Models of direct-interactions-driven collective economic phenomena

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The papers marked by * are not included in the electronic document.

Introduction and Overview

The present thesis consists of seven papers specifying and analysing models of *direct-interactions*-driven *collective* economic phenomena. Within theories and models of coordination in large decentralised market economies (and analogously also within theories and models of coordination in natural systems, for instance models of magnetism) the notion of *direct interactions* is to be contrasted with the notion of *indirect interaction via aggregate signals*, such as the price system. The latter notion has been at the core of the Walrasian concept of decentralised economic coordination (see Walras (1898)) which developed into the Arrow-Debreu formulation of General Equilibrium Theory (see Debreu (1959) and Arrow and Hahn (1969)). Unlike indirect interactions – for which realistic micro-foundations (for instance, how a macroscopic signal emerges in a decentralised way) are absent in the Arrow-Debreu model and related models – direct interactions between individuals are a straightforwardly realistic notion: they correspond to our daily experience of social life, affecting attitudes, opinions, beliefs and decisions in all areas of life. Yet despite its intuitive appeal, the consideration of direct interactions is a relatively new development in large-system coordination models in economics, with its roots going back to Föllmer (1974).[1] Blume and Durlauf define the (direct) interactions-based approach as one "focusing on direct interdependencies between economic actors rather than those indirect interdependencies that arise through the joint participation of economic agents in a set of markets" (Blume and Durlauf, 2001, p.16).[2]

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[1]The literature on direct-interactions based phenomena includes Bisin et al. (2006), Blume (1993, 1995), Cont and Bouchaud (2000), Horst and Scheinkman (2006), Kim and Markowitz (1989), Kirman (1991, 1993), Levy et al. (1994) and Lux (1995). For a recent overview of interactions-based models of financial markets, see Samanidou et al. (2007). Interactions-based models are also often *called agent-based models*.

[2]It should be noted that strategic interactions which are modelled in Game Theory in many cases fit into the category of direct interactions as defined by Blume and Durlauf (2001).

One can distinguish two polar types of direct-interactions models: those with a fixed topology of interactions and those with random matching. In models with a fixed topology, a graph represents the structure of inter-agent influence or communication channels. Any given agent always interacts with the same subset of other agents (called his local neighbourhood) - consisting of those agents to whom a direct link exists.[3] As a result, for any particular specification of variables associated with the agents the macroscopic properties of the system (obtained from averaging over large subsets of these variables) might look quite different in different "areas" of the graph, at least temporarily. In contrast, in random matching models for each instance of an interaction the two agents interacting are selected randomly from the entire population of agents. Thereby the probability to be matched with a certain type of agent corresponds to the empirical distribution of types in the entire population. As a result, no "spatial" patterns emerge in such models, and an approximation by indirect-interaction models becomes correct. Clearly, real-world matching schemes feature aspects of both of the above polar cases: interaction partners are neither fixed, nor is it the case that they are completely random, as past partners and future partners are in many cases not independent.[4] In the present thesis, Papers 4 and 5 are random matching models, whereas all other papers involve a fixed topology.

An important conceptual issue by which many direct-interaction large-economy models differ from the neo-classical modelling tradition is the following: Rather than modelling individual decisions according to the economic concepts of rationality – such that these decisions obtain endogenously in the model – large-economy models with direct interactions are in many cases phenomenological on the individual level. That is to say, some basic tendency driving the interactions between agents, for instance the tendency to conform with others, is not derived from a rationality concept *within the model* (in other words, it is exogenous to the model). Instead, a behavioural pattern is introduced based on empirical knowledge by specifying appropriate stochastic transition kernels which produce the observed

[3]More generally, interactions of decreasing strength can be defined for those agents who are "farther away" (i.e. who can be reached via two links, three links etc.

[4]For a mixture of both mechanisms, see Stauffer et al. (2006).

statistical regularity in large samples of individual agents.[5] But it is important to keep in mind that the stochastic specification does not represent intrinsic randomness in individual behaviour but rather outside effects not considered in the model.

Let us now turn to the second main notion emphasised in the title of the thesis, namely the notion of *collective phenomena*. Loosely speaking, a collective phenomenon is one in which the macroscopic properties of the system in no way mirror the individual behaviour of the system. Rather, they emerge from the interactions between the constituents. In contrast, in much of the neo-classical modelling of large economies, a single "representative" agent allegedly representing an entire sector of the economy is assumed (for a critique of this concept, see Kirman (1992)). Thereby, the distinction between the micro-variables and macro-variables of a system is effectively destroyed.

A reason for the prevalence of the representative-agent approach in neo-classical economics is the complexity of the endogenous, rationality-based modelling of individual decisions, which would lead in many cases to insurmountable computational difficulties in models with many heterogeneous agents. Recently, however, models with many heterogeneous agents which are "simple" at the individual level and which aim at collective phenomena have become increasingly popular among economists (see, for instance, Galegatti and Kirman (1999)). This due to two factors: first, the emerging strong evidence of the crucial role of heterogeneity in modelling economic systems coming from empirical research, and, second, results of experimental research in economics pointing to the conclusion that the rationality-based approach to individual decision making is insufficient or positively wrong (see, for instance, Selten (2001)).

Let us now turn to the methods of analysis of stochastic models of multi-component (multi-agent) systems with direct interactions. In a few exceptional cases, analytical techniques developed in mathematical physics are available, which are applicable also to models of very large systems (these are models with an infinite number of agents). Still, analytical techniques typically yield results of a

[5]In papers 1-6 the specification of interactions are such that agents tend to conform to the expectations of others. There exists ample evidence that agents do so (see, for instance, Asch (1951, 1956) and Festinger (1956)). The neo-classical tradition would require that this type of behaviour is derived from some rationality principles.

principle nature (existence of an equilibrium measure, conditions for its uniqueness vs. non-uniqueness etc.). As an alternative, Monte-Carlo techniques have become widespread in the investigation of stochastic interactions-driven models. Monte Carlo simulations readily provide a clear picture of the typical behaviour of models with a finite set of agents.[6] In the present thesis, all papers except Paper 6 feature Monte-Carlo results of models with a finite set of agents.

In the remainder of this introductory chapter, an overview of the papers constituting the present thesis is given. Except Paper 7, in all papers of the present thesis direct interactions affect individual expectations of agents. Traditionally, expectations are modelled as autonomous, either rational – implying that they are defined as being correct rather than emerging from certain cognitive procedures – or derived by an individual agent by sampling the path of macroscopic variables and forming adaptive, extrapolative or other types of expectations. However, there is well-established empirical evidence from social psychology that expectations – just as any other type of individual judgements – are, at least to some degree, affected by interactions among human beings. The papers 1-6 of this thesis demonstrate how interdependent expectations can shed new light on various economic phenomena.

In particular, **Part 1** of the thesis, consisting of Papers 1 and 2, features two models of an interactive dynamics of economic sentiment in which no link between economic sentiment and real economic variables is established. Such an approach is admissible if one can neglect feedbacks from the real economic variables to economic sentiment, such that the dynamics of economic sentiment can be modelled separately from that of real economic variables. In these two models, economic sentiment of a certain individual is affected by the consumer sentiment of others and possibly by exogenous macroscopic shocks of an economic, social or political nature. Each paper of **Part 2** of the thesis – consisting of Papers 3, 4, 5 and 6 – does establish a link between the interactive sentiment dynamics and real economic variables. This link can be provided in various economic frameworks determining real economic variables. In particular, Papers 4 and 5 are in the context of the

[6] Moreover, concepts have been developed how to generalise results by induction from finite models to infinite models (this approach is called Thermodynamic Limit in Physics). However, in economic applications the extension of results to infinite systems is often not desired, as the number of agents is only modestly large.

standard Keynesian multiplier macro-model of consumption, output and saving, while Paper 6 is in the context of General Equilibrium Theory. Paper 3 features a model of precautionary saving which can be integrated within Keynesian, General Equilibrium and other models. By considering feedbacks from real economic variables to economic sentiment, these papers integrate the notion of interactively-driven consumer sentiment within a complete economic model framework, being a quite new theoretical development. Finally, **Part 3** containing Paper 7, addresses the issue of new-product diffusion. In that context, direct interactions trigger the formation of individual valuations for a new product. This part demonstrates the flexibility of the stochastic interactions-based approach by showing that phenomenologically quite different phenomena can by explained in an essentially same conceptual approach to modelling interactions.

In the following, a brief summary of each of the seven papers is given. Paper 1, titled **Socioeconomic interaction and swings in business confidence indicators** (being a joint paper with S. Pittnauer, S. Solomon and D. Stauffer)[7] is motivated by certain patterns in the German Ifo-Index data which were first investigated from the perspective of collective phenomena by Flieth and Foster (2002). To account for such patterns in our paper, collective economic sentiment formation is modelled as emerging from direct interaction between managers within an industry. The neighbourhood structure of managers is assumed to be fixed (each agent interacts with the same four neighbours over the entire time horizon). Following the structure of the Ifo-survey, each manager can have three opinions: optimistic, pessimistic or neutral. Each manager tends to conform with the opinions prevalent in his/her reference group, with the strength of the tendency being different for different types of opinion. The main result of the paper is the appearance of large abrupt swings in the mean consumer sentiment. This phenomenon is well-known in the study of collective phenomena in physics. It corresponds to the appearance of spontaneous changes of the phase of the Ising model or the Blume-Capel model of ferro-magnetism.

Paper 2, titled **The impact of external events on the emergence of social herding of consumer sentiment**, (which is a joint paper with S. Pittnauer and D. Stauffer) incorporates the impact of external events on the dynamics of

[7]The paper has been published in Physica A (2005) 646-656.

economic sentiment in a model very similar to that featured in Paper 1. A direct-interactions-driven dynamics of economic sentiment is represented by an Ising model on a large (two-dimensional) square lattice. The individual states of agents are called *optimism* and *pessimism*. The exogenous environment is modelled as a sequence of random events, which might have a positive or negative influence on economic sentiment. These exogenous events can be frequent or rare, have a lasting impact or a non-lasting impact. Impact of events is inhomogeneous over the lattice, as individuals might fail to perceive particular events. Events have a probabilistic relation with the state of each individual agent – technically specified in the same manner as the impact of states of neighbouring agents. The aim of the paper is to investigate the effect of an exogenous environment on the emergence of social herding of economic sentiment. We introduce two notions of social herding: *permanent herding* refers to the situation where an ordered state (i.e. a state with an overwhelming majority of optimists or pessimists) persists over an infinite time horizon, while *temporary herding* refers to the situation where ordered states appear, persist for some time and decay. The parameter of the inter-agent interaction strength is such as to engender permanent herding without the influence of the environment. We consider two cases: in the first case positive and negative events have both the same empirical frequencies and strengths (i.e. the environment is "neutral" in the long term), while in the second case events have the same empirical frequencies but different strengths. In the neutral case we find temporary herding if events are sufficiently "strong" and/or perceived by a sufficiently large proportion of agents, and our results suggest that permanent herding occurs for small values of the parameters. In the "non-neutral" case we find only temporary herding.

Paper 3, titled **Statistical economics on multi-variable layered networks** (which is a joint paper with T. Erez and S. Solomon)[8] extends the framework of the above two papers by incorporating individual saving rate as a second type of individual variable. In our model, the individual sentiment variables can assume two values (optimism, pessimism), while the saving variables can assume any of the Q budget shares $\frac{1}{Q}, \frac{2}{Q}, \ldots, \frac{Q-1}{Q}, 1$. The topology of interactions is that

[8]This paper has been published in the edited volume *Economics: Complex Windows*, ed. M. Salzano and A. Kirman, Springer, 2005.

of a (large but finite) one-dimensional lattice with respect to each of the variable types. Using the statistical approach, we introduce the regularity that an agent who is optimistic about the prospects of the economy is likely to save more than a pessimistic agent.

In addition to the economic issues just discussed, the paper makes the following contribution to the general issue on network architectures: Since the local neighbourhood of each agent related to each of his/her individual variables need not be identical (agents who influence an agent's economic sentiment might not be the same as those who influence his saving), agents are effectively linked in a more complex way than what would be the case without the linkage between individual saving and economic sentiment. A network structure representing that situation (which arises quite naturally whenever an agent is characterised by more than one interdependent variables) is sometimes called a Solomon-Network. Along with Malarz (2003), the paper is one of the first to define and analyse stochastic models on such network structures (see also Billari et al. (2006) for additional motivation to consider such network structures in modelling socio-economic systems).

Paper 4, titled **A note on interactions-driven business cycles** (which is a joint paper with F. Westerhoff)[9] explores the impact of an interactive dynamics of economic sentiment on real economic variables in the standard Keynesian multiplier framework. Unlike in the above Papers 1-3, the interactions are direct but not local (random matching). That is to say, each consumer meets another consumer at random, independently of whom he/she met before or what his/her individual variable values are. The underlying micro-model of direct interactions is a version of Kirman's (1993) generic opinion formation model, with an additional feedback effect from macroscopic variables on the transition probabilities. As in the above model, pessimism (optimism) is assumed to engender higher (lower) saving. Our model engenders cyclical fluctuations of economic variables, despite the fact that neither the Keynesian multiplier model nor Kirman's model does so on its own.

Paper 5, titled **Business cycle synchronisation in a simple Keynesian model with socially-transmitted economic sentiment and international sentiment spill-overs** (which is a joint paper with F. Westerhoff) is based on

[9]The paper appears in Journal of Economic Interaction and Coordination 2(1), 2007.

the Keynesian multiplicator-accelerator model (see Samuelson 1939) with the novel feature that national income expectations affect investment and drive the Samuelsonian accelerator dynamics (see Westerhoff 2006). The role of the interactive sentiment dynamics is the same as in the previous paper, while the influence channel of economic sentiment is complementary: it is assumed that investors who hold optimistic views about the future state of the economy expect a higher aggregate demand in the following period and thus invest more than pessimistic ones. Simulations show that the model can generate a complex business cycle dynamics. Based on that framework, a three-country model of business cycle synchronisation is provided, in which international spill-over effects on the level of economic sentiment synchronise the national cycles, provided that investors believe that the economies are indeed coupled.

Paper 6, titled **Local-interactions-based coordination of equilibrium expectations and the emergence of sunspot equilibria**[10] differs from all other papers of the thesis in several respects: it features a model with an infinite number of agents (representing the dynamics of a very large system), it investigates the model analytically (using well-known results in Probability Theory) and it is set in the context of General Equilibrium Theory.[11] As all of the previous papers, it introduces a stochastic direct-interactions-based dynamics of agents' expectations. Unlike in the previous papers, the expectations refer to different Walrasian equilibria which can prevail in the model economy in a future point in time. The paper demonstrates that the stochastic direct-interactions-based dynamics can explain how in General Equilibrium Theory one equilibrium (out of multiple) can be decentrally agreed upon. Moreover the model naturally engenders coordination into sunspot equilibria (with the sunspot events endogenously arising in the coordination process).[12]

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[10]A previous version is available online as *Bonn Econ Discussion Paper* 2005/23

[11]For those who are sceptical about the positive content of the assumptions underlying General Equilibrium Theory, such as market clearing, one could perceive the paper as applying a direct-interactions mechanism to General Equilibrium Theory to alleviate some inherent problems of it rather than to positive modelling.

[12]A sunspot-equilibrium in General Equilibrium models is one in which a random event with no a-priori effect on the economy (called *sunspot*) nevertheless affects the economy because agents believe that it does. Then due to the so arising macroscopic uncertainty the structure

The particular specification of transition rates chosen in the present paper is known as the (two-dimensional) Voter Model. The composite process has two extremal invariant measures and a continuum of non-extremal invariant measures, to each of which it can converge. Convergence to either one of the extremal invariant measures corresponds to a deterministic coordination outcome, selecting a non-sunspot equilibrium of the underlying economy. Convergence to a convex mixture of invariant measures corresponds to a fully coordinated but random coordination outcome of individual expectations. The latter has precisely the structure of equilibrium expectations in a sunspot equilibrium, and it is the uncertainty of the coordination outcome which determines the set of sunspot equilibria of the underlying economy. In effect, it is the non-ergodicity of the coordination process which facilitates the occurrence of sunspot equilibria in the proposed model.

In the final Paper 7, titled **Explaining delayed take-off in new-product diffusion** (which is a joint paper with S. Pittnauer and D. Stauffer)[13] direct interactions affect a quite different type of variable as compared to the previous six papers. Instead of economic expectations, it is now the formation of the individual valuation of a new product which is triggered by local direct interactions between consumers: it is only upon the observation of a neighbour's usage of a product that a consumer forms the valuation of the product. In that model – despite the formation of the valuation being triggered by the neighbours – the valuations themselves are stochastically independent among agents. A consumer buys the new product if her valuation of it is not below the price of the product in a given period. From a technical perspective, the dynamics of the model amounts to a stochastic process which is well-known in physics as site-percolation dynamics.[14] It is known that there exists a real number called the percolation threshold P_c which constitutes a value for the "transmission probability" above which "active" sites spread over the entire graph with a significant probability, but "die out" below of it, unless for extremely rare instances. The model presented in Paper 7 attributes the empirical finding of a delayed "take-off" of a new product to a

of the economy might change in such a way that the belief in the effect of the sunspot becomes justified.

[13]The paper was accepted for publication in *Industrial and Corporate Change* subject to minor extension and some data testing.

[14]See Stauffer and Aharony (1994)

drift of the percolation dynamics from a non-percolating regime to a percolating regime. This drift is caused by learning effects lowering the price of the product, or by network effects increasing its valuation by consumers, with an increasing number of buyers.

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The impact of external events on the emergence of social herding of economic sentiment

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Abstract

We investigate the impact of an exogenous environment on the emergence of social herding of economic sentiment. An interactions-driven dynamics of economic sentiment is modeled by an Ising model on a large (two-dimensional) square lattice. The individual states are called *optimism* and *pessimism*. The exogenous environment is modeled as a sequence of random events, which might have a positive or negative influence on economic sentiment. These exogenous events can be frequent or rare, have a lasting impact or a non-lasting impact. Impact of events is inhomogeneous over the lattice, as individuals might fail to perceive particular events. We introduce two notions of social herding: *permanent herding* refers to the situation where an ordered state (i.e. a state with an overwhelming majority of optimists or pessimists) persists over an infinite time horizon, while *temporary herding* refers to the situation where ordered states appear, persist for some time and decay. The parameter of the inter-agent interaction strength is such as to engender permanent herding without the influence of the environment. To investigate the impact of an environment we determine whether an initially ordered state decays. We consider two cases: in the first case positive and negative events have both the same empirical frequencies and strengths, while in second case events have the same empirical frequencies but different strengths. (In the first case the environment is "neutral"in the long term), In the neutral case we find temporary herding if events are sufficiently "strong" and/or perceived by a sufficiently

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large proportion of agents, and our results suggest that permanent herding occurs for small values of the parameters. In the "non-neutral" we find only temporary herding.

Keywords: herding, economic sentiment, consumer confidence, endogenous vs. exogenous dynamics, local interactions, social interactions

1 Introduction

Recently, there has been renewed strong interest among scholars of economics in the notion of *consumer sentiment* – a vague concept operationalized in particular surveys as a bundle of consumer expectations and assessments of the economic prospects – on individual and aggregate economic activity [11, 27, 35]. While consumer sentiment has been considered a relevant indicator by practitioners of economic policy [20], economic modeling is primarily concerned with other, more specific types of expectations (such as income and price expectations for a particular point in time), and there is much less agreement among theorists in what way – if any – the concept of consumer sentiment – or, more generally, the concept of economic sentiment – should enter economic modeling.

The basic assumption of the present paper is that economic sentiment is prone to imitative social influence, in that, say, a consumer is more likely to hold an optimistic (pessimistic) expectation about the economic prospects if his peers do. That assumption is well substantiated: according to experimental social psychology, an individual is the more likely to conform to the judgment of others the less he is able to form his own judgment in a rational and informed manner [1, 2, 13]. Since experimental studies in the human perception of complex dynamic systems [16, 36] suggest that a typical consumer has only a limited perception about the functioning of the economy and the political system in which it is embedded (arguably, this limitation applies to a considerable extent even to specialists), one is indeed led to the conclusion that the formation of economic sentiment is prone to social imitation. Social imitation of consumer sentiment might or might not result in herding of economic sentiment at the macroscopic level. The conditions for the emergence of sentiment herding at the macroscopic level are investigated for a particular model in the present paper.

The general phenomenon of social herding has attracted much interest in economic theory over the last two decades, particularly in the wake of disturbances on financial markets (see [3, 5, 25, 26, 29, 6] for some seminal contributions). A principle question in models of social herding is whether or not individual behavior should be derived from the principles of economic rationality. The first two of the above cited papers do so. Our paper does not, as it is based on the *statistical modeling approach*[1] which directly applies to sentiment formation the empirical evidence of *social comparison* processes (see [1, 2, 13]) rather than explaining economic sentiment formation from rationality principles. We believe that the statistical approach is particularly appropriate for modeling the dynamics of economic sentiment because economic sentiment, if considered as a particular instance, or at least part, of a consumer's *mental model* [8, 33], is a premise of individual reasoning and decision-making rather than the subject of it. Most directly, our present paper belongs to the recent literature on socially-driven economic sentiment formation [14, 21, 12, 37, 38].

The aim of this paper is to investigate the impact of an exogenous environment on the emergence of social herding of consumer sentiment. Indeed, economic traders react as much to the news coming from the broader geo-political environment – whether or not these news items are objectively interpreted – as to the behavior/advice of others. Consumer sentiment subject to social imitation is modeled in our paper as a large Ising field with nearest-neighbor interactions on a (two-dimensional) square lattice. The individual states are called *optimism* and *pessimism*. Social herding corresponds to the emergence of coordination states of the Ising model, i.e. states with predominantly optimistic or predominantly pessimistic individual entities. The environment is modeled as a a sequence of exogenous events (external influences), stochastically fluctuating over time. The exogenous events can be frequent or rare, have a lasting impact or a non-lasting impact. The field of events is not homogeneous, as individual actors might fail to perceive events. Though the environment does have an impact in existing models of financial herding, for instance in chartists-fundamentalists models [25, 29] as changes in the fundamental value, or as news affecting traders [30], our model – due to its simple abstract structure – is particularly suitable for analyzing the interplay of social (local) interactions and environmental (global) influences in a more abstract way suitable for computer simulations. [2]

Motivated by our results, we introduce two notions of social herding in our model: *permanent herding*, the stronger notion, refers to an ordered state (i.e. a state with an overwhelming majority of optimists or pessimists) which persists over an infinite time horizon, while

[1]See [15, 31, 19] for early formulations of the statistical modeling approach in economics and sociology.

[2]In socio-economic applications of the Ising model, an external environment has been previously considered by [19, 18, 22]. Relatedly, there has been much interest recently in the more general issue of disentangling endogenous and exogenous dynamics in complex systems [9, 34].

temporary herding refers to a state in which ordered states appear, persist for some time and decay. The parameter of the inter-agent interaction strength in the underlying Ising field is such as to engender a persistent ordered phase of the infinite model (permanent herding in our terminology) *without* the influence of the environment. To investigate the impact of an environment we determine whether an initially ordered state decays for the two cases that positive and negative events have both the same empirical frequencies and strengths (i.e. the environment is "neutral" in the long term) and that the latter property does not hold. In the neutral case we find temporary herding if events are sufficiently "strong" and/or perceived by a sufficiently large proportion of agents, and our results suggest that permanent herding occurs for small values of the parameters. In the "non-neutral" case we find only temporary herding.

In the present paper we concentrate on the interplay of endogenous and exogenous influences on economic sentiment, neglecting its link with real economic variables. We do so because that link has not been investigated at the behavioral level; see [12, 37, 38], however, for attempts to account for that link in similar or related models).

2 The model

We let the Ising model on a two-dimensional square lattice with nearest-neighbor interactions represent socially-driven collective dynamics of economic sentiment.[3] The variable x_i denotes the economic sentiment of agent i. Individual states $x_i = -1$ and $x_i = 1$ represent the individual states of pessimism and optimism respectively. It is well-known that for interactions between agents stronger than some critical value J_c (leaving aside any external influences) there exist on the infinite lattice two phases of the sentiment field ("coordination states"), with the economic actors in each of them being either predominantly pessimistic or predominantly optimistic [10]. These phases are stable states which emerge – in an appropriate formal sense – already in a large enough finite system [23] ("permanent herding"

[3]We must point out the limitations of our basic model. First, the topology of social interactions is hardly as simple as a square lattice, yet we are not aware of empirical investigations of network structures for our particular subject of social interactions, and network structures found for other contexts (see e.g. [4]) are not necessarily transferable to our context [32]. Second, interactions need not be symmetrical with respect to the individuals involved, as is the case in our model. Third, the individual states of economic sentiment should be more rich, possibly even continuous. However, we do believe that the Ising model provides a first approximation to the type of systems we aim to analyze.

in our terminology).

The events affecting consumer sentiment ("the environment") at a given point in time are modeled in the present paper as realizations of a random variable B with the possible values $B = b$ ("positive" event), or $B = -b$ ("negative" event), or $B = 0$ (no event) in the case of a neutral environment, and $B = -2b$ ceteris paribus in the case of a biased environment. We assume an agent perceives the event correctly with probability p, while ignoring the event with probability $1 - p$. Perception of an event is independent among agents. We introduce a variable ϵ_i such that $\epsilon_i = 1$ represents the situation that agent i perceives the event and $\epsilon_i = 0$ that he does not.

According to principles of statistical modeling, the following interaction potential[4] appropriately characterizes the interaction structure of our model in a finite square lattice Λ

$$H(x) = -\frac{J}{2} \sum_{i,j \in \Lambda : ||i-j||=1} x_i x_j - B \sum_{i \in \Lambda} \epsilon_i x_i, \tag{1}$$

with periodic boundary conditions specified in our simulations. The strictly positive parameter J characterizes the interaction-to-noise ratio. The first sum accounts for local interaction between individual agents, while the second accounts for the impact of the exogenous events.

In Monte-Carlo Statistical Physics equilibrium states are obtained from an appropriate algorithm (which can be interpreted as a stochastic dynamics of the system), whereby individual sites are sequentially updated according to the probabilities proportional to $\exp(-H)$, using the prevailing configuration of next-neighbors. In doing so, we use the following specifications of the process B_t representing the environment: external events can be frequent (time scales of the Ising field and the external field are comparable) (see Figure 1, top), lasting but rare, (e.g. the environment may change only once in T updates of all individual variables) (see Figure 1, top), and rare transitory (i.e. shocks; see Figure 1 bottom). Positive and negative events/shocks occur equally frequent (on average) in all cases.

3 Results

Figure 2 summarizes our simulation results on the persistence of an initial ordered state of the consumer sentiment field for the case of a large system and a neutral environment.

[4] In physics, Eq. 1 has the interpretation of energy, and the sum of individual energy contributions is called Hamiltonian. In social sciences, we do not have a quantity corresponding to energy, such that Eq. 1 is merely a representation of interactions between people and events.

The curves in Figure 2 separate areas of the parameter space – the parameters being the proportion p of agents perceiving the event[5] and event "strength" b – for which the initial ordered state persists over 4000 Monte Carlo time steps (these areas are below a curve), and in which the initial ordered states do not survive over 4000 MCTS (these areas are above a curve). For a fixed b, the proportion p was diminished until for half of the four simulated samples no change from the initial optimistic majority to a slight majority of pessimists was observed during 4000 iterations (sweeps through the lattice). This border point then was put into Fig.2.

As 4000 is a somewhat arbitrary time scale, we also investigated stability of an initial ordered state over very long times for frequent news and $J_c/J = 0.9$. Figure 3 depicts the dependence on p of the median time at which the ordered state is destroyed for a fixed exemplary value $b = 1$. It turns out that this time tends to infinity for $p \simeq 0.16$, close to the corresponding $p \simeq 0.20$ for 4000 iterations in Figure 2. This suggests that there is an area of the parameter space (presumably separated by a curve running slightly below the curve in Fig. 2) for which an ordered state is stable over infinite time horizons for a very large lattice. (Small lattices do not have shar transitions.) In this case, the environment does not have a destructive impact on collective consumer sentiment.

What are the properties of the model for parameters b and p above a curve in Fig. 2? The inspection of figures displaying time-paths of the proportion of optimists/pessimists for parameter values above a curve in Fig. 2 (available from the authors on request) shows that the typical time-path above a curve in Figure 2 is irregular: Periods of collective pessimism emerge, persist for some time and decay, as do periods of collective optimism. Periods of collective pessimism and optimism change much more slowly than the exogeneous environment, thus collective pessimism (optimism) often persist while and despite positive (negative) events occur. Periods of collective pessimism and optimism occur – as positive and negative events – equally frequently over time for the neutral environment.

These results lead us to introduce two notions of social herding in our model: *permanent herding*, the stronger notion, refers to an ordered state (i.e. a state with an overwhelming majority of optimists or pessimists) which persists over an infinite time horizon, while *temporary herding* refers to a state in which ordered states appear, persist for some time and decay. (Note that the parameter of the inter-agent interaction strength in the underlying Ising field is such as to engender a persistent ordered phase of the infinite model – perma-

[5] Clearly, for large enough systems this fraction equals the probability p of an agent perceiving the event.

nent herding in our terminology – *without* the influence of the environment.) Thus, for the neutral environment, our results strongly suggest that permanently stable ordered states of collective pessimism/optimism do not occur if events are too strong and/or are considered by a sufficiently large proportion of agents. This is due to a "competition" between the social mechanism tending to produce coordination, and the disorder of the external environment. Given that positive and negative events have equal empirical frequencies, such that the environment is "neutral" with respect to sentiment, it is quite intuitive that a widespread perception of external events destroys endogenous collective states: the disorder of the environment then prevails over the tendency to herding in economic sentiment.

We also considered an environment which is biased in favor of pessimism. We take the strength of negative event as $-2b$, i.e. twice as strong as the positive event b. Analogously to Fig. 2, the curves in Figure 4 separate areas of the parameter space in which the initial ordered state persists for a simulation length of 4000 Monte Carlo time steps (these areas are below a curve), and in which the initial ordered states does not survive over 4000 MCTS (these areas are above a curve). Fig.4 shows that the b or p values required for this transition are now drastically smaller than for the neutral environments in Fig.2. Analogously to Fig.3 we show in Fig.5 the variation of the median destruction time, but now for this biased case of Fig.4. Roughly the data follow a straight line on this log-log plot, suggesting a finite decay time for all finite p, going to infinity for $p \to 0$ only. We see upward deviations for small p but the larger the lattice is the smaller are these deviations. Thus in the biased case, in contrast to the unbiased one, the initial order is always destroyed if we only wait long enough.

4 Discussion

States of "collective pessimism" – if this social phenomenon indeed occurs – might be detrimental to the efficiency of allocation of economic resources. Indeed, "explanations" to that effect can often be heard in the public discussion about the state of the economy and economic policy. We believe that economic-sentiment-based arguments are relevant despite the lack of proper theoretical foundations, and the present paper is an exploratory step toward formulating relevant models. Our results confirm an intuitive presumption: attention to news reduces the prevalence of collective economic sentiment. This result appears to suggest that our model might be a useful starting point, though the present paper does not cover several important issues. In particular, the role of the graph structure of the underlying network should be investigated. Also, we have not specified the actual "transmission mechanism" of economic sentiment into economic variables necessary for a welfare analysis of the impact of collective economic sentiment.

A more general problem lies in the fact that what we called environment is only in part exogenous, as the economy itself produces relevant news which is interpreted by the decision-makers – albeit not necessarily in a correct way. For instance, prolonged investor pessimism might lead to a reduction of GDP, which in turn negatively affects investor sentiment. Such collective expectational biases turning into real economic forces have been qualitatively described by Keynes [24], but are largely neglected in modern macroeconomic theory. (In contrast, recent modeling of financial markets does incorporate expectational biases (see [28, 29] for seminal contributions to this research direction). The present model does not include a feedback from real macro-variables ("endogenous environment") to the economic sentiment (see [38] for an attempt in this direction) as we do not specify a link of economic sentiment to real variables.

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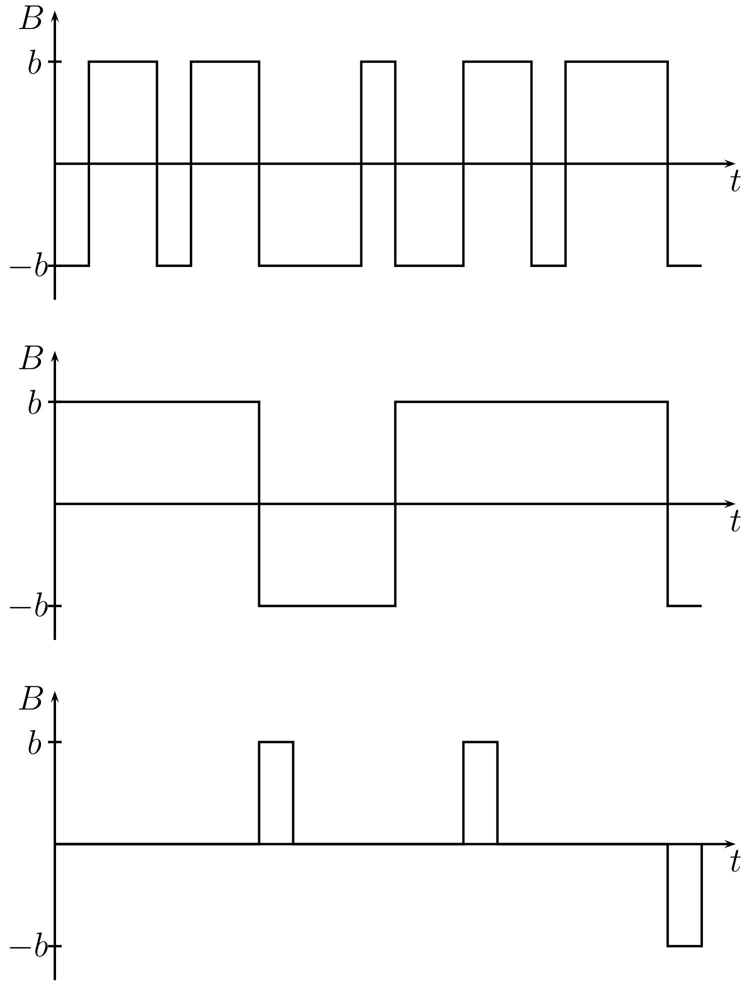

Figure 1: Time-patterns of the process B_t representing the environment: time scales of the Ising field and the external field are comparable (top), lasting but rare events (middle) and rare transitory events (shocks) (bottom). The proportion of positive and negative events is equal on average for the neutral environment.

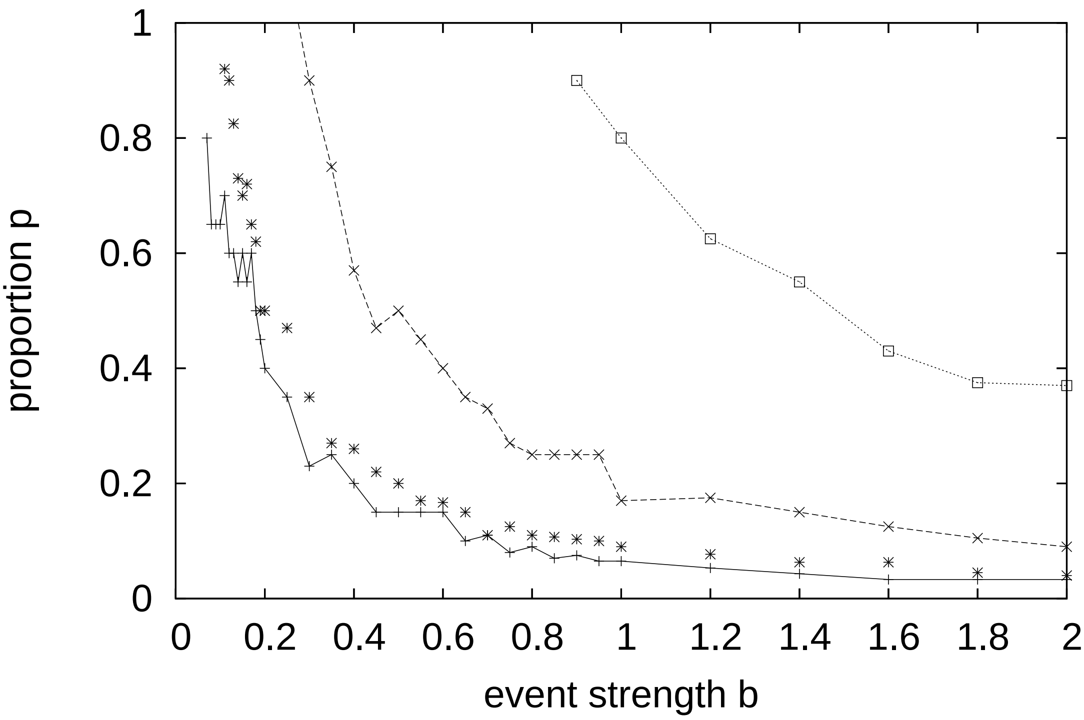

Figure 2: Areas of the parameter space of our model where ordered states persist and do not persist for a neutral environment for 4000 sweeps through a 3001 x 3001 lattice: ordered states of collective sentiment exist for vectors of the parameter values below a boundary curve, but do not exist for vectors of parameter values above; the curves are as follows: (+) frequent events, $J_c/J = 0.99$, ($\times$) frequent events, $J_c/J = 0.9$; ($*$) rare persistent events, $J_c/J = 0.9$, (sq.) shocks $J_c/J = 0.9$.

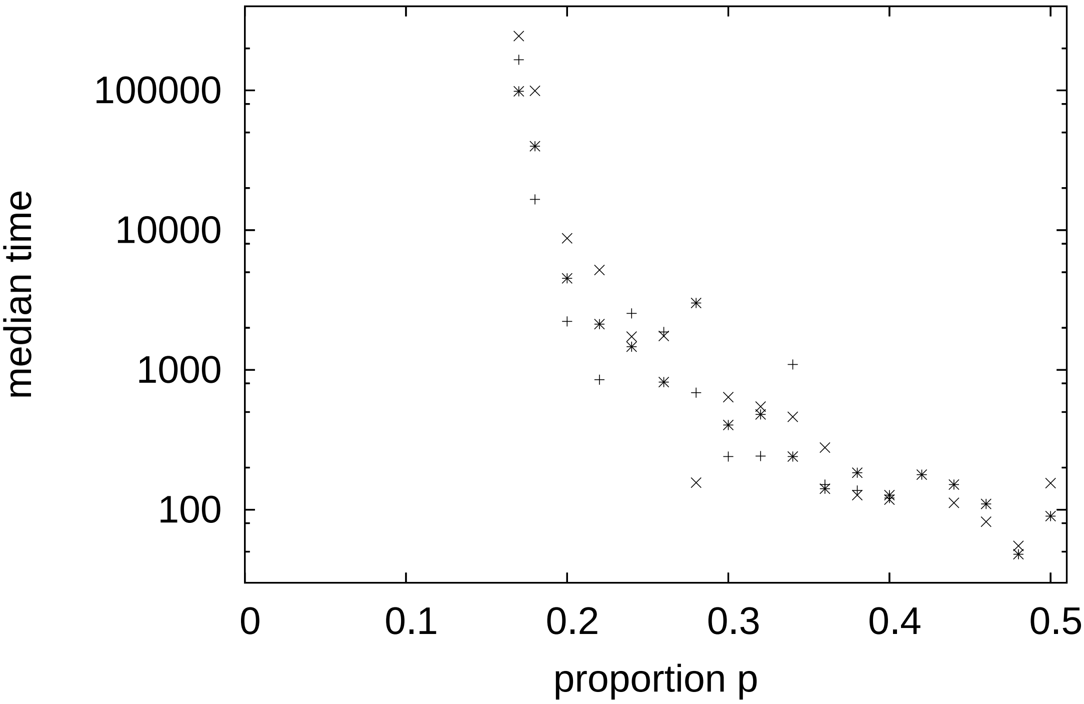

Figure 3: The dependence on p of the median time at which the ordered state is destroyed for the exemplary value $b = 1$; frequent events, $b = 1$, side of the square lattice d= 301(+), 1001(x), 3001(*).

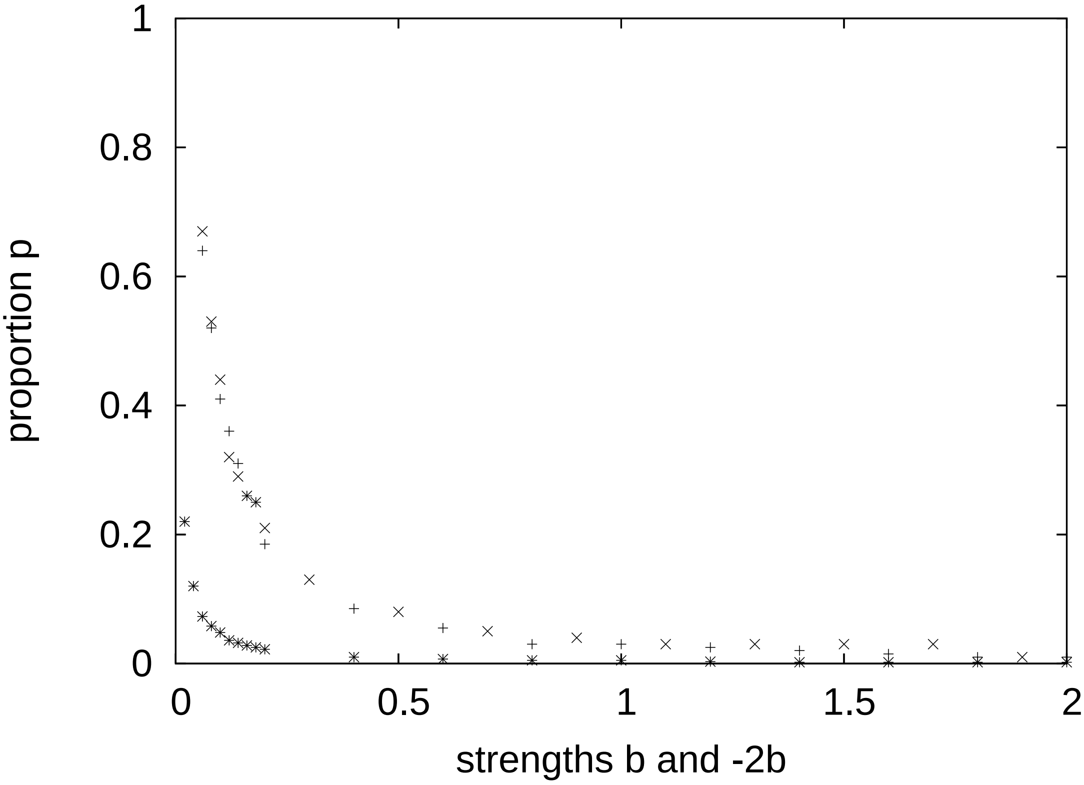

Figure 4: As Figure 2, but for biased (non-neutral) environment. We made 4000 sweeps for $J_c/J = 0.90$ (+,x) and 0.99 (*), for frequent events only, for 1001×1001 (+) and 3001×3001 (x,*) lattices. For 1001×1001 at $J_c/J = 0.99$ the symbols would overlap with those for the larger lattice and are thus not shown.

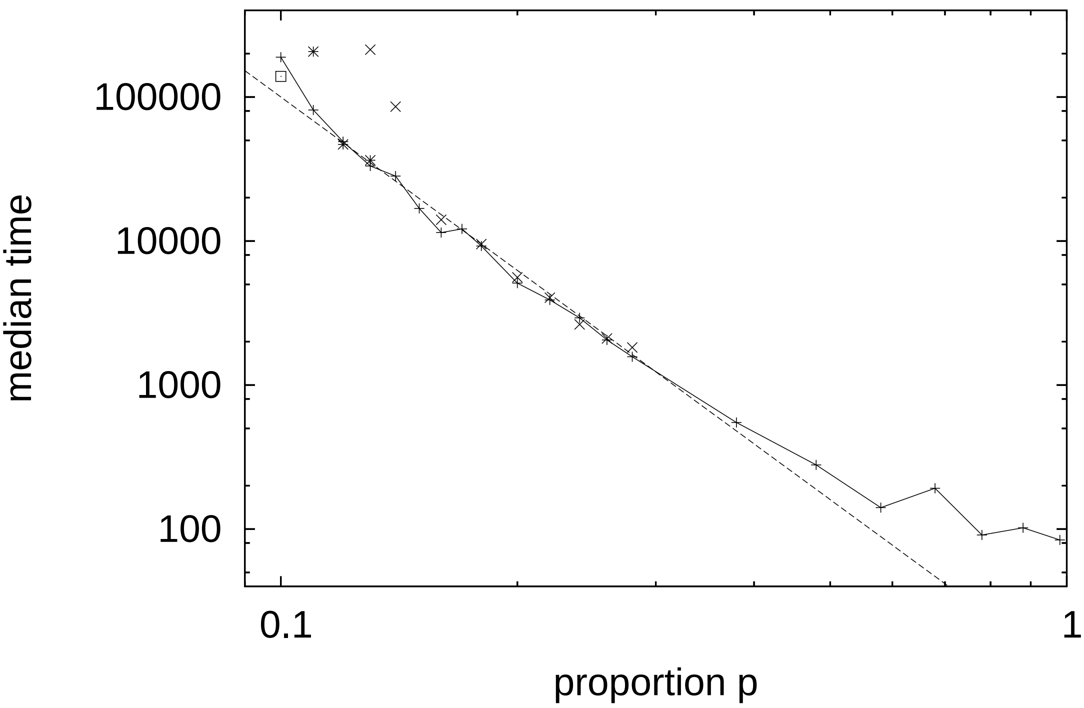

Figure 5: As Figure 3, but for biased (non-neutral) environment. The side of the square lattice d is 301 (+), 1001 (x), 3001 (solid line) und 10,001 (square). The dashed line corresponds to a power law time $= 10/p^4$.

Business cycle synchronization in a simple Keynesian macro model with socially-transmitted economic sentiment and international sentiment spill-over

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Abstract

We propose a simple Keynesian business cycle model in which national income expectations of heterogeneous interacting investors affect their investment decisions. The expectation formation of the investors is influenced by economic sentiment: Investors who hold optimistic views about the future state of the economy expect a higher aggregate demand in the following period and thus invest more than pessimistic ones. The sentiment of the investors evolves subject to socio-economic interactions and to macroscopic feedbacks from the national income movements. Simulations show that our model has the potential to generate complex business cycle dynamics. Based on that framework, we provide a three-country model of business cycle synchronization in which spill-over effects on the level of economic sentiment synchronize the national cycles, provided that investors believe that the economies are indeed coupled.

Keywords

Business cycles; expectation formation; socio-economic interactions; optimism and pessimism; synchronization of business cycles.

JEL Classification

D11; E11; E32.

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1 Introduction

Recently, there has been much interest among economists in the role of interactions-driven socio-economic processes which may lead to important phenomena such as herding in financial markets (Kirman 1993, Lux 1995) or the formation of economic sentiment (Franke 2007a, Westerhoff and Hohnisch 2007). Some of the basic principles of these models have already been proposed by Weidlich and Haag (1983). Not reliant on strong rationality concepts, such models are well-suited for formalizing notions such as Keynesian "animal spirits" (Keynes 1936).

Following that general approach, the present paper defines and analyzes a modified Keynesian multiplier-accelerator model[1] which is intertwined with an interactions-driven process of individual economic sentiment. In our model, economic sentiment affects investment expenditures via the investors' national income expectations, and, in turn, is itself affected by social interactions and national income movements. More precisely, we replace the Samuelsonian accelerator term $i(Y_t - Y_{t-1})$ by an expectation-based term $i(E[Y_t] - Y_t)$, i.e. investment expenditures do not depend on past changes in national income but on the current expected change in national income. Then, we assume that each investor expects an increase in national income in the following period if he is optimistic and a decrease if he is pessimistic. The investor is more likely to become optimistic if national income is increasing, and is more likely to become pessimistic if national income is decreasing, thereby providing a macroscopic feedback from national income movements to expectation formation.

[1] We use a simple Keynesian goods market setup to be able to pin down some of the causalities acting inside our model. Note that the Keynesian multiplier-accelerator model or related Keynesian models are frequently applied in economics (e.g. Chiarella, Flaschel and Franke 2005, Puu and Sushko 2006).

Simulations reveal that our simple model is able to produce intricate oscillations in national income. We then use the model to propose a sentiment-based explanation of international business cycle synchronization in a three-country model. The economies become coupled and their cycles synchronized through "sentiment-spillover effects" across the economies if investors believe that the economies are indeed coupled. This effect occurs in our model even without trade between the economies, pointing to the potential role of economic-sentiment-based effects in business cycle synchronization. Our work complements real-sector (trade-based) arguments explaining business cycle synchronization as put forward by Frankel and Rose (1998), among others.

The structure of the paper is as follows: Section 2 specifies the basic model and analyzes its dynamics. Section 3 extends the basic model into a three-country model in which the emergence of business cycle synchronization due to business-sentiment spill-over between countries is demonstrated. Section 4 concludes with some additional comments.

2 The basic model

Our model extends the multiplier-accelerator model of Samuelson (1939). Accordingly, national income at time step $t+1$ is given by

$$Y_{t+1} = C_{t+1} + I_{t+1}, \tag{1}$$

where C and I denote aggregate consumption and aggregate investment expenditures, respectively.

Consumption depends on the last period's national income

$$C_{t+1} = c\,Y_t. \tag{2}$$

The marginal propensity to consume is denoted by $0 < c < 1$.

As in Westerhoff (2006), we do not assume that induced investments are proportional to the observed change in national income between period *t* and period *t*-1 (as in Samuelson 1939), but are proportional to the expected change in national income between period *t* and period *t*+1. Thus, we specify that

$$I_{t+1} = \bar{I} + i(E[Y_{t+1}] - Y_t), \tag{3}$$

where the first component stands for autonomous investments and the second component reflects the accelerator principle which now entails expectations.

Expectation formation is sentiment-driven: If an investor is optimistic (pessimistic), he expects that national income will increase (decrease) by an exogenous amount *e>0*. The average value of the expected national income may thus be written as

$$E[Y_{t+1}] = W_t(Y_t + e) + (1 - W_t)(Y_t - e), \tag{4}$$

where W_t and $(1 - W_t)$ are the fractions of optimistic and pessimistic investors, respectively.

Inserting (2)-(4) in (1) yields

$$Y_{t+1} = cY_t + \bar{I} + i(eW_t - e(1 - W_t)). \tag{5}$$

Note that for fixed fractions of optimistic and pessimistic agents, (5) would turn into a first-order linear difference equation. This observation will become important later on.

Let us next give an informal description of how the fraction of optimistic (pessimistic) investors evolves over time. We assume that there are N investors in total, with each investor being in either an optimistic or a pessimistic state of mind. We follow Kirman (1993) in specifying that in each time period two investors meet at random and the first will adopt the state of mind of the other with a probability $1 - \delta(\cdot)$. In addition, there is a small probability ε that an investor will change his attitude independently. Unlike in Kirman's (1993) model, the probability that one agent may convince another agent is asymmetric in that an optimist is more likely to convert a pessimist when national income is increasing then a pessimist is likely to convert an optimist. Similarly, when national income decreases, chances are higher that a pessimist will convert an optimist than the other way around.

Formally, let $K_t = W_t N$ denote the number of optimistic agents at time step *t*. The transition probability for the birth-and-death process of *K* is specified as follows

$$K_t = \begin{cases} K_{t-1}+1 & \text{with probability} \quad p_{t-1}^{+} = \dfrac{N-K_{t-1}}{N}(\varepsilon + (1-\delta_{t-1}^{P\to O})\dfrac{K_{t-1}}{N-1}) \\ K_{t-1}-1 & \text{with probability} \quad p_{t-1}^{-} = \dfrac{K_{t-1}}{N}(\varepsilon + (1-\delta_{t-1}^{O\to P})\dfrac{N-K_{t-1}}{N-1}) \\ K_{t-1} & \text{with probability} \quad 1-p_{t-1}^{+}-p_{t-1}^{-} \end{cases}, \tag{6}$$

where the probability that a pessimist converts into an optimist is

$$\delta_{t-1}^{P\to O} = \begin{cases} 0.5+\gamma & \text{for} \quad Y_{t-1}-Y_{t-2}<0 \\ 0.5-\gamma & \text{otherwise} \end{cases} \tag{7}$$

and the probability that an optimists converts into a pessimist is

$$\delta_{t-1}^{O\to P} = \begin{cases} 0.5+\gamma & \text{for} \quad Y_{t-1}-Y_{t-2}>0 \\ 0.5-\gamma & \text{otherwise} \end{cases}, \tag{8}$$

respectively.

We now turn to the dynamics of our model. The simulation run displayed in figure 1 has been produced with the following parameter setting:

$$c = 0.9,\ i = 3.5,\ \bar{I} = 10,\ e = 0.5,\ N = 100,\ \gamma = 0.45,\ \varepsilon = 0.05.$$

The top panel of figure 1 shows the evolution of national income while the bottom panel depicts the number of optimistic agents. The figure demonstrates that our model is able to generate complex swings in economic activity. Both the duration and the amplitude of the business cycles vary strongly over time.[2]

---------- Figure 1 goes about here ----------

Let us now discuss the origin of these business cycles in our model. For fixed fractions of optimistic and pessimistic agents, the evolution of national income is due to the iteration of a first-order linear difference equation. In this case, the level of autonomous expenditures is determined by the relation between optimistic and pessimistic investors. If all agents are optimistic, then total autonomous expenditures are $\bar{I} + i\,e$ so that equilibrium income is $(\bar{I} + i\,e)/(1-c)$. Should there be as many optimistic as pessimistic agents, then the autonomous expenditures amount to $\bar{I}$ and equilibrium income is $\bar{I}/(1-c)$. Finally, if all agents are pessimistic, autonomous expenditures decrease to $\bar{I} - i\,e$ and national income is equal to $(\bar{I} - i\,e)/(1-c)$. Since the marginal propensity to consume is below one, our model implies that national income always monotonically approaches equilibrium income and thus may not leave the lower and upper boundaries $(\bar{I} + i\,e)/(1-c)$ and $(\bar{I} - i\,e)/(1-c)$.

However, the equilibrium income itself undergoes a rather cyclical movement within these upper and lower boundaries. This is illustrated in figure 2 in which we have plotted national income in period *t*+1 against national income in period *t*. The line segments K^{min} and K^{max} stand for the maps in which all agents are either optimistic or pessimistic. The corresponding equilibrium incomes are given by the intersections of these maps with the 45-degree line. Suppose now that we are in period τ and that the fraction of optimistic agents results in the map K_τ. Obviously, national income converges within two time steps from Y_τ to $Y_{\tau+\Delta t}$. Note that if the fraction of optimistic

[2] Recall that for *c*=0.9 and *i*=3.5, the multiplier-accelerator model of Samuelson (1939) generates unstable oscillations. Our present setup may cope with such parameter values.

agents increases, the map K_τ shifts upwards, for instance to $K_{\tau+\Delta t}$. Again, we have drawn the evolution of national income for two additional time steps. Once national income is increasing (decreasing) the bias of the social interactions and the income equilibrium movements jointly manage to sustain that movement. A turnaround may set in either randomly or when the upper (lower) boundary of national income is eventually reached.

---------- Figure 2 goes about here ----------

A high time-resolution excerpt of the dynamics is presented in figure 3. On the left-hand side, the number of optimistic investors is depicted. Note that this number does often not change from one time step to the next. On the right-hand side, the evolution of national income for the corresponding time slot is presented. It changes in each time step according to (5). We remark that national income is computed in our model in each time step, i.e. after each instance of an interaction between two agents. However, running the sentiment dynamics for several time steps between two consecutive computations of national income may yield quite similar fluctuations in sentiment and economic variables.

---------- Figure 3 goes about here ----------

3 A multi-country model with business cycle synchronization

In the present section, we consider three national economies, indexed by D, F and I. To demonstrate that a mechanism based on international coupling of economic sentiment may in itself be sufficient to produce business cycle synchronization, we abstract in our three-country model from goods transfers between the countries.

The equations determining the national economy remain the same as in the basic model, apart from being indexed by the respective national economy. For instance, for D we have

$$Y_{t+1}^D = C_{t+1}^D + I_{t+1}^D, \tag{9}$$

$$C_{t+1}^D = c\,Y_t^D, \tag{10}$$

$$I_{t+1}^D = \bar{I} + i(E[Y_{t+1}^D] - Y_t^D), \tag{11}$$

$$E[Y_{t+1}^D] = W_t^D (Y_t^D + e) + (1 - W_t^D)(Y_t^D - e), \tag{12}$$

and

$$K_t^D = W_t^D N, \tag{13}$$

respectively.

We introduce a trans-national coupling of economic sentiment in the following way: For the transition probabilities, we now specify that the bias towards optimism (pessimism) in times when national income is rising (falling) also includes spill-over effects from the other two national economies (with p denoting a parameter representing the degree of coupling between the economy of D with the two other countries, as perceived by investors in D). Therefore, we obtain

$$K_t^D = \begin{cases} K_{t-1}^D + 1 & \text{with probability} \quad p_{t-1}^{D,+} = \dfrac{N - K_{t-1}^D}{N}(\varepsilon + (1 - \delta_{t-1}^{D,P\to O})\dfrac{K_{t-1}^D}{N-1}) \\ K_{t-1}^D - 1 & \text{with probability} \quad p_{t-1}^{D,-} = \dfrac{K_{t-1}^D}{N}(\varepsilon + (1 - \delta_{t-1}^{D,O\to P})\dfrac{N - K_{t-1}^D}{N-1}) \\ K_{t-1}^D & \text{with probability} \quad 1 - p_{t-1}^{D,+} - p_{t-1}^{D,-} \end{cases}, \tag{14}$$

$$\delta_{t-1}^{D,P\to O} = \begin{cases} 0.5 + \gamma & \text{for} \quad Y_{t-1}^D - Y_{t-2}^D + p(Y_{t-1}^F - Y_{t-2}^F + Y_{t-1}^I - Y_{t-2}^I) < 0 \\ 0.5 - \gamma & \text{otherwise} \end{cases} \tag{15}$$

and

$$\delta_{t-1}^{D,O\to P} = \begin{cases} 0.5+\gamma & \textit{for} \quad Y_{t-1}^{D} - Y_{t-2}^{D} + p(Y_{t-1}^{F} - Y_{t-2}^{F} + Y_{t-1}^{I} - Y_{t-2}^{I}) > 0 \\ 0.5-\gamma & \textit{otherwise} \end{cases}. \tag{16}$$

Analogous equations hold for the other countries.

Note that the coupling of the three national economies is a self-fulfilling prophecy in our model: If investors believe that the countries are coupled (that is that the rise or decline of national income in the other countries is relevant for the prospect in the own country) then our simulations discussed below show that business cycles resulting from Samuelson's multiplier-accelerator principle indeed will be synchronized (in an amount dependent on the parameter *p*). However if the investors do not believe in such a coupling, then no mechanism in our model would account for business cycle synchronization.

Let us now present some simulation results of the dynamics of our three-country model. Figures 4-7 show the time-parallel evolution of national income in each of three countries for the values of the sentiment coupling parameter of p=0, p=0.125, p=0.25 und p=0.5, respectively. A casual investigation of the diagrams reveals that the business cycles resulting from our model become increasingly synchronized with increasing values of *p*.[3] The synchronization is even clearer visible in the correlation plots presented in figure 8 in which business cycle correlation between two of the three national economies (F and D) are depicted. The plot in the top left quadrant shows the case of no coupling (p=0), and the series continues with p=0.125 (top right), p=0.25 (bottom left) and p=0.5 (bottom right).

---------- Figures 4-8 go about here ----------

[3] Although investors match randomly in our model, local phenomena may nevertheless show up in individual countries.

4 Discussion

The aim of our paper is twofold. First, we show how social interactions on the level of individual economic sentiment may lead to rather stable business cycles in Samuelson's (1939) multiplier-accelerator model. The mechanism which engenders such cycles in the present paper is complementary to that suggested and analyzed by Westerhoff and Hohnisch (2007) who apply a pure multiplier model. In their paper, consumer sentiment affects individual consumption propensities while in our paper economic sentiment affects individual expectations which, in turn, impact upon investment. The present paper is also related to Westerhoff (2006) in which investors switch between extrapolative and regressive predictors with respect to market circumstances. In his model, nonlinear deterministic interactions between the investors may lead to chaotic business cycles. In the present model, we abstract from such expectation formation rules to highlight which ingredients are *sufficient* to produce endogenous business cycles in our framework.[4] In that sense, our model is minimalist.

Second, with this basic model producing endogenous business cycles we illustrate in a three-country framework how business cycle synchronization may arise if an investor believes that information concerning the other national economies is relevant for the prospects of his own national economy. Again, the model is minimalist in the sense that we abstract from goods transfers between the countries which are presumably of importance for the phenomenon of business cycle synchronization.

It would be tempting to test to what extent economic sentiment is synchronized over various countries. This empirical issue would presumably add to results on

[4] However, it should be noted that a kind of extrapolative expectations may endogenously emerge within our model. Suppose that national income is increasing. In such a situation, more and more agents become optimistic and at some point in time, aggregate expectations and national income changes become positively correlated.

business cycle synchronization itself. However, it appears to be difficult to answer a basic question implied by the present paper: Is the synchronization of business sentiment causal for the emergence of business cycle synchronization, or does the causality go in a different direction – if business cycle synchronization arises for different reasons then business sentiment synchronization, then synchronization of business sentiment – if indeed it is evident from the data – might be a by-product of the former. Of course, it is very well possible that both phenomena are endogenously intertwined and no clear direction of causality prevails. However, recent progress in estimating models with heterogeneous interacting agents has been achieved by Alfarano, Lux, and Wagner (2005), Lux (2007) and Franke (2007b).

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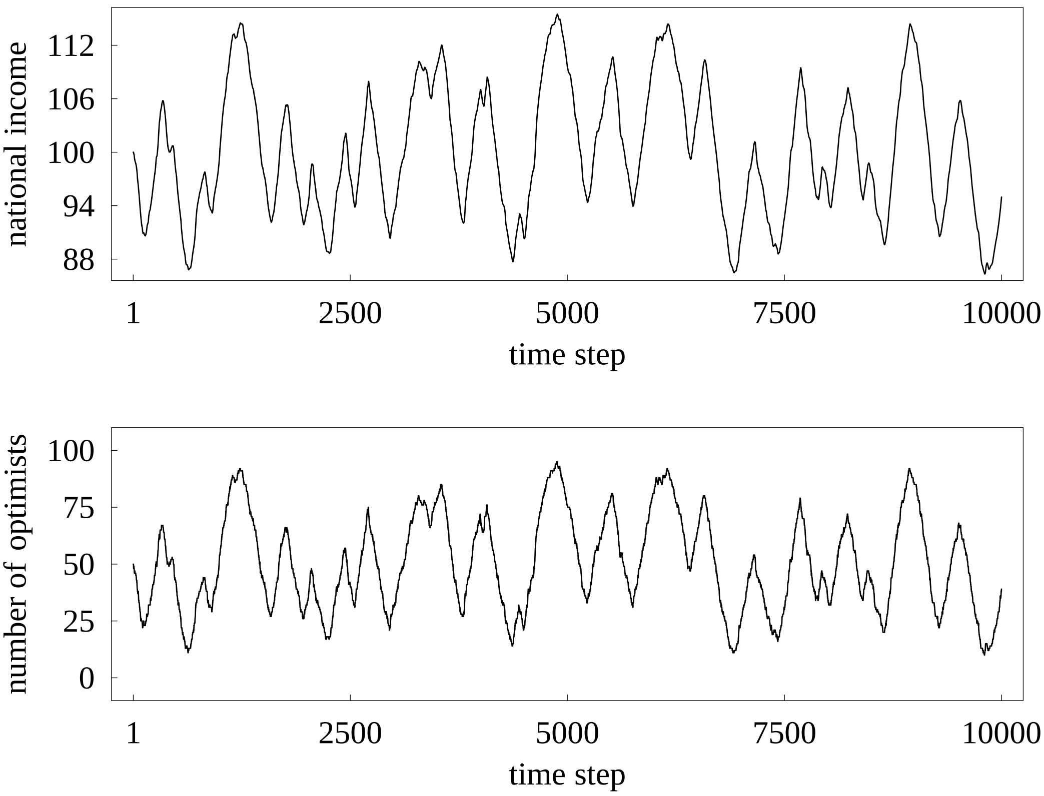

Figure 1: Illustration of the dynamics of the basic model in section 2. The panels present the evolution of national income (top) and the number of optimistic investors in the time domain with 10000 observations. Parameter values are given in section 2.

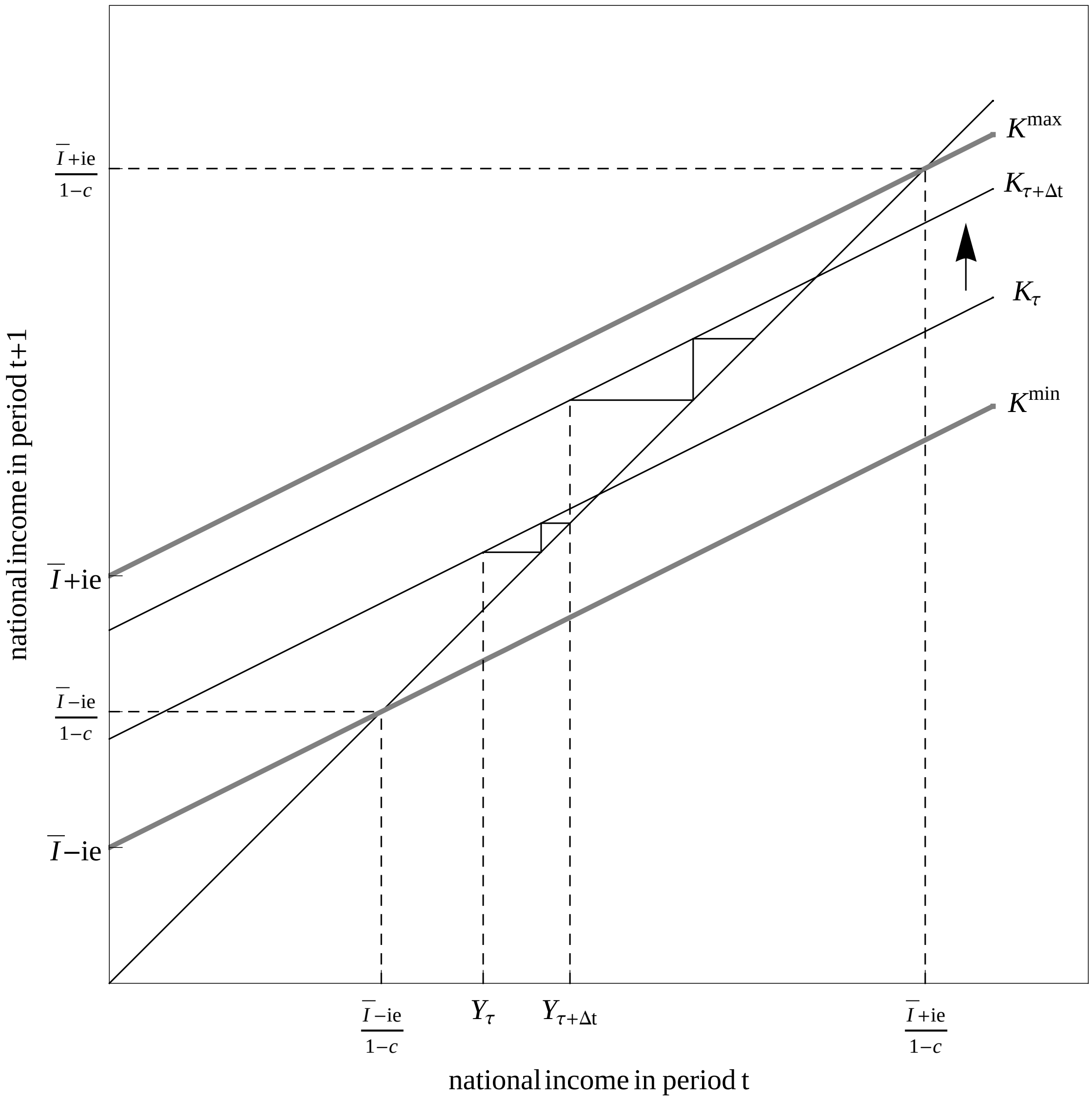

Figure 2: The dynamics of the model for a fixed number of optimistic investors. We plot national income in period t+1 against national income in period t.

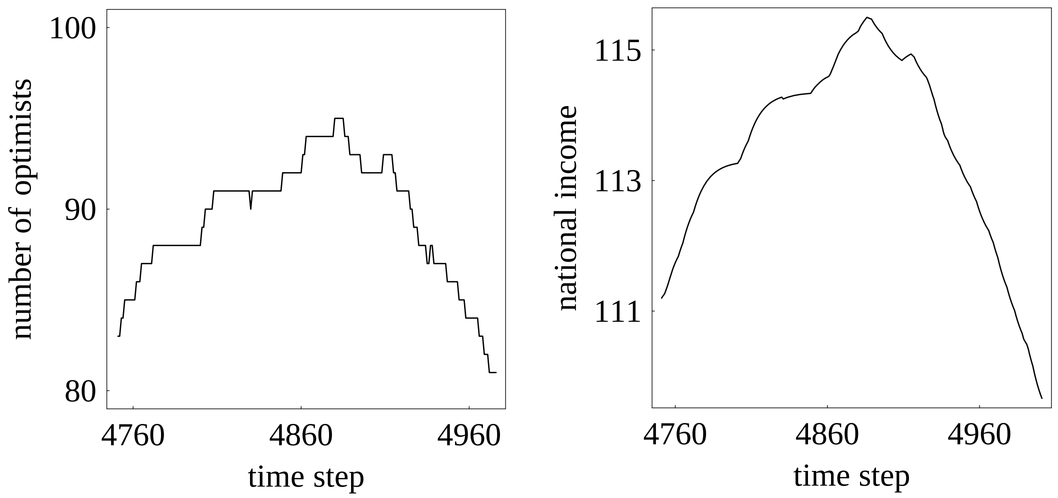

Figure 3: The number of optimistic investors (left) and national income (right) for a short time interval. Parameters as in section 2.

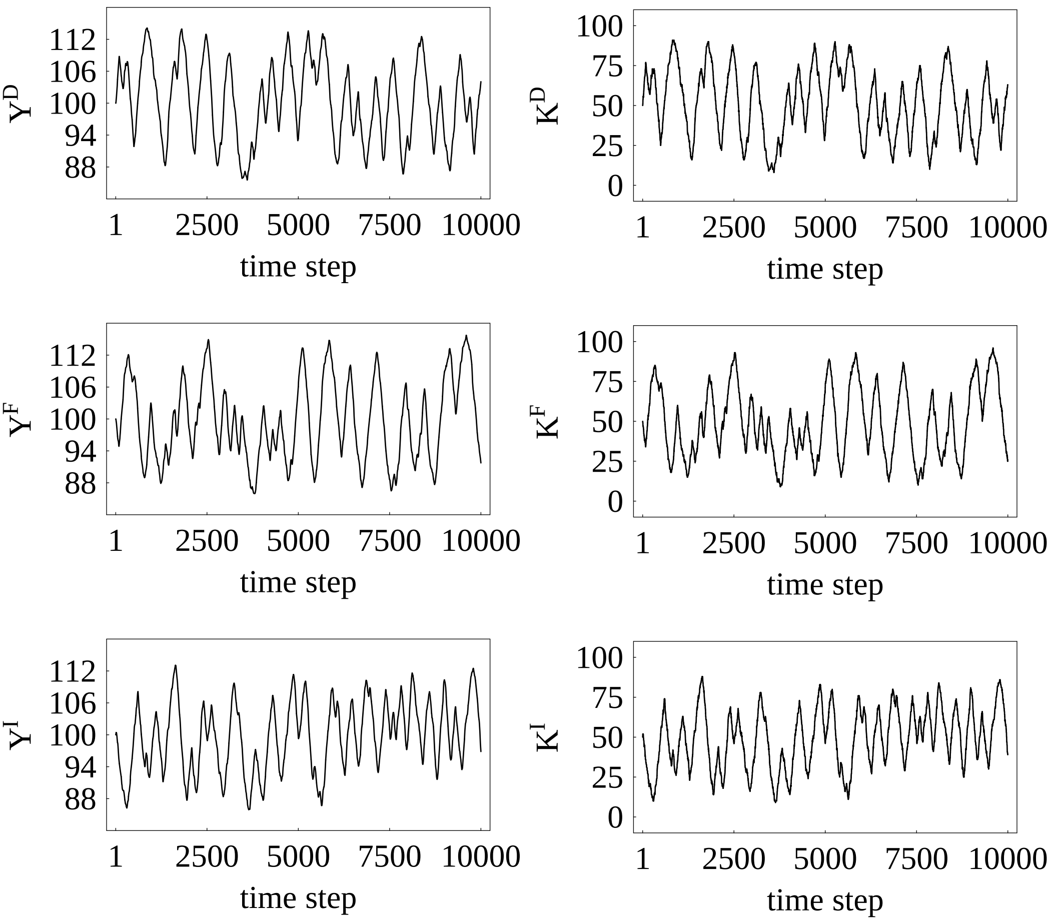

Figure 4: Illustration of the dynamics of the three-country model of section 3. The panels show the evolution of national income (left) and of the number of optimistic investors (right). The countries are D (top), F (central) and I (bottom). The sentiment-spillover parameter is set p=0. The other parameter values are as in section 2.

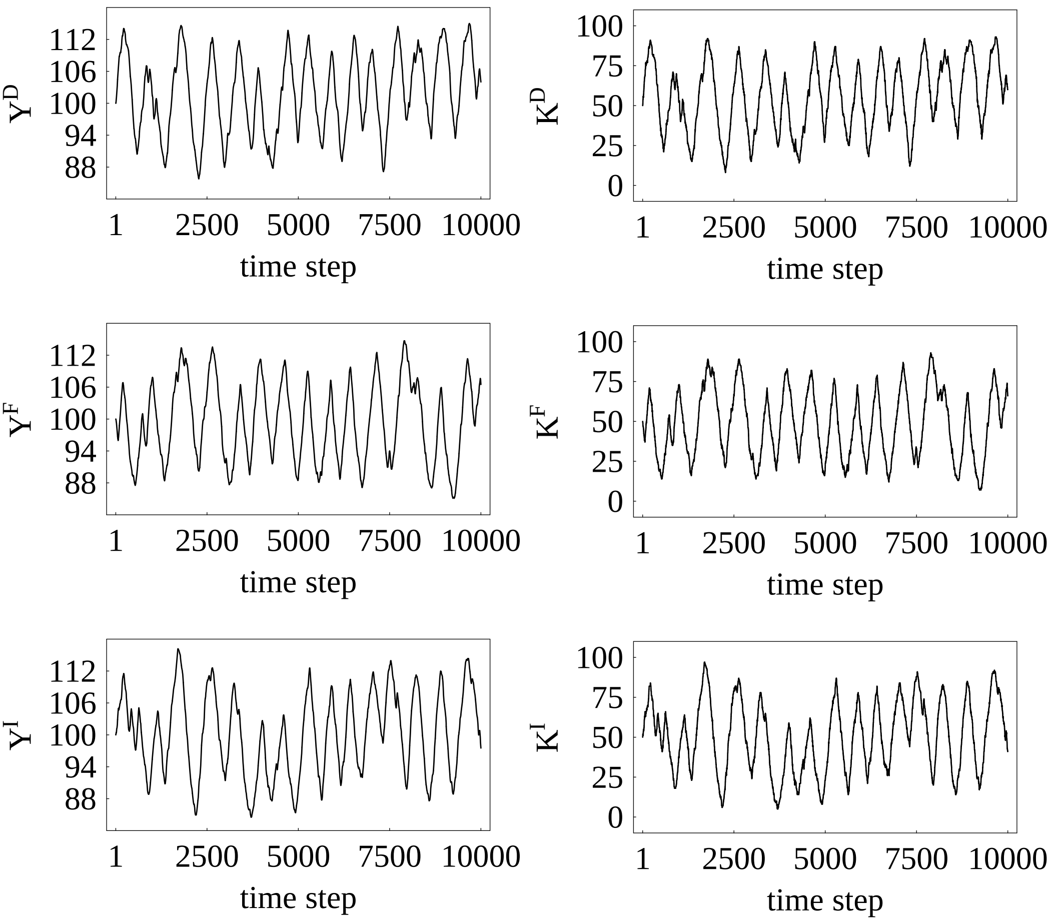

Figure 5: Illustration of the dynamics of the three-country model of section 3. The same simulation design as in figure 4 but now p=0.125.

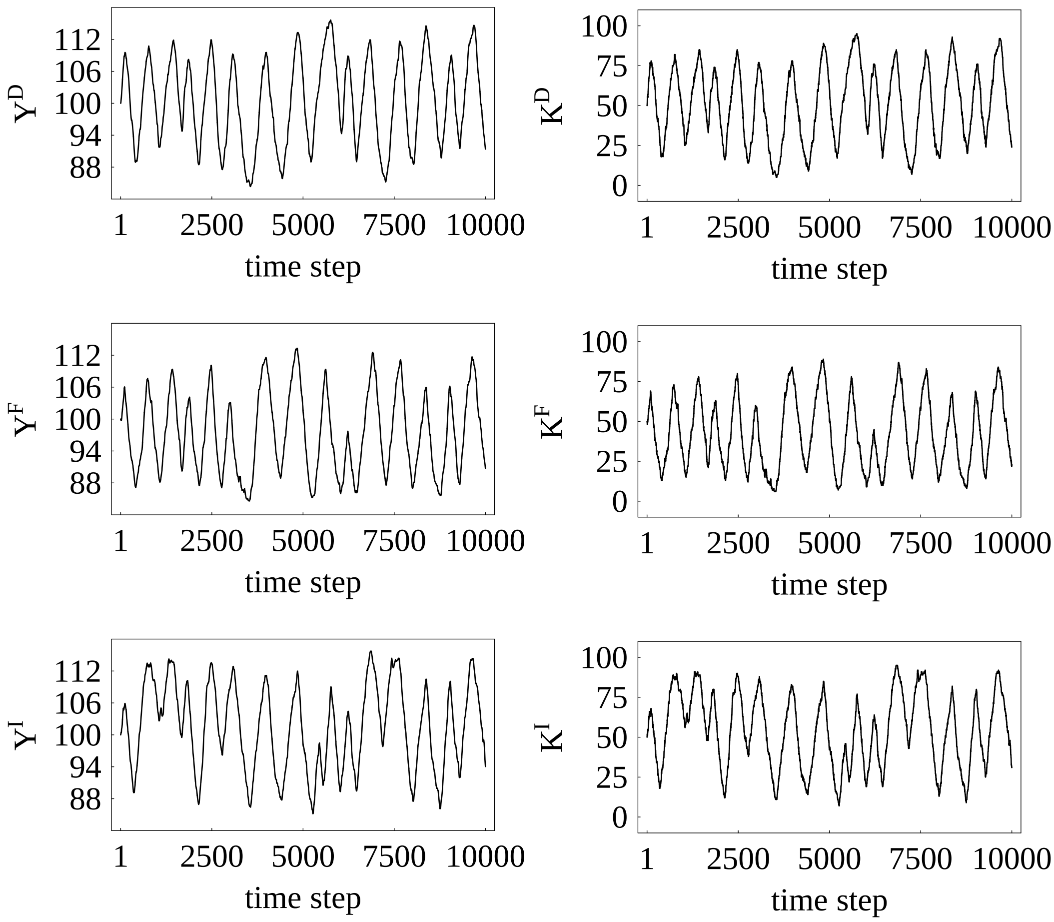

Figure 6: Illustration of the dynamics of the three-country model of section 3: The same simulation design as in figure 4 but now p=0.25.

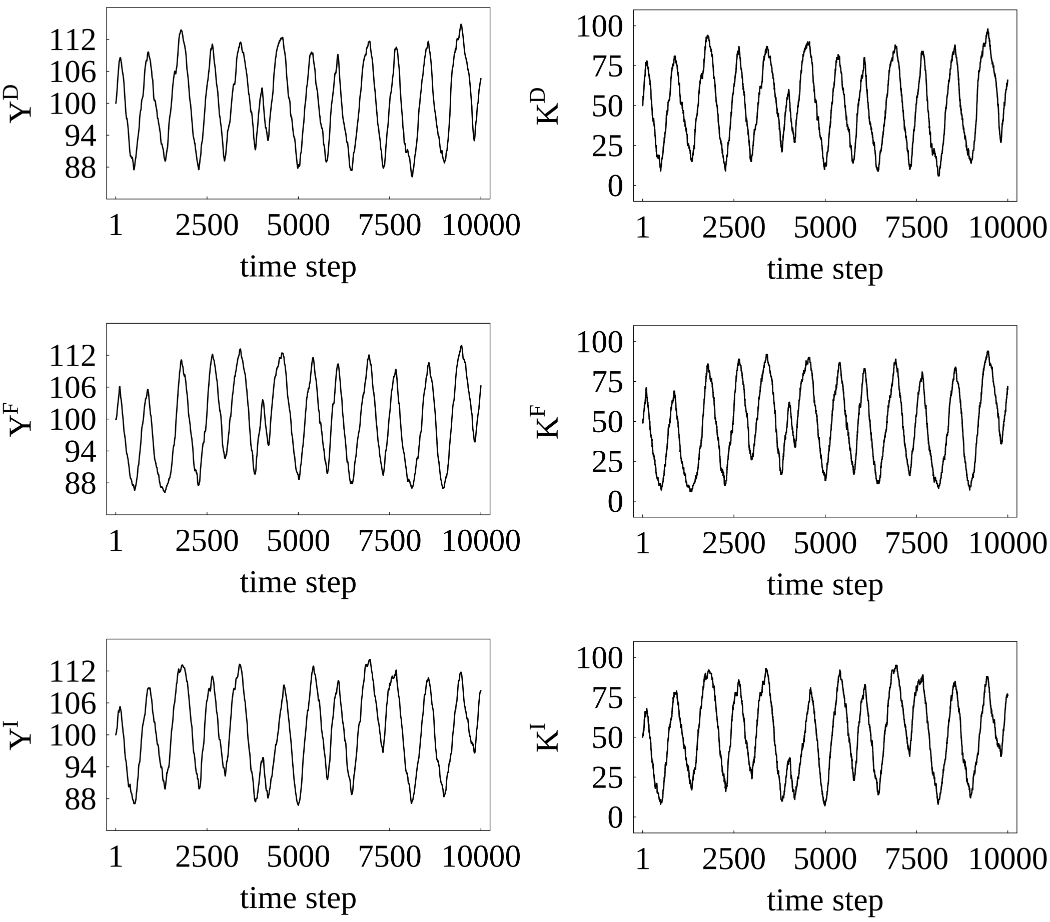

Figure 7: Illustration of the dynamics of the three-country model of section 3. The same simulation design as in figure 4 but now p=0.5.

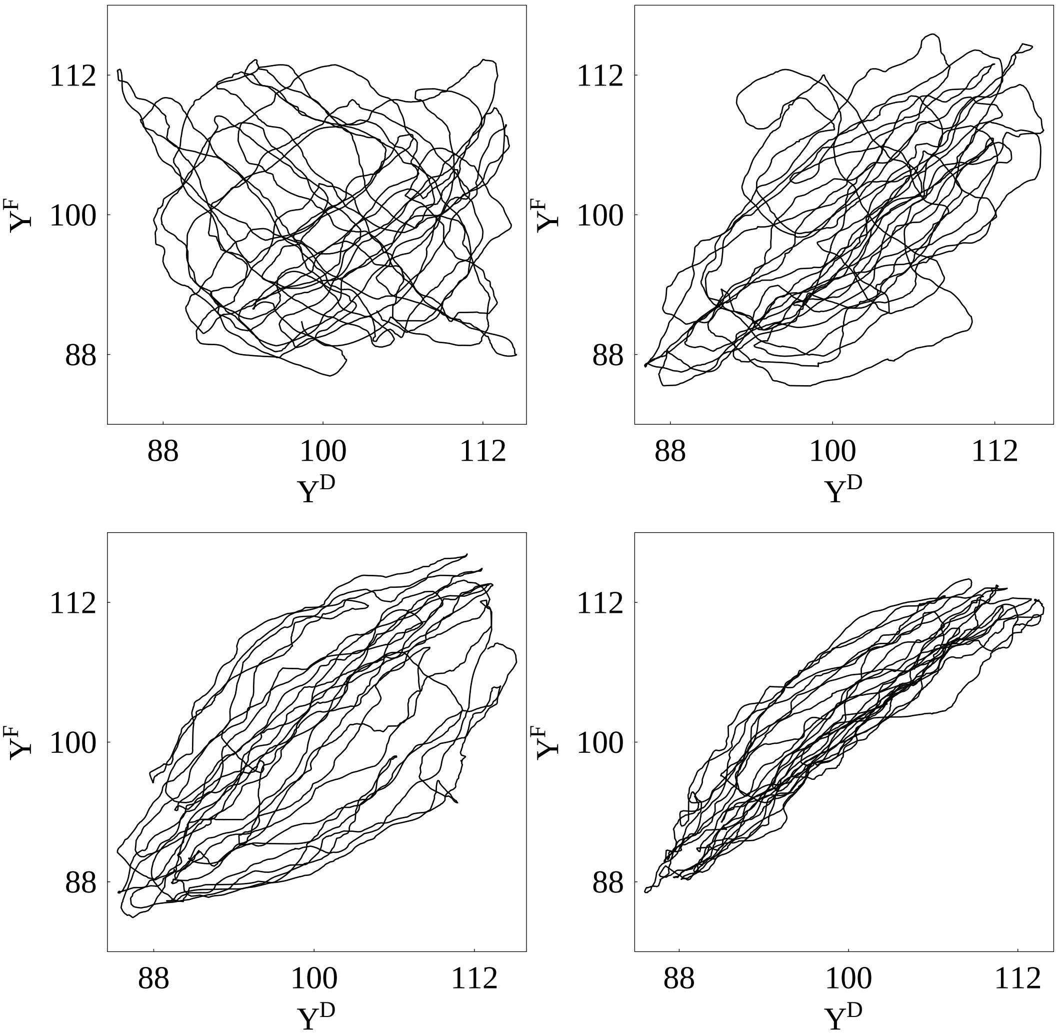

Figure 8: A series of scatter plots depicting business cycle correlation between two of the three national economies (F and D) emerging in our model. The plot in the top left quadrant shows the case of no coupling (p=0), and the series continues with p=0.125 (top right), p=0.25 (bottom left) and p=0.5 (bottom right).

LOCAL-INTERACTIONS-BASED COORDINATION OF EQUILIBRIUM EXPECTATIONS AND THE EMERGENCE OF SUNSPOT EQUILIBRIA

M. HOHNISCH

Abstract. The aim of this paper is to introduce a stochastic model of decentralized coordination of individual equilibrium expectations in a large sequential-trade economy, and to demonstrate that this coordination model naturally engenders sunspot equilibria (with the sunspot events endogenously arising in the coordination process), thereby providing a positive rationale for the emergence of such equilibria. Decentralized coordination of equilibrium-expectations is facilitated by local interactions between agents. Interactions are modeled by specifying a locally interdependent family of individual stochastic processes on a two-dimensional integer lattice $\mathbb{Z}^2$ in continuous time. The particular specification of transition rates chosen in the present paper is known as the (two-dimensional) Voter Model. The composite process has two extremal invariant measures and a continuum of non-extremal invariant measures, to each of which it can converge. While convergence to either one of the extremal invariant measures corresponds to a deterministic coordination outcome, selecting a non-sunspot equilibrium of the underlying economy, convergence to a convex mixture of invariant measures corresponds to a fully coordinated but random coordination outcome of individual expectations. The latter has precisely the structure of equilibrium expectations in a sunspot equilibrium, and it is the uncertainty of the coordination outcome which determines the set of sunspot equilibria of the underlying economy. In effect, it is the non-ergodicity of the coordination process which facilitates the occurrence of sunspot equilibria in the proposed model.

JEL classification: D50, D51, D52, D80, D84

Keywords: coordination, local interactions, non-ergodicity, sunspot equilibrium, voter model

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1. Introduction

It is well-known that in Walrasian economies sunspot equilibria can arise if agents come to expect that the market equilibrium depends on the occurrence of some random event which is otherwise irrelevant to the economy and if the occurrence of this event is not insurable (see Cass (1989) for the particular type of economy referred to in the present paper, and Cass and Shell (1983) for the general issue of sunspot equilibria). Yet from a positive perspective, relatively little is known about how the structure of expectations prevailing in a sunspot equilibrium can be facilitated by some decentralized mechanism. More generally, the question of how decentralized coordination of equilibrium expectations can be facilitated is an open problem in any rational-expectations model with multiple equilibria. In that context, the aim of this paper is to introduce a stochastic model of decentralized coordination of individual equilibrium expectations in a large sequential-trade economy, and to demonstrate that this coordination model naturally leads to sunspot equilibria (with the sunspot events endogenously arising in the coordination process), providing a positive rationale for the emergence of such equilibria.

In the present paper, the coordination of equilibrium expectations is facilitated by direct local interactions between agents. The interactions dynamics is modeled as a continuous-time Markov process. The process consists of a family of individual processes, indexed by the two-dimensional integer lattice $\mathbb{Z}^2$. The index set $\mathbb{Z}^2$ corresponds to the set of agents in the economy. Each individual process has a common state space consisting of two distinct particular equilibrium expectations. The individual processes are locally interdependent in that the transition rate of the individual expectation held by an agent also depends on the configuration of individual expectations of his nearest neighbors (with respect to $\mathbb{Z}^2$). The particular specification of transition rates is known as the (two-dimensional) Voter Model.

The basic properties of the above process are the following (see Liggett (1985) for a comprehensive treatment): It is non-ergodic with two extremal invariant measures. It can converge to either one of its extremal invariant measures, in which mass is concentrated on a "full-consensus" configuration (a configuration in which all agents hold the same expectation), or to an element of the continuum-set of non-extremal invariant measures (being convex mixtures of the extremal invariant measures), each of which has both full-consensus configurations as its support. Any particular limiting distribution of the process can be traced back to a set of initial configurations or more general initial distributions of expectations.

The above described interaction dynamics of individual equilibrium expectations is embedded in simple Walrasian two-period sequential-trade economies as a decentralized coordination mechanism for individual equilibrium expectations. Due to the particular structure of the limiting distributions of the expectations dynamics, the following interpretation in terms of the coordination outcome applies to the respective types of its limiting behavior: While convergence of the process to either one of the extremal invariant measures corresponds to a sunspot-free coordination state of expectations in an underlying economy (the process being in this case a mechanism of equilibrium selection, see Section 3.1), convergence to a non-extremal invariant measure engenders a sunspot equilibrium in an underlying economy: the limiting distribution represents a fully coordinated but random coordination outcome on the level of individual expectations, and the randomness of the coordination outcome itself constitutes the sunspot event (see Section 3.2 where the underlying sunspot equilibrium is that analyzed by Cass (1989)).

The principle behavioral assumption in the proposed coordination-mechanism is that an agent who must form an expectation about the occurrence of a future event but lacks relevant information to do so tends to align himself with the expectations held by other agents in his "reference group". The reader is referred to empirical results in social psychology which provide evidence that there is a tendency to socially-driven alignment of opinion in humans. Pioneering results in that field of research are those of Asch (1951, 1956) and Festinger (1954). The present paper does not attempt to provide an *explanatory* model for such reference-group influence in expectation formation (for an instance of such a model see Banerjee (1992).)

Also, it should be stressed that a stochastic modelling of expectations does not imply that expectations are necessarily to be thought of as random in a behavioral sense. One might well propose deterministic interactive behavioral mechanisms on the micro-level, possibly more explanatory in character. However, when applied in modelling large systems, such deterministic models would most likely produce a degree of complexity which is solvable neither analytically nor numerically for long-enough periods of time. Therefore, the stochastic modelling approach to large economies should be considered as a statistical one[1] - a descriptive shortcut providing the possibility of analyzing analytically the aggregate behavior of a large economy with the structure of direct interactions between agents deduced empirically.

[1]Such a modelling approach corresponds to a concept called *Statistical Economics*. The terminus has been suggested by J.M. Grandmont (1992) alluding to Statistical Physics. Seminal work in this direction was done by Hildenbrand (1971) and Föllmer (1974).

The organization of the present paper is as follows. Section 2 outlines the basic specifications and properties of the proposed stochastic process of expectations based on well-known mathematical results concerning the Voter Model. Section 3 provides two examples of how the process can be incorporated into concrete GET-models. In Section 3.1, an example of equilibrium selection facilitated by the expectation-coordination model in decoupled spot-markets ("trivial sunspot-equilibria" in the terminology of Mas-Colell (1992)) is outlined. In Section 3.2, the expectations dynamics endogenously creates sunspot equilibria. The particular setting is based on Cass's (1989) leading example of sunspot equilibria in a two-period economy with nominal assets. Finally, Section 4 contains a few additional comments on the results presented, in particular it relates the present model to existing economic work applying similar types of stochastic processes.

2. A stochastic model of interactions-driven dynamics of expectations

This section introduces a family of locally interacting stochastic processes, representing in the present paper the time-evolution of individual equilibrium expectations. Such a family can be viewed as a single process on the corresponding product space. Assume that at any given time $t \in \mathbb{R}_+$, an agent can expect one of two equilibria to obtain in the following trading period. These individual expectation states are denoted by e_1 and e_2 and the individual state space of the process by $S = \{e_1, e_2\}$.[2] The countably infinite set of agents is denoted by A, and by $X := S^A$ the set of expectation configurations of agents in A. (The space X is compact in the product topology.) Let $\eta_{a,t} \in S$ denote the expectation of agent a at time $t \in \mathbb{R}_+$ and $\eta_t \in X$ a configuration of expectations at time t.

To specify a topology of local interactions, the set of agents A is endowed with a time-invariant graph structure (implying that each agent a always interacts with the same subset $N(a) \subset A$ of other agents). In the present paper we identify A with the two-dimensional integer lattice $\mathbb{Z}^2$ and set $N(a) = \{b \in \mathbb{Z}^2 : |b - a| = 1\}$ with $|\cdot|$ denoting the Euclidean distance in $\mathbb{R}^2$. The particular specification of transition rates chosen in this paper is known as the (two-dimensional) Voter Model.[3] The transition mechanism is given by the

[2]In general, the individual allocations and prices in equilibrium become known to the agents only "after" the expectation coordination process has converged to a limiting distributions. This is because in the case when the process engenders a coordination state of expectations corresponding to a sunspot equilibrium, the set of equilibria will depend on the limiting distribution of the process (because it determines the probability of the "sunspot event"; see Section 3.2 of this paper). In contrast, when the coordination process engenders a deterministic full-consensus coordination outcome the expectations correspond to a-priori given market equilibria of the underlying economy (see Section 3.1).

[3]The Voter Model was introduced independently by Clifford and Sudburry (1973) and Holley and Liggett (1975). For an extensive discussion, see Chapter V of Liggett (1985) and Part II of Liggett (1999).

assumption that $\eta_{a,t} \in S$ changes to the other type of expectation at a rate

$$c(a, \eta_t) = \frac{1}{4} \sum_{b \in \mathbb{Z}^2 : |b-a|=1} 1_{\{\eta_{a,t} \neq \eta_{b,t}\}} \tag{1}$$

The transition rate of an agent's equilibrium expectation is thus proportional to the number of neighboring agents holding a different equilibrium expectation. Clearly, if the expectations of agents were independent, the individual transition rate $c(a, \cdot)$ would depend only on $\eta_{a,t}$; in the case specified by the above rate, expectations are interdependent, but direct interaction is restricted to next-neighbors.

It is to be shown that there exists a process with path-space distribution P^{η_0} (with the superscript indicating that the process starts at the initial configuration η_0 in time 0) such that for each $a \in A$ and each initial configuration η_0

$$P^{\eta_0}[\eta_{a,t} \neq \eta_{a,0}] = c(a, \eta_0)t + o(t) \tag{2}$$

for $t \to 0$. Moreover, in continuous time it is natural to require that the process P^{η_0} is such that at most one coordinate changes in a point in time, i.e. for each $a, b \in A$ and each η_0

$$P^{\eta_0}[\eta_{a,t} \neq \eta_{a,0}, \eta_{b,t} \neq \eta_{b,0}] = o(t) \tag{3}$$

for $t \to 0$. The existence and uniqueness of such a process has been shown, among others, by means of the Markov semigroup approach and the Hille-Yosida theorem. The reader is referred to Liggett (1985) for a thorough treatment of these issues.

Let $\mathcal{P}(X)$ denote the set of all probability measures on the set X of expectation-configurations, equipped with the topology of weak convergence, i.e.

$$\mu_n \Rightarrow \mu \in \mathcal{P}(X) \Leftrightarrow \int f d\mu_n \to \int f d\mu \quad \forall f \in C(X)$$

with $C(X)$ denoting the space of continuous functions on X. Suppose $\mu \in \mathcal{P}(X)$ as an initial distribution of the process. The probability measure of the process at time t, denoted by $\mu S(t) \in \mathcal{P}(X)$ is implicitly defined via

$$\int f d[\mu S(t)] = \int S(t) f d\mu \quad \forall f \in C(X),$$

with $(S_t)_{t \in \mathbb{R}_+}$ denoting the Markov semigroup of the process.

A measure $\mu \in \mathcal{P}(X)$ is called *invariant* for $(S_t)_{t\in\mathbb{R}_+}$ if $\mu S_t = \mu$ for all $t \geq 0$. The set of all invariant measures will be denoted by $\mathcal{I}$. Invariant measures represent equilibrium-states for the underlying stochastic process. Only invariant measures can obtain as limiting distributions of the process (see Liggett, 1985, p.10).[4]

The property that a process converges to a single limit distribution from any initial measure (implying that $\mathcal{I}$ is a singleton) is called *probabilistic ergodicity* of a process. For the Voter Model, non-ergodicity is obvious. Because transition rate is zero for an agent in agreement with all his next-neighbors, the Voter Model has as invariant measures at least the point-mass measures concentrated on the full-consensus configurations η_{e_i} with $(\eta_{e_i})_a = e_i \; \forall a$, with $i = 1, 2$. These measures are denoted by δ_{e_i}. Every convex combination $\delta_\alpha = \alpha\delta_{e_1} + (1-\alpha)\delta_{e_2}$ with $\alpha \in (0, 1)$ is also an invariant measure, but for these measures macroscopic events remain random. In particular, the empirical distribution is random, with probability α for occurrence of a full-consensus on e_1, and $1 - \alpha$ for e_2.

For the interpretation of the expectations dynamics as a coordination process a question to be asked is whether there exists stable coexistence of opinions, i.e. whether there exist extremal invariant measures other than δ_{e_1} and δ_{e_2}. For the present model, the answer is negative[5] (see Liggett (1985, Sect. V 1)). The following property, called *clustering*, holds for the two-dimensional Voter Model (see Liggett (1999), Th. 1.3, p.141)

$$\lim_{t\to\infty} P^{\eta_0}(\eta_t(a) \neq \eta_t(b)) = 0 \tag{4}$$

for all $a, b \in \mathbb{Z}^2$ and all initial conditions η_0. Thus, for any arbitrary large finite volume, after a long enough period of time one observes (almost) all agents having assumed expectations of the same type.

Theorem 1.9 with Corollary 1.13 from Liggett (1985, p.231) provide necessary and sufficient conditions for convergence for a wide class of initial measures. An example is the following statement *Let $\mu \in \mathcal{P}(X)$ denote a translation-invariant measure with marginals $\mu\{\eta : \eta(a) = e_1\} = \alpha$. Then*

$$\lim_{n\to\infty} \mu S(t) = \alpha\delta_{e_1} + (1-\alpha)\delta_{e_2}.$$

However, there are initial conditions for which convergence does not occur.

Having reviewed some facts about the limiting behavior of the considered expectations dynamics, we shall now relate them to the "limiting expectation" *of each individual.* The

[4]The formal statement is: *If $\nu = \lim_{t\to\infty} \mu S_t$ exists for some initial measure $\mu \in \mathcal{P}(X)$, then $\nu \in \mathcal{I}$.*

[5]The answer is positive for lattice-dimension greater than or equal three (for details, see Liggett (1985, Sect. V 1)).

reader is reminded that by the definition of the expectations dynamics only the two point expectations e_1 *will occur with certainty* and e_2 *will occur with certainty* constitute the individual state space of the process. However, if an agent is assumed to sample the path of his own equilibrium expectation over time, or the path of equilibrium expectations in a small subset of agents "close" to him, the notion of a "limiting expectation" based on his individual limiting statistics arises, allowing a more general type of expectations to appear. The question is how such a "limiting expectation" is related to the limiting distribution of the compound process.

With Equation 4 in mind, one can think of two local scenarios for the convergence path of the considered process when it converges to a non-extremal measure, with different implications for the "limiting expectations" of individual agents. First, in any large but finite volume the process "settles randomly on one extremal invariant measure"[6] (with probabilities α and $1-\alpha$ respectively). Second, in any such volume the process oscillates infinitely often "between the two invariant measures". Cox and Griffeath (1986) and, in a more general setting, Cox and Klenke (2000) have shown that the second scenario actually obtains for the present model. Moreover, the weights α and $1-\alpha$ determine the proportion of time spent "close" to either one of the extremal measures (again, restricted to a large finite volume).

By the above result, each agent arrives at a "limiting expectation" being a distribution on $\{e_1, e_2\}$ with exactly the same weights as the limiting distribution of the process places on δ_{e_1} and δ_{e_2}. This leads to the following interpretation to be elaborated in the following section: If the limiting distribution is one of the extremal measures δ_{e_1} and δ_{e_2}, each agent will expect a determinate full-coordination outcome, either on e_1 or e_2. (It is then natural to assume that the corresponding equilibrium of the underlying economy will be selected; see Section 3.1.) If the limiting distribution is a non-extremal measure, each agent will expect a full-coordination outcome, but with remaining randomness of whether it will be e_1 or e_2. The main point, which will underlie Section 3.2, is that it is exactly this structure of expectations which prevails in a sunspot equilibrium of an underlying economy.

3. Coordination of Expectations and the Emergence of Sunspot Equilibria

The present section places the stochastic dynamics of equilibrium expectations, the mathematical properties of which were outlined in the previous section, in the context of two

[6]More precisely, what is meant is the convergence in a finite volume to the projection of an infinite-volume invariant measure to that finite-volume.

simple example economies, each a two-period ($t = 0, 1$) sequential-trade pure-exchange economy. Yet before turning to the respective functions of the expectations dynamics in these example economies, we shall provide some explanations applying to both situations.

The expectation dynamics is assumed to evolve, and its limiting distribution to emerge, prior to trading decisions being made in period $t = 0$. It is important to remark that the time scales of (the mathematical models of) the market exchange process in discrete time and the expectations dynamics in continuous time are inherently incomparable. Therefore it is up to the modeller to specify a relation between them. As a result, the convergence of the process (the limit $t \to \infty$) need not "take longer" than the time interval between two consecutive trading periods.[7] The limiting distribution of the expectations dynamics will be called a *coordination state*. A realization of a configuration of individual equilibrium expectations from this limiting distribution will be called a *coordination outcome*. The coordination outcome is assumed to become known to the agents between $t = 0$ and $t = 1$.

Two principle types of coordination states can be distinguished according to whether or not macroscopic uncertainty prevails in them. First, the process converges to one of its extremal invariant measures, say δ_{e_1}. Then there is no uncertainty as to the coordination-outcome, and all agents correctly assign the probability $\pi_1 = 1$ to the occurrence of η_{e_1} and the probability $\pi_2 = 0$ to the occurrence of η_{e_2}. Second, the limiting distribution of the process is a convex combination $\delta_\alpha = \alpha\delta_{e_1} + (1 - \alpha)\delta_{e_2}$ of its extremal invariant measures. When a realization is drawn from the measure δ_α, it is either the full-consensus configuration η_{e_1} (with probability α), or the full-consensus configuration η_{e_2} (with probability $1 - \alpha$).[8] Importantly, it follows from the properties of the convergence of the process to a mixed invariant measure, as described in the final part of Section 2, that each agent can learn - by sampling the realizations of the process in any finite subset of agents - the correct probabilities of the full-consensus outcomes. Thus, each agent will correctly expect an occurrence of the coordination outcome η_{e_1} with probability $\pi_1 = \alpha$ and of η_{e_2} with probability $\pi_2 = 1-\alpha$. Since in each non-extremal coordination-state $\delta_\alpha = \alpha\delta_{e_1} + (1-\alpha)\delta_{e_2},\ \alpha \in (0, 1)$ the possible coordination-outcomes η_{e_1} and η_{e_2} both have non-zero probabilities (and because agents are automatically in agreement about the associated equilibrium in each of

[7]For that reason, no conceptual problems would arise in extending the model to a multiperiod setting, with the convergence of the coordination process taking place between consecutive periods.

[8]With a more general specification of the expectations dynamics, it would no longer be the case that only full-consensus configurations appear in the support of the limiting distribution of the process. In that more general case, the empirical distribution would characterize the coordination state. The latter is a random measure on S (see further comments on the specific choice of the Voter model on $\mathbb{Z}^2$ in Section 4).

them), these coordination outcomes can play the role of the sunspot-events (see Section 3.2).

3.1. **Coordination between multiple equilibria and trivial sunspots.** In this section, the underlying market structure is a pair of decoupled spot markets (i.e there are no financial assets to transfer wealth between them) with multiple equilibria in the second period. This structure ensures that possibly remaining randomness of the coordination outcome has no impact on the set of market equilibria.

Consider an economy with a countably infinite set of agents. Assume that the spot market of the economy in $t = 1$ allows for multiple – say two – equilibria, while no actions relevant for period 1 can be taken in period $t = 0$. Each element of the individual state-space S is the expectation that a specific market equilibrium will occur in $t = 1$. Say the process is started at some configuration η_0 or at some initial distribution μ_0 for which convergence to an invariant measure $\delta \in \mathcal{I}$ is known to occur. According to the distinction made above, the following coordination states can be distinguished. If the limiting measure is extremal, then there is no macroscopic uncertainty, and the equilibrium which is expected will occur (due to the rational assumption hypothesis or some mechanism). If the limiting measure is non-extremal, say with weights α and $1 - \alpha$ of the full-consensus states δ_{e_1} and δ_{e_2} in the mixture, the coordination outcome is random. The expectations π of the full-consensus coordination outcomes η_{e_1} and η_{e_2} will be correct, i.e. $\pi_1 = \alpha$ and $\pi_2 = 1 - \alpha$. This case corresponds to what Mas-Colell (1992) calls a trivial sunspot equilibrium.

In the above situation, the expectations dynamics explains coordination of expectations among the a-priori given equilibria and thereby equilibrium selection (due to the assumption of rational expectations). However, it is obvious that since in this context the coordination process has no real influence on the economy, the equilibrium selection role is a *deus ex machina* (unless, of course, one is ready to consider a causal link via some real mechanism from the expectations of agents to the equilibrium actually obtaining).

3.2. **Coordination with remaining macroscopic uncertainty and the emergence of sunspot-equilibria.** With its basic lines already explicated, we shall now complete the statement of the link between certain coordination outcomes of the expectations dynamics and the emergence of sunspot equilibria in Walrasian models. For simplicity, the market structure selected in this section is that of Cass' (1989) leading example of sunspot equilibria in Walrasian sequential economies with nominal assets. (For generalizations of that model see Cass (1992) and Siconolfi (1991).) Assume a two-period ($t = 0, 1$) pure-exchange economy with a single good available in each period, its quantity denoted by y^t, traded on a spot market in period t. In period 0, there is also a financial instrument (bond), with its quantities denoted by denoted by b. There is no uncertainty of fundamentals in the economy, but agents might come to believe that the equilibrium depends on the outcome of some external two-state ($s = 1, 2$) lottery (sunspot). The exogenous nominal returns r_1 and r_2 of the bond may depend on the sunspot outcome. The price of the good in period t is denoted by p^t. The spot price of the good in the second period may also depend on a sunspot outcome. The price vector is denoted by $p = (p^0, p^{1,1}, p^{1,2})$, with the second superscript referring to the sunspot outcome. The price of the bond is set to one.

There are two agents, each characterized by a differentiable, strictly increasing and strictly concave Neumann-Morgenstern utility function $u_a : \mathbb{R}^2_+ \to \bar{\mathbb{R}}$ with $u_a(0) = -\infty$, and an endowment vector $w_a = (w^0_a, w^1_a, w^1_a) \in \mathbb{R}^3_{++}$. Neither the utility function nor the endowment vector depend on the realization of the expectation coordination process. The demand of agent a solves the maximization problem

$$\max_{(y^0_a, b_a, y^{1,1}_a, y^{1,2}_a)} \pi_1 u_a(y^0_a, y^{1,1}_a) + \pi_2 u_a(y^0_a, y^{1,2}_a) \tag{5}$$

subject to the constraint that the budget be balanced in both periods and states

$$(i) \quad p^0 y^0_a + b^0_a = p^0 w^0_a \tag{6}$$

$$(ii) \quad p^{1s} y^{1s}_a = p^{1s} w^1_a + r^s b_a \qquad s = 1, 2. \tag{7}$$

Let f_y and f_b denote the demand function for the good in the respective states and for the bond in the zero period, respectively, defined on an appropriate set of prices. Cass (1989) shows that sunspot equilibria generically exist in this model.

To relate the expectations dynamics specified above to the Cass (1989) example, our economy is assumed to be constituted by countably infinite "copies" of the Cass (1989) economy. We use the Hildenbrand's (1974) distribution approach with the characteristics distribution ν concentrated on two preference-endowment vectors (ϕ_i, w_i), $i = 1, 2$ with each having the weight of one-half. The mean good-demand then obtains as

$$\bar{f}_y(p) = \int_{\mathcal{P}\times T} f_y(p, \phi, w) d\nu(\phi, w) = \frac{1}{2} f_y(p, \phi_1, w_1) + \frac{1}{2} f_y(p, \phi_2, w_2) \tag{8}$$

while mean asset-demand obtains as

$$\bar{f}_b(p) = \int_{\mathcal{P}\times T} f_b(p, \phi, w) d\nu(\phi, w) = \frac{1}{2} f_b(p, \phi_1, w_1) + \frac{1}{2} f_b(p, \phi_2, w_2). \tag{9}$$

Thus, the equilibrium market clearing conditions for the good-markets and the bond market correspond to the two-agent economy conditions

$$f_y(p^*, \phi_1, w_1) + f_y(p^*, \phi_2, w_2) = w_1 + w_2, \tag{10}$$

$$f_b(p^*, \phi_1, w_1) + f_b(p^*, \phi_2, w_2) = 0. \tag{11}$$

Now, while the results on the Voter model discussed in Section 2 assure the existence of limiting distributions corresponding to expectation coordination outcomes with complete but random coordination on either e_1 (with probability $\pi \in (0, 1)$) or e_2 (with probability $1-\pi$) and each agent correctly foreseeing this outcome, the results of Cass (1989) imply that Walrasian market equilibria (sunspot equilibria) with precisely that expectation structure exist for the infinite economy just specified. The random coordination outcome of the process provides by its very nature the specification of a sunspot event with no additional requirements on some agreement procedure required in the standard formulation of sunspot equilibrium. Thus, the suggested expectation coordination mechanism can be considered as a complement to the sunspot literature by explaining how sunspot expectations can actually emerge in a decentralized way.

Since the set of equilibria emerges only upon convergence of the process, it seems unreasonable to assume that agents know equilibrium prices and their respective allocations before the limiting distribution has emerged. Yet for a consistent definition of the interacting process, agents need not to know the details of the equilibrium while the process is unfolding, they just must be able to distinguish whether or not they have the same expectation as each of their neighbors (because of the specific formulation of transition rates in Equation 1). For practical purposes, the concrete content of e_1 and e_2 might be thought of being provided by two competing "forecast agencies" and it might well be time-varying while the coordination process unfolds.

Concluding this section, we emphasize the difference of the respective roles the expectation coordination process plays in Sections 3.1 and 3.2, respectively: the expectation coordination process as defined in the present paper can facilitate equilibrium selection (as it does in Section 3.1) only if, first, the set of market equilibria is a-priori given rather than determined by the limiting distribution of process and, second, the set of market equilibria is finite. (The second condition is due to the fact that the Voter model has a finite individual state space.[9]) In contrast, in the present Section 3.2 the state space $\{e_1, e_2\}$ emerges as representing components of equilibrium in different states (defined by the coordination outcome), jointly constituting *a single* sunspot equilibrium, rather than different equilibria. Therefore, equilibrium selection is exogenous (i.e. left to the "Walrasian auctioneer" or "forecast agencies") in the sunspot context considered in Section 3.2. This clearly is somewhat unfortunate, all the more because the set of sunspot equilibria turns out to be a continuum set (see Cass (1989, 1992)).[10]

4. Discussion

This final section of the paper contains additional remarks on the presented ideas and relates them to economic literature featuring similar types of stochastic processes. We begin by addressing the following possible objection to one of the paper's statements: one could argue that since the expectation coordination process is not affected by the agents' consumption and saving decisions, it is as much exogenous to the economy as the vaguely specified sunspot event, in effect being no different from a biased coin. However, while the structure of the limiting distribution of the two-dimensional Voter model indeed corresponds to that of a biased coin, the associated coordination mechanism is fully decentralized and endogenously both providing the sunspot event and ensuring the proper correlation of events and equilibria, none of which is provided by the biased-coin mechanism. Nevertheless, it would presumably be a worthwhile endeavour to specify a feedback from individual or macroscopic variables of the economy to the coordination process of expectations (in a multiple-period setting). Conceptually, a feedback from macroscopic to microscopic expectational variables was recently proposed by Hahn (2003). Technically, it can be introduced and analyzed using mathematical concepts and results of Föllmer (1979) and Föllmer and Horst (2001).

An issue deserving elaboration is the specific role of the two-dimensional Voter Model as the underlying stochastic process. In fact, the property that the set of extremal invariant measures consists of just the two full-consensus point-mass configurations is specific to the Voter Model for lattice-dimension less or equal two. For lattice-dimension three or more,

[9]Equilibrium selection for economies with a continuum of equilibria can be facilitated by analogous processes with a continuum individual state space, but such models are beyond the scope of the present paper.

[10]A immediate solution to this problem would be a two-stage combination of a the mechanisms of Section 3.2 (sunspot emergence) and 3.1 (equilibrium selection), with the latter facilitated by an appropriate multicomponent process with a continuum individual state space.

there appear additional extremal invariant measures not concentrated on full-consensus configurations. The same is true if the transition probabilities are slightly changed. For instance, if the transition rate for an agent is slightly positive despite all his neighbors being in agreement with him, the resulting invariant measures (there are multiple such measures for lattice-dimension two or more) no longer have an empirical distribution concentrated on full-consensus outcomes. Then, though there is no macroscopic uncertainty for translation-invariant extremal measures, a fixed percentage of agents deviates from the majority expectation in these states. There are interesting economic implications of the emergence of such residual heterogeneity, but they call for a separate treatment and discussion.

Finally, we shall comment on the relation of the present paper to the existing economic literature featuring similar types of interacting stochastic processes. Two important instances of that literature are the papers of Blume (1993) and Kosfeld (2005). The main point of the comparison turns out to be that it is the non-ergodicity of the stochastic process which is essential in the present paper, while that appears not to be the case for Blume's (1993) and Kosfeld's (2005) work.

Blume (1993) defines a process of myopic individual strategy revision in a countably infinite population of players. Each player plays a series of independent games against a number of players in his "neigborhood" (the latter being defined by the topology of the integer lattice). He changes his strategy at random times according to a combination of best-response and random factors. Beyond the obvious contextual aspects, the main structural difference between Blume's (1993) model and the present one lies in the fact that the players in Blume's (1993) model do not respond to macroscopic variables of the process, as do players in the present model. Therefore, non-ergodicity and – associated with it – remaining macroscopic uncertainty do not appear to play a particular role in Blume's (1993) model.

In Kosfeld's (2005) model the individual state space consists of two beliefs about the prevailing state of nature: agents may either hold a default belief (which is specified as the true one), or they can believe in a rumor (which is objectively wrong, but still able to alter the economic variable if enough agents come to believe in it). While the structure of the transition rates specified by Kosfeld (corresponding to the so-called Contact process, see Liggett (1985), Chapter VI) differs from that of the Voter model (for instance, transition between the idividual states is not symmetric), the Kosfeld (2005) model is based on the same behavioral assumptions with regard to interactions as our model (agents tend to believe in the rumor if their neighbors do). The Kosfeld (2005) paper does not, however, address the economic implications of the convergence of the belief dynamics to a limiting distribution with remaining macroscopic uncertainty.

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A Percolation-Based Model Explaining Delayed Take-Off in New-Product Diffusion

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Abstract

A model of new-product diffusion is proposed in which a site-percolation dynamics represents socially-driven diffusion of knowledge about the product's characteristics in a population of potential buyers. A consumer buys the new product if her valuation of it is not below the price of the product announced in a given period. Our model attributes the empirical finding of a delayed "take-off" of a new product to a drift of the percolation dynamics from a non-percolating regime to a percolating regime. This drift is caused by learning effects lowering the price of the product, or by network effects increasing its valuation by consumers, both with an increasing number of buyers.

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1 Introduction

The understanding of innovation is central in the study of capitalistic economies (see Schumpeter (1911, 1942)). In particular, there exists a rich literature analyzing the incentives for industrial innovation, starting with Arrow (1962). In the present paper, however, we take as given that a new product has emerged and concentrate on the time-profile of its spread in a population of consumers.

The analysis of the process of adoption of a new product (in the following termed *new-product diffusion*) constitutes an important research area in both marketing science and economics. From a practitioner's perspective, relevant questions are, for instance, how to forecast whether the new product will "take-off" (see Garber et al. (2004)), or, once it did, the level of its future sales depending on the use of elements of the marketing mix (see e.g. Bass et al. (2000); see also Chandrasekaran and Tellis (2005) for a general overview). From a more theoretical perspective, one is interested, for instance, in why consumers develop preferences for new products (see Witt (2001)), or whether adoption processes tend to be "path-dependent" or "ergodic" (see David (1985)).

Three main approaches to quantitative modeling of the time-profile of new-product diffusion can be distinguished: First, the essentially phenomenological models of new-product diffusion, starting with the contribution of Bass (1969). The Bass-model has seen numerous refinements over the years (for an overview, see Mahajan et al. (1990, 1995)), and can reproduce the evolution of sales over a wide range of the product life cycle employing appropriate parameter fits. Second, micro-models of new-product diffusion focusing on rational individual decision-making (see, for instance, David and Olsen (1986, 1992)). These models typically ascribe to consumers a high degree of sophistication, in particular consumers correctly foresee the future evolution of the market. The dynamics of diffusion is driven by the interplay of expectations and utility maximization. Third, stochastic micromodels of new-product diffusion which focus on collective effects, often with a simplistic specification of decision making. These most recently proposed models are variants of the spatial stochastic process called *percolation*[2] (see e.g. Allen (1982), Mort (1991), David and Foray (1994), Solomon et al. (2000), Goldenberg et al. (2000), Silverberg and Verspagen (2005)).

Our present model falls in the percolation-based category (yet it features a model of the individual adoption decision). It aims to explain the empirical phenomenon that in the early stages of new-product diffusion low levels of sales often persist over a prolonged period of time before a "take-off" occurs (for a detailed discussion of this phenomenon see Golder and Tellis (1997) and Geroski (2003)). Serving as a prototypical example of this phenomenon, Figure 1 (top) depicts the cumulative frequency of adopters of a novel agricultural technique in Iowa in the first half of the last century. The data in Figure 1 (top) is adapted from Ryan and Gross (1943). (More examples of long-tailed diffusion curves along with a discussion of the phenomenon of a delayed "take-off" of new products can be found in Mort (1991) and Golder and Tellis (1997).) Our model provides a possible analytical explanation for delayed take-off in new-product diffusion with an underlying myopic individual decision-making model, i.e. avoiding a self-fulfilling-prophecy mechanism relying on rational expectations. Our model attributes the empirical finding of a delayed "take-off" of a new product to a drift of the percolation dynamics from a non-percolating regime to a percolating regime. This drift is caused by learning effects lowering the price of the product, or by network effects increasing its valuation by consumers, both with an increasing number of buyers. Up to our knowledge, the proposed model is the first one capable of explaining delayed take-off as a purely collective coordination phenomenon.[3]

The structure of the paper is as follows: Section 2 specifies the basic model. Section 3 introduces macroscopic feedbacks and shows by Monte Carlo simulations that the latter can lead to a diffusion-dynamics exhibiting a delayed take-off. The paper concludes with a brief discussion of some additional aspects of our model.

2 The basic model

We model the process of diffusion of a new product[4] (the emergence of which is *assumed* rather than explained in our model) among a large population of consumers. Time is discrete. In any period t, a consumer may buy either one unit of the product or none, with at most one unit bought over the entire time horizon. The individual decision model of a consumer consists of three steps: firstly, learning the product's characteristics, secondly, forming an individual (subjective) valuation of it, and thirdly, comparing one's individual valuation with the price set by the producer.

An essential ingredient of our model is a local "spatial"[5] stochastic dynamics triggering the individual assessment of the product's value by each consumer (i.e. potential buyer). Underlying such a dynamics is a social network – left *exogenous* in the present paper[6] – modeled as a graph. We specify this graph as a two-dimensional square lattice. In the infinite version of the model, $\mathbb{Z}^2$ represents the set of nodes (consumers), with a link between any two $a, b \in \mathbb{Z}^2$ existing if and only if $||a - b|| = 1$, with $|| \cdot ||$ denoting the Euclidean distance in $\mathbb{R}^2$. Two consumers who are directly linked are called *nearest neighbors.* Each consumer has thus four nearest neighbors. Our particular choice of the network model is presumably not a realistic one. Alas, we are not aware of empirical studies investigating topologies of interactions in our particular context, while results of studies investigating social network topologies related with other types of human interactions do not appear to be a-priori transferable (see Schnegg (2006)). Yet the principle mechanism by which a "take-off dynamics" is generated in our model with feedbacks (see Section 3) does not depend on the specific topology of the underlying network (with an exceptional case to be discussed in the last section of this paper).

While the benchmark percolation model is defined on the infinite lattice, to investigate properties of our model by Monte-Carlo simulations the set of potential buyers is represented by a finite square-shaped subset $\Lambda \subset \mathbb{Z}^2$. Consumers at the boundary of Λ are also linked with consumers at the opposite boundary of the square.

In the first period, the dynamics is initialized by the introduction of a fixed number of early buyers located randomly in the population. The origin of such "early birds" is exogenous to our model. In each of the subsequent periods, the nearest neighbors of those consumers who bought the product in the immediately preceding period acquaint themselves with the product by observing or being exposed to its usage.[7] Based on that experience they form their individual valuations of the product reflected in the reservation price θ_a (i.e. the highest price at which consumer a would buy). The formation of the individual valuation θ_a is made only once – thus it is not reassessed if in a later period another nearest neighbor of consumer a buys the product. We assume in our model that the transfer of "experience of the product" from one consumer to another is "neutral" in the sense that the valuation formed by consumer a does not depend on the valuation of that buyer who triggered the formation of a's valuation. Thus we specify that θ_a is a realization of the random variable Θ_a with the family $(\Theta_a)_{a\in\Lambda}$ independently identically distributed. To directly relate our basic model to the standard percolation model which will be briefly explicated below, each random variable Θ_a is equi-distributed on $[0, 1]$.

Finally, the consumer's decision to buy the new product is the following: consumer a buys the product if her individual valuation θ_a exceeds or equals the price p of the product. Thus, the decision to buy the new product is "socially transmitted" only in the sense that it is by experiencing the product via one's immediate social environment that a consumer forms the valuation of it.

We employ a simple specification of the supply side as consisting of a "non-maximizing"[8] monopolist using mark-up pricing i.e. the price p is given by the formula

$$p = (1 + m)c \tag{1}$$

with c denoting the unit production costs and m a positive number, the time-constant mark-up. (See Blinder (1991) and Hall et al. (1997) for empirical evidence that firms indeed use mark-up pricing.) In accordance with the specification of the range of individual valuations we assume that $p \in [0, 1]$.

The model specified so far can be easily mapped on a well-known mathematical model called *site percolation*, and we shall briefly describe some of its basic properties. In the simplest case of an (atemporal) site-percolation model with some underlying countably infinite graph structure, say a square lattice $\mathbb{Z}^2$, each site of the graph is randomly assigned a value from $\{0, 1\}$, with probability P for a realization of the value 1. A site to which the value 0 is assigned, can, for instance, represent a site which is in some sense "non-active", while a site to which the value 1 is assigned is "active". The assignment of each value is stochastically independent of the values assigned to other sites. *Percolation* is said to occur if there appears at least one infinite unbounded cluster[9] of "active" sites. It turns out that there is a threshold-value for the probability P, denoted by P_c, such that such an infinite cluster of "active" sites occurs with probability 1 for $P > P_c$ and with probability 0 for $P < P_c$ (see Grimmett (1999) and Stauffer and Aharony (1994)). For the particular graph structure specified in the paper (two-dimensional square lattice) we have approximately $P_c = 0.592743$.

To apply Monte-Carlo techniques for the analysis of percolation models, several instances of stochastic processes on finite subsets of $\mathbb{Z}^n$ were proposed enabling to decide whether or not percolation occurs in a given (possibly infinite) model based on samples of the process. Such processes are typically initiated with an "active" site located in the center of the graph, and in each time step nearest neighbors of an active site may themselves become active with probability P, independently of each other, with the state of each site realized only once in time (see Leath (1976)). For such processes, the percolation threshold P_c constitutes a value for the "transmission probability" above which "active" sites spread over the entire graph with a significant probability, but "die out" below of it, unless for extremely rare instances. Indeed, the dynamics of our basic model specified above corresponds to the Leath-algorithm of percolation.

Let us now return to our particular model context. The probability for a consumer to buy the product, given she comes to form her valuation (the latter condition is referred to as C), is the probability that her valuation θ_a falls into the interval $[p, 1]$. Thus $\text{Prob}(a \text{ buys}|C) = 1 - p$. The above quoted results imply that in the new-product diffusion model specified so far there is a threshold value for the price p such that for $p > p_c$ the diffusion of the product will "die out" but will spread over the population for $p < p_c$. We have $p_c = 1 - P_c$, the numerical value being approximately $1 - P_c = 0.407$. A generic time profile of the adoption dynamics of the basic model is illustrated in Figure 2: percolation occurs for $p = 0.39$, but does not occur for $p = 0.52$. Figure 3 (top) depicts the final share of buyers as a function of the price. A drastic decrease of that share occurs at $p_c = 0.407$.

Note that while the functional form of the time-profile of sales in our model depends on the particular network structure, the occurrence of spread over the entire population of consumers depends only on whether the prevailing price p is above or below p_c (with p_c depending on the particular type of network).

In the next section we will extend our basic model by macroscopic feedbacks which can affect the price or the valuation (or both). It turns out that this feature can produce a "drift" of the percolation dynamics from a "non-percolating regime" to a "percolating regime", thereby facilitating a dynamics corresponding to a delayed "take-off".

3 New-product diffusion with macroscopic feedbacks

In the following we introduce two types of macroscopic feedback mechanisms. We specify a feedback affecting the supply side by assuming that unit production costs decrease with the cumulative quantity of units already produced. The decrease of unit production costs is empirically well established and explained by learning within the firm. Decreasing unit production costs are associated with the "learning curve" (see e.g. Yelle (1979)) and with the related notion of "economies of scale" (see e.g. Scherer and Ross (1990)). In our model, the "learning curve" is represented by a functional relationship $c_t = f(N_{t-1})$ with N_{t-1} denoting the number of consumers who bought the product up to period $t-1$. To represent empirical data, the function f (assumed to be at least twice differentiable) should satisfy $f(x) > 0, f'(x) < 0$ and $f''(x) > 0$ for the non-negative real numbers. The time-dependent price obtains from Eq. 1 as

$$p_t = (1+m)f(N_{t-1}). \tag{2}$$

We specify a feedback affecting the demand side based on the notion of "network externalities" (David (1985), Katz and Shapiro (1992)): For each consumer a, the initial valuation θ_a increases by an amount proportional to the fraction of buyers in the population. We have thus a time-dependent individual valuation

$$\theta_{a,t} = \theta_a + \mu \frac{N_{t-1}}{N}, \tag{3}$$

with some constant μ which we assume to be independent of a; N denotes the total number of potential buyers. Note that with the above assumption on the distribution of Θ_a it is not the case that $\theta_{a,t} \in [0,1]$ always holds, see Eq. 4.

Depending on the nature of the product considered, either one of the feedback effects might vanish. For instance, computer software presumably exhibits only the second kind of feedback effect, while household electronics exhibit only the first.

Because in the extended model the price and the individual valuation may be time-dependent the decision rule reads: consumer a buys in t with $t \geq t_a$ (with t_a denoting the time period in which consumer a learns the product's characteristics and forms an initial valuation) if

$$\theta_{a,t} \geq p_t \quad \text{and} \quad \theta_{a,t} < p_\tau \quad \text{for all } \tau : t_a \leq \tau < t,$$

i.e. a consumer buys the product as soon as her individual valuation equals or exceeds the price. Thus, unlike in the basic model, a consumer may buy in a period subsequent to the one in which she formed her (initial) valuation. The probability of buying thus increases over time. For a consumer who forms her evaluation in period t (condition C), the probability to buy in period t obtains as

$$Prob(a \text{ buys in } t|C) = \begin{cases} 1 - p_t + \mu\frac{N_{t-1}}{N} & \text{if } 0 \leq 1 - p_t + \mu\frac{N_{t-1}}{N} \leq 1 \\ 0 & \text{if } 1 - p_t + \mu\frac{N_{t-1}}{N} < 0 \\ 1 & \text{if } 1 - p_t + \mu\frac{N_{t-1}}{N} > 1. \end{cases} \tag{4}$$

In each period subsequent to the period in which valuation is formed, the decision of a consumer who has not yet bought might be revised.

Note that the two types of macroscopic feedback effects are mathematically equivalent in the sense that with increasing N_{t-1} the existing gaps between the price of the product and individual valuations of consumers who have not yet bought the product tend to vanish. For that reason, many qualitative results do not depend upon which type of feedback is considered.

Using Monte-Carlo simulations, we find for the model with feedback(s) the general result that there exists a range of initial prices (for instance, for the model with supply-side feedback and the particular parameter specification as depicted in Figure 3 (bottom) between approximately 0.407 and 0.529) for which the product "takes off" eventually, despite it would not take-off in the basic model of Section 2. For this range of initial prices, the per-period sales curve exhibits two specific phases. First, a very low sales level persists corresponding to the system being in the non-percolating regime. The dynamics may exhibit a temporary decrease of per-period sales resulting from local diffusion seeds (initiated by exogenous initial buyers) which "die out" before the system reaches the percolating regime. Second, a "take-off" phase occurring when diffusion seeds which "survived" long enough enter the percolating regime. The general principle underlying that delayed take-off behavior is thus that the diffusion dynamics may "drift" from the non-percolating regime to the percolating regime. This drift occurs because the probability of buying increases over time with the cumulative number of buyers and because consumers who did not buy initially can revise their decision later on.

In the remainder of this section, we present Monte Carlo simulations of the model with feedback for particular parameter values. We maintain the assumption that the initial individual valuation θ_a is equi-distributed on $[0, 1]$.[10] Figure 1 (bottom) depicts a diffusion curve, averaged over 500 simulation runs, in the model with macroscopic feedback affecting only the demand side with one initial buyer in period $t = 1$, a 501×501 lattice, a time-independent price $p = 0.433$ and the parameter μ equal to 0.4. The reader may think of this averaged curve as modeling new-product diffusion in a population located in many cities with network externalities affecting the population within a single city only. A comparison of Figure 1 (top) and (bottom) illustrates that our model can explain long flat tails empirically observed in the early stages of new-product diffusion.

Figure 4 depicts the averaged evolution of per-period sales (left-hand side) and cumulative sales (right-hand side) resulting from a specification with one initial buyer in period $t = 1$ on a 501×501 lattice. Macroscopic feedback affects the demand side only; the time-independent price is set to $p = 0.435$ (top) and $p = 0.421$ (bottom) and the constant μ describing the influence of network externalities equals 0.4. For both prices, the curves are obtained by averaging over 500 simulation runs. Note that the threshold product price being approximately 0.407, for both the new product would not spread over the population in the basic model without macroscopic feedbacks. However, because individual valuations increase with the number of buyers N_t, some simulation runs persist up to the point where the additional term in Eq. 3 closes the gap between the average evaluation and the price, so that spread of the product occurs. The length of the long left tail increases with p and decreases with μ ceteris paribus. A comparison of Figure 4 (top) with Figure 4 (bottom) exemplifies the first part of this statement. Furthermore, the decrease of per period sales in the first phase as visible in Figure 4 (top, left-hand side) increases with increasing price.

Figure 5 depicts three curves corresponding to per-period sales, cumulative share of buyers and the evolution of the product price resulting from a single simulation run in a setting with macroscopic feedbacks affecting the supply side only. We specify the time-dependent price p_t (see Eq. 2) as

$$p(n_{t-1}) = p_0 - qn_{t-1} + \alpha n_{t-1}^2, \tag{5}$$

with the fraction of buyers $n_{t-1} = \frac{N_{t-1}}{N}$, the initial price $p_0 \in [0, 1]$ and $q > 0$ and $\alpha > 0$ constant parameters. Figure 5 corresponds to the parameter values $q = 0.5$, $p_0 = 0.52$ and $\alpha = 0.095$. The initial number of buyers equals 3000 and lattice size is 1501×1501. Initial price p_0 is set to 0.52. Figure 5 shows that the characteristic take-off dynamics displayed by the averaged curves of Figure 4 can be obtained from a single simulation run. This is significant because in the case of macroscopic feedbacks affecting the price, averaging over multiple simulation runs is difficult to justify (as it would involve a scenario with different price sequences in different "cities", so that the issue of arbitrage opportunities would become relevant).

4 Discussion

We conclude the paper with two comments. First, while the functional form of the time-profile of sales in our model might depend on the particular underlying topology, the phenomenon of delayed "take-off" itself does not (except for possible changes of the parameter range for which it occurs). This is because it relies solely on the existence of a percolation threshold p_c separating a percolating regime from a non-percolating regime of the dynamics. It is the passage of the dynamics from the former to the latter that facilitates the "take-off" phenomenon. However, for a certain type of graph structures – called *scale-free* networks – the percolation threshold tends to zero with a growing number of sites (see Cohen et al. (2000)). Thus, delayed "take-off" would not occur in our model with a scale-free network representing the topology of local interactions. It is tempting to empirically test this implication of our model, once comparative studies on interaction/communication topologies related with different product categories or technologies are available.

Second, though in our model individual valuations of the new product made by consumers are not subject to *local* social influence (by which a consumer's valuation is directly affected by those of her next neighbors), we do believe that such effects are realistic (for interesting phenomena resulting from such local interdependencies in valuation see, for instance, Goldenberg at al. (2000), Solomon et al. (2000) and Erez et al. (2006)). However, the present paper aims at explaining the occurrence of delayed take-off as simply as possible.

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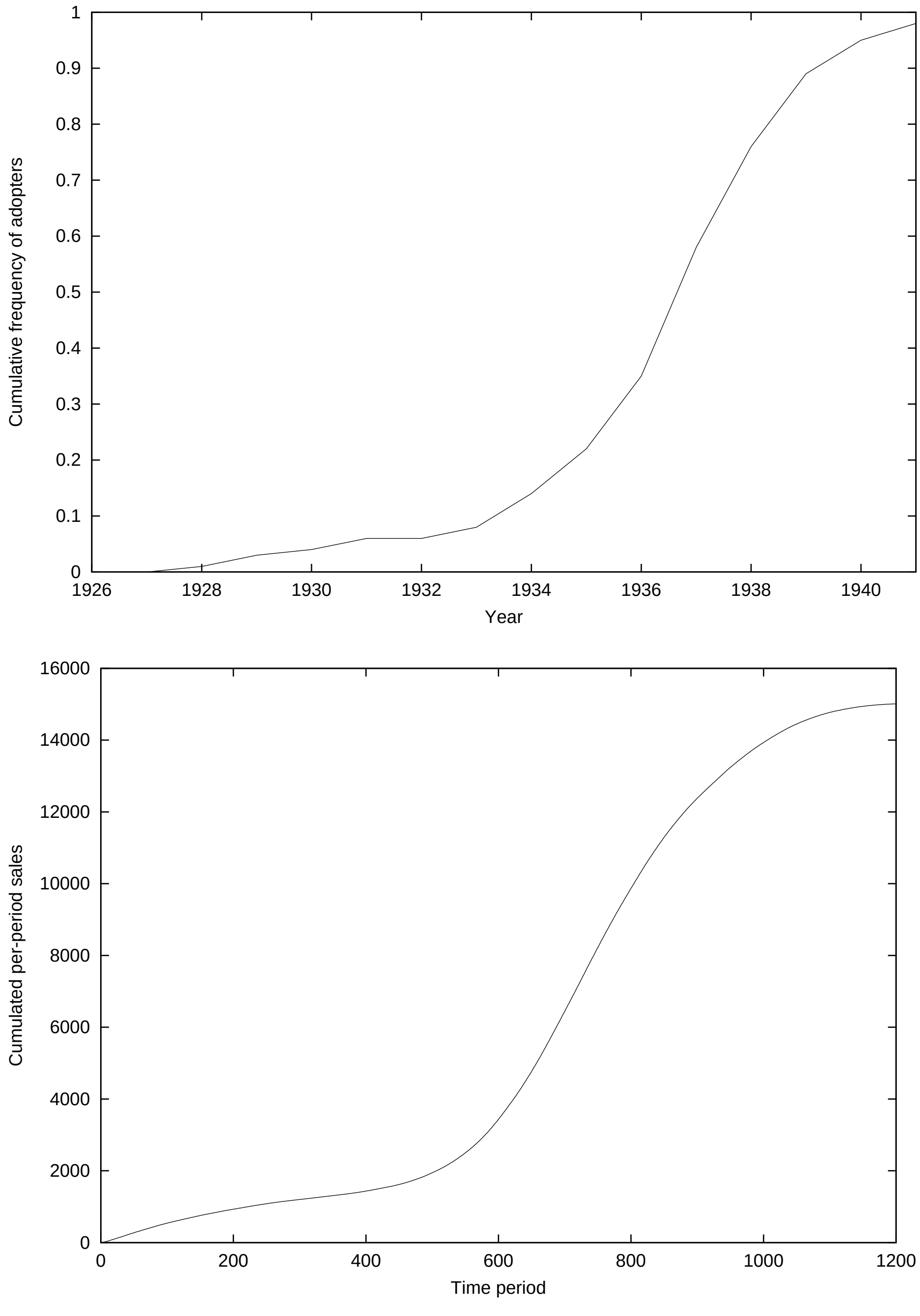

Figure 1: (top): Cumulative frequency of adopters for the diffusion of hybrid corn seed in two Iowa farming communities adapted from [25]; (bottom): cumulative number of buyers in our model with demand-side feedback and the parameter values $p = 0.433$ and $\mu = 0.4$; data averaged over 500 simulation runs on a 501×501 lattice (bottom) with one initial buyer in each simulation run.

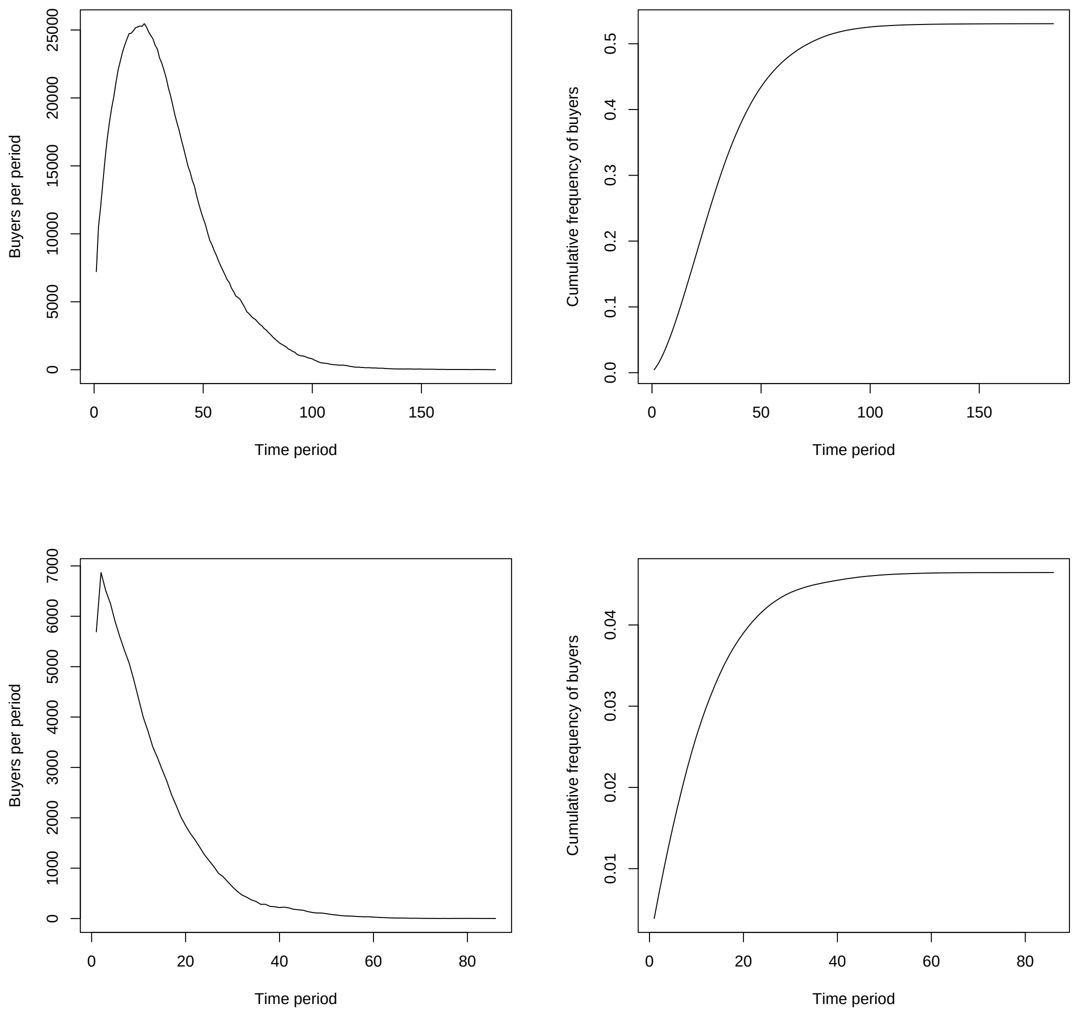

Figure 2: Per-period number of buyers (left-hand side) and cumulative shares of buyers over time (right-hand side) in the basic model of Section 2; curves correspond to a single simulation run on a 1501×1501 lattice with 3000 initial buyers; percolation occurs for a price p=0.39 (top) but does not occur for p=0.52 (bottom).

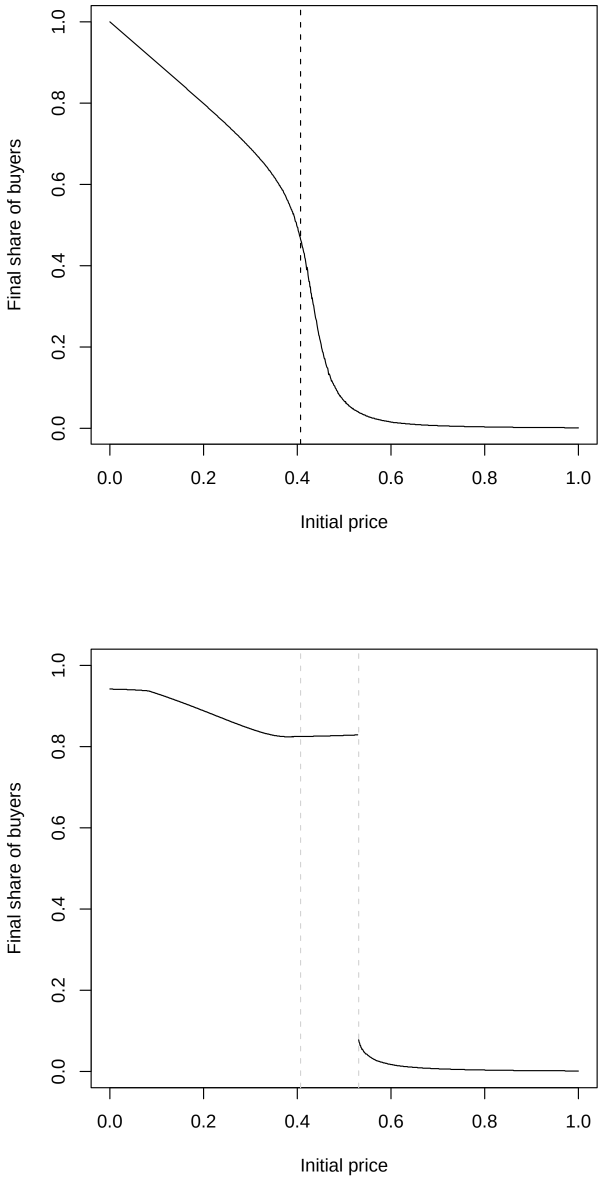

Figure 3: (top): Total (final) share of buyers as a function of price in the basic model (without macroscopic feedback) of Section 2; each point of the curve corresponds to a single simulation run on a 1501×1501 lattice with 3000 initial buyers; (bottom): total (final) share of buyers as a function of initial price in the model with supply-side-feedbacks and parameters as in Figure 5; in the bottom figure delayed take-off occurs for initial prices in the range enclosed by the two dashed lines, that is for initial prices between 0.407 and 0.529.

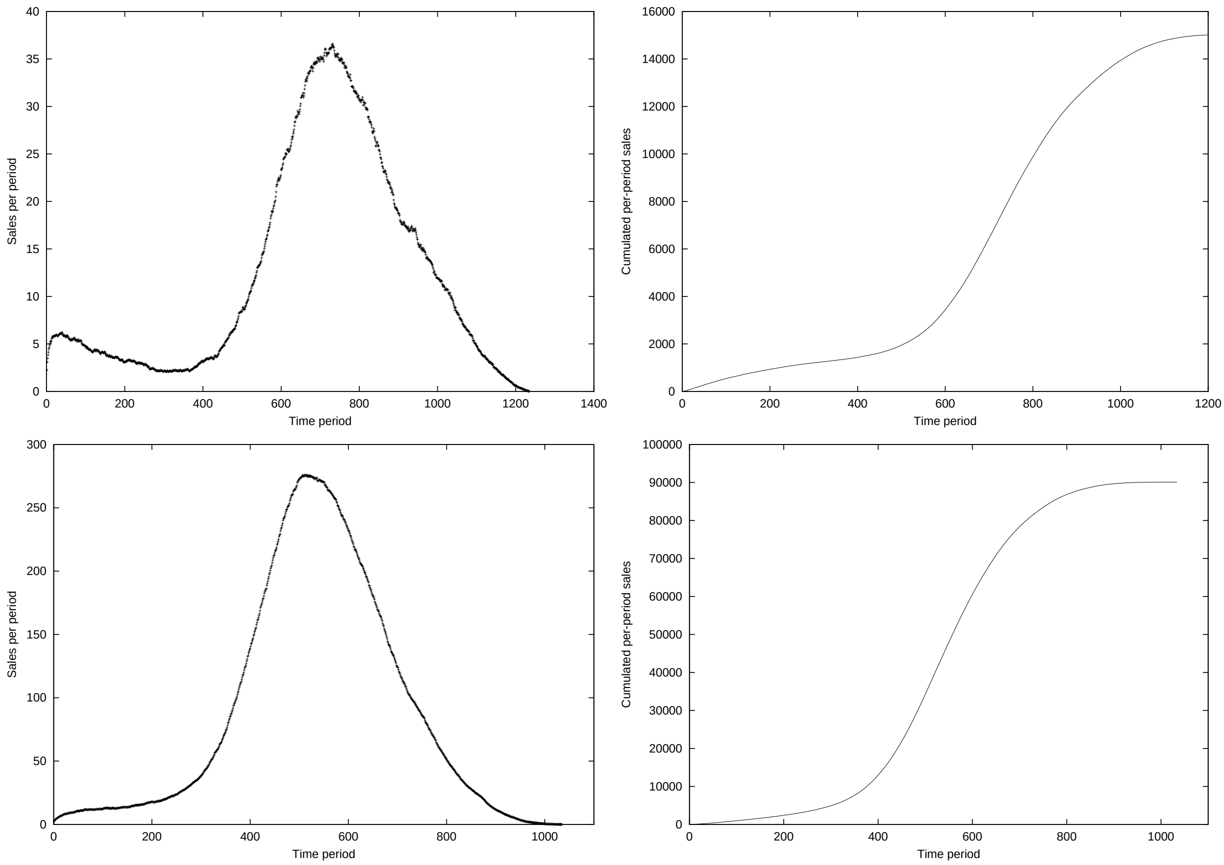

Figure 4: Per-period number of buyers (left-hand side) and cumulative number of buyers (right-hand side) in our model with parameter values $p = 0.435$ and $\mu = 0.4$ (top) and $p = 0.421$ and $\mu = 0.4$ (bottom) on a 501×501 lattice; data points correspond to the average over 500 simulation runs each with one initial buyer.

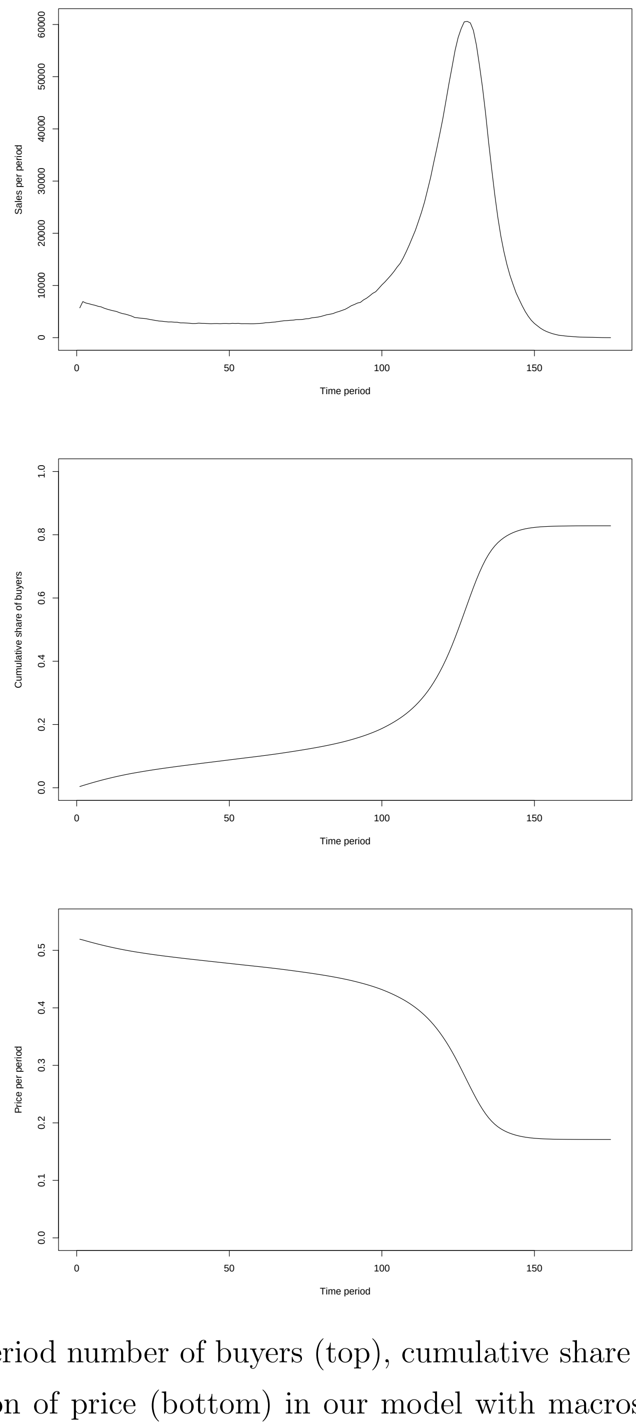

Figure 5: Per-period number of buyers (top), cumulative share of buyers (middle) and the evolution of price (bottom) in our model with macroscopic feedback affecting supply side only; the curves corrrespond to a single simulation run with 3000 initial buyers, lattice size 1501×1501 and initial price $p_0 = 0.52$.

Notes

[1] We are grateful to Paul A. David for many valuable comments on an earlier version of this paper. Any conceptual and technical shortcomings of the paper belong to us.

[2]For an introduction to percolation and its applications see Stauffer and Aharony (1994). An advanced mathematical treatment of percolation can be found in the monograph by Grimmett (1999).

[3]Delayed take-off of new products has been explained in the model of David and Olsen (1986, 1992) in the context of rational expectations.

[4]We believe that our diffusion model can be applied to the more general issue of diffusion of innovations. However, that more general context would require a more specific analysis of the question of why and how innovations get adopted (see, for instance, Nelson et al. (2004)). In the present paper in the context of a new product, we confine ourselves to specifying an abstract framework using the notion valuation which is popular in abstract decision models in economics.

[5]The type of stochastic process which is considered in the present paper is usually referred to as a *spatial* or *spatio-temporal process*, though the distance between individual entities may have very different interpretations in particular aplications. In the present paper, the distance is a social distance between consumers, which is not necessarily related to the geographical distance between them.

[6]An interesting question is how such networks emerge in social systems. This question is beyond the scope of our present investigation, but see, for instance, the paper of Schnegg (2006) for an investigation of this question in a different social context.

[7]We assume that buyers enable all their nearest neighbors to experience the product corresponding to the case of pure site-percolation, i.e. bonds are always "open".

[8] The standard textbook model of a profit-maximizing monopolist is based on the assumption that the monopolist knows the demand function. In the situation where the demand function is strongly time-dependent, this assumption is not a realistic one, and different approaches – such as mark-up pricing – appear well suited.

[9]A cluster is a set of connected sites all of them featuring a certain property.

[10]With this distribution being, for instance, a truncated normal distribution on $[0, 1]$ all qualitative results were reestablished.