surplus (weakly) increases until it reaches its maximum at $\tau = 5/(18c)$ and (weakly) decreases thereafter.

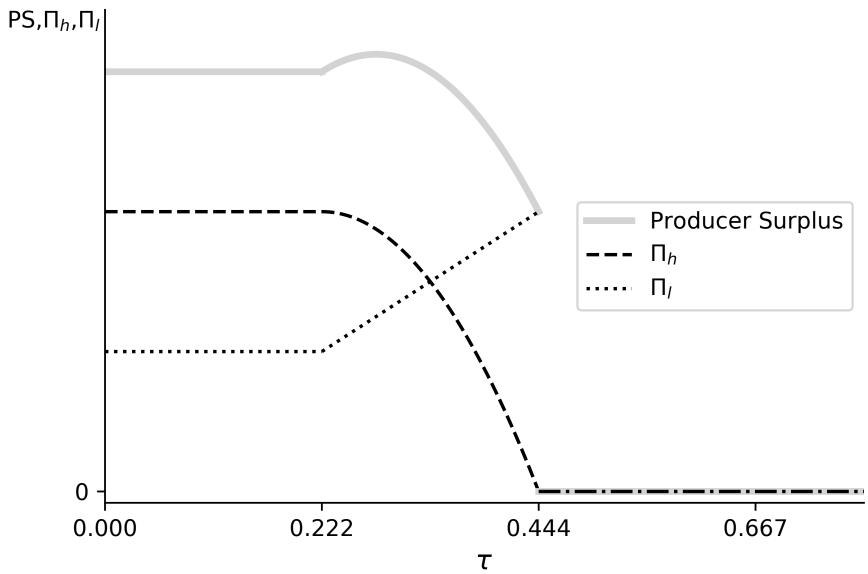

Figure 2.3: Producer Surplus (PS, solid), profit of high-quality firm (Π_h, dashed), and profit of low-quality firm (Π_l, dotted) as a function of τ for $c = 1$.

Consumers also benefit from imperfect perception. Figure 2.4 illustrates the consumer surplus and the surplus of consumers who buy from the low- and the high-quality firm respectively. For $\tau \leq 4/(9c)$, firms choose different qualities such that 2/3 of the consumers buy from the high-quality firm and 1/3 from the low-quality firm. For $\tau > 4/(9c)$, both firms choose the same quality ($q_1 = q_2 = 0$) and the same price such that consumers randomize. Overall, the consumer surplus is driven by the consumers who buy from the low-quality firm. As the low-quality firm always produces zero quality in equilibrium, but for $\tau \leq 4/(9c)$ charges a strictly positive price, its consumers on average have a lower utility than the consumers who buy from the high-quality firm. As the price depends on the quality difference, if the difference increases, the price also increases. Thus for $2/(9c) \leq \tau \leq 4/(9c)$, the utility of the consumers buying from the low-quality firm decreases. High perception thresholds, i.e., $\tau > 4/(9c)$, induce both firms to set zero quality which leads to Bertrand competition. Thus with high perception thresholds, firms are unable to extract high prices from consumers: Both firms set prices equal to marginal costs, i.e., prices equal to zero. In particular, the firm that used to produce zero quality but did charge positive prices, has to set prices equal to zero, which is beneficial to its consumers. In sum, consumers who buy from the high-quality firm receive an average utility that is independent of the threshold.[3] But consumers who buy from the low-quality firm benefit from high perception thresholds.

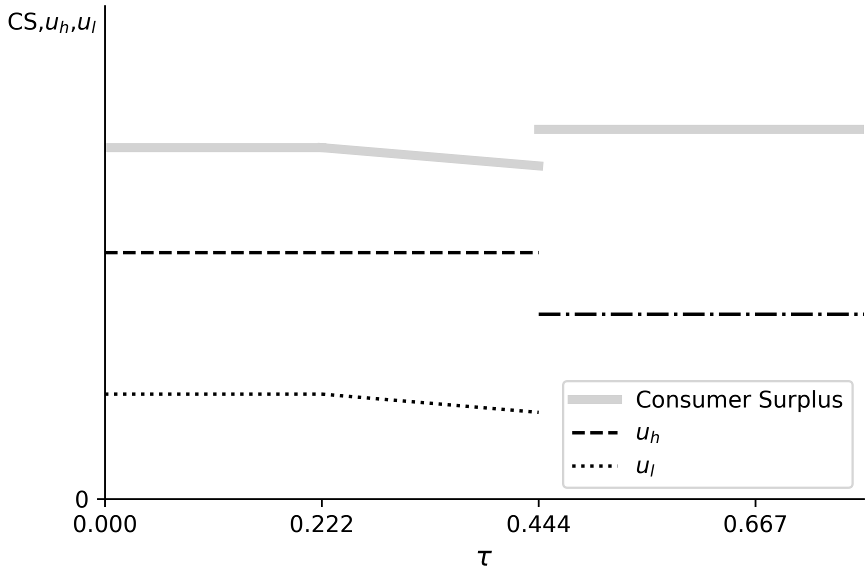

Figure 2.4: Consumer Surplus (CS, solid) and surplus of consumers buying the high- (u_h, dashed), low-quality (u_l, dotted) good as a function of the absolute perception threshold τ for $c = 1$ and $v = 1/2$.

In contrast, welfare is higher under (close to) perfect perception, as long as the benchmark equilibrium occurs, than under imperfect perception. Under intermediate thresholds, the benefits to firms do not outweigh the costs to consumers, and under high thresholds, the benefits to consumers do not outweigh the costs to firms. Proposition 2 summarizes the results of the comparative statics of the consumer surplus, producer surplus, and welfare with respect to the perception threshold τ.

[3]The drop in u_h at $\tau = 4/(9c)$ results from less people purchasing the good from the high-quality firm as both firms produce the same quality of zero.

Proposition 2

The consumer surplus reaches its maximum for any $\tau \in (4/(9c), \infty)$*, the producer surplus reaches its maximum at* $\tau = 5/(18c)$*, and the welfare reaches its maximum for any* $\tau \in [0, 2/(9c)]$*.*

The proof is in the appendix.

2.4 Discussion and conclusion

In this article, I introduce an absolute perception threshold into a model of vertical product differentiation. I show that an absolute perception threshold influences equilibria and welfare. In contrast, in this model a relative perception threshold has no effect on equilibria and welfare. In the benchmark of perfect perception one firm sets zero quality and the other a strictly positive quality. Thus the relative quality difference is infinite. Then, even if consumers have a relative perception threshold, the relative quality difference in the benchmark is larger than any relative perception threshold. That means, a relative perception threshold cannot influence the benchmark result; neither with simultaneous nor with sequential quality-setting.[4]

With an absolute perception threshold, the welfare analysis shows that whether firms choose quality simultaneously or sequentially has no effect on consumer surplus, producer surplus, and welfare. Overall welfare is higher under perfect (and close to perfect) perception than under imperfect perception. Nevertheless, consumer and producer surplus are higher under imperfect than under perfect perception.

In this model, consumers actually benefit from high thresholds, because firms then choose zero qualities. Such situations arise, for example, when the quality of goods is determined by information that is difficult to evaluate, such as when food quality is determined by ingredients that are chemical abbreviations. A lay person may then be unable to discern which good has the higher quality. Or, generally, shifts in thresholds may occur that change the equilibrium outcomes. For instance, as people become more likely to buy goods than to produce goods themselves, identifying the quality of bought goods becomes more difficult: When people bake less frequently, they are less aware of ingredients and what makes up high quality pastries. In other words, they are less able to distinguish between high and low quality pastries; their perception threshold increases. In consequence, if consumers cannot perceive whether a firm produces higher quality, they

[4] Webb (2017) also finds product differentiation in the benchmark. But, as his model setup differs in that consumers do not always buy, both firms set a strictly positive quality. Thus the relative difference is large, but not infinite. Consequently, in Webb (2017), the benchmark result exists for a large range of thresholds. But, if the threshold is larger than the benchmark equilibrium difference, firms just increase the quality difference. Thus compared to Webb (2017), this article exacerbates the problems of the relative perception threshold.

are unwilling to pay for the higher quality and firms have no incentive to produce higher quality than their competitors.

To keep the model tractable, I make a number of assumptions that limit the scope of the analysis. First of all, I assume that the utility from buying a good is large enough such that consumers always buy in equilibrium. This assumption actually helps to illustrate the difference between a relative and an absolute perception threshold more clearly: In this model, a relative perception threshold has no effect on the subgame-perefect equilibrium qualities, whereas an absolute perception threshold has an effect on the subgame-perefect equilibrium qualities. Nevertheless, this assumption also influences the qualities in the subgame-perefect equilibrium and, consequently, influences the consumer surplus, producer surplus, and welfare. Without such a utility v, consumers would only buy if the utility of consuming the good exceeds the utility of the outside option (see, e.g., Webb, 2017). I leave the detailed analysis to future research. Yet, this would probably induce the firm with the lower quality to increase its quality in equilibrium but would not influence the general trend that the quality difference first weakly increases until, for sufficiently high thresholds, both firms produce goods with zero quality. Then, changing this assumption should also increase consumer surplus and decrease producer surplus.

To keep the model tractable, I, furthermore, assume that all consumers have the same perception threshold. From a psychological perspective it seems reasonable to assume that perception thresholds differ among consumers, for example, with respect to age. Moreover, I assume that the threshold represents an abrupt cutoff such that at the threshold all consumers abruptly switch from noticing the difference to not noticing the difference. Changing these assumptions might influence the subgame-perfect equilibrium qualities. I leave these questions to future research. Moreover, I assume a cost function where the costs of quality are independent of the quantity a firm sells. This captures, for example, situations where firms invest in research and development to increase the quality of the goods such as technological advances that can be applied to all goods a firm sells at no extra costs.

Appendix

Proof of Proposition 1

The analysis of the quality-setting of firm 2 is the same for the simultaneous and the sequential game.

The profit of firm 2 is

$$\Pi_2(q_1, q_2) = -cq_2^2 + \begin{cases} \frac{1}{9}(q_1 - q_2) & \text{if } q_2 \leq q_1 - \tau \\ 0 & \text{if } q_1 - \tau < q_2 < q_1 + \tau \\ \frac{4}{9}(q_2 - q_1) & \text{if } q_2 \geq q_1 + \tau. \end{cases}$$

For $q_2 < q_1 + \tau$, the profit function is strictly decreasing in q_2: For $q_2 \leq q_1 - \tau$ and $q_1 - \tau < q_2 < q_1 + \tau$, $\partial \Pi_2(q_1, q_2)/\partial q_2 < 0$. In addition, at the discontinuity at $q_2 = q_1 - \tau$ the profit strictly decreases by $\tau/9$. Thus the only candidate for the best reply of firm 2 in the range $[0, q_1 + \tau)$ is $q_2 = 0$.

At $q_2 = q_1 + \tau$, the profit jumps up by $4\tau/9$. For $q_2 > q_1 + \tau$, $\partial \Pi_2(q_1, q_2)/\partial q_2 = -2cq_2 + 4/9$. Thus the profit on $q_2 \geq q_1 + \tau$ is strictly increasing for all $q_2 < 2/(9c)$ and strictly decreasing for all $q_2 > 2/(9c)$. Thus if $2/(9c) \in [q_1 + \tau, \infty)$, i.e., $q_1 \leq 2/(9c) - \tau$, $q_2 = 2/(9c)$ is the candidate for best reply on $q_2 \geq q_1 + \tau$, otherwise the boundary $q_2 = q_1 + \tau$ is the candidate for best reply.

Overall then, two candidates for the best reply exist: $q_2 = 0$ and

$$q_2' = \begin{cases} \frac{2}{9c} & \text{if } q_1 \leq \frac{2}{9c} - \tau \\ q_1 + \tau & \text{if } q_1 > \frac{2}{9c} - \tau. \end{cases}$$

With $q_2 = 0$ firm 2 makes a profit of

$$\Pi_2(q_1, q_2 = 0) = \begin{cases} 0 & \text{if } q_1 < \tau \\ \frac{1}{9} q_1 & \text{if } q_1 \geq \tau. \end{cases}$$

With $q_2 = q_2'$ firm 2 makes a profit of

$$\Pi_2(q_1, q_2 = q_2') = \begin{cases} \frac{4}{9}\left(\frac{1}{9c} - q_1\right) & \text{if } q_1 \leq \frac{2}{9c} - \tau \\ \frac{4}{9}\tau - c(q_1 + \tau)^2 & \text{if } q_1 > \frac{2}{9c} - \tau. \end{cases}$$

The best reply is the quality that yields the highest profit:

(i) $q_2 = \frac{2}{9c}$ is the best reply if $q_1 \le 2/(9c) - \tau$ and $\Pi_2(q_1, q_2 = 2/(9c)) \ge \Pi_2(q_1, q_2 = 0)$ which is equivalent to

$$q_1 \le \frac{2}{9c} - \tau \qquad \text{and} \qquad \begin{cases} q_1 \le \frac{1}{9c} & \text{if } q_1 < \tau \\ q_1 \le \frac{4}{45c} & \text{if } q_1 \ge \tau. \end{cases}$$

That is, $q_2 = 2/(9c)$ is the best reply if

- either $q_1 \le 2/(9c) - \tau$, $q_1 < \tau$, and $q_1 \le 1/(9c)$, i.e., if $\tau \le 1/(9c)$ the binding condition is $q_1 < \tau$, and if $\tau > 1/(9c)$ the binding condition is $q_1 \le 2/(9c) - \tau$
- or $q_1 \le 2/(9c) - \tau$, $q_1 \ge \tau$, and $q_1 \le 4/(45c)$, i.e., if $\tau \le q_1 \le 4/(45c)$ which implies that $\tau \le 4/(45c)$. In addition, if $\tau \le 4/(45c)$, $q_1 \le 2/(9c) - \tau$ is always fulfilled because then $4/(45c) < 2/(9c) - \tau$.

$q_2 = 2/(9c)$ is thus the best reply whenever

- $\tau \le \frac{4}{45c}$ and $q_1 \le \frac{4}{45c}$
- $\frac{4}{45c} < \tau \le \frac{1}{9c}$ and $q_1 < \tau$
- $\tau > \frac{1}{9c}$ and $q_1 \le \frac{2}{9c} - \tau$.

(ii) $q_2 = q_1 + \tau$ is the best reply if $q_1 > 2/(9c) - \tau$ and $\Pi_2(q_1, q_2 = q_1 + \tau) \ge \Pi_2(q_1, q_2 = 0)$ which is equivalent to

$$q_1 > \frac{2}{9c} - \tau \qquad \text{and} \qquad \begin{cases} q_1 \le \sqrt{\frac{4\tau}{9c}} - \tau & \text{if } q_1 < \tau \\ q_1 \le -\tau - \frac{1}{18c} + \sqrt{\frac{5\tau}{9c} + \frac{1}{(18c)^2}} & \text{if } q_1 \ge \tau. \end{cases}$$

That is, $q_2 = q_1 + \tau$ is the best reply if

- $q_1 > 2/(9c) - \tau$ and $q_1 < \tau$ and $q_1 \le \sqrt{4\tau/(9c)} - \tau$, i.e., if $2/(9c) - \tau < q_1 \le \sqrt{4\tau/(9c)} - \tau$ which implies $\tau > 1/(9c)$. In addition, if $\tau > 1/(9c)$, then $\sqrt{\tau} > 1/(3\sqrt{c})$ and $2\tau > 2\sqrt{\tau}/(3\sqrt{c})$. Thus $\tau > \sqrt{4\tau/(9c)} - \tau$. Consequently, $q_1 < \tau$ is always fulfilled.
- $q_1 > 2/(9c) - \tau$ and $q_1 \ge \tau$ and $q_1 \le -\tau - 1/(18c) + \sqrt{5\tau/(9c) + 1/(18c)^2}$. These conditions are never fulfilled at the same time, because if $\tau \le 1/(9c)$, then $2/(9c) - \tau < q_1 \le -\tau - 1/(18c) + \sqrt{5\tau/(9c) + 1/(18c)^2}$ exists only if $\tau > 2/(15c)$. And if $\tau > 1/(9c)$, $\tau \le q_1 \le -\tau - 1/(18c) + \sqrt{5\tau/(9c) + 1/(18c)^2}$ exists only if $\tau \le 1/(12c)$.

$q_2 = q_1 + \tau$ is the best reply if $\tau > 1/(9c)$ and $2/(9c) - \tau < q_1 \le \sqrt{4\tau/(9c)} - \tau$.

(iii) $q_2 = 0$ has to be the best reply in all remaining cases.

Thus the best reply of firm 2 can be summarized as

for $\tau \leq \frac{4}{45c}$:

$$q_2^*(q_1) = \begin{cases} \frac{2}{9c} & \text{if } q_1 \leq \frac{4}{45c} \\ 0 & \text{if } q_1 > \frac{4}{45c}. \end{cases}$$

for $\frac{4}{45c} < \tau \leq \frac{1}{9c}$:

$$q_2^*(q_1) = \begin{cases} \frac{2}{9c} & \text{if } q_1 < \tau \\ 0 & \text{if } q_1 \geq \tau. \end{cases}$$

for $\tau > \frac{1}{9c}$:

$$q_2^*(q_1) = \begin{cases} \frac{2}{9c} & \text{if } q_1 \leq \frac{2}{9c} - \tau \\ q_1 + \tau & \text{if } \frac{2}{9c} - \tau < q_1 \leq \sqrt{\frac{4\tau}{9c}} - \tau \\ 0 & \text{if } q_1 > \sqrt{\frac{4\tau}{9c}} - \tau. \end{cases} \tag{2.3}$$

Simultaneous quality choice

If both firms choose their quality simultaneously, by symmetry the best reply of firm 1 is

for $\tau \leq \frac{4}{45c}$:

$$q_1^*(q_2) = \begin{cases} \frac{2}{9c} & \text{if } q_2 \leq \frac{4}{45c} \\ 0 & \text{if } q_2 > \frac{4}{45c}. \end{cases}$$

for $\frac{4}{45c} < \tau \leq \frac{1}{9c}$:

$$q_1^*(q_2) = \begin{cases} \frac{2}{9c} & \text{if } q_2 < \tau \\ 0 & \text{if } q_2 \geq \tau. \end{cases}$$

for $\tau > \frac{1}{9c}$:

$$q_1^*(q_2) = \begin{cases} \frac{2}{9c} & \text{if } q_2 \leq \frac{2}{9c} - \tau \\ q_2 + \tau & \text{if } \frac{2}{9c} - \tau < q_2 \leq \sqrt{\frac{4\tau}{9c}} - \tau \\ 0 & \text{if } q_2 > \sqrt{\frac{4\tau}{9c}} - \tau \end{cases} \tag{2.4}$$

Overall, if the perception threshold is absolute and the firms choose quality simultaneously, the subgame-perfect equilibria are ($i \in \{1,2\}$, $j \in \{1,2\}$, and $i \neq j$)

- if $\tau \leq \frac{4}{45c}$: $q_i^* = \frac{2}{9c}$ and $q_j^* = 0$ with $p_i^* = \frac{4}{27c}$ and $p_j^* = \frac{2}{27c}$.
- if $\frac{4}{45c} < \tau \leq \frac{1}{9c}$: $q_i^* = \frac{2}{9c}$ and $q_j^* = 0$ with $p_i^* = \frac{4}{27c}$ and $p_j^* = \frac{2}{27c}$.
- if $\frac{1}{9c} < \tau$:
  - and $\tau \leq \frac{2}{9c}$: $q_i^* = \frac{2}{9c}$ and $q_j^* = 0$ with $p_i^* = \frac{4}{27c}$ and $p_j^* = \frac{2}{27c}$.
  - and $\frac{2}{9c} < \tau \leq \frac{4}{9c}$: $q_i^* = \tau$ and $q_j^* = 0$ with $p_i^* = \frac{2}{3}\tau$ and $p_j^* = \frac{1}{3}\tau$
    (If $\frac{2}{9c} < \tau \leq \frac{4}{9c}$, the first case of (2.3) and (2.4) does not exist as $\frac{2}{9c} - \tau < 0$. Thus the best reply against zero is now $q_i = q_j + \tau = \tau$).
  - and $\frac{4}{9c} < \tau$: $q_i^* = q_j^* = 0$ with $p_i^* = p_j^* = 0$
    (If $\frac{4}{9c} < \tau$, the first and the second case of (2.3) and (2.4) do not exist as $\sqrt{\frac{4\tau}{9c}} - \tau < 0$. Thus the best reply is always $q_i^*(q_j) = 0$).

Thus summarizing the cases $\tau \leq \frac{4}{45c}$, $\frac{4}{45c} < \tau \leq \frac{1}{9c}$, and $\frac{1}{9c} < \tau \leq \frac{2}{9c}$, because they yield the same subgame-perfect equilibria: As long as $\tau \leq \frac{2}{9c}$, the subgame-perfect equilibria are $q_i^* = \frac{2}{9c}$ and $q_j^* = 0$ with $p_i^* = \frac{4}{27c}$ and $p_j^* = \frac{2}{27c}$ (see Proposition 1).

Sequential quality choice

The profit of firm 1 is:

$$\Pi_1(q_1, q_2) = -cq_1^2 + \begin{cases} \frac{1}{9}(q_2 - q_1) & \text{if } q_1 \leq q_2 - \tau \\ 0 & \text{if } q_2 - \tau < q_1 < q_2 + \tau \\ \frac{4}{9}(q_1 - q_2) & \text{if } q_1 \geq q_2 + \tau. \end{cases}$$

If both firms choose their quality sequentially, firm 1 accounts in the first stage for the quality that firm 2 chooses as a best reply.

Assume $\tau \leq 4/(45c)$. If firm 1 chooses a quality $q_1 \leq 4/(45c)$, firm 2 responds with $q_2 = 2/(9c)$. That means, firm 2 chooses a higher quality than firm 1 and the quality difference exceeds the perception threshold: $2/(9c) - q_1 \geq \tau$. If firm 1 chooses a quality $q_1 > 4/(45c)$, firm 2 responds with $q_2 = 0$. That means, firm 2 chooses a lower quality than firm 1 and the quality difference exceeds the perception threshold: $q_1 - 0 \geq \tau$. Thus firm 2 always chooses quality such that all consumers notice the quality difference. Thus the profit of firm 1 is

$$\Pi_1(q_1) = -cq_1^2 + \begin{cases} \frac{1}{9}(\frac{2}{9c} - q_1) & \text{if } q_1 \leq \frac{4}{45c} \\ \frac{4}{9}(q_1 - 0) & \text{if } q_1 > \frac{4}{45c}. \end{cases}$$

For $q_1 < 4/(45c)$, $\partial \Pi_1(q_1)/\partial q_1 < 0$. Thus for $q_1 < 4/(45c)$, the profit of firm 1 is strictly decreasing. For $q_1 > 4/(45c)$, $\partial \Pi_1(q_1)/\partial q_1 = 0 \Leftrightarrow q_1 = 2/(9c)$ with $\partial^2 \Pi_1(q_1)/(\partial q_1)^2 < 0$. At $q_1 = 4/(45c)$ the profit of firm 1 jumps up by $2/(81c)$, but the profit of firm 1 at $q_1 = 4/(45c)$ is lower than the profit at $q_1 = 2/(9c)$. I.e., firm 1 either chooses $q_1 = 0$ and earns $\Pi_1 = 2/(81c)$ or chooses $q_1 = 2/(9c)$ and earns $\Pi_1 = 4/(81c)$. As $2/(81c) < 4/(81c)$, for $\tau \leq 4/(45c)$, the unique subgame-perfect equilibrium is $q_1^* = 2/(9c)$ and $q_2^* = 0$.

Assume $4/(45c) < \tau \leq 1/(9c)$. If firm 1 chooses a quality $q_1 < \tau$, firm 2 responds with $q_2 = 2/(9c)$. That means, firm 2 chooses a higher quality than firm 1 and the quality difference exceeds the perception threshold: $|2/(9c) - q_1| \geq \tau$. If firm 1 chooses a quality $q_1 \geq \tau$, firm 2 responds with $q_2 = 0$. That means, firm 2 chooses a lower quality than firm 1 and the quality difference exceeds the perception threshold: $|q_1 - 0| \geq \tau$. Thus firm 2 always chooses quality such that all consumers notice the quality difference. Thus the profit of firm 1 is

$$\Pi_1(q_1) = -cq_1^2 + \begin{cases} \frac{1}{9}(\frac{2}{9c} - q_1) & \text{if } q_1 < \tau \\ \frac{4}{9}(q_1 - 0) & \text{if } q_1 \geq \tau. \end{cases}$$

For $q_1 < \tau$, $\partial \Pi_1(q_1)/\partial q_1 < 0$. At $q_1 = \tau$ the profit of firm 1 jumps up. For $q_1 > \tau$, $\partial \Pi_1(q_1)/\partial q_1 = 0 \Leftrightarrow q_1 = 2/(9c)$ with $\partial^2 \Pi_1(q_1)/(\partial q_1)^2 < 0$. The profit at $q_1 = 2/(9c)$ is larger than the profit at $q_1 = \tau$: $4/(81c) > (4\tau)/9 - c\tau^2 \Leftrightarrow (\tau - 2/(9c))^2 > 0$. Therefore, firm 1 either chooses $q_1 = 0$ and earns $\Pi_1 = 2/(81c)$ or chooses $q_1 = 2/(9c)$ and earns $\Pi_1 = 4/(81c)$. As $2/(81c) < 4/(81c)$, for $4/(45c) < \tau \leq 1/(9c)$, the unique subgame-perfect equilibrium is $q_1^* = 2/(9c)$ and $q_2^* = 0$.

Assume $\tau > 1/(9c)$. If firm 1 chooses a quality $q_1 \leq 2/(9c) - \tau$, firm 2 responds with $q_2 = 2/(9c)$. That means firm 2 chooses a larger quality than firm 1 and the quality difference exceeds the perception threshold: $2/(9c) - q_1 \geq \tau$. If firm 1 chooses a quality $2/(9c) - \tau < q_1 \leq \sqrt{4\tau/(9c)} - \tau$, firm 2 responds with $q_2 = q_1 + \tau$. That means firm 2 chooses a larger quality than firm 1 and the quality difference exceeds the perception threshold: $q_1 + \tau - q_1 \geq \tau$. If firm 1 chooses a quality $q_1 > \sqrt{4\tau/(9c)} - \tau$, firm 2 responds with $q_2 = 0$. That means firm 2 responds with a lower quality than firm 1. The quality difference exceeds the perception threshold if firm 1 chooses a quality $q_1 \geq \tau$. Otherwise, the quality difference does not exceed the perception threshold and firm 1 receives zero revenue. Thus the profit of firm 1 is

$$\Pi_1(q_1) = -cq_1^2 + \begin{cases} \frac{1}{9}(\frac{2}{9c} - q_1) & \text{if } q_1 \leq \frac{2}{9c} - \tau \\ \frac{1}{9}((q_1 + \tau) - q_1) & \text{if } \frac{2}{9c} - \tau < q_1 \leq \sqrt{\frac{4\tau}{9c}} - \tau \\ 0 & \text{if } \sqrt{\frac{4\tau}{9c}} - \tau < q_1 < \tau \\ \frac{4}{9}(q_1 - 0) & \text{if } \tau \leq q_1. \end{cases}$$

The first and the second case only exist if τ is sufficiently small, i.e., $\tau \leq 2/(9c)$ and $\tau \leq 4/(9c)$ respectively. For $q_1 \leq 2/(9c) - \tau$, $\partial\Pi_1(q_1)/\partial q_1 < 0$. At $q_1 = 2/(9c) - \tau$, the profit decreases. For $2/(9c) - \tau < q_1 \leq \sqrt{4\tau/(9c)} - \tau$, $\partial\Pi_1(q_1)/\partial q_1 < 0$. At $q_1 = \sqrt{4\tau/(9c)} - \tau$, the profit jumps down by $\tau/9$. For $\sqrt{4\tau/(9c)} - \tau < q_1 < \tau$, $\partial\Pi_1(q_1)/\partial q_1 < 0$. Thus for $q_1 < \tau$, the profit of firm 1 is strictly decreasing. At $q_1 = \tau$, the profit jumps up by $4\tau/9$. For $q_1 > \tau$, $\partial\Pi_1(q_1)/\partial q_1 = 0 \Leftrightarrow q_1 = 2/(9c)$ with $\partial^2\Pi_1(q_1)/(\partial q_1)^2 < 0$; however, this potential maximum only lies in the range $q_1 \geq \tau$ if $\tau \leq 2/(9c)$. For $\tau \leq 2/(9c)$ the profit at $q_1 = 2/(9c)$ is larger than the profit at $q_1 = \tau$: $4/(81c) \geq 4\tau/9 - c\tau^2$.

For $\tau \leq 2/(9c)$, firm 1 maximizes $\Pi_1(q_1)$ over all four cases. The firm achieves the highest profit either at $q_1 = 0$ or at $q_1 = 2/(9c)$. The profit is higher with $q_1 = 2/(9c)$: $4/(81c) > 2/(81c)$. Therefore, $q_1^* = 2/(9c)$ and $q_2^* = 0$.

For $2/(9c) < \tau \leq 4/(9c)$, the firm maximizes $\Pi_1(q_1)$ over the last three cases. The firm achieves the highest profit either with $q_1 = 0$ or with $q_1 = \tau$. The firm achieves the highest profit with

$$q_1^* = \begin{cases} \tau & \text{if } \tau \leq \frac{1}{3c} \\ 0 & \text{if } \tau > \frac{1}{3c}. \end{cases}$$

For $\tau > 4/(9c)$, the firm maximizes $\Pi_1(q_1)$ over the last two cases. The firm achieves the highest profit either with $q_1 = 0$ or with $q_1 = \tau$. The firm achieves the highest profit with $q_1^* = 0$. Then, $q_2^* = 0$.

Overall, if the firms choose quality sequentially, the subgame-perfect equilibrium is

- if $\tau \leq \frac{2}{9c}$: $q_1^* = \frac{2}{9c}$ and $q_2^* = 0$ with $p_1^* = \frac{4}{27c}$ and $p_2^* = \frac{2}{27c}$.
- if $\frac{2}{9c} < \tau \leq \frac{3}{9c}$: $q_1^* = \tau$ and $q_2^* = 0$ with $p_1^* = \frac{2}{3}\tau$ and $p_2^* = \frac{1}{3}\tau$.
- if $\frac{3}{9c} < \tau \leq \frac{4}{9c}$: $q_1^* = 0$ and $q_2^* = \tau$ with $p_1^* = \frac{1}{3}\tau$ and $p_2^* = \frac{2}{3}\tau$.
- if $\frac{4}{9c} < \tau$: $q_1^* = q_2^* = 0$ with $p_1^* = p_2^* = 0$.

Proof of Proposition 2

The only difference between simultaneous and sequential quality-setting is which firm produces the higher quality, but this has no effect on welfare. Let u_l and u_h be the surplus the consumers receive who buy from the low-quality and high-quality firm. Let Π_h (Π_l) be the profit of the firm that produces the higher (lower) quality.

If $\tau \leq 2/(9c)$, $q_h = 2/(9c)$, $q_l = 0$, $p_h = 4/(27c)$, $p_l = 2/(27c)$, and thus $\hat{\theta} = 1/3$. Then, the consumer surplus (CS), producer surplus (PS), and welfare (W=CS+PS) are

$$
\begin{aligned}
CS &= u_l + u_h = \int_0^{\hat{\theta}} v + \theta \cdot 0 - \frac{2}{27c} d\theta + \int_{\hat{\theta}}^1 v + \theta \frac{2}{9c} - \frac{4}{27c} d\theta = \frac{v}{3} - \frac{2}{81c} + \frac{2v}{3} = v - \frac{2}{81c} \\
PS &= \Pi_l + \Pi_h = \hat{\theta} p_l^* - c(q_l^*)^2 + (1-\hat{\theta}) p_h^* - c(q_h^*)^2 = \frac{2}{81c} + \frac{4}{81c} = \frac{6}{81c} \\
W &= CS + PS = v - \frac{2}{81c} + \frac{6}{81c} = v + \frac{4}{81c}.
\end{aligned}
$$

If $2/(9c) < \tau \leq 4/(9c)$, $q_h = \tau$, $q_l = 0$, $p_h = 2\tau/3$, $p_l = \tau/3$, and thus $\hat{\theta} = 1/3$. Then,

$$
\begin{aligned}
CS &= u_l + u_h = \int_0^{\hat{\theta}} v + \theta \cdot 0 - \frac{1}{3}\tau d\theta + \int_{\hat{\theta}}^1 v + \theta\tau - \frac{2}{3}\tau d\theta = \frac{v}{3} - \frac{1}{9}\tau + \frac{2v}{3} = v - \frac{1}{9}\tau \\
PS &= \Pi_l + \Pi_h = \frac{1}{9}\tau + \frac{4}{9}\tau - c\tau^2 = \frac{5}{9}\tau - c\tau^2 \\
W &= CS + PS = v - \frac{1}{9}\tau + \frac{5}{9}\tau - c\tau^2 = v + \frac{4}{9}\tau - c\tau^2.
\end{aligned}
$$

If $4/(9c) < \tau$, $q_h = 0$, $q_l = 0$, $p_h = 0$, $p_l = 0$. Thus all consumers are indifferent and randomize. Then,

$$
\begin{aligned}
CS &= \frac{v}{2} + \frac{v}{2} = v \\
PS &= \Pi_l + \Pi_h = 0 + 0 = 0 \\
W &= CS + PS = v + 0 = v.
\end{aligned}
$$

The consumers surplus reaches its maximum for all $\tau > 4/(9c)$, because $0 > -2/(81c)$ and $0 > -\tau/9$. The producer surplus reaches its unique maximum at $\tau = 5/(18c)$, because $0 < 6/(81c) < 5\tau/9 - c\tau^2$ for $\tau = 5/(18c)$ and $5/(18c) = \arg\max_\tau 5\tau/9 - c\tau^2$. The welfare reaches its maximum for all $\tau \in [0, 2/(9c)]$, because $v + 4/(81c) > v$ and $2/(9c) = \arg\max_\tau v + 4\tau/9 - c\tau^2$, and $v + 4\tau/9 - c\tau^2 = v + 4/(81c)$ for $\tau = 2/(9c)$.

Chapter 3

Competition with constrained consumer perception

Abstract

I present a model where two horizontally differentiated firms compete for consumers with imperfect perception. Consumers pay limited attention to quality differences between goods. In particular, consumers do not perceive quality differences between goods that are below their perception thresholds. I show that firms only choose qualities that consumers can distinguish for extremely low thresholds and, otherwise, choose indistinguishable qualities. I demonstrate that firms' profits decrease and consumer surplus increases with increasing attention of consumers. Overall, within this model, welfare increases with attention.

KEYWORDS: Limited Attention, Perception, Product Differentiation.

JEL CODES: D43, D91, L13.

3.1 Introduction

In this article, I present a model where horizontally differentiated firms compete for consumers who are constrained in their perception of quality differences. I analyze the implications of consumers' imperfect perception on equilibrium outcomes and welfare. In particular, I explore whether consumers' perception constraints influence the quality distribution in the market and whether consumers are harmed by imperfectly attending to quality differences between goods.

Evidence indicates that consumers imperfectly attend to information about goods. For example, consumers imperfectly account for odometer mileage (Busse, Lacetera, Pope, Silva-Risso, and Syndor, 2013; Lacetera, Pope, and Syndor, 2012) or age (Englmaier, Schmöller, and Stowasser, 2018) of used cars. Thus if consumers imperfectly attend to information about goods, consumers are also constrained in their ability to compare goods. Of all the characteristics goods possess, quality is usually the most difficult to compare. The quality of, for example, meals and beverages depends on the ingredients, for instance, included food additives. Such information is often difficult to observe or interpret. For example, at restaurants the menu often does not directly list the ingredients of each dish. In addition, firms often list food additives as chemical abbreviations, which complicates inferring the quality of the good.

I model such imperfect perception as a threshold: Consumers only perceive a quality difference between two goods, if the quality difference exceeds the consumers' perception threshold.[1] For example, consumers might notice a quality difference between yogurts if one yogurt lists no food additives and the other yogurt lists ten food additives, whereas they do not notice a quality difference, if one yogurt lists eight and the other ten food additives. Following Schmitt (2019), I assume that consumers have an absolute perception threshold. That means, consumers notice that goods differ in quality, if the goods' absolute quality difference exceeds the consumers' perception thresholds. The consumers' perception thresholds only indicate that consumers do not notice the quality difference between goods; consumers still have a preference for more quality. If the consumers would receive more detailed information about the quality difference, for example, through well-designed labels, they would, ceteris paribus, prefer the good with the higher quality.

The focus of this article is to explore the behavior of two horizontally differentiated firms, firm A and firm B, that compete in qualities and prices for the consumers. I analyze a three-stage game. In the first stage, firm A chooses its quality. In the second stage, firm B observes the quality of firm A and chooses its own quality. In the third stage, firm A observes the quality of firm B and both firms simultaneously set prices. Subsequently, consumers decide between buying from firm A and firm B. This setup

[1]The perception of other characteristics, such as prices, is sometimes also imperfect. Nevertheless, consumers can usually easily compare price tags, but quality differences are much harder to compare. Therefore, in this article, I focus on quality differences

captures, for example, situations where an existing firm and an entrant compete for consumers. The entrant can observe the good of the existing firm before entering the market. Once both firms serve the market, firms adjust prices.

I demonstrate that firms that face (sufficiently) inattentive consumers produce goods with indistinguishable qualities.[2] This finding conflicts with earlier models by Webb (2017) and Schmitt (2019), who find that firms prefer to produce qualities that consumers can distinguish. Thus I show that the horizontal product differentiation changes firms' incentives. When firms are not horizontally differentiated and consumers are unable to perceive a quality difference, consumers choose the good with the lower price. Then, firms face Bertrand competition and make zero profits. However, horizontal product differentiation changes the incentives of firms. Firms do not have to fear price competition as much, because the horizontal differentiation reduces competition and allows firms to set prices above marginal costs.

Therefore, in this model with horizontal product differentiation, in equilibrium, firms produce goods with indistinguishable qualities, sell the goods at the same price, and split the market equally. Then, both firms make the same revenue. Nevertheless, firms produce goods with different qualities (as long as the consumers' perception thresholds are not too high). Firm A produces higher quality than firm B. Firm A as the first mover has to choose its quality first and firm B can optimally respond to this quality. For firm B it is never optimal to choose a higher quality than firm A that is indistinguishable from firm A's quality. If firms have indistinguishable qualities, they receive the same revenues but still incur quality costs. Thus if firm B chooses a quality that is indistinguishable from firm A's quality, firm B chooses the lowest quality that is still indistinguishable from firm A's quality and such a quality is always lower than the quality of firm A.

Thus firm A has to decide among (i) producing a quality so high that firm B responds by setting a noticeably lower quality, (ii) setting a lower quality such that firm B chooses an unnoticeably lower quality, and (iii) setting an even lower quality such that firm B responds by choosing a noticeably higher quality. To avoid being the firm with noticeably lower quality, firm A has to choose its quality to discourage firm B from setting a noticeably higher quality. To make the quality difference noticeable, firm B would always have to set a quality that exceeds the quality of firm A by the perception threshold. If firm A already sets a high quality, to make its quality noticeably higher, firm B would incur high quality costs. Thus if firm A sets a high enough quality in the first stage, it can discourage firm B from setting a noticeably higher quality. Then, firm B always chooses a lower quality than firm A. In the subgame-perfect equilibrium, firm B prefers to set the lowest possible quality that is still indistinguishable from the quality of firm A. That means, firm B free rides: firm B benefits from the imperfect perception of the consumers,

[2]I show that firms only choose to produce goods with distinguishable qualities in the benchmark with perfect perception or if consumers' perception is (extremely) close to perfect.

who think that firm B offers the good at the same quality than firm A, without incurring the same quality costs as firm A.

As the thresholds increase, firm A can reduce its quality and still discourage firm B from choosing a noticeably higher quality. As firm A reduces its quality and the thresholds increase, firm B who undercuts the quality of firm A by the size of the thresholds also reduces its quality until firm B produces zero quality. Thus with increasing thresholds, both firms reduce their quality until both firms produce zero quality.

Therefore, as firms produce goods with indistinguishable qualities, firms choose the same prices, split the market equally, and receive the same profit. As firm A produces higher quality, it incurs higher costs. Consequently, firm A makes less profit and has a first-mover *disadvantage*. Nevertheless, as with increasing thresholds firms choose (weakly) lower quality, quality costs are (weakly) decreasing. Thus as revenues are constant, firms' profits (weakly) increase with increasing thresholds. In contrast, as firms sell at the same price and consumers are harmed by lower quality, consumer surplus (weakly) decreases with increasing thresholds. Overall, the increasing profits of firms do not balance the decreasing surplus of consumers such that overall welfare (weakly) decreases with increasing thresholds.

The remainder of this article is structured as follows: Section 3.2 gives an overview of the related literature. Section 3.3 introduces the model. Section 3.4 derives the results and discusses the welfare implications. Section 3.5 concludes.

3.2 Related literature

The assumption that individuals perceive similar options as identical raises problems for models of decision-making. Luce describes these problems as follows:

> "Find a subject who prefers a cup of coffee with one cube of sugar to one with five cubes (this should not be difficult). Now prepare 401 cups of coffee with $\left(1 + \frac{i}{100}\right) x$ grams of sugar, $i = 0, 1, ..., 400$, where x is the weight of one cube of sugar. It is evident that he will be indifferent between cup i and cup $i + 1$, for any i, but by choice he is not indifferent between $i = 0$ and $i = 400$." (Luce, 1956, p.179)[3]

This means, a perception constraint proves troublesome for the transitivity of preferences. Consequently, a number of decision-making models try to account for such similarity perception constraints (see, for example, Luce, 1956; Rubinstein, 1988). Current models on decision-making under limited attention also attempt to capture the influence of similarity on choices. Bordalo, Gennaioli, and Shleifer (2012) and Köszegi and Szeidl (2013) analyze

[3]The subject might still prefer the cup with less sugar but is incapable of discriminating between the cups.

the implications of salience on decision-making. The salience of a dimension depends on how much the options differ in this dimension. For options with multiple dimensions (e.g., lotteries with various outcomes or goods with different payment plans), the more similar options are in one dimension, the less weight this dimension receives in the evaluation.

In contrast, in this article, I focus on the interactions of firms that face consumers with constrained perception. Thus my model is more closely related to the literature on market interactions with boundedly rational consumers; specifically, to models with consumers who only notice sufficiently large differences between goods. The implications of such perception-constrained consumers on market outcomes are largely unexplored. Exceptions are Allen and Thisse (1992), Bachi (2016), Webb (2017), and Schmitt (2019). Allen and Thisse (1992) and Bachi (2016) analyze price competition in duopolies where consumers' perception of prices is subject to a perception threshold. If the prices are too similar, consumers perceive them as identical. Both models show that consumers' perception thresholds lead to prices above marginal costs and positive profits. In other words, the imperfect perception of consumers allows firms to overcome the Bertrand paradox. The focus of Allen and Thisse (1992) and Bachi (2016) lies exclusively on price competition. Yet, among the different dimensions of goods, prices are usually easier to compare than, for instance, qualities. Therefore, I focus on a perception constraint on quality.

Most closely related to my model are, therefore, Webb (2017) and Schmitt (2019) who also explore the implications of a threshold on quality perception. Webb (2017) analyzes firms' strategic interactions when consumers have a relative perception threshold. The use of a relative perception threshold is often justified by Weber's Law.[4] Webb (2017) shows that, if firms have fixed costs for quality, firms always choose qualities that consumers can perfectly distinguish and that consumer surplus is only affected (negatively) for high perception thresholds.[5] Yet, a relative perception threshold is only sensible for quality values that are sufficiently large. Suppose one firm chooses to produce a good with zero quality. A relative perception threshold implies that if the other firm chooses any positive quality, even an extremely low quality, the relative difference is infinite. Thus all consumers notice the quality difference between the goods—even if they have a high perception threshold. This explains why Weber's law does not hold at the extremes (see, e.g., Hunt, 2007). Consequently, Schmitt (2019) explores the implications of an absolute perception threshold. Schmitt (2019) shows that consumers and firms benefit from imperfect perception, but that welfare is highest under perfect (and close to perfect) perception. I follow Schmitt (2019) in modeling the perception threshold as absolute.

[4]Weber's Law states that the difference between two stimuli which is just noticeable depends on the overall level of the stimuli (Hunt, 2007).

[5]Webb (2017) assumes that firms choose quality sequentially. In a companion paper, Webb (2014) analyzes the implications if firms choose qualities simultaneously and finds that firms always set qualities such that consumers are able to discriminate between the goods.

In both Webb (2017) and Schmitt (2019), firms have an incentive to set qualities such that the quality difference is noticeable. In both models, setting similar qualities implies that consumers choose from the firm with the lower price and Bertrand competition yields zero profits. Thus firms try to avoid producing similar qualities. However, goods are usually also horizontally differentiated. I contribute to the literature by investigating the implications of a perception threshold when goods include a horizontally differentiated characteristic. With horizontal product differentiation, the price competition is less intense when firms produce goods with the same level of quality. I show that assuming such horizontal product differentiation changes the incentives of the firms and thus considerably changes market outcomes. Instead of producing distinguishable quality as in Webb (2017) and Schmitt (2019), in this model, firms produce goods with indistinguishable qualities.

3.3 Model

Consider two firms, firm A and firm B, that both produce a horizontally differentiated good and compete in qualities and prices for a unit mass of consumers. Firms play a three-stage game depicted in Figure 3.1: In the first stage, firm A sets its quality. In the second stage, firm B observes the quality of firm A and sets its quality. In the third stage, firm A observes the quality of firm B and both firms, independently and simultaneously, set prices. Subsequently, consumers perceive the goods and buy either from firm A or from firm B.

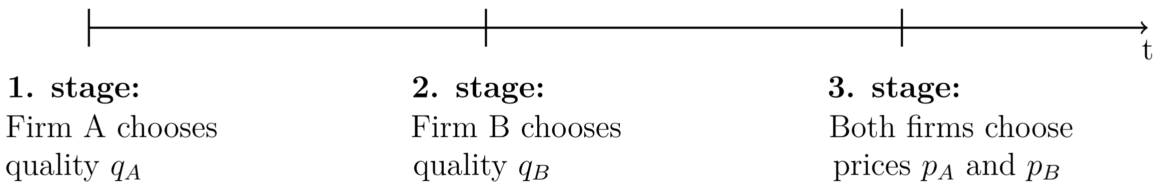

Figure 3.1: Timeline.

I follow Hotelling (1929) in modeling horizontal product differentiation as a real line. Consumers are uniformly distributed on $[0, 1]$. The position $x \in [0, 1]$ of a consumer denotes the consumer's ideal version of the good. Firm A is located at 0 and firm B is located at 1. Consumers buy exactly one unit of the good. By buying that unit from firm $i \in \{A, B\}$, the consumer at position x receives utility

$$
\begin{aligned}
u_x(A) &= v + q_A - p_A - x^2 \\
u_x(B) &= v + q_B - p_B - (1 - x)^2,
\end{aligned}
$$

where $v \gg 0$ is the gross utility of the good, $q_i \in \mathbb{R}_0^+$ is the quality, and $p_i \in \mathbb{R}_0^+$ is the price of the good of firm i. I assume that v is large enough such that consumers always buy the good in equilibrium. I follow d'Aspremont, Gabszewicz, and Thisse (1979) in modeling disutility from consuming a non-ideal good as quadratic: $(x - y_i)^2$, where y_i is the location of firm i in the product space.

Consumers are constrained in their perception of quality. Consumers only perceive that firms offer the good at different qualities if the quality difference is sufficiently large. The perceived quality $\hat{q}_i$ is thus:

$$\hat{q}_i = \begin{cases} q_i & \text{if } q_B < q_A - \tau \\ q(q_A, q_B) & \text{if } q_A - \tau \leq q_B < q_A + \tau \\ q_i & \text{if } q_B \geq q_A + \tau \end{cases} \tag{3.1}$$

where $q(q_A, q_B) > 0$ and the perception threshold is $\tau > 0$. As firm B chooses its quality after observing the quality of firm A, firm B decides whether the quality difference is noticeable and the condition on whether the quality difference is noticeable is expressed in terms of the quality of firm B. If the quality difference is larger than the perception threshold τ, consumers perceive the quality of each firm perfectly. If the quality difference is smaller than τ, the consumers perceive the quality of firm i as $\hat{q}_i = q(q_A, q_B)$ for both $i \in \{A, B\}$. In other words, the consumers perceive the quality of firm A and the quality of firm B as identical.[6] That means, I assume that if firm B wants to make its quality indistinguishable from the quality of firm A it has to choose a $q_A - \tau \leq q_B < q_A + \tau$. But if firm B wants to make its quality distinguishable from the quality of firm A it has to choose $q_B < q_A - \tau$ or $q_B \geq q_A + \tau$.[7] The perception threshold τ is identical for all consumers. The model includes perfect perception as the limiting case $\tau = 0$.

Consumers decide between buying the good from firm A and buying the good from firm B by comparing the perceived utilities $\hat{u}_x(A)$ and $\hat{u}_x(B)$. The perceived utilities differ from the true utilities if consumers do not perceive quality perfectly. That is, the consumer has a decision utility $\hat{u}_x(i)$ and an experience utility $u_x(i)$ for each good $i \in \{A, B\}$. A consumer is indifferent between buying from firm A and firm B if she believes the utilities

[6] For instance, $q(q_A, q_B)$ could be the average of q_A and q_B. Yet, for the analysis it is not necessary to specify $q(q_A, q_B)$, it only has to be identical for both firms.

[7] Expression (3.1) captures:

$$\hat{q}_i = \begin{cases} q_i & \text{if } |q_A - q_B| > \tau \\ q(q_A, q_B) & \text{if } |q_A - q_B| < \tau. \end{cases} \tag{3.2}$$

That is, if the quality difference exceeds the threshold, i.e., $|q_A - q_B| > \tau$, the true quality is observed and if the quality difference is below the threshold, i.e., $|q_A - q_B| < \tau$, the consumers observe both qualities as identical. Because of the discontinuities of profits at $|q_A - q_B| = \tau$, I have to be careful to make choices such that an equilibrium exists. The supremum of the profit function needs to be the maximum. The choice that the consumers do not notice a quality difference if $q_A - \tau \leq q_B < q_A + \tau$ but notice a quality difference if $q_B < q_A - \tau$ or if $q_B \geq q_A + \tau$ ensures this.

are identical:

$$\hat{u}_x(A) = \hat{u}_x(B) \Leftrightarrow v + \hat{q}_A - p_A - x^2 = v + \hat{q}_B - p_B - (1-x)^2.$$

Denote the indifferent consumer by

$$\bar{x} \equiv \frac{1 + \hat{q}_A - \hat{q}_B + p_B - p_A}{2}.$$

Then, all consumers $x \leq \bar{x}$ buy from firm A and all consumers $x > \bar{x}$ buy from firm B. As long as $\bar{x} \in (0,1)$, both firms receive some demand. However, if firms choose very different prices and/or qualities, it is possible that one firm receives the demand of all consumers and the other firm receives zero demand.

If firm B has chosen its quality to make the quality difference noticeable, i.e., if $q_B < q_A - \tau$ or $q_B \geq q_A + \tau$, all consumers perceive the qualities perfectly ($\hat{q}_A = q_A$ and $\hat{q}_B = q_B$). Then, the indifferent consumer is

$$\bar{x}^a = \frac{1 + q_A - q_B + p_B - p_A}{2}.$$

The demand for the good of firm A is, therefore,

$$x_A^{attentive} = \begin{cases} 1 & \text{if } \bar{x}^a > 1 \\ \frac{1+q_A-q_B+p_B-p_A}{2} & \text{if } 0 \leq \bar{x}^a \leq 1 \\ 0 & \text{if } \bar{x}^a < 0 \end{cases} \tag{3.3}$$

and the demand for the good of firm B is $x_B^{attentive} = 1 - x_A^{attentive}$.

In contrast, if firm B has chosen its quality to make the quality difference unnoticeable, i.e., if $q_A - \tau \leq q_B < q_A + \tau$, consumers perceive the qualities as identical ($\hat{q}_A = \hat{q}_B = q(q_A, q_B)$). Then, the indifferent consumer is

$$\bar{x}^i = \frac{1 + q(q_A, q_B) - q(q_A, q_B) + p_B - p_A}{2} = \frac{1 + p_B - p_A}{2}$$

and the demand for the good of firm A is

$$x_A^{inattentive} = \begin{cases} 1 & \text{if } \bar{x}^i > 1 \\ \frac{1+p_B-p_A}{2} & \text{if } 0 \leq \bar{x}^i \leq 1 \\ 0 & \text{if } \bar{x}^i < 0. \end{cases}$$

Consequently, the demand for the good of firm B is $x_B^{inattentive} = 1 - x_A^{inattentive}$.

I assume identical (fixed) cost functions $C(q_i) = \frac{1}{2}q_i^2$ for the quality provision of both firms; all other marginal quantity costs are set to 0. The profit functions are thus

$$\Pi_A(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_A^2 + p_A \begin{cases} x_A^{attentive} & \textit{if } q_B < q_A - \tau \\ x_A^{inattentive} & \textit{if } q_A - \tau \leq q_B < q_A + \tau \\ x_A^{attentive} & \textit{if } q_B \geq q_A + \tau \end{cases}$$

$$\Pi_B(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_B^2 + p_B \begin{cases} x_B^{attentive} & \textit{if } q_B < q_A - \tau \\ x_B^{inattentive} & \textit{if } q_A - \tau \leq q_B < q_A + \tau \\ x_B^{attentive} & \textit{if } q_B \geq q_A + \tau. \end{cases}$$

I solve for the subgame-perfect equilibria by backward induction.

3.4 Results

In the price setting stage, firms are in the subgame where both qualities are set and firms simultaneously set prices to maximize profits dependent on the qualities set in stage 1 and stage 2. The prices depend on the consumers' perception of quality. Consumers are only willing to pay for a quality difference that they perceive. Therefore, to solve for the prices in the price-setting stage, I distinguish two classes of subgames: First, firms have chosen qualities such that the consumers are inattentive to the quality difference, i.e., $q_A - \tau \leq q_B < q_A + \tau$, and, second, firms have chosen qualities such that the consumers are attentive to the quality difference, i.e., $q_B < q_A - \tau$ or $q_B \geq q_A + \tau$.

Inattention ($q_A - \tau \leq q_B < q_A + \tau$): If the qualities are so similar that consumers do not notice the difference, firms maximize $\Pi_i(p_A, p_B, q_A, q_B) = -1/2q_i^2 + p_i x_i^{inattentive}$. In equilibrium, both firms choose prices such that both firms receive some demand.[8] Then firm A and firm B choose their prices p_A and p_B to maximize their profits

$$\Pi_A(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_A^2 + p_A\frac{1 + p_B - p_A}{2}$$
$$\Pi_B(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_B^2 + p_B\left(1 - \frac{1 + p_B - p_A}{2}\right).$$

The best replies of firm A and firm B are then

$$p_A^*(p_B) = \frac{1 + p_B}{2} \qquad \text{and} \qquad p_B^*(p_A) = \frac{1 + p_A}{2}.$$

The resulting price equilibrium is thus: $p_A^* = 1$ and $p_B^* = 1$. If $q_A - \tau \leq q_B < q_A + \tau$, consumers think the firms offer goods with identical quality. Therefore, they are not

[8] Under inattention, if firms set prices such that one firm receives the full demand, the other firm receives zero demand and makes zero revenue. This firm can always deviate to increase its profit by setting a lower price. Thus prices where one firm receives zero demand cannot exist in the inattention equilibrium.

willing to pay a mark-up for quality. Consequently, in equilibrium, both firms have identical incentives to set prices and such set identical prices. However, because the goods are horizontally differentiated which dampens price competition, firms can charge prices above marginal costs (here above zero).

Attention ($q_B < q_A - \tau$ or $q_B \geq q_A + \tau$): If the qualities are sufficiently different so that consumers notice the quality difference, firms maximize

$$\Pi_i(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_i^2 + p_i x_i^{attentive}.$$

The best replies of firm A and B are then (for a detailed derivation, see appendix A):

$$\begin{cases} p_A^*(p_B) \in [0, \infty) & \text{if } p_B < q_B - q_A - 1 \\ p_A^*(p_B) = \frac{1}{2}(q_A - q_B + p_B + 1) & \text{if } q_B - q_A - 1 \leq p_B \leq q_B - q_A + 3 \\ p_A^*(p_B) = q_A - q_B + p_B - 1 & \text{if } p_B > q_B - q_A + 3 \end{cases}$$

$$\begin{cases} p_B^*(p_A) \in [0, \infty) & \text{if } p_A < q_A - q_B - 1 \\ p_B^*(p_A) = \frac{1}{2}(q_B - q_A + p_A + 1) & \text{if } q_A - q_B - 1 \leq p_A \leq q_A - q_B + 3 \\ p_B^*(p_A) = q_B - q_A + p_A - 1 & \text{if } p_A > q_A - q_B + 3. \end{cases}$$

The resulting price equilibrium depends on the quality difference:

$$\begin{array}{lllll} \text{I)} & \text{if} & 3 < q_A - q_B: & p_A^* = q_A - q_B - 1 & p_B^* = 0 \\ \text{II)} & \text{if} & -3 < q_A - q_B \leq 3: & p_A^* = 1 + \frac{q_A - q_B}{3} & p_B^* = 1 + \frac{q_B - q_A}{3} \\ \text{III)} & \text{if} & q_A - q_B \leq -3: & p_A^* = 0 & p_B^* = q_B - q_A - 1. \end{array}$$

In the first case, i.e., $3 < q_A - q_B$, firm A receives the full demand and firm B receives no demand. In the third case, i.e., $q_A - q_B \leq -3$, firm B receives the full demand and firm A receives no demand. In both cases, the firm that receives the full demand makes strictly negative profits. This cannot be the result of a subgame-perfect equilibrium, because a firm can always choose a quality of zero to make at least zero profit. Therefore, the firms will avoid these cases in the quality-setting stages and these prices never occur in the subgame-perfect equilibrium.

In stage 2, firm B chooses its quality to maximize its updated profit given the quality choice of firm A. For a given quality of firm A, firm B decides whether to produce indistinguishable or distinguishable quality. If firm B would choose a quality that is indistinguishable from the quality of firm A, i.e., $q_A - \tau \leq q_B < q_A + \tau$, no consumer would notice a quality difference. As firms would then choose identical prices $p_A^* = p_B^* = 1$ in the price-setting stage, firms would split the demand equally and, consequently, for

$q_A - \tau \leq q_B < q_A + \tau$, firm B would receive a profit of

$$\Pi_B^{inattentive}(q_A, q_B) = \frac{1}{2} - \frac{1}{2}q_B^2. \tag{3.4}$$

Consequently, firm B never chooses a quality $\max\{0, q_A - \tau\} < q_B < q_A + \tau$. If firm B decides to produce a quality $q_B \in [\max\{0, q_A - \tau\}, q_A + \tau)$, consumers do not notice a difference and firm B makes the same revenue with any $q_B \in [\max\{0, q_A - \tau\}, q_A + \tau)$. But, the higher the quality, the higher the costs. Thus firm B would choose the lowest quality such that consumers do not perceive the quality difference; this quality is $q_B = \max\{0, q_A - \tau\}$. In other words, it is impossible that firm A has a quality that is lower than the quality of firm B and that is indistinguishable from the quality of firm B.

If firm B would choose a quality such that $q_B < q_A - \tau$ or $q_B \geq q_A + \tau$, all consumers would notice the quality difference. Then, the prices would depend on the quality difference and, consequently, the profit of firm B would also depend on the quality difference:

$$\Pi_B^{attentive}(q_A, q_B) = -\frac{1}{2}q_B^2 + \begin{cases} 0 & \text{if } q_B < q_A - 3 \\ \frac{1}{2}\left(1 + \frac{q_B - q_A}{3}\right)^2 & \text{if } q_A - 3 \leq q_B < q_A + 3 \\ q_B - q_A - 1 & \text{if } q_B \geq q_A + 3. \end{cases} \tag{3.5}$$

The last case, $q_B \geq q_A + 3$ never occurs in equilibrium. If firm B would choose a quality $q_B \geq q_A + 3$, the firm would receive a negative profit. The firm can always avoid a negative profit by choosing $q_B = 0$, then the firm has no costs and thus its profits cannot be negative. Overall, firm B maximizes over (3.4) and (3.5) to derive the best response to the quality of firm A. For a detailed derivation of the best reply, see appendix B, part 1.

In the first stage, firm A chooses its quality taking the subsequent decision of firm B into account. As firm A can never produce unnoticeably lower quality than firm B, firm A is either the firm with noticeably higher, noticeably lower, or unnoticeably higher quality. To avoid being the firm with noticeably lower quality, firm A has to choose a sufficiently high quality in the first stage. Then, firm B would have to exceed the quality of firm A by the threshold to make the quality difference noticeable. If the quality of firm A is already high, this implies too high costs for firm B. Thus in the subgame-prefect equilibrium, firm A provides a quality that is high enough to discourage firm B from noticeably overbidding firm A's quality. The necessary quality for this depends on the size of the threshold. The lower the threshold, the higher the necessary quality. As quality is costly, firm A prefers the lowest such quality to discourage firm B from noticeably overbidding its quality. Thus the quality of firm A (weakly) decreases with the threshold. As a best response, firm B undercuts the quality of firm A such that the quality difference is just not noticeable, i.e., $q_B = \max\{0, q_A - \tau\}$. Thus also the quality of firm B (weakly) decreases with increasing thresholds. Proposition 3 characterizes the resulting subgame-perfect equilibria.

For the following proposition, define

$$\begin{aligned}
\tau_a &\equiv \tau \leq \sqrt{6/55} - 1/2\sqrt{3\sqrt{6/55} - 61/110} \approx 0.00002 \\
\tau_b &\equiv 3 - \sqrt{8} \approx 0.17 \\
\tau_c &\equiv 3/4(3\sqrt{2} - 4) \approx 0.18.
\end{aligned}$$

Proposition 3 *Characterization of the pure-strategy sugame-perfect equilibria dependent on τ:*

(i) If $\tau \leq \tau_a$, in the unique pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = 21/55$ and $q_B^* = 18/55$ and $p_A^* = 56/55$ and $p_B^* = 54/55$.

(ii) If $\tau_a < \tau < \tau_b$, in the unique pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = (3+8\tau)/9 - \sqrt{48\tau - 8\tau^2}/9$ and $q_B^* = (3-\tau)/9 - \sqrt{48\tau - 8\tau^2}/9$ and $p_A^* = p_B^* = 1$.

(iii) If $\tau_b \leq \tau \leq \tau_c$, in the unique pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = 3 - \sqrt{8}$ and $q_B^* = 0$ and $p_A^* = p_B^* = 1$.

(iv) If $\tau_c < \tau \leq 3/4$, in the unique pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = -\tau + \sqrt{2/3\tau + 1/9\tau^2}$ and $q_B^* = 0$ and $p_A^* = p_B^* = 1$.

(v) If $\tau > 3/4$, in the unique pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = q_B^* = 0$ and $p_A^* = p_B^* = 1$.

The proof is in appendix B. Figure 3.2 illustrates Proposition 3 graphically.

In the benchmark under perfect perception, i.e., $\tau = 0$, both firms produce goods with positive quality: $q_A^* = 21/55$ and $q_B^* = 18/55$. Firm A has a first mover advantage. Firm A takes the best reply of firm B, which is here $q_B = (3 - q_A)/8$, into account. That means by increasing its quality, firm A has a direct and a strategic effect: Firm A directly increases its profit if it increases its quality. In addition, if firm A increases its quality, firm B reduces its quality which, in equilibrium, also increases firm A's profit. Therefore, in equilibrium, the sequential game structure results in firm A producing goods with higher quality than firm B. This subgame-perfect equilibrium also results for extremely low levels of τ, i.e., for all $\tau \leq \tau_a$. Consumers notice this quality difference which allows firm A to set a higher price than firm B. In addition, because of the quality difference more consumers buy from firm A and firm A makes a higher profit than firm B. Thus for $\tau \leq \tau_a$, firm A has a first-mover advantage.

In contrast, for $\tau > \tau_a$, it becomes profitable for firm B to undercut the quality of firm A unnoticeably. That is, firms set qualities such that consumers do not notice the quality difference. Then, the firms set identical prices and split the demand equally. Yet,

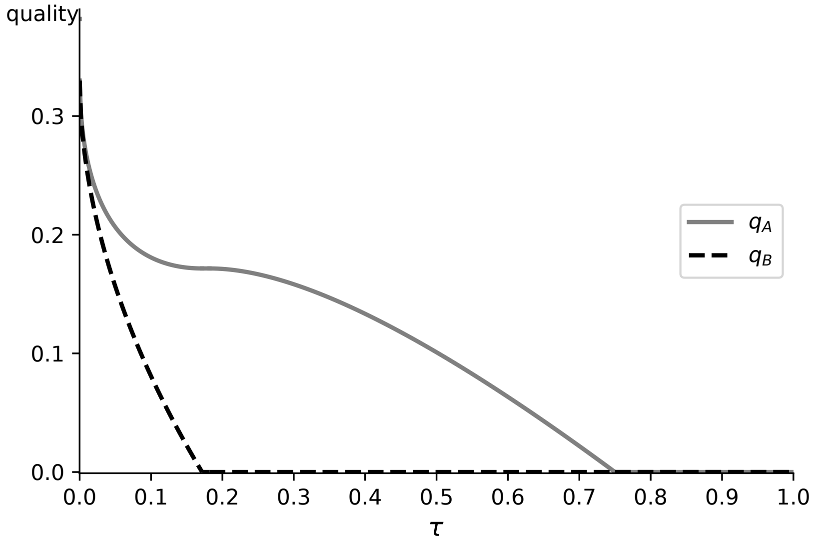

Figure 3.2: Equilibrium qualities of firm A and firm B as a function of τ.

for $\tau \leq 3/4$, firm A produces goods with strictly higher quality than firm B: By setting a sufficiently high quality, firm A wants to discourage firm B from producing a noticeably higher quality than firm A. As a consequence, the best reply of firm B is to unnoticeably undercut the quality of firm A. As τ increases, firm A can reduce its quality without firm B producing noticeably higher quality. Firm B, which produces $q_B = \max\{0, q_A - \tau\}$, then also reduces its quality until $q_B = 0$. Firm A reduces its quality further with increasing τ until, for $\tau > 3/4$, both firms produce zero quality.

Overall then, the horizontal differentiation changes the incentives of the firms compared to Webb (2017) and Schmitt (2019). Firms do not want to avoid indistinguishable qualities as much as in Webb (2017) and Schmitt (2019). Firms actually produce indistinguishable quality for all $\tau > \tau_a \approx 0.00002$ and with increasing τ both firms reduce their quality. Figure 3.3 illustrates the subgame-perfect equilibrium profits of firm A and firm B dependent on τ. Figure 3.3 shows that firm A makes less profit than firm B for all $\tau_a < \tau \leq 3/4$. Because the quality difference is unnoticeable, both firms sell at the same price and split the market equally. Thus both firms receive the same revenue, but firm A has higher quality costs. Consequently, firm A makes less profit than firm B and has a first-mover disadvantage. As τ increases firms A and B reduce their quality until at $\tau > 3/4$ both firms produce zero quality. Then, as the firms receive the same revenue and have zero quality costs, firms make the same profits.

Because with increasing τ firms (weakly) reduce quality and because consumers buy exactly one unit of the good independent of quality, firms' profits (weakly) increase in τ. Thus the producer surplus (weakly) increases in τ. In contrast, as consumers prefer higher quality to lower quality, consumer surplus (weakly) decreases in τ. In sum, as τ increases, the gain of the firms does not balance the loss of the consumers and welfare

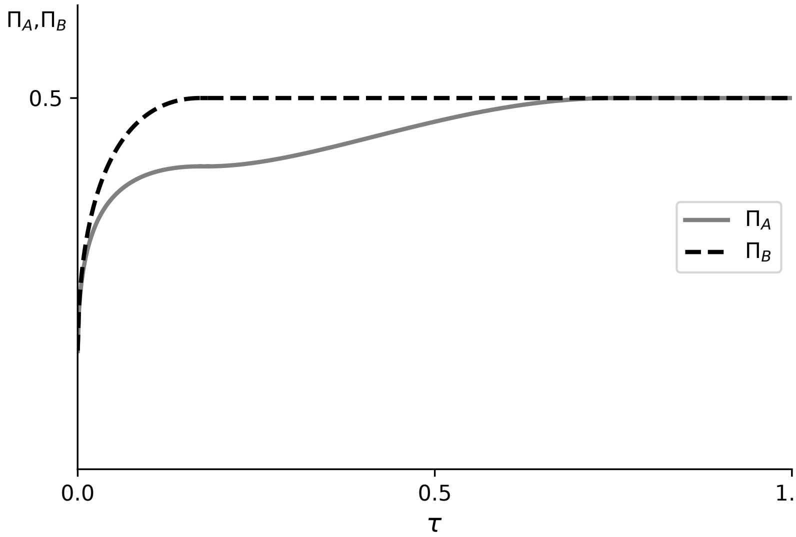

Figure 3.3: Equilibrium profits of firm A and firm B as a function of τ.

(weakly) decreases in τ. That is, consumer surplus and welfare are highest under close to perfect perception, i.e., as long as the benchmark equilibrium results, whereas, firms prefer inattentive consumers. Proposition 4 summarizes the consumer surplus, producer surplus, and welfare and Figure 3.4 illustrates the consumer surplus, producer surplus, and the welfare dependent on τ.

Proposition 4 *The producer surplus reaches its maximum for $\tau \geq 3/4$. The consumer surplus and the welfare reach their maxima for $\tau \leq \tau_a$.*

The proof is in appendix C.

3.5 Discussion and conclusion

In this article, I present a model where horizontally differentiated firms compete in qualities and prices for consumers with imperfect perception. Consumers do not perceive quality differences that are below their perception thresholds. I demonstrate that the perception thresholds influence the quality distribution in the market. I show that in the pure-strategy subgame-perfect equilibrium, firms choose distinguishable qualities only for very low thresholds. For most values of the thresholds, firms choose qualities just so that consumers cannot distinguish the qualities. When firms choose indistinguishable qualities, both firms sell the good at the same price and split the market equally. Then, the firm with the higher quality—firm A—has higher costs and thus makes less profit than firm B. Firm A thus has a first-mover disadvantage. Furthermore, the higher the threshold, the lower the qualities the firms produce. Then with increasing thresholds, firms have less

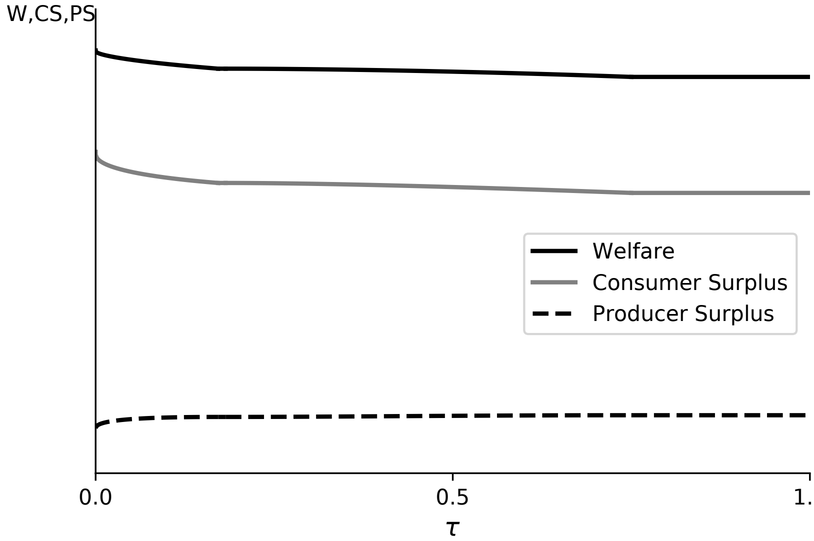

Figure 3.4: Welfare, consumer surplus, and producer surplus as a function of τ for $v = 4$.

costs and make larger profits. Consequently, producer surplus (weakly) increases with increasing thresholds. In other words, the more inattentive consumers are to quality differences, the higher the firms' profits. In contrast, as consumers benefit from quality, consumer surplus (weakly) decreases with inattention. Overall, the increase of firms' profits does not balance the decline in consumer surplus and welfare (weakly) decreases with inattention. Thus consumer surplus and welfare are highest under perfect, and close to perfect, perception, whereas producer surplus is highest under imperfect perception.

To keep the model tractable, I make a number of assumptions which limit the scope of the analysis. In particular, I adopt specific cost functions for the firms. Webb (2017) and Schmitt (2019) show that the costs matters for the resulting subgame-perfect equilibria. Allowing for a more general cost function turned out to complicate the analysis substantially and is, therefore, left to future research. In addition, a homogeneous threshold for all consumers neglects that some consumers might be better at observing quality differences; for example, because they are better qualified or more attentive. A fruitful avenue for future research might be to discuss how such heterogeneous thresholds influence the results. Furthermore, I assume that all consumers value quality equally. Yet, some consumers might value quality less than others, for instance, due to income constraints. Including such heterogeneous quality preferences might yield interesting dynamics. Additionally, for reasons of tractability, I assume that firms choose quality sequentially. I leave the question of how the game structure changes the results to future research. Moreover, I assume that all consumers always buy exactly one unit of the good. Generalizing this assumption changes the demand for the goods of the firms and might, therefore, also impact the equilibria. I leave the detailed analysis to future research.

My results markedly differ from Webb (2017) and Schmitt (2019). In Webb (2017) and

Schmitt (2019), firms prefer to produce goods with distinguishable qualities, whereas I show in this article that the introduction of horizontal product differentiation changes the incentives of firms; they produce indistinguishable qualities. The driver behind this result is the introduction of horizontal product differentiation. This product differentiation gives firms market power which holds even if they produce goods with indistinguishable quality. As this market power decreases, producing indistinguishable qualities would become less profitable as price competition increases. Firms then should prefer distinguishable qualities at some point.

Appendix

A Derivation of the best replies in the price-setting subgame

If $q_B < q_A - \tau$ or $q_B \geq q_A + \tau$, consumers notice the quality difference. Thus firms maximize

$$\Pi_i(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_i^2 + p_i x_i^{attentive}, \tag{3.6}$$

where $x_i^{attentive}$ is defined in (3.3). Note that

$$0 \leq \bar{x}^a \leq 1 \Leftrightarrow q_A - q_B + p_B - 1 \leq p_A \leq q_A - q_B + p_B + 1$$

Then, the profit of firm A from (3.6) can be written as

$$\Pi_A(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_A^2 + p_A \begin{cases} 1 & \text{if } p_A < q_A - q_B + p_B - 1 \\ \frac{1+q_A-q_B+p_B-p_A}{2} & \text{if } q_A - q_B + p_B - 1 \leq p_A \\ & \qquad \leq q_A - q_B + p_B + 1 \\ 0 & \text{if } p_A > q_A - q_B + p_B + 1. \end{cases} \tag{3.7}$$

Firm A chooses its price p_A to maximize (3.7). However, as firm A needs to choose a $p_A \geq 0$, the first case of (3.7) only exists if $q_A - q_B + p_B - 1 > 0 \Leftrightarrow p_B > q_B - q_A + 1$ and the second case only exists if $q_A - q_B + p_B + 1 \geq 0 \Leftrightarrow p_B \geq q_B - q_A - 1$.

(I) Assume $p_B > q_B - q_A + 1$, then firm A maximizes (3.7).

(i) If firm A chooses any $0 \leq p_A < q_A - q_B + p_B - 1$, firm A's profit is $\Pi_A(p_A, p_B, q_A, q_B) < -\frac{1}{2}q_A^2 + q_A - q_B + p_B - 1$.

(ii) If firm A chooses any $q_A - q_B + p_B - 1 \leq p_A \leq q_A - q_B + p_B + 1$,

$$\frac{\partial \Pi_A(p_A, p_B, q_A, q_B)}{\partial p_A} = \frac{1 + q_A - q_B + p_B - 2p_A}{2} \geq 0 \Leftrightarrow p_A \leq \frac{1 + q_A - q_B + p_B}{2}$$
$$\frac{\partial \Pi_A(p_A, p_B, q_A, q_B)}{\partial p_A} < 0 \Leftrightarrow p_A > \frac{1 + q_A - q_B + p_B}{2}.$$

Thus the candidate for best reply on $q_A - q_B + p_B - 1 \leq p_A \leq q_A - q_B + p_B + 1$ is either the interior solution $p_A = \frac{1+q_A-q_B+p_B}{2}$ or a boundary. As

$$q_A - q_B + p_B - 1 \leq \frac{1+q_A-q_B+p_B}{2} \leq q_A - q_B + p_B + 1 \Leftrightarrow p_B \leq q_B - q_A + 3,$$

the candidate for best reply is

$$p_A = \begin{cases} \frac{1+q_A-q_B+p_B}{2} & \text{if } p_B \leq q_B - q_A + 3 \\ q_A - q_B + p_B - 1 & \text{if } p_B > q_B - q_A + 3 \end{cases}$$

with profits

$$\Pi_A(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_A^2 + \begin{cases} \frac{1}{2}\left(\frac{1+q_A-q_B+p_B}{2}\right)^2 & \text{if } p_B \leq q_B - q_A + 3 \\ q_A - q_B + p_B - 1 & \text{if } p_B > q_B - q_A + 3 \end{cases}$$

(iii) If firm A chooses any $p_A > q_A - q_B + p_B + 1$, the profit of firm A is $\Pi_A(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_A^2$.

Note that

$$\begin{aligned} -\frac{1}{2}q_A^2 &< -\frac{1}{2}q_A^2 + \frac{1}{2}\left(\frac{1+q_A-q_B+p_B}{2}\right)^2 && \text{if } p_B \leq q_B - q_A + 3 \\ -\frac{1}{2}q_A^2 &< -\frac{1}{2}q_A^2 + q_A - q_B + p_B - 1 && \text{if } p_B > q_B - q_A + 3 \end{aligned}$$

$$\begin{aligned} &\Pi_A(p_A < q_A - q_B + p_B - 1, p_B, q_A, q_B) < -\frac{1}{2}q_A^2 + q_A - q_B + p_B - 1 \\ &\text{and } -\frac{1}{2}q_A^2 + q_A - q_B + p_B - 1 < -\frac{1}{2}q_A^2 + \frac{1}{2}\left(\frac{1+q_A-q_B+p_B}{2}\right)^2 \text{ if } p_B \leq q_B - q_A + 3 \end{aligned}$$

Therefore, the best reply of firm A if $p_B > q_B - q_A + 1$ is

$$p_A^*(p_B) = \begin{cases} \frac{1+q_A-q_B+p_B}{2} & \text{if } p_B \leq q_B - q_A + 3 \\ q_A - q_B + p_B - 1 & \text{if } p_B > q_B - q_A + 3. \end{cases} \tag{3.8}$$

(II) Assume $q_B - q_A - 1 \leq p_B \leq q_B - q_A + 1$, then firm A maximizes

$$\Pi_A(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_A^2 + p_A \begin{cases} \frac{1+q_A-q_B+p_B-p_A}{2} & \text{if } 0 \leq p_A \leq q_A - q_B + p_B + 1 \\ 0 & \text{if } p_A > q_A - q_B + p_B + 1. \end{cases}$$

(i) If firm A chooses any $0 \leq p_A \leq q_A - q_B + p_B + 1$,

$$\frac{\partial \Pi_A(p_A, p_B, q_A, q_B)}{\partial p_A} = \frac{1 + q_A - q_B + p_B - 2p_A}{2} \geq 0 \Leftrightarrow p_A \leq \frac{1 + q_A - q_B + p_B}{2}$$
$$\frac{\partial \Pi_A(p_A, p_B, q_A, q_B)}{\partial p_A} < 0 \Leftrightarrow p_A > \frac{1 + q_A - q_B + p_B}{2}.$$

Thus the candidate for best reply on $q_A - q_B + p_B - 1 \leq p_A \leq q_A - q_B + p_B + 1$ is $p_A = \frac{1+q_A-q_B+p_B}{2}$ if

$$0 \leq \frac{1 + q_A - q_B + p_B}{2} \leq q_A - q_B + p_B + 1,$$

which is always fulfilled.

(ii) If firm A chooses any $p_A > q_A - q_B + p_B + 1$, the profit of firm A is $\Pi_A(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_A^2$.

Note that

$$-\frac{1}{2}q_A^2 < -\frac{1}{2}q_A^2 + \frac{1}{2}\left(\frac{1 + q_A - q_B + p_B}{2}\right)^2.$$

Therefore, the best reply of firm A if $q_B - q_A - 1 \leq p_B \leq q_B - q_A + 1$ is

$$p_A^*(p_B) = \frac{1 + q_A - q_B + p_B}{2}. \tag{3.9}$$

(III) Assume $p_B < q_B - q_A - 1$, then firm A maximizes

$$\Pi_A(p_A, p_B, q_A, q_B) = -\frac{1}{2}q_A^2 + p_A \cdot 0.$$

Thus the firm receives the same profit with any $p_A \in [0, \infty)$. Therefore, the best reply of firm A if $p_B < q_B - q_A - 1$ is any

$$p_A^*(p_B) \in [0, \infty). \tag{3.10}$$

Bringing, (3.8), (3.9), and (3.10) together, the best reply of firm A is

$$\begin{cases} p_A^*(p_B) \in [0, \infty) & \text{if } p_B < q_B - q_A - 1 \\ p_A^*(p_B) = \frac{1}{2}(q_A - q_B + p_B + 1) & \text{if } q_B - q_A - 1 \leq p_B \leq q_B - q_A + 3 \\ p_A^*(p_B) = q_A - q_B + p_B - 1 & \text{if } p_B > q_B - q_A + 3. \end{cases}$$

By symmetry, the best reply of firm B is

$$\begin{cases} p_B^*(p_A) \in [0, \infty) & \text{if } p_A < q_A - q_B - 1 \\ p_B^*(p_A) = \frac{1}{2}(q_B - q_A + p_A + 1) & \text{if } q_A - q_B - 1 \leq p_A \leq q_A - q_B + 3 \\ p_B^*(p_A) = q_B - q_A + p_A - 1 & \text{if } p_A > q_A - q_B + 3. \end{cases}$$

B Proof of Proposition 3

The proof proceeds in two parts. In the first part, I derive the best reply of firm B. In the second part, I derive the quality of firm A.

Part 1: Best reply of firm B

In the second stage, firm B chooses its quality given the quality of firm A to maximize its profit given by (3.4) and (3.5). As the overall profit depends on the size of τ, the proof separately considers the cases $\tau \geq 3$ and $\tau < 3$.

Case $\tau \geq 3$: For $\tau \geq 3$, the overall profit of firm B is

$$\Pi_B(q_A, q_B) = -\frac{1}{2}q_B^2 + \begin{cases} 0 & \text{if } q_B < q_A - \tau \\ \frac{1}{2} & \text{if } q_A - \tau \leq q_B < q_A + \tau \\ q_B - q_A - 1 & \text{if } q_B \geq q_A + \tau. \end{cases}$$

(1) Firm B never chooses any $0 < q_B < q_A - \tau$ (only exists if $q_A > \tau$). Suppose firm B chooses any $0 \leq q_B < q_A - \tau$, then its profit is $\Pi_B(q_A, q_B) = -\frac{1}{2}q_B^2$ and all $q_B \in (0, q_A - \tau)$ are strictly dominated by $q_B = 0$. Thus the only remaining candidate for best reply from $0 \leq q_B < q_A - \tau$ is $q_B = 0$.

(2) Firm B never chooses any $\max\{0, q_A - \tau\} < q_B < q_A + \tau$. Suppose firm B chooses any $\max\{0, q_A - \tau\} \leq q_B < q_A + \tau$, then its profit is $\Pi_B(q_A, q_B) = \frac{1}{2} - \frac{1}{2}q_B^2$ and all $q_B \in (\max\{0, q_A - \tau\}, q_A + \tau)$ are strictly dominated by $q_B = \max\{0, q_A - \tau\}$. Thus the only remaining candidate for best reply from $\max\{0, q_A - \tau\} \leq q_B < q_A + \tau$ is $q_B = \max\{0, q_A - \tau\}$.

(3) Firm B never chooses any $q_B \geq q_A + \tau$. Suppose firm B would choose any $q_B \geq q_A + \tau$, then its profit is $\Pi_B(q_A, q_B) = q_B - q_A - 1 - \frac{1}{2}q_B^2 < 0$, but with $q_B = 0$ firm B would make at least zero profit. Thus no $q_B \geq q_A + \tau$ can be a best reply.

Thus for $q_A \leq \tau$, the only remaining candidate for best reply is $q_B = 0$. For $q_A > \tau$, the only remaining candidates for best reply are $q_B = q_A - \tau$ and $q_B = 0$. Firm B prefers $q_B = q_A - \tau$ to $q_B = 0$ if $\Pi_B(q_A, q_B = q_A - \tau) = \frac{1}{2} - \frac{1}{2}(q_A - \tau)^2 > 0 = \Pi_B(q_A, q_B = 0) \Leftrightarrow \tau - 1 < q_A < 1 + \tau$. The condition $q_A > \tau - 1$ is always fulfilled because $q_A > \tau$.

The best reply of firm B, for $\tau \geq 3$, is thus:

$$q_B^*(q_A) = \begin{cases} 0 & \text{if } q_A \leq \tau \\ q_A - \tau & \text{if } \tau < q_A < 1 + \tau \\ 0 & \text{if } q_A \geq 1 + \tau. \end{cases}$$

Case $\tau < 3$: (I) Assume $q_A > 3$. Then the overall profit of firm B is

$$\Pi_B(q_A, q_B) = -\frac{1}{2}q_B^2 + \begin{cases} 0 & \text{if } q_B < q_A - 3 \\ \frac{1}{2}\left(1 + \frac{q_B - q_A}{3}\right)^2 & \text{if } q_A - 3 \leq q_B < q_A - \tau \\ \frac{1}{2} & \text{if } q_A - \tau \leq q_B < q_A + \tau \\ \frac{1}{2}\left(1 + \frac{q_B - q_A}{3}\right)^2 & \text{if } q_A + \tau \leq q_B < q_A + 3 \\ q_B - q_A - 1 & \text{if } q_B \geq q_A + 3. \end{cases} \tag{3.11}$$

(1) Firm B never chooses any $0 < q_B < q_A - 3$. Suppose firm B chooses any $0 \leq q_B < q_A - 3$, then its profit is $\Pi_B(q_A, q_B) = -\frac{1}{2}q_B^2$ and all $q_B \in (0, q_A - 3)$ are strictly dominated by $q_B = 0$. Thus the only remaining candidate for best reply from $0 \leq q_B < q_A - 3$ is $q_B = 0$.

(2) For $q_A - 3 \leq q_B < q_A - \tau$, $\frac{\partial \Pi_B(q_A, q_B)}{\partial q_B} < 0$. Thus the candidate for best reply on $q_A - 3 \leq q_B < q_A - \tau$ is $q_B = q_A - 3$. But, $\Pi_B(q_A, q_B = q_A - 3) = -\frac{1}{2}(q_A - 3)^2 < 0$. Thus $q_B = q_A - 3$ is never a best reply because firm B can always make zero profit with $q_B = 0$.

(3) Firm B never chooses any $q_A - \tau < q_B < q_A + \tau$. Suppose firm B chooses any $q_A - \tau \leq q_B < q_A + \tau$, then its profit is $\Pi_B(q_A, q_B) = \frac{1}{2} - \frac{1}{2}q_B^2$ and all $q_B \in (q_A - \tau, q_A + \tau)$ are strictly dominated by $q_B = q_A - \tau$. Thus the only remaining candidate for best reply from $q_A - \tau \leq q_B < q_A + \tau$ is $q_B = q_A - \tau$.

(4) For $q_A + \tau \leq q_B < q_A + 3$, $\frac{\partial \Pi_B(q_A, q_B)}{\partial q_B} < 0$. Thus the candidate for best reply on $q_A + \tau \leq q_B < q_A + 3$ is $q_B = q_A + \tau$.

(5) Firm B never chooses any $q_B \geq q_A + 3$. Suppose firm B would choose any $q_B \geq q_A + \tau$, then its profit is $\Pi_B(q_A, q_B) = q_B - q_A - 1 - \frac{1}{2}q_B^2 < 0$, but with $q_B = 0$ firm B would make zero profit. Thus no $q_B \geq q_A + \tau$ can be a best reply.

As $\Pi_B(q_A, q_B = 0) = 0 > \frac{1}{2}\left(1 + \frac{\tau}{3}\right)^2 - \frac{1}{2}(q_A + \tau)^2 = \Pi_B(q_A, q_B = q_A + \tau)$ and $\Pi_B(q_A, q_B = 0) = 0 \geq \frac{1}{2} - \frac{1}{2}(q_A - \tau)^2 = \Pi_B(q_A, q_B = q_A - \tau) \Leftrightarrow q_A \geq 1 + \tau$, the best reply of firm B for $\tau < 3$ and $q_A > 3$ is:

- if $\tau \leq 2$: $q_B^*(q_A) = 0$
- if $2 < \tau < 3$:

$$q_B^*(q_A) = \begin{cases} q_A - \tau & \text{if } q_A < 1 + \tau \\ 0 & \text{if } q_A \geq 1 + \tau \end{cases}$$

(II) Assume $\tau < q_A \leq 3$. Then, the first case of (3.11) does not exist and the profit of firm B becomes

$$\Pi_B(q_A, q_B) = -\frac{1}{2}q_B^2 + \begin{cases} \frac{1}{2}\left(1 + \frac{q_B - q_A}{3}\right)^2 & \text{if } q_B < q_A - \tau \\ \frac{1}{2} & \text{if } q_A - \tau \leq q_B < q_A + \tau \\ \frac{1}{2}\left(1 + \frac{q_B - q_A}{3}\right)^2 & \text{if } q_A + \tau \leq q_B < q_A + 3 \\ q_B - q_A - 1 & \text{if } q_B \geq q_A + 3. \end{cases}$$

(1) For $0 \leq q_B < q_A - \tau$, $\frac{\partial \Pi_B(q_A, q_B)}{\partial q_B} > 0$ for all $q_B < \frac{3 - q_A}{8}$ and $\frac{\partial \Pi_B(q_A, q_B)}{\partial q_B} < 0$ for all $q_B > \frac{3 - q_A}{8}$. But, $\frac{3 - q_A}{8} \in [0, q_A - \tau) \Leftrightarrow q_A > \frac{3 + 8\tau}{9}$. Thus the candidate for best reply on $0 \leq q_B < q_A - \tau$ is

$$q_B = \begin{cases} \frac{3 - q_A}{8} & \text{if } q_A > \frac{3 + 8\tau}{9} \\ q_A - \tau - \varepsilon & \text{if } q_A \leq \frac{3 + 8\tau}{9} \end{cases}$$

where $0 < \varepsilon \leq q_A - \tau$. That is either firm B chooses $q_B = \frac{3 - q_A}{8}$ or if $\frac{3 - q_A}{8} \notin [0, q_A - \tau)$, I show in the following that no $q_B \in [0, q_A - \tau)$ can be a best reply.

(2) Firm B never chooses any $q_A - \tau < q_B < q_A + \tau$. Suppose firm B chooses any $q_A - \tau \leq q_B < q_A + \tau$, then its profit is $\Pi_B(q_A, q_B) = \frac{1}{2} - \frac{1}{2}q_B^2$ and all $q_B \in (q_A - \tau, q_A + \tau)$ are strictly dominated by $q_B = q_A - \tau$. Thus the only remaining candidate for best reply from $q_A - \tau \leq q_B < q_A + \tau$ is $q_B = q_A - \tau$.

(3) For $q_A + \tau \leq q_B < q_A + 3$, $\frac{\partial \Pi_B(q_A, q_B)}{\partial q_B} > 0$ for all $q_B < \frac{3 - q_A}{8}$ and $\frac{\partial \Pi_B(q_A, q_B)}{\partial q_B} < 0$ for all $q_B > \frac{3 - q_A}{8}$. But, $\frac{3 - q_A}{8} \in [q_A + \tau, q_A + 3) \Leftrightarrow q_A \leq \frac{3 - 8\tau}{9}$. As $\tau < q_A \leq 3$ has to hold, if $\frac{3 - 8\tau}{9} \geq 3$, $q_B = \frac{3 - q_A}{8}$ would be the candidate for best reply. However, this case never exists. In addition, if $\frac{3 - 8\tau}{9} \leq \tau \Leftrightarrow \tau \geq \frac{3}{17}$, then the boundary $q_B = q_A + \tau$ is the candidate for best reply. And if $\frac{3 - 8\tau}{9} \in (\tau, 3]$, $q_B = \frac{3 - q_A}{8}$ is the candidate for best reply if $q_A \leq \frac{3 - 8\tau}{9}$, and otherwise $q_B = q_A + \tau$ is the candidate for best reply. Thus the candidate for best reply on $q_A + \tau \leq q_B < q_A + 3$ is

- if $\tau < \frac{3}{17}$:

$$q_B = \begin{cases} \frac{3 - q_A}{8} & \text{if } q_A \leq \frac{3 - 8\tau}{9} \\ q_A + \tau & \text{if } q_A > \frac{3 - 8\tau}{9} \end{cases}$$

- if $\tau \geq \frac{3}{17}$: $q_B = q_A + \tau$.

(4) Firm B never chooses any $q_B \geq q_A + 3$. Suppose firm B would choose any $q_B \geq q_A + \tau$, then its profit is $\Pi_B(q_A, q_B) = q_B - q_A - 1 - \frac{1}{2}q_B^2 < 0$, but with $q_B = 0$ firm B would make at least zero profit. Thus no $q_B \geq q_A + \tau$ can be a best reply.

For firm B $q_B = q_A - \tau - \varepsilon$ is never a best reply: for all $\varepsilon > 0$ and as $q_a \leq \frac{3+8\tau}{9}$,

$$\begin{aligned}
&\Pi_B(q_A, q_B = q_A - \tau) > \Pi_B(q_A, q_B = q_A - \tau - \varepsilon) \\
\Leftrightarrow\ & \frac{1}{2} - \frac{1}{2}(q_A - \tau)^2 > \frac{1}{2}\left(1 - \frac{\tau + \varepsilon}{3}\right)^2 - \frac{1}{2}(q_A - \tau - \varepsilon)^2.
\end{aligned}$$

In addition, $q_B = q_A + \tau$ is never a best reply:

$$\begin{aligned}
&\Pi_B(q_A, q_B = q_A - \tau) > \Pi_B(q_A, q_B = q_A + \tau) \\
\Leftrightarrow\ & \frac{1}{2} - \frac{1}{2}(q_A - \tau)^2 > \frac{1}{2}\left(1 + \frac{\tau}{3}\right)^2 - \frac{1}{2}(q_A + \tau)^2 \Leftrightarrow q_A > \frac{6 + \tau}{36}.
\end{aligned}$$

Then, if $\tau \geq \frac{3}{17}$, $\frac{6+\tau}{36} < \tau$ and if $\tau < \frac{3}{17}$, $\frac{6+\tau}{36} < \frac{3-8\tau}{9}$ and thus $q_A > \frac{6+\tau}{36}$ is always fulfilled. Thus either $q_B = \frac{3-q_A}{8}$ or $q_B = q_A - \tau$ is the best reply. Note that $q_B = \frac{3-q_A}{8}$ can only be a best reply if either $q_A > \frac{3+8\tau}{9}$ or $\tau < \frac{3}{17}$ and $q_A \leq \frac{3-8\tau}{9}$. Furthermore,

$$\begin{aligned}
&\Pi_B(q_A, q_B = q_A - \tau) \geq \Pi_B(q_A, q_B = \frac{3 - q_A}{8}) \\
\Leftrightarrow\ & \frac{1}{2} - \frac{1}{2}(q_A - \tau)^2 \geq 4\left(\frac{3 - q_A}{8}\right)^2 \\
\Leftrightarrow\ & \frac{3 + 8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq \frac{3 + 8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2}.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\frac{3 + 8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} > \frac{3 + 8\tau}{9} \text{ for all } \tau \\
\tau < &\frac{3 + 8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq 3 \ \Leftrightarrow\ \tau \leq 2 \\
&\frac{3 + 8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2} < \frac{3 + 8\tau}{9} \text{ for all } \tau \\
&\frac{3 + 8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2} < \frac{3 - 8\tau}{9} \ \Leftrightarrow\ \tau < \frac{2}{11} \\
\tau < &\frac{3 + 8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq 3 \ \Leftrightarrow\ \tau < 3 - \sqrt{8}
\end{aligned}$$

Therefore, the best reply for firm B for $\tau < 3$ and $\tau < q_A \leq 3$ is:

- if $\tau < 3 - \sqrt{8}$: $q_B^*(q_A) = \begin{cases} \frac{3-q_A}{8} & \text{if } q_A < \frac{3+8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2} \\ q_A - \tau & \text{if } \frac{3+8\tau}{9} - \frac{\sqrt{48\tau-8\tau^2}}{9} \leq q_A < \frac{3+8\tau}{9} + \frac{\sqrt{48\tau-8\tau^2}}{9} \\ \frac{3-q_A}{8} & \text{if } q_A \geq \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \end{cases}$

- if $3 - \sqrt{8} \leq \tau \leq 2$: $q_B^*(q_A) = \begin{cases} q_A - \tau & \text{if } q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \\ \frac{3-q_A}{8} & \text{if } q_A \geq \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \end{cases}$

- if $2 < \tau$: $q_B^*(q_A) = q_A - \tau$

(III) Assume $0 \leq q_A \leq \tau$. Then, the first and the second case of (3.11) do not exist and the profit of firm B becomes

$$\Pi_B(q_A, q_B) = -\frac{1}{2}q_B^2 + \begin{cases} \frac{1}{2} & \text{if } q_B < q_A + \tau \\ \frac{1}{2}\left(1 + \frac{q_B - q_A}{3}\right)^2 & \text{if } q_A + \tau \leq q_B < q_A + 3 \\ q_B - q_A - 1 & \text{if } q_B \geq q_A + 3. \end{cases}$$

(1) Firm B never chooses any $0 < q_B < q_A + \tau$. Suppose firm B chooses any $0 \leq q_B < q_A + \tau$, then its profit is $\Pi_B(q_A, q_B) = \frac{1}{2} - \frac{1}{2}q_B^2$ and all $q_B \in (0, q_A + \tau)$ are strictly dominated by $q_B = 0$. Thus the only remaining candidate for best reply from $0 \leq q_B < q_A + \tau$ is $q_B = 0$.

(2) For $q_A + \tau \leq q_B < q_A + 3$, $\frac{\partial \Pi_B(q_A, q_B)}{\partial q_B} > 0$ for all $q_B < \frac{3 - q_A}{8}$ and $\frac{\partial \Pi_B(q_A, q_B)}{\partial q_B} < 0$ for all $q_B > \frac{3 - q_A}{8}$. But, $\frac{3 - q_A}{8} \in [q_A + \tau, q_A + 3) \Leftrightarrow q_A \leq \frac{3 - 8\tau}{9}$. At the same time it has to hold that $0 \leq q_A \leq \tau$. Thus the candidate for best reply on $q_A + \tau \leq q_B < q_A + 3$ is either $q_B = \frac{3 - q_A}{8}$ or the boundary $q_B = q_A + \tau$. Therefore, the candidate for best reply is

- if $\tau \leq \frac{3}{17}$: $q_B = \frac{3 - q_A}{8}$
- if $\frac{3}{17} < \tau \leq \frac{3}{8}$: $q_B = \begin{cases} \frac{3 - q_A}{8} & \text{if } q_A \leq \frac{3 - 8\tau}{9} \\ q_A + \tau & \text{if } q_A > \frac{3 - 8\tau}{9} \end{cases}$
- if $\tau > \frac{3}{8}$: $q_B = q_A + \tau$.

(3) Firm B never chooses any $q_B \geq q_A + 3$. Suppose firm B would choose any $q_B \geq q_A + \tau$, then its profit is $\Pi_B(q_A, q_B) = q_B - q_A - 1 - \frac{1}{2}q_B^2 < 0$, but with $q_B = 0$ firm B would make at least zero profit. Thus no $q_B \geq q_A + \tau$ can be a best reply.

Note that

$$\Pi_B(q_A, q_B = 0) = \frac{1}{2} \geq 4\left(\frac{3 - q_A}{8}\right)^2 = \Pi_B\left(q_A, q_B = \frac{3 - q_A}{8}\right) \Leftrightarrow q_A \geq 3 - \sqrt{8} \tag{3.12}$$

and

$$\begin{aligned} \Pi_B(q_A, q_B = 0) = \frac{1}{2} &\geq \frac{1}{2}\left(1 + \frac{\tau}{3}\right)^2 - \frac{1}{2}(q_A + \tau)^2 = \Pi_B(q_A, q_B = q_A + \tau) \\ \Leftrightarrow q_A &\geq -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}. \end{aligned} \tag{3.13}$$

If $\tau \leq \frac{3}{17}$, either $q_B = 0$ or $q_B = \frac{3 - q_A}{8}$ is the best reply. According to condition (3.12), $q_B = \frac{3 - q_A}{8}$ is the best reply if $q_A < 3 - \sqrt{8}$ and $q_B = 0$ is the best reply if $q_A \geq 3 - \sqrt{8}$ (does not exist if $\tau < 3 - \sqrt{8}$).

If $\frac{3}{17} < \tau \leq \frac{3}{8}$ and $q_A \leq \frac{3-8\tau}{9}$, either $q_B = 0$ or $q_B = \frac{3-q_A}{8}$ is the best reply. According to condition (3.12), $q_B = \frac{3-q_A}{8}$ is the best reply if $q_A < 3 - \sqrt{8}$ and $q_B = 0$ is the best reply if $q_A \geq 3 - \sqrt{8}$. But $3 - \sqrt{8} \leq \frac{3-8\tau}{9} \Leftrightarrow \tau \leq \frac{3}{4}\left(3\sqrt{2} - 4\right)$.

If $\frac{3}{17} < \tau \leq \frac{3}{8}$ and $q_A > \frac{3-8\tau}{9}$, either $q_B = 0$ or $q_B = q_A + \tau$ is the best reply. According to condition (3.13), $q_B = q_A + \tau$ is the best reply if $q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$ and $q_B = 0$ is the best reply if $q_A \geq -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$. But $q_B = q_A + \tau$ can only be the best reply if $q_A > \frac{3-8\tau}{9}$ and $q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$; these conditions are only jointly satisfied if $\tau > \frac{3}{4}\left(3\sqrt{2} - 4\right)$.

If $\tau > \frac{3}{8}$, either $q_B = 0$ or $q_B = q_A + \tau$ is the best reply. According to condition (3.13), $q_B = q_A + \tau$ is the best reply if $q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$ and $q_B = 0$ is the best reply if $q_A \geq -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$. But $-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \in [0, \tau) \Leftrightarrow \tau \leq \frac{3}{4}$.

Therefore, the best reply for firm B for $\tau < 3$ and $0 \leq q_A \leq \tau$ is:

- if $\tau < 3 - \sqrt{8}$: $q_B^*(q_A) = \frac{3-q_A}{8}$

- if $3 - \sqrt{8} \leq \tau \leq \frac{3}{17}$: $q_B^*(q_A) = \begin{cases} \frac{3-q_A}{8} & \text{if } q_A < 3 - \sqrt{8} \\ 0 & \text{if } q_A \geq 3 - \sqrt{8} \end{cases}$

- if $\frac{3}{17} < \tau \leq \frac{3}{4}(3\sqrt{2} - 4)$: $q_B^*(q_A) = \begin{cases} \frac{3-q_A}{8} & \text{if } q_A < 3 - \sqrt{8} \\ 0 & \text{if } q_A \geq 3 - \sqrt{8} \end{cases}$

- if $\frac{3}{4}(3\sqrt{2} - 4) < \tau \leq \frac{3}{8}$: $q_B^*(q_A) = \begin{cases} \frac{3-q_A}{8} & \text{if } q_A \leq \frac{3-8\tau}{9} \\ q_A + \tau & \text{if } \frac{3-8\tau}{9} < q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \\ 0 & \text{if } -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \leq q_A \end{cases}$

- if $\frac{3}{8} < \tau \leq \frac{3}{4}$: $q_B^*(q_A) = \begin{cases} q_A + \tau & \text{if } q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \\ 0 & \text{if } q_A \geq -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \end{cases}$

- if $\frac{3}{4} < \tau$: $q_B^*(q_A) = 0$

The cases $3 - \sqrt{8} \leq \tau \leq \frac{3}{17}$ and $\frac{3}{17} < \tau \leq \frac{3}{4}(3\sqrt{2} - 4)$ can be summarized into one case $3 - \sqrt{8} \leq \tau \leq \frac{3}{4}(3\sqrt{2} - 4)$.

To derive the best reply of firm B fully, it is necessary to bring together the cases $q_A > 3$, $\tau < q_A \leq 3$, and $0 \leq q_A \leq \tau$. Instead of summarizing the best reply of firm B here, I will in the analysis of firm A's quality decision start each case with the best reply of firm B.

Part 2: Quality of firm A

Generally, the profit of firm A is

$$\Pi_A(q_A, q_B) = -\frac{1}{2}q_A^2 + \begin{cases} 0 & \text{if } q_B \geq q_A + 3 \\ \frac{1}{2}\left(1 + \frac{q_A - q_B}{3}\right)^2 & \text{if } q_A + \tau \leq q_B < q_A + 3 \\ \frac{1}{2} & \text{if } q_A - \tau \leq q_B < q_A + \tau \\ \frac{1}{2}\left(1 + \frac{q_A - q_B}{3}\right)^2 & \text{if } q_A - 3 \leq q_B < q_A - \tau \\ q_A - q_B - 1 & \text{if } q_B < q_A - 3. \end{cases}$$

The profit of firm A depends on the response of firm B. In the first stage, firm A maximizes its profit taking the response of firm B into account. The following proof uses the fact that

$$\frac{21}{55} = \text{argmax}_{q_A} \frac{1}{2}\left(1 + \frac{q_A - \frac{3-q_A}{8}}{3}\right)^2 - \frac{1}{2}q_A^2$$

and that the corresponding profit is $\Pi_A(q_A = \frac{21}{55}) = \frac{49}{110}$. That means if the profit of firm A is $\Pi_A(q_A) = \frac{1}{2}\left(1 + \frac{q_A - \frac{3-q_A}{8}}{3}\right)^2 - \frac{1}{2}q_A^2$, then with any q_A, $\Pi_A(q_A) \leq \frac{49}{110}$.

(i) Case $\tau < 3 - \sqrt{8}$:

If $\tau < 3 - \sqrt{8}$, the best reply of firm B is

$$q_B^*(q_A) = \begin{cases} \frac{3-q_A}{8} & \text{if } q_A < \frac{3+8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2} \\ q_A - \tau & \text{if } \frac{3+8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \\ \frac{3-q_A}{8} & \text{if } \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3 \\ 0 & \text{if } q_A > 3. \end{cases}$$

Then, if firm A chooses any $q_A < \frac{3+8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2}$, firm B responds with $q_B = \frac{3-q_A}{8}$. That is, $q_B \geq q_A + \tau$ which means the quality difference is noticeable. If firm A chooses any $\frac{3+8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2}$, firm B responds with $q_B = q_A - \tau$, which means the quality difference is not noticeable. If firm A chooses any $\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3$, firm B responds with $q_B = \frac{3-q_A}{8}$. That is, $q_B < q_A - \tau$ which means the quality difference is noticeable. If firm A chooses any $q_A > 3$, firm B responds with $q_B = 0$ which means the quality difference is noticeable.

Thus the profit of firm A is

$$\Pi_A(q_A) = -\frac{1}{2}q_A^2 + \begin{cases} \frac{1}{2}\left(1+\frac{q_A-\frac{3-q_A}{8}}{3}\right)^2 & \text{if } q_A < \frac{3+8\tau}{9} - \frac{1}{9}\sqrt{48\tau-8\tau^2} \\ \frac{1}{2} & \text{if } \frac{3+8\tau}{9} - \frac{\sqrt{48\tau-8\tau^2}}{9} \leq q_A < \frac{3+8\tau}{9} + \frac{\sqrt{48\tau-8\tau^2}}{9} \\ \frac{1}{2}\left(1+\frac{q_A-\frac{3-q_A}{8}}{3}\right)^2 & \text{if } \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2} \leq q_A \leq 3 \\ q_A - q_B - 1 & \text{if } q_A > 3. \end{cases}$$

(1) For $q_A < \frac{3+8\tau}{9} - \frac{1}{9}\sqrt{48\tau-8\tau^2}$, $\frac{21}{55} = \text{argmax}_{q_A}\Pi_A(q_A)$ with $\Pi_A(q_A = \frac{21}{55}) = \frac{49}{110}$. Furthermore, $\frac{21}{55} \in [0, \frac{3+8\tau}{9} - \frac{1}{9}\sqrt{48\tau-8\tau^2}) \Leftrightarrow \tau > \frac{21+\sqrt{433}}{55}$, which is never the case, because $\tau < 3-\sqrt{8}$.

(2) For $\frac{3+8\tau}{9} - \frac{\sqrt{48\tau-8\tau^2}}{9} \leq q_A < \frac{3+8\tau}{9} + \frac{\sqrt{48\tau-8\tau^2}}{9}$, $\frac{\partial\Pi_A(q_A)}{\partial q_A} < 0$, i.e., firm A makes the highest profit with $q_A = \frac{3+8\tau}{9} - \frac{\sqrt{48\tau-8\tau^2}}{9}$:
$\Pi_A(q_A = \frac{3+8\tau}{9} - \frac{\sqrt{48\tau-8\tau^2}}{9}) = \frac{1}{2} - \frac{1}{2}\left(\frac{3+8\tau}{9} - \frac{\sqrt{48\tau-8\tau^2}}{9}\right)^2$.

(3) For $\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2} \leq q_A \leq 3$, the profit reaches its maximum at $q_A = \frac{21}{55}$ with $\Pi_A(q_A = \frac{21}{55}) = \frac{49}{110}$. Furthermore, $\frac{21}{55} \in [\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2}, 3] \Leftrightarrow \tau \leq \frac{21-\sqrt{433}}{55}$

(4) For $q_A > 3$, $\Pi_A(q_A) = q_A - q_B - 1 - \frac{1}{2}q_A^2 < 0$.

Note that $\sqrt{\frac{6}{55}} - \frac{1}{2}\sqrt{3\sqrt{\frac{6}{55}} - \frac{61}{110}} < \frac{21-\sqrt{433}}{55}$. Thus for $\tau \leq \sqrt{\frac{6}{55}} - \frac{1}{2}\sqrt{3\sqrt{\frac{6}{55}} - \frac{61}{110}}$, $\frac{21}{55} \in [\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2}, 3]$. In addition,

$$\frac{49}{110} < \frac{1}{2} - \frac{1}{2}\left(\frac{3+8\tau}{9} - \frac{\sqrt{48\tau-8\tau^2}}{9}\right)^2 \Leftrightarrow \tau > \sqrt{6/55} - 1/2\sqrt{3\sqrt{6/55} - 61/110}$$

Thus for $\sqrt{6/55} - 1/2\sqrt{3\sqrt{6/55} - 61/110} < \tau < 3-\sqrt{8}$, in the pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = \frac{3+8\tau}{9} - \frac{\sqrt{48\tau-8\tau^2}}{9}$ and $q_B^* = \frac{3-\tau}{9} - \frac{\sqrt{48\tau-8\tau^2}}{9}$ and, for $\tau \leq \sqrt{6/55} - 1/2\sqrt{3\sqrt{6/55} - 61/110}$, in the pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = \frac{21}{55}$ and $q_B^* = \frac{18}{55}$.

(ii) Case $3-\sqrt{8} \leq \tau \leq \frac{3}{4}(3\sqrt{2}-4)$:

If $3-\sqrt{8} \leq \tau \leq \frac{3}{4}(3\sqrt{2}-4)$, the best reply of firm B is

$$q_B^*(q_A) = \begin{cases} \frac{3-q_A}{8} & \text{if } q_A < 3-\sqrt{8} \\ 0 & \text{if } 3-\sqrt{8} \leq q_A \leq \tau \\ q_A - \tau & \text{if } \tau < q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \\ \frac{3-q_A}{8} & \text{if } \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2} \leq q_A \leq 3 \\ 0 & \text{if } q_A > 3. \end{cases}$$

Then, if firm A chooses any $q_A < 3-\sqrt{8}$, firm B responds with $q_B = \frac{3-q_A}{8}$. That is, $q_B \geq q_A+\tau$ which means the quality difference is noticeable. If firm A chooses any $3-\sqrt{8} \leq q_A \leq \tau$, firm B responds with $q_B = 0$. If firm A chooses any $\tau < q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2}$, firm B responds with $q_B = q_A - \tau$. In both cases $q_A - \tau \leq q_B < q_A + \tau$ which means the quality difference is unnoticeable. If firm A chooses any $\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2} \leq q_A \leq 3$, firm B responds with $q_B = \frac{3-q_A}{8}$. That is, $q_B < q_A - \tau$ which means the quality difference is noticeable. If firm A chooses any $q_A > 3$, firm B responds with $q_B = 0$ which means the quality difference is noticeable. Thus the profit of firm A is

$$\Pi_A(q_A) = -\frac{1}{2}q_A^2 + \begin{cases} \frac{1}{2}\left(1 + \frac{q_A - \frac{3-q_A}{8}}{3}\right)^2 & \text{if } q_A < 3-\sqrt{8} \\ \frac{1}{2} & \text{if } 3-\sqrt{8} \leq q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2} \\ \frac{1}{2}\left(1 + \frac{q_A - \frac{3-q_A}{8}}{3}\right)^2 & \text{if } \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2} \leq q_A \leq 3 \\ q_A - q_B - 1 & \text{if } q_A > 3. \end{cases}$$

(1) For $q_A < 3-\sqrt{8}$, $\Pi_A(q_A) \leq \Pi_A(q_A = \frac{21}{55}) = \frac{49}{110}$.

(2) For $3-\sqrt{8} \leq q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2}$, $\frac{\partial \Pi_A(q_A)}{\partial q_A} < 0$, i.e., firm A makes the highest profit with $q_A = 3-\sqrt{8}$: $\Pi_A(q_A = 3-\sqrt{8}) = \frac{1}{2} - \frac{1}{2}\left(3-\sqrt{8}\right)^2$.

(4) For $\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau-8\tau^2} \leq q_A \leq 3$, $\Pi_A(q_A) \leq \Pi_A(q_A = \frac{21}{55}) = \frac{49}{110}$.

(5) For $q_A > 3$, $\Pi_A(q_A) = q_A - q_B - 1 - \frac{1}{2}q_A^2 < 0$.

As $0 < \frac{49}{110} < \frac{1}{2} - \frac{1}{2}\left(3-\sqrt{8}\right)^2$, firm A chooses $q_A = 3-\sqrt{8}$. Thus for $3-\sqrt{8} \leq \tau \leq \frac{3}{4}(3\sqrt{2}-4)$, in the pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = 3-\sqrt{8}$ and $q_B^* = 0$.

(iii) Case $\frac{3}{4}(3\sqrt{2}-4) < \tau \leq \frac{3}{8}$:

If $\frac{3}{4}(3\sqrt{2}-4) < \tau \leq \frac{3}{8}$, the best reply of firm B is

$$q_B^*(q_A) = \begin{cases} \frac{3-q_A}{8} & \text{if } q_A \leq \frac{3-8\tau}{9} \\ q_A + \tau & \text{if } \frac{3-8\tau}{9} < q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \\ 0 & \text{if } -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \leq q_A \leq \tau \\ q_A - \tau & \text{if } \tau < q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \\ \frac{3-q_A}{8} & \text{if } \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3 \\ 0 & \text{if } q_A > 3. \end{cases}$$

Then, if firm A chooses any $q_A \leq \frac{3-8\tau}{9}$, firm B responds with $q_B = \frac{3-q_A}{8}$. That is $q_B \geq q_A + \tau$ which means the quality difference is noticeable. If firm A chooses any $\frac{3-8\tau}{9} < q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$, firm B responds with $q_B = q_A + \tau$, which means the quality difference is noticeable. If firm A chooses any $-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \leq q_A \leq \tau$, firm B responds with $q_B = 0$. If firm A chooses any $\tau < q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2}$, firm B responds with $q_B = q_A - \tau$. In both cases, $q_A - \tau \leq q_B < q_A + \tau$ which means the quality difference is unnoticeable. If firm A chooses any $\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3$, firm B responds with $q_B = \frac{3-q_A}{8}$. That is, $q_B < q_A - \tau$ which means the quality difference is noticeable. If $q_A > 3$, firm B responds with $q_B = 0$, which means the quality difference is noticeable. Thus the profit of firm A is

$$\Pi_A(q_A) = -\frac{1}{2}q_A^2 + \begin{cases} \frac{1}{2}\left(1 + \frac{q_A - \frac{3-q_A}{8}}{3}\right)^2 & \text{if } q_A \leq \frac{3-8\tau}{9} \\ \frac{1}{2}\left(1 - \frac{\tau}{3}\right)^2 & \text{if } \frac{3-8\tau}{9} < q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \\ \frac{1}{2} & \text{if } -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \leq q_A < \frac{3+8\tau}{9} + \frac{\sqrt{48\tau - 8\tau^2}}{9} \\ \frac{1}{2}\left(1 + \frac{q_A - \frac{3-q_A}{8}}{3}\right)^2 & \text{if } \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3 \\ q_A - q_B - 1 & \text{if } q_A > 3. \end{cases}$$

(1) For $q_A \leq \frac{3-8\tau}{9}$, $\Pi_A(q_A) \leq \Pi_A(q_A = \frac{21}{55}) = \frac{49}{110}$.

(2) For $\frac{3-8\tau}{9} < q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$, the profit is $\Pi_A(q_A) = \frac{1}{2}\left(1 - \frac{\tau}{3}\right)^2 - \frac{1}{2}q_A^2$

(3) For $-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \leq q_A < \frac{3+8\tau}{9} + \frac{\sqrt{48\tau - 8\tau^2}}{9}$, $\frac{\partial \Pi_A(q_A)}{\partial q_A} < 0$, i.e., firm A makes the highest profit with $q_A = -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$:
$\Pi_A(q_A = -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}) = \frac{1}{2} - \frac{1}{2}\left(-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}\right)^2$.

(4) For $\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3$, $\Pi_A(q_A) \leq \Pi_A(q_A = \frac{21}{55}) = \frac{49}{110}$.

(5) For $q_A > 3$, $\Pi_A(q_A) = q_A - q_B - 1 - \frac{1}{2}q_A^2 < 0$.

As $0 < \frac{49}{110} < \frac{1}{2} - \frac{1}{2}\left(-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}\right)^2$ and $\frac{1}{2}\left(1 - \frac{\tau}{3}\right)^2 - \frac{1}{2}q_A^2 < \frac{49}{110}$, firm A chooses $q_A = -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$. Thus for $\frac{3}{4}(3\sqrt{2} - 4) < \tau \leq \frac{3}{8}$, in the pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$ and $q_B^* = 0$.

(iv) Case $\frac{3}{8} < \tau \leq \frac{3}{4}$:

If $\frac{3}{8} < \tau \leq \frac{3}{4}$, the best reply of firm B is

$$q_B^*(q_A) = \begin{cases} q_A + \tau & \text{if } q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \\ 0 & \text{if } -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \leq q_A \leq \tau \\ q_A - \tau & \text{if } \tau < q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \\ \frac{3-q_A}{8} & \text{if } \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3 \\ 0 & \text{if } q_A > 3. \end{cases}$$

Then, if firm A chooses any $q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$, firm B responds with $q_B = q_A + \tau$. That is, $q_B \geq q_A + \tau$ which means the quality difference is noticeable. If firm A chooses any $-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \leq q_A \leq \tau$, firm B responds with $q_B = 0$. That is, $q_A - \tau \leq q_B < q_A + \tau$ which means the quality difference is unnoticeable. If firm A chooses any $\tau < q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2}$, firm B responds with $q_B = q_A - \tau$ which means the quality difference is unnoticeable. If firm A chooses any $\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3$, firm B responds with $q_B = \frac{3-q_A}{8}$. That is, $q_B < q_A - \tau$ which means the quality difference is noticeable. If firm A chooses any $q_A > 3$, firm B responds with $q_B = 0$. That is, $q_B < q_A - \tau$ and $q_A - q_B > 3$ which means the quality difference is noticeable. Thus the profit of firm A is

$$\Pi_A(q_A) = -\frac{1}{2}q_A^2 + \begin{cases} \frac{1}{2}\left(1 - \frac{\tau}{3}\right)^2 & \text{if } q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \\ \frac{1}{2} & \text{if } -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \leq q_A < \frac{3+8\tau}{9} + \frac{\sqrt{48\tau - 8\tau^2}}{9} \\ \frac{1}{2}\left(1 + \frac{q_A - \frac{3-q_A}{8}}{3}\right)^2 & \text{if } \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3 \\ q_A - q_B - 1 & \text{if } q_A > 3. \end{cases}$$

(1) For $q_A < -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$, $\frac{\partial \Pi_A(q_A)}{\partial q_A} < 0$. Thus on $[0, -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2})$, firm A makes the highest profit with $q_A = 0$: $\Pi_A(q_A = 0) = \frac{1}{2}\left(1 - \frac{\tau}{3}\right)^2$.

(2) For $-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} \leq q_A < \frac{3+8\tau}{9} + \frac{\sqrt{48\tau - 8\tau^2}}{9}$, $\frac{\partial \Pi_A(q_A)}{\partial q_A} < 0$, i.e., firm A makes the highest profit with $q_A = -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$:
$\Pi_A(q_A = -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}) = \frac{1}{2} - \frac{1}{2}\left(-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}\right)^2$.

(3) For $\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3$, $\Pi_A(q_A) \leq \Pi_A(q_A = \frac{21}{55}) = \frac{49}{110}$,

(4) For $q_A > 3$, $\Pi_A(q_A) = q_A - q_B - 1 - \frac{1}{2}q_A^2 < 0$.

As $0 < \frac{49}{110} < \frac{1}{2} - \frac{1}{2}\left(-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}\right)^2$ and $\frac{49}{110} > \frac{1}{2}\left(1 - \frac{\tau}{3}\right)^2$, firm A chooses $q_A = -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$. Thus for $\frac{3}{8} < \tau \leq \frac{3}{4}$, in the pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = -\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}$ and $q_B^* = 0$.

(v) Case $\frac{3}{4} < \tau \leq 2$:

If $\frac{3}{4} < \tau \leq 2$, the best reply of firm B is

$$q_B^*(q_A) = \begin{cases} 0 & \text{if } q_A \leq \tau \\ q_A - \tau & \text{if } \tau < q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \\ \frac{3-q_A}{8} & \text{if } \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3 \\ 0 & \text{if } q_A > 3. \end{cases}$$

Then, if firm A chooses any $q_A \leq \tau$, firm B responds by choosing $q_B = 0$. That is, $q_A - \tau \leq q_B < q_A + \tau$ which means the quality difference is unnoticeable. If firm A chooses any $\tau < q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2}$, firm B will respond with $q_B = q_A - \tau$. That is, $q_A - \tau \leq q_B < q_A + \tau$ which means the quality difference is unnoticeable. If firm A chooses any $\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3$, firm B will respond with $q_B = \frac{3-q_A}{8}$. That is, $q_B < q_A - \tau$ which means the quality difference is noticeable. If firm A chooses any $q_A > 3$, firm B responds with $q_B = 0$. That is, $q_B < q_A - \tau$ which means the quality difference is noticeable. Thus the profit of firm A is

$$\Pi_A(q_A) = -\frac{1}{2}q_A^2 + \begin{cases} \frac{1}{2} & \text{if } q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \\ \frac{1}{2}\left(1 + \frac{q_A - \frac{3-q_A}{8}}{3}\right)^2 & \text{if } \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3 \\ q_A - q_B - 1 & \text{if } q_A > 3. \end{cases}$$

(1) For $q_A < \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2}$, $\frac{\partial \Pi_A(q_A)}{\partial q_A} < 0$. Thus on $[0, \frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2})$, firm A makes the highest profit with $q_A = 0$: $\Pi_A(q_A = 0) = \frac{1}{2}$.

(2) For $\frac{3+8\tau}{9} + \frac{1}{9}\sqrt{48\tau - 8\tau^2} \leq q_A \leq 3$, $\Pi_A(q_A) \leq \Pi_A(q_A = \frac{21}{55}) = \frac{49}{110}$.

(3) For $q_A > 3$, $\Pi_A(q_A) = q_A - q_B - 1 - \frac{1}{2}q_A^2 < 0$.

As $\frac{1}{2} > \frac{49}{110} > 0$, firm A chooses $q_A = 0$. Thus if $\frac{3}{4} < \tau \leq 2$, in the pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = q_B^* = 0$.

(vi) Case $\tau > 2$:

If $\tau > 2$, the best reply of firm B is

$$q_B^*(q_A) = \begin{cases} 0 & \text{if } q_A \leq \tau \\ q_A - \tau & \text{if } \tau < q_A < 1 + \tau \\ 0 & \text{if } q_A \geq 1 + \tau \end{cases}$$

Then, if firm A chooses any $q_A \leq \tau$, firm B responds by choosing $q_B = 0$. That means, $q_A - \tau \leq q_B < q_A + \tau$ which means the quality difference is unnoticeable. If firm A chooses any $\tau < q_A < 1 + \tau$, firm B will respond with $q_B = q_A - \tau$. That is, firm B will unnoticeable undercut firm A. If firm A chooses any $q_A \geq 1 + \tau$, firm B will respond with $q_B = 0$. That is, firm B will noticeably undercut firm A. Thus the profit of firm A is

$$\Pi_A(q_A) = -\frac{1}{2}q_A^2 + \begin{cases} \frac{1}{2} & \text{if } q_A < 1 + \tau \\ q_A - q_B - 1 & \text{if } q_A \geq 1 + \tau \end{cases}$$

(1) For $q_A < 1 + \tau$, $\frac{\partial \Pi_A(q_A)}{\partial q_A} < 0$. Thus firm A receives the highest profit on $[0, 1 + \tau)$ with $q_A = 0$: $\Pi_A(q_A = 0) = \frac{1}{2}$.

(2) For $q_A \geq 1 + \tau$, $\Pi_A(q_A) = q_A - q_B - 1 - \frac{1}{2}q_A^2 < 0$.

As $\frac{1}{2} > 0$, firm A chooses $q_A = 0$. Thus if $\tau > 2$, in the pure-strategy subgame-perfect equilibrium, firms choose $q_A^* = q_B^* = 0$.

C Proof of Proposition 4

i) If $0 \leq \tau \leq \sqrt{6/55} - 1/2\sqrt{3\sqrt{6/55} - 61/110} \equiv \tau_a$, in the unique subgame-perfect equilibrium firms choose $q_A^* = 21/55$ and $q_B^* = 18/55$ and $p_A^* = 56/55$ and $p_B^* = 54/55$. Then, the indifferent consumer is $\bar{x} = 28/55$. The producer surplus, consumer surplus, and welfare are

$$
\begin{aligned}
PS &= \Pi_A + \Pi_B = \frac{56}{55}\frac{28}{55} - \frac{1}{2}\left(\frac{21}{55}\right)^2 + \frac{54}{55}\left(1 - \frac{28}{55}\right) - \frac{1}{2}\left(\frac{18}{55}\right)^2 = \frac{49}{110} + \frac{1296}{3025} = \frac{5287}{6050} \\
CS &= \int_0^{\frac{28}{55}} v + \frac{21}{55} - \frac{56}{55} - x^2 dx + \int_{\frac{28}{55}}^1 v + \frac{18}{55} - \frac{54}{55} - (1-x)^2 dx = v - \frac{6613}{9075} \\
W &= PS + CS = v + \frac{527}{3630}.
\end{aligned}
$$

ii) If $\tau_a < \tau < 3 - \sqrt{8}$, in the unique subgame-perfect equilibrium firms choose $q_A^* = (3 + 8\tau)/9 - \sqrt{48\tau - 8\tau^2}/9$ and $q_B^* = (3 - \tau)/9 - \sqrt{48\tau - 8\tau^2}/9$ and $p_A^* = p_B^* = 1$. Then, the indifferent consumer is $\bar{x} = 1/2$. The producer surplus, consumer surplus, and welfare are

$$
\begin{aligned}
PS &= \Pi_A + \Pi_B = \frac{1}{2} - \frac{1}{2}\left(\frac{3+8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2}\right)^2 + \frac{1}{2} - \frac{1}{2}\left(\frac{3-\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2}\right)^2 \\
CS &= \int_0^{\frac{1}{2}} v + \frac{3+8\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2} - 1 - x^2 dx \\
&\quad + \int_{\frac{1}{2}}^1 v + \frac{3-\tau}{9} - \frac{1}{9}\sqrt{48\tau - 8\tau^2} - 1 - (1-x)^2 dx = v - \frac{3}{4} + \frac{7}{18}\tau - \frac{1}{9}\sqrt{48\tau - 8\tau^2} \\
W &= PS + CS = v + \frac{5}{36} - \frac{25}{54}\tau - \frac{49}{162}\tau^2 + \left(\frac{7\tau}{81} - \frac{1}{27}\right)\sqrt{48\tau - 8\tau^2}.
\end{aligned}
$$

iii) If $3 - \sqrt{8} \leq \tau \leq 3/4(3\sqrt{2} - 4)$, in the unique subgame-perfect equilibrium firms choose $q_A^* = 3 - \sqrt{8}$ and $q_B^* = 0$ and $p_A^* = p_B^* = 1$. Then, the indifferent consumer is $\bar{x} = 1/2$. The producer surplus, consumer surplus, and welfare are

$$
\begin{aligned}
PS =& \Pi_A + \Pi_B = \frac{1}{2} - \frac{1}{2}\left(3 - \sqrt{8}\right)^2 + \frac{1}{2} \\
CS =& \int_0^{\frac{1}{2}} v + 3 - \sqrt{8} - 1 - x^2 dx + \int_{\frac{1}{2}}^1 v - 1 - (1-x)^2 dx = v + \frac{5}{12} - \sqrt{2} \\
W =& PS + CS = v - \frac{85}{12} + 5\sqrt{2}.
\end{aligned}
$$

iv) If $3/4(3\sqrt{2} - 4) < \tau \leq 3/4$, in the unique subgame-perfect equilibrium firms choose $q_A^* = -\tau + \sqrt{2/3\tau + 1/9\tau^2}$ and $q_B^* = 0$ and $p_A^* = p_B^* = 1$. Then, the indifferent consumer is $\bar{x} = 1/2$. The producer surplus, consumer surplus, and welfare are

$$PS = \Pi_A + \Pi_B = \frac{1}{2} - \frac{1}{2}\left(-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}\right)^2 + \frac{1}{2}$$

$$CS = \int_0^{\frac{1}{2}} v - \tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2} - 1 - x^2 dx + \int_{\frac{1}{2}}^1 v - 1 - (1-x)^2 dx$$

$$= v - \frac{13}{12} + \frac{1}{2}\left(-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}\right)$$

$$W = PS + CS = v - \frac{1}{12} - \frac{1}{2}\left(-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}\right)^2 + \frac{1}{2}\left(-\tau + \sqrt{\frac{2}{3}\tau + \frac{1}{9}\tau^2}\right).$$

vi) If $\tau > 3/4$, in the unique subgame-perfect equilibrium firms choose $q_A^* = q_B^* = 0$ and $p_A^* = p_B^* = 1$. Then, the indifferent consumer is $\bar{x} = 1/2$. The producer surplus, consumer surplus, and welfare are

$$PS = \Pi_A + \Pi_B = \frac{1}{2} + \frac{1}{2} = 1$$

$$CS = \int_0^{\frac{1}{2}} v - 1 - x^2 dx + \int_{\frac{1}{2}}^1 v - 1 - (1-x)^2 dx = v - \frac{13}{12}$$

$$W = PS + CS = v - \frac{1}{12}.$$

Producer Surplus: For $\tau > \tau_a$, each firm's revenue is $1/2$; the joint revenue is 1. In addition, for $\tau_a < \tau < 3/4$, the firms have costs but for $\tau > 3/4$ the firms have no costs. Then, firms reach the highest producer surplus ($PS = 1$) in the range $\tau > \tau_a$ for $\tau > 3/4$. In addition, as $1 > 5289/6050$, firms reach the overall highest producer surplus for $\tau > 3/4$.

Consumer Surplus: For $\tau > \tau_a$, the firms qualities are indistinguishable and the firms sell the goods at the same price so that all consumers $x < \bar{x} = 1/2$ buy from firm A and all $x > \bar{x} = 1/2$ buy from firm B. Thus for $\tau > \tau_a$, the highest consumer surplus is achieved when firms produce the goods with the highest quality. As for $\tau > \tau_a$ firms reduce their qualities with increasing τ, firms produce the highest qualities in the range $\tau_a < \tau < 3 - \sqrt{8}$. In addition,

$$v - \frac{6613}{9075} > v - \underbrace{\frac{3}{4}}_{> \frac{6613}{9075}} + \underbrace{\frac{7}{18}\tau - \frac{1}{9}\sqrt{48\tau - 8\tau^2}}_{<0}.$$

Thus the consumer surplus reaches its highest value for $\tau \leq \tau_a$.

Welfare: As

$$
\begin{aligned}
v+\frac{527}{3630} &> v+\underbrace{\frac{5}{36}}_{<\frac{527}{3630}}\underbrace{-\frac{25}{54}\tau-\frac{49}{162}\tau^2}_{<0}+\underbrace{\left(\frac{7\tau}{81}-\frac{1}{27}\right)\sqrt{48\tau-8\tau^2}}_{<0}\\
v+\frac{527}{3630} &> v-\frac{85}{12}+5\sqrt{2}\\
v+\frac{527}{3630} &> v+\underbrace{\frac{1}{36}}_{<\frac{527}{3630}}\underbrace{-\frac{4}{27}\tau-\frac{32}{81}\tau^2}_{<0}\\
v+\frac{527}{3630} &> v\underbrace{-\frac{1}{12}-\frac{1}{2}\left(-\tau+\sqrt{\frac{2}{3}\tau+\frac{1}{9}\tau^2}\right)^2)}_{<0}\underbrace{+\frac{1}{2}\underbrace{\left(-\tau+\sqrt{\frac{2}{3}\tau+\frac{1}{9}\tau^2}\right)}_{<\frac{1}{4}}}_{<\frac{1}{8}<\frac{527}{3630}}\\
v+\frac{527}{3630} &> v-\frac{1}{12}
\end{aligned}
$$

the welfare reaches its highest value for $\tau \leq \tau_a$.

Chapter 4

Horizontal product differentiation with limited attentive consumers (joint with Marc P. Saur and Markus G. Schlatterer)

Abstract

We analyze the effects of consumers' limited attention on welfare in a model of horizontal product differentiation. We present a novel approach of modeling limited attention: an *attention radius*. Each consumer only notices goods that are within her attention radius, i.e., goods that are sufficiently similar to her preferred version of the good. Limited attention induces firms to differentiate their products in a way that is beneficial to consumers. In addition, prices may be lower under limited than under full attention. Consumer surplus and welfare are not maximized under full attention but increase for some degree of limited attention.

Keywords: Attention, Horizontal Product Differentiation, Hotelling, Price Discrimination.

JEL Codes: D43, D91, L13.

4.1 Introduction

This article proposes a model of horizontal product differentiation that captures preference-dependent attention allocation of consumers. We investigate the effects of consumers' limited attention on consumer surplus, firms' profits, and overall welfare.

We construct a novel method of modeling limited attention. We model attention as a spotlight that only highlights the section of the product space around the consumer's preferred version of the good. For example, a consumer who prefers minivans only notices cars that are similar to minivans, like SUVs, and does not notice smaller cars, like compacts or roadsters, when she looks for a new car. Alternatively, a consumer who prefers blue focuses on blue t-shirts. Cyan or turquoise t-shirts also capture her attention as these colors are similar to blue, but red and brown t-shirts do not capture her attention as these colors are too far from blue. That means, we model attention allocation as preference-dependent; the consumer's preference primes her perception. Consumers only notice options that are similar to their preferred option and do not necessarily notice all available options.

Experiments on inattentional blindness demonstrate that it is reasonable to assume that consumers are not necessarily aware of all available goods in the market and that consumers are more likely to notice goods that are sufficiently similar to their target good. Inattentional blindness experiments show that by focusing on some events, people fail to perceive other events (see, e.g., Simons and Chabris, 1999; Most, Simons, Scholl, Jimenez, Clifford, and Chabris, 2001). In particular, inattentional blindness experiments show that similarity matters: For instance, if people focus on events in a particular color, they are more likely to notice other events if those events have the same color (e.g., Simons and Chabris, 1999; Most, Simons, Scholl, Jimenez, Clifford, and Chabris, 2001; Drew and Stothart, 2016).

We follow Hotelling (1929) in modeling horizontal product differentiation as a real line $[0, 1]$. Consumers are uniformly distributed on $[0, 1]$. The position $x \in [0, 1]$ of a consumer describes the consumer's preferred version of the good. Consumers are constrained in their attention: Each consumer only notices goods that are inside her *attention radius* κ. The *attention radius* highlights the section of the product space around the consumer's preferred version of the horizontally differentiated good, i.e., $[x - \kappa, x + \kappa]$. Figure 4.1 illustrates the attention radius of a consumer whose preferred version of the good is at $x \in [0, 1]$. Suppose two versions of the good exist at y_1 and y_2. As y_1 is inside the consumer's attention radius, the consumer at x is aware of good 1. As y_2 is not inside the consumer's attention radius, the consumer at x is not aware of good 2.

In this article, we investigate the effects of such attention allocation of consumers on product differentiation. We follow d'Aspremont, Gabszewicz, and Thisse (1979) in modeling transportation costs as quadratic. In our model, transportation costs describe

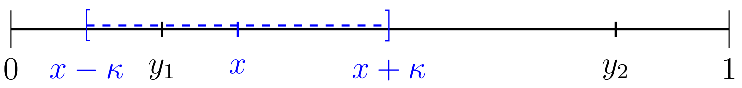

Figure 4.1: Example of the attention radius of the consumer at x.

the disutility of consumers who consume a good which does not perfectly fit their preferred version. We analyze the strategic considerations of two firms. Our analysis consists of two parts. In the first part, we assume that firms have to sell the good at an exogenously fixed price to derive the direct effects of consumers' limited attention on product differentiation. Then, firms independently and simultaneously choose the optimal location in the product space to maximize profits. We demonstrate that some degree of limited attention can have positive implications for consumers and overall welfare. In contrast to the standard Hotelling model where firms locate at the median consumer (see, e.g., Tirole, 1988), we show that firms choose to further differentiate their products. For lower levels of attention, the attention radii of consumers induce firms to locate closer to the efficient locations. Otherwise they miss demand. Thus consumers benefit from (some degree of) limited attention.

In the second part of the analysis, we allow firms to choose locations and prices. Firms, then, play a two-stage game: In the first stage, firms, independently and simultaneously, choose their locations in the product space. In the second stage, firms observe the location of their competitor and, independently and simultaneously, choose prices. Subsequently, consumers make a consumption decision. Three groups of consumers exist: Consumers who notice neither, one, or both firms. Firms compete for the consumers who notice both firms, but are monopolists for consumers who notice only one firm. We assume that a firm can price discriminate between consumers who notice only one firm and consumers who notice both firms. This captures situations where sellers can infer whether consumers are aware of competitors.

We show that very low levels of attention are not beneficial to consumers: Firms act as monopolists for all consumers, who then pay a high monopoly price. However, full attention is also not optimal for consumers. Under full attention, firms maximally differentiate their products and exploit this market power by setting higher prices. There exist intermediate levels of attention, where consumers pay lower prices than under full attention. Under these intermediate levels of attention, firms locate closer to the efficient locations than under full attention. This effect of limited attention on product differentiation also prevails under low levels of attention. Thus even under low levels of attention, consumers at least benefit on average from better product differentiation. Overall then, full attention is not optimal for consumers, instead consumer surplus is higher under some degree of limited attention.

The remainder of the article is structured as follows. Section 4.2 provides an overview of the related literature. Section 4.3 introduces the model. Section 4.4 discusses the results if prices are exogenously fixed and contrasts these results with the standard model with fully attentive consumers. In section 4.5, we allow firms to set prices and locations and analyze the resulting subgame-perfect equilibrium and the resulting welfare. Section 4.6 discusses the results and concludes. All proofs are in the appendix.

4.2 Related literature

Horizontal product differentiation is an extensively discussed topic in economics and, although limited attention is a growing strand of the economic literature, few articles discuss limited attention in the context of a Hotelling (1929) model. Exceptions are Schultz (2004) and Polo (1991). Yet, these articles exogenously distinguish between attentive and inattentive consumers. The uninformed consumers are inattentive to, for example, prices and/or locations (e.g., Polo, 1991; Schultz, 2004).[1] Then, instead of making a consumption choice with perfect information, inattentive consumers form expectations (Schultz, 2004) or buy from the nearest or cheapest firm (Polo, 1991). These models show that the fraction of consumers who are inattentive, distinctly influences market outcomes. Schultz (2004), for instance, shows that product differentiation, prices, and profits decrease in the number of attentive consumers. Yet, in these models the distinction who is informed and who is uninformed is random and consumers are generally aware of the existence of all firms. However, the distinction who is attentive and who is inattentive can also arise endogenously because of horizontal product differentiation. We add to this strand of the literature by analyzing this preference-dependent allocation of attention.

In addition, in our model, consumers are only aware of firms inside their attention radius. With this modeling choice, we also add to the literature on consideration sets by proposing a novel formation criterion for consideration sets in models of horizontal product differentiation. Generally, the consideration set literature utilizes a two-stage framework: In the first stage, the decision maker forms the consideration set, i.e., a subset of the set of all available options. In the second stage, the decision maker chooses one element from the consideration set. In our model, consumers can only buy from firms inside their attention radius. The literature usually assumes that inside the consideration set the decision is made rationally (e.g., Manzini and Mariotti, 2018). We adhere to this assumption. Yet, the literature differs on the formation of the consideration set: Eliaz and Spiegler (2011a,b), for example, assume that the formation is deterministic, whereas, for example, Manzini and Mariotti (2018) assume that it is probabilistic.

Eliaz and Spiegler (2011a,b) assume that in a market with two firms, consumers are

[1] This approach can also be generalized to demand functions independent of the Hotelling real line and by further assuming that a fraction of consumers are aware of only one firm (Cosandier, Garcia, and Knauff, 2018).

only aware of their default firm. This default firm is firm 1 for half of the consumers and firm 2 for the other half of the consumers. This allocation is random. Firms produce goods or menus of goods and can induce the rival's consumers to consider them via marketing strategies (Eliaz and Spiegler, 2011a) or via producing attention grabbers (Eliaz and Spiegler, 2011b). The formation criteria of Eliaz and Spiegler (2011b) also utilize similarity. The authors discuss the case that attention grabbers only grab attention if they are similar to the rival's menu. In contrast, in our model similarity to the consumer's taste is the driving factor behind attention. Both models (Eliaz and Spiegler, 2011a,b) abstract from price setting and consider only homogeneous consumer preferences. In addition, allocation of consumers to the default is random. Eliaz and Spiegler (2011a,b) find that profits are the same as with fully rational consumers, but that consumers are worse off.

Manzini and Mariotti (2018) also discuss similarity as a formation criteria. Nevertheless, Manzini and Mariotti (2018) assume that an option makes it into the consideration set of the decision maker probabilistically: The higher the salience of the option, the more likely that the option enters the consideration set. Options can invest in their salience to increase this probability. Salience, for example, means standing out. Being similar to other options in the choice set thus decreases salience. As in Eliaz and Spiegler (2011a), similarity is measured against the other available options, whereas in our model, similarity is measured against the consumer's preferences. Specifically, we assume that consumers only notice that a particular good of a firm exists if that good is inside the consumers' attention radius.[2] One way to interpret this assumption is by assuming—as Manzini, Mariotti, and Tyson (2013) do in their choice-theoretic model—that a threshold exists.

The attention radii of consumers thus suggest that each firm potentially faces two groups of consumers: One group notices only one firm, the other group notices both firms. We assume that firms can distinguish between those groups. The firms are thus able to offer the good at different prices to the two groups. Price discrimination between informed and uninformed consumers is also, for example, discussed in Heidhues and Köszegi (2017) and Armstrong and Vickers (2018). Yet, in contrast to our model, Heidhues and Köszegi (2017) and Armstrong and Vickers (2018) focus on a distinction of informed and uninformed consumers that is independent of the consumers' preferences. Generally, the literature on price discrimination in spatial models is very broad and includes price differentiation with respect to valuation, preference for differentiation, or location of consumers (see, e.g., Armstrong, 2006, for a survey).

[2]This attention radius implies that the firms may be unable to reach the whole market. The literature discusses similar constraints besides limited attention. For instance, Cancian, Bills, and Bergstrom (1995) assume that firms can only sell to consumers who are located on one side of them.

4.3 The model

We consider a market for a horizontally differentiated product where two firms, firm 1 and firm 2, compete for a unit mass of consumers. We assume that firms have identical marginal costs that we set to 0. The consumers are uniformly distributed on the interval $[0, 1]$. The location $x \in [0, 1]$ of a consumer describes the consumer's most preferred version of the good. Initially, firms decide which version of the good to produce by choosing their positions $y_1, y_2 \in [0, 1]$. Without loss of generality, we assume $y_1 \leq y_2$.

Each consumer wants to buy exactly one unit of the good. If a consumer does not buy the good, her utility is normalized to 0. If the consumer located at $x \in [0, 1]$ buys the good from firm $i \in \{1, 2\}$, the consumer's utility is

$$u_x(i) = v - p_i - (x - y_i)^2,$$

where p_i is the price at which firm i sells the good, $y_i \in [0, 1]$ is the location of firm i, and v is the gross utility of the good. We assume $v > 3$; this ensures that, in equilibrium, all consumers who notice at least one firm buy from one of these firms.

However, in our model, consumers' attention is limited and this constraint may prevent purchase: Each consumer only considers firms within her *attention radius* κ. The consumer at position x, then, only notices firm i on position y_i if $|x - y_i| \leq \kappa$, where $0 < \kappa \leq 1$. Firms thus make it into the consideration set[3] of a consumer, if they produce a version of the good that fits the consumer's taste well enough. If $|x - y_i| > \kappa$, the consumer does not even know (or remember) that firm i exists and, consequently, does not consider buying from firm i. Thus limited attention may prevent purchase from a firm that, potentially, has the overall better offer. Generally, if $\kappa = 1$, every consumer on $[0, 1]$ observes any point in $[0, 1]$. Therefore, this limiting case represents the standard Hotelling model where the choice set is identical to the consideration set.

From the perspective of firm $i \in \{1, 2\}$, the attention radii of consumers suggest that the firm can only reach consumers who are close enough. That means, the firm can only reach consumers that are inside its *radius of attentive consumers*, i.e., within the interval $[y_i - \kappa, y_i + \kappa]$. Consumers outside the radius of attentive consumers of firm i do not perceive firm i and thus never buy from firm i. Thus consumers' limited attention restricts the demand firms can capture.

In the following, we analyze how consumers' limited attention influences market outcomes. First, in Section 4.4, we discuss the effects of limited attention on product differentiation if the price is exogenously fixed at some price p. Second, in Section 4.5, we discuss

[3]The consideration set is a subset of the choice set. The choice set includes all available options (here, buying from firm 1 or firm 2, or not buying). The consideration set includes only those elements the consumer actively considers (here, not buying and buying from any of the firms inside the consumer's attention radius).

the effects of limited attention if prices are endogenously set by the non-cooperative firms.

4.4 Exogenous price

In this section, we analyze the direct effects of consumers' limited attention on product differentiation if prices are exogenously fixed such that $p_i = p$ for all $i \in \{1, 2\}$. We assume $0 < p \leq v - \kappa^2$. This assumption ensures that all consumers who notice at least one firm are willing to buy from one of these firms. To derive the demand of the firms, we have to distinguish two cases: Either the radii of attentive consumers of firm 1 and firm 2 overlap or do not overlap.

If the firms' radii do not overlap, i.e., $[y_1 - \kappa, y_1 + \kappa] \cap [y_2 - \kappa, y_2 + \kappa] = \emptyset$ or, equivalently, $y_1 + \kappa \leq y_2 - \kappa$, no consumer who notices firm 1 notices firm 2 and vice versa. Thus each firm is a monopolist in its radius. All consumers $x \in [y_1 - \kappa, y_1 + \kappa]$ have a utility $v - p_1 - (x - y_1)^2 \geq 0$ and buy from firm 1; everyone else does not buy from firm 1. Then, firm 1's demand is

$$x_1^m = y_1 + \kappa - \max\{0, y_1 - \kappa\}.$$

Similarly, for firm 2

$$x_2^m = \min\{y_2 + \kappa, 1\} - (y_2 - \kappa).$$

Figure 4.2 illustrates such a situation for $y_1 > \kappa$ and $y_2 > 1 - \kappa$: Then, the demand of firm 1 is $x_1^m = y_1 + \kappa - (y_1 - \kappa)$ and the demand of firm 2 is $x_2^m = 1 - (y_2 - \kappa)$.

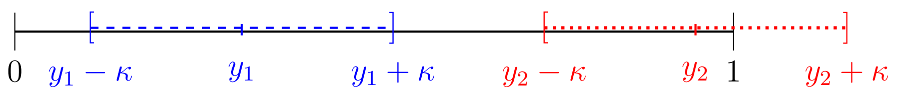

Figure 4.2: Example of non-overlapping radii of attentive consumers of firm 1 (blue/dashed) and firm 2 (red/dotted).

If the radii overlap, i.e., $[y_1 - \kappa, y_1 + \kappa] \cap [y_2 - \kappa, y_2 + \kappa] \neq \emptyset$ or, equivalently, $y_1 + \kappa > y_2 - \kappa$, some consumers notice both firms (see Figure 4.3 for an example). In particular, all consumers $x \in [0, 1]$ such that $y_1 - \kappa \leq x < y_2 - \kappa$ notice only firm 1. All consumers $x \in [0, 1]$ such that $y_2 - \kappa \leq x \leq y_1 + \kappa$ notice both firms. All consumers $x \in [0, 1]$ such that $y_1 + \kappa < x \leq y_2 + \kappa$ notice only firm 2. Consumers buy from firm 1 if they see only firm 1 or see both firms and prefer firm 1, i.e., $v - p_1 - (x - y_1)^2 \geq v - p_2 - (x - y_2)^2$. Similarly, consumers buy from firm 2 if they see only firm 2 or see both firms and prefer firm 2. We denote the consumer who is indifferent between buying from firm 1 and buying from firm 2 by

$$\hat{x} = \frac{p_2 - p_1}{2(y_2 - y_1)} + \frac{y_1 + y_2}{2}. \tag{4.1}$$

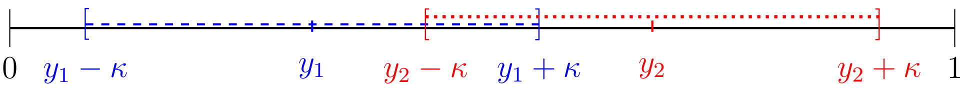

Figure 4.3: Example of overlapping radii of attentive consumers of firm 1 (blue/dashed) and firm 2 (red/dotted).

As prices are identical, the indifferent consumer (4.1) is given by

$$\hat{x} = \frac{y_1 + y_2}{2}.$$

Firms play a one-stage game in which they choose their location in the product space to maximize their profits. Overall then, firm 1's demand is

$$x_1^{FP}(y_1, y_2) = \min\{y_1 + \kappa, \hat{x}\} - \max\{0, y_1 - \kappa\}.$$

Similarly, for firm 2

$$x_2^{FP}(y_1, y_2) = \min\{y_2 + \kappa, 1\} - \max\{\hat{x}, y_2 - \kappa\}.$$

With marginal costs set to 0, the profit of firm 1 is

$$\Pi_1^{FP}(y_1, y_2) = p\ x_1^{FP}$$

and the profit of firm 2 is

$$\Pi_2^{FP}(y_1, y_2) = p\ x_2^{FP}.$$

Firms choose their locations to maximize profits. Proposition 5 characterizes the equilibrium locations of firm 1 and firm 2 dependent on κ.

Proposition 5 *Characterization of the Nash equilibria in the model with exogenous prices dependent on the attention radius* κ*:*

(i) For $0 < \kappa \leq 1/4$*, any pair of locations* $(y_1^*, y_2^*) \in \{(y_1, y_2) | \kappa \leq y_1 \leq 1 - 3\kappa, 3\kappa \leq y_2 \leq 1 - \kappa, y_2 - y_1 \geq 2\kappa\}$ *is an equilibrium. In any equilibrium, the profits are* $\Pi_1^* = \Pi_2^* = 2\kappa p$*.*

(ii) For $1/4 < \kappa \leq 1/2$*, the unique equilibrium locations are* $(y_1^* = \kappa, y_2^* = 1 - \kappa)$*. The equilibrium profits are* $\Pi_1^* = \Pi_2^* = p/2$*.*

(iii) For $\kappa > 1/2$, the unique equilibrium locations are $(y_1^* = 1/2, y_2^* = 1/2)$. The equilibrium profits are $\Pi_1^* = \Pi_2^* = p/2$.

Figure 4.4 illustrates the equilibrium locations for different values of κ. For $\kappa < 1/4$, a continuum of equilibrium locations exists. The gray area illustrates the locations of firm 1 and firm 2. For $\kappa \geq 1/4$, the equilibrium locations are unique.

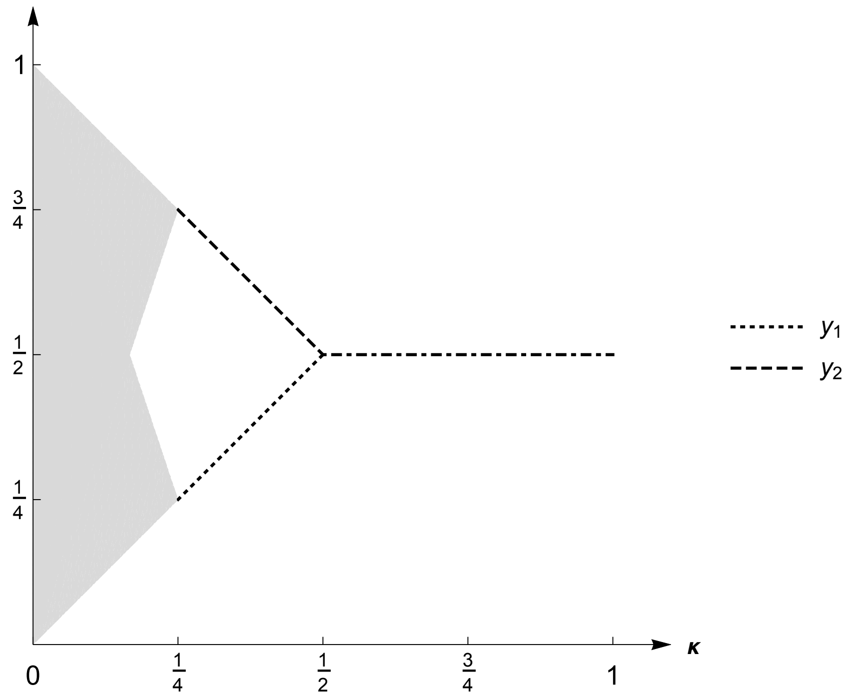

Figure 4.4: Equilibrium locations of firm 1 (dotted) and firm 2 (dashed) as a function of κ. For $\kappa \leq 1/4$ a continuum of equilibria exists, which is illustrated by the gray area.

Firms never choose locations such that their radii of attentive consumers overshoot the interval $[0, 1]$. Therefore, firm 1 never chooses a location $y_1 < \kappa$ and firm 2 never chooses a location $y_2 > 1 - \kappa$. Furthermore, both firms want to avoid an overlap of their radii of attentive consumers. As long as $\kappa \leq 1/4$, firms are able to choose locations to avoid an overlap. For $\kappa < 1/4$, a range of such locations exists. When $\kappa > 1/4$, firms are not able to avoid an overlap but choose locations that reduce the extent of the overlap. Firm 1, therefore, never chooses a location $y_1 > \kappa$ and firm 2 never chooses a location $y_2 > 1 - \kappa$ as long as $1/4 < \kappa \leq 1/2$. When $\kappa > 1/2$, both firms choose the median position to ensure that their radii of attentive consumers cover the entire product space $[0, 1]$ and the market is equally split among the firms. See the appendix for a complete proof.

For $0 < \kappa \leq 1/4$, firms locate such that no consumer notices both firms. Then, the consumers in $[y_1 - \kappa, y_1 + \kappa]$ buy from firm 1, the consumers in $[y_2 - \kappa, y_2 + \kappa]$ buy from firm 2, and some consumers notice neither firm and are unable to buy the good. Thus the consumer surplus is

$$CS = \int_{y_1-\kappa}^{y_1+\kappa} v - p - (x - y_1)^2 dx + \int_{y_2-\kappa}^{y_2+\kappa} v - p - (x - y_2)^2 dx = 4\kappa(v - p - \frac{\kappa^2}{3})$$

and the producer surplus is

$$PS = \Pi_1^* + \Pi_2^* = 4\kappa p.$$

For $\kappa < 1/4$, as some consumers notice neither firm, those consumers do not participate in the market. As κ increases, the number of consumers who notice neither firm decreases. Consequently, the consumer surplus and the producer surplus, and thus the overall welfare, are increasing in κ as long as $0 < \kappa \leq 1/4$.

For $1/4 < \kappa \leq 1/2$, all consumers notice at least one firm and thus buy a good. In equilibrium, firms locate such that they split the market equally. The consumer surplus becomes

$$CS = \int_{0}^{1/2} v - p - (x - \kappa)^2 dx + \int_{1/2}^{1} v - p - (x - (1 - \kappa))^2 dx = v - p - \kappa^2 + \frac{\kappa}{2} - \frac{1}{12}$$

and the producer surplus becomes

$$PS = \Pi_1^* + \Pi_2^* = p.$$

Because the locations are such that each firm always captures half of the consumers and prices are fixed, firms have no possibility to further increase their profits. Therefore, producer surplus is constant in κ. As κ increases, in equilibrium, firms choose to locate closer to the median consumer and thus increase the mean distance between consumers' and firms' locations. Consequently, consumer surplus decreases in κ. As producer surplus is constant and consumer surplus is decreasing in κ, welfare decreases in κ.

For $\kappa > 1/2$, the consumer surplus is

$$CS = \int_{0}^{1/2} v - p - \left(x - \frac{1}{2}\right)^2 dx + \int_{1/2}^{1} v - p - \left(x - \frac{1}{2}\right)^2 dx = v - p - \frac{1}{12}$$

and the producer surplus is

$$PS = \Pi_1^* + \Pi_2^* = p.$$

As long as $\kappa > 1/2$, the equilibrium locations are fixed at the location of the median consumer $y_1^* = y_2^* = 1/2$. This corresponds to the standard Hotelling result (Tirole, 1988). Consumer and producer surplus and, therefore, welfare, are constant in κ. Proposition 6 summarizes the welfare analysis.

Proposition 6 *Welfare analysis for exogenous prices:*

(i) Consumer surplus reaches its maximum at $\kappa = 1/4$.

(ii) Producer surplus reaches its maximum for all $\kappa \in [1/4, 1]$.

(iii) Welfare reaches its maximum at $\kappa = 1/4$.

Proposition 6 shows that the highest welfare level is achieved at $\kappa = 1/4$, where all consumers notice exactly one firm and participate in the market and the average distance between consumers' and firms' locations is minimized (firms choose $y_1 = 1/4$ and $y_2 = 3/4$). At $\kappa = 1/4$, consumer surplus and producer surplus also reach their maxima. In the standard Hotelling model, which our model captures at $\kappa = 1$, all consumers always notice both firms. This increases competition and induces firms to locate at the median consumer. In contrast, in our model, limited attention of consumers weakens competition as the number of consumers who notice both firms and for which firms compete is constrained. For low levels of attention, firms have an incentive to differentiate their products to capture more consumers who otherwise would not participate in the market as they notice neither firm. Therefore, firms locate closer to the efficient locations $y_1 = 1/4$ and $y_2 = 3/4$ under limited attention. Thus under exogenously fixed prices some level of inattention is actually beneficial to consumers.

4.5 Endogenous prices with price discrimination

In this section, we analyze the effects of limited attention on product differentiation when firms are also able to set prices. Then, the two firms play a two-stage game: In stage one, firms simultaneously and independently choose locations in the product space; in stage two, each firm observes the location of its competitor and, then, the firms simultaneously and independently set prices. Each firm (potentially) faces two groups of consumers. Consumers who notice one firm and consumers who notice both firms. Firms are monopolists for consumers who notice only one firm, but have to compete for the consumers who notice both firms. By choosing their location in the product space, firms can influence the size of their two groups of consumers. We assume that firms can distinguish between those two groups of consumers and thus charge different prices from the two groups. Then, firms charge a monopoly price p_i^m from the consumers who notice only one firm, and a competition price p_i^c from the consumers who notice both firms. We solve for subgame-perfect equilibria by backward induction.

In the price-setting stage, firms set prices to maximize profits given the locations chosen in the first stage. Profits can be split into two parts; the profits from the monopoly and the profits from competition:

$$\Pi_1(p_1^m, p_1^c, p_2^c, y_1, y_2) = \Pi_1^m(p_1^m, y_1, y_2) + \Pi_1^c(p_1^c, p_2^c, y_1, y_2)$$
$$\Pi_2(p_2^m, p_2^c, p_1^c, y_1, y_2) = \Pi_2^m(p_2^m, y_1, y_2) + \Pi_2^c(p_2^c, p_1^c, y_1, y_2).$$

As firms set two different prices, we can solve for the two prices separately. Firm 1's monopoly demand consists of all consumers who notice only firm 1, i.e., $x \in [y_1 - \kappa, y_1 + \kappa] \cap [0, 1]$ and $x \notin [y_2 - \kappa, y_2 + \kappa]$, and whose utility exceeds zero: $u_1(x) = v - p_1^m - (x - y_1)^2 \geq 0 \Leftrightarrow y_1 - \sqrt{v - p_1^m} \leq x \leq y_1 + \sqrt{v - p_1^m}$. Thus as long as $p_1^m \leq v - \kappa^2$, all consumers who notice only firm 1 have a positive utility and buy from firm 1. If $v > p_1^m > v - \kappa^2$, all consumers who notice only firm 1 and are in $[y_1 - \sqrt{v - p_1^m}, y_1 + \sqrt{v - p_1^m}]$ have a positive utility and buy from firm 1. If $p_1^m > v$, the monopoly price exceeds the gross utility of all consumers and no consumer buys from firm 1. Thus the profit of firm 1 from the monopoly is

$$\Pi_1^m(p_1^m, y_1, y_2) = p_1^m \begin{cases} (\min\{y_2 - \kappa, y_1 + \kappa\} - \max\{0, y_1 - \kappa\}) & \text{if } p_1^m \leq v - \kappa^2 \\ (\min\{y_2 - \kappa, y_1 + \sqrt{v - p_1^m}\} - \max\{0, y_1 - \sqrt{v - p_1^m}\}) & \text{if } v - \kappa^2 < p_1^m \leq v \\ 0 & \text{if } v < p_1^m. \end{cases}$$

Similarly, the profit of firm 2 from the monopoly is

$$\Pi_2^m(p_2^m, y_1, y_2) = p_2^m \begin{cases} (\min\{y_2 + \kappa, 1\} - \max\{y_2 - \kappa, y_1 + \kappa\}) & \text{if } p_2^m \leq v - \kappa^2 \\ (\min\{y_2 + \sqrt{v - p_2^m}, 1\} - \max\{y_2 - \sqrt{v - p_2^m}, y_1 + \kappa\}) & \text{if } v - \kappa^2 < p_2^m \leq v \\ 0 & \text{if } v < p_2^m. \end{cases}$$

In general, the maximum monopoly demand that firm 1 can receive is given by $\min\{y_2 - \kappa, y_1 + \kappa\} - \max\{0, y_1 - \kappa\}$. For $v > 3$, firm 1 has an incentive to set its monopoly price such that all consumers who notice only firm 1 are willing to buy from firm 1. The detailed derivation is in the appendix.

If the firms' radii of attentive consumers do not overlap (i.e., $y_1 + \kappa \leq y_2 - \kappa$), the last consumer who notices only firm 1 is at $x = y_1 + \kappa$ and firm 1 sets a price $p_1^m = v - \kappa^2$. If the firms' radii overlap (i.e., $y_1 + \kappa > y_2 - \kappa$) and $y_1 \geq \kappa$, the last consumer who notices only firm 1 is at $x = y_1 - \kappa$ and firm 1 sets a price $p_1^m = v - \kappa^2$. Thus when firm 1 can fully exploit one side of its radius, firm 1 sets the monopoly price such that all of these consumers are willing to buy from firm 1. Otherwise, firm 1 sets its monopoly price to capture the last consumer who notices only firm 1. Then, if the radius of firm 1 yields more monopoly demand on the left side than on the right side of firm 1 (i.e., $y_1 - 0 \geq y_2 - \kappa - y_1$), the last consumer who notices just firm 1 is at $x = 0$ and firm 1 sets a price $p_1^m = v - y_1^2$. If the radius yields more demand on the right side (i.e.,

$y_1 - 0 < y_2 - \kappa - y_1$), the last consumer who notices just firm 1 is at $x = y_2 - \kappa$ and firm 1 sets a price $p_1^m = v - (y_1 - y_2 + \kappa)^2$.

The monopoly price of firm 1 is, therefore,

$$p_1^{m*} = \begin{cases} v - \kappa^2 & \text{if } y_1 + \kappa \leq y_2 - \kappa \text{ or } y_1 + \kappa > y_2 - \kappa \text{ with } y_1 \geq \kappa \\ v - y_1^2 & \text{if } y_1 + \kappa > y_2 - \kappa \text{ with } y_2 - \kappa - y_1 \leq y_1 < \kappa \\ v - (y_1 - y_2 + \kappa)^2 & \text{if } y_1 + \kappa > y_2 - \kappa \text{ with } y_1 < y_2 - \kappa - y_1 \text{ with } y_1 < \kappa. \end{cases}$$

Similarly, the monopoly price of firm 2 is

$$p_2^{m*} = \begin{cases} v - \kappa^2 & \text{if } y_1 + \kappa \leq y_2 - \kappa \text{ or } y_1 + \kappa > y_2 - \kappa \text{ with } y_2 \leq 1 - \kappa \\ v - (1 - y_2)^2 & \text{if } y_1 + \kappa > y_2 - \kappa \text{ with } 1 - y_2 \geq y_2 - y_1 - \kappa \text{ with } y_2 > 1 - \kappa \\ v - (y_2 - y_1 - \kappa)^2 & \text{if } y_1 + \kappa > y_2 - \kappa \text{ with } 1 - y_2 < y_2 - y_1 - \kappa \text{ with } y_2 > 1 - \kappa. \end{cases}$$

If the firms' radii of attentive consumers overlap, firms also face consumers who notice both firms. That means, firms compete for consumers in the interval $[y_2 - \kappa, y_1 + \kappa] \cap [0, 1] = [\max\{0, y_2 - \kappa\}, \min\{y_1 + \kappa, 1\}]$. All consumers in this interval located to the left of the indifferent consumer $\hat{x}$ buy from firm 1, all others from firm 2. In equilibrium, firms set prices such that both firms receive some demand.[4] If $y_1 \neq y_2$, the competition profits of firm 1 and firm 2 are

$$\begin{aligned} \Pi_1^c(p_1^c, p_2^c, y_1, y_2) &= p_1^c \left(\hat{x} - \max\{0, y_2 - \kappa\}\right) \\ &= p_1^c \left(\frac{p_2^c - p_1^c}{2(y_2 - y_1)} + \frac{y_1 + y_2}{2} - \max\{0, y_2 - \kappa\} \right) \\ \Pi_2^c(p_2^c, p_1^c, y_1, y_2) &= p_2^c \left(\min\{y_1 + \kappa, 1\} - \hat{x}\right) \\ &= p_2^c \left(\min\{y_1 + \kappa, 1\} - \frac{p_2^c - p_1^c}{2(y_2 - y_1)} - \frac{y_1 + y_2}{2} \right). \end{aligned}$$

Firms set their prices p_1^c and p_2^c to maximize profits. If $y_1 + \kappa > y_2 - \kappa$, the best replies of firm 1 and firm 2 are

$$\begin{aligned} p_1^{c*}(p_2^c) &= \frac{p_2^c}{2} + (y_1 - y_2) \left(\max\{0, y_2 - \kappa\} - \frac{y_1 + y_2}{2} \right) \\ p_2^{c*}(p_1^c) &= \frac{p_1^c}{2} + (y_1 - y_2) \left(-\min\{y_1 + \kappa, 1\} + \frac{y_1 + y_2}{2} \right). \end{aligned}$$

[4] If both firms would set prices such that one firm receives the full competition demand and the other firm receives zero competition demand, the firm that receives zero demand can strictly increase its profit by choosing the (strictly positive) price of its competitor. Thus such prices cannot exist in equilibrium.

The equilibrium prices are, then,

$$p_1^{c*} = \frac{1}{3}(y_1 - y_2)(4\max\{0, y_2 - \kappa\} - 2\min\{y_1 + \kappa, 1\} - y_1 - y_2)$$
$$p_2^{c*} = \frac{1}{3}(y_1 - y_2)(2\max\{0, y_2 - \kappa\} - 4\min\{y_1 + \kappa, 1\} + y_1 + y_2).$$

The prices are increasing in the distance between firm 1 and firm 2. If firms have chosen the same location in the first stage, i.e., $y_1 = y_2$, price competition will ensure that $p_1^{c*} = p_2^{c*} = 0$. Taking these equilibrium prices, p_1^{m*}, p_2^{m*}, p_1^{c*}, and p_2^{c*}, the updated profits are

$$\Pi_1(y_1, y_2) = \begin{cases} p_1^{m*} x_1^m & \text{if } y_1 + \kappa \le y_2 - \kappa \\ p_1^{m*} x_1^m + p_1^{c*} x_1^c & \text{if } 0 < y_2 - \kappa < y_1 + \kappa \\ p_1^{c*} x_1^c & \text{if } y_2 - \kappa \le 0 < y_1 + \kappa \end{cases} \tag{4.2}$$

$$\Pi_2(y_1, y_2) = \begin{cases} p_2^{m*} x_2^m & \text{if } y_1 + \kappa \le y_2 - \kappa \\ p_2^{m*} x_2^m + p_2^{c*} x_2^c & \text{if } y_2 - \kappa < y_1 + \kappa < 1 \\ p_2^{c*} x_2^c & \text{if } y_2 - \kappa < 1 \le y_1 + \kappa \end{cases} \tag{4.3}$$

$$\begin{aligned} \text{where} \quad x_1^m &= \min\{y_2 - \kappa, y_1 + \kappa\} - \max\{0, y_1 - \kappa\} \\ x_2^m &= \min\{y_2 + \kappa, 1\} - \max\{y_2 - \kappa, y_1 + \kappa\} \\ x_1^c(p_1^{c*}, p_2^{c*}) &= -\frac{1}{6}\left(4\max\{0, y_2 - \kappa\} - 2\min\{y_1 + \kappa, 1\} - y_1 - y_2\right) \\ x_2^c(p_1^{c*}, p_2^{c*}) &= -\frac{1}{6}\left(2\max\{0, y_2 - \kappa\} - 4\min\{y_1 + \kappa, 1\} + y_1 + y_2\right) \end{aligned}$$

In the first stage, firms maximize profits by choosing their location in the product space. The structure of the profit functions (4.2) and (4.3) gives rise to a multitude of case distinctions. The first case of each profit function captures the situation that no consumer notices both firms. Thus both firms operate as pure monopolists. The second case captures the situation that firm i faces a subgroup of consumers who only notice firm i and a subgroup of consumers who also notice firm j. Therefore, the profit function consists of two terms: The profit from operating as a monopolist and the profit from competition. The third case captures that all consumers of firm i also notice firm j. Thus firm i only serves a competitive market. The size of the demand depends on the locations of the firms. Firms maximize profits over all cases to derive their best replies. Figure 4.5 illustrates the subgame-perfect equilibrium locations of firm 1 and firm 2.

If $0 < \kappa \le 1/4$, the firms are able to choose locations such that both firms are monopolists in their complete radii of attentive consumers and firms will do so in all subgame-perfect equilibria. Therefore, firms' radii of attentive consumers do not overlap.

Assume the firms' locations induce an overlap of their radii, i.e., $y_1 + \kappa > y_2 - \kappa$. Then, for $0 < \kappa \leq 1/4$, either $y_1 > \kappa$, $y_2 < 1 - \kappa$, or both. This means, at least one firm is able to move farther away from the opponent and thereby gain additional monopoly demand by simultaneously losing competition demand. As the additional monopoly profit exceeds the lost competition profit, the firm will move farther away until it has reached a full monopoly. Then, if the other firm does not have a full monopoly, because its outer boundary overshoots the product range, e.g., $y_2 + \kappa > 1$, it will move closer to its opponent as it trades no demand for competition demand. This induces the other firm to move farther outwards again until both firms have full monopolies. Consequently, in all subgame-perfect equilibria, both firms have only monopoly demand and all pairs of locations that induce two full monopolies are subgame-perfect equilibria. See the appendix for a formal proof.

If $\kappa > 1/4$, firms are unable to capture two full separate monopolies and competition becomes attractive for firms and is not avoided anymore. Nevertheless, as monopoly prices are higher than competition prices, firms prefer monopoly demand to competition demand. As κ increases, for fixed locations, more consumers notice both firms and the firms have to compete for these consumers. Generally, if the overlap of the radii of attentive consumers is small, few consumers notice both firms. For these consumers, the distance to the locations of both firms is about equally large. Therefore, for the choice of these consumers, the price is more relevant than the distance. Then, firms face price competition, which leads to lower competition prices. As the overlap increases, more consumers notice both firms. Therefore, the fraction of consumers for whom the distance is important for the consumption choice increases. This allows firms to extract higher surplus by setting higher prices. Nevertheless, competition prices are always lower than monopoly prices. Thus firms prefer to serve consumers as monopolists.

To dampen the effect that with increasing κ more consumers notice both firms, firms have an incentive to move outwards. Thus both firms only compete for a small number of consumers in the center of the product space and prefer to exploit as much monopoly rent as possible. However, as firms move outwards, a part of the radii of attentive consumers is outside $[0, 1]$. Thus the firms make no profit from $[y_1 - \kappa, 0)$ and $(1, y_2 + \kappa]$. When κ increases, these areas from which firms make no profits become larger and, despite firms moving outwards, more consumers notice both firms. As this also increases competition prices, competition becomes more tempting for firms. Finally, at $\kappa = (7 - 3\sqrt{3})/4$ competition is more attractive. Thus with increasing κ, firms move inwards to steal the business of their competitor and to receive a larger share of the competitive market.

As κ increases further, the competition demand increases as well and locating close to the center increases price competition among the firms. This reduces profits. Therefore, for $\kappa \geq (3\sqrt{3} - 4)/2$, firms move outwards to avoid competition which increases profits due to higher competition prices. At $\kappa = 3/4$, all consumers notice both firms, which means that the monopoly profit disappears. Nevertheless, as long as consumers are not fully attentive, not all consumers notice every part of $[0, 1]$. Thus firms have no incentive to directly locate at the extremes as this would enable the competitor to steal some fraction of the firm's demand and reduce its profits. In the limit as $\kappa = 1$, the classical Hotelling result of maximum product differentiation occurs. Figure 4.5 illustrates the subgame-perfect equilibrium locations of firm 1 and firm 2. See the appendix for a formal proof.

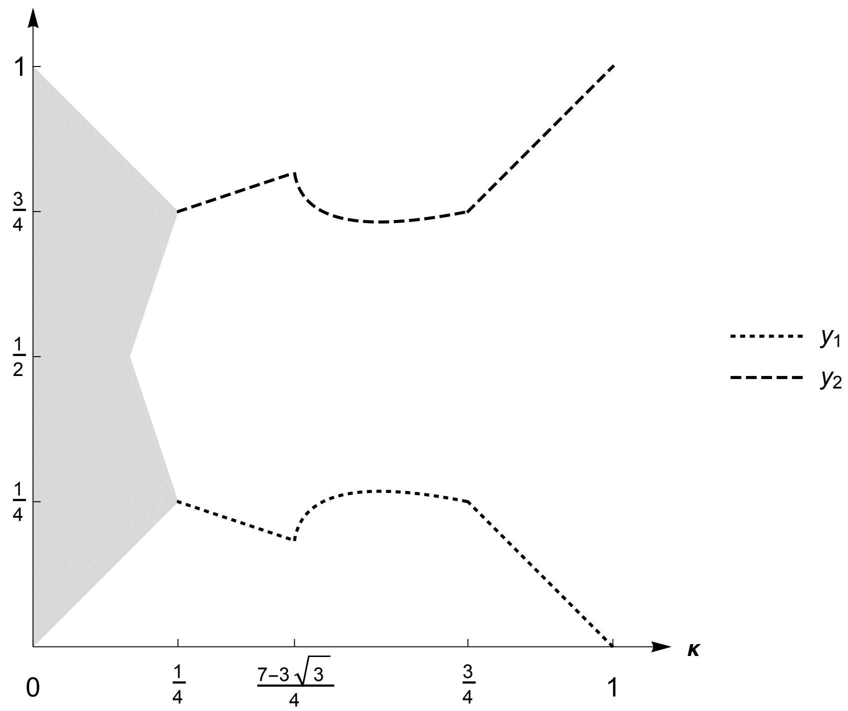

Figure 4.5: Subgame-perfect equilibrium locations of firm 1 (dotted) and firm 2 (dashed) as a function of κ. For $\kappa \leq 1/4$ a continuum of subgame-perfect equilibria exist, which is illustrated by the gray area .

Proposition 7 characterizes the subgame-perfect equilibria for all values of κ.

Proposition 7 *Characterization of the subgame-perfect equilibria in the model with endogenous prices dependent on the attention radius* κ*:*

(i) If $0 < \kappa \leq 1/4$*, any pair of locations* $(y_1^*, y_2^*) \in \{(y_1, y_2) | \kappa \leq y_1 \leq 1 - 3\kappa, 3\kappa \leq y_2 \leq 1 - \kappa, y_2 - y_1 \geq 2\kappa\}$ *is a subgame-perfect equilibrium. The corresponding equilibrium prices are* $p_1^{m*} = p_2^{m*} = v - \kappa^2$*. In any subgame-perfect equilibrium, the profits are* $\Pi_1^* = \Pi_2^* = (v - \kappa^2)2\kappa$*.*

(ii) If $1/4 < \kappa \leq (7 - 3\sqrt{3})/4$*, the unique subgame-prefect equilibrium locations are* $y_1^* = (1 - \kappa)/3$ *and* $y_2^* = (2 + \kappa)/3$*. The corresponding equilibrium prices are* $p_1^{m*} = p_2^{m*} = v - ((1 - \kappa)/3)^2$ *and* $p_1^{c*} = p_2^{c*} = 1/9(1 + 2\kappa)(4\kappa - 1)$*. The profits are* $\Pi_1^* = \Pi_2^* = p_1^{m*}(2 - 2\kappa)/3 - 1/6(1 - 4\kappa)p_1^{c*}$*.*

(iii) If $(7-3\sqrt{3})/4 < \kappa \leq 3/4$*, the unique subgame-perfect equilibrium locations are* $y_1^* = 1/4(2-3\kappa+\sqrt{\kappa^2+4\kappa-2})$ *and* $y_2^* = 1/4(2+3\kappa-\sqrt{\kappa^2+4\kappa-2})$*. The corresponding equilibrium prices are* $p_1^{m*} = p_2^{m*} = v - 1/16(2-3\kappa+\sqrt{\kappa^2+4\kappa-2})^2$ *and* $p_1^{c*} = p_2^{c*} = 1/2(1-2\kappa+\kappa^2+\kappa\sqrt{\kappa^2+4\kappa-2})$*. The profits are* $\Pi_1^* = \Pi_2^* = 1/4(2-\kappa-\sqrt{\kappa^2+4\kappa-2})p_1^{m*} + 1/16(3\kappa-\sqrt{\kappa^2+4\kappa-2})(\kappa+\sqrt{\kappa^2+4\kappa-2})^2$.

(iv) If $\kappa > 3/4$*, the unique subgame-perfect equilibrium locations are* $y_1^* = 1-\kappa$ *and* $y_2^* = \kappa$*. The corresponding equilibrium prices are* $p_1^{c*} = p_2^{c*} = 2\kappa-1$*. The profits are* $\Pi_1^* = \Pi_2^* = (2\kappa-1)/2$.

Proposition 7 shows that for $0 < \kappa \leq 1/4$, the subgame-perfect equilibrium locations are equivalent to the fixed price case (see Proposition 5). But if firms face exogenous prices (see Section 4.4), they tend towards the median location as κ increases. In contrast, if prices are endogenous, as $\kappa \to 1$, we approach maximum product differentiation ($y_1 \to 0$ and $y_2 \to 1$). Our model thus captures the standard result of d'Aspremont, Gabszewicz, and Thisse (1979)[5] as the limiting case of fully attentive consumers. For $\kappa = 1/4$, $\kappa = 1/2$, and $\kappa = 3/4$, firms choose the efficient locations, i.e., the locations that minimize the average distance between consumers' and firms' locations $y_1 = 1/4$ and $y_2 = 3/4$. Figure 4.6 illustrates the consumer surplus, the producer surplus, and the overall welfare for different levels of κ.

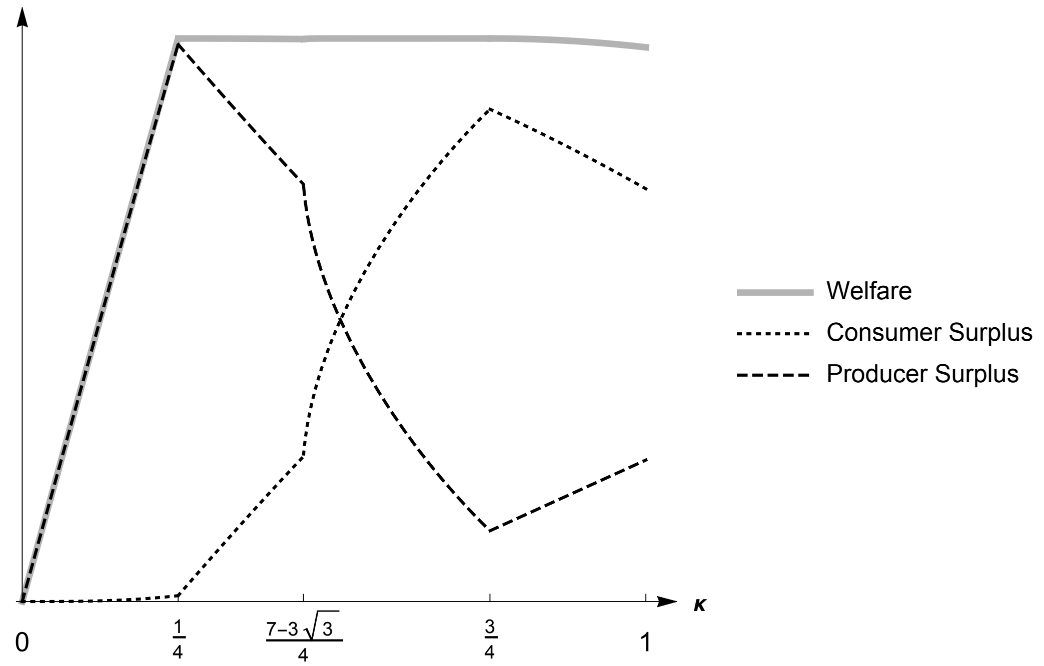

Figure 4.6: Welfare (solid), consumer surplus (dotted), and producer surplus (dashed) as a function of κ for $v = 4$.

[5]d'Aspremont, Gabszewicz, and Thisse (1979) analyze a Hotelling model where firms choose locations and prices and firms have quadratic transportation costs. They find, that firms maximally differentiate their products.

For $\kappa \leq 1/4$, in the subgame-perfect equilibrium, firms choose locations such that all consumers notice at most one firm. Thus both firms serve the market as monopolists. For $\kappa < 1/4$, some consumers notice neither firm and do not participate in the market. All consumers who notice a firm have to pay the monopoly price. The consumer surplus is, then,

$$CS = \int_{y_1-\kappa}^{y_1+\kappa} v - (v - \kappa^2) - (x - y_1)^2 dx + \int_{y_2-\kappa}^{y_2+\kappa} v - (v - \kappa^2) - (x - y_2)^2 dx = \frac{8}{3}\kappa^3$$

and the producer surplus is

$$PS = \Pi_1^* + \Pi_2^* = (v - \kappa^2)4\kappa.$$

As long as $\kappa \leq 1/4$, an increase in κ implies that firms can reach more consumers without facing competition. In addition, the fraction of consumers who do not participate in the market decreases. Consequently, consumer surplus, producer surplus, and welfare are increasing in κ. As all consumers pay the same (monopoly) price, the logic is similar to Section 4.4.

For $\kappa > 1/4$, all consumers buy the good in the subgame-perfect equilibrium. Thus equilibrium prices are only relevant for the division of surplus between firms and consumers, but are irrelevant for total welfare. Welfare is only affected by equilibrium locations and the corresponding disutility consumers receive from buying a non-ideal version of the good. For $1/4 < \kappa \leq (7-3\sqrt{3})/4$, in the subgame-perfect equilibrium, firms choose locations such that all consumers notice at least one firm. Therefore, all consumers participate in the market. Some consumers notice only one firm and have to pay the monopoly price, whereas, the other consumers notice both firms and pay a lower competition price. Thus the consumer surplus is

$$\begin{aligned}
CS = &\int_{0}^{y_2^*-\kappa} v - \left(v - (y_1^*)^2\right) - (x - y_1^*)^2 dx + \int_{y_2^*-\kappa}^{\hat{x}} v - \frac{1}{9}(1+2\kappa)(4\kappa-1) - (x - y_1^*)^2 dx \\
&+ \int_{\hat{x}}^{y_1^*+\kappa} v - \frac{1}{9}(1+2\kappa)(4\kappa-1) - (x - y_2^*)^2 dx \\
&+ \int_{y_1^*+\kappa}^{1} v - \left(v - (1-y_2^*)^2\right) - (x - y_2^*)^2 dx \\
=&v\frac{4\kappa-1}{3} - \frac{4}{3}\kappa^3 + \frac{1}{3}\kappa^2 - \frac{1}{6}\kappa + \frac{1}{12}
\end{aligned}$$

and the producer surplus is

$$PS = \Pi_1^* + \Pi_2^* = 2\left(\left(v - \left(\frac{1-\kappa}{3}\right)^2\right)\frac{2-2\kappa}{3} + \frac{(1+2\kappa)(1-4\kappa)^2}{54}\right).$$

As κ increases, more consumers notice both firms, such that more consumers pay the lower competition price. Consequently, producer surplus is decreasing and consumer surplus is increasing in κ. In total, overall welfare is decreasing.

For $(7-3\sqrt{3})/4 < \kappa \leq 3/4$, the consumer surplus is

$$
\begin{aligned}
CS =& \int_0^{y_2^*-\kappa} v - \left(v - (y_1^*)^2\right) - (x-y_1^*)^2 dx + \int_{y_2^*-\kappa}^{\hat{x}} v - p_1^{c*} - (x-y_1^*)^2 dx \\
&+ \int_{\hat{x}}^{y_1^*+\kappa} v - p_2^{c*} - (x-y_2^*)^2 dx + \int_{y_1^*+\kappa}^{1} v - \left(v - (1-y_2^*)^2\right) - (x-y_2^*)^2 dx \\
=& \frac{1}{48}\left(24v\kappa - 30\kappa^3 + 18\kappa^2 - 51\kappa + 20 + \sqrt{\kappa^2+4\kappa-2}(24v - 30\kappa^2 + 30\kappa - 9)\right)
\end{aligned}
$$

and the producer surplus is

$$
\begin{aligned}
PS = \Pi_1^* + \Pi_2^* =& \frac{1}{2}\left(v - \frac{1}{16}\left(2 - 3\kappa + \sqrt{\kappa^2+4\kappa-2}\right)^2\right)(2 - \kappa - \sqrt{\kappa^2+4\kappa-2}) \\
&+ \frac{1}{8}\left(3\kappa - \sqrt{\kappa^2+4\kappa-2}\right)\left(\kappa + \sqrt{\kappa^2+4\kappa-2}\right)^2.
\end{aligned}
$$

For $(7-3\sqrt{3})/4 < \kappa \leq 3/4$, in the subgame-perfect equilibrium, the distance between the locations of firm 1 and firm 2 decreases for $\kappa < (3\sqrt{3}-4)/2$ and increases for $\kappa > (3\sqrt{3}-4)/2$. The locations approach the efficient locations ($y_1 = 1/4$ and $y_2 = 3/4$) as $\kappa \to 1/2$ and $\kappa \to 3/4$. This is beneficial to consumers and increases consumer surplus. Yet, increasing product differentiation decreases competition between firms and thus increases competition prices which reduces consumer surplus. However, as κ increases, more consumers notice both firms and more consumers pay the lower competition price. Overall, consumer surplus is increasing in κ. Firms exchange monopoly demand for competition demand. Overall therefore, producer surplus is decreasing in κ. Between $(7-3\sqrt{3})/4 < \kappa \leq 3/4$ welfare is reallocated from firms to consumers. In addition, at $\kappa = 1/2$ and at $\kappa = 3/4$ the firms choose locations that minimize the mean distance between consumers' and firms' locations. Therefore, the overall welfare reaches its maximum at $\kappa = 1/2$ and at $\kappa = 3/4$.

For $\kappa > 3/4$, firms locate such that all consumers see both firms and as κ increases $y_1 \to 0$ and $y_2 \to 1$. The consumer and the producer surplus are

$$
\begin{aligned}
CS &= \int_0^{\hat{x}} v - (2\kappa - 1) - (x - (1-\kappa))^2 dx + \int_{\hat{x}}^{1} v - (2\kappa-1) - (x-\kappa)^2 dx \\
&= v - \kappa^2 - \frac{1}{2}\kappa + \frac{5}{12} \\
PS &= \Pi_1^* + \Pi_2^* = 2\kappa - 1.
\end{aligned}
$$

For $\kappa > 3/4$, in the subgame-perfect equilibrium, the distance between the firms increases in κ, which allows firms to increase prices. This harms consumers and benefits firms. Therefore, consumer surplus is decreasing and producer surplus is increasing in κ. The overall welfare is decreasing.

Proposition 8 summarizes the welfare analysis.

Proposition 8 *Welfare analysis for endogenous prices:*

(i) Producer surplus reaches its maximum at $\kappa = 1/4$.

(ii) Consumer surplus reaches its maximum at $\kappa = 3/4$.

(iii) Welfare reaches its maximum at $\kappa = 1/4$, $\kappa = 1/2$, and $\kappa = 3/4$.

In summary, some degree of inattention is actually beneficial to consumers, because the consumers' inattention induces firms to decrease the average distance between consumers' and firms' location. In addition, limited attention also influences the prices consumers have to pay. The smaller κ, the more consumers have to pay the monopoly price instead of the lower competition price. Producer surplus is maximized at $\kappa = 1/4$, where the firms operate as independent monopolists; each firm for exactly half of the consumers. Thus firms cannot increase demand and sell at the monopoly price to all consumers. At $\kappa = 1/4$ consumers actually benefit from product differentiation as firms choose locations $y_1 = 1/4$ and $y_2 = 3/4$ which minimize the mean distance between consumers' and firms' locations. However, all consumers have to pay the monopoly price. Consumer surplus is maximized at $\kappa = 3/4$, where firms also locate at $y_1 = 1/4$ and $y_2 = 3/4$, but all consumers pay the lower competition price. In addition, the competition price is lower at $\kappa = 3/4$ than under full attention.

Under full attention, firms maximally differentiate their products to increase their market power which allows them to set higher prices. Therefore, consumers benefit from limited attention as limited attention induces more efficient product differentiation that is favorable to consumers and reduces firms' market power. Consumer surplus is maximized under limited attention and not under full attention. Between $\kappa = 1/4$ and $\kappa = 3/4$, welfare is reallocated from firms to consumers as more consumers pay the lower competition price instead of the monopoly price. In addition, at $\kappa = 1/4$, $\kappa = 1/2$, and $\kappa = 3/4$ as firms choose the efficient locations consumer surplus increases. Therefore, the overall welfare reaches its maximum at $\kappa = 1/4$, $\kappa = 1/2$, and $\kappa = 3/4$. That is, welfare is higher under limited than under full attention.

4.6 Conclusion

In this article, we demonstrate the effects of limited attention on horizontal product differentiation and the implications for welfare. To capture the effects of limited attention, we develop a novel method to model limited attention: An attention radius for each consumer. This radius restricts the consumers' focus to the fraction of the product space that is close to the consumers' preferred version of the good. Therefore, limited attention reduces competition among firms and thus it might classically be expected that limited attention is harmful to the consumers. However, we find that limited attention is only harmful to consumers for very low levels of attention, but that an intermediate level of attention is actually beneficial to consumers. At low levels of attention, some consumers notice neither firm and are, therefore, unable to participate in the market. But as attention increases, all consumers notice at least one firm. Then, consumers benefit from limited attention, because limited attention induces firms to differentiate their products. Overall, we find that welfare is higher for some degrees of limited attention than under full attention.

We make a number of limiting assumptions. We assume price discrimination between fully and partially informed consumers to keep the model tractable. Future research might address the question, how robust our results are to other forms of price setting such as uniform pricing or other degrees of price discrimination. In addition, we assume homogeneous attention radii with a cut-off, where consumers abruptly turn from attentive to inattentive. From a psychological perspective, the size of the attention radii might differ among consumers. For example, experts might have a different attention radius than lay persons. Alternatively, a consumer might have a different attention radius when she is fully awake than when she is tired. Adding such heterogeneity might change the behavior of firms and thus yield additional insights. Furthermore, relaxing the assumption of an abrupt cut-off towards a smoother transition between attention and inattention can be a fruitful avenue for future research.

Additionally, we frame our model in terms of horizontal product differentiation. Nevertheless, our model can easily be applied to other contexts, for example, political or spatial competition. Another interesting extension might be to identify other areas where our attention radius can be applied. For instance it might prove interesting to analyze the effects of our attention radius in other models of horizontal or vertical product differentiation.

Appendix

Proof Proposition 5

For $0 < \kappa \leq 1/2$, both firms locate in the interval $[\kappa, 1-\kappa]$. If a firm deviates to a location outside $[\kappa, 1-\kappa]$, its radius of attentive consumers overshoots the product range and it loses demand without any gain.

(i) For $0 < \kappa \leq 1/4$, in equilibrium, firms' radii of attentive consumers never overlap. Suppose radii would overlap, i.e., $y_1 + \kappa > y_2 - \kappa$, then, $y_1 > \kappa$ and/or $y_2 < 1 - \kappa$. Then, at least one of the two firms can strictly increase its profits by moving closer to κ or $1 - \kappa$ respectively and trading competition demand for monopoly demand. For $y_1^* \in [\kappa, 1 - 3\kappa]$ and $y_2^* \in [y_1^* + 2\kappa, 1 - \kappa]$, the radii of attentive consumers do not overlap and both firms earn their highest possible profits. All of these locations are Nash equilibria.

(ii) For $1/4 < \kappa \leq 1/2$, as in equilibrium $y_1, y_2 \in [\kappa, 1 - \kappa]$, the firms' radii always overlap. Within this range, firm 1 minimizes the overlap by setting $y_1 = \kappa$ and firm 2 minimizes the overlap by setting $y_2 = 1 - \kappa$. This maximizes each firms' profit and thus forms the unique Nash equilibrium.

(iii) For $\kappa > 1/2$, firms are able to choose locations that ensure that all consumers in the market are within their radii. Firms locate at the median consumer's position: $y_1 = y_2 = 1/2$. This is a Nash equilibrium as any deviation by $\varepsilon < 1/2$ lowers the demand by $|\varepsilon|/2$. Further, there is no other equilibrium. Each firm must receive at least half of the demand, otherwise it would relocate to the median location. Both firms receive half of the demand either if they choose symmetric locations with $y_1 < 1/2$ and $y_2 > 1/2$ (but then each firm would benefit from relocating to 1/2) or if they choose the same location $y_1 = y_2 \neq 1/2$ (but then each firm has an incentive to move closer to 1/2).

Derivation of the monopoly prices

Assume firm 1's monopoly demand on one side is larger than the monopoly demand on the other side. Without loss of generality, we assume that the left side is the larger side. The distance from firm 1's location to the right end of the monopoly area can be denoted as $y_2 - \kappa - y_1$ (as the right side is constrained by the radius of attentive consumers of firm 2). Note that this value can also be negative such that the monopoly area is only on the left side of the firm. We can define $d \in [0, \min\{y_1, y_2 - \kappa - y_1, \kappa\}]$ as the distance between the consumer who is indifferent between buying the good at the monopoly price from firm 1 and not buying. Then, we can express the monopoly price and the monopoly profit as a function of the distance d:[6] $p_1^m = v - d^2$ and

$$\Pi_1(d) = (d + y_2 - \kappa - y_1)(v - d^2).$$

The optimal distance is

$$d^* \equiv -\frac{y_2 - y_1 - \kappa}{3} + \frac{1}{3}\sqrt{(y_2 - y_1 - \kappa)^2 + 3v} = \arg\max_d \Pi_1(d).$$

We find that the profit of firm 1 is strictly increasing for $d \in [0, d^*)$. Then, firm 1 is always willing to exploit the whole monopoly range if

$$d^* \geq \kappa \Leftrightarrow v \geq \kappa^2 + 2\kappa(y_2 - y_1).$$

As $0 \leq y_2 - y_1 \leq 1$, $0 < \kappa \leq 1$ and $v > 3$, firm 1 always exploits the whole market. By symmetry, the same holds true for monopolies where the larger part of the monopoly is on the right side of firm 1.[7] Thus in the asymmetric case, the monopoly price is always set to fully exploit the monopoly demand. This must then also be true in the symmetric case (when the monopoly demand on the left side is as large as the monopoly demand on the right side), as now by setting a higher price, the firm would not only loose demand on one but on both sides.

As we have shown, firms have an incentive to always exploit the full monopoly demand. The monopoly price of firm 1 is, therefore,

$$p_1^{m*} = \begin{cases} v - \kappa^2 & \text{if } y_1 + \kappa \leq y_2 - \kappa \text{ or } y_1 + \kappa > y_2 - \kappa \text{ with } y_1 \geq \kappa \\ v - y_1^2 & \text{if } y_1 + \kappa > y_2 - \kappa \text{ with } y_2 - \kappa - y_1 \leq y_1 < \kappa \\ v - (y_1 - y_2 + \kappa)^2 & \text{if } y_1 + \kappa > y_2 - \kappa \text{ with } y_1 < y_2 - \kappa - y_1 \text{ with } y_1 < \kappa. \end{cases}$$

By symmetry, firm 2 also always exploits its whole monopoly market. Thus the monopoly

[6] $u_1(d) = v - p_1^m - d^2 = 0 \Leftrightarrow p_1^m = v - d^2$.

[7] If the right demand side of firm 1 is larger, the profit changes to $\Pi_1 = (d + y_1)(v - d^2)$. However, $d^* \geq \kappa$ and the firms are willing to exploit the whole monopoly market.

price of firm 2 is

$$p_2^{m*} = \begin{cases} v - \kappa^2 & \text{if } y_1 + \kappa \leq y_2 - \kappa \text{ or } y_1 + \kappa > y_2 - \kappa \text{ with } y_2 \leq 1 - \kappa \\ v - (1 - y_2)^2 & \text{if } y_1 + \kappa > y_2 - \kappa \text{ with } 1 - y_2 \geq y_2 - y_1 - \kappa \text{ with } y_2 > 1 - \kappa \\ v - (y_2 - y_1 - \kappa)^2 & \text{if } y_1 + \kappa > y_2 - \kappa \text{ with } 1 - y_2 < y_2 - y_1 - \kappa \text{ with } y_2 > 1 - \kappa. \end{cases}$$

Proof of Proposition 7

Subgame-perfect equilibria for $0 < \kappa \leq 1/4$

Assume $0 < \kappa \leq 1/4$. The proof proceeds in three steps: First, we show that any pair of locations (y_1, y_2) such that the firms' radii of attentive consumers overlap (i.e., $y_1 + \kappa > y_2 - \kappa$) cannot be a subgame-perfect equilibrium. Second, we show that in the subgame-perfect equilibrium firms do not choose locations such that $y_i < \kappa$ or $y_i > 1 - \kappa$. Third, we show that the remaining pairs of locations (y_1, y_2) such that $y_1 \in [\kappa, 1 - 3\kappa]$ and $y_2 \in [y_1 + 2\kappa, 1 - \kappa]$ are the locations in the subgame-perfect equilibria.

Step 1: Any pair of locations such that $y_1 + \kappa > y_2 - \kappa$ can never be a subgame-perfect equilibrium. Suppose $y_1 + \kappa > y_2 - \kappa$, then one firm has an incentive to move away from the opponent without overshooting $[0, 1]$, which increases that firm's profit. With $y_1 + \kappa > y_2 - \kappa$ and $0 < \kappa \leq 1/4$, either $y_1 > \kappa$, $y_2 < 1 - \kappa$, or both. Suppose $y_1 > \kappa$,

$$\Pi_1(y_1, y_2) = (v - \kappa^2)(y_2 - y_1) + \frac{1}{18}(y_2 - y_1)(3y_2 - 4\kappa - 2\min\{y_1 + \kappa, 1\} - y_1)^2$$

$$\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1} = \begin{cases} \underbrace{-v - \kappa^2}_{<-3} + \underbrace{4\kappa(y_2 - y_1)}_{<8\kappa^2 \leq \frac{1}{2}} \underbrace{- \frac{3}{2}(y_2 - y_1)^2}_{\leq 0} < 0 & \text{if } y_1 + \kappa \leq 1 \\ \underbrace{\kappa^2 - v}_{<0} - \underbrace{\frac{(3y_2 - y_1 - 4\kappa - 2)(5y_2 - 3y_1 - 4\kappa - 2)}{18}}_{\geq 0} < 0 & \text{if } y_1 + \kappa > 1 \end{cases}$$

and, by symmetry, if $y_2 < 1 - \kappa$, $\partial \Pi_2(y_1, y_2)/\partial y_2 > 0$. Thus if the firms' radii of attentive consumers overlap, at least one of the two firms has an incentive to deviate until the distance between y_1 and y_2 is large enough such that $y_1 + \kappa \leq y_2 - \kappa$.

Step 2: Any pair of locations such that $y_i < \kappa$ can never be a subgame-perfect equilibrium. Suppose $y_1 < \kappa$, then a part of the attention radius of firm 1 lies outside $[0, 1]$. Thus firm 1 can profitable deviate to $y_1 = \kappa$ to increase its profit. This either strictly increases monopoly profit or weakly increases monopoly profit and strictly increases competition profit. Suppose $y_2 < \kappa$, the radii of attentive consumers would overlap, which is excluded in the first step of this proof. Thus neither firm chooses a location $y_i < \kappa$. By symmetry, neither firm chooses a location $y_i > 1 - \kappa$.

Step 3: All remaining pairs of locations (y_1, y_2) such that $y_1 \in [\kappa, 1 - \kappa]$ and $y_2 \in [\kappa, 1 - \kappa]$ with $y_1 + \kappa \leq y_2 - \kappa$ are subgame-perfect equilibria. With each of these pairs of locations, firms receive the highest possible profit $\Pi_1 = \Pi_2 = (v - \kappa^2)2\kappa$. Thus neither firm has an incentive to deviate.

Subgame-perfect equilibria for $1/4 < \kappa \leq 1/2$

Assume $1/4 < \kappa \leq 1/2$. The proof proceeds in four steps: First, we show that any pair of locations (y_1, y_2) where the firms' radii of attentive consumers do not overlap (i.e., $y_1 + \kappa \leq y_2 - \kappa$) cannot be a subgame-perfect equilibrium. Second, we show that in the subgame-perfect equilibrium firms do not choose locations such that $y_1 > \kappa$ and/or $y_2 < 1 - \kappa$. Third, we show that firm 1 never chooses a location $y_1 < (y_2 - \kappa)/2$ and firm 2 never chooses a location $y_2 > (1 + y_1 + \kappa)/2$. Fourth, we specify the best replies and the subgame-perfect equilibria.

Step 1: Any pair of locations such that $y_1 + \kappa \leq y_2 - \kappa$ can never be a subgame-perfect equilibrium. Suppose $y_1 + \kappa \leq y_2 - \kappa$, then a part of the radius of at least one firm lies outside $[0, 1]$. This firm can profitable deviate to increase its profit by forcing an overlap. This increases monopoly profit and competition profit. Therefore, firms always choose locations such that $y_1 + \kappa > y_2 - \kappa$.

Step 2: Firm 1 never chooses a location $y_1 > \kappa$. Suppose $y_1 > \kappa$,

$$\Pi_1(y_1, y_2) = (v - \kappa^2)(y_2 - y_1) + \frac{1}{18}(y_2 - y_1)(3y_2 - 4\kappa - 2\min\{y_1 + \kappa, 1\} - y_1)^2$$

$$\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1} = \begin{cases} -\underbrace{v}_{>3} \underbrace{-\kappa^2 + 4\kappa(y_2 - y_1)}_{<8\kappa^2 - \kappa^2 < 3} \underbrace{-\frac{3}{2}(y_2 - y_1)^2}_{\leq 0} < 0 & \text{if } y_1 + \kappa < 1 \\ \underbrace{\kappa^2 - v}_{<0} - \underbrace{\frac{(3y_2 - y_1 - 4\kappa - 2)(5y_2 - 3y_1 - 4\kappa - 2)}{18}}_{>0} < 0 & \text{if } y_1 + \kappa \geq 1. \end{cases}$$

The first derivative is strictly negative and firm 1 always has an incentive to move to the left. Therefore, firm 1 never chooses a location $y_1 > \kappa$. By symmetry, firm 2 never chooses a location $y_2 < 1 - \kappa$. Consequently, a potential subgame-perfect equilibrium must involve $y_1 \leq \kappa$ and $y_2 \geq 1 - \kappa$.

Step 3: As $y_1 \leq \kappa$ and $y_2 \geq 1 - \kappa$ with $y_1 + \kappa > y_2 - \kappa$, both firms locate close to the boundaries of the product space but also compete for consumers who notice both firms in the center. Profits for both firms become

$$\Pi_1(y_1, y_2) = \frac{1}{2}(y_2 - y_1)(y_2 - y_1 - 2\kappa)^2 + (y_2 - \kappa) \begin{cases} (v - (y_2 - y_1 - \kappa)^2) & \text{if } y_1 < \frac{y_2 - \kappa}{2} \\ (v - y_1^2) & \text{if } y_1 \geq \frac{y_2 - \kappa}{2} \end{cases}$$

$$\begin{aligned} \Pi_2(y_1, y_2) = &\frac{1}{2}(y_2 - y_1)(y_2 - y_1 - 2\kappa)^2 \\ &+ (1 - y_1 - \kappa) \begin{cases} (v - (y_2 - y_1 - \kappa)^2) & \text{if } y_2 > \frac{1 + y_1 + \kappa}{2} \\ (v - (1 - y_2)^2) & \text{if } y_2 \leq \frac{1 + y_1 + \kappa}{2} \end{cases} \end{aligned}$$

First, suppose firm 1 would choose a location $y_1 < (y_2-\kappa)/2$. As

$$\frac{\partial \Pi_1(y_1,y_2)}{\partial y_1} = 2\underbrace{(y_2-y_1-\kappa)}_{>0 \text{ as } y_1<\frac{y_2-\kappa}{2}}\underbrace{(y_2-\kappa)}_{>0} - \underbrace{\frac{3}{2}(y_2-y_1-2\kappa)}_{<0}\underbrace{\left(y_2-y_1-\frac{2}{3}\kappa\right)}_{>0 \text{ as } y_1<\frac{y_2-\kappa}{2}} > 0,$$

firm 1 always has an incentive to move inwards for $y_1 < (y_2-\kappa)/2$. By symmetry, the same holds for firm 2 choosing $y_2 > (1+y_1+\kappa)/2$. Then, a potential subgame-perfect equilibrium must involve locations such that $y_1 \in [(y_2-\kappa)/2, \kappa]$ and $y_2 \in [1-\kappa, (1+y_1+\kappa)/2]$.

Step 4: Now, we derive the best replies of firm 1 and firm 2 with $y_1 \in [(y_2-\kappa)/2, \kappa]$ and $y_2 \in [1-\kappa, (1+y_1+\kappa)/2]$ and, subsequently, specify the subgame-perfect equilibria. The first derivative of the profit functions of both firms is

$$\begin{aligned}
\frac{\partial \Pi_1(y_1,y_2)}{\partial y_1} &= -2y_1(y_2-\kappa) - \frac{3}{2}(y_2-y_1-2\kappa)\left(y_2-y_1-\frac{2}{3}\kappa\right) = 0\\
&\Leftrightarrow y_1(y_2) = \frac{1}{3}\left(y_2-2\kappa \pm 2\sqrt{-2y_2^2+5\kappa y_2-2\kappa^2}\right)\\
\frac{\partial \Pi_2(y_1,y_2)}{\partial y_2} &= 2(1-y_2)(1-y_1-\kappa) + \frac{3}{2}(y_2-y_1-2\kappa)\left(y_2-y_1-\frac{2}{3}\kappa\right) = 0\\
&\Leftrightarrow y_2(y_1) = \frac{1}{3}\left(2+y_1+2\kappa \pm 2\sqrt{-2y_1^2+y_1(4-5\kappa)+\kappa(5-2\kappa)-2}\right).
\end{aligned}$$

Checking the second order condition, we find that the potential maxima are[8]

$$\begin{aligned}
y_1(y_2) &= \frac{1}{3}\left(y_2-2\kappa + 2\sqrt{-2y_2^2+5\kappa y_2-2\kappa^2}\right)\\
y_2(y_1) &= \frac{1}{3}\left(2+y_1+2\kappa - 2\sqrt{-2y_1^2+y_1(4-5\kappa)+\kappa(5-2\kappa)-2}\right).
\end{aligned}$$

Note that these potential maxima must fulfill the conditions $y_1 \in [(y_2-\kappa)/2, \kappa]$ and $y_2 \in [1-\kappa, (1+y_1+\kappa)/2]$ to be a best reply. For simplicity, let us first focus on the derivation of the best reply function for firm 1. Consequently, for $y_1 \in [(y_2-\kappa)/2, \kappa]$ we must have

$$\frac{y_2-\kappa}{2} \leq \frac{1}{3}\left(y_2-2\kappa+2\sqrt{-2y_2^2+5\kappa y_2-2\kappa^2}\right) \leq \kappa.$$

If the best reply lies outside the range, the firm chooses the boundary solution. Checking

[8] For firm 1 the potential maximum only exists if $y_2 \leq 2\kappa$. Suppose $y_2 > 2\kappa$, then $\partial\Pi_1(y_1,y_2)/\partial y_1 < 0$ and firm 1 chooses $y_1 = (y_2-\kappa)/2$. For firm 2 the potential maximum only exists if $y_1 \geq 1-2\kappa$. Suppose $y_1 < 1-2\kappa$, then $\partial\Pi_2(y_1,y_2)/\partial y_2 > 0$ and firm 2 chooses $y_2 = (1+y_1+\kappa)/2$.

both conditions we find that

$$y_1(y_2) = \begin{cases} \frac{y_2-\kappa}{2} & \text{if } y_2 < \frac{13-4\sqrt{3}}{11}\kappa \\ \frac{y_2-2\kappa+2\sqrt{-2y_2^2+5\kappa y_2-2\kappa^2}}{3} & \text{if } \frac{13-4\sqrt{3}}{11}\kappa \leq y_2 \leq \frac{13+4\sqrt{3}}{11}\kappa \\ \frac{y_2-\kappa}{2} & \text{if } y_2 > \frac{13+4\sqrt{3}}{11}\kappa. \end{cases} \tag{4.4}$$

Next, we need to check whether the conditions of (4.4) satisfy $[1-\kappa,(1+y_1+\kappa)/2]$ or are partly outside. First, we check

$$\begin{aligned} \frac{13-4\sqrt{3}}{11}\kappa \leq 1-\kappa \leq \frac{13+4\sqrt{3}}{11}\kappa \\ \frac{11}{24+4\sqrt{3}} \leq \kappa \leq \frac{11}{24-4\sqrt{3}} \end{aligned} \tag{4.5}$$

Later, we check the conditions for $(1+y_1+\kappa)/2$, when we analyze whether potential subgame-perfect equilibria are in the range of the best reply function. Using (4.4) and (4.5), we can rewrite the best reply function of firm 1:

If $\kappa < \frac{11}{24+4\sqrt{3}}$,

$$y_1^*(y_2) = \frac{y_2-\kappa}{2}.$$

If $\frac{11}{24+4\sqrt{3}} \leq \kappa \leq \frac{1}{2}$,

$$y_1^*(y_2) = \begin{cases} \frac{y_2-2\kappa+2\sqrt{-2y_2^2+5\kappa y_2-2\kappa^2}}{3} & \text{if } y_2 \leq \frac{13+4\sqrt{3}}{11}\kappa \\ \frac{y_2-\kappa}{2} & \text{if } y_2 > \frac{13+4\sqrt{3}}{11}\kappa \end{cases}$$

Checking the same conditions for firm 2, if $\kappa < \frac{11}{24+4\sqrt{3}}$,

$$y_2^*(y_1) = \frac{1+y_1+\kappa}{2}$$

and if $\frac{11}{24+4\sqrt{3}} \leq \kappa \leq \frac{1}{2}$,

$$y_2^*(y_1) = \begin{cases} \frac{1+y_1+\kappa}{2} & \text{if } y_1 < \frac{11-13\kappa-\sqrt{48}\kappa}{11} \\ \frac{2+y_1+2\kappa-2\sqrt{-2y_1^2+y_1(4-5\kappa)+\kappa(5-2\kappa)-2}}{3} & \text{if } \frac{11-13\kappa-\sqrt{48}\kappa}{11} \leq y_1 \leq \kappa \end{cases}$$

The intersections of the best replies gives the subgame-perfect equilibria.

Thus if $1/4 < \kappa \leq (7-3\sqrt{3})/4$, the subgame-perfect equilibrium locations are

$$y_1^* = \frac{1-\kappa}{3}$$
$$y_2^* = \frac{2+\kappa}{3}$$

and if $(7-3\sqrt{3})/4 < \kappa \leq 1/2$ the subgame-perfect equilibrium locations are

$$y_1^* = \frac{1}{4}\left(2-3\kappa+\sqrt{\kappa^2+4\kappa-2}\right)$$
$$y_2^* = \frac{1}{4}\left(2+3\kappa-\sqrt{\kappa^2+4\kappa-2}\right).$$

Subgame-perfect equilibria for $1/2 < \kappa \leq 1$

Assume $1/2 < \kappa \leq 1$, then $\kappa > 1-\kappa$. Then, even if firms maximally differentiate their products, the firms' radii of attentive consumers will always overlap, i.e., $y_1+\kappa > y_2-\kappa$. The proof proceeds in three steps: First, we show that a pair (y_1, y_2) such that $y_1 > 1-\kappa$ and/or $y_2 < \kappa$ cannot constitute a subgame-perfect equilibrium. Second, we show that firm 1 never chooses a location $y_1 < (y_2-\kappa)/2$ and firm 2 never chooses a location $y_2 > (1+y_1+\kappa)/2$. Third, we specify the best replies and the subgame-perfect equilibria.

Step 1: Suppose $y_1 \geq 1-\kappa$ and $y_2 > \kappa$:

$$\Pi_1(y_1, y_2) = \frac{1}{18}(y_2-y_1)(3y_2-y_1-4\kappa-2)^2 + \begin{cases} (v-\kappa^2)(y_2-y_1) & \text{if } y_1 \geq \kappa \\ (v-y_1^2)(y_2-\kappa) & \text{if } y_1 < \kappa \end{cases}$$

$$\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1} = \underbrace{\frac{1}{18}\underbrace{(3y_2-y_1-4\kappa-2)}_{<0 \text{ as } 4\kappa+2>3y_2}\underbrace{(-5y_2+3y_1+4\kappa+2)}_{>0 \text{ as } 3y_1+4\kappa+2>5y_2} + \begin{cases} \underbrace{-(v-\kappa^2)}_{<0} & \text{if } y_1 \geq \kappa \\ \underbrace{-2y_1(y_2-\kappa)}_{<0} & \text{if } y_1 < \kappa \end{cases}}_{<0}$$

Suppose $y_1 \geq 1-\kappa$ and $y_2 \leq \kappa$. Then,

$$\Pi_1(y_1, y_2) = \frac{1}{18}(y_2-y_1)(2+y_1+y_2)^2$$
$$\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1} = \frac{1}{18}\underbrace{(2+y_1+y_2)}_{>0}\underbrace{(y_2-3y_1-2)}_{<0} < 0.$$

Therefore, firm 1 never chooses a location $y_1 > 1-\kappa$. Those locations are strictly dominated by $y_1 = 1-\kappa$. By symmetry, firm 2 never chooses a location $y_2 < \kappa$. Those locations are strictly dominated by $y_2 = \kappa$.

Step 2: Thus $y_1 \leq 1-\kappa$ and $y_2 \geq \kappa$. Consequently, the profits of firm 1 and firm 2 are

$$
\Pi_1(y_1,y_2) = \frac{1}{2}(y_2-y_1)(y_2-y_1-2\kappa)^2 + (y_2-\kappa)\begin{cases}(v-(y_2-y_1-\kappa)^2) & \text{if } y_1 < \frac{y_2-\kappa}{2}\\ (v-y_1^2) & \text{if } y_1 \geq \frac{y_2-\kappa}{2}\end{cases}
$$

$$
\begin{aligned}
\Pi_2(y_1,y_2) =& \frac{1}{2}(y_2-y_1)(y_2-y_1-2\kappa)^2\\
&+ (1-y_1-\kappa)\begin{cases}(v-(y_2-y_1-\kappa)^2) & \text{if } y_2 > \frac{1+y_1+\kappa}{2}\\ (v-(1-y_2)^2) & \text{if } y_2 \leq \frac{1+y_1+\kappa}{2}\end{cases}
\end{aligned}
$$

The profit of firm 1 is strictly increasing for $y_1 < (y_2-\kappa)/2$ and the profit of firm 2 is strictly decreasing for $y_2 > (1+y_1+\kappa)/2$:

$$
\begin{aligned}
\frac{\partial \Pi_1(y_1,y_2)}{\partial y_1} &= 2\underbrace{(y_2-y_1-\kappa)}_{>0 \text{ as } y_1<\frac{y_2-\kappa}{2}}\underbrace{(y_2-\kappa)}_{\geq 0}+\frac{3}{2}\underbrace{(y_1-y_2+2\kappa)}_{>0}\underbrace{(y_2-y_1-\frac{2}{3}\kappa)}_{>0 \text{ as } y_1<\frac{y_2-\kappa}{2}} > 0\\
\frac{\partial \Pi_2(y_1,y_2)}{\partial y_2} &= 2\underbrace{(y_1-y_2+\kappa)}_{<0}\underbrace{(1-y_1-\kappa)}_{\geq 0}+\frac{3}{2}\underbrace{(y_2-y_1-2\kappa)}_{<0}\underbrace{(y_2-y_1-\frac{2}{3}\kappa)}_{>0 \text{ as } y_2>\frac{1+y_1+\kappa}{2}} < 0
\end{aligned}
$$

Thus firm 1's optimal location has to be in the interval $[(y_2-\kappa)/2, 1-\kappa]$ and firm 2's optimal location has to be in the interval $[\kappa, (1+y_1+\kappa)/2]$.

Step 3: Next, we derive the best replies for firm 1 and firm 2 with $y_1 \in [(y_2-\kappa)/2, 1-\kappa]$ and $y_2 \in [\kappa, (1+y_1+\kappa)/2]$:

$$
\begin{aligned}
\frac{\partial \Pi_1(y_1,y_2)}{\partial y_1} =& -2y_1(y_2-\kappa) - \frac{3}{2}(y_2-y_1-2\kappa)\left(y_2-y_1-\frac{2}{3}\kappa\right) = 0\\
&\Leftrightarrow y_1(y_2) = \frac{1}{3}\left(y_2-2\kappa \pm 2\sqrt{-2y_2^2+5y_2\kappa-2\kappa^2}\right)\\
\frac{\partial \Pi_2(y_1,y_2)}{\partial y_2} =& 2(1-y_2)(1-y_1-\kappa) + \frac{3}{2}(y_2-y_1-2\kappa)\left(y_2-y_1-\frac{2}{3}\kappa\right) = 0\\
&\Leftrightarrow y_2(y_1) = \frac{1}{3}\left(2+y_1+2\kappa \pm 2\sqrt{-2y_1^2+y_1(4-5\kappa)+\kappa(5-2\kappa)-2}\right).
\end{aligned}
$$

Checking the second order condition, we find that the potential maxima are:

$$
\begin{aligned}
y_1(y_2) &= \frac{1}{3}\left(y_2-2\kappa+2\sqrt{-2y_2^2+5y_2\kappa-2\kappa^2}\right)\\
y_2(y_1) &= \frac{1}{3}\left(2+y_1+2\kappa-2\sqrt{-2y_1^2+y_1(4-5\kappa)+\kappa(5-2\kappa)-2}\right).
\end{aligned}
$$

But to be the best replies, the potential maxima have to lie inside the interval

$[(y_2-\kappa)/2, 1-\kappa]$ and $[\kappa, (1+y_1+\kappa)/2]$. For firm 1:

$$y_1(y_2) \le 1-\kappa \Leftrightarrow y_2 \le \frac{1+3\kappa-2\sqrt{3\kappa-2}}{3} \text{ or } \frac{1+3\kappa+2\sqrt{3\kappa-2}}{3} \le y_2$$
$$\frac{y_2-\kappa}{2} \le y_1(y_2) \Leftrightarrow \frac{13-\sqrt{48}}{11}\kappa \le y_2 \le \frac{13+\sqrt{48}}{11}\kappa.$$

In addition $y_2 \in [\kappa, (1+y_1+\kappa)/2]$. Therefore, the best reply of firm 1 is

- if $\kappa > \frac{3}{4}$: $y_1^*(y_2) = 1-\kappa$
- if $\frac{2}{3} < \kappa \le \frac{3}{4}$:

$$y_1^*(y_2) = \begin{cases} \frac{1}{3}\left(y_2 - 2\kappa + 2\sqrt{-2y_2^2+5y_2\kappa-2\kappa^2}\right) & \text{if } y_2 \le \frac{1+3\kappa-2\sqrt{3\kappa-2}}{3} \\ 1-\kappa & \text{if } y_2 > \frac{1+3\kappa-2\sqrt{3\kappa-2}}{3} \end{cases}$$

- if $\frac{11}{13+\sqrt{48}} < \kappa \le \frac{2}{3}$: $y_1^*(y_2) = \frac{1}{3}\left(y_2 - 2\kappa + 2\sqrt{-2y_2^2+5y_2\kappa-2\kappa^2}\right)$
- if $\frac{1}{2} < \kappa \le \frac{11}{13+\sqrt{48}}$:

$$y_1^*(y_2) = \begin{cases} \frac{1}{3}\left(y_2 - 2\kappa + 2\sqrt{-2y_2^2+5y_2\kappa-2\kappa^2}\right) & \text{if } y_2 \le \frac{13+\sqrt{48}}{11}\kappa \\ \frac{y_2-\kappa}{2} & \text{if } y_2 > \frac{13+\sqrt{48}}{11}\kappa. \end{cases}$$

Similarly, the best reply of firm 2 is, then,

- if $\kappa > \frac{3}{4}$: $y_2^*(y_1) = \kappa$
- if $\frac{2}{3} < \kappa \le \frac{3}{4}$:

$$y_2^*(y_1) = \begin{cases} \kappa & \text{if } y_1 < \frac{2-3\kappa+2\sqrt{3\kappa-2}}{3} \\ \frac{2+y_1+2\kappa-2\sqrt{-2y_1^2+y_1(4-5\kappa)+\kappa(5-2\kappa)-2}}{3} & \text{if } y_1 \ge \frac{2-3\kappa+2\sqrt{3\kappa-2}}{3} \end{cases}$$

- if $\frac{11}{13+\sqrt{48}} < \kappa \le \frac{2}{3}$: $y_2^*(y_1) = \frac{2+y_1+2\kappa-2\sqrt{-2y_1^2+y_1(4-5\kappa)+\kappa(5-2\kappa)-2}}{3}$
- if $\frac{1}{2} < \kappa \le \frac{11}{13+\sqrt{48}}$:

$$y_2^*(y_1) = \begin{cases} \frac{1+y_1+\kappa}{2} & \text{if } y_1 < \frac{11-13\kappa-\sqrt{48}\kappa}{11} \\ \frac{2+y_1+2\kappa-2\sqrt{-2y_1^2+y_1(4-5\kappa)+\kappa(5-2\kappa)-2}}{3} & \text{if } y_1 \ge \frac{11-13\kappa-\sqrt{48}\kappa}{11}. \end{cases}$$

The intersections of the best replies gives the subgame-perfect equilibria.

Thus if $1/2 < \kappa \leq 3/4$, the subgame-perfect equilibrium locations are

$$y_1^* = \frac{1}{4}\left(2 - 3\kappa + \sqrt{\kappa^2 + 4\kappa - 2}\right)$$
$$y_2^* = \frac{1}{4}\left(2 + 3\kappa - \sqrt{\kappa^2 + 4\kappa - 2}\right)$$

and if $3/4 < \kappa \leq 1$, the subgame-perfect equilibrium locations are $y_1^* = 1 - \kappa$ and $y_2^* = \kappa$.

Chapter 5

Rational allocation of attention in decision-making

Abstract

I propose a model of attention as selecting information for costly processing. I assume that a decision-maker allocates attention rationally. I show that the resulting attention allocation is context-dependent. A change in context can change attention allocation and thus trigger changes in behavior. Furthermore, the optimal attention strategies of the decision-maker exhibit similarities to the psychological concepts of top-down and bottom-up attention. In an extension, I analyze the incentives of a firm to influence a worker's attention allocation. I show that this ability offers the firm a device to screen for productive workers.

Keywords: Limited attention, processing costs, salience, shrouding.

JEL Codes: D91, L13, L15.

5.1 Introduction

This article proposes a model of attention allocation in decision-making. Across the economic literature attention has different definitions. I understand attention as selecting information for costly processing. The article investigates how a decision-maker rationally allocates attention to information relevant for a decision. In particular, the paper examines the impact of processing costs on the resulting attention allocation and on the quality of decision-making. In addition, the paper links the different ways of allocating attention in the model to observed attention mechanisms in the psychological literature (top-down and bottom-up attention). Furthermore, taking the resulting attention allocation as a premise of how decision-makers allocate attention, I analyze firms that influence this attention allocation.

The model focuses on the choice of a decision-maker between two options. Decisions can be of different importance. In a more important decision more utility is at stake. The decision-maker has to choose between a default option and an alternative option. The decision-maker knows the value of the default, but does not know the value of the alternative option. In addition, the decision-maker does not know the importance of the decision problem. However, the decision-maker can pay attention, i.e., select these variables for costly processing. If the decision-maker does not pay attention to a variable, she has to choose between the options according to her expectations. If the decision-maker pays attention to a variable, the decision-maker eliminates all uncertainty about the realization of that variable.

This setup captures a number of situations. For instance, Hanna, Mullainathan, and Schwartzstein (2014) show in an experiment that seaweed farmers have to make various decisions to increase profits. For instance, a farmer has to choose pot size or length of lines. Some dimensions are more important compared to other dimensions. Hanna, Mullainathan, and Schwartzstein (2014) show that farmers are unaware of the importance of pot size and are, therefore, also unaware of which pot size is the best option. I capture such problems in my model: A farmer can pay attention to the importance of a dimension, for instance, how important pot size is, and a farmer can pay attention to which pot size gives the highest yield. Other examples include interviewing candidates for a job. The selection of a qualified candidate is more important for some jobs than for others. In addition, applicants differ in their qualifications. An employer can pay attention to the qualifications of the applicants to determine the best applicant. But an employer can also pay attention to how important that job is for the overall firm and thus potentially pay more attention to applicants for more critical jobs. This setup also captures choices between two goods. In this case, the consumer prefers to buy the good with the higher utility. The importance of the decision problem then represents the weight with which the good enters the consumer's overall consumption utility that consists of a basket of goods. Both information need not be available to the consumer. The consumer can pay attention to ingredients to figure out which good is the better option and by introspection the consumer can figure out how important this choice is to her.

The objective of the decision-maker is to decide which variable(s) to pay attention to and then to choose between the options. Thus the model consists of a two-stage decision problem for the decision-maker. In the first decision, the decision-maker chooses the variable(s) she wants to process and the processing order. A selection and corresponding order constitute an *attention strategy*. The set of attention strategies includes all possible ways of selecting the two variables for processing. If the decision-maker selects the variables simultaneously, she can select none, only one, or both variables. If the decision-maker selects the variables sequentially, she can select conditional, e.g., first check how important the decision is, and then—only if the decision is important—check which option has a higher value. After the decision-maker chooses an attention strategy, she processes the selected variable as specified in the chosen attention strategy. This is costly. In the second decision, the decision-maker chooses between the available options given the previously processed information. The decision-maker is always rational in the sense that she maximizes her expected utility when choosing an attention strategy and when choosing an option.

I assume that attention is the selection of information for costly processing. If the processing of information were for free, the decision about how to allocate attention would be trivial. A rational decision-maker would gather all pieces of information, calculate the utilities of both options precisely, and always choose the option with the highest utility. However, at the very least, the processing of information entails opportunity costs. Experimental and empirical evidence indicates that the costs for processing information are not zero and that these costs have an impact on decisions. Chetty, Looney, and Kroft (2009), for instance, find that—although consumers are mostly aware of sales taxes if asked—if it becomes easier to process the tax-inclusive price of an option, consumption behavior changes; consumers reduce consumption.[1]

In this paper, results demonstrate that only three attention strategies can prevail: Complete inattention, a direct selection of the decisive information, i.e., the value of the alternative option, and a conditional attention strategy. With this conditional attention strategy, the decision-maker first processes the importance of the decision and then, only if the decision is sufficiently important, checks the value of the alternative. Generally, the model shows that attention is context-dependent, i.e., the selection of an attention strategy depends on the processing costs and the beliefs about the value of the alternative. If the costs for processing the decisive variable are low, the decision-maker processes the decisive

[1]Following the psychological literature which assumes a capacity limit on processing information (see e.g., Nobre and Kastner, 2014, for a brief overview), the costs for processing some information represent the inability to process other information.

variable directly. The decision-maker then chooses between the options knowing which option is better. If the costs for processing the decisive variable are high, the decision-maker stays inattentive and decides according to expectations. If the costs for processing the decisive variable are intermediate, such that the benefits of knowing which option is better exceed the costs of processing in high-stake, but not in low-stake situations, the decision-maker selects the conditional attention strategy and processes the value of the alternative only in high-stake situations. The thresholds that determine which attention strategy is optimal depend on the decision problem, the distribution of the importance of decision problems, and the expectations of the decision-maker. Therefore, even if costs are constant, a change of the decision-maker's expectations can trigger a change in the decision-maker's attention allocation.

In addition, processing costs control which attention strategy is optimal and thus whether decisions are made with information or with expectations. As processing costs change, another attention strategy may become optimal and thus the decision between the options may be made with information instead of expectations (if costs drop) or vice versa (if costs increase). Then, whenever expectations do not align with reality, choice reversals occur. This finding is in line with evidence that shows that salience affects behavior (see, e.g., Chetty, Looney, and Kroft, 2009). Consequently, although the decision-maker allocates attention rationally, she does not always choose the utility-maximizing option.

The resulting attention allocation shares similarities with *top-down* and *bottom-up attention*—concepts reported in the psychological literature. Top-down attention describes an endogenous selection of information that requires more cognitive effort (see, e.g., Chun and Wolfe, 2005). In contrast, bottom-up attention describes an exogenous selection where the salience of the information draws attention automatically (see, e.g., Chun and Wolfe, 2005). I argue that the conditional selection shares similarities with top-down and the direct selection with bottom-up attention. That the rational considerations of the decision-maker result in the use of attention strategies that share similarities with top-down and bottom-up attention offers an explanation for the existence of top-down and bottom-up attention. Furthermore, processing costs alone—or whether a variable is salient—do not determine whether a variable is processed. In the model only the decisive variable is directly processed. Importance is never directly processed, even if it is very easy to process. The decision-maker only processes importance with the higher aim to condition the processing of the decisive variable on importance. Thus processing in the model accounts for type and relevance of a variable. This is in line with psychological research that demonstrates that top-down and bottom-up attention are not two fully separate concepts: In particular, the top-down goals—here to choose the option with the higher value—influence which stimuli are captured bottom-up—here the relevant stimuli, and the only one that is directly processed, is the value of the alternative (see, e.g., Pashler, Johnston, and Ruthruff, 2001, for an overview).

In Section 5.5, I analyze the strategy of a firm that wants to hire a worker. The firm cannot observe the productivity of a worker, but the firm can influence the worker's attention allocation. I show that the firm can use this ability to screen for productive workers.

The remainder of the article is structured as follows. Section 5.2 gives an overview of the related literature. Section 5.3 describes the model and introduces the attention strategies. Section 5.4 derives the optimal attention allocation and discusses the implications of such an attention allocation. Section 5.5 extends the model to include a firm being able to influence a worker's attention allocation. Section 5.6 concludes.

5.2 Related literature

A growing literature attempts to model limited attention. Two strands of that literature are related to my model: General models of attention, particularly models assuming rational attention allocation, and models emphasizing firms' strategic considerations about influencing the attention of consumers or workers. Models of attention are extremely varied; highlighting different aspects and methods of allocating attention. One strand of the literature, for example, focuses on the formation of consideration sets (see, e.g., Eliaz and Spiegler, 2011a; Manzini and Mariotti, 2018). This strand assumes that the decision-makers are not aware of the existence of all options in the choice set, but build a consideration set. The consideration set includes the options the decision-maker is aware of and the decision-maker knows the utility of all options inside the consideration set. Attention thus operates to form the consideration set.[2] However, models of attention often abstract from how consideration sets are formed and analyze the effects of attention inside the consideration set. Bordalo, Gennaioli, and Shleifer (2013), for instance, discuss the effects of salience on the evaluation of all perceived items.

My model is characterized by modeling attention as selecting information for costly processing. Therefore, I add to the long literature of economic models that follow Stigler (1961) in modeling information-processing as costly. More current approaches focus on attention allocation (see Gabaix, 2019, for a detailed survey). For instance, Gabaix (2014) analyzes how much attention a decision-maker pays to different dimensions of a decision problem when attention to dimension is costly. Alternative approaches to model costly processing, are rational inattention models developed originally by Sims (2003).[3] In these models of rational inattention, decision-makers choose an action and a signal optimally by comparing the benefits of information with the costs of information. Specific about

[2]Eliaz and Spiegler (2011a) also assume that all decision-makers have a default option and the competitor attempts to capture these consumers.

[3]Rational inattention has been applied to various different problems. For example, Matějka and McKay (2015) consider discrete choices, Maćkowiak and Wiederholt (2009) discuss price-setting, and van Nieuwerburgh and Veldkamp (2010) analyze investment decisions.

rational inattention models is that they model costs with entropy. That is more information reduces entropy, i.e., uncertainty about the state of the world, and a reduction in entropy is costly. Rational allocation models (as Gabaix, 2019; Sims, 2003) are often criticized, because the decision-maker needs to know the benefits and the costs of attention (see, e.g., Hanna, Mullainathan, and Schwartzstein, 2014). Hanna, Mullainathan, and Schwartzstein (2014) show that people do not necessarily know what is important and thus do not process beneficial information, although the costs for processing this information is sufficiently low. Therefore, I assume that the importance of a decision is unknown. I allow for conditional attention allocation and show that as a consequence the distinction of top-down and bottom-up attention arises naturally.[4] To my knowledge, my model is the first to capture this.

In Section 5.5, I allow firms to influence the processing costs of a worker, i.e., whether to shroud or highlight information about wages. In the industrial organization literature, different approaches of how to model shrouding (or obfuscation or educating consumers) exist. For example, Gabaix and Laibson (2006) model shrouding as a decision on whether to educate consumers about the existence of add-on prices and show that firms may have no incentive to educate myopic consumers. More closely related to my model, Ellison and Wolitzky (2012) model obfuscation as increasing consumers' cost to search for the price of a good. Ellison and Wolitzky (2012) show that a firm has an incentive to increase the search costs for its good to prevent consumers from searching further for better offers. Heidhues and Köszegi (2018) provide a detailed overview. However, these models focus specifically on competition between firms, which means that besides choosing their shrouding strategy firms also choose, for example, prices. Whereas, in my model the value of the option to the decision-makers is exogenously fixed.

Thus my model is most closely related to Manzini and Mariotti (2018). Manzini and Mariotti (2018) analyze consideration set formation. Decision-makers do not consider all options. Decision-makers only consider options that make it into their consideration set. In Manzini and Mariotti (2018), options with exogenously fixed quality invest in salience to increase the probability to make it into decision-makers' consideration sets. Thus consideration set formation is stochastic. Manzini and Mariotti (2018) show that if an increase of an option's salience has no or positive effects on the probability that other options enter the consideration set, the most salient option is also the best option and is chosen more often. However, if an increase of an option's salience also decreases the probability that other options are considered, then, cases exist where decision-makers

[4]Gabaix (2014) also argues that a relationship between psychology and his model exist. Yet, he assumes that in his two stage approach, the first stage, setting attention optimally, relates to Kahneman's sytem 1—a bottom-up approach—and the second stage, i.e., choosing an option, relates to system 2—a top-down approach. In contrast, I assume that in the first stage, when an attention strategy is chosen, the attention strategies differ in that some are top-down and others bottom-up.

choose the worse option more often.[5] I show that the firm with the higher wage is not necessarily salient. This depends on the expectations and costs of the firm.

In addition, I assume that the decision-makers know about the existence of the options, but do not know the value. Whereas in Manzini and Mariotti (2018), decision-makers only know about the existence of an option when it enters the consideration set. Expectations play no role, as decision-makers only choose options from inside their consideration set and these options are perfectly perceived. Additionally, in Manzini and Mariotti (2018) informing decision-makers works via salience, which is not specifically modeled via processing costs.

5.3 Model

A decision-maker chooses between two options, for example, between two goods. The decision-maker is randomly allocated to an option such that option 1 is the default option 50% of the cases and option 2 is the default in the other 50% of the cases. The objective of the decision-maker is to decide whether to switch from the default option to the alternative. When a decision-maker chooses an option, she receives utility

$$U_i = \theta u_i,$$

where $\theta \geq 0$ describes the importance of the decision problem.[6] θ is drawn from a twice-continuously differentiable distribution $F(\cdot)$ on $[0, \bar{\theta}]$ with pdf $f(\cdot) > 0$ for all $\theta \in [0, \bar{\theta}]$ and $F(0) = 0$. This is known to the decision-maker. Let $E[\theta] \equiv \int_0^{\bar{\theta}} \theta f(\theta) d\theta$. u_a is distributed according to the twice-continuously differentiable distribution $G(\cdot)$ with pdf $g(\cdot)$ on a subset of $\mathbb{R}_0^+$, which corresponds to the expectations of the decision-maker. Let $E[u_a] \equiv \int_0^{\infty} u_a g(u_a) du_a$. θu_i is the total utility of option i. Assume that θ and u_a are independently distributed. Then, the expected utility of option i is $E[U_i] = E[\theta]E[u_i]$.

The variable θ represents the importance of the decision problem: a higher θ ceteris paribus means that more utility is at stake in the decision. This setup captures two situations: First, a decision-maker faces different decision problems that are of different importance. Second, different decision-makers face the same decision problem but for some decision-makers this is more important than for others. Independent of which option is the default, the values u_1 and u_2 determine whether the utility of option 1 is higher or lower than the utility of option 2. If $u_1 > u_2$, the utility of option 1 is higher and the decision-maker prefers option 1. If $u_1 < u_2$, the utility of option 1 is lower and the decision-maker prefers option 2. If $u_1 = u_2$, the decision-maker is indifferent between option 1 and option 2. I assume that the decision-maker always breaks indifference in

[5] Such equlibria can also occur when salience is not defined as more being better but as being different.

[6] For example, θ gives the weight with which this decision problem enters in the decision-maker's overall life or consumption utility.

favor of the default.

Denote the decision-maker's default option by $d \in \{1, 2\}$ and the decision-maker's alternative option by $a \in \{1, 2\}$ with $a \neq d$. I assume that the decision-maker knows the value of the default u_d. In addition, I assume that the decision-maker knows the setup of the decision problem (i.e., the probabilities, possible values of the random variables, etc.). Yet, the decision-maker does not automatically process the value of the alternative u_a or the realization of θ. Only if the decision-maker pays attention to θ or u_a, does she process the realizations of those variables. If the decision-maker does not process u_a, the decision-maker chooses according to expectations.

I understand attention as selecting variables for costly processing. The model includes two variables whose realizations are unknown to the decision-maker: θ and u_a. Let $c_\theta > 0$ and $c_{u_a} > 0$ describe the costs of processing θ and the costs of processing u_a. I assume that the processing costs are additive separable to the utility derived from the choice between the two options:

$$U_i - C_i.$$

The decision-maker has to decide which variable(s) to pay attention to and then processes these variables, before she decides between the two options. The decision-maker can process these variables simultaneously or sequentially to find out the realizations of the variables. If she chooses to process the variables simultaneously, with two variables four strategies are possible: processing neither variable, processing only θ, processing only u_a, or processing both θ and u_a. In contrast, sequential processing allows the decision-maker to process one variable first and then condition the processing of the other variable on the realization of the first processed variable. For example, the decision-maker can first check how important the decision is by processing θ and, if it is an sufficiently important decision problem (θ is high), process u_a. Sequential processing can be summarized in two ways: (i) processing θ first and processing u_a if $\theta \in A_\theta$, where A_θ is a non-empty measurable subset of $[0, \bar{\theta}]$. That means, if the decision-maker observes a $\theta \in A_\theta$, she then processes u_a. But, if she processes a $\theta \notin A_\theta$, she does not process u_a; (ii) processing u_a first and processing θ if $u_a \in A_{u_a}$, where A_{u_a} is a non-empty measurable subset of $\mathbb{R}_0^+$. That means, if the decision-maker observes a $u_a \in A_{u_a}$, she then goes on to process θ. But, if she processes a $u_a \notin A_{u_a}$, she does not process θ.

Each of these possibilities to process the two variables specifies an *attention strategy*. I define an attention strategy by the set of variables that are processed with the attention strategy. The set of all possible attention strategies is thus

$$\Big\{\{\}, \{\theta\}, \{u_a\}, \{\theta, u_a\}, (\theta, \{u_a | \theta \in A_\theta\}), (u_a, \{\theta | u_a \in A_{u_a}\})\Big\}.$$

The first four elements of the set refer to the four simultaneous attention strategies. The attention strategy {} describes that the decision-maker processes neither θ nor u_a. The attention strategy $\{\theta\}$ describes that the decision-maker processes θ, but does not process u_a. The attention strategy $\{u_a\}$ describes that the decision-maker processes u_a, but does not process θ. The attention strategy $\{\theta, u_a\}$ describes that the decision-maker processes both θ and u_a. The last two elements refer to the sequential attention strategies. As with sequential processing, the order of processing matters, the sequential attention strategies are expressed as ordered pairs. The attention strategy $(\theta, \{u_a|\theta \in A_\theta\})$ describes that the decision-maker first processes θ and then, if she observes a $\theta \in A_\theta$, the decision-maker also processes u_a, but if she processes a $\theta \notin A_\theta$, the decision-maker does not process u_a. The attention strategy $(u_a, \{\theta|u_a \in A_{u_a}\})$ describes that the decision-maker first processes u_a. Then, if the decision-maker observes a $u_a \in A_{u_a}$, she also processes θ, but if she processes a $u_a \notin A_{u_a}$, she does not process θ. These last two elements summarize a continuum of attention strategies, as A_{u_a} and A_θ can be any measurable subset of $\mathbb{R}_0^+$ and $[0, \bar{\theta}]$.

The decision-maker has to choose between the attention strategies before she chooses between the two options. Consequently, the model consists of a two-stage decision problem for the decision-maker. At the beginning, the decision-maker is aware of the setup, but uncertain about θ and u_a. In the first decision, the decision-maker chooses an attention strategy. Given her choice, she processes the selected variable(s) as specified in the attention strategy. In the second decision, she chooses an option. I assume that the decision-maker is rational in the sense that she maximizes her expected utility in both choices taking the processing costs into account.

5.4 Optimal attention allocation

Before the decision-maker decides which option to choose, the decision-maker decides which attention strategy to use. So at the time the decision-maker chooses between the options, she either has processed the utility of the alternative or not. If the decision-maker processes the alternative's utility before the decision, the decision-maker chooses the option with the higher utility. But if the decision-maker does not process the alternative's utility, she chooses according to expectations. The decision-maker chooses the attention strategy that maximizes her expected utility. Some attention strategies are never optimal, because, compared to other attention strategies, they imply higher processing costs without improving the decision. The decision-maker never chooses such an attention strategy. Lemma 1 summarizes the attention strategies that are never optimal.

Lemma 1 *An decision-maker with default $i \in \{1, 2\}$ never chooses the attention strategies $\{\theta\}$, $\{\theta, u_a\}$, or $(u_a, \{\theta|u_a \in A_{u_a}\})$.*

All proofs are in appendix B.

Compared to other attention strategies, the strategies listed in Lemma 1 entail higher processing costs without increasing the quality of the decision. With $\{\}$ the decision-maker processes neither θ nor u_a and with $\{\theta\}$ she processes only θ. Then, with $\{\}$ as well as with $\{\theta\}$, the decision-maker does not process u_a and thus chooses according to expectations. In addition, with $\{\}$ the decision-maker has no processing costs. Yet, with $\{\theta\}$ the decision-maker processes θ which does not help her to find the option with the higher utility, but entails processing costs. Thus $\{\theta\}$ can never be an optimal strategy. Furthermore, with $\{\theta, u_a\}$ the decision-maker processes θ and u_a, with $\{u_a\}$ she processes only u_a, and with $(u_a, \{\theta | u_a \in A_{u_a}\})$ she processes u_a first and if she observes a $u_a \in A_{u_a}$ she also processes θ. Then, as the decision-maker expects higher processing costs with $\{\theta, u_a\}$ and $(u_a, \{\theta | u_a \in A_{u_a}\})$ than with $\{u_a\}$ but processes u_a with all three strategies and thus chooses the utility-maximizing option with all three strategies, $\{\theta, u_a\}$ and $(u_a, \{\theta | u_a \in A_{u_a}\})$ are never optimal.

In addition not all strategies $(\theta, \{u_a | \theta \in A_\theta\})$ are optimal:

Lemma 2 $(\theta, \{u_a | \theta \in A_\theta\})$ *can only be optimal with* $A_\theta = A^*_\theta \equiv [\theta^*, \bar{\theta}]$ *and*

$$\theta^* \equiv \begin{cases} \frac{c_{u_a}}{\int_{u_d}^{\infty} u_a g(u_a) du_a - (1-G(u_d))u_d} & \text{if } u_d \geq E[u_a] \\ \frac{c_{u_a}}{G(u_d)u_d - \int_0^{u_d} u_a g(u_a) du_a} & \text{if } u_d < E[u_a]. \end{cases} \tag{5.1}$$

$(\theta, \{u_a | \theta \in A_\theta\})$ summarizes a continuum of attention strategies. But, $(\theta, \{u_a | \theta \in A_\theta\})$ can only be an optimal attention strategy if $A_\theta = A^*_\theta \equiv [\theta^*, \bar{\theta}]$ with θ^* specified in (5.1). As $(\theta, \{u_a | \theta \in A_\theta\})$ is a sequential strategy, at the time the decision-maker decides whether to process u_a, the decision-maker already knows θ and the costs of processing θ are sunk. To process u_a is optimal if the expected utility of processing u_a exceeds the expected utility of not-processing u_a. Because the utility is increasing in θ but the costs for processing u_a are fixed, this comparison yields a unique cut-off θ^* as specified in (5.1) such that, for all $\theta < \theta^*$, the decision-maker does not process u_a and, for all $\theta \geq \theta^*$, the decision-maker does process u_a.

Overall, Lemma 1 and 2 exclude allocation of attention to redundant or useless information. For each decision-maker three strategies prevail: Inattention, $\{\}$; direct selection of u_a, $\{u_a\}$; and a conditional allocation, where the decision-maker selects u_a only if the decision is important, $(\theta, \{u_a | \theta \geq \theta^*\})$. Each of the three strategies is strictly optimal for some cost combinations. Proposition 9 specifies the optimal attention allocation.

Proposition 9 *Consider a decision-maker whose default is option $d \in \{1,2\}$ and let*

$$\bar{c}_{u_a} \equiv \begin{cases} E[\theta]\left(\int_{u_d}^{\infty} u_a g(u_a)du_a - (1-G(u_d))u_d\right) & \text{if } u_d \geq E[u_a] \\ E[\theta]\left(G(u_d)u_d - \int_0^{u_d} u_a g(u_a)du_a\right) & \text{if } u_d < E[u_a] \end{cases}$$

$$\bar{c}_\theta(c_{u_a}) \equiv \begin{cases} (1-F(\theta^*))\left(E[\theta|\theta \geq \theta^*]\left(\int_{u_d}^{\infty} u_a g(u_a)du_a - (1-G(u_d))u_d\right) - c_{u_a}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u_d \geq E[u_a] \\ (1-F(\theta^*))\left(E[\theta|\theta \geq \theta^*]\left(G(u_d)u_d - \int_0^{u_d} u_a g(u_a)du_a\right) - c_{u_a}\right) \quad \text{if } u_d < E[u_a] \end{cases}$$

$$\underline{c}_\theta(c_{u_a}) \equiv \begin{cases} F(\theta^*)\left(E[\theta|\theta < \theta^*]\left((1-G(u_d))u_d - \int_{u_d}^{\infty} u_a g(u_a)du_a\right) + c_{u_a}\right) & \text{if } u_d \geq E[u_a] \\ F(\theta^*)\left(E[\theta|\theta < \theta^*]\left(\int_0^{u_d} u_a g(u_a)du_a - G(u_d)u_d\right) + c_{u_a}\right) & \text{if } u_d < E[u_a] \end{cases}$$

(i) $\{\}$ is the optimal attention strategy if and only if $c_{u_a} \geq \bar{c}_{u_a}$ and $c_\theta \geq \bar{c}_\theta(c_{u_a})$.

(ii) $\{u_a\}$ is the optimal attention strategy if and only if $c_{u_a} \leq \bar{c}_{u_a}$ and $c_\theta \geq \underline{c}_\theta(c_{u_a})$.

(iii) $(\theta, \{u_a|\theta \in [\theta^*, \bar{\theta}]\})$ with θ^* specified in (5.1) is the optimal attention strategy if and only if $c_{u_a} \leq \bar{c}_{u_a}$ and $c_\theta \leq \underline{c}_\theta(c_{u_a})$ or $c_{u_a} \geq \bar{c}_{u_a}$ and $c_\theta \leq \bar{c}_\theta(c_{u_a})$.

Figure 5.1 provides an example of an attention allocation as specified in Proposition 9.

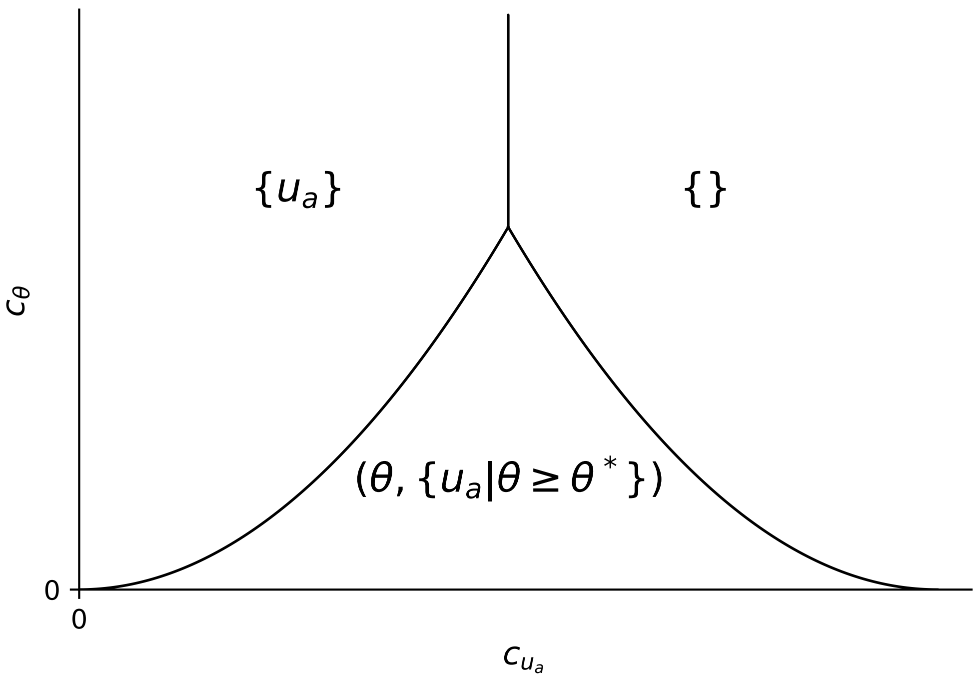

Figure 5.1: Illustration of Proposition 9, for $u_d = 1$, θ being uniformly distributed on $[0,1]$, and the decision-maker believes that u_a is uniformly distributed on $[0,2]$

Proposition 9 specifies the conditions for the optimal attention allocation. $\bar{c}_{u_a}$ determines the costs of processing u_a such that the attention strategies $\{\}$ and $\{u_a\}$ yield the same expected utility. Therefore, if $c_{u_a} < \bar{c}_{u_a}$, the decision-maker never uses $\{\}$ and, if $c_{u_a} > \bar{c}_{u_a}$, the decision-maker never uses $\{u_a\}$. Consequently, for all $c_{u_a} < \bar{c}_{u_a}$, the decision-maker either uses $\{u_a\}$ or $(\theta, \{u_a|\theta \geq \theta^*\})$. A unique cost threshold function for the costs of processing θ, $\underline{c}_\theta(c_{u_a})$, determines which strategy the decision-maker uses. If $c_\theta < \underline{c}_\theta(c_{u_a})$, the decision-maker uses $(\theta, \{u_a|\theta \geq \theta^*\})$ and if $c_\theta > \underline{c}_\theta(c_{u_a})$, the decision-maker uses $\{u_a\}$. Similarly, if $c_{u_a} > \bar{c}_{u_a}$, the decision-maker either uses $\{\}$ or $(\theta, \{u_a|\theta \geq \theta^*\})$. The threshold $\bar{c}_\theta(c_{u_a})$ on the costs of processing θ uniquely specifies which attention strategy is optimal. For all $c_\theta < \bar{c}_\theta(c_{u_a})$, the decision-maker uses $(\theta, \{u_a|\theta \geq \theta^*\})$ and for all $c_\theta > \bar{c}_\theta(c_{u_a})$ the decision-maker uses $\{\}$. More generally, the decision-maker prefers to stay inattentive if it is too costly to process u_a directly. The decision-maker prefers to directly process u_a if it is cheap to process u_a, and the decision-maker prefers the conditional attention strategy if the costs of processing u_a are intermediate such that the expected benefits of knowing u_a exceed the costs in important decision problems but not in unimportant decision problems.

The optimal attention allocation of the decision-maker, as specified in Proposition 9, shows that the allocation of attention is context-dependent. Which attention strategy a decision-maker uses depends on the costs of processing θ and u_a. The thresholds $\bar{c}_\theta(c_{u_a})$, $\underline{c}_\theta(c_{u_a})$, and $\bar{c}_{u_a}$ are functions of the default utility, the beliefs about the alternative's utility, and the distribution of the importance of the decision problem. Therefore, given the costs c_θ and c_{u_a} if, for example, the beliefs about u_a change, the thresholds $\bar{c}_\theta(c_{u_a})$, $\underline{c}_\theta(c_{u_a})$, and $\bar{c}_{u_a}$ shift, which might make another attention strategy optimal.

Although the attention allocation is optimal, the attention allocation does not imply that the decision-maker always chooses the better option. For example, with attention strategy $\{\}$ the decision-maker always chooses according to expectations. Then, whenever expectations and realizations do not align, i.e., $u_d < u_a$ and $u_d \geq E[u_a]$ or $u_d \geq u_a$ and $u_d < E[u_a]$, the decision-maker chooses the option with the lower utility. With the attention strategy $(\theta, \{u_a|\theta \geq \theta^*\})$, whenever $\theta \geq \theta^*$, the decision-maker processes u_a and knows which option is better. Thus whenever $\theta \geq \theta^*$, the decision-maker always chooses the better option. However, when $\theta < \theta^*$, the decision-maker chooses according to expectations and thus chooses the better option only when expectations and reality align. Only with attention strategy $\{u_a\}$ does the decision-maker always process u_a and thus always know which option is better. Consequently, the decision quality depends on the attention strategy and as the attention strategy depends on the processing costs and pdf of the decision-maker, the decision quality is also highly context-dependent.

Therefore, a change in the decision context can change the attention strategy and the subsequent decision. For example, consider the situation depicted in Figure 5.2 and assume $u_d < u_a$ and $u_d \geq E[u_a]$. Starting at point A the processing costs are such that it is optimal to use attention strategy $\{\}$, i.e., the decision-maker chooses according to expectations and chooses the default, which is the option with the lower utility. If costs for processing u_a drop until they reach level B, $\{\}$ is no longer optimal. The optimal attention strategy is now $(\theta, \{u_a|\theta \geq \theta^*\})$ and the decision-maker processes u_a if $\theta \geq \theta^*$, i.e., with probability $1 - F(\theta^*)$. If $\theta < \theta^*$, the decision-maker does not process u_a and chooses the default. However, if $\theta \geq \theta^*$, the decision-maker processes u_a and chooses the better option. Thus if $\theta \geq \theta^*$, the decision-maker chooses a different option with $\{\}$ than with $(\theta, \{u_a|\theta \geq \theta^*\})$ and if $\theta < \theta^*$ she chooses the same option with $\{\}$ and with $(\theta, \{u_a|\theta \geq \theta^*\})$. By moving from point B to point C, the costs of processing θ do not change. However, the costs of processing u_a drop such that the optimal attention strategy becomes $\{u_a\}$. Thus θ is no longer processed although its processing costs have not changed. If $\theta < \theta^*$, decisions change because with $(\theta, \{u_a|\theta \geq \theta^*\})$ the decision-maker chooses the default and now with $\{u_a\}$ the decision-maker chooses the better alternative. Thus by moving from A over B to C, decisions changes if reality and expectations do not align: The decision change occurs either at the switch from $\{\}$ to $(\theta, \{u_a|\theta \geq \theta^*\})$ or at the switch to $\{u_a\}$.

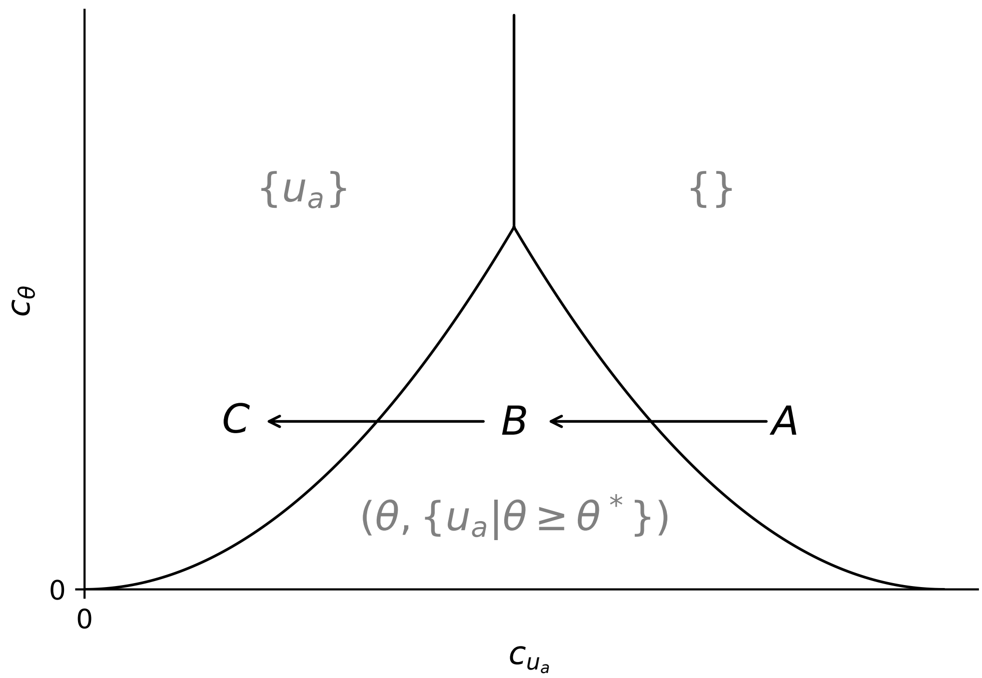

Figure 5.2: Example of crowding out and choice reversals.

Overall, this example illustrates that, the processing of a variable does not only depend on its own processing costs. Similarly, for given costs c_θ and c_{u_a}, if, for instance, the beliefs $G(\cdot)$ of the decision-maker with default i change, the cut-offs $\underline{c}_\theta(c_{u_a})$, $\bar{c}_\theta(c_{u_a})$, and $\bar{c}_{u_a}$ shift and thus another attention strategy can become optimal. In addition, this model shows that decisions change as a consequence of changing attention allocation which is triggered, for example, by a change in processing costs or beliefs. Table 5.1 summarizes all possible situations and the corresponding decisions between option 1 and option 2. Table 5.1 shows that decisions change when expectations do not align with reality: I.e., either if $u_d < u_a$ and $u_d \geq E[u_a]$ or if $u_d \geq u_a$ and $u_d < E[u_a]$. Then, changes occur as a consequence of changes in information. Consequently, the model offers an explanation for changes in observed behavior that occur as salience of information increases (see, e.g., Chetty, Looney, and Kroft, 2009, where demand decreases as tax-inclusive prices become more salient). The increase in salience induces processing of information that was previously not processed.

		$u_d \geq u_a$ $\theta < \theta^*$	$u_d \geq u_a$ $\theta \geq \theta^*$	$u_d < u_a$ $\theta < \theta^*$	$u_d < u_a$ $\theta \geq \theta^*$
	$\{\}$	default	default	default	default
$u_d \geq E[u_a]$	$\{u_a\}$	default	default	alternative	alternative
	$(\theta, \{u_a \vert \theta \geq \theta^*\})$	default	default	default	alternative
	$\{\}$	alternative	alternative	alternative	alternative
$u_d < E[u_a]$	$\{u_a\}$	default	default	alternative	alternative
	$(\theta, \{u_a \vert \theta \geq \theta^*\})$	alternative	default	alternative	alternative

Table 5.1: Choice of the decision-maker for each attention strategy and each possible situation. The gray cells highlight the cases, when due to changing attention, decisions change.

Proposition 9 demonstrates that—in addition to inattention—two attention strategies prevail: direct attention $\{u_a\}$ and a more complicated conditional allocation of attention $(\theta, \{u_a|\theta \geq \theta^*\})$. I argue that these two attention strategies show similarities to the two attention mechanisms *bottom-up* and *top-down* attention described in the psychological literature.[7] Bottom-up attention refers to information selection which is driven by exogenous factors; the salience of the stimuli draws attention automatically (see, e.g., Chun and Wolfe, 2005; Pashler, Johnston, and Ruthruff, 2001). In contrast, top-down attention is driven by cognitive processes and goals, i.e., it is endogenous, requires more effort, and is slower (see, e.g., Chun and Wolfe, 2005; Pashler, Johnston, and Ruthruff, 2001).

Strategy $(\theta, \{u_a|\theta \geq \theta^*\})$, the conditional allocation of attention, shares characteristics with top-down attention. Being conditional makes $(\theta, \{u_a|\theta \geq \theta^*\})$ more complex than a direct selection. In addition, as $(\theta, \{u_a|\theta \geq \theta^*\})$ requires a decision about what implications the processing of θ has on the processing of u_a, i.e., $(\theta, \{u_a|\theta \geq \theta^*\})$ is more endogenous. Furthermore, the processing of two variables (instead of one variable with

[7]The mechanism of bottom-up attention is comparable with that of *system 1* and the mechanism of top-down attention is comparable with that of *system 2* (Kahneman, 2003, 2011). I will exclusively refer to the concepts as top-down and bottom-up attention.

$\{u_a\}$) makes $(\theta, \{u_a|\theta \geq \theta^*\})$ slower. This is true especially if time costs are part of processing costs. The processing induced by $(\theta, \{u_a|\theta \geq \theta^*\})$ includes higher processing costs if u_a is processed (i.e., if $\theta \geq \theta^*$). Strategy $\{u_a\}$ shares characteristics with bottom-up attention. The processing of u_a is faster with $\{u_a\}$ than with $(\theta, \{u_a|\theta \geq \theta^*\})$ because the decision-maker processes u_a directly without detour via θ. Bottom-up attention is induced because of the salience of the information which automatically draws attention. In other words, if it is easy to process the information, the decision-maker uses $\{u_a\}$.

Bottom-up and top-down attention are not two fully separate concepts. In particular, Pashler, Johnston, and Ruthruff (2001) show that bottom-up capture of information is not only driven by low processing costs (i.e., by salience), but by the top-down goals of the decision-maker. That is, if the properties of the stimuli are in line with the top-down goals of the decision-maker, the stimuli capture attention. This is in line with my finding that only u_a is directly processed: The goal of the decision-maker is to choose the option with the higher utility. The variable θ determines the importance of the decision problem, but does not give any information about which option is better. Therefore the decision-maker has no incentive to process θ directly without conditioning the processing of u_a on θ (Lemma 1). To attain the goal, the decision-maker needs to process u_a. Overall the model thus accounts for different types of information and the costs alone do not determine whether the information are processed.

That rational considerations of decision-maker in the model result in the use of two attention strategies that share similarities with attention mechanisms reported in psychology offers a rational explanation for the existence of top-down and bottom-up attention. Changing the perspective, by assuming that a link between $\{u_a\}$ and bottom-up and $(\theta, \{u_a|\theta \geq \theta^*\})$ and top-down attention exists, the model offers predictions about when a decision-maker uses bottom-up attention, top-down attention, and inattention: A decision-maker uses bottom-up attention in a low cost environment, when benefits exceed the costs, e.g., when the decisive variable is salient. If costs are intermediate, the decision-maker uses top-down attention and in a high cost environment she uses inattention. Relevant for the selection are thus the processing costs. Options might benefit from such knowledge and influence whether the decision-maker can observe her utilities by increasing or decreasing the processing costs accordingly, for example, via shrouding or advertising.

5.5 Influencing processing costs as screening

In this section, I extend the model developed in Section 5.3 to allow the options to influence the processing costs. If firms know how people allocate attention, they can manipulate this attention allocation by influencing the processing costs. In the following, I discuss the incentives of firms to choose processing costs that potential employees have to observe wages. I show that firms might use their ability to influence processing costs to induce a specific attention allocation of workers to screen for very productive workers. Thus I show that the model of attention allocation developed in Section 5.3 captures interesting effects.

Consider two firms, firm d and firm a. Firms each produce a homogeneous good that they can sell for a fixed price normalized to 1. The amount that firms produce depends on the productivity of their worker θ and the wage they pay their worker. That means the profit of firm i is

$$\Pi_i = \theta(1 - u_i) - C,$$

where $\theta \in [0, \bar{\theta}]$ is the productivity of the worker, $0 < u_i < 1$ is the piece wage firm i pays the worker, and C are the fixed costs for employing the worker. To make it potentially profitable to hire a worker, let $\bar{\theta} > C/(1 - u_i)$. The profitability of a worker is drawn from a twice-continuously differentiable distribution $F(\cdot)$ on $[0, \bar{\theta}]$ with pdf $f(\cdot) > 0$ for all $\theta \in [0, \bar{\theta}]$ and $F(0) = 0$. This is known to the worker and the firm.

Consider a worker who has to choose between working at firm d and firm a. Her utility for working at firm i is

$$U_i = \theta u_i.$$

Assume the worker knows the piece wage u_d of firm d, for example, because an acquaintance works at firm d, but does not know the piece wage of firm a. The worker has expectations about the wage given by the twice-continuously differentiable distribution $G(\cdot)$ with pdf $g(\cdot)$ on a subset of $\mathbb{R}_0^+$. Let $E[u_a] \equiv \int_0^\infty u_a g(u_a) du_a$. For simplicity, assume $u_a > u_d$ and $u_d \geq E[u_a]$, which is known to the firm. This means, the firm knows that it is the better option for the worker, because it offers the higher wage, but that the worker does not know this. The worker expects that firm d has the higher wage. In addition, the worker does not know her productivity θ. But, the worker can find out θ and u_a by paying attention. This is costly as specified in Section 5.3. For example, by thinking about past working experiences, the worker can find out the productivity θ and by researching the worker can find out the piece wage u_a. Assume the attention allocation of the worker is given by Proposition 9.

Firm a can influence the costs the worker incurs to process the wage u_a: c_{u_a}. For example, the firm can advertise its wage or shroud this information. The firm might incur costs for this, However, these costs to influence the worker's processing costs are negligible. Influencing the processing costs includes implementation costs. For example, to construct the websites in a way that wages are difficult to find or that wages are advertised prominently. In both cases the firm has setup costs, for instance, for constructing a website. Compared to these setup costs, the costs to highlight or shroud information is negligible.

Therefore, I assume that these costs to influence the decision-maker's processing costs are zero. This provides a good benchmark.

Assume the firm cannot observe θ before hiring a person, but the firm knows the distribution of θ in the population. In addition, c_θ is exogenously given and I assume that the setup is such that all three attention strategies can be targeted by the firm. Whenever the worker does not observe the wage of the alternative, she chooses between the firms according to her expectations.

Firm a chooses the costs c_{u_a} to maximize its profit. The firm can either choose c_{u_a} such that the worker uses attention strategy $\{\}$, such that the worker uses attention strategy $\{u_a\}$, or such that the worker uses attention strategy $(\theta, \{u_a | \theta \geq \theta^*\})$.

Assume the firm chooses the costs for processing u_a so high that the worker uses attention strategy $\{\}$. Then, the worker chooses between firm d and firm a according to her expectations. That is, she chooses d because $u_d \geq E[u_a]$. The profit of firm a is then

$$\Pi_a = 0.$$

Assume the firm chooses the costs for processing u_a so low that the worker uses attention strategy $\{u_a\}$. Then, the worker chooses between firm d and firm a by comparing the actual wages. That is, because $u_a > u_d$ she chooses firm a. The expected profit of firm a is

$$\Pi_a = E[\theta](1 - u_a) - C.$$

Assume the firm chooses the costs for processing u_a such that the worker uses attention strategy $(\theta, \{u_a | \theta \geq \theta^*\})$. Then, the worker first observes her productivity θ, and if $\theta \geq \theta^*$ she chooses by comparing the actual wages which means she chooses firm a. In contrast, if $\theta < \theta^*$, she chooses according to expectations and chooses firm d. The firm does not know θ, so the expected profit of firm a is then

$$\Pi_a = (1 - F(\theta^*))\Big(E[\theta | \theta \geq \theta^*](1 - u_a) - C\Big) + F(\theta^*)0.$$

The firm can influence θ^* by choosing c_{u_a} accordingly (see (5.1)). To maximize its profits, firm a chooses c_{u_a} such that

$$\theta^* = C/(1 - u_a).$$

In other words in the case that firm a chooses processing costs such that the worker uses attention strategy $(\theta, \{u_a | \theta \geq \theta^*\})$, the firm chooses c_{u_a} such that $\theta^* = C/(1 - u_a)$.

Firm a should choose the processing costs to maximize its profits over all three possible attention strategies. Firm a chooses costs c_{u_a} such that $(\theta, \{u_a | \theta \geq \theta^*\})$ is optimal if and only if

$$\begin{aligned}
&(1-F(\theta^*))\Big(E[\theta|\theta \geq \theta^*](1-u_a)-C\Big) \geq E[\theta](1-u_a)-C \\
\Leftrightarrow\ & C \geq E[\theta|\theta < \theta^*](1-u_a) \\
\Leftrightarrow\ & \frac{C}{(1-u_a)} \geq \underbrace{E[\theta|\theta < \frac{C}{(1-u_a)}]}_{<\frac{C}{(1-u_a)}}
\end{aligned}$$

and

$$\begin{aligned}
(1-F(\theta^*))\Big(E[\theta|\theta \geq \theta^*](1-u_a)-C\Big) \geq 0 \ \Leftrightarrow\ & C \leq E[\theta|\theta \geq \theta^*](1-u_a) \\
\Leftrightarrow\ & \frac{C}{(1-u_a)} \leq \underbrace{E[\theta|\theta \geq \frac{C}{1-u_a}]}_{\geq\frac{C}{1-u_a}}.
\end{aligned}$$

Thus choosing the processing costs c_{u_a} such that the worker uses the attention strategy $(\theta, \{u_a|\theta \geq \theta^*\})$ is always optimal for firm a. Although, the firm cannot directly sort workers according to their productivity, the attention strategy does this for the firm. All workers do observe their productivity with this attention strategy but only very productive workers (i.e., workers with $\theta \geq \theta^*$) also observe u_a. Then, if the worker has a $\theta < \theta^*$, she does not process u_a and chooses according to her expectations. Thus she chooses firm d. If the worker has a $\theta \geq \theta^*$, she processes u_a and chooses to work for firm a. Consequently, by targeting attention strategy $(\theta, \{u_a|\theta \geq \theta^*\})$, the firm only receives a worker who is generating a surplus for the firm.

Overall, this example demonstrates that the optimal attention allocation developed in Section 5.4 can capture important problems. The firm can use its ability to influence the workers' attention allocation as a screening device that screens out unproductive workers.

5.6 Discussion and conclusion

In this article, I propose a model of rational allocation of attention in decision-making. Exploring the resulting attention allocation reveals that the selection of an attention strategy and, consequently, the quality of the choice between the two options are highly context-dependent. A decision-maker is never fully informed about the situation. The always fully informed decision-maker only shows up in the limiting case as costs c_θ and c_{u_a} are zero. Generally, the decision-maker does not always gather all available information. Therefore, the decision-maker is not always informed about which option she prefers. This result is in line with economic findings that decision-makers do not always fully process all information and that salience is relevant to induce processing (see, e.g., Chetty, Looney, and Kroft, 2009). In my model, as the processing costs change sufficiently, the optimal attention strategy changes such that the decision-maker chooses given information instead of expectations. Consequently, the behavior of the decision-maker can change if attention allocation changes.

In the resulting attention allocation, (next to inattention) two attention strategies prevail. These attention strategies share similarities with top-down and bottom-up attention, concepts from the psychological literature. The model thus offers an explanation for the existence of top-down and bottom-up attention; as the result of optimization processes. The resulting attention allocation becomes the premise of how the worker allocates attention in Section 5.5. That the attention strategies share similarities with observed mechanisms is advantageous as it provides evidence for the premise of how the worker allocates attention and thus strengthens the soundness of the argument. I investigate how a firm influences the attention allocation of a worker by influencing the worker's processing costs. The results demonstrate that the firm uses attention allocation of the worker as a screening device.

Some issues are beyond the scope of this paper and may serve as ideas for future research. For example, I assume that the decision-makers always observe the true value of the option, i.e., options cannot mislead decision-makers through false advertising. Future research might discuss the consequences of such false advertising. In addition, I also implicitly assume that decision-makers are aware of the costs c_{u_a} and c_θ, before choosing an attention strategy. An interesting avenue for further research would be to discuss the implications of alternative assumptions.

Furthermore, the model includes only two variables to which the decision-maker can allocate attention. Nevertheless, the model can be extended easily to allow for options with multiple dimensions. One possibility is to interpret u_i as a summary that captures all dimensions. Another possibility is to consider that the two options are mostly identical and differ only in one dimension. For example, two notebooks are identical, but differ in weight. For some decision-makers weight is extremely important (θ is high), for instance, because they travel frequently, for others weight matters little compared to other dimensions (θ is low). Then, u_i represents the utility of the weight of notebook i. A further alternative is that the dimensions enter with weight θ into the overall utility. Then, this model captures decision-making between options with multiple dimensions such that every dimension is a separate attention allocation problem.

Appendix

A Expected utilities of attention strategies

The expected utility of an attention strategy corresponds to the expected utility of the chosen option and the costs for processing incurred with that strategy. The utility is $U = \theta u_i$. The costs for processing are always strictly positive, $c_\theta > 0$ and $c_{u_a} > 0$, and additive separable to that utility. A decision-maker's default option is denoted by $d \in \{1, 2\}$ and the alternative by $a \in \{1, 2\}$ with $a \neq d$. Consider a decision-maker whose default is option $i \in \{1, 2\}$.

Attention strategy $\{\}$: The decision-maker processes neither θ nor u_a. Thus she chooses according to expectations: If $u_d \geq E[u_a]$, the decision-maker chooses the default. If $u_d < E[u_a]$, the decision-maker chooses the alternative. The expected utility of attention strategy $\{\}$ is then:

$$E[U|\{\}] = E[\theta] \begin{cases} u_d & \text{if } u_d \geq E[u_a] \\ E[u_a] & \text{if } u_d < E[u_a]. \end{cases}$$

Attention Strategy $\{\theta\}$: The decision-maker always processes θ, but never processes u_a. Thus the decision-maker does not know which option is better and chooses according to expectations. Nevertheless, she still has costs for processing θ. The expected utility is:

$$E[U|\{\theta\}] = -c_\theta + E[\theta] \begin{cases} u_d & \text{if } u_d \geq E[u_a] \\ E[u_a] & \text{if } u_d < E[u_a]. \end{cases}$$

Attention strategy $\{u_a\}$: The decision-maker processes only u_a. Thus the decision-maker has costs for processing u_a. As the decision-maker always knows the value of u_a, she knows whether the default yields a higher or lower utility than the alternative. If $u_d \geq u_a$, which the decision-maker expects to occur with probability $Prob(u_d \geq u_a) = G(u_d)$, she chooses the default. If $u_d < u_a$, which the decision-maker expects to occur with probability $1 - G(u_d)$, she chooses the alternative. The expected utility associated with strategy $\{u_a\}$ is thus:

$$\begin{aligned} E[U|\{u_a\}] &= -c_{u_a} + (G(u_d)E[\theta]u_d + \int_{u_d}^{\infty} E[\theta]u_a g(u_a) du_a \\ &= -c_{u_a} + E[\theta] \left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a) du_a \right). \end{aligned}$$

Attention Strategy $\{\theta, u_a\}$: The decision-maker processes θ and u_a. Thus she always choose the utility-maximizing option. If $u_d \geq u_a$, she chooses the default and if $u_d < u_a$, she chooses the alternative. Nevertheless, the decision-maker also always has costs for processing θ and u_a. Thus the expected utility of strategy $\{\theta, u_a\}$ is:

$$E[U|\{\theta, u_a\}] = -c_{u_a} - c_\theta + E[\theta]\left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a) du_a\right).$$

Attention Strategy $(u_a, \{\theta | u_a \in A_{u_a}\})$**:** The decision-maker processes θ only if $u_a \in A_{u_a}$, but always processes u_a and thus always knows which option is better. If $u_d \geq u_a$, she chooses the default. If $u_d < u_a$, she chooses the alternative. As the decision-maker processes θ only if $u_a \in A_{u_a}$, she incurs costs for processing θ when $u_a \in A_{u_a}$; denote this probability with $Prob(u_a \in A_{u_a}) \in [0,1]$. In addition, the decision-maker always has costs for processing u_a:

$$\begin{aligned} E[U|(u_a, \{\theta | u_a \in A_{u_a}\})] = &- c_{u_a} - Prob(u_a \in A_{u_a}) c_\theta \\ &+ E[\theta]\left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a) du_a\right). \end{aligned}$$

Attention Strategy $(\theta, \{u_a | \theta \in A_\theta\})$**:** The decision-maker always processes θ, but processes u_a only if $\theta \in A_\theta$. Thus for $\theta \in A_\theta$: If $u_d \geq u_a$, she chooses the default and if $u_d < u_a$, she chooses the alternative. If $\theta \notin A_\theta$, the decision-maker does not know u_a and chooses given her expectations. The probability that $\theta \in A_\theta$ is $Prob(\theta \in A_\theta)$. Thus the expected utility of strategy $(\theta, \{u_a | \theta \in A_\theta\})$ is

$$\begin{aligned} E[U|(\theta, \{u_a | \theta \in A_\theta\})] = &- c_\theta - Prob(\theta \in A_\theta) c_{u_a} \\ &+ Prob(\theta \notin A_\theta) E[\theta | \theta \notin A_\theta] \begin{cases} u_d & \text{if } u_d \geq E[u_a] \\ E[u_a] & \text{if } u_d < E[u_a] \end{cases} \\ &+ Prob(\theta \in A_\theta) E[\theta | \theta \in A_\theta]\left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a) du_a\right). \end{aligned}$$

B Proofs

Proof of Lemma 1

As the decision-maker is an expected utility maximizer, she chooses the attention strategy with which she receives the highest expected utility (the expected utilities are summarized in Appendix A). Consider a decision-maker with default $i \in \{1, 2\}$.

(i) $\{\theta, u_a\}$ and $(u_a, \{\theta | u_a \in A_{u_a}\})$ give a lower expected utility than $\{u_a\}$: With all three attention strategies the decision-maker processes u_a and thus knows which option is better. Thus with all three strategies the decision-maker always chooses the option with the higher utility. However, with $\{\theta, u_a\}$ and $(u_a, \{\theta | u_a \in A_{u_a}\})$ the decision-maker expects additional costs for processing θ. Consequently, the decision-maker never chooses $\{\theta, u_a\}$ or $(u_a, \{\theta | u_a \in A_{u_a}\})$ as these attention strategies are strictly dominated by $\{u_a\}$.

(ii) $\{\theta\}$ gives a lower utility than $\{\}$: With both attention strategies the decision-maker never processes u_a and thus chooses according to expectations. However, with $\{\theta\}$ the decision-maker has additional costs for processing θ. Consequently, the decision-maker never chooses $\{\theta\}$ as this attention strategy is strictly dominated by $\{\}$.

Proof of Lemma 2

$(\theta, \{u_a | \theta \in A_\theta\})$ is a sequential strategy. I.e., at the time the decision is made whether to process u_a, θ is already known and c_θ are thus sunk costs. If the decision-maker does not process u_a, she chooses according to expectations and her expected utility is

$$\begin{cases} \theta u_d & \text{if } u_d \geq E[u_a] \\ \theta E[u_a] & \text{if } u_d < E[u_a]. \end{cases}$$

If the decision-maker does process u_a, she chooses the default if $u_d \geq u_a$ and she chooses the alternative if $u_d < u_a$. As the decision-maker does not know ex-ante which utility is higher, she uses her beliefs about the distribution of u_a. Thus her expected utility is

$$\theta \left(G(u_d) u_d + \int_{u_d}^{\infty} u_a g(u_a) du_a \right) - c_{u_a}.$$

Then, the decision-maker chooses to process u_a for a given θ if

- for $u_d \geq E[u_a]$:

$$\theta \left(G(u_d) u_d + \int_{u_d}^{\infty} u_a g(u_a) du_a \right) - c_{u_a} \geq \theta u_d \Leftrightarrow \theta \geq \frac{c_{u_a}}{\int_{u_d}^{\infty} u_a g(u_a) du_a - (1 - G(u_d)) u_d}.$$

- for $u_d < E[u_a]$:

$$\theta\left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a)du_a\right) - c_{u_a} \geq \theta E[u_a] \Leftrightarrow \ \theta \geq \frac{c_{u_a}}{G(u_d)u_d - \int_0^{u_d} u_a g(u_a)du_a}.$$

Thus a decision-maker who has observed a given θ chooses to process u_a if

$$\theta \geq \theta^* \equiv \begin{cases} \frac{c_{u_a}}{\int_{u_d}^{\infty} u_a g(u_a)du_a - (1-G(u_d))u_d} & \text{if } u_d \geq E[u_a] \\ \frac{c_{u_a}}{G(u_d)u_d - \int_0^{u_d} u_a g(u_a)du_a} & \text{if } u_d < E[u_a]. \end{cases}$$

Proof of Proposition 9

A decision-maker chooses the attention strategy that maximizes her expected utility. Consider a decision-maker with default $i \in \{1, 2\}$.

(i) A decision-maker chooses strategy $\{\}$ if and only if $E[U|\{\}] \geq E[U|(\theta, \{u_a|\theta \geq \theta^*\})]$ and
$E[U|\{\}] \geq E[U|\{u_a\}]$.

(a) Assume $u_d \geq E[u_a]$. Then,

$$\begin{aligned} E[U|\{\}] \geq E[U|\{u_a\}] &\Leftrightarrow E[\theta]u_d \geq -c_{u_a} + E[\theta]\left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a)du_a\right) \\ &\Leftrightarrow c_{u_a} \geq E[\theta]\left(\int_{u_d}^{\infty} u_a g(u_a)du_a - (1 - G(u_d))\, u_d\right) \end{aligned}$$

$$\begin{aligned} & E[U|\{\}] \geq E[U|(\theta, \{u_a|\theta \geq \theta^*\})] \\ \Leftrightarrow\ & E[\theta]u_d \geq -c_\theta - (1 - F(\theta^*))\, c_{u_a} + F(\theta^*)E[\theta|\theta < \theta^*]u_d \\ & \qquad + (1 - F(\theta^*))E[\theta|\theta \geq \theta^*]\left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a)du_a\right) \\ \Leftrightarrow\ & c_\theta \geq (1 - F(\theta^*))\left(E[\theta|\theta \geq \theta^*]\left(\int_{u_d}^{\infty} u_a g(u_a)du_a - (1 - G(u_d))u_d\right) - c_{u_a}\right) \end{aligned}$$

(b) Assume $u_d < E[u_a]$. Then,

$$\begin{aligned} E[U|\{\}] \geq E[U|\{u_a\}] &\Leftrightarrow \ E[\theta]E[u_a] \geq -c_{u_a} + E[\theta]\left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a)du_a\right) \\ &\Leftrightarrow \ c_{u_a} \geq E[\theta]\left(G(u_d)u_d - \int_0^{u_d} u_a g(u_a)du_a\right) \end{aligned}$$

$$
\begin{aligned}
&E[U|\{\}] \geq E[U|(\theta, \{u_a|\theta \geq \theta^*\})] \\
\Leftrightarrow\ &E[\theta]E[u_a] \geq -c_\theta - (1 - F(\theta^*))\, c_{u_a} + F(\theta^*)E[\theta|\theta < \theta^*]E[u_a] \\
&\qquad\qquad\qquad + (1 - F(\theta^*))E[\theta|\theta \geq \theta^*] \left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a) du_a \right) \\
\Leftrightarrow\ &c_\theta \geq (1 - F(\theta^*)) \left(E[\theta|\theta \geq \theta^*] \left(G(u_d)u_d - \int_0^{u_d} u_a g(u_a) du_a \right) - c_{u_a} \right)
\end{aligned}
$$

Thus a decision-maker chooses $\{\}$ if and only if $c_{u_a} \geq \bar{c}_{u_a}$ and $c_\theta \geq \bar{c}_\theta(c_{u_a})$ with

$$
\bar{c}_{u_a} \equiv \begin{cases} E[\theta] \left(\int_{u_d}^{\infty} u_a g(u_a) du_a - (1 - G(u_d))u_d \right) & \text{if } u_d \geq E[u_a] \\ E[\theta] \left(G(u_d)u_d - \int_0^{u_d} u_a g(u_a) du_a \right) & \text{if } u_d < E[u_a] \end{cases}
$$

and

$$
\bar{c}_\theta(c_{u_a}) \equiv \begin{cases} (1 - F(\theta^*)) \left(E[\theta|\theta \geq \theta^*] \left(\int_{u_d}^{\infty} u_a g(u_a) du_a - (1 - G(u_d))u_d \right) - c_{u_a} \right) & \text{if } u_d \geq E[u_a] \\ (1 - F(\theta^*)) \left(E[\theta|\theta \geq \theta^*] \left(G(u_d)u_d - \int_0^{u_d} u_a g(u_a) du_a \right) - c_{u_a} \right) & \text{if } u_d < E[u_a]. \end{cases}
$$

(ii) A decision-maker chooses strategy $\{u_a\}$ if and only if $E[U|\{u_a\}] \geq E[U|(\theta, \{u_a|\theta \geq \theta^*\})]$ and $E[U|\{u_a\}] \geq E[U|\{\}]$.

(a) Assume $u_d \geq E[u_a]$. Then,

$$
\begin{aligned}
&E[U|\{u_a\}] \geq E[U|(\theta, \{u_a|\theta \geq \theta^*\})] \\
\Leftrightarrow\ &-c_{u_a} + E[\theta] \left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a) du_a \right) \\
&\geq -c_\theta - (1 - F(\theta^*))\, c_{u_a} + F(\theta^*)E[\theta|\theta < \theta^*]u_d \\
&\qquad\qquad + (1 - F(\theta^*))E[\theta|\theta \geq \theta^*] \left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a) du_a \right) \\
\Leftrightarrow\ &c_\theta \geq F(\theta^*) \left(E[\theta|\theta < \theta^*] \left((1 - G(u_d))u_d - \int_{u_d}^{\infty} u_a g(u_a) du_a \right) + c_{u_a} \right)
\end{aligned}
$$

$$
E[U|\{u_a\}] \geq E[U|\{\}] \ \Leftrightarrow\ c_{u_a} \leq E[\theta] \left(\int_{u_d}^{\infty} u_a g(u_a) du_a - (1 - G(u_d))u_d \right).
$$

(b) Assume $u_d < E[u_a]$. Then,

$$\begin{aligned}
&E[U|\{u_a\}] \geq E[U|(\theta, \{u_a|\theta \in [\theta^*, \bar{\theta}]\})] \\
\Leftrightarrow\ & -c_{u_a} + E[\theta]\left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a)du_a\right) \geq \\
& -c_\theta - (1-F(\theta^*))\,c_{u_a} + F(\theta^*)E[\theta|\theta<\theta^*]E[u_a] \\
&\qquad\qquad + (1-F(\theta^*))E[\theta|\theta \geq \theta^*]\left(G(u_d)u_d + \int_{u_d}^{\infty} u_a g(u_a)du_a\right) \\
\Leftrightarrow\ & c_\theta \geq F(\theta^*)\left(E[\theta|\theta<\theta^*]\left(\int_0^{u_d} u_a g(u_a)du_a - G(u_d)u_d\right) + c_{u_a}\right)
\end{aligned}$$

$$E[U|\{u_a\}] \geq E[U|\{\}] \ \Leftrightarrow\ c_{u_a} \leq E[\theta]\left(G(u_d)u_d - \int_0^{u_d} u_a g(u_a)du_a\right)$$

Thus a decision-maker chooses $\{u_a\}$ if and only if $c_{u_a} \leq \bar{c}_{u_a}$ and $c_\theta \geq \underline{c}_\theta(c_{u_a})$ with

$$\underline{c}_\theta(c_{u_a}) \equiv \begin{cases} F(\theta^*)\left(E[\theta|\theta<\theta^*]\left((1-G(u_d))u_d - \int_{u_d}^{\infty} u_a g(u_a)du_a\right) + c_{u_a}\right) & \text{if } u_d \geq E[u_a] \\ F(\theta^*)\left(E[\theta|\theta<\theta^*]\left(\int_0^{u_d} u_a g(u_a)du_a - G(u_d)u_d\right) + c_{u_a}\right) & \text{if } u_d < E[u_a]. \end{cases}$$

(iii) A decision-maker chooses $(\theta, \{u_a|\theta \geq \theta^*\})$ if and only if

$$\begin{aligned}
&E[U|(\theta, \{u_a|\theta \geq \theta^*\})] > E[U|\{\}] \\
\text{and } &E[U|(\theta, \{u_a|\theta \geq \theta^*\})] > E[U|\{u_a\}].
\end{aligned}$$

Thus a decision-maker chooses $(\theta, \{u_a|\theta \geq \theta^*\})$ if and only if

$$c_{u_a} \leq \bar{c}_{u_a} \text{ and } c_\theta \leq \underline{c}_\theta(c_{u_a}) \qquad \text{or} \qquad c_{u_a} \geq \bar{c}_{u_a} \text{ and } c_\theta \leq \bar{c}_\theta(c_{u_a}).$$

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